



Capital Accumulation with Constraint: A Mean Field Type Control Framework

Alain Bensoussan^{1,*}, SingRu (Celine) Hoe² and
Zhongfeng Yan^{3,†}

¹ International Center for Decision and Risk Analysis, Jindal School of Management,
University of Texas at Dallas

² Texas A&M University – Commerce

³ Jinan University, Guangzhou, China

Received August 22, 2018, revised March 28, 2019

Abstract. To succeed in a competitive business environment, optimal capital investment plays a significant role. A firm cannot ignore the penalty associated with carrying excessive or insufficient production capacity. We provide a model of the optimal rate of capital investment under uncertainty incorporating a penalty to study the key impact. The penalty is modeled as a squared deviation between the expected and the desired levels. The payoff functional thus incorporates a nonlinear function of the expected capital level. This control problem is of the mean field type. We obtain a closed form solution by a direct method. As expected for mean field type control problems, the optimal feedback depends not only on the current states, but also on the initial conditions. We perform numerical studies to quantitatively address how risk control in capital level deviation affects the optimal investment policy.

KEYWORDS: mean field type control, capital investment, time-inconsistent solution, Gâteaux differential

*also with the College of Science and Engineering, Systems Engineering and Engineering Management, City University Hong Kong. Research supported by the National Science Foundation under grant DMS-1905449 and the Research Grants Council of the Hong Kong Special Administrative Region (CityU 11303316). *He has the pleasure to dedicate this work to Professor Guy Fayolle.*

[†]Department of Mathematics, Jinan University, Guangzhou, PR China (zfyang0310@gmail.com). This author is supported by National Natural Science Foundation of China (Grant Nos. 11601186, 11426115). Part of the work was done during visiting KdV Institute for Mathematics, University of Amsterdam (Supported by China Scholarship Council, No. 201706785021).

AMS SUBJECT CLASSIFICATION: 91G50, 91G80, 93E20, 49J55, 49K99

1. Introduction

When firms make capital investment decisions under uncertainty, they must deal with risk management. Carrying excessive capital capacity would endanger the return on assets of a firm due to poor asset turnover. Insufficient capital stock harms the profitability of a firm such as failing to seize a growth opportunity or undertaking a costly expansion. It is thus tempting for a firm to operate by maximizing operating revenue while minimizing risk (i.e., curtailing penalty) that may generate form undesirable capital deviations over time. Although there is impressive volume of work studying how adjustment costs and irreversibility affect a firm's optimal investment under uncertainty (e.g. Cooper and Haltiwanger [11], Bo and Lensin [7], and Caballero [10]...etc.), little work exists in the literature on how risk originating from fluctuations in the capital level affects the optimal capital investment (eg. Bensoussan, Hoe and Yan [6]).

We study the optimal rate of capital investment of a firm facing uncertainty with respect to risk control in undesirable capital level fluctuations. To accomplish the goal, we extend the study of Abel and Eberly [2] by adding a penalty term to the payoff functional. The penalty term, accounting for undesirable deviations from the desired capital level, is modeled through a squared deviation between the expected and the desired levels. The control problem presented is no longer a stochastic control problem, but rather a mean field type control problem, which is time inconsistent. As a consequence, the dynamic programming method for the classical control cannot be applied here.

Due to the simplifying feature that the control density does not exist in our model formulation, we use the direct method to solve our mean field type control problem¹ explicitly in which the solution is sought as in the calculus of variations, computing the Gâteaux differential. The direct method that we use yields the optimal control and the optimal value directly. We show that this method leads to the same result as that obtained from an HJB equation for the standard stochastic control problem, and it is extendable to the mean field type control problem. We are able to arrive at a closed form solution for our mean field type control problem. Unlike Abel and Eberly [2], the optimal feedback (i.e., the optimal rate of capital investment) depends not only on the current state of the output price process, but also on the initial conditions of the output price and capital levels.

In the following, we first revisit the model of Abel and Eberly [2] with the direct method, which bypasses dynamic programming and thus avoids writing

¹There are several methods for solving mean field type control problems, for example, the maximum principle (e.g. Andersson and Djehiche [3], Buckdahn Djehiche and Li [9], Li [14], Buckdahn, Li and Ma [8], among others), the HJB-FP (Hamilton-Jacobi-Bellman-Fokker-Planck) system introduced by Bensoussan, Frehse and Yam [5]...etc.

an HJB equation. We then present our mean field type control problem. We modify the payoff functional studied in Abel and Eberly [2] to account for the penalty due to the deviation of the expected capital level from the desired level, which is the main interest of our study. We present in detail the implementation of the direct method to obtain the closed form optimal policy. Finally, we demonstrate the significance of risk control in capital level deviation on the optimal investment policy through numerical studies. Managerial insights are inferred accordingly.

2. The model of Abel and Eberly

2.1. Setting of the problem

We first briefly review the model of Abel and Eberly [2]. A firm decides its rate of capital investment (the control) denoted by $v(t)$. The corresponding capital $x(t)$ is defined by

$$\begin{aligned}\frac{dx(t)}{dt} &= v(t) - \delta x(t), \\ x(0) &= x,\end{aligned}\tag{2.1}$$

where the coefficient δ represents the depreciation unit rate. The price of output $y(t)$ follows a geometric Brownian motion described by

$$\begin{aligned}dy(t) &= \mu y dt + \sigma y dw(t), \\ y(0) &= y,\end{aligned}\tag{2.2}$$

where $w(t)$ is a standard Wiener process, and coefficients $\mu > 0$, $\sigma > 0$ represent the growth rate and volatility respectively. The price of output combined with the capital contributes to the profit of the firm. Therefore, the manager considers the evolution of the value $y(t)$ to decide the investment, $v(t)$. So the control, $v(t)$, is random and adapted to the filtration $\mathcal{F}^t = \{\sigma(y(s), s \leq t)\}$. The state of the system is the pair $(x(t), y(t))$ in spite of the fact that $y(t)$ evolves by itself.

The output technology is assumed to be of the Cobb-Douglas form $P(x_t, l_t) = l_t^\xi x_t^{1-\xi}$, where x is the capital input, l is the labor input, and $0 < \xi < 1$. A firm chooses l_t to maximize the instantaneous profit at time t because the labor is assumed to be costless and instantaneously adjustable with a fixed wage, W . Therefore, the instantaneous operating profit is $cy(t)^\gamma x(t)$ where $\gamma = 1(1 - \xi)$ and $c = \gamma^{-\gamma}(\gamma - 1)^{\gamma-1}W^{1-\gamma}$.

In addition, investment in capital stock incurs an additional adjustment cost, which is assumed to be quadratic in the control variable, i.e., $v(t)^2/2$. The payoff is defined by

$$J_{x,y}(v(\cdot)) = E \int_0^{+\infty} e^{-\alpha t} \left[cy(t)^\gamma x(t) - \frac{v^2(t)}{2} \right] dt, \quad (2.3)$$

where α is the discount rate.

The firm maximizes its expected discounted operating revenue over time through capital investment, so the value function is

$$u(x, y) = \sup_{v(\cdot)} J_{x,y}(v(\cdot)). \quad (2.4)$$

We assume $x, y > 0$. The control $v(t)$ is positive, so the state $x(t)$ is positive. Since $v(t) = 0$ is admissible, the value function $u(x, y) > 0$.

The following assumptions and notations will be in force throughout this paper. We assume

$$\rho = \alpha - 2\gamma\mu - \gamma(2\gamma - 1)\sigma^2 > 0. \quad (2.5)$$

2. The control $v(t)$ belongs to the class of adapted processes such that

$$E \int_0^{+\infty} e^{-\alpha t} \frac{v^2(t)}{2} dt < +\infty. \quad (2.6)$$

We denote

$$A = \frac{c}{\alpha + \delta - \gamma\mu - \gamma(\gamma - 1)\sigma^2/2} > 0, \quad \text{and} \quad B = \frac{A^2}{2\rho}. \quad (2.7)$$

2.2. Solution of the problem by direct approach

The classical approach for solving the standard stochastic optimal problem of (2.3) is to write down the HJB equation and to solve it explicitly. However, we proceed differently here by presenting a direct approach. The direct approach is implementable in this model because $x(t)$ is the solution of a simple first order ordinary differential equation. We can write

$$x(t) = xe^{-\delta t} + \int_0^t e^{-\delta(t-s)} v(s) ds. \quad (2.8)$$

We state the result of Abel and Eberly [2] in the following proposition.

Proposition 2.1. *With the assumption (2.5), the value function $u(x, y)$ is finite and given by*

$$u(x, y) = Axy^\gamma + By^{2\gamma}. \quad (2.9)$$

The optimal feedback is given by

$$\hat{v}(t) = Ay(t)^\gamma, \quad (2.10)$$

and therefore is independent of the state $x(t)$.

Proof. We prove (2.9) directly without using dynamic programming. We can, in fact, compute directly $J_{x,y}(v(\cdot))$ for any $v(\cdot)$. Using (2.8) we can write

$$\begin{aligned} J_{x,y}(v(\cdot)) &= E \int_0^{+\infty} e^{-\alpha t} \left[cy(t)^\gamma \left(xe^{-\delta t} + \int_0^t e^{-\delta(t-s)} v(s) ds \right) - \frac{v^2(t)}{2} \right] dt \\ &= cxE \int_0^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dt - E \int_0^{+\infty} e^{-\alpha t} \frac{v^2(t)}{2} dt \\ &\quad + cE \int_0^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma \left(\int_0^t e^{\delta s} v(s) ds \right) dt. \end{aligned}$$

Inverting the order of integration of the last term, we have

$$\begin{aligned} cE \int_0^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma \left(\int_0^t e^{\delta s} v(s) ds \right) dt \\ = cE \int_0^{+\infty} e^{\delta s} v(s) \left(\int_s^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dt \right) ds. \end{aligned} \quad (2.11)$$

From Ito's formula, we get (recalling the definition of c given in (2.7))

$$de^{-(\alpha+\delta)t} y(t)^\gamma = -\frac{c}{A} e^{-(\alpha+\delta)t} y(t)^\gamma dt + \gamma \sigma e^{-(\alpha+\delta)t} y(t)^\gamma dw(t). \quad (2.12)$$

We integrate over t between s and T large, and get

$$\begin{aligned} e^{-(\alpha+\delta)T} y(T)^\gamma - e^{-(\alpha+\delta)s} y(s)^\gamma \\ = -\frac{c}{A} \int_s^T e^{-(\alpha+\delta)t} y(t)^\gamma dt + \gamma \sigma \int_s^T e^{-(\alpha+\delta)t} y(t)^\gamma dw(t). \end{aligned} \quad (2.13)$$

Moreover, from (2.12), we easily observe

$$e^{-(\alpha+\delta)T} y(T)^\gamma = y^\gamma \exp \left[-\left(\frac{c}{A} + \frac{1}{2} \gamma^2 \sigma^2 \right) T + \gamma \sigma w(T) \right].$$

Therefore,

$$E e^{-2(\alpha+\delta)T} y(T)^{2\gamma} = y^{2\gamma} e^{(-2\frac{c}{A} + \gamma^2 \sigma^2)T} = y^{2\gamma} e^{(-2\delta - \alpha - \rho)T},$$

which tends to 0 as $T \rightarrow +\infty$. Thus $\exp\{-(\alpha + \delta)T\} y(T)^\gamma \rightarrow 0$ in L^2 . Since $\int_s^T e^{-(\alpha+\delta)t} y(t)^\gamma dt \rightarrow \int_s^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dt$ a.s., we can pass to the limit in L^2 in equation (2.13) and obtain

$$-e^{-(\alpha+\delta)s} y(s)^\gamma = -\frac{c}{A} \int_s^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dt + \gamma \sigma \int_s^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dw(t).$$

Using this relation in (2.11), we get

$$\begin{aligned}
 cE \int_0^{+\infty} e^{\delta s} v(s) \left(\int_s^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dt \right) ds \\
 &= AE \int_0^{+\infty} e^{\delta s} v(s) \left[e^{-(\alpha+\delta)s} y(s)^\gamma + \gamma \sigma \int_s^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dw(t) \right] ds \\
 &= AE \int_0^{+\infty} v(s) e^{-\alpha s} y(s)^\gamma ds,
 \end{aligned}$$

recalling that $v(s)$ is adapted. Collecting results, we obtain

$$\begin{aligned}
 J_{x,y}(v(\cdot)) &= cxE \int_0^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dt + AE \int_0^{+\infty} v(t) e^{-\alpha t} y(t)^\gamma dt \quad (2.14) \\
 &\quad - E \int_0^{+\infty} e^{-\alpha t} \frac{v^2(t)}{2} dt.
 \end{aligned}$$

We note also that $cE \int_0^{+\infty} e^{-(\alpha+\delta)t} y(t)^\gamma dt = Ay^\gamma$; hence

$$J_{x,y}(v(\cdot)) = Axy^\gamma + E \int_0^{+\infty} e^{-\alpha t} \left[Av(t)y(t)^\gamma - \frac{v^2(t)}{2} \right] dt. \quad (2.15)$$

From this expression, it is immediate that the optimal control is $\hat{v}(t) = Ay(t)^\gamma$ and thus

$$u(x, y) = Axy^\gamma + \frac{A^2}{2} E \int_0^{+\infty} e^{-\alpha t} y(t)^{2\gamma} dt = Axy^\gamma + \frac{A^2}{2\rho} y^{2\gamma},$$

which completes the proof of (2.9). $\square$

3. Mean field type control model

3.1. Setting of the model

We now study the optimal rate of capital investment for a firm which faces uncertainty and gets penalized if its expected capital level deviates from its fixed desired level. We define the following payoff functional

$$J_{x,y}^\varepsilon(v(\cdot)) = J_{x,y}(v(\cdot)) - \frac{\varepsilon}{2} \int_0^{+\infty} e^{-\alpha t} |Ex(t) - \bar{x}|^2 dt. \quad (3.1)$$

We have added to the functional $J_{x,y}(v(\cdot))$ (cf. (2.3)) a penalty term to minimize the distance between the expected capital level, $Ex(t)$, and a desired capital level, $\bar{x}$. This penalty term accounts for the costs of the undesirable capital level deviation. The squared distance is necessary since the deviation from

the desired fixed capital level in either directions harms the performance of a firm. In (3.1), ε measures the magnitude of the penalty. It can be viewed as a surrogate of a manager's risk aversion. The more risk averse a manager is, the larger is the ε . The penalty term is a nonlinear function of the expected value of $x(t)$. Because of this term, this is not a standard control problem, but a mean field type control problem which is time inconsistent. In sections that follow, we show that the direct method of Proposition 2.1 extends to this type of problem.

Remark 1. When we say that feedback is time consistent or inconsistent, we refer to a function of x and time t only ($x(s)$ is the state process with $x(t) = x$). Introducing the probability law of the state process in the feedback will make it time consistent, but for a different state space.

Remark 2. It is true that a large crowd with a benevolent dictator leads to a mean field type control. However, in our work, we are dealing with a risk management problem. It turns out that it is a mean field type control.

3.2. Computation of Gâteaux differential

The functional $J_{x,y}^\varepsilon(v(\cdot))$ of (3.1) can be written explicitly as

$$\begin{aligned} J_{x,y}^\varepsilon(v(\cdot)) &= Axy^\gamma + E \int_0^{+\infty} e^{-\alpha t} \left[Av(t)y(t)^\gamma - \frac{v^2(t)}{2} \right] dt \\ &\quad - \frac{\varepsilon}{2} \int_0^{+\infty} e^{-\alpha t} \left(xe^{-\delta t} - \bar{x} + \int_0^t e^{-\delta(t-s)} Ev(s)ds \right)^2 dt. \end{aligned} \quad (3.2)$$

We next compute the Gâteaux differential at argument $\hat{v}(\cdot)$. We have

$$\begin{aligned} \frac{d}{d\theta} J_{x,y}^\varepsilon(\hat{v}(\cdot) + \theta v(\cdot))|_{\theta=0} &= E \int_0^{+\infty} e^{-\alpha t} v(t) (Ay(t)^\gamma - \hat{v}(t)) dt \\ &\quad - \varepsilon \int_0^{+\infty} e^{-\alpha t} \left(xe^{-\delta t} - \bar{x} + \int_0^t e^{-\delta(t-s)} E\hat{v}(s)ds \right) \int_0^t e^{-\delta(t-s)} Ev(s)ds dt, \end{aligned} \quad (3.3)$$

and by inverting the order of integration in the last integral, we get

$$\begin{aligned} \frac{d}{d\theta} J_{x,y}^\varepsilon(\hat{v}(\cdot) + \theta v(\cdot))|_{\theta=0} &= E \int_0^{+\infty} e^{-\alpha t} v(t) (Ay(t)^\gamma - \hat{v}(t)) dt \\ &\quad - \varepsilon E \int_0^{+\infty} e^{-\alpha t} v(t) \end{aligned} \quad (3.4)$$

$$\times \left(\int_t^{+\infty} e^{-(\alpha+\delta)(s-t)} (xe^{-\delta s} - \bar{x} + \int_0^s e^{-\delta(s-\tau)} E\hat{v}(\tau) d\tau) ds \right) dt.$$

3.3. Optimality condition

The functional (3.2) is strictly concave in the Hilbert space

$$V = \left\{ v(\cdot) \mid v(t) \text{ is adapted to } \mathcal{F}^t, E \int_0^{+\infty} e^{-\alpha t} |v(t)|^2 dt < +\infty \right\}.$$

We note that, because of the penalty term, it is worthwhile to introduce the possibility of disinvesting, i.e., when x is much larger than $\bar{x}$. So we drop the constraint $v(t) \geq 0$. We assume, of course, that this is physically possible. Otherwise, the solution will be considered as an ideal benchmark.

The optimal $\hat{v}(\cdot)$ satisfies

$$\frac{d}{d\theta} J_{x,y}^\varepsilon(\hat{v}(\cdot) + \theta v(\cdot)) \Big|_{\theta=0} = 0, \forall v(\cdot) \in V. \quad (3.5)$$

It follows from formula (3.4) that $\hat{v}(t)$ must satisfy the equation

$$\hat{v}(t) = Ay(t)^\gamma - \varepsilon \left(\int_t^{+\infty} e^{-(\alpha+\delta)(s-t)} (xe^{-\delta s} - \bar{x} + \int_0^s e^{-\delta(s-\tau)} E\hat{v}(\tau) d\tau) ds \right), \quad (3.6)$$

whose solution is derived in the next section.

4. Optimal control

4.1. Equation for the mean

Since the term in ε is deterministic in equation (3.6), we get easily by taking the expected value

$$E\hat{v}(t) = Ay^\gamma e^{\gamma(\mu + \frac{1}{2}(\gamma-1)\sigma^2)t} - \varepsilon \left(\int_t^{+\infty} e^{-(\alpha+\delta)(s-t)} \left(xe^{-\delta s} - \bar{x} + \int_0^s e^{-\delta(s-\tau)} E\hat{v}(\tau) d\tau \right) ds \right). \quad (4.1)$$

After integration by parts, we verify the formula

$$\begin{aligned} & \int_t^{+\infty} e^{-(\alpha+\delta)(s-t)} \left(xe^{-\delta s} + \int_0^s e^{-\delta(s-\tau)} E\hat{v}(\tau) d\tau \right) ds \\ &= \frac{1}{\alpha + 2\delta} \left[xe^{-\delta t} + \int_0^t e^{-\delta(t-s)} E\hat{v}(s) ds + \int_t^{+\infty} e^{-(\delta+\alpha)(s-t)} E\hat{v}(s) ds \right]. \end{aligned}$$

Therefore (4.1) becomes

$$E\hat{v}(t) = Ay^\gamma \exp\left[\gamma\left(\mu + \frac{1}{2}(\gamma-1)\sigma^2\right)t\right] - \varepsilon\left(\frac{xe^{-\delta t}}{\alpha+2\delta} - \frac{\bar{x}}{\alpha+\delta}\right) \\ - \frac{\varepsilon}{\alpha+2\delta} \left[\int_0^t e^{-\delta(t-s)} E\hat{v}(s)ds + \int_t^{+\infty} e^{-(\delta+\alpha)(s-t)} E\hat{v}(s)ds \right].$$

If we introduce the function $z(t) = E\hat{v}(t)e^{-(\alpha+\delta)t}$, we obtain the functional equation

$$z(t) = Ay^\gamma \exp\left[-\frac{c}{A}t\right] - \varepsilon\left(\frac{xe^{-(\alpha+2\delta)t}}{\alpha+2\delta} - \frac{\bar{x}e^{-(\alpha+\delta)t}}{\alpha+\delta}\right) \\ - \frac{\varepsilon}{\alpha+2\delta} \left[\int_0^t e^{-(\alpha+2\delta)(t-s)} z(s)ds + \int_t^{+\infty} z(s)ds \right]. \quad (4.2)$$

By computing the first and the second derivative, we can check that $z(t)$ is the solution of the second order differential equation

$$-z'' - (\alpha+2\delta)z' + \varepsilon z = c\left(\delta + \mu\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1)\right)y^\gamma \exp\left[-\frac{c}{A}t\right] + \varepsilon\delta\bar{x}e^{-(\alpha+\delta)t}, \quad (4.3)$$

with the boundary conditions

$$z'(0) = -cy^\gamma + \varepsilon(x - \bar{x}), \quad z'(+\infty) = 0.$$

It can be easily verified that (4.2) and (4.3) are equivalent. Note that, since $z(t)$ is smooth and $z'(+\infty) = 0$, we must have $z''(+\infty) = 0$. The equation then implies $z(+\infty) = 0$.

4.2. Solution of the differential equation

We can solve explicitly the boundary value problem (4.3). We introduce the solutions $\beta > 0$ and $\beta^* < 0$ of the algebraic equation

$$\beta^2 - (\alpha+2\delta)\beta - \varepsilon = 0; \quad (4.4)$$

namely

$$\beta = \frac{\alpha+2\delta + \sqrt{(\alpha+2\delta)^2 + 4\varepsilon}}{2}, \quad \beta^* = \frac{\alpha+2\delta - \sqrt{(\alpha+2\delta)^2 + 4\varepsilon}}{2}. \quad (4.5)$$

Following standard but tedious calculations, we obtain

$$z(t) = \frac{\varepsilon}{\beta} e^{-\beta t} \left[\frac{cy^\gamma}{\varepsilon + (c/A)(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)} - (x - \bar{x}) - \frac{\delta(\alpha+\delta)\bar{x}}{\delta(\alpha+\delta) + \varepsilon} \right] \\ - \frac{c(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)}{(\beta - (c/A))(\beta^* - (c/A))} y^\gamma e^{-ct/A} + \frac{\varepsilon\delta\bar{x}}{\delta(\alpha+\delta) + \varepsilon} e^{-(\alpha+\delta)t}. \quad (4.6)$$

4.3. Optimal feedback

We return to formula (3.6) which we write as follows

$$\begin{aligned} \hat{v}(t) = Ay(t)^\gamma - \varepsilon \left(\frac{xe^{-\delta t}}{\alpha + 2\delta} - \frac{\bar{x}}{\alpha + \delta} \right) \\ - \frac{\varepsilon}{\alpha + 2\delta} \left[\int_0^t e^{-\delta(t-s)} E\hat{v}(s)ds + \int_t^{+\infty} e^{-(\delta+\alpha)(s-t)} E\hat{v}(s)ds \right]. \end{aligned} \quad (4.7)$$

Taking the mathematical expectation, we obtain

$$\begin{aligned} \hat{v}(t) = Ay(t)^\gamma + E\hat{v}(t) - Ay^\gamma \exp \left[\left(\gamma\mu + \frac{1}{2}\gamma(\gamma-1)\sigma^2 \right) t \right] \\ = Ay(t)^\gamma + e^{(\alpha+\delta)t} z(t) - Ay^\gamma \exp \left[\left(\gamma\mu + \frac{1}{2}\gamma(\gamma-1)\sigma^2 \right) t \right]. \end{aligned} \quad (4.8)$$

Using formula (4.6) and rearranging, we obtain the following proposition:

Proposition 4.1. *The optimal rate of capital investment is given by:*

$$\begin{aligned} \hat{v}(t) = Ay(t)^\gamma + \frac{\varepsilon}{\beta} e^{(\beta^* - \delta)t} \\ \times \left[\frac{cy^\gamma}{\varepsilon + (c/A)(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)} - (x - \bar{x}) - \frac{\delta\bar{x}(\alpha + \delta)}{\delta(\alpha + \delta) + \varepsilon} \right] \\ - \frac{\varepsilon Ay^\gamma}{\varepsilon + (c/A)(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)} \exp \left[\left(\gamma\mu + \frac{1}{2}\gamma(\gamma-1)\sigma^2 \right) t \right] \\ + \frac{\varepsilon\delta\bar{x}}{\delta(\alpha + \delta) + \varepsilon}. \end{aligned} \quad (4.9)$$

From Proposition 4.1, we see that the optimal control depends not only on the current state $y(t)$ (the output price process), but also on the initial conditions x and y (capital level and output price level).

4.4. Computation of the value function

Using the optimal rate of capital investment given in Proposition 4.1, we generate a formula for the optimal value $u_\varepsilon(x, y) = J_{x,y}^\varepsilon(\hat{v}(\cdot))$ in the following theorem.

Theorem 4.1. *The optimal value of the firm is given by*

$$\begin{aligned} u_\varepsilon(x, y) = J_{x,y}^\varepsilon(\hat{v}(\cdot)) = Axy^\gamma + By^{2\gamma} + \frac{\varepsilon}{2} ACy^{2\gamma} \\ + \frac{\varepsilon}{2} Ay^\gamma [D_1x + D_2\bar{x} + \varepsilon(D'_1x + D'_2\bar{x})] \\ + \varepsilon[E_1x^2 + E_2\bar{x}^2 + E_3x\bar{x} + \varepsilon(E'_1x^2 + E'_2\bar{x}^2 + E'_3x\bar{x})], \end{aligned} \quad (4.10)$$

where $C, D_1, D_2, D'_1, D'_2, E_1, E_2, E_3, E'_1, E'_2, E'_3$ are constants.

Proof. Using the optimality condition, we first state

$$J_{x,y}^\varepsilon(\hat{v}(\cdot)) = Axy^\gamma + \frac{1}{2}E \int_0^{+\infty} e^{-\alpha t} \hat{v}(t) Ay(t)^\gamma dt - \frac{\varepsilon}{2} \int_0^{+\infty} e^{-\alpha t} (xe^{-\delta t} - \bar{x})^2 dt \\ - \frac{\varepsilon}{2} \int_0^{+\infty} e^{-\alpha t} (xe^{-\delta t} - \bar{x}) \int_0^t e^{-\delta(t-s)} E\hat{v}(s) ds dt,$$

which can be rewritten as

$$J_{x,y}^\varepsilon(\hat{v}(\cdot)) = Axy^\gamma - \frac{\varepsilon}{2} \int_0^{+\infty} e^{-\alpha t} (xe^{-\delta t} - \bar{x})^2 dt \\ + \frac{1}{2}E \int_0^{+\infty} e^{-\alpha t} \hat{v}(t) \left[Ay(t)^\gamma - \varepsilon \left(\frac{xe^{-\delta t}}{\alpha + 2\delta} - \frac{\bar{x}}{\alpha + \delta} \right) \right] dt,$$

and, using (4.9), we obtain

$$J_{x,y}^\varepsilon(\hat{v}(\cdot)) \\ = Axy^\gamma - \frac{\varepsilon}{2} \int_0^{+\infty} e^{-\alpha t} (xe^{-\delta t} - \bar{x})^2 dt + By^{2\gamma} \\ + \frac{\varepsilon}{2} Ay^\gamma \int_0^{+\infty} \exp \left[- \left(\alpha - \gamma\mu - \frac{1}{2}\gamma(\gamma-1)\sigma^2 \right) t \right] \left\{ - \left(\frac{xe^{-\delta t}}{\alpha + 2\delta} - \frac{\bar{x}}{\alpha + \delta} \right) \right. \\ + \frac{1}{\beta} e^{(\beta^* - \delta)t} \left[\frac{cy^\gamma}{\varepsilon + (c/A)(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)} - (x - \bar{x}) - \frac{\delta\bar{x}(\alpha + \delta)}{\delta(\alpha + \delta) + \varepsilon} \right] \\ - \frac{Ay^\gamma}{\varepsilon + (c/A)(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)} \exp \left[\left(\gamma\mu + \frac{1}{2}\gamma(\gamma-1)\sigma^2 \right) t \right] \\ \left. + \frac{\delta\bar{x}}{\delta(\alpha + \delta) + \varepsilon} \right\} dt \\ - \frac{\varepsilon^2}{2} \int_0^{+\infty} e^{-\alpha t} \left(\frac{xe^{-\delta t}}{\alpha + 2\delta} - \frac{\bar{x}}{\alpha + \delta} \right) \\ \times \left\{ \frac{1}{\beta} e^{(\beta^* - \delta)t} \left[\frac{cy^\gamma}{\varepsilon + (c/A)(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)} - (x - \bar{x}) - \frac{\delta\bar{x}(\alpha + \delta)}{\delta(\alpha + \delta) + \varepsilon} \right] \right. \\ - \frac{Ay^\gamma}{\varepsilon + (c/A)(\delta + \mu\gamma + \gamma(\gamma-1)\sigma^2/2)} \exp \left[\left(\gamma\mu + \gamma(\gamma-1)\sigma^2/2 \right) t \right] \\ \left. + \frac{\delta\bar{x}}{\delta(\alpha + \delta) + \varepsilon} \right\} dt.$$

We can compute these integrals and obtain the formula

$$J_{x,y}^\varepsilon(\hat{v}(\cdot)) = Axy^\gamma + By^{2\gamma} + \frac{\varepsilon}{2} ACy^{2\gamma} \\ + \frac{\varepsilon}{2} Ay^\gamma [D_1x + D_2\bar{x} + \varepsilon(D'_1x + D'_2\bar{x})]$$

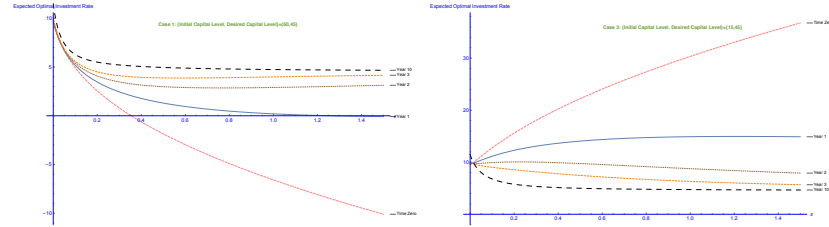


Figure 1. The expected optimal rate of investment with respect to ε for various t

$$+ \varepsilon[E_1 x^2 + E_2 \bar{x}^2 + E_3 x\bar{x} + \varepsilon(E'_1 x^2 + E'_2 \bar{x}^2 + E'_3 x\bar{x})],$$

where $C, D_1, D_2, D'_1, D'_2, E_1, E_2, E_3, E'_1, E'_2, E'_3$ are constants. This completes the proof. $\square$

Remark 3. We can recover the optimal control and the value function of Abel and Eberly's model [2] by letting $\varepsilon = 0$ in (4.9) and (4.10)

5. Numerical analyses

In this section, we explore how a risk-averse manager's penalty term, ε , affects the expected optimal rate of capital investment over time.

We use the following parameter values as the base case scenario:

$$\mu = 0.01; \sigma = 0.2; \delta = 0.1; \xi = 0.2; W = 5; \alpha = 0.2; y = 5.$$

In addition, we choose the following two different sets of the initial capital level and the desired capital level,

$$\{x, \bar{x}\} = \{15, 45\} \text{ and } \{60, 45\},$$

to explore two scenarios, that is, a firm's initial capital level is either well below or well above the desired capital level.

We have the following observations.

1. The expected optimal rate of capital investment is sensitive to the penalty, ε , that a risk averse entrepreneur assigns. Moreover, the sensitivity decreases as the investment progresses through time. That is, the expected optimal rate of capital investment is less affected by the penalty, ε , assigned as the investment progresses. Figure 1 illustrates this observation.

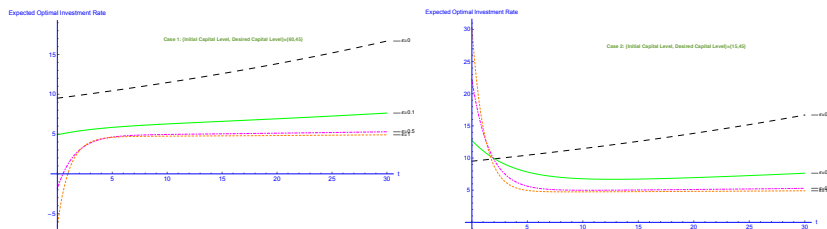


Figure 2. The expected optimal rate of investment with respect to t for various ε

2. The penalty, ε , determines how fast a manager adjusts the firm's investment rate towards the desired level, where the slope of the expected optimal rate of capital investment is equal to zero. The larger the ε is, the larger the investment rate (both investing and disinvesting) is made at the beginning phase toward the desired level. Obviously, a firm has an incentive to accelerate the investment/disinvestment toward the desired level in order to avoid large losses in firm value due to the large penalty imposed. A firm, however, may be able to adjust the rate of investment/disinvestment at a slower pace toward the desired level when the loss of the firm's value followed by the penalty is small.

To sum up, the risk term matters for the optimal rate of capital investment. Furthermore, from both graphics, we see that the naive model, ignoring the capital level deviation penalty (risk), leads to myopic decisions far from the optimal policy.

6. Conclusion

We study the optimal rate of capital investment of a firm under uncertainty penalized by the deviation of its expected capital level from the fixed desired level. The control problem is of the mean field type.

Using the direct approach, we are able to arrive at a closed form solution. The optimal rate of capital investment depends not only on the current state of the output price process, but also on the initial conditions of the output price level and the capital level. The optimal policy is significantly different from one that ignores the penalty caused by undesirable capital level deviations. The optimal rate of capital investment for a firm without the penalty is independent of the initial capital level as well as the initial output price level. It depends only on the current state of the output price process. We thus advise managers

to employ the appropriate formulation of the optimal investment problem. If a firm is exposed to the risk associate with undesirable capital deviations, the risk (penalty) should be appropriately accounted for in its control problem. Otherwise, myopic decisions may be made at the expense of a firm's value. In other words, incorporating risk control into a firm's investment model is too important to ignore.

References

- [1] A.B. ABEL (1983) Optimal Investment under Uncertainty. *The American Economic Review* **73** (1), 228–233.
- [2] A.B. ABEL AND J.C. EBERLY (1997) An Exact Solution for the Investment and Value of a Firm Facing Uncertainty, Adjustment Costs and Irreversibility. *Journal of Economic Dynamics and Control* **21**, 831–852.
- [3] D. ANDERSSON AND B. DJEHICHE (2011) A maximum principle for SDEs of mean field type. *Appl. Math. Optim.* **63**, 341–356.
- [4] S. ANKIRCHNER AND A. DERMOUNE (2011) Multiperiod Mean-Variance Portfolio Optimaziton via Market Cloning. *Appl. Math. Optim.* **64** (1), 135–154.
- [5] A. BENSOUSSAN, J. FREHSE AND P. YAM (2013) *Mean-field Games and Mean-field Type Control*. Springer Briefs in Mathematics.
- [6] A. BENSOUSSAN, C. HOE AND Z. YAN (2019) Mean-Variance Approach to Capital Investment Optimization. *SIAM J. Financial Math.* **10** (1), 156–180.
- [7] H. BO AND R. LENSIN (2005) Is the Investment-Uncertainty Relationship Nonlinear? An Empirical Analysis for the Netherlands. *Economica* **72** (286), 307–331.
- [8] R. BUCKDAHN, J. LI AND J. MA (2016) A Stochastic Maximum Principle for General Mean-Field System. *Appl. Math. Optim.* **74**, 507–534.
- [9] R. BUCKDAHN, B. DJEHICHE AND J. LI (2011) A General Stochastic Maximum Principle for SDES of Mean-field Type. *Appl. Math. Optim.* **64**, 197–216.
- [10] R.J. CABALLERO (1991) On the Sign of the Investment-Uncertainty Relationship. *The American Economic Review* **81** (1), 279–288.
- [11] R.W. COOPER AND J.C. HALTIWANGER (2006) On the Nature of Capital Adjustment Costs. *The Review of Economic Studies* **73** (3), 611–633.
- [12] A. DIXIT AND R.S. PINDYCK (1994) *Investment Under Uncertainty*. Princeton University Press, Princeton, NJ.
- [13] M. FISCHER AND G. LIVIERI (2016) Continuous time mean-variance portfolio optimization through the mean field approach. *ESAIM: PS* **20**, 30–44.
- [14] J. LI (2012) Stochastic Maximum Principle in the mean-field controls. *Automatic* **48**, 366–373.