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Linear cycles of consecutive lengths

Tao Jiang [a,1], Jie Ma [b,2], Liana Yepremyan [c]

[a] Department of Mathematics, Miami University, Oxford, OH 45056, USA
[b] School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, PR China
[c] Department of Mathematics, Emory University, Atlanta, GA 30322, USA

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A well-known result of Verstraëte [23] shows that for each integer $k \geq 2$ every graph G with average degree at least $8k$ contains cycles of k consecutive even lengths, the shortest of which is of length at most twice the radius of G. We establish two extensions of Verstraëte's result for linear cycles in linear r-uniform hypergraphs.

We show that for any fixed integers $r \geq 3$ and $k \geq 2$, there exist constants $c_1 = c_1(r)$ and $c_2 = c_2(r)$, such that every n-vertex linear r-uniform hypergraph G with average degree $d(G) \geq c_1 k$ contains linear cycles of k consecutive even lengths, the shortest of which is of length at most $2\lceil \frac{\log n}{\log(d(G)/k) - c_2} \rceil$. In particular, as an immediate corollary, we retrieve the current best known upper bound on the linear Turán number of C^r_{2k} with improved coefficients.

Furthermore, we show that for any fixed integers $r \geq 3$ and $k \geq 2$, there exist constants $c_3 = c_3(r)$ and $c_4 = c_4(r)$ such that every n-vertex linear r-uniform hypergraph with average degree $d(G) \geq c_3 k$, contains linear cycles of k consecutive lengths, the shortest of which has length at most $6\lceil \frac{\log n}{\log(d(G)/k) - c_4} \rceil + 6$. In both cases for given average degree

E-mail addresses: jiangt@miamioh.edu (T. Jiang), jiema@ustc.edu.cn (J. Ma), lyeprem@emory.edu (L. Yepremyan).

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d, the length of the shortest cycles cannot be improved up to the constant factors c_2, c_4.

1. Introduction

For $r \geq 2$, an r-uniform hypergraph (henceforth, r-graph) is *linear* if any two edges share at most one vertex. For $r = 2$, linear r-graphs are just the usual simple graphs. An r-uniform *linear cycle* of length k, denoted by C_k^r, is a linear r-graph on $(r-1)k$ vertices whose edges can be ordered as $e_1, e_2, ..., e_k$ such that $|e_i \cap e_j| = 1$ if $j = i \pm 1$ (indices taken modulo k) and $|e_i \cap e_j| = 0$ otherwise. Motivated by known results for graphs, we study sufficient conditions for the existence of linear cycles of given lengths in linear r-graphs for $r \geq 3$. Our results apply to linear r-graphs of broad edge density, covering both sparse and dense hypergraphs.

1.1. History

The line of research about the distribution of cycle lengths in graphs was initiated by Burr and Erdős (see [7]) who conjectured that for every odd number k, there is a constant c_k such that for every natural number m, every graph of average degree at least c_k contains a cycle of length m modulo k. This conjecture was confirmed in its full generality by Bollobás [1] for $c_k = 2((k+1)^k - 1)/k$, although earlier partial results were obtained by Erdős and Burr [7] and Robertson [7]. The constant c_k was improved to $8k$ by Verstraëte [23]. Thomassen [21,22] strengthened the result of Bollobás by proving that for every k (not necessarily odd), every graph with minimum degree at least $4k(k+1)$ contains cycles of all even lengths modulo k.

On a similar note, Bondy and Vince [3] proved a conjecture of Erdős in a strong form showing that any graph with minimum degree at least three contains two cycles whose lengths differ by one or two. Since then there has been extensive research (such as [13,10,20,17,16]) on the general problem of finding k cycles of consecutive (even or odd) lengths under minimum degree or average degree conditions in graphs. Very recently, the optimal minimum degree condition assuring the existence of such k cycles was proved in [12].

The problem of finding consecutive length cycles in r-graphs is related to another classical problem in extremal graph theory, namely Turán numbers for cycles in graphs and hypergraphs. For $r \geq 2$, the *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of a family $\mathcal{F}$ of r-graphs is the maximum number of edges in an n-vertex r-graph which does not contain any member of $\mathcal{F}$ as its subgraph. If $\mathcal{F}$ consists of a single graph F, we write $\mathrm{ex}(n, F)$ for $\mathrm{ex}(n, \{F\})$. A well-known result of Erdős (unpublished) and independently of Bondy and Simonovits [2] states that for any integer $k \geq 2$, there exists some absolute constant

$c > 0$ such that $\mathrm{ex}(n, C_{2k}) \le ckn^{1+1/k}$. The value of c was further improved by the results of Verstraëte [23] and Pikhurko [18], and the current best known upper bound is $\mathrm{ex}(n, C_{2k}) \le 80\sqrt{k}\log kn^{1+1/k} + O(n)$, due to Bukh and Jiang [4,5]. Verstraëte's main result from [23] is as follows.

Theorem 1.1 *(Verstraëte, [23]). Let $k \ge 2$ be an integer and G a bipartite graph of average degree at least $4k$ and girth g. Then there exist cycles of $(g/2-1)k$ consecutive even lengths in G, the shortest of which has length at most twice the radius of G.*

In Theorem 1.1, in addition to finding k cycles of consecutive even lengths Verstraëte also gave a tight upper bound on the length of the shortest cycle among these which in turn immediately yields $\mathrm{ex}(n, C_{2k}) \le 8kn^{1+1/k}$, thus improving the coefficients in the theorems of Erdős and of Bondy-Simonovits. Notice that Verstraëte's theorem is applicable to both sparse and dense host graphs while arguments establishing bounds on $\mathrm{ex}(n, F)$ directly usually address relatively dense host graphs. For example, for $F = C_{2k}$, these would typically be graphs with average degree at least $\Omega(n^{1/k})$.

For hypergraphs, Verstraëte [24] conjectured that for $r \ge 3$ any r-graph with average degree $\Omega(k^{r-1})$ contains Berge cycles of k consecutive lengths where an r-uniform *Berge cycle* of length k is a hypergraph containing k vertices $v_1, ..., v_k$ and k distinct edges $e_1, ..., e_k$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for each i, where the indices are taken modulo k. This conjecture was confirmed by Jiang and Ma in [14]. As an intermediate step, they proved the following result.

Theorem 1.2 *(Jiang and Ma, [14]). For all $r \ge 3$, any linear r-graph with average degree at least $7r(k+1)$ contains Berge cycles of k consecutive lengths.*

One of the two main results of this paper strengthens this result by replacing Berge cycles by linear cycles, and also obtaining optimal bounds (up to a constant factor) on the length of the shortest cycle. The study of the emergence of linear cycles in linear host hypergraphs is related to so-called linear Turán numbers. The *linear Turán number* $\mathrm{ex}_L(n, H)$ of a linear r-graph H is the maximum number of edges in an n-vertex linear r-graph G that does not contain H as a subgraph. Collier-Cartaino, Graber, and Jiang [6] proved that for all integers $r, k \ge 2$ there exist positive constants $c(r, k)$ and $d(r, k)$ such that $\mathrm{ex}_L(n, C^r_{2k}) \le c(r, k)n^{1+1/k}$ and $\mathrm{ex}_L(n, C^r_{2k+1}) \le d(r, k)n^{1+1/k}$. For fixed r, the constants $c(r, k)$ and $d(r, k)$ they establish are exponential in k. As a corollary, one of the main results we prove, Theorem 1.3 implies that $c(r, k)$ can be taken linear in k, improving the results in [6]. Note that these results on linear Turán numbers of linear even cycles can be viewed as a generalization of the Bondy-Simonovits' even cycle theorem, while the result on linear odd cycles demonstrates a phenomenon that is very different from the graph case.

1.2. Our results

We establish two extensions of Theorem 1.1 for linear cycles in linear r-uniform hypergraphs. First, we give a generalization of Theorem 1.1 for even linear cycles in linear r-graphs along with a near optimal control on the shortest length of the even cycles obtained.

Theorem 1.3. *Let $r \geq 3$ and $k \geq 2$ be integers, define $c_1 = 180r^{2r+2}$ and $c_2 = \log(64r^{2r+2})$. If G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive even lengths, the shortest of which is at most*

$$2\left\lceil \frac{\log n}{\log(d(G)/k) - c_2} \right\rceil.$$

Theorem 1.3 implies an improved upper bound on the linear Turán number of C_{2k}^r which previously was $ex_L(n, C_{2k}^r) \leq cn^{1+1/k}$ for some c exponential in k [6]. We now prove that c can be taken to be linear in k.

Corollary 1.4. *Let $r \geq 3$ and $k \geq 2$ be integers, denote $c = 180r^{2r+1}$. For all positive integers n, we have*

$$ex_L(n, C_{2k}^r) \leq ckn^{1+1/k}.$$

Our next main result shows that under analogous degree conditions as in Theorem 1.3, we can in fact ensure the existence of linear cycles of k consecutive lengths (even and odd both included). Furthermore, the length of the shortest cycle in the collection is within a constant factor of being optimal. Note that such a phenomenon can only exist in r-graphs with $r \geq 3$, as for graphs, one needs more than $n^2/4$ edges in an n-vertex graph just to ensure the existence of any odd cycle.

Theorem 1.5. *Let $r \geq 3$ and $k \geq 1$ be integers. There exist constants c_1, c_2 depending on r such that if G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive lengths, the shortest of which is at most*

$$6\left\lceil \frac{\log n}{\log(d(G)/k) - c_2} \right\rceil + 6.$$

When viewed as a result on the average degree needed to ensure the existence of cycles of consecutive lengths, Theorem 1.5 is a substantial strengthening of both Theorem 1.2 and Theorem 1.3. However, the upper bound on the shortest length of a cycle in the collection is weaker than the one in Theorem 1.3 by roughly a factor of 3. As a result, while Theorem 1.3 yields $\mathrm{ex}_L(n, C_{2k}^r) = O(n^{1+1/k})$, Theorem 1.5 would only give us $\mathrm{ex}(n, C_{2k+1}^r) = O(n^{1+3/k})$, and hence it does not imply the bound on $\mathrm{ex}_L(n, C_{2k+1}^r)$ given in [6].

Finally, note that the shortest lengths of linear cycles that we find in Theorem 1.3 and Theorem 1.5 are within a constant factor of being optimal, due to the following proposition which can be proved using a standard deletion argument. We delay its proof to the appendix.

Proposition 1.6. *Let $r \geq 2$ be an integer. For sufficiently large n and all d satisfying $(2r)^{\frac{1}{\varepsilon^2}} \leq d \leq n/2r$, there exists an n-vertex linear r-graph with average degree at least d and containing no linear cycles of length at most $\lfloor (1-\varepsilon)\log_d n \rfloor$.*

The rest of the paper is organized as follows. In Section 2, we introduce some notation. In Section 3, we prove Theorem 1.5. In Section 4, we prove Theorem 1.3, whose proof is more involved than that of Theorem 1.5 due to the tighter control on the shortest lengths of the cycles. In Section 5, we conclude with some remarks and problems for future study on related topics.

2. Notation

Let $r \geq 2$ be an integer. Given an r-graph G, we use $\delta(G)$ and $d(G)$ to denote the minimum degree and the average degree of G, respectively. Given a graph G and a set S, an *edge-colouring* of G using subsets of S is a function $\chi : E(G) \to 2^S$. We say that χ is *strongly proper* if $V(G) \cap S = \emptyset$ and whenever e, f are two distinct edges in G that share an endpoint we have $\chi(e) \cap \chi(f) = \emptyset$. We say that χ is *strongly rainbow* if $V(G) \cap S = \emptyset$ and whenever e, f are distinct edges of G we have $\chi(e) \cap \chi(f) = \emptyset$.

For $r \geq 2$, an r-graph G is *r-partite* if there exists a partition of $V(G)$ into r subsets $A_1, A_2 \dots, A_r$ such that each edge of G contains exactly one vertex from each A_i; we call such $(A_1, \dots, A_r)$ an *r-partition* of G. For any $1 \leq i \neq j \leq r$, we define the (A_i, A_j)-*projection* of G, denoted by $P_{A_i,A_j}(G)$ to be the graph with edge set $\{e \cap (A_i \cup A_j) | \, e \in E(G)\}$. It is easy to see that for linear r-partite r-graphs the following mapping $f : E(G) \to E(P_{A_i,A_j}(G))$ defined by $f(e) = e \cap (A_i \cup A_j)$ is bijective.

In this paper, logarithms are base 2 and $[k]$ denotes the set $\{1, 2, ..., k\}$ for all positive integers k.

3. Linear cycles of consecutive lengths

In this section, we prove Theorem 1.5. Given a linear r-graph G and two vertices x, y in G, we define the *distance* $d_G(x, y)$ to be the length of a shortest linear path between x and y. We drop the index G whenever the context is clear. For any vertex $x \in V(G)$, we define $S_0^G(x) = \{x\}$ and for all $i \geq 1$ define

$$S_i^G(x) = \{y \in V(G) : d_G(x, y) = i\}.$$

When G is clear from the context we will drop the superscript.

We first prove some auxiliary lemmas that are used in the proof of Theorem 1.5. Our first lemma is folklore.

Lemma 3.1. *Let $r \geq 2$ be an integer and $d > 0$ a real. Every r-graph G of average degree d contains a subgraph of minimum degree at least d/r.*

Lemma 3.2. *Let $r \geq 3$ be an integer. Let G be a linear r-graph. Let d be a real satisfying $1 \leq d \leq \delta(G)/2$. Let $x \in V(G)$. Then there exist a positive integer $m \leq \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$ and a subgraph H of G satisfying*

(A1) H has average degree at least $d/4$, and
(A2) each edge of H contains at least one vertex in $S_m(x)$ and no vertex from $\bigcup_{j<m} S_j(x)$.

Proof. For each $i \geq 0$, let $S_i = S_i(x)$. By the definition of the S_i's, for each $e \in E(G)$, there exists $j \geq 0$ such that $e \subseteq S_j \cup S_{j+1}$. For each $i \geq 1$ let G_i be the subgraph of G induced by the edges that contain some vertex in S_i. Then $V(G_i) \subseteq S_{i-1} \cup S_i \cup S_{i+1}$. Let $t = \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$. First we show that for some $i \in [t]$, G_i has average degree at least $d/2$. Suppose for contradiction that for each $i \in [t]$, G_i has average less than $d/2$. Then for each $i \in [t]$, $e(G_i) < (d/2)|V(G_i)|/r \leq (d/2r)(|S_{i-1}| + |S_i| + |S_{i+1}|)$. On the other hand, by minimum degree condition we have $e(G_i) \geq \delta(G)|S_i|/r$. Combining the two inequalities, we get

$$|S_{i-1}| + |S_i| + |S_{i+1}| \geq \frac{2\delta(G)}{d}|S_i|. \tag{1}$$

Claim 3.3. *For each $i \in [t]$, we have $|S_i| > (\delta(G)/d)|S_{i-1}|$.*

Proof. The claim holds for $i = 1$ since $|S_1| \geq \delta(G)$ and $|S_0| = 1$. Let $1 \leq j < t$ and suppose the claim holds for $i = j$. We prove the claim for $i = j + 1$. By (1) and the induction hypothesis that $|S_{j-1}| \leq (d/\delta(G))|S_j|$, we have

$$\frac{d}{\delta(G)}|S_j| + |S_j| + |S_{j+1}| \geq \frac{2\delta(G)}{d}|S_j|.$$

Hence

$$|S_{j+1}| \geq \left(\frac{2\delta(G)}{d} - \frac{d}{\delta(G)} - 1\right)|S_j| > \frac{\delta(G)}{d}|S_j|,$$

where the last inequality uses $d \leq \delta(G)/2$. □

By the claim, $|S_t| > (\delta(G)/d)^t \geq n$, which is a contradiction. So there exists $i \in [t]$ such that G_i has average degree at least $d/2$. By our earlier discussion, each edge of G_i

contains a vertex in S_i and lies inside $S_{i-1} \cup S_i \cup S_{i+1}$. If at least half of the edges of G_i contain some vertex in S_{i-1} then let H be the subgraph of G_i consisting of these edges and let $m = i-1$. Otherwise, let H be the subgraph of G_i consisting of edges that do not contain vertices of S_{i-1} and let $m = i$. In either case, H and m satisfy (A1) and (A2). □

Lemma 3.4. *Let $r \geq 3$. Let G be a linear r-graph. Let d be a real satisfying $1 \leq d \leq \delta(G)/2$. Let $x \in V(G)$. For each $v \in V(G)$, let P_v be a fixed shortest (x,v)-path in G and let $\mathcal{P} = \{P_v : v \in V(G)\}$. Then there exist a positive integer $m \leq \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$, $A \subseteq S_m(x)$ and a subgraph F of G such that the following hold:*

(A3) $\delta(F) \geq d/r2^{3r+2}$,
(A4) each edge of F contains exactly one vertex from A and no vertex from the set $\bigcup_{j<m} S_j(x)$,
(A5) for each $v \in V(F) \cap A$, P_v intersects $V(F)$ only in v.

Proof. By Lemma 3.2, there exist a subgraph H of G and a positive integer $m \leq \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$ satisfying properties (A1)-(A2). So, in particular, $d(H) \geq d/4$. For any vertex $v \in S_m(x)$, let e_v be the edge in the path P_v which contains v. For every $f \in E(H)$, we know that $|f \cap S_m(x)| \geq 1$. Let v_f be one of these vertices in $f \cap S_m(x)$.

Now randomly colour the vertices of $S_m(x)$ with three colours as follows; the colour of a vertex is red or blue each with probability p, and it is green with probability $1-2p$. We call an edge $f \in E(H)$ *good* if v_f is coloured green, all vertices in $f \cap S_m(x) \setminus \{v_f\}$ are red, and all other vertices in $e_{v_f} \cap S_m(x)$ are blue. It is easy to see that the probability of f being good is exactly

$$(1-2p)p^{|f\cap S_m(x)|-1}p^{|e_{v_f}\cap S_m(x)|-1} \geq (1-2p)p^{2r-2}.$$

For our purposes it is enough to choose $p = 2/5$, however, a more optimal choice is possible, which will give better constants. So in expectation, there are at least

$$\frac{e(H)}{5}\left(\frac{2}{5}\right)^{2r-2} \geq \frac{e(H)}{2^{3r}}$$

many good edges. Fix such a colouring and let $H' \subseteq H$ be induced by all the good edges, and let A to be the collection of all v_f, for which f is good. Now it is easy to check that both (A4) and (A5) are satisfied. The first one is very easy to check; notice that every f good has a unique green vertex, v_f, all the other vertices are red. But by definition, all the vertices in A have green colour so f cannot intersect A in more than one vertex. As for (A5), notice that P_v for any $v \in A$ can intersect $V(F)$ only in the vertices of e_v. However, if $v \in A$ then all the vertices of e_v except v are coloured blue. But all the vertices in A have colour green since they are all of form v_f for some f good, so P_v cannot intersect $V(F)$ except at the vertex v. Finally, by Lemma 3.1, H' has a subgraph

F of minimum degree at least $d(H')/r \geq d(H)/r2^{3r} \geq d/r2^{3r+2}$. Notice that A and F satisfy (A3)-(A5). $\square$

Lemma 3.5. *Let $r \geq 3$, $k \geq 1$ be integers. Let F be a linear r-graph and $A \subset V(F)$ be such that each edge of F contains exactly one vertex of A. If $\delta(F) \geq rk+1$ then F contains a linear path P of length $k+2$ such that P intersects A only at degree one vertices of P.*

Proof. Let P be a longest linear path in F with the property that P intersects A only at degree one vertices of P. The length of P is at least one. Let e be the last edge of P. There exists a vertex $v \in e \setminus A$ that has degree one in P since $|e \cap A| = 1$. There are at least $\delta(G) \geq rk+1$ edges of G containing v. Since G is linear, there are at most $|V(P)|-r+1$ edges in G that contain v and another vertex of P. Suppose $|V(P)|-r+1 \leq rk$. Then there is an edge f in G that contains v and no other vertex of P. But now $P \cup f$ is a longer linear path than P and each vertex in $(V(P) \cup f) \cap A$ has degree one in $P \cup f$, contradicting our choice of P. Hence $|V(P)| \geq rk+r$, which implies $|P| \geq k+2$. $\square$

Lemma 3.6. *Let $r \geq 3, k \geq 1$ and $d = kr^2 2^{3r+1}$. Let F be a linear r-graph with $\delta(F) \geq 2d$ and x be any vertex in F. Then there exist edges e and f and some integer $t \leq \lceil \frac{\log n}{\log(\delta(F)/d)} \rceil$ such that for each $i \in [k]$, there is a linear path of length $t+2+i$ starting at x and having e and f as its last two edges.*

Proof. For each vertex v in F, let P_v be a shortest (x,v)-linear path in F. By Lemma 3.4 (with F playing the role of G) there exist a positive integer $t \leq \lceil \frac{\log n}{\log(\delta(F)/d)} \rceil$, a subset $A \subseteq S_t(x)$ and a subgraph F' of F that satisfy (A3)-(A5). In particular, $\delta(F') \geq d/r2^{3r} = 2rk \geq rk+1$. Applying Lemma 3.5 to F', we obtain a linear path P of length $k+2$ in F' such that each vertex in $V(P) \cap A$ has degree one in P. Suppose the edges of P are ordered as $e_1, \dots, e_k, e, f$. For each $i \in [k]$, let v_i be the unique vertex in $e_i \cap A$. For each $i \in [k]$ since P_{v_i} intersects $V(F')$ only in v_i, $P_{v_i} \cup \{e_i, \dots, e_k, e, f\}$ is a linear path of length $(k+2)-(i-1)+t$ that starts at x and ends with e, f. Since this holds for each $i = 1, \dots, k$, the claim follows. $\square$

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. We will show the statement holds for $c_1 = 2^{6r+6}r^4, c_3 = 2^{6r+4}r^4$ and $c_2 = \log(c_3)$. Let G be an n-vertex linear r-graph with $d(G) \geq c_1 k$. By Lemma 3.1 G contains a subgraph G' with $\delta(G') \geq d(G)/r$. Set $\delta = d(G)/r$. Then $\delta(G') \geq \delta$. Set

$$d' = kr^2 2^{3r+1}, \quad d = r^{3/2} 2^{3r+2} \sqrt{\delta k}.$$

Note that when k, r are fixed, d' is a constant, but $d = \Theta(\sqrt{d(G)})$. Our d is chosen to approximately optimize the upper bound we obtain later on the lengths of the cycles. As $\delta = d(G)/r \geq c_1 k/r = 2^{6r+6} r^3 k$, we have $d = r^{3/2} 2^{3r+2} \sqrt{\delta k} \geq kr^3 2^{6r+5} \geq r2^{3r+3} \cdot 2d'$.

Let x_0 be any vertex in G'. We apply Lemma 3.4 to G'. For this one needs to check that $\delta(G') \geq \delta \geq 2d$ which holds by the choice of c_1. Thus there exist $m \leq \lceil \frac{\log n}{\log(\delta(G')/d)} \rceil \leq \lceil \frac{\log n}{\log(\delta/d)} \rceil$, a subset $A \subseteq S_m(x_0)$ and a subgraph F of G' such that

(P1) $\delta(F) \geq \frac{d}{r2^{3r+2}}$,
(P2) each edge of F contains exactly one vertex in A but no vertex in $\bigcup_{j<m} S_j(x_0)$, and
(P3) for each $v \in V(F) \cap A$, P_v intersects $V(F)$ only in v.

Now let x be any vertex in $V(F) \cap A$. Since $\delta(F) \geq d/r2^{3r+2} \geq 2d'$, by Lemma 3.6, there exist two edges e and f in F and some integer $t \leq \lceil \frac{\log n}{\log(\delta(F)/d')} \rceil$ such that for each $i \in \{t+3, t+4, \ldots, t+k+2\}$ there is a linear path Q_i in F of length i which starts at x and has e and f as the last two edges.

Let y be the unique vertex in $A \cap f$. By (P3), P_x and P_y intersect $V(F)$ only in x and y, respectively. Therefore, $P_x \cup P_y$ must contain a linear (x,y)-path of length $q \leq 2m$ that intersects $V(F)$ only in x and y. Let us denote this subpath by P_{xy}.

If $y \notin e \cap f$, then $P_{xy} \cup Q_1, \ldots, P_{xy} \cup Q_k$ are linear cycles of lengths $q+t+3, \ldots, q+t+k+2$, respectively. If $y \in e \cap f$ then $P_{xy} \cup (Q_1 \setminus f), \ldots, P_{xy} \cup \{Q_k \setminus f)$ are linear cycles of lengths $q+t+2, \ldots, q+t+k+1$, respectively. In either case we find linear cycles of k consecutive lengths, the shortest of which has length at most

$$q+t+3 \leq 2m+t+3 \leq 2\left\lceil \frac{\log n}{\log(\delta/d)} \right\rceil + \left\lceil \frac{\log n}{\log(\delta(F)/d')} \right\rceil + 3 \leq 3\left\lceil \frac{\log n}{\log(\delta/d)} \right\rceil + 3,$$

where the last inequality holds since $\delta(F)/d' \geq d/kr^3 2^{6r+3} \geq \delta/d$. By our choice of d and c_3, we can check that $\delta/d \geq (d(G)/kc_3)^{1/2}$. Hence, the above upper bound is at most

$$3\left\lceil \frac{\log n}{(1/2)\log(d(G)/c_3 k)} \right\rceil + 3 \leq 3\left\lceil \frac{2\log n}{\log(d(G)/k) - c_2} \right\rceil + 3 \leq \frac{6\log n}{\log(d(G)/k) - c_2} + 6.$$

This completes the proof of Theorem 1.5. □

4. Sharper results for linear cycles of even consecutive lengths

For linear cycles of even consecutive lengths, we obtain much tighter control on the shortest length of a cycle in the collection, which as a byproduct also gives us an improvement on the current best known upper bound on the linear Turán number $\mathrm{ex}_L(n, C^r_{2k})$ of an r-uniform linear cycle of a given even length $2k$. The previous best known upper bound is $c_{r,k}n^{1+1/k}$, where $c_{r,k}$ is exponential in k for fixed r. For fixed r, we are now able to improve the bound on $c_{r,k}$ to a linear function of k.

4.1. A useful lemma on long paths with special features

One of the key ingredients of our proof of the main result in this section is Lemma 4.2. The lemma is about the existence of a long path with special features in a properly edge-coloured graph with high average degree. It may be viewed a strengthening of two lemmas used in [14] (Lemma 2.6 and Lemma 2.7).

Lemma 4.1. *Let G be an n-vertex graph with average degree at least $2d$, for $d \geq 1$. Then there exists a linear ordering σ of $V(G)$ as $x_1 < x_2 < \cdots < x_n$ and some $0 \leq m < n$ such that for each $1 \leq i \leq m$ $|N_G(x_i) \cap \{x_{i+1}, \ldots, x_n\}| < d$ and that the subgraph F of G induced by $\{x_{m+1}, \ldots, x_n\}$ has minimum degree at least d.*

Proof. As long as G contains a vertex whose degree in the remaining subgraph is less than d we delete it from G. We continue until no such vertex exists. Let F denote the remaining subgraph. Suppose this terminates after m steps. Then we have deleted at most $dm \leq d(n-1) < e(G)$ edges. Hence F is nonempty. Let $x_1 < x_2 < \cdots < x_m$ be the vertices deleted in that order. Let $x_{m+1} < \ldots < x_n$ be an arbitrary linear ordering of the remaining vertices. Then the ordering $\sigma := x_1 < \ldots < x_n$ and F satisfy the requirements. □

The following lemma is written in terms of colourings of graphs, but in our applications the graph H will be some (A_i, A_j)-projection of an r-partite r-graph G where the colouring is obtained by colouring the edge $e \cap (A_i \cup A_j)$ in H by the $(r-2)$-set $e \setminus (A_i \cup A_j)$ for each $e \in E(G)$.

Lemma 4.2. *Let $r \geq 3$ and $\ell \geq 2$. Let H be a connected graph with minimum degree at least $4r\ell$. Let χ be a strongly proper edge-colouring of H using $(r-2)$-sets. Let E_1, E_2 be any partition of $E(H)$ into two nonempty sets such that $|E_1| \leq |E_2|$. Then there exists a strongly rainbow path of length at least ℓ in H such that the first edge of P is in E_1 and all the other edges are in E_2.*

Proof. Let us call a strongly rainbow path P in H a *good path* if it can be ordered such that its first edge is in E_1, and all the other edges are in E_2, if the path is of length at least two. To prove the lemma, we need to show that H has a good path of length at least ℓ.

For $i = 1, 2$, let H_i be the subgraph of H induced by the edge set E_i. Note that $d(H_2) \geq 2r\ell$. Let L be a connected component of H_2 with $d(L) \geq 2r\ell$. Let us denote $n_L = |V(L)|$. We can apply Lemma 4.1 to L and obtain that there exist some integer $0 \leq m < n_L$ and a linear ordering $\sigma := x_1 < x_2 < \cdots < x_{n_L}$ of $V(L)$ such that for each $1 \leq i \leq m$, $|N_L(x_i) \cap \{x_{i+1}, \ldots x_{n_L}\}| < r\ell$ and that the subgraph F of L induced by $\{x_{m+1}, \ldots, x_{n_L}\}$ has minimum degree at least $r\ell$.

Claim 4.3. *For any edge e which has an endpoint in F there exists a strongly rainbow path P of length ℓ whose first edge is e and $P \setminus e \subseteq F$.*

Proof. Suppose $e = uv_1$, where $v_1 \in V(F)$. Note that u may or may not be in F. Let $P = uv_1v_2 \dots v_j$ be a longest strongly rainbow path such that $v_i \in V(F)$, for all $2 \le i \le j$. For contradiction, assume that $j \le \ell - 1$. Since $\delta(F) \ge r\ell$, there are at least $r\ell$ edges of F incident to v_j. Among these edges, more than $\ell r - j > \ell(r-1)$ of them join v_j to a vertex outside $V(P)$. Since the colouring χ is strongly proper, the colours of these edges form a matching of $(r-2)$-sets of size more than $\ell(r-1)$. For any subgraph H' of H, denote

$$C(H') = \bigcup_{e \in E(H')} \{c | c \in \chi(e)\}.$$

Note that $|C(P)| \le j(r-2) < \ell(r-2)$. Recall that by the definition of strongly proper colouring we have $C(H) \cap V(H) = \emptyset$ thus there must exist a vertex $v_{j+1} \in V(F) \setminus V(P)$ such that $\chi(v_jv_{j+1}) \cap C(P) = \emptyset$. Now $P \cup v_jv_{j+1}$ is a longer strongly rainbow path than P, a contradiction. $\square$

Case 1: $m = 0$. In this case, $L = F$. If $n_L = |v(H)|$ then since E_1 is not empty there must be an edge $e \in E_1$ going between two vertices of $V(H) = V(L)$. If $n_L < |v(H)|$ then since L is a connected component of H_2, all the edges leaving L to $V(H) \setminus V(L)$ must be from E_1. And there must be at least one such edge $f \in E_1$ since H itself is connected. In both cases, e or f have at least one endpoint in F. Thus we can apply Claim 4.3 to either the edge e or f and obtain a good path of length ℓ whose first edge is either e or f.

Case 2: $m \ge 1$. Let $\mathcal{P}$ be the collection of all strongly rainbow paths of length ℓ such that their first vertex is among $\{x_1, x_2, \dots, x_m\}$, and all the edges along the path belong to E_2. We may assume $\mathcal{P} \ne \emptyset$. Indeed, since L is connected, there is an edge e going across $\{x_1, x_2, \dots, x_m\}$ to F. By Claim 4.3, we can extend e via a strongly rainbow path P of length ℓ such that $P \setminus e \subseteq F$. If $e \in E_1$, P will be the desired good path of length ℓ. So we may assume $e \in E_2$. Thus, $P \in \mathcal{P}$.

Among all the paths in $\mathcal{P}$, let $P = x_{j_1}x_{j_2} \dots x_{j_\ell}$ be such that j_1 is minimum. By our assumption, $j_1 \in [m]$ and so $|N_L(x_{j_1}) \cap \{x_{j_1+1}, \dots, x_{n_L}\}| < r\ell$. If $|N_L(x_{j_1}) \cap \{x_1, \dots, x_{j_1-1}\}| > \ell(r-2)$ then by a similar argument as in the proof of Claim 4.3 we can find $j_0 < j_1$ such that $\chi(x_{j_0}x_{j_1})$ is disjoint from $C(P)$ and $x_{j_0}x_{j_1} \in L$. In this case, $x_{j_0}x_{j_1} \dots x_{j_{\ell-1}}$ would contradict to our choice of P. Hence, $|N_L(x_{j_1}) \cap \{x_1, \dots, x_{j_1-1}\}| \le \ell(r-2)$. This shows that $d_L(x_{j_1}) < \ell(r-2) + r\ell$. Since $\delta_H(x_{j_1}) \ge 4r\ell$, x_{j_1} is incident to more than $4\ell r - \ell r - \ell(r-2) = 2\ell(r+1)$ many edges in E_1. Among them, more than $2\ell r + \ell$ of them join x_{j_1} to a vertex outside $V(P)$. Since χ is strongly proper, the colours on these edges form a matching of size more than $2\ell r + \ell$. Since $|C(P)| < \ell r$, there must exist at least one edge of E_1 that joins x_{j_1} to a vertex x_{j_0} outside $V(P)$ such that $\chi(x_{j_0}x_{j_1})$ is disjoint from $C(P)$. Then $x_{j_0}x_{j_1} \dots x_{j_\ell}$ is a good path of length $\ell + 1$. $\square$

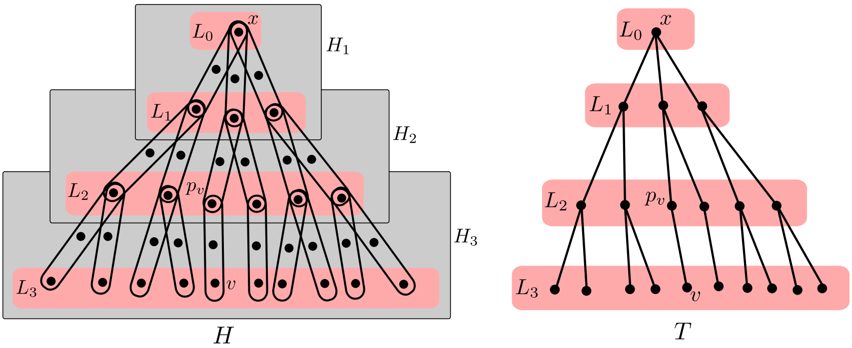

Fig. 1. $H = H_1 \cup H_2 \cup H_3$ and the corresponding tree T.

The following cleaning lemma is similar to part of Lemma 3.4.

Lemma 4.4. *Let H be a linear r-partite r-graph with an r-partition $(A_1, \cdots, A_r)$. Let M be an $(r-1)$-uniform matching where for each $f \in M$, f contains one vertex of each of $A_2, \ldots, A_r$. Then there exists a subgraph $H' \subseteq H$ such that*

(1) $e(H') \geq [1/(r-1)]^{r-1} e(H)$,
(2) each edge of M intersects $V(H')$ in at most one vertex.

Proof. Let us independently colour each edge of M using a colour in $\{2, \ldots, r\}$ chosen uniformly at random. Denote the colouring c. For each $i \in \{2, \ldots, r\}$, let $M_i = \{f \in M : c(f) = i\}$ and let $B_i = \{f \cap A_i : f \in M_i\}$. Let H' be the subgraph of H induced by the edge set $\{e \in E(H) : e \cap V(M) \subseteq B_2 \cup \cdots \cup B_r\}$. By definition, H' satisfies (2), it remains to show (1) holds.

We claim that for any edge e in H, the probability that e is in H' is at least $[1/(r-1)]^{r-1}$. Let $s = |e \cap V(M)|$. If $s = 0$ then e is in H' with probability 1. So we may assume that $1 \leq s \leq r-1$. Without loss of generality, suppose $e \cap V(M) = \{a_2, \ldots, a_{s+1}\}$, where $a_i \in A_i$ for each $i = 2, \ldots, s+1$. Since M is matching, for each $i = 2, \ldots, s+1$, there is a unique edge $f_i \in M$ that contains a_i. The probability that $a_i \in B_i$ is the probability that f_i is coloured i, which is $1/(r-1)$. Hence, the probability that $a_i \in B_i$ for each $i \in \{2, \ldots, s+1\}$ is $[1/(r-1)]^s$. In other words, the probability that e is in H' is $[1/(r-1)]^s \geq [1/(r-1)]^{r-1}$, as claimed. So the expectation of $e(H')$ is at least $[1/(r-1)]^{r-1} e(H)$. Therefore, there exists a colouring c for which $e(H') \geq [1/(r-1)]^{r-1} e(H)$. $\square$

4.2. Rooted expanded trees and linear cycles of consecutive even lengths

In this subsection, we introduce some of the key notions we use, in particular, a variant of a breadth-first-search tree in a linear r-partite r-graph G, and prove some auxiliary results we need for the proof of the main theorem (see Fig. 1 for the representation of $H = H_1 \cup H_2 \cup H_3$ and the corresponding tree T).

Definition 4.5. Let $r \geq 3$ be an integer. Let G be a graph. Let χ be any edge-colouring of G $\chi : E(G) \to C^{(r-2)}$, where the set C is the *colour* set and disjoint from $V(G)$. Given such χ, we define the (χ, r)-*expansion* of G, denoted by G^χ, to be the r-graph on vertex set $V(G) \cup C$ obtained from G by expanding each edge e of G into the r-set $e \cup \chi(e)$.

Note that if χ is a strongly rainbow, then (χ, r)-expansion is isomorphic to what is known in the literature, as the r-*expansion* of G, defined as follows. The r-*expansion* G^r of G is an r-graph obtained from G by expanding each edge e of G into an r-set using pairwise distinct $(r-2)$-sets disjoint from $V(G)$. Note that the $(r-2)$-sets used for the r-expansion naturally define a strongly rainbow edge-colouring on G.

Algorithm 4.6 *(Maximal expanded rooted tree - MERT).*

Input: A linear r-partite r-graph G with a fixed r-partition $(A_1, \dots, A_r)$ and a vertex x in A_1.

Output: A triple (H, T, χ) where H is some subgraph of G, T is a tree with $V(T) \subseteq V(H)$, rooted at x, such that H is the r-expansion of T and furthermore, for each $i \geq 0$, there exists some $j \in [r]$ such that $L_i(x) \subseteq A_j$, where $L_i(x)$ is the ith level in T, and χ is a strongly rainbow edge-colouring of T, such that $\chi : E(T) \to [V(G) \setminus V(T)]^{(r-2)}$ and $H = T^\chi$.

We will also obtain a collection of trees $T_0 \subseteq T_1 \subseteq T_2 \subseteq T_m = T$, subgraphs of H, $\{H_i\}_{i=0}^m$ where each H_i is called the ith segment of H and a collection of $(r-1)$-uniform matchings $\{M_i\}_{i=1}^{m-1}$ where $V(M_i) = V(H_{i+1}) \setminus V(H_i)$ and M_i is called the ith matching of H, these are described further below.

Initialization: Let H_0 be the r-graph with the single vertex x and T_0 the tree with the single vertex x. Let $L_0 = \{x\}$. Let H_1 be the subgraph of G consisting of all edges of G containing x. For every $v \in V(H_1) \setminus \{x\}$, let $p_v = \{x\}$.

Iteration - $i \geq 1$: Let E_i denote the set of edges in G that contain exactly one vertex in $V(H_i)$ and no vertices in $\cup_{j<i} V(H_j)$.

If $E_i = \emptyset$ then let $L_i = (V(H_i) \setminus V(H_{i-1})) \cap A_j$, where A_j is any part of $(A_1, \dots, A_r)$ that doesn't contain L_{i-1}. Let T_i be the tree obtained from T_{i-1} by joining every $v \in L_i$ to $p_v \in L_{i-1}$. Let $H = \cup_{0 \leq j \leq i} H_i$, $T = T_i$ and terminate.

If $E_i \neq \emptyset$ then do the following. Suppose $L_{i-1} \subseteq A_\ell$. For each $j \in [r] \setminus \{\ell\}$, let E_i^j be the set of edges in $e \in E_i$ such that the unique vertex in $e \cap (V(H_i) \setminus V(H_{i-1}))$ lies in A_j. Then $E_i = \bigcup_{j \in [r] \setminus \{\ell\}} E_i^j$. Let $s(i)$ be some $j \in [r] \setminus \{\ell\}$ that maximizes $|E_i^j|$. Let $L_i = E_i^{s(i)} \cap A_{s(i)}$. Let T_i be the tree obtained from T_{i-1} by joining every $v \in L_i$ to $p_v \in L_{i-1}$. Let M_i be a largest matching of $(r-1)$-tuples in $\{e \setminus L_i : e \in E_i^{s(i)}\}$. For each $I \in M_i$ we do the following. Since the graph G is linear, there is a unique $v_I \in L_i$ such that $I \cup v_I \in E_i^{s(i)}$. For each $u \in I$, we define p_u to be v_I and call it the *parent* of u in H. (Note that while u may not belong to $V(T)$, $p_u \in L_i \subseteq V(T)$.) Let H_{i+1} be the subgraph of G induced by the edge set $\{I \cup v_I | I \in M_i\}$. Increase i by one and repeat.

Termination: Suppose the algorithm stopped after m steps then we call m the *height* of H. Note that m is also the height of the tree T. Let χ be the following colouring on

$E(T)$: For every edge $uv \in E(T)$ there is a unique $(r-2)$-tuple I such that $uv \cup I \in E(H)$, we let $\chi(uv) = I$. By construction of H, χ is strongly rainbow. We will interchangeably call both H and the triple (H, T, χ) an *MERT* of G rooted at x of height m. □

Lemma 4.7. *Let $r \geq 3$ and $k \geq 1$ be integers. Let G be an r-partite r-graph with an r-partition $(A_1, A_2, \ldots, A_r)$. Let x be a vertex in G. Let (H, T, χ) be an MERT of G rooted at x of height m. Fix some integer $t \leq m$, and let L_i be the ith level of T. Let D be the subgraph of G consisting of all the edges in G that contain a vertex in L_{t-1}, at least one vertex in $V(H_t) \setminus L_{t-1}$ and no vertices from $\left(\bigcup_{j<t} V(H_j)\right) \setminus L_{t-1}$. If $e(D) \geq 8kr(r-1)(|L_{t-1}| + |L_t|)$ then G contains linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$ for some $\ell \leq t-1$.*

Proof. By definition of MERT, without loss of generality we may suppose $L_{t-1} \subseteq A_1$. Since $A_2 \cap V(H_t), A_3 \cap V(H_t), \ldots, A_r \cap V(H_t)$ partition $V(H_t) \setminus L_{t-1}$, by the pigeonhole principle, for some $i \in \{2, \ldots, r\}$, at least $e(D)/(r-1)$ of the edges of D contain a vertex from $A_i \cap V(H_t)$. Without loss of generality, suppose $i = 2$.

Let $X = L_{t-1}$ and $Y = A_2 \cap V(H_t)$. By definition of MERT, we have $|Y| = |V(H_t) \cap A_2| = |L_t|$. Let D' be the subgraph of D consisting of the edges that contain a vertex in X and a vertex in Y. By the previous discussion,

$$e(D') \geq e(D)/(r-1). \tag{2}$$

Let B be the (X, Y)-projection of D'. Since G is linear, $e(B) = e(D')$. Also, $|V(B)| \leq |X| + |Y| = |L_{t-1}| + |L_t|$. By our assumption about $e(D)$ and (2),

$$e(B) = e(D') \geq 8kr(|L_{t-1} + |L_t|) \geq 8kr|V(B)|.$$

So B has average degree at least $16kr$. By Lemma 3.1, B contains a connected subgraph B' with minimum degree at least $8kr$.

Let $S = V(B') \cap X$. Suppose x' is the closest common ancestor of S in the tree T. The union of the paths of T joining vertices of S to x' forms a subtree T' of T rooted at x'. Suppose that $x' \in L_j$ for some $j \geq 0$. Then $V(T') \subseteq L_j \cup \cdots \cup L_{t-1}$, and x' is the only vertex in $V(T') \cap L_j$. For each $v \in S$, let Q_v denote the unique (v, x')-path in T'.

Since x' is the closest common ancestor of S in T, x' has at least two children in T'. Let x_1 be one of the children of x in T'. We define a vertex labelling f on S as follows. For each $v \in S$, if Q_v contains x_1 then let $f(v) = 1$, and otherwise let $f(v) = 2$. Note that since x' has at least two children, there will be some $u, v \in S$ with $f(u) = 1$ and $f(v) = 2$. The following claim, despite being simple, is one of the key ingredients used by Bondy and Simonovits in proving their results in [2].

Claim 4.8. *Let $u, v \in S$. If $f(u) = 1$ and $f(v) = 2$ then $Q_u \cup Q_v$ is a path of length $2(t-1-j)$ in T' that intersects S only in u and v.*

Now, we define a partition of $E(B')$ into E_1 and E_2 as follows. Let ab be any edge in $E(B')$ where $a \in X$ and $b \in Y$. For $i = 1, 2$, we put ab in E_i if $f(a) = i$. We define an edge-colouring φ on B' using $(r-2)$-sets by letting $\varphi(ab)$ be the unique $(r-2)$-set such that $ab \cup \varphi(ab) \in E(D')$ for all $ab \in E(B')$. Since G is r-partite, $\varphi(B')$ is disjoint from $V(B')$. Since G is linear, φ is strongly proper. Without loss of generality, we may assume that $|E_1| \le |E_2|$. By Lemma 4.2, with $\ell = 2k$, B' contains a strongly rainbow path $P = a_1b_1a_2b_2 \dots a_kb_ka_{k+1}$ of length $2k$ such that the first edge of P is in E_1 and all other edges are in E_2. Note that we must have $a_1 \in S$. Otherwise if $b_1 \in S$ instead then the first two edges of P would have the same colour, contradicting our definition of P. Hence, $a_1, a_2, \dots, a_{k+1} \in S$ and by our assumption about P, $f(a_1) = 1$ and $f(a_2) = \dots = f(a_{k+1}) = 2$. For each $i \in [k]$, let P_i be the subpath P from a_1 to a_i. Let χ be the colouring in (H, T, χ) produced by Algorithm 4.6.

Claim 4.9. *For each $i \ge 2$, let R_i be the union of the r-uniform paths P_i^{φ}, $Q_{a_1}^{\chi}$ and $Q_{a_i}^{\chi}$. Then R_i is a linear cycle of length $2(t-1-j) + 2(i-1)$ in G.*

Proof. Since $f(a_1) = 1$ and $f(a_i) = 2$, by Claim 4.8, $Q_{a_1} \cup Q_{a_i}$ is a path of length $2(t-1-j)$ in T' that intersects S only in a_1 and a_i. On the other hand, P_i is a path of length $2(i-1)$ in B', which intersects $Q_{a_1} \cup Q_{a_i}$ only at a_1 and a_i. So $P_i \cup Q_{a_1} \cup Q_{a_i}$ is a cycle of length $2(t-1-j)+2(i-1)$ in $T' \cup B'$. By our assumptions, φ is strongly rainbow on P_i and χ is strongly rainbow on $Q_{a_1} \cup Q_{a_i}$. Furthermore, for any $e \in P_i$ and $f \in Q_{a_1} \cup Q_{a_i}$, $\varphi(e)$ has no vertices in $\left(\bigcup_{j<t} V(H_j)\right) \setminus L_{t-1}$, while $\chi(f) \subseteq \left(\bigcup_{j<t} V(H_j)\right) \setminus L_{t-1}$. So $\varphi(e) \cap \chi(f) = \emptyset$. Therefore, R_i is a linear cycle of length $2(t-1-j) + 2(i-1)$ in G. $\square$

By Claim 4.9, we see Lemma 4.7 holds for some $\ell = t-1-j \le t-1$. $\square$

Lemma 4.10. *Let $r \ge 3$, and $k \ge 1$ be integers. Let G be an r-partite r-graph with an r-partition $(A_1, A_2, \dots, A_r)$. Let x be a vertex in G. Let (H, T, χ) be an MERT rooted at x of height m. For any integer $t \le m$, let*

$$F = \{e \in E(G) : e \cap \bigcup_{i<t} V(H_i) = \emptyset \text{ and } |e \cap V(H_t)| \ge 2\}.$$

If $e(F) \ge 8kr^{r+2}|L_t|$, then G contains linear cycles of even lengths $2\ell+2, 2\ell+4, \dots, 2\ell+2k$ for some $1 \le \ell \le t$.

Proof. By our assumption L_{t-1} is contained in one partite set of G. Without loss of generality suppose that $L_{t-1} \subseteq A_1$. By definition of the MERT, $M = \{e \setminus L_{t-1} : e \in H_t\}$ is an $(r-1)$-uniform $(r-1)$-partite matching with partition $(A_2, A_3, \dots, A_r)$. By Lemma 4.4, there exists a subgraph F' of F such that

(1) $e(F') \ge (1/(r-1))^{r-1} e(F)$,
(2) each edge of M intersects $V(F')$ in at most one vertex.

Since $V(F')$ is disjoint from L_{t-1}, item (2) above ensures that

$$\forall e \in H_t,\ |e \cap V(F')| \leq 1. \tag{3}$$

By the definition of F and the fact that $F' \subseteq F$, any edge in F' contains at least two vertices of $V(H_t)\backslash L_{t-1} = V(M)$.

By the pigeonhole principle, there exist some $i, j \in \{2, \ldots, r\}$ such that the subgraph F'' of F' with edge set $E(F'') := \{e \in F' : |e \cap V(M) \cap A_i| = |e \cap V(M) \cap A_j| = 1\}$ satisfies

$$e(F'') \geq e(F')/\binom{r-1}{2} \geq (2/r^{r+1})e(F) \geq 16kr|L_t|, \tag{4}$$

where the last inequality holds as $e(F) \geq 8kr^{r+2}|L_t|$.

Without loss of generality, suppose that $\{i, j\} = \{2, 3\}$. Let B be the (A_2, A_3)-projection of F''. Since G is linear, $e(B) = e(F'')$. Also, note that $|V(B)| \leq |V(M) \cap (A_2 \cup A_3)| \leq 2|L_t|$. Hence, by (4),

$$e(B) = e(F'') \geq 16kr|L_t| \geq 8kr|V(B)|.$$

So B has average degree at least $16kr$. By Lemma 3.1, B contains a connected subgraph B' such that

$$\delta(B') \geq 8kr.$$

Let us define an edge-colouring φ of B' using $(r-2)$-sets as follows. For all $ab \in E(B')$, let $\varphi(ab)$ be the unique $(r-2)$-set such that $ab \cup \varphi(ab) \in E(F'')$. It is easy to see that φ is strongly proper.

Recall that for every $a \in V(H_t)$ p_a denotes the parent of a in H as in Algorithm 4.6. Let $S = \{p_a | a \in V(B') \cap A_2\}$. Note that $S \subseteq L_{t-1}$. If S is a single vertex, let x' be that single vertex and $T' = \{x'\}$. Otherwise, let x' be the closest common ancestor in T of vertices in S, and let T' be the subtree of T with the root x' and having vertices of S as leaves.

Case 1: $|S| = 1$. Let $P = a_1 a_2 \ldots a_{2k+1}$ with $a_1 \in A_2, a_{2k+1} \in A_2$ be any strongly rainbow path of length $2k+1$ in B' under the colouring φ. It easily exists because the minimum degree in B' is at least $8kr$. For every $1 \leq i \leq k$, let P_{2i+1} be the subpath of P between a_1 and a_{2i+1}. Since φ is strongly rainbow on P and $\varphi(P) \subseteq V(F'')$, P^{φ}_{2i+1} is a linear path of length $2i$ in F''. For $j = 1, 3, \ldots, 2k+1$, let e_{a_j} be the unique edge of H_t containing both a_j and p_{a_j}.

By (3), for all $i \geq 1$, $e_{a_{2i+1}}$ intersects P_{2i+1} only at a_{2i+1}, and e_{a_1} intersects P_{2i+1} only at a_1. By definition of MERT, $e_{a_1} \cap e_{a_{2i+1}} = \{x'\}$. Therefore, for all $1 \leq i \leq k$, $P^{\varphi}_{2i+1} \cup e_{a_1} \cup e_{a_{2i+1}}$ is a linear cycle of length $2i+2$. Thus, we find linear cycles of even lengths $4, \ldots, 2k+2$, thus the lemma follows for $\ell = 1$.

Case 2: $|S| \geq 2$. Then T' has at least two children, let x_1 be one of them. Suppose that $x' \in L_j$. For each $v \in S$, let Q_v denote the unique (v, x')-path in T'. We label vertices of $A_2 \cap V(B')$ as follows: for $a \in A_2 \cap V(B')$ if Q_{p_a} contains x_1 set $f(a) = 1$, otherwise $f(a) = 2$. The next claim is easy to see.

Claim 4.11. *Let $u, v \in A_2 \cap V(B')$. If $f(u) = 1$ and $f(v) = 2$, then $Q_{p_u} \cup Q_{p_v}$ is a path of length $2(t-1-j)$ in T' that intersects S only in p_u and p_v.* □

Now define a partition of the edges $E(B') = E_1 \cup E_2$ as follows. For $ab \in E(B')$ put $ab \in E_i$ if $f(a) = i$. Note that this is well defined partition since B' is bipartite with bipartition $(V(B') \cap A_2, V(B') \cap A_3)$.

Claim 4.12. $E_1 \neq \emptyset$, $E_2 \neq \emptyset$.

Proof. Since T' has at least two children, there must be $a, a' \in A_2 \cap V(B')$ such that $f(a) = 1$ and $f(a') = 2$, therefore any edge adjacent to a lies in E_1, any edge adjacent to a' lies in E_2, and recall that the minimum degree in B' is positive. □

We may assume that $|E_1| \leq |E_2|$. The other case is similar so we skip the analysis. By Lemma 4.2, with $\ell = 2k$, B' contains a strongly rainbow path $P = a_1 b_1 a_2 b_2 \dots a_k b_k a_{k+1}$ of length $2k$ such that the first edge of P is in E_1 and all other edges are in E_2. For each $i \in [k]$, let P_i be the subpath P from a_1 to a_i. Since φ is strongly rainbow on P_i and $\varphi(P_i) \subseteq V(F'')$, P_i^{φ} is a linear path of length $2i-2$ in F''. Note that $f(a_1) = 1$ and $f(a_i) = 2$, for all $i \geq 2$. Thus, for all $i \geq 2$, by Claim 4.11, $Q_{p_{a_1}} \cup Q_{p_{a_i}}$ is a path of length $2(t-1-j)$ in T'. Since χ is strongly rainbow on T', $Q^{\chi}_{p_{a_1}} \cup Q^{\chi}_{p_{a_i}}$ is a linear path of length $2(t-1-j)$ in $\bigcup_{j<t} H_j$. In particular, we see that P_i^{φ} and $Q^{\chi}_{p_{a_1}} \cup Q^{\chi}_{p_{b_i}}$ are vertex disjoint. For $i \in [k+1]$, let e_{a_i} be the unique edge of H_t containing both a_i and p_{a_i}.

By (3), for all $i \geq 1$, e_{a_i} intersects P_i only at a_i, and e_{a_1} intersects P_i only at a_1. Recall that $e_{a_1}, e_{a_i} \in H_t$ are disjoint as well therefore one can easily check that $R_i := P_i^{\varphi} \cup Q^{\chi}_{p_{a_1}} \cup Q^{\chi}_{p_{a_i}} \cup \{e_{a_1}, e_{a_i}\}$ is a linear cycle of length $2(t-j)+2i-2$ in G, for all $2 \leq i \leq k+1$. This gives us linear cycles of $2(t-j)+2$, $2(t-j)+4, \dots, 2(t-j)+2k$ in G. So the lemma follows with $\ell = t-j \leq t$, as desired. □

4.3. Linear cycles of even consecutive lengths in linear r-graphs

In this section we prove Theorem 1.3 and derive Corollary 1.4 from it. Theorem 4.13 is the main result of this section, as Theorem 1.3 will follow from it by a standard argument of selecting a large r-partite subgraph.

Theorem 4.13. *Let k, r be integers where $k \geq 1$ and $r \geq 3$. Let $c_3 = 180r^{r+2}$ and $c_4 = \log(64r^{r+2})$. If G is an n-vertex r-partite linear r-graph with average degree $d(G) \geq c_3 k$,* *then G contains linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$, for some positive integer $\ell \leq \lceil \frac{\log n}{\log(d(G)/k)-c_4} \rceil - 1$.*

We need the following definition of "dense" graphs which helps us to facilitate the inductive argument used to prove Theorem 4.13.

Definition 4.14. Given a positive real d, an r-graph G is said to be *d-minimal*, if $d(G) \geq d$ but for every proper induced subgraph H we have $d(H) < d(G)$.

Lemma 4.15. *Let $r \geq 2$ be an integer and $d > 0$ a real. If G is an r-graph satisfying that $d(G) \geq d$ then G contains a d-minimal subgraph G'.*

Proof. Among all induced subgraphs H of G satisfying $d(H) \geq d$, let G' be one that minimizes $|V(G')|$. Then G' is d-minimal. $\square$

Lemma 4.16. *Let $r \geq 3$ be an integer and d a positive real. Let G be a d-minimal r-graph. For any proper subset S of $V(G)$, the number of edges of G that contain a vertex in S is at least $d|S|/r$.*

Proof. Otherwise, suppose there is a proper subset S of $V(G)$ such that the number of edges of G that contain a vertex in S is less than $d|S|/r$. Then the subgraph G' of G induced by $V(G) \setminus S$ satisfies

$$e(G') \geq e(G) - d|S|/r \geq d|V(G)|/r - d|S|/r = d(|V(G')|/r.$$

Hence $d(G') \geq d$, contradicting G being d-minimal. $\square$

Proof of Theorem 4.13. Let $d = d(G)$ and let $p = \lceil \frac{\log n}{\log(d/k)-c_4)} \rceil$. By Lemma 4.15, G contains a d-minimal subgraph G'. Suppose G' does not contain a collection of linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$, where $\ell \leq p-1$. We will derive a contradiction. Let us apply Algorithm 4.6 to G' with a fixed vertex x and let (H, T, χ) be the produced MERT. Let m denote the height of H and T.

For each $i \in [m]$, let

$$G_i = \{e \in E(G') \setminus E(H) : e \cap V(H_i) \neq \emptyset, e \cap \bigcup_{j<i} V(H_j) = \emptyset\},$$

$$G_i^1 = \{e \in E(G_i) : |e \cap V(H_i)| = 1\}, \quad \text{and} \quad F_i = \{e \in E(G_i) : |e \cap V(H_i)| \geq 2\}.$$

Note that $G_m^1 = \emptyset$, as otherwise Algorithm 4.6 would have produced non-empty L_{m+1}, instead of stopping at step m, L_m being the last level. For convenience, define $L_{m+1} = \emptyset$.

Claim 4.17. *For each $1 \leq i \leq \min\{m, p\} - 1$, we have $e(G_i^1) \leq 8kr^3(|L_i| + |L_{i+1}|)$.*

Proof. Let D_i be the set of edges in G_i^1 that intersect $V(H_i)$ in L_i. By Algorithm 4.6,

$$e(D_i) \geq e(G_i^1)/r.$$

Let $e \in D_i$. By definition, e intersects $V(H_i)$ in exactly one vertex and that vertex lies in L_i. Furthermore, e contains no vertex in $\bigcup_{j<i} V(H_j)$. If $e \setminus L_i$ is vertex disjoint from $V(H_{i+1}) \setminus L_i$, then e would have been added to H_{i+1} by Algorithm 4.6, contradicting $e \notin E(H)$. Hence e must contain at least one vertex in $V(H_{i+1} \setminus L_i)$. If $e(D_i) \geq 8kr(r-1)(|L_i| + |L_{i+1}|)$ then by Lemma 4.7 (with $t = i+1$) G contains linear cycles of lengths $2\ell+2, 2\ell+4, \dots, 2\ell+2k$ for some $\ell \leq i \leq \min\{m,p\} - 1 \leq p-1$, contradicting our assumption. Hence,

$$e(D_i) \leq 8kr(r-1)(|L_i| + |L_{i+1}|) < 8kr^2(|L_i| + |L_{i+1}|).$$

Therefore, we have $e(G_i^1) \leq 8kr^3(|L_i| + |L_{i+1}|)$. □

Claim 4.18. *For each $1 \leq i \leq \min\{m, p-1\}$ we have $e(F_i) \leq 8kr^{r+2}|L_i|$.*

Proof. Suppose $e(F_i) \geq 8kr^{r+2}|L_i|$. Then by Lemma 4.10 (with $t = i$), we can find in G linear cycles of length $2\ell+2, 2\ell+4, \dots, 2\ell+2k$ for some $\ell \leq i \leq p-1$, contradicting our assumption. Hence, $e(F_i) \leq 8kr^{r+2}|L_i|$ hods for each $1 \leq i \leq \min\{m, p-1\}$. □

By Claims 4.17 and 4.18, we have that for any $1 \leq i \leq \min\{m,p\} - 1$,

$$e(G_i) = e(G_i^1) + e(F_i) \leq 16kr^{r+2}(|L_i| + |L_{i+1}|). \tag{5}$$

Claim 4.19. *For each $1 \leq i \leq \min\{m,p\} - 1$, $e(\bigcup_{j=1}^{i} G_j) \geq (d/2)\sum_{j=1}^{i} |L_j| - |L_{i+1}|$.*

Proof. Let $S = \bigcup_{j=0}^{i} V(H_j)$. Since $i \leq m-1$, S is a proper subset of $V(G')$. Let E_S denote the set of edges of G' that contains a vertex in S. By our definitions, $E_S \subseteq \bigcup_{j=1}^{i+1} E(H_j) \cup \bigcup_{j=1}^{i} G_j$. Since G' is d-minimal, by Lemma 4.16,

$$|E_S| \geq d|S|/r = d\left(1 + \sum_{j=1}^{i}(r-1)|L_j|\right)/r.$$

On the other hand, by the definition of H, $|\bigcup_{j=1}^{i+1} E(H_j)| = \sum_{j=1}^{i+1} |L_j|$. Hence, we have

$$\begin{aligned} e\left(\bigcup_{j=1}^{i} G_j\right) &= |E_S| - |\bigcup_{j=1}^{i+1} E(H_j)| \geq d\left(1 + \sum_{j=1}^{i}(r-1)|L_j|\right)/r - \sum_{j=1}^{i+1}|L_j| \\ &\geq \sum_{j=1}^{i} |L_j|\left(d\left(1 - \frac{1}{r}\right) - 1\right) + d/r - |L_{i+1}| \geq (d/2)\sum_{j=1}^{i}|L_j| - |L_{i+1}|, \end{aligned}$$

completing the proof. □

For each $1 \le i \le m$, let $U_i = \bigcup_{j=1}^{i} L_j$. By (5) and Claim 4.19, for each $1 \le i \le \min\{m,p\} - 1$,

$$32kr^{r+2}|U_{i+1}| - |L_{i+1}| \ge \sum_{j=1}^{i} 16kr^{r+2}\left(|L_j| + |L_{j+1}|\right) \ge e\left(\bigcup_{j=1}^{i} G_j\right) \ge (d/2)|U_i| - |L_{i+1}|.$$

Hence, for each $1 \le i \le \min\{m,p\} - 1$ we have

$$|U_{i+1}| \ge (d/64kr^{r+2})|U_i|. \tag{6}$$

Case 1: $m \le p-1$. We will see that if Algorithm 4.6 stops after at most $p-1$ steps then L_m must be relatively small but that would contradict to (6) for $i = m-1$. Let $S = V(H_m) \setminus (L_{m-1} \cup L_m)$. Then S is a proper subset of $V(G')$ with $|S| = (r-2)|L_m|$. Let E_S denote the set of edges of G' that contain a vertex in S. Since G' is d-minimal, we have

$$|E_S| \ge d|S|/r = d|L_m|(r-2)/r.$$

On the other hand, since L_m is the last level of H, by the definitions and the fact that $G_m^1 = \emptyset$, we have $E_S \subseteq E(H_m) \cup \bigcup_{i=1}^{m-1} E(G_i) \cup F_m$. By (5), Claim 4.18 and the fact that $e(H_m) = |L_m|$, we have

$$|E_S| \le |L_m| + \sum_{j=1}^{m-1} 16kr^{r+2}(|L_j| + |L_{j+1}|) + 8kr^{r+2}|L_m| \le 32kr^{r+2}|U_{m-1}| + 16kr^{r+2}|L_m|.$$

Combining the lower and upper bounds above on $|E_S|$, we get

$$(r-2)d|L_m|/r \le 32kr^{r+2}|U_{m-1}| + 16kr^{r+2}|L_m|.$$

As $d \ge c_3 k = 180kr^{r+2}$ and $r \ge 3$, this inequality above implies $|L_m| < |U_{m-1}|$. Therefore, $|U_m| = |U_{m-1}| + |L_m| < 2|U_{m-1}|$. But by (6), we have

$$|U_m| \ge (d/64kr^{r+2})|U_{m-1}| \ge 2|U_{m-1}|,$$

a contradiction.

Case 2: $m \ge p$**.** In this case we show that the expansion rate is so fast that $|U_p| \ge n$, which is a contradiction with $x \notin U_p$. Since G' is d-minimal, by Lemma 4.16, the minimum degree of G' is at least d/r, and therefore, $|U_1| = |L_1| = d_{G'}(x) \ge d/r$. Thus by (6), we have

$$|U_p| \ge (d/64kr^{r+2})^p.$$

Taking logarithm of both sides of the inequality and using $c_4 = \log(64r^{r+2})$ and $p = \lceil \frac{\log n}{\log(d/k)-c_4} \rceil$, we get

$$\log |U_p| \geq p(\log(d/k) - c_4) \geq \log n.$$

So $|U_p| \geq n$. This is a contradiction as $x \notin U_p$, completing the proof of the theorem. $\square$

Finally we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let c_3, c_4 be the constants obtained in Theorem 4.13. Let $c_1 = c_3 r^r = 180r^{2r+2}$ and $c_2 = c_4 + \log(r^r) = \log(64r^{2r+2})$. Let G be an n-vertex r-graph with $d(G) \geq c_1 k$. By a well-known result of Erdős and Kleitman [8] G contains an r-partite subgraph G' with $d(G') \geq d(G)(r!/r^r) \geq d(G)/r^r \geq c_3 k$. By Theorem 4.13, G' (and thus G also) contains linear cycles of lengths $2\ell + 2, 2\ell + 4, \dots, 2\ell + 2k$, for some positive integer

$$\begin{aligned} \ell &\leq \left\lceil \frac{\log n}{\log(d(G')/k) - c_4} \right\rceil - 1 \\ &\leq \left\lceil \frac{\log n}{\log(d(G)/k) - \log r^r - c_4} \right\rceil - 1 = \left\lceil \frac{\log n}{\log(d(G)/k) - c_2} \right\rceil - 1. \end{aligned}$$

This finishes the proof of Theorem 1.3. $\square$

As mentioned in the introduction, as a quick application of Theorem 1.3, we obtain an improvement on the upper bound given in [6] on $ex_L(n, C^r_{2k})$ by reducing the coefficient from at least exponential in k to a function linear in k (for fixed r).

Proof of Corollary 1.4. Let $c_1 = 180r^{2r+2}, c_2 = \log 64r^{2r+2}$ be as in the proof of Theorem 1.3. Let G be an n-vertex linear r-graph with $e(G) \geq ckn^{1+1/k}$, where $c = c_1/r = 180r^{2r+1}$. Then $d(G) = re(G)/n \geq c_1 kn^{1/k} \geq c_1 k$, thus we can apply Theorem 1.3 to G and conclude that it contains linear cycles of lengths $2\ell, 2\ell+2, \dots, 2\ell+2(k-1)$ for some positive integer

$$\ell \leq \left\lceil \frac{\log n}{\log(d(G)/k) - c_2} \right\rceil \leq \left\lceil \frac{\log n}{\log c_1 + \log n^{1/k} - c_2} \right\rceil \leq k,$$

where in the last inequality we used that $c_2 < \log c_1$. Since the even numbers in the interval $[2\ell, \dots, 2\ell + 2(k-1)]$ contain the number $2k$ it follows that G' must contain a linear cycle of length exactly $2k$. Hence the corollary holds. $\square$

5. Concluding remarks

In Theorem 1.5, we can slightly improve the leading coefficient 6 but we do not know if one can further improve the leading coefficient to 2 as in Theorem 1.3. In particular, we pose the following two questions, the second being a weakening of the first.

Question 5.1. *Let $r \geq 3$ and $k \geq 2$ be integers. Do there exist constants $c_1 = c(r), c_2 = c(r)$ such that if G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive lengths, the shortest of which is at most $2\lceil \frac{\log n}{\log d(G)/k - c_2} \rceil + 1$?*

Question 5.2. *Let $r \geq 3$ and $k \geq 2$ be integers. Do there exist constants $c_1 = c(r), c_2 = c(r)$ such that if G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive odd lengths, the shortest of which is at most $2\lceil \frac{\log n}{\log(d(G)/k) - c_2} \rceil + 1$?*

Ergemlidze, Győri and Methuku [9] proved that for $m \in \{2, 3, 4, 6\}$, $\mathrm{ex}_L(n, \{C_3^3, C_5^3, \cdots, C_{2m+1}^3\}) = \Omega(n^{1+1/m})$. On the other hand, it follows from the main result of [6] that $\mathrm{ex}_L(n, \{C_{2m+1}^r\}) = O(n^{1+1/m})$, for any $r \geq 3$ and $m \geq 1$. Hence, for $m \in \{2, 3, 4, 6\}$ and all sufficiently large n, there are n-vertex linear 3-graphs G with $d(G) = \Theta(n^{1/m})$ that contain no odd linear cycles of length at most $2m + 1$. Thus these graphs exhibit that in general the bound on the shortest odd cycle length in Questions 5.1 and 5.2 is best possible for $m \in \{2, 3, 4, 6\}$, up to the constants c_1, c_2.

It is natural to consider the analogous problem in general r-graphs. Interestingly, for sufficiently large n the answer is implied by exact results on the (usual) Turán number $\mathrm{ex}(n, C_k^r)$, obtained in [11], for all $r \geq 5$ and [15], for all $r \geq 3$, for all $k \geq 3$. These results combined show that for all $r \geq 3, t \geq 1$, $\mathrm{ex}(n, C_{2t+1}^r) = \binom{n}{r} - \binom{n-t}{r}$ and for all $(r, t) \neq (3, 1)$, $\mathrm{ex}(n, C_{2t+2}^r) = \binom{n}{r} - \binom{n-t}{r} + \binom{n-t-2}{r-2}$, while $\mathrm{ex}(n, C_4^3) = \binom{n}{3} - \binom{n-1}{3} + \max\{n-3, 4\lfloor \frac{n-1}{4} \rfloor\}$. Note that by these results, for all $k \geq 3$, $r \geq 3$, $\mathrm{ex}(n, C_k^r)$ is strictly increasing as a function of k. Furthermore, they show there are extremal r-graphs with $ex(n, C_k^r)$ many edges with no C_k^r which also don't contain C_ℓ^r, for all $\ell \geq k$. Indeed, a largest C_{2t+1}^r-free r-graph can be obtained by taking all the r-sets in $[n]$ that contain some vertex in a fixed t-set S. For $(r, t) \neq (3, 1)$, one can obtain a largest C_{2t+2}^r-free r-graph by adding to the above-mentioned extremal construction for C_{2t+1}^r all the r-sets in $[n] \setminus S$ that contain some two fixed vertices. As for C_4^3, one extremal construction is to take all the triples in $[n]$ containing a fixed vertex x and a largest P_2^3-free 3-graph on $[n] \setminus \{x\}$, where P_2^3 is the linear 3-uniform path of length two.

Hence, for all $r \geq 3, k \geq 2$ and sufficiently large n, the maximum number of edges in an n-vertex r-graph that does not contain linear cycles of k consecutive lengths is precisely $\mathrm{ex}(n, C_{k+2}^r)$. Indeed, if G is an r-graph and has more than $\mathrm{ex}(n, C_{k+2}^r)$ edges, then since $\mathrm{ex}(n, C_t^r)$ is an increasing function in t, G must contain cycles of all lengths up to $k+2$, which gives in total k cycles of consecutive lengths. On the other hand, there is an r-graph G with exactly $\mathrm{ex}(n, C_{k+2}^r)$ edges, in which the length of the longest cycle is $k+1$, hence we can only hope to find at most $k-1$ many cycles of consecutive lengths.

Data availability

No data was used for the research described in the article.

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Appendix A

Proof of Proposition 1.6. In the proof we use the fact that d is chosen such that $d \geq \max\{r^{1/\varepsilon}, 2^{\frac{1-\varepsilon}{\varepsilon^2}}\}$. By results of Rödl [19], for sufficiently large n there exists a linear n-vertex r-graph G of size at least $0.9\binom{n}{2}/\binom{r}{2}$. Set $p = 2rd/n$ and let F be a random subgraph of G obtained by independently including each edge of G with probability p. Let $\mathbb{X}$ denote the number of edges in F and $\mathbb{Y}$ the number of linear cycles of length at most m in F. Then

$$\mathbb{E}[\mathbb{X}] \geq 0.9\binom{n}{2}/\binom{r}{2} \cdot (2rd/n) > 1.8dn/r.$$

On the other hand, observe that for any fixed ℓ, since G is linear, there are at most n^ℓ linear cycles of length ℓ in G. Hence, using $d \geq (2r)^{\frac{1}{\epsilon^2}}$ and $m \leq (1-\epsilon)\log_d n$, we have

$$\mathbb{E}[\mathbb{Y}] \leq \sum_{\ell=3}^{m} n^\ell p^\ell = \sum_{\ell=3}^{m} (2rd)^\ell < 2(2rd)^m \leq 2^{m+1} d^{(1+\epsilon)m} \leq 2^{m+1} n^{1-\epsilon^2},$$

where the second to last inequality holds since $d \geq r^{1/\varepsilon}$. Therefore,

$$\mathbb{E}[\mathbb{X} - \mathbb{Y}] > \frac{1.8dn}{r} - 2^{m+1} n^{1-\epsilon^2} > \left(\frac{1.8d}{r} - \frac{2^{m+1}}{n^{\epsilon^2}}\right) n \geq \frac{dn}{r},$$

where the last inequality follows since $\frac{0.8d}{r} - \frac{2^{m+1}}{n^{\varepsilon^2}} \geq 0$ which holds because $0.8d \geq 2r$ and $d \geq 2^{\frac{1-\varepsilon}{\varepsilon^2}}$.

Hence there exists an r-graph F for which $\mathbb{X} - \mathbb{Y} \geq \frac{dn}{r}$. We can delete one edge from each linear cycle of length at most m in F. The remaining graph is an n-vertex linear r-graph that has average degree at least d and has no linear cycles of length at most $\lfloor (1-\varepsilon)\log_d n \rfloor$. $\square$

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