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TREE-DEGENERATE GRAPHS AND NESTED DEPENDENT RANDOM CHOICE*

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Abstract. The celebrated dependent random choice lemma states that in a bipartite graph, an average vertex (weighted by its degree) has the property that almost all small subsets S in its neighborhood have a common neighborhood almost as large as in the random graph of the same edge-density. There are two well-known applications of this lemma. The first is a theorem of Füredi [*Combinatorica*, 11 (1991), pp. 75–79] and Alon, Krivelevich, and Sudakov [*Combin. Probab. Comput.*, 12 (2003), pp. 477–494] showing that the maximum number of edges in an n-vertex graph not containing a fixed bipartite graph with maximum degree at most r on one side is $O(n^{2-1/r})$. This was recently extended by Grzesik, Janzer, and Nagy [*J. Combin. Theory Ser. B*, 156 (2022), pp. 299–309] to the family of so-called (r,t)-blowups of a tree. A second application is a theorem of Conlon, Fox, and Sudakov [*Geom. Funct. Anal.,* 20 (2010), pp. 1354–1366], confirming a special case of a conjecture of Erdős and Simonovits and of Sidorenko, showing that if H is a bipartite graph that contains a vertex that is completely joined to the other part and G is a graph, then the probability that the uniform random mapping from $V(H)$ to $V(G)$ is a homomorphism is at least $\left[\frac{2|E(G)|}{|V(G)|^2}\right]^{|E(H)|}$. In this paper, we introduce a nested variant of the dependent random choice lemma, which might be of independent interest. We then apply it to obtain a common extension of the theorem of Conlon, Fox, and Sudakov and the theorem of Grzesik, Janzer, and Nagy regarding Turán and Sidorenko properties of so-called tree-degenerate graphs.

Key words. extremal, Sidorenko, Turán

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1. Introduction. Given a graph G, let $|G|$ denote its number of vertices. A *homomorphism* from a graph H to a graph G is a mapping $f : V(H) \to V(G)$ such that for each edge uv in H, $f(u)f(v)$ is an edge in G. Let $h_H(G)$ denote the number of homomorphisms from H to G, and let $t_H(G) = h_H(G)/|G|^{|H|}$. Thus $t_H(G)$ represents the fraction of mappings from $V(H)$ to $V(G)$ that are homomorphisms. Viewed probabilistically, $t_H(G)$ is the probability that the uniform random mapping from $V(H)$ to $V(G)$ is a homomorphism. A beautiful conjecture of Sidorenko [24] is as follows.

CONJECTURE 1.1 (Sidorenko [24]). *Let H be any bipartite graph. For every graph G, we have*

$$t_H(G) \geq [t_{K_2}(G)]^{e(H)}.$$

Since $[t_{K_2}(G)]^{e(H)} = (\frac{2e(G)}{n^2})^{e(H)}$, one may view Sidorenko's conjecture as saying that the number of homomorphic copies of H in an n-vertex graph G is asymptotically at least as large as in the n-vertex random graph with the same edge-density. The following lemma, based on tensor products (see, e.g., Remark 2 in the English version

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of [23]), is commonly known and used in many earlier papers (see, e.g., [1], [5], [20]). It reduces the conjecture to a slightly weaker statement.

LEMMA 1.2 ([23]). *Let H be a bipartite graph. If there exists a positive constant c depending only on H such that for all graphs G, $t_H(G) \geq c[t_{K_2}(G)]^{e(H)}$ holds, then for all G, $t_H(G) \geq [t_{K_2}(G)]^{e(H)}$.*

When the edge-density of G is sufficiently high, it is expected that many of the homomorphisms from H to G are injective. Erdős and Simonovits [12] made several conjectures regarding the number of injective homomorphisms. As usual, let $\mathrm{ex}(n, H)$ denote the Turán number of H, which is the maximum number of edges in an n-vertex graph not containing H. Let $h^*_H(G)$ denote the number of injective homomorphisms from H to G, and let $t^*_H(G) = h^*_H(G)/|G|^{|H|}$. The first conjecture of Erdős and Simonovits from [12] states that for every $c > 0$ there is a $c' > 0$ such that if $e(G) > (1+c)\mathrm{ex}(n, H)$, then $t^*_H(G) \geq c' t_{K_2}(G)^{e(H)}$. The second, weaker, conjecture from [12] says that if $\mathrm{ex}(n, H) = O(n^{2-\alpha})$, then there exist constants $0 < \tilde{\alpha} \leq \alpha, c, c' > 0$ such that if $e(G) > cn^{2-\tilde{\alpha}}$, then $t^*_H(G) \geq c' t_{K_2}(G)^{e(H)}$. It is known (see [23]) that this weaker version is equivalent to Sidorenko's conjecture. However, compared to the stronger conjecture of Erdős and Simonovits, Sidorenko's conjecture does not give an explicit sharp edge-density threshold for when to guarantee the stated number of injective homomorphisms. There is yet another version of the Erdős–Simonovits conjecture, given in [21], that is equivalent to saying that there exist two constants $c, c' > 0$ such that if G is an n-vertex graph with $e(G) > c\mathrm{ex}(n, H)$, then $t^*_H(G) \geq c'[t_{K_2}(G)]^{e(H)}$.

Sidorenko [24] verified his own conjecture when H is a complete bipartite graph, an even cycle, a tree, or a bipartite graph with at most four vertices on one side. Hatami [17] proved that hypercubes satisfy Sidorenko's conjecture by developing a concept of norming graphs. The first major progress on Sidorenko's conjecture was made by Conlon, Fox, and Sudakov [5], who used the celebrated dependent random choice method (see [13] for a survey) to show the following.

THEOREM 1.3 (Conlon, Fox, and Sudakov [5]). *If H is a bipartite graph that has contains a vertex that is completely joined to the other part, then H satisfies Sidorenko's conjecture.*

In fact, Conlon, Fox, and Sudakov proved the stronger theorem that if H is a bipartite graph with bipartition (A, B), and contains vertices $v_1, \ldots, v_r$ in A complete to B, and degree of all u in A is at least d, then $t_H(G) \geq [t_{K_{r,d}}(G)]^{\frac{e(H)}{rd}}$. From Theorem 1.3, Conlon, Fox, and Sudakov [5] also deduced an approximate version of Sidorenko's conjecture. Since the work of Conlon, Fox, and Sudakov, there has been much further progress on Sidorenko's conjecture. Li and Szegedy [22] used the entropy method (presented in the form of logarithmic convexity inequalities) to extend the result of Conlon, Fox, and Sudakov to a more general family of graphs H, which they refer to as reflection trees. These ideas were further developed by Kim, Lee, and Lee [20], who proved the conjecture for what they called tree-arrangeable graphs and showed that if T is a tree and H is a bipartite graph that satisfies Sidorenko's conjecture, then the Cartesian product of T and H also satisfies Sidorenko's conjecture. Subsequently, Conlon et al. [8, 9] and independently Szegedy [25] established more families of bipartite graphs H for which Sidorenko's conjecture holds. These include bipartite graphs that admit a certain type of tree decomposition, subdivisions of certain graphs including cliques, certain Cartesian products, etc. More recently, Conlon and Lee [10] showed that Sidorenko's conjecture holds for any bipartite graph H with a bipartition (A, B) where the number of vertices in B of degree k satisfies a

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certain divisibility condition for each k. As a corollary, for every bipartite graph H with a bipartition (A,B) there is a positive integer p such that the blowup H_A^p formed by taking p vertex-disjoint copies of H and gluing all copies of A along corresponding vertices satisfies Sidorenko's conjecture.

Another line of work that motivates our result is related to a long-standing conjecture of Erdős regarding the Turán number of the so-called r-degenerate graphs. Given a positive integer r, a graph H is *r-degenerate* if its vertices can be linearly ordered such that each vertex has back degree at most r.

CONJECTURE 1.4 (Erdős [11]). *Let r be a fixed positive integer. Let H be any r-degenerate bipartite graph. Then* $\mathrm{ex}(n,H)=O(n^{2-1/r})$.

The first major progress on Conjecture 1.4 was the following theorem, which was first obtained by Füredi [14] in an implicit form and then later reproved by Alon, Krivelevich, and Sudakov [2] using the dependent random choice method.

THEOREM 1.5 (Füredi [14]; Alon, Krivelevich, and Sudakov [2]). *Let r be a positive integer. Let H be a bipartite graph with maximum degree at most r on one side. Then* $\mathrm{ex}(n,H)=O(n^{2-1/r})$.

The family of graphs satisfying the condition of Theorem 1.5 forms a very special family of r-degenerate bipartite graphs, which we will refer to as *one-side r-bounded* bipartite graphs. Recently, Grzesik, Janzer, and Nagy [16] extended Theorem 1.5 to a broader family of graphs called (r,t)-blowups of a tree.

DEFINITION 1.6 ((r,t)-blowups of a tree). *Let $r\le t$ and m be positive integers. A bipartite graph H is an (r,t)-blowup of a tree (or (r,t)-blowup for short) with root block B_0 and nonroot blocks $B_1,\dots,B_m$ if $B_0,B_1,\dots,B_m$ partition $V(H)$, $|B_0|=r, |B_1|=\cdots=|B_m|=t$ and H can be constructed by joining B_1 completely to B_0 and for each $2\le i\le m$ joining B_i completely to an r-subset of $B_{\gamma(i)}$ for some $\gamma(i)\le i-1$.*

THEOREM 1.7 (Grzesik, Janzer, and Nagy [16]). *Let $r\le t$ be positive integers. If H is an (r,t)-blowup of a tree, then* $\mathrm{ex}(n,H)=O(n^{2-1/r})$.

Since every one-side r-bounded graph is a subgraph of an (r,t)-blowup, Theorem 1.7 substantially generalizes Theorem 1.5.

In this paper, we give a common strengthening of Theorems 1.3 and 1.7 by proving a general theorem on the Turán and Sidorenko properties of so-called tree-degenerate graphs.

DEFINITION 1.8 (tree-degenerate graphs). *A bipartite graph H is called tree-degenerate with root block B_0 and nonroot blocks $B_1,\dots,B_m$ if $B_0,B_1,\dots,B_m$ partition $V(H)$ and H can be constructed by letting $P(B_1)=B_0$ and joining B_1 completely to B_0 and, for each $2\le i\le m$, joining B_i completely to a subset $P(B_i)$ of $B_{\gamma(i)}$ for some $1\le\gamma(i)\le i-1$, such that for all $i\ge 2$, $|P(B_{\gamma(i)})|\le|P(B_i)|$. We call $P(B_i)$ the parent set of B_i, call $B_{\gamma(i)}$ the parent block of B_i, and call P the parent function. We call $(B_0,\dots,B_m,P)$ a block representation of H.*

We present our main result in terms of so-called s-norm density. We will explain the advantage of doing so after presenting the theorem.

DEFINITION 1.9 (s-norm density). *Let G be a graph with n vertices. For each positive integer s, we define the s-norm density of G, denoted by $p_s(G)$, as*

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$$p_s(G) := t_{K_{1,s}}(G)^{1/s}.$$

Note that $p_1(G) = t_{K_2}(G) = \frac{2e(G)}{n^2}$ *is the usual edge-density of* G.

In general, one may view $p_s(G)$ as a modified measure of edge-density of G that takes the degree distribution into account. Using convexity, one can show that $p_r(G) \geq p_s(G)$ whenever $r \geq s$ (see Lemma 2.3).

THEOREM 1.10 (main theorem). *Let* H *be a tree-degenerate graph with a block representation* $(B_0, B_1, \ldots, B_m, P)$. *Let* $s = |B_0|$ *and* $r = \max_{i\geq 0} |P(B_i)|$. *There exist positive constants* $c_1 = c_1(H), c_2 = c_2(H), c_3 = c_3(H)$ *depending only on* H *such that for any graph* G,

$$t_H(G) \geq c_1[p_s(G)]^{e(H)} \geq c_1[t_{K_2}(G)]^{e(H)}.$$

Furthermore, if $h_{K_{1,s}}(G) > c_2 n^{s+1-\frac{s}{r}}$, *where* $n = |G|$, *then*

$$t^*_H(G) \geq c_3[p_s(G)]^{e(H)} \geq c_3[t_{K_2}(G)]^{e(H)}.$$

The first part of the theorem and Lemma 1.2 imply the following.

COROLLARY 1.11. *Let* H *be a tree-degenerate graph. Then* H *satisfies Sidorenko's conjecture.*

Since a bipartite graph H contains a vertex that is completely joined to the other part is tree-degenerate with $|B_0| = 1$ and $\gamma(i) = 1$ for all $i \geq 2$, Corollary 1.11 generalizes Theorem 1.3.

A special case of the second part of Theorem 1.10 yields the following.

COROLLARY 1.12. *Let* $r \leq t$ *be positive integers. Let* H *be an* (r,t)*-blowup of a tree with* h *vertices. Then* $\mathrm{ex}(n,H) = O(n^{2-1/r})$. *Furthermore, there exist constants* $c, c' > 0$ *such that every* n*-vertex graph* G *with* $h_{K_{1,r}}(G) \geq cn^r$ *contains at least* $c'n^h(\frac{2e(G)}{n^2})^{e(H)}$ *copies of* H.

Corollary 1.12 strengthens Theorem 1.7 in two ways. First, it relaxes the density requirement on G from $e(G) = \Omega(n^{2-1/r})$ to $h_{K_{1,r}}(G) = \Omega(n^r)$ (i.e., from $p_1(G) = \Omega(n^{-1/r})$ to $p_r(G) = \Omega(n^{-1/r})$). Second, it gives not only at least one copy of H but also an optimal number (up to a multiplicative constant) of copies of H. A closer examination of the proof of Theorem 1.7 given by Grzesik, Janzer, and Nagy in [16] shows that their proof can be strengthened to also give Corollary 1.12. However, Theorem 1.10 is more general than Corollary 1.12, as the counting statement applies to any tree-degenerate graph H, where parent set sizes can vary, instead of just to (r,t)-blowups. The relaxation of $p_1(G) = \Omega(n^{-1/r})$ to $p_r(G) = \Omega(n^{-1/r})$ is also a useful feature, as in bipartite Turán problems sometimes we need to handle cases where the host graph has very uneven degree distribution and hence high s-norm density, despite having relatively low 1-norm density (see [19] for an instance of this kind). It is also worth noting that the following proposition is implicit in several recent papers, such as [6] and [7]. (Lemma 2.8 of [6] is a bit more general than the proposition stated below, but the proof idea is the same.)

PROPOSITION 1.13 ([6], [7]). *Let* H *be a bipartite graph in which each vertex in one part has degree at most* r. *There exists a constant* $c = c(H)$ *such that if* G *is an* n*-vertex graph with* $h_{K_{1,r}}(G) \geq cn^r$, *then* G *contains* H *as a subgraph.*

Note that Corollary 1.12 generalizes and strengthens Proposition 1.13 in the sense that it applies to a broader family and also gives a counting result. To prove Theorem 1.10, we introduce a notion of goodness and prove a lemma that might be viewed

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as a nested variant of the dependent random choice lemma. Once we establish the lemma, the proof of Theorem 1.10 readily follows. Conceivably, this variant could find more applications.

We organize our paper as follows. In section 2, we introduce some preliminary lemmas. In section 3, we establish the nested goodness lemma. In section 4, we prove Theorem 1.10. In section 5, we give some concluding remarks.

2. Preliminary lemmas. In this section, we first give some useful lemmas. They will be used to motivate some definitions and will also be used in the proofs in later sections. We start with a standard convexity-based inequality, which is sometimes referred to as the power means inequality. We include a proof for completeness.

LEMMA 2.1. *Let n be a positive integer. Let $1 \le a \le b$ be reals. Let $x_1, \dots, x_n$ be nonnegative reals. Then*

$$\sum_{i=1}^{n} x_i^a \le n^{1-a/b} \cdot \left(\sum_{i=1}^{n} x_i^b \right)^{a/b}.$$

Equivalently,

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i^a \right)^{1/a} \le \left(\frac{1}{n} \sum_{i=1}^{n} x_i^b \right)^{1/b}.$$

Proof. Since the function $x^{b/a}$ is either linear or convex, by Jensen's inequality, we have $\sum_{i=1}^{n} x_i^b = \sum_{i=1}^{n} (x_i^a)^{b/a} \ge n[\frac{1}{n}\sum_{i=1}^{n} x_i^a]^{b/a}$. Rearranging, we obtain the desired inequalities. ∎

Let G be a graph with n vertices. Let s be a positive integer. Recall that $p_s(G) := t_{K_{1,s}}(G)^{1/s}$.

LEMMA 2.2. *For any n-vertex graph G and positive integer s,*

$$p_s(G) = \frac{1}{n} \left(\frac{1}{n} \sum_{v \in V(G)} d(v)^s \right)^{1/s}.$$

Proof. Recall that $t_{1,s}(G) = h_{K_{1,s}(G)}/n^{s+1}$, where $h_{K_{1,s}}(G)$ is the number of homomorphisms from $K_{1,s}$ to G. It is easy to see that $h_{K_{1,s}}(G) = \sum_{v \in V(G)} d(v)^s$. Hence,

$$p_s(G) = t_{K_{1,s}}(G)^{1/s} = (\frac{1}{n^{s+1}} \sum_{v \in V(G)} d(v)^s)^{1/s} = \frac{1}{n} (\frac{1}{n} \sum_{v \in V(G)} d(v)^s)^{1/s}.$$ ∎

Lemmas 2.2 and 2.1 imply the following useful fact.

LEMMA 2.3. *For any graph G and positive integers $r \ge s$, we have $p_r(G) \ge p_s(G)$.*

3. Nested goodness lemma. Given a set W and a sequence S of elements of W, we call S a sequence in W for brevity. The length of S is defined to be the number of elements in the sequence S (multiplicity counted) and is denoted by $|S|$. Given a positive integer k, we let W^k denote the set of sequences of length k in W and let W_k denote the set of sequences of length k in W in which the k elements are all different. Given a graph G and a sequence S in $V(G)$, the *common neighborhood* $N(S)$ is the set of vertices adjacent to every vertex in S.

We now introduce a goodness notion that is inspired by Lemma 2.1 of [5]. A more specialized version of it was introduced in [18].

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DEFINITION 3.1 (s-norm i-good sequences). *Let $0 < \alpha, \beta < 1$ be reals. Let s, h be positive integers. Let G be an n-vertex graph. Let $p_s = p_s(G)$. For each $0 \le i \le h$, we define an s-norm i-good sequence in $V(G)$ relative to (α, β, h) (or simply (s,i)-good for short) as follows. We say that a sequence T in $V(G)$ is $(s,0)$-good if $|N(T)| \ge \alpha p_s^{|T|} n$. For all $1 \le i \le h$, we say that a sequence S of length at most h in $V(G)$ is (s,i)-good if S is $(s,0)$-good and for each $|S| \le k \le h$, the number of $(s, i-1)$-good sequences of length k in $N(S)$ is at least $(1-\beta)|N(S)|^k$.*

Below is our main theorem on the goodness notion.

THEOREM 3.2 (nested goodness lemma). *Let $h \ge s$ be positive integers. Let $0 < \beta < 1$ be a real. There is a positive real α depending on h, s, and β such that the following is true. Let G be any graph on n vertices. Let $p_s = p_s(G)$. For each $0 \le i \le h, 1 \le j \le h$, let $\mathcal{A}^s_{i,j}$ denote the set of (s,i)-good sequences of length j relative to (α, β, h) in $V(G)$. Then for each $0 \le i \le h$ and $s \le \ell \le j \le h$,*

$$\sum_{S \in \mathcal{A}^s_{i,j}} |N(S)|^\ell \ge (1-\beta) n^{j+\ell} p_s^{j\ell}.$$

In particular, there exists an (s,i)-good sequence S of size j such that $|N(S)| \ge (1-\beta)^{1/s} p_s^j n$.

Applying Theorem 3.2 with $s = 1, \ell = 1$, we get the following, which may be of independent interest.

THEOREM 3.3. *Let h be a positive integer. Let $0 < \beta < 1$ be a real. There is a positive real α depending on h and β such that the following is true. Let G be any graph on n vertices. Let $p_1 = \frac{2e(G)}{n^2}$. For any $i, j \in [h]$, let $\mathcal{A}^1_{i,j}$ denote the set of $(1,i)$-good sequences of length j relative to (α, β, h) in $V(G)$. Then, for all $i \in [h]$,*

$$\sum_{S \in \mathcal{A}^1_{i,j}} |N(S)| \ge (1-\beta) n^{j+1} p_1^j.$$

In particular, there exists an $(1,i)$-good sequence S of size j such that $|N(S)| \ge (1-\beta) p_1^j n$.

Loosely speaking, one may think of the usual dependent random choice lemma as saying that for any positive integers j, h and real $0 < \beta < 1$, there is a $(1,1)$-good sequence S of size j relative to (α, β, h) for some appropriate $\alpha > 0$; that is, most of the subsets in $N(S)$ of size at most h have their common neighborhood fractionally as large as expected in the random graph of the same edge-density. Theorem 3.2 follows from the next, more technical, lemma.

LEMMA 3.4. *Let $h \ge s$ be positive integers. Let $0 < \beta < 1$ be a real. There exists a positive real α depending on h, s, and β such that the following is true. Let G be any graph on n vertices. For each $0 \le i \le h$ and $1 \le j \le h$, let $\mathcal{A}^s_{i,j}$ denote the set of (s,i)-good sequences of length j relative to (α, β, h) in $V(G)$, and let $\mathcal{B}^s_{i,j} = [V(G)]^j \setminus \mathcal{A}_{i,j}$. Then for each $0 \le i \le h, 1 \le j \le h$, and $1 \le \ell \le j$,*

$$\sum_{S \in \mathcal{B}^s_{i,j}} |N(S)|^\ell \le \beta n^{j+\ell} p_s^{j\ell}.$$

Proof. Suppose α has been specified; we define a sequence α_i, $0 \le i \le h$, by letting $\alpha_0 = \alpha$ and $\alpha_i = \alpha + h(\alpha_{i-1}/\beta)^{1/h}$ for each $i \in [h]$. For fixed h and β, it is easy to

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see that by choosing α to be small enough, $\alpha_h < \beta$. Note also that the α_i are an increasing function in terms of i. Let us fix such an α. Now, let $\mathcal{A}^s_{i,j}$ and $\mathcal{B}^s_{i,j}$ be defined as stated. We use induction on i to prove that for all $0 \le i \le h, j \in [h]$, and $1 \le \ell \le j$,

$$\sum_{S \in \mathcal{B}^s_{i,j}} |N(S)|^\ell \le \alpha_i n^{j+\ell} p_s^{j\ell}.$$

For the basis step, let $i = 0$. Let $j, \ell \in [h]$ where $\ell \le j$. By the definition of $(s,0)$-goodness,

$$\sum_{S \in \mathcal{B}^s_{0,j}} |N(S)|^\ell \le n^j (\alpha p_s^j n)^\ell \le \alpha n^{j+\ell} p_s^{j\ell} = \alpha_0 n^{j+\ell} p_s^{j\ell}. \tag{3.1}$$

Hence the claim holds for $i = 0$. For the induction step, let $i \ge 1$, and suppose the claim holds when i is replaced with $i-1$. Let $j \in [h]$. For each $j \le k \le h$, let $\mathcal{C}^s_{i,j,k}$ denote the set of sequences S in $\mathcal{B}^s_{i,j}$ such that the number of sequences of length k in $N(S)$ that are not $(s, i-1)$-good is at least $\beta|N(S)|^k$. By definition, $\mathcal{B}^s_{i,j} = \mathcal{B}^s_{0,j} \cup \bigcup_{k=j}^h \mathcal{C}^s_{i,j,k}$. Let $\mathcal{F}_k$ be the collection of pairs (S,T), where $S \in \mathcal{C}^s_{i,j,k}$ and T is a sequence of length k in $N(S)$ that is not $(s, i-1)$-good. By our definition,

$$|\mathcal{F}_k| \ge \sum_{S \in \mathcal{C}^s_{i,j,k}} \beta |N(S)|^k = \beta \cdot \sum_{S \in \mathcal{C}^s_{i,j,k}} |N(S)|^k.$$

On the other hand, for each sequence T of length k in $V(G)$ that is not $(s, i-1)$-good, the number of sequences S of length j that satisfy $(S,T) \in \mathcal{F}_k$ is most $|N(T)|^j$. Hence,

$$|\mathcal{F}_k| \le \sum_{T \in \mathcal{B}^s_{i-1,k}} |N(T)|^j \le \alpha_{i-1} n^{j+k} p_s^{jk},$$

where the last inequality follows from the induction hypothesis. Combining the lower and upper bounds on $|\mathcal{F}_k|$, we get

$$\sum_{S \in \mathcal{C}^s_{i,j,k}} |N(S)|^k \le (\alpha_{i-1}/\beta) n^{j+k} p_s^{jk}. \tag{3.2}$$

Let $\ell \in [h]$ such that $\ell \le j$. Since $j \le k$, we have $\ell \le k$. Applying Lemma 2.1 with $a = \ell, b = k$ and using $|\mathcal{C}^s_{i,j,k}| \le n^j$, we get

$$\sum_{S \in \mathcal{C}^s_{i,j,k}} |N(S)|^\ell \le (n^j)^{1-\ell/k} (\alpha_{i-1}/\beta)^{\ell/k} (n^{j+k} p_s^{jk})^{\ell/k} \le (\alpha_{i-1}/\beta)^{1/h} n^{j+\ell} p_s^{j\ell},$$

where we used the fact that $\alpha_{i-1}/\beta < 1$. By (3.1) and (3.2), we have

$$\begin{aligned}\sum_{S \in \mathcal{B}^s_{i,j}} |N(S)|^\ell &\le \sum_{S \in \mathcal{B}^s_{0,j}} |N(S)|^\ell + \sum_{k=j}^h \sum_{S \in \mathcal{C}^s_{i,j,k}} |N(S)|^\ell \\ &\le [\alpha + h(\alpha_{i-1}/\beta)^{1/h}] n^{j+\ell} p_s^{j\ell} \le \alpha_i n^{j+\ell} p_s^{j\ell}.\end{aligned}$$

This completes the induction and the proof. ∎

Remark 3.5. In Lemma 3.4, the reason we impose the condition $\ell \le j$ is because its proof uses Lemma 2.1, which holds only when $a \le b$. Also, due to this restriction, in our definition of tree-degenerate graphs (Definition 1.8), we impose the condition that for each $i \ge 2$, $|P(B_{\gamma(i)})| \le |P(B_i)|$.

We need another quick lemma. Given two positive integers n, j, let $n_j = n(n-1)\cdots(n-j+1)$.

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LEMMA 3.6. *Let G be a graph on n vertices, and let $\ell \geq s$ and j be positive integers. Let $p_s = p_s(G)$. Then $\sum_{S\in[V(G)]^j} |N(S)|^\ell \geq n^{j+\ell}p_s^{j\ell}$. If $p_s \geq 4jn^{-1/\ell}$, then $\sum_{S\in[V(G)]_j} |N(S)|^\ell \geq \frac{1}{2^{j+1}}n^{j+\ell}p_s^{j\ell}$.*

Proof. First, note that $\sum_{T\in[V(G)]^\ell} |N(T)| = h_{K_{1,\ell}}(G) = n^{\ell+1}t_{K_{1,\ell}}(G) = n^{\ell+1}p_\ell^\ell$. Hence, by convexity,

$$\sum_{T\in[V(G)]^\ell} |N(T)|^j \geq n^\ell \left(\frac{\sum_{T\in[V(G)]^\ell}|N(T)|}{n^\ell}\right)^j = n^\ell(np_\ell^\ell)^j = n^{j+\ell}p_\ell^{j\ell} \geq n^{j+\ell}p_s^{j\ell}.$$

If $p_s > 4jn^{-1/\ell}$, then

$$\sum_{T\in[V(G)]^\ell, |N(T)|\geq 2j} |N(T)|^j \geq n^{j+\ell}p_s^{j\ell} - n^\ell(2j)^j \geq \frac{1}{2}n^{j+\ell}p_s^{j\ell}.$$

Hence,

$$\sum_{T\in[V(G)]^\ell, |N(T)|\geq 2j} |N(T)|_j \geq \sum_{T\in[V(G)]^\ell, |N(T)|\geq 2j} (|N(T)|/2)^j \geq \frac{1}{2^{j+1}}n^{j+\ell}p_s^{j\ell}.$$

To prove the first statement, note that $\sum_{S\in[V(G)]^j} |N(S)|^\ell$ counts pairs (S,T), where S is a sequence of length j and T is a sequence of length ℓ in $N(S)$. By double counting, we have $\sum_{S\in[V(G)]^j} |N(S)|^\ell = \sum_{T\in[V(G)]^\ell} |N(T)|^j \geq n^{j+\ell}p_s^{j\ell}$.

For the second statement, note that $\sum_{S\in[V(G)]_j} |N(S)|^\ell$ counts pairs (S,T), where S is a sequence of length j with no repetition and T is a sequence of length ℓ in $N(S)$. By a similar double counting, we have

$$\sum_{S\in[V(G)]_j} |N(S)|^\ell = \sum_{T\in[V(G)]^\ell} |N(T)|_j \geq \frac{1}{2^{j+1}}n^{j+\ell}p_s^{j\ell}. \qquad \square$$

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $h \geq s$ be positive integers and $0 < \beta < 1$ be a real. Let α be defined as in Lemma 3.4. Let $0 \leq i \leq h$ and $s \leq \ell \leq j \leq h$. Let $\mathcal{A}_{i,j}^s$ denote the set of (s,i)-good sequences of length j relative to (α,β,h) in $V(G)$, and let $\mathcal{B}_{i,j}^s = [V(G)]^j \setminus \mathcal{A}_{i,j}^s$. By Lemma 3.6,

$$\sum_{S\in[V(G)]^j} |N(S)|^\ell \geq n^{j+\ell}p_s^{j\ell}.$$

By Lemma 3.4, $\sum_{S\in\mathcal{B}_{i,j}^s} |N(S)|^\ell \leq \beta n^{j+\ell}p_s^{j\ell}$. Hence, $\sum_{S\in\mathcal{A}_{i,j}^s} |N(S)|^\ell \geq (1-\beta)n^{j+\ell}p_s^{j\ell}$, as desired. This proves the first part of the theorem. Now, since $|\mathcal{A}_{i,j}^s| \leq n^j$, applying the first part of the theorem with $\ell = s$ and via averaging, there exists an $S \in \mathcal{A}_{i,j}^s$ such that $|N(S)|^s \geq (1-\beta)p_s^{js}n^s$ and hence $|N(S)| \geq (1-\beta)^{1/s}p_s^j n$. This proves the second part of the theorem. $\square$

In order to prove the second part of Theorem 1.10, we need the following variant of Theorem 3.2. We omit the proof since it is almost identical to that of Theorem 3.2, except that we use the second statement of Lemma 3.6 instead of the first statement.

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LEMMA 3.7. *Let $h \geq \ell \geq s$ be positive integers. Let $0 < \beta < 1$ be a real. There is a positive real α depending on h, s, and β such that the following is true. Let G be any graph on n vertices. Let $p_s = p_s(G)$. For each $0 \leq i \leq h, 1 \leq j \leq h$, let $\widetilde{\mathcal{A}}^s_{i,j}$ denote the set of (s,i)-good sequences of length j relative to (α, β, h) in $V(G)$ that have no repetition. If $p_s > 4jn^{-1/\ell}$, then for each $0 \leq i \leq h$ and $s \leq \ell \leq j \leq h$,*

$$\sum_{S \in \widetilde{\mathcal{A}}^s_{i,j}} |N(S)|^\ell \geq \left(\frac{1}{2^{j+1}} - \beta\right) n^{j+\ell} p_s^{j\ell}.$$

4. Proof of Theorem 1.10.

Proof of Theorem 1.10. Let $n = |G|$ and $h = |H|$. Let $\beta = \frac{1}{h2^{h+2}}$. Recall that $s = |B_0|$ and $r = \max_{i \geq 0} |P(B_i)|$. Since $P(B_1) = B_0$, we have $r \geq s$. Let α be the positive constant given by Theorem 3.2 for the given h, s, and β. Let

$$c_1 = c_1(H) = (1-\beta)^{|B_1|/|B_0|} \alpha^{\sum_{i=2}^m |B_i|} (1 - h\beta)^{m-1}.$$

Let

$$c_2 = c_2(H) = 4h/\alpha \quad \text{and} \quad c_3 = c_3(H) = c_1/2^{h^2+4h}.$$

Suppose H has root block B_0 and nonroot blocks $B_1, \ldots, B_m$ such that B_1 is completely joined to its parent set $P(B_1) = B_0$ and for each $i = 2, \ldots, m$, B_i is completely joined to its parent set $P(B_i)$ where $P(B_i) \subseteq B_{\gamma(i)}$ for some $1 \leq \gamma(i) < i$ and $|P(B_i)| \geq |P(B_{\gamma(i)})|$. For each $i \in [m]$, let $\mathcal{F}_i$ denote the collection of all the parent sets $P(B_j)$ that are contained in B_i. Let T be a tree with $V(T) := \{v_0, v_1, \ldots, v_m\}$ and edge set $E(T) := v_0v_1 \cup \{v_i v_{\gamma(i)} :\in [m]\}$. We call T the *auxiliary tree* for H. For each $i \in [m]$, define the *depth* of B_i, denoted by d_i, to be the distance from v_0 to v_i in T. Let q denote the maximum depth of a block. Then clearly $q \leq m \leq h-1$.

Let G be any graph. For convenience, we say that a sequence in $V(G)$ is (s,i)-good if it is (s,i)-good relative to (α, β, h). As in Theorem 3.2, for each $0 \leq i \leq h$ and $s \leq j \leq h$, let $\mathcal{A}^s_{i,j}$ be the set of (s,i)-good sequences of length j in $V(G)$. Let $\mathcal{B}^s_{i,j} = [V(G)]^j \setminus \mathcal{A}^s_{i,j}$. Let $\widetilde{\mathcal{A}}^s_{i,j}$ be the set of (s,i)-good sequences of length j in $V(G)$ that contain no repetition. Let f be the uniform random mapping from $V(H)$ to $V(G)$.

Let

$E_1 =$ the event that $f(B_0) \in \mathcal{A}^s_{q,|B_0|}$ and $f(B_1) \in [N(f(B_0))]^{|B_1|}$,
$F_1 =$ the event that each sequence in $\mathcal{F}_1$ is mapped to an $(s, q-1)$-good sequence,
$E_1^* =$ the event that $f(B_0) \in \widetilde{\mathcal{A}}^s_{q,|B_0|}$ and $f(B_1) \in [N(f(B_0))]^{|B_1|}$, and
$L_1 =$ the event that f is injective on $B_0 \cup B_1$.

For each $i \in \{2, \ldots, m\}$, let

$E_i =$ the event that $f(B_i) \in [N(f(P(B_i)))]^{|B_i|}$,
$F_i =$ the event that each sequence in $\mathcal{F}_i$ is mapped to an $(s, q-d_i)$-good sequence, and
$L_i =$ the event that f is injective on $B_0 \cup B_1 \cup \cdots \cup B_i$.

Since $s = |B_0|$, by Theorem 3.2,

$$\sum_{S \in \mathcal{A}^s_{q,|B_0|}} |N(S)|^s \geq (1-\beta) n^{2s} p_s^{s^2}. \tag{4.1}$$

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Furthermore, if $p_s \geq 4jn^{-1/r}$, then since $r \geq s$, we have $p_s \geq 4jn^{-1/s}$, and by Lemma 3.7,

$$\sum_{S \in \widetilde{\mathcal{A}}^s_{q,|B_0|}} |N(S)|^s \geq \left(\frac{1}{2^{j+1}} - \beta\right) n^{2s} p_s^{s^2}. \tag{4.2}$$

Hence, since $|B_1| \geq |B_0| = s$, using (4.1) and convexity we get

$$\begin{aligned}\sum_{S \in \mathcal{A}^s_{q,|B_0|}} |N(S)|^{|B_1|} &= \sum_{S \in \mathcal{A}^s_{q,|B_0|}} (|N(S)|^s)^{\frac{|B_1|}{s}} \\ &\geq n^s \left(\frac{1}{n^s}(1-\beta) n^{2s} p_s^{s^2}\right)^{\frac{|B_1|}{s}} \\ &= (1-\beta)^{\frac{|B_1|}{|B_0|}} n^{|B_0|+|B_1|} p_s^{|B_0||B_1|},\end{aligned}$$

and if $p_s \geq 4|B_0|n^{-1/r}$, then

$$\sum_{S \in \widetilde{\mathcal{A}}^s_{q,|B_0|}} |N(S)|^{|B_1|} \geq \left(\frac{1}{2^{j+1}} - \beta\right)^{\frac{|B_1|}{|B_0|}} n^{|B_0|+|B_1|} p_s^{|B_0||B_1|}.$$

Hence,

$$\begin{aligned}\mathbb{P}(E_1) &= \sum_{S \in \mathcal{A}^s_{q,|B_0|}} \frac{1}{n^{|B_0|}} \cdot \frac{|N(S)|^{|B_1|}}{n^{B_1}} = \frac{1}{n^{|B_0|+|B_1|}} \sum_{S \in \mathcal{A}^s_{q,|B_0|}} |N(S)|^{|B_1|} \\ &\geq (1-\beta)^{\frac{|B_1|}{|B_0|}} p_s^{|B_0||B_1|},\end{aligned} \tag{4.3}$$

and if $p_s \geq 4|B_0|n^{-1/r}$, then

$$\begin{aligned}\mathbb{P}(E_1^*) &= \sum_{S \in \widetilde{\mathcal{A}}^s_{q,|B_0|}} \frac{1}{n^{|B_0|}} \cdot \frac{|N(S)|^{|B_1|}}{n^{B_1}} \\ &\geq \left(\frac{1}{2^{j+1}} - \beta\right)^{\frac{|B_1|}{|B_0|}} p_s^{|B_0||B_1|} \geq \left(\frac{1}{2^{h+2}}\right)^{|B_1|} p_s^{|B_0||B_1|}.\end{aligned} \tag{4.4}$$

We now bound $\mathbb{P}(F_1|E_1)$. Recall that $\mathcal{F}_1$ consists of parent sets $P(B_j)$ that are contained in B_1. By requirement, these sets have size at least $|P(B_1)| = |B_0|$. Let S be any fixed sequence in $\mathcal{A}^s_{q,|B_0|}$. By the definition of $\mathcal{A}^s_{q,|B_0|}$, for each $|B_0| \leq k \leq h$, the number of $(s,q-1)$-good sequences of length k in $N(S)$ is at least $(1-\beta)|N(S)|^k$. So, conditioning on f mapping B_0 to S and B_1 to $N(S)$, the probability that f maps any particular sequence in $\mathcal{F}_1$ to an $(s,q-1)$-good sequence is at least $(1-\beta)$. Since there are clearly at most h sequences in $\mathcal{F}_1$, the probability that f maps every sequence in $\mathcal{F}_1$ to an $(s,q-1)$-good sequence is at least $1-h\beta$. Hence,

$$\mathbb{P}(F_1|E_1) \geq 1 - h\beta. \tag{4.5}$$

Furthermore, as B_0 is mapped to an (s,q)-good sequence and thus is an $(s,0)$-good sequence, we have that

$$|N(f(B_0))| \geq \alpha p_s^{|B_0|} n \geq \alpha[(4h/\alpha)^{1/s} n^{-1/r}]^s n \geq 4h.$$

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Since $|N(f(B_0)) \setminus f(B_0)| \geq 3h$, we have that

$$\mathbb{P}(L_1|E_1^*) > (3h)_{|B_1|}/(4h)^{|B_1|} > (1/2)^{|B_1|}.$$

Now, by definition,

$$\mathbb{P}(L_1|E_1^* F_1) = \frac{\mathbb{P}(L_1 F_1|E_1^*)}{\mathbb{P}(F_1|E_1^*)}.$$

As $h\beta = \left(\frac{1}{2}\right)^{h+2} < \left(\frac{1}{2}\right)^{|B_1|+2}$,

$$\mathbb{P}(L_1 F_1|E_1^*) \geq \mathbb{P}(L_1|E_1^*) - \mathbb{P}(\overline{F_1}|E_1^*) \geq (1/2)^{|B_1|} - h\beta \geq (1/2)^{|B_1|+1}.$$

Finally, as $\mathbb{P}(F_1|E_1^*) \leq 1$, we have

$$\mathbb{P}(L_1|E_1^* F_1) \geq (1/2)^{|B_1|+1}. \tag{4.6}$$

For each $i = 2, \ldots, h$, we estimate $\mathbb{P}(E_i|E_1F_1 \ldots E_{i-1}F_{i-1})$. Assume the event $E_1F_1 \cdots E_{i-1}F_{i-1}$. Since $P(B_i) \subseteq B_{\gamma(i)}$, where $\gamma(i) < i$, by our assumption, $P(B_i)$ is mapped to an $(s, q-d_{\gamma(i)})$-good sequence. Since an $(s, q-d_{\gamma(i)})$-sequence is $(s, 0)$-good, by definition, we have $|N(f(P(B_i)))| \geq \alpha p_s^{|P(B_i)|} n$. Hence,

$$\begin{aligned} \mathbb{P}(E_i|E_1F_1 \ldots E_{i-1}F_{i-1}) &= \frac{|N(f(P(B_i)))|^{|B_i|}}{n^{|B_i|}} \\ &\geq \frac{(\alpha p_s^{|P(B_i)|} n)^{|B_i|}}{n^{|B_i|}} = \alpha^{|B_i|} p_s^{|P(B_i)||B_i|}. \end{aligned} \tag{4.7}$$

Now assume $E_1F_1 \ldots E_{i-1}F_{i-1}E_i$. Since $S := f(P(B_i))$ is an $(s, q-d_{\gamma(i)})$-good sequence, by definition, for each $|S| \leq k \leq h$ the number of $(s, q-1-d_{\gamma(i)})$-good sequences of length k in $N(S)$ is at least $(1-\beta)|N(S)|^k$. Recall that $\mathcal{F}_i$ consists of parent sets $P(B_j)$ for all $j > 1$ that are contained in B_i. By requirement, these sets have size at least $|P(B_i)| = |S|$. By our discussion above, given $E_1F_1 \ldots E_{i-1}F_{i-1}E_i$, the probability of any member of $\mathcal{F}_i$ not mapped to an $(s, q-1-d_{\gamma(i)})$-good sequence is at most β. Since there are at most h sequences in $\mathcal{F}_i$, we have

$$\mathbb{P}(F_i|E_1F_1 \ldots E_{i-1}F_{i-1}E_i) \geq 1 - h\beta. \tag{4.8}$$

By (4.3), (4.5), (4.7), and (4.8), we have

$$\begin{aligned} &\mathbb{P}(f \text{ is a homomorphism}) \geq \mathbb{P}(E_1F_1 \ldots E_{m-1}F_{m-1}E_m) \\ &\geq (1-\beta)^{\frac{|B_1|}{|B_0|}} \alpha^{\sum_{i=2}^m |B_i|} (1-h\beta)^{m-1} p_s^{|B_0||B_1| + \sum_{i=2}^m |P(B_i)||B_i|} \\ &= c_1 p_s^{e(H)}. \end{aligned} \tag{4.9}$$

This proves the first (main) part of the theorem.

For the second statement, suppose $h_{K_{1,s}}(G) > c_2 n^{s+1-\frac{s}{r}}$. Then

$$p_s = (h_{K_{1,s}}(G)/n^{s+1})^{1/s} \geq c_2^{1/s} n^{-1/r} = (4h/\alpha)^{1/s} n^{-1/r}.$$

For each $i \geq 2$, we bound $\mathbb{P}(L_i|E_1^*F_1L_1E_2F_2L_2 \ldots L_{i-1}E_iF_i)$. Assume the event $E_1^*F_1L_1E_2F_2L_2 \ldots L_{i-1}E_iF_i$. By our assumption, $P(B_i)$ is mapped to an $(s, q-d_{\gamma(i)})$-good sequence, and B_i is mapped into $N(f(P(B_i)))$. Since $f(P(B_i))$ is $(s, 0)$-good, $|P(B_i)| \leq r$, and $r \geq s$,

$$|N(f(P(B_i)))| \geq \alpha p_s^{|P(B_i)|} n \geq \alpha[(4h/\alpha)^{1/s} n^{-1/r}]^r n \geq 4h.$$

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Given $E_1^*F_1L_1E_2F_2L_2\dots L_{i-1}E_iF_i$, the probability that f maps B_i injectively into $N(F(P(B_i))$ and avoids $f(B_0\cup B_1\cup\dots\cup B_{i-1})$ is at least $(3h)_{|B_i|}/(4h)^{|B_i|} > (1/2)^{|B_i|}$. Following the reasoning leading to (4.6), we have that

$$\mathbb{P}(L_i|E_1^*F_1L_1E_2F_2L_2\dots L_{i-1}E_iF_i) > (1/2)^{|B_i|+1}. \tag{4.10}$$

By (4.4), (4.10), and a calculation similar to (4.9), we have

$$\begin{aligned}\mathbb{P}(f \text{ is an injective homomorphism}) &\geq \mathbb{P}(E_1^*F_1L_1E_2F_2L_2\dots E_mF_mL_m)\\ &\geq \left(\frac{1}{2^{h+2}}\right)^{|B_1|}\left(\frac{1}{2}\right)^{m+|B_1|+\dots+|B_m|} c_1p_s^{e(H)}\\ &\geq \left(\frac{1}{2^{h+2}}\right)^{|B_1|}\left(\frac{1}{2}\right)^{2h} c_1p_s^{e(H)}\\ &\geq \frac{1}{2^{h^2+4h}}c_1p_s^{e(H)} = c_3p_s^{e(H)}.\end{aligned}$$

This proves the second part of the theorem. ∎

5. Concluding remarks. In this paper, we used a nested variant of the dependent random choice to not only embed an appropriate tree-degenerate bipartite graph H in a host graph G, but also to give tight (up to a multiplicative factor) bound on the number of copies of H in G. In this variant, we are able to show that the standard notion of goodness extends to the iterated version almost for free. It will be interesting to find more applications of it.

Another interesting feature of Theorem 1.10 is that the condition of the host graph is relaxed from 1-norm density to s-norm density, which makes the result more flexible for applications. In principle, one could study the so-called *s-norm* Turán problem for bipartite graphs, where one wants to determine the maximum s-norm density of an H-free graph on n vertices for a given bipartite graph H. The problem seems particularly natural for the family of s-degenerate graphs. For hypergraph codegree problems, such a study has recently been initiated by Balogh, Clemen, and Lidický [3], [4].

Last but not least, it will be highly desirable to make more progress on Conjecture 1.4 beyond the following general bound obtained by Alon, Krivelevich, and Sudakov [2], which has stood as the best known upper bound for the last two decades.

THEOREM 5.1 ([2]). *If H is an r-degenerate bipartite graph, then* $\mathrm{ex}(n,H) = O(n^{2-1/4r})$.

REFERENCES

[1] N. Alon and I. Z. Ruzsa, *Non-averaging subsets and non-vanishing transversals*, J. Combin. Theory Ser. A, 86 (1999), pp. 1–13.
[2] N. Alon, M. Krivelevich, and B. Sudakov, *Turán numbers of bipartite graphs and related Ramsey-type questions*, Combin. Probab. Comput., 12 (2003), pp. 477–494.
[3] J. Balogh, F. C. Clemen, and B. Lidický, *Hypergraph Turán Problems in ℓ_2-Norm*, preprint, arXiv:2108.10406, 2021.
[4] J. Balogh, F. C. Clemen, and B. Lidický, *Solving Turán's tetrahedron problem for the ℓ_2-norm*, London J. Math., to appear.
[5] D. Conlon, J. Fox, and B. Sudakov, *An approximate version of Sidorenko's conjecture*, Geom. Funct. Anal., 20 (2010), pp. 1354–1366.
[6] D. Conlon and O. Janzer, *Rational exponents near two*, Adv. Combin., 9 (2022), https://doi.org/10.19086/aic.2022.9.

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[7] D. Conlon, O. Janzer, and J. Lee, *More on the extremal number of subdivisions*, Combinatorica, 41 (2021), pp. 465–494.
[8] D. Conlon, J. H. Kim, C. Lee, and J. Lee, *Some advances on Sidorenko's conjecture*, J. Lond. Math. Soc., 2 (2018), pp. 98–608.
[9] D. Conlon, J. H. Kim, C. Lee, and J. Lee, *Sidorenko's Conjecture for Higher Tree Decompositions*, preprint, https://arxiv.org/abs/1805.02238, 2018.
[10] D. Conlon and J. Lee, *Sidorenko's conjecture for blowups*, Discrete Anal., (2021), 2, https://doi.org/10.19086/da.21472.
[11] P. Erdős, *Some recent results on extremal problems in graph theory*, in Theory of Graphs (Rome, 1966), Gordon and Breach, New York, pp. 117–123.
[12] P. Erdős and M. Simonovits, *Cube-supersaturated graphs and related problems*, in Progress in Graph Theory (Waterloo, Ont., 1982), Academic Press, Toronto, 1984, pp. 203–218.
[13] J. Fox and B. Sudakov, *Dependent random choice*, Random Structures Algorithms, 38 (2011), pp. 68–99.
[14] Z. Füredi, *On a Turán type problem of Erdős*, Combinatorica, 11 (1991), pp. 75–79.
[15] Z. Füredi and M. Simonovits, *The history of the degenerate (bipartite) extremal graph problems*, in Erdős Centennial, Bolyai Soc. Math. Stud. 25, János Bolyai Math. Soc., Budapest, 2013, pp. 169–264.
[16] A. Grzesik, O. Janzer, and Z. Nagy, *The Turán number of blow-ups of trees*, J. Combin. Theory Ser. B, 156 (2022), pp. 299–309.
[17] H. Hatami, *Graph norms and Sidorenko's conjecture*, Israel J. Math., 175 (2010), pp. 125–150.
[18] T. Jiang and A. Newman, *Small dense subgraphs of a graph*, SIAM J. Discrete Math., 31 (2017), pp. 124–142, https://doi.org/10.1137/15M1007598.
[19] O. Janzer, A. Methuku, and Z. L. Nagy, *On the Turán number of the blow-up of the hexagon*, SIAM J. Discrete Math., 36 (2022), pp. 1187–1199, https://doi.org/10.1137/21M1428510.
[20] J. H. Kim, C. Lee, and J. Lee, *Two approaches to Sidorenko's conjecture*, Trans. Amer. Math. Soc., 368 (2016), pp. 5057–5074.
[21] M. Simonovits, *Extremal graph problems, degenerate extremal problems and super-saturated graphs*, in Progress in Graph Theory (Waterloo, Ont., 1982), Academic Press, Toronto, 1984, pp. 419–437.
[22] J. L. Li and B. Szegedy, *On the logarithmic calculus and Sidorenko's conjecture*, Combinatorica, to appear.
[23] A. F. Sidorenko, *Inequalities for functionals generated by bipartite graphs*, Diskret. Mat., 3 (1991), pp. 50–65, (in Russian); Discrete Math. Appl., 2 (1992), pp. 489–504 (English translation).
[24] A. F. Sidorenko, *A correlation inequality for bipartite graphs*, Graph. Combin., 9 (1993), pp. 201–204.
[25] B. Szegedy, *An Information Theoretic Approach to Sidorenko's Conjecture*, preprint, https://doi.org/10.48550/arXiv.1406.6738, 2015.

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