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From regression rank scores to robust inference for censored quantile regression

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Abstract: Quantile regression for right- or left-censored outcomes has attracted attention due to its ability to accommodate heterogeneity in regression analysis of survival times. Rank-based inferential methods have desirable properties for quantile regression analysis, but censored data poses challenges to the general concept of ranking. In this article, we propose a notion of censored quantile regression rank scores, which enables us to construct rank-based tests for quantile regression coefficients at a single quantile or over a quantile region. A model-based bootstrap algorithm is proposed to implement the tests. We also illustrate the advantage of focusing on a quantile region instead of a single quantile level when testing the effect of certain covariates in a quantile regression framework.

Résumé: En analyse de survie pour données censurées à gauche ou à droite, la régression quantile a suscité beaucoup d'attention en raison de sa capacité à accommoder l'hétérogénéité dans le cadre d'analyses de régression. Les méthodes inférentielles fondées sur les rangs ont de nombreuses propriétés attrayantes pour des analyse de régression quantile, mais elles sont confrontées à des défis posés par le concept d'ordre en présence de censure. Pour contourner cette difficulté, les auteurs de ce travail proposent un concept de régression quantile à scores de rangs censurés. Cette approche permet de construire des tests de rangs pour les coefficients du modèle de régression quantile en question, que ce soit pour un seul quantile ou tout un ensemble de quantiles. L'implantation de tels tests repose sur un algorithme bootstrap fondé sur un modèle. Enfin, les auteurs illustrent l'avantage de cibler une région quantile plutôt qu'un quantile unique lorsqu'il est question de tester l'effet de certaines covariables dans un cadre de régression quantile.

1. INTRODUCTION

Since first proposed by Koenker & Bassett (1978), quantile regression has emerged as a powerful tool to study the relationship between a response variable and a set of covariates. By modelling the regression coefficients as a function of quantile level $\tau \in (0, 1)$, quantile regression is particularly useful when the effect of the covariates on the response varies across different regions of τ. This phenomenon is often observed in biomedical studies, where the effect of certain treatments is heterogeneous in the population and dependant on certain unobserved characteristics of the subjects.

In biomedical studies, the responses/outcomes (commonly known as the survival times) are often censored from the right because the participating subjects may drop out of the study or a

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clinical trial has to terminate after a certain period of time. The accelerated failure time (AFT) model and the Cox proportional hazards model are popular regression models to study censored outcomes. However, the AFT model works on the conditional mean of the responses and is unable to capture the possible changes in the treatment effect across τ. The Cox proportional hazards model requires the conditional quantiles of the response to follow a specific form as discussed in Section 3 of Portnoy (2003), which precludes certain types of heterogeneity of the treatment effect (e.g., negative or zero effect at low τ but positive effect at high τ). In this article, we focus on statistical inference for quantile regression with censored outcomes, allowing the quantile treatment effect to vary with the quantile level.

Censored quantile regression was studied in Powell (1984, 1986), where the censoring time is assumed to be known. Wang & Fygenson (2009) developed methods for longitudinal data with fixed censoring. Ying, Jung & Wei (1995), Zhou (2006), and Bang & Tsiatis (2002), among others, proposed different estimating equations assuming the censored time is independent of the outcome. A less stringent assumption in the literature assumes the censored time is conditionally independent of the outcome given the covariates. Several estimation methods have been proposed under this assumption, and they can be classified into two groups by whether the linear (or more generally, parametric) quantile regression model is assumed to hold locally at one τ or globally at all τ.

Under the local linear quantile regression assumption, Wang & Wang (2009), Leng & Tong (2013), and Backer, Ghouch & Keilegom (2019) constructed various forms of adapted loss functions for censored quantile regression, but they all share the same feature that the conditional distribution of either the survival time or the censored time needs to be estimated non-parametrically to carry out the estimation.

Under the global linear quantile regression model, Portnoy (2003) proposed an iterative self-consistency algorithm based on the idea of redistribution of mass, while Peng & Huang (2008) constructed their estimation equation through a Martingale related to the counting process of the signs of the residuals. Under the global linear quantile assumption, the conditional distribution of the survival time is determined by the quantile regression coefficient function of τ, so explicit estimation of the conditional distributions is not needed.

The focus of the present article is statistical inference on quantile regression coefficients over a given quantile region, with a single quantile level as a special case. To distinguish it from modelling and inference at just a single quantile level, we call the former "regional quantile regression inference". In applications, it is typically more meaningful to evaluate the impact of a covariate on the outcome at many quantile levels than at one specific quantile level. He, Hsu & Hu (2010) discussed the detection of treatment effects in the upper quantiles (e.g., $\tau \geq 0.75$) of the outcome in clinical trials. Park & He (2017), Sun & He (2021), and Chen et al. (2021) showed that regional quantile regression inferences are often more stable and powerful than hypothesis testing at a single quantile level.

A natural approach to regional quantile regression inference is to consider the quantile process in that region. A test statistic that characterizes the quantile coefficients over the region can take various forms, including averaging or taking the supremum norm. As has been demonstrated in the earlier work on quantile regression (Gutenbrunner & Jurečková, 1992; Kocherginsky, He & Mu, 2005), regression rank-based statistics are particularly useful for their robustness in quantile regression analysis. This motivates us to consider rank-based tests under the global linear quantile regression model with random right censoring.

We address two major challenges in developing rank-based tests in this article. First, for quantile regression without censoring, the rank-based test is constructed with the regression rank scores, which are the solutions to the dual problem of optimizing the quantile loss function. However, the regression rank score is not naturally defined for censored observations. We propose a (generalized) regression rank score for censored quantile regression by utilizing the idea of redistribution of mass for the censored observations as used in Portnoy (2003) and Wang & Wang (2009). Second, rank-based test statistics do not have simple limiting distributions and their critical values are not easily available. If we use the bootstrap method in the hypothesis testing framework, the bootstrap sample should be generated from the null hypothesis to ensure bootstrap consistency. Resampling schemes like the paired bootstrap or the generalized bootstrap (done by perturbing the weights to each observation) generate data from the full model and are therefore not applicable to our method. We propose a new bootstrap procedure that approximates the data generating procedure under a global linear quantile model satisfying the null hypothesis.

In other words, the main contributions of our work are the construction of regression rank scores for censored quantile regression and the development of a model-based bootstrap method that can be used in a general hypothesis testing framework for regional censored quantile regression inference.

The remaining parts of the article are organized as follows. Section 2 presents our main results. More specifically, we introduce the problem setting in Section 2.1, construct the censored regression rank scores in Section 2.2, and then propose the rank-based tests, the model-based bootstrap algorithm, and study the relevant large-sample properties of the proposed tests in Sections 2.3, 2.4, and 2.5, respectively. We give simulated results in Section 3 to demonstrate the robustness and power of the proposed test. In Section 4, we apply our method to study the natural mortality of bighorn sheep as an illustrative example for our proposed methodology. In Section 5, we conclude the article with summary comments.

2. METHODOLOGY AND MAIN RESULTS

2.1. Problem Setting

We consider a random sample of size n that follows the linear quantile model

$$Q_{T_i}\left(\tau|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right)=\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(\tau)+\boldsymbol{x}_{i2}^T\boldsymbol{\beta}_2(\tau),\quad \forall\tau\in(0,\tau_U],\quad i=1,2,\ldots,n, \tag{1}$$

for a given value τ_U between 0 and 1, where T_i denotes the survival times (or responses in general) that are subject to right censoring, and $Q_{T_i}(\tau|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2})$ denotes the τth conditional quantile of T_i given the covariates of interest $\boldsymbol{x}_{i1}\in\mathbb{R}^p$ and $\boldsymbol{x}_{i2}\in\mathbb{R}^q$. The first component of $\boldsymbol{x}_{i1}$ is taken to be 1 to represent the intercept. Note that model (1) can be equivalently written as $T_i=\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(\tau)+\boldsymbol{x}_{i2}^T\boldsymbol{\beta}_2(\tau)+e_{i,\tau}$, where the conditional τth quantile of $e_{i,\tau}$ given $\boldsymbol{x}_i=(\boldsymbol{x}_{i1},\boldsymbol{x}_{i2})$ is assumed to be 0 to make $\boldsymbol{\beta}(\tau)=\left(\boldsymbol{\beta}_1(\tau),\boldsymbol{\beta}_2(\tau)\right)$ identifiable.

One implication from right censoring is that $\boldsymbol{\beta}(\tau)$ may not be identifiable for some upper quantiles. At the population level, we consider τ_U to be the highest quantile level at which $\boldsymbol{\beta}(\tau)$ is identifiable. Regional quantile regression inference will obviously be limited to any subset of τ within $(0,\tau_U]$.

Due to censoring, the observed data are given by the triples $(Y_i,\boldsymbol{x}_i,\Delta_i)$, where $Y_i=\min(T_i,C_i)$ and $\Delta_i=\mathbb{I}(T_i\leq C_i)$, with C_i denoting the censoring time. We further assume that T_i and C_i are independent for a given $\boldsymbol{x}_i$, which is commonly assumed in the survival analysis literature.

For such global censored quantile regression models, the estimation methods proposed in Portnoy (2003) and Peng & Huang (2008) operate sequentially on a set of $M+1$ grid points $\mathcal{S}=(t_0,t_1,\ldots,t_M)$, where $0<t_0<t_M=\tau_U$. In this article, we utilize those methods for the estimation of $\boldsymbol{\beta}(\tau)$, but we are interested in testing the hypothesis

$$H_0\,:\,\boldsymbol{\beta}_2(\tau)=0\text{ for all }\tau\in(0,\tau_U]\text{ versus }H_1\,:\,\boldsymbol{\beta}_2(\tau)\neq0\text{ for some }\tau\in[\tau_a,\tau_b],$$

where $[\tau_a, \tau_b]$ is a user-specific proper subset of $(0, t_U]$, reflecting the alternative hypothesis of interest in a particular problem.

For quantile regression without censoring, rank-based tests have been shown to be robust and effective; we refer to Gutenbrunner et al. (1993), Koenker & Machado (1999), Wang (2009), and Sun & He (2021) for details. Regression rank scores, however, need to be appropriately constructed for censored observations.

2.2. Censored Regression Rank Scores

To motivate and construct the censored regression rank scores, we first give a brief review of regression rank scores for complete data.

When times T_i in model (1) are fully observed, Koenker & Bassett (1978) proposed the quantile regression estimator $\boldsymbol{\beta}(\tau)$ by solving the following optimization problem

$$\hat{\boldsymbol{\beta}}(\tau) = \underset{\boldsymbol{t}\in\mathbb{R}^{p+q}}{\operatorname{argmin}} \sum_{i=1}^{n} \rho_\tau \left(T_i - \boldsymbol{x}_i^T \boldsymbol{t}\right), \tag{2}$$

where $\rho_\tau(u) = u(\tau - \mathbb{I}(u < 0))$ is the quantile loss function. The optimization problem (2) corresponds to its dual problem

$$\hat{\boldsymbol{a}}(\tau) = \underset{\boldsymbol{a}\in[0,1]^n}{\operatorname{argmax}} \left\{\boldsymbol{a}^T \boldsymbol{T} \mid X^T \boldsymbol{a} = (1-\tau)\boldsymbol{X}^T \mathbf{1}_n\right\}, \tag{3}$$

where $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$, $\boldsymbol{X} = \left(\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_n^T\right)^T$, and $\mathbf{1}_n$ denotes the n-dimensional vector of 1s. By the duality between Equations (2) and (3), we have the following relationship,

$$\hat{a}_i(\tau) = \begin{cases} 1, & T_i > \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau), \\ \in (0,1), & T_i = \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau), \\ 0, & T_i < \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau), \end{cases} \tag{4}$$

which implies $\hat{a}_i(\tau)$ can be seen as the indicator of whether the ith observation is above the τth fitted quantile function. Gutenbrunner & Jurečková (1992) studied the properties of $\hat{a}_i(\tau)$ and named it the regression rank score for the ith observation.

However, the formulation of the censored quantile regression is not simply an optimization problem with a similar duality to Equation (3), and thus we do not have a direct generalization of the regression rank scores to the censored data.

We observe from Equation (4) that $\hat{a}_i(\tau) = \mathbb{I}\left(T_i > \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau)\right)$ unless T_i is exactly equal to the fitted τ-quantile. For the uncensored cases where T_i has a continuous distribution, the number of observations fit exactly by the estimated quantile function is $p+q$ for any τ with probability 1. Therefore the difference between $\hat{a}_i(\tau)$ and $\mathbb{I}\left(T_i > \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau)\right)$ is of a small order and most asymptotic properties are not influenced if $\hat{a}_i(\tau)$ is replaced with $\mathbb{I}\left(T_i > \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau)\right)$.

Also note that only the sign of $T_i - \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau)$, rather than the magnitude of the residual, is needed to determine the rank scores, so any censored observation that is above $\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau)$ would have the same sign no matter what the unobserved T_i is. On the other hand, $T_i - \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau)$ for a censored observation below $\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau)$ can be either positive or negative. With these observations in mind, we take the following steps.

Based on the censored quantile regression estimate $\hat{\boldsymbol{\beta}}(\tau)$ for $\tau \in \mathcal{S} = (t_0, t_1, \ldots, t_M)$, we construct $\tilde{\boldsymbol{\beta}}(\tau)$ as a function of τ by taking the linear interpolation between the grid points in $\mathcal{S}$, and set $\tilde{\boldsymbol{\beta}}(\tau) = \hat{\boldsymbol{\beta}}(t_M)$ for $\tau > t_M$. Then, let

$$\hat{\tau}_i = \inf_{\tau \geq t_0} \left\{ \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau) \geq Y_i \right\},$$

with the understanding that $\hat{\tau}_i = t_M$ if $Y_i > \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau)$. Now for each observation, we define the weight $\hat{w}_i(\tau)$ for $\tau \in (0, 1)$ as

$$\hat{w}_i(\tau) = \begin{cases} \frac{\tau - \hat{\tau}_i}{1 - \hat{\tau}_i}, & \Delta_i = 0, \tau \geq \hat{\tau}_i, \\ 1, & \Delta_i = 0, \tau < \hat{\tau}_i, \\ 1, & \Delta_i = 1, \end{cases} \tag{5}$$

and the censored regression rank score as

$$\hat{a}_i^c(\tau) = 1 - \hat{w}_i(\tau)\mathbb{I}\left(Y_i - \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau) < 0\right). \tag{6}$$

To understand the weights for the censored observations with $\Delta_i = 0$, we note the following. If $\tau < \hat{\tau}_i$, which implies $C_i > \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau)$, and therefore $T_i > \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau)$, it does not matter exactly what T_i is for the determination of the τ-quantile. In this case, we can simply treat the observation as uncensored. If $\tau \geq \hat{\tau}_i$, we have $C_i < \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau)$ and in this case, T_i could be above or below $\boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau)$. The weight assigned to this point is $\hat{w}_i(\tau) = \mathbb{P}\left(T_i < \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau) | T_i > C_i\right) = (\tau - \hat{\tau}_i)/(1 - \hat{\tau}_i)$. This is essentially redistributing the censored point into two values, one above and the other below the τ-quantile.

The idea of the redistribution of mass for each censored observation was used in Portnoy (2003) and Wang & Wang (2009) in quantile regression. To be more specific, each censored observation $(Y_i, \boldsymbol{x}_i, \Delta_i)$ can be equivalently viewed as a weighted combination of two fully observed pseudo points: one point at $(Y_i, \boldsymbol{x}_i)$ with weight $\hat{w}_i(\tau)$ and the other at $(Y_\infty, \boldsymbol{x}_i)$ with weight $1 - \hat{w}_i(\tau)$, where Y_∞ is a sufficiently large time value (which could be taken as infinity). For $\tau \in \mathcal{S}$, Portnoy's censored quantile regression estimator can be written as

$$\hat{\boldsymbol{\beta}}(\tau) = \underset{\boldsymbol{b} \in \mathbb{R}^{p+q}}{\operatorname{argmin}} \sum_{i=1}^{n} \left(\hat{w}_i(\tau)\rho_\tau(Y_i - \boldsymbol{x}_i^T \boldsymbol{b}) + (1 - \hat{w}_i(\tau))\rho_\tau(Y_\infty - \boldsymbol{x}_i^T \boldsymbol{b}) \right). \tag{7}$$

The following proposition shows that the censored regression rank score $\hat{a}_i^c(\tau)$ is approximately the weighted sum of the regression rank scores of the two pseudo points.

Proposition 1. *For any $\tau \in \mathcal{S}$, the optimization problem (7) can be transformed into a dual problem*

$$\check{\boldsymbol{a}}(\tau) = \underset{\boldsymbol{a} \in [0,1]^{2n}}{\operatorname{argmax}} \left\{ \left(\check{\boldsymbol{W}}(\tau)\boldsymbol{a}\right)^T \check{\boldsymbol{Y}} \mid \check{\boldsymbol{X}}^T \left(\check{\boldsymbol{W}}(\tau)\boldsymbol{a}\right) = (1 - \tau)\check{\boldsymbol{X}}^T \check{\boldsymbol{w}}(\tau) \right\}, \tag{8}$$

where $\check{\boldsymbol{Y}} = \left(Y_1, Y_\infty, \ldots, Y_n, Y_\infty\right) \in \mathbb{R}^{2n}$, $\check{\boldsymbol{X}} = \left(\boldsymbol{x}_1^T, \boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_n^T, \boldsymbol{x}_n^T\right)^T \in \mathbb{R}^{2n \times (p+q)}$, $\check{\boldsymbol{w}}(\tau) = \left(\hat{w}_1(\tau), 1 - \hat{w}_1(\tau), \ldots, \hat{w}_n(\tau), 1 - \hat{w}_n(\tau)\right)$ *is a $2n$-dimensional function, and* $\check{\boldsymbol{W}} \in \mathbb{R}^{2n \times 2n}$ *is a diagonal matrix with diagonal term* $\check{\boldsymbol{w}}(\tau)$.

Because of the duality between Equations (7) and (8), $\check{a}_{2i}(\tau)$ is always 1 and

$$\check{a}_{2i-1}(\tau) = \begin{cases} 1, & Y_i > \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau), \\ \in (0,1), & Y_i = \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau), \\ 0, & Y_i < \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}(\tau). \end{cases}$$

Therefore, for any $\tau \in \mathcal{S}$ and $i = 1, 2, \ldots, n$,

$$\begin{aligned} \hat{a}_i^c(\tau) &= \hat{w}_i(\tau)\mathbb{I}\left(Y_i \geq \boldsymbol{x}_i^T \tilde{\boldsymbol{\beta}}(\tau)\right) + 1 - \hat{w}_i(\tau) \\ &\approx \hat{w}_i(\tau)\check{a}_{2i-1}(\tau) + \left(1 - \hat{w}_i(\tau)\right)\check{a}_{2i}(\tau). \end{aligned}$$

Similar to the regression rank scores for the complete data, $\hat{a}_i^c(\tau)$ can be interpreted as a generalization of ranks to the regression setting. For two observations i and j, if $\int_0^1 \hat{a}_i^c(\tau)d\tau > \int_0^1 \hat{a}_j^c(\tau)d\tau$, observation i is considered to have higher rank than observation j after adjusting for the covariates.

Remark 1. When Portnoy's censored quantile regression is used for estimation, we may define $\hat{a}_i^c(\tau) := \hat{w}_i(\tau)\check{a}_{2i-1}(\tau) + \left(1 - \hat{w}_i(\tau)\right)\check{a}_{2i}(\tau)$. However, our definition (6) may accommodate other estimation procedures.

2.3. Test Statistics

We utilize $\hat{a}_i^c(\tau)$ to construct the rank-based test statistics. To do so, we fix some notations first. We write the design matrix of model (1) as $\boldsymbol{X} = [\boldsymbol{X}_1, \boldsymbol{X}_2]$, and then let $\hat{\boldsymbol{X}}_2 = \boldsymbol{X}_1(\boldsymbol{X}_1^T\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1^T\boldsymbol{X}_2$ be the projection of $\boldsymbol{X}_2$ into the spaces spanned by the columns of $\boldsymbol{X}_1$. We define

$$\boldsymbol{S}(\tau) = n^{-1/2}\sum_i \left(\boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2}\right)\hat{a}_i^c(\tau),$$

where $\hat{a}_i^c(\tau)$ is given in Equation (6) but calculated under the restricted model (under the null hypothesis) that only includes $\boldsymbol{X}_1$ as the covariates. Intuitively, $\hat{a}_i^c(\tau)$ here represents the position of observation i relative to the τth quantile after adjusting for $\boldsymbol{X}_1$. If the null hypothesis is true, no more variation in $\hat{a}_i^c(\tau)$ can be further explained by $\boldsymbol{X}_2 - \hat{\boldsymbol{X}}_2$, so we expect $\boldsymbol{S}(\tau)$ to be close to 0.

With $\boldsymbol{Q}_n = n^{-1}(\boldsymbol{X}_2 - \hat{\boldsymbol{X}}_2)^T(\boldsymbol{X}_2 - \hat{\boldsymbol{X}}_2)$, we construct the following two test statistics,

$$\mathcal{T}_1 = \left(\sum_{t_m \in \mathcal{S}\cap[\tau_a,\tau_b]} \boldsymbol{S}(t_m)\left(t_m - t_{m-1}\right)\right)^T \boldsymbol{Q}_n^{-1}\left(\sum_{t_m \in \mathcal{S}\cap[\tau_a,\tau_b]} \boldsymbol{S}(t_m)\left(t_m - t_{m-1}\right)\right), \tag{9}$$

$$\mathcal{T}_2 = \sum_{t_m \in \mathcal{S}\cap[\tau_a,\tau_b]} \left(\boldsymbol{S}(t_m)^T\boldsymbol{Q}_n^{-1}\boldsymbol{S}(t_m)\right)\left(t_m - t_{m-1}\right). \tag{10}$$

The first test statistic $\mathcal{T}_1$ takes a weighted sum of $\boldsymbol{S}(\tau)$ over all the grid points in $[\tau_a, \tau_b]$, aiming at detecting the effect of $\boldsymbol{X}_2$ over the $[\tau_a, \tau_b]$ region. It is a generalization of the regional quantile regression rank test for fully observed data proposed in Sun & He (2021).

When the effect of $\boldsymbol{X}_2$ is positive at some quantile levels but negative at the other quantile levels, the use of averaging in $\mathcal{T}_1$ could reduce power at detecting the effect because of the cancellation of effects in different directions. Alternatively, we consider the test statistic $\mathcal{T}_2$

where the weighted sum is taken over a quadratic form of $\boldsymbol{S}(\tau)$. We expect $\mathcal{T}_2$ to have better power in the aforementioned scenario. The performances of $\mathcal{T}_1$ and $\mathcal{T}_2$ are compared numerically in Section 3.

As shown in Section 2.5, the large-sample null distribution of the rank-based test statistics are either mixed chi-squares or an integral of a squared Gaussian process, so we resort to an appropriate bootstrap method to approximate the critical values of the tests.

2.4. Model-Based Bootstrap

To approximate the null distribution of $\mathcal{T}_1$ or $\mathcal{T}_2$, we propose a new model-based bootstrap algorithm in this subsection.

We generate bootstrap samples of T_i^* and C_i^* independently while keeping $\boldsymbol{x}_{i1}$ and $\boldsymbol{x}_{i2}$ fixed. To generate T_i^*, notice that under H_0, we have $Q_{T_i}\left(\tau|\boldsymbol{x}_{i1}\right) = \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(\tau)$. Therefore, it is natural to generate u_i independently from $U(0,1)$ and set $T_i^* = \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(u_i)$. One complication here is that the quantile estimate $\hat{\boldsymbol{\beta}}_1(\tau)$ may be available only for $\tau < \tau_U$. However, our test focuses on the interval $[\tau_a, \tau_b]$, and the exact value of $\hat{\boldsymbol{\beta}}_1(\tau)$ when $\tau > \tau_U$ has no influence on $\mathcal{T}_1$ and $\mathcal{T}_2$; therefore, we can simply let $\tilde{\boldsymbol{\beta}}_1(\tau) = \hat{\boldsymbol{\beta}}_1(\tau_U)$ for $\tau > \tau_U$.

To resample C_i^*, we estimate $G\left(\cdot|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right)$, the conditional distribution of C_i, using the local Kaplan–Meier (KM) estimator. Specifically, let

$$\hat{G}(y|\boldsymbol{x}) = 1 - \prod_{k=1}^{n}\left(1 - \frac{B_{nk}(\boldsymbol{x})}{\sum_j \mathbb{I}\left(Y_k \le Y_j\right) B_{nj}(\boldsymbol{x})}\right)^{\mathbb{I}(Y_k<y,\Delta_k=0)}, \tag{11}$$

where $B_{nj}(\boldsymbol{x}) = \frac{K((\boldsymbol{x}-\boldsymbol{x}_j)/h_n)}{\sum_k K((\boldsymbol{x}-\boldsymbol{x}_k)/h_n)}$, $K(\cdot)$ is a selected kernel density function and h_n is a sequence of bandwidths that tends to 0 as n increases. We then generate $v_i \sim U(0,1)$ independent of u_i to obtain $C_i^* = \hat{G}^{-1}(v_i|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2})$.

We need to address two challenges with this method of generating C_i^*. First, it is likely that $\sup_y \hat{G}\left(y|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right) < 1$, so $\hat{G}^{-1}\left(\tau|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right)$ is undefined for large values of τ. We can simply assign a very large value to C_i^* when $\hat{G}^{-1}\left(v_i|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right)$ is undefined. If we look at any $(\boldsymbol{x}_{i1},\boldsymbol{x}_{i2})$ in the covariate space, $G\left(y|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right)$ is not fully identifiable when the largest attainable value of T_i is smaller than the largest attainable value of C_i. In this case, we have no information about the distribution of C_i for $C_i > \sup T_i$, and $G\left(y|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right)$ is unidentifiable for $y > \sup T_i$. The above statement also holds in the bootstrap space with probability tending to 1. Therefore for v_i where $\hat{G}^{-1}(v_i|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2})$ is undefined, since T_i^* is the outcome of interest, the exact value of C_i^* is not important, as long as T_i^* can be observed with a sufficiently large C_i^*.

Second, it is difficult to get an accurate estimate of $G\left(\cdot|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right)$ using the local KM estimator unless $p+q$ is small. Alternatively, we could fit a censored quantile regression $Q_{C_i}\left(\tau|\boldsymbol{x}_{i1},\boldsymbol{x}_{i2}\right) = \boldsymbol{x}_{i1}^T\boldsymbol{\gamma}_1(\tau) + \boldsymbol{x}_{i2}^T\boldsymbol{\gamma}_2(\tau)$ and let $C_i^* = \boldsymbol{x}_{i1}^T\hat{\boldsymbol{\gamma}}_1(v_i) + \boldsymbol{x}_{i2}^T\hat{\boldsymbol{\gamma}}_2(v_i)$. This approach requires the additional assumption that the linear quantile model also holds for C_i.

We are now ready to provide the proposed algorithm for the model-based bootstrap. We do this for the test statistic $\mathcal{T}_1$ and use the local KM to resample C_i^*; the algorithm for $\mathcal{T}_2$ because of the test statistic or for using censored quantile regression to resample C_i^* is similar.

Step 1: Fit the censored quantile regression under H_0 (using Portnoy's method, for example). Calculate $\mathcal{T}_1$ from Equation (9).

Step 2: For $i = 1, \dots, n$, generate random numbers $u_i \sim U(0,1)$. Let $T_i^* = \boldsymbol{x}_{i1}\tilde{\boldsymbol{\beta}}(u_i)$, where $\tilde{\boldsymbol{\beta}}(\tau)$ is the linear interpolation of $\{\hat{\boldsymbol{\beta}}_1(\tau_m), m \in \mathcal{S}\}$ calculated in Step 1. Set $\tilde{\boldsymbol{\beta}}_1(\tau) = \hat{\boldsymbol{\beta}}_1(\tau_U)$ for $\tau > \tau_U$.

Step 3: For $i = 1, \dots, n$, generate random numbers $v_i \sim U(0,1)$ independent of u_i. Let $C_i^* = \hat{G}^{-1}(v_i|\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2})$, where $\hat{G}(\cdot|\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2})$ is estimated using the local KM estimator described in Equation (11). Set C_i^* to be an arbitrarily large number if $\hat{G}^{-1}(\cdot|\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2})$ is undefined at v_i.

Step 4: Construct a bootstrap sample $\left(Y_i^*, \Delta_i^*, \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}\right)$, where $Y_i^* = \min\{T_i^*, C_i^*\}$, $\Delta_i^* = \mathbb{I}\left(T_i^* < C_i^*\right)$. Calculate $\mathcal{T}_1^*$ with this bootstrap sample.

Step 5: Repeat Steps 2–4 for B iterations to get $\{\mathcal{T}_{11}^*, \mathcal{T}_{12}^*, \dots, \mathcal{T}_{1B}^*\}$. The resulting P-value is calculated by $B^{-1}\sum_b \mathbb{I}\left(\mathcal{T}_1 > \mathcal{T}_{1b}^*\right)$.

2.5. Asymptotic Properties

In this subsection, we first study the asymptotic properties of $\mathcal{T}_1$ and $\mathcal{T}_2$, and then show the validity of the proposed bootstrap inference. Detailed proofs of the results in this subsection are provided in the Appendix.

Let $f(\cdot|\boldsymbol{x}_i)$ and $g(\cdot|\boldsymbol{x}_i)$ be the conditional densities of T_i and C_i for a given $\boldsymbol{x}_i$, respectively. We assume the following regularity conditions.

(C1) Let $0 < \epsilon = t_0 < 2\epsilon = t_1 < \cdots < t_M = \min(1-\epsilon, \tau_U)$ be a set of grid points where $0 < \epsilon < 1/2$ is a small constant and $n^{-1/2} \ll t_j - t_{j-1} \ll n^{-1/4}$, $j = 2, \dots, M$.

(C2) Norms $\|\boldsymbol{x}_i\|$ are bounded uniformly in $i = 1, \dots, n$.

(C3) The conditional densities $f\left(\boldsymbol{x}_i^T\boldsymbol{\beta}(\tau)|\boldsymbol{x}_i\right)$ and $g\left(\boldsymbol{x}_i^T\boldsymbol{\beta}(\tau)|\boldsymbol{x}_i\right)$ are strictly positive and have uniformly bounded derivatives with respect to τ, for any $\tau \in [\epsilon, \tau_U]$.

(C4) The limit of $\frac{1}{n}\sum_i \boldsymbol{x}_i\boldsymbol{x}_i^T$ exists and is positive definite.

(C5) The functions $F\left(s|\boldsymbol{x}_i\right)$ and $G\left(s|\boldsymbol{x}_i\right)$ have second-order partial derivatives with respect to $\boldsymbol{x}_i$ and are bounded uniformly in s and $\boldsymbol{x}_i$.

(C6) The kernel density function $K(\cdot)$ used in Equation (11) is positive, with compact support, and Lipschitz continuous of order 1. Furthermore, $\int K(u)du = 1$, $\int uK(u)du = 0$, $\int K^2(u)du < \infty$ and $\int |u|^2 K(u)du < \infty$.

(C7) The bandwidth satisfies $h_n = c_n n^{-1/2+\gamma_0}$, with $c_n \to c$, where c is a constant, and $0 < \gamma_0 < 1/4$.

(C8) There is no censoring below the 2ϵ-quantile. That is, for any $\tau < 2\epsilon$, $\boldsymbol{x}_i^T\boldsymbol{\beta}(\tau) < C_i$.

(C9) The censored quantile regression estimator satisfies the following asymptotic property for $\hat{\boldsymbol{\beta}}_1(\tau)$: $\boldsymbol{B}_n(\tau) := \sqrt{n}\left(\hat{\boldsymbol{\beta}}_1(\tau) - \boldsymbol{\beta}_1(\tau)\right)$ converges weakly, as $n \to \infty$, to a zero-mean Gaussian process for $\tau \in [\epsilon, \tau_U]$.

Condition (C1) restricts the grid size (the distance between two adjacent grid points) on the quantile levels to between $n^{-1/2}$ and $n^{-1/4}$. The same condition is required in Portnoy & Lin (2010) to establish the asymptotic normality of $\hat{\boldsymbol{\beta}}(\tau)$ with Portnoy's method. The main requirement is that there are sufficiently fine grids to ensure good approximation of the quantile estimates at all τ, and there is no benefit of using grids finer than $n^{-1/2}$, because finer grids do not further improve approximation accuracy but add more complexity to both the theory and the computations. In (C2), we assume that the covariates are bounded, which may appear restrictive. However, under the regional or global linear quantile model, the quantile functions $\boldsymbol{x}^T\boldsymbol{\beta}(\tau_1)$ and $\boldsymbol{x}^T\boldsymbol{\beta}(\tau_2)$ will cross if $\boldsymbol{x}$ is allowed to go unbounded, unless $\boldsymbol{x}^T\boldsymbol{\beta}(\tau_1)$ and $\boldsymbol{x}^T\boldsymbol{\beta}(\tau_2)$ are parallel. Therefore, to accommodate heterogeneity in the model, it is necessary to restrict the linear quantile model to a bounded region for the covariates. Conditions (C3) and (C4) are common assumptions assumed when studying the asymptotic properties of censored quantile regression. Conditions (C5), (C6), and (C7) are needed in Theorem 2.1 of Gonzalez-Manteiga & Cadarso-Suarez (1994) where the asymptotic behaviour of $\hat{G}(y|\boldsymbol{x})$ is studied. Condition (C8) is required by Portnoy's method to ensure that it is valid to use quantile regression without censoring to estimate $\hat{\boldsymbol{\beta}}(t_0)$. This assumption could be relaxed to allow the number of censored points below the 2ϵ-quantile to be of a smaller order than $n^{1/2}$.

If $\hat{\boldsymbol{\beta}}_1(\tau)$ is estimated with Portnoy's method, then (C9) is satisfied under Conditions (C1)–(C4) and (C8), by Theorem 3.1 of Portnoy & Lin (2010). If Peng and Huang's estimator is used, (C9) holds under slightly different assumptions by Theorem 2 of Peng & Huang (2008).

Theorem 1. *Assume regularity Conditions (C1)–(C4) as well as (C9). We have, under H_0, that $\mathcal{T}_1$ converges in distribution to $\overline{\chi}^2$, a mixed chi-square distribution as a weighted sum of q chi-square variables of one degree of freedom, and that $\mathcal{T}_2$ converges in distribution to an integral of a squared Gaussian process.*

To establish Theorem 1, we provide an asymptotic representation for $\boldsymbol{S}(\tau)$. To do so, let $\tau_i = \inf_\tau\{\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(\tau) \geq Y_i\}$ and w_i be the "true" weight where $\hat{\tau}_i$ in Equation (5) is replaced with τ_i. Furthermore, let $a_i^c(\tau) := 1 - w_i(\tau)\mathbb{I}(Y_i - \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(\tau) < 0)$. To show that $\boldsymbol{S}(\tau)$ converges weakly to a zero-mean Gaussian process for $\tau \in [\tau_a, \tau_b]$, we derive the following representation

$$\begin{aligned}\boldsymbol{S}(\tau) = \frac{1}{\sqrt{n}}\sum_i(\boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2})a_i^c(\tau) + \boldsymbol{K}_{\boldsymbol{x}_{i2}-\hat{\boldsymbol{x}}_{i2}}(\tau)\boldsymbol{B}_n(\tau)\\ + \int_0^\tau \boldsymbol{\Gamma}_{\boldsymbol{x}_{i2}-\hat{\boldsymbol{x}}_{i2}}(u)\boldsymbol{B}_n(u)du + o_p(1)\end{aligned}$$

uniformly for $\tau \in [\tau_a, \tau_b]$ using empirical process theory, where $\boldsymbol{K}_{\boldsymbol{x}_{i2}-\hat{\boldsymbol{x}}_{i2}}(\tau)$ and $\boldsymbol{\Gamma}_{\boldsymbol{x}_{i2}-\hat{\boldsymbol{x}}_{i2}}(\tau)$ are $q \times p$ matrices defined in the proof of this result in the Appendix.

To show the validity of the proposed bootstrap inference, we further require that $\boldsymbol{B}_n^*(\tau) := \sqrt{n}\big(\hat{\boldsymbol{\beta}}_1^*(\tau) - \hat{\boldsymbol{\beta}}_1(\tau)\big)$ is bootstrap consistent.

Proposition 2. *Using Portnoy's censored quantile regression estimator, and under regularity Conditions (C1)–(C8), $\boldsymbol{B}_n^*(\tau)$ is bootstrap consistent. That is, conditional on the data, $\boldsymbol{B}_n^*(\tau)$ converges weakly to the same Gaussian process as $\boldsymbol{B}_n(\tau)$ does for $\tau \in [\tau_a, \tau_b]$.*

Proposition 2 can be verified by establishing the Bahadur representation for $\boldsymbol{B}_n^*(\tau)$ and $\boldsymbol{B}_n(\tau)$ using results from the product-integration theory of Gill & Johansen (1990) and comparing the expansions term by term. The next theorem shows the proposed bootstrap inference is valid.

Theorem 2. *Assume regularity Conditions (C1)–(C7) and (C9). If $\boldsymbol{B}_n^*(\tau)$ is bootstrap consistent, then we have, under H_0, that the conditional distributions of $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ converge to the same limiting distributions as $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.*

3. SIMULATIONS

In this section, we evaluate the performance of the proposed tests in finite samples using Monte Carlo simulations.

We generate the data from the following model

$$T_i = \beta_0(u_i) + z_{i1}\beta_1(u_i) + z_{i2}\beta_2(u_i) + z_{i3}\beta_3(u_i), \quad i = 1, \ldots, n, \tag{12}$$

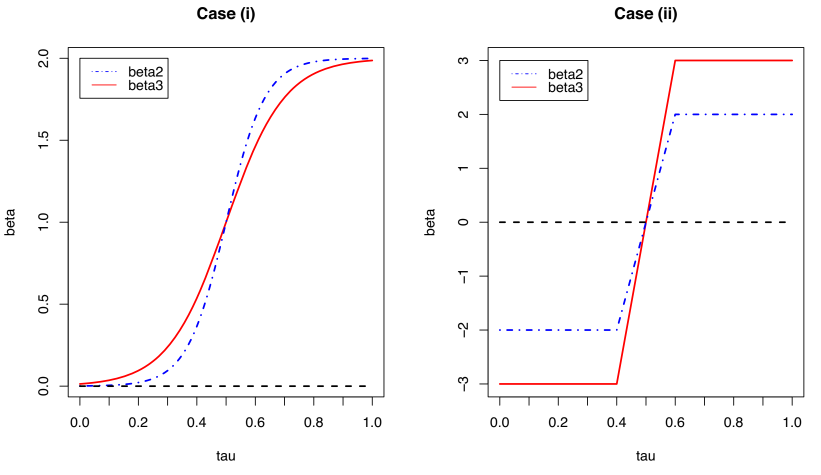

FIGURE 1: Curves of quantile coefficients of Cases (i) and (ii) under the alternative.

where $u_i \sim U(0,1)$, $z_{i2} \sim U(0,2)$, $z_{i3} \sim U(0,2)$, $z_{i1}|z_{i2} \sim U(1,3)$ when $z_{i2} < 1$ but $z_{i1}|z_{i2} \sim U(0,2)$ when $z_{i2} \geq 1$. Furthermore, we have $\beta_0(\tau) = \Phi^{-1}(\tau)$, the inverse of the standard normal distribution function, and $\beta_1(\tau) = \tau^2$. For the rest of the model specifications, we consider two cases, and the null hypothesis is $H_0 : \beta_2(\tau) = \beta_3(\tau) = 0$ in both cases.

For Case (i), let $\beta_2(\tau) = \frac{2\exp(15(\tau-0.5))}{1+\exp(15(\tau-0.5))}$ and $\beta_3(\tau) = \frac{2\exp(10(\tau-0.5))}{1+\exp(10(\tau-0.5))}$ under the alternative hypothesis H_1 (Figure 1). The censoring variable is $C_i|z_i \sim U\left(-0.5z_{i1}, 5-0.5z_{i1}\right)$ under H_0 and $C_i|z_i \sim U(2-0.5z_{i1}, 7-0.5z_{i1})$ under H_1, where $z_i = (z_{i1}, z_{i2}, z_{i3})$. Case (i) is designed to represent a scenario where the effects of z_{i1} and z_{i2} are mostly expected at upper quantile levels, and the interval $[\tau_a, \tau_b]$ is chosen to focus on the upper tail (see He, Hsu & Hu, 2010 for a motivating example).

For Case (ii), let $\beta_2(\tau) = -2\mathbb{I}(\tau < 0.4) + 20(\tau - 0.4)\mathbb{I}(0.4 < \tau < 0.6) + 2\mathbb{I}(\tau > 0.6)$, and $\beta_3(\tau) = -3\mathbb{I}(\tau < 0.4) + 30(\tau - 0.4)\mathbb{I}(0.4 < \tau < 0.6) + 3\mathbb{I}(\tau > 0.6)$ under the alternative hypothesis (Figure 1). The censoring variable is $C_i|z_i \sim U(-z_{i1}, 5-z_{i1})$ under H_0 and $C_i|z_i \sim U(2-z_{i1}, 7-z_{i1})$ under H_1. In this case, the effects of z_{i2} and z_{i3} change from negative to positive as τ increases. Case (ii) is designed to represent a scenario when the goal is to detect an overall effect of z_{i2} and z_{i3}, and $[\tau_a, \tau_b]$ is chosen to cover a relatively large quantile region.

In Cases (i) and (ii), we compare the performance of $\mathcal{T}_1$, $\mathcal{T}_2$, and the following test statistic that focuses on a single quantile level τ,

$$\mathcal{T}_3 = \boldsymbol{S}(\tau)^T \boldsymbol{Q}_n^{-1} \boldsymbol{S}(\tau).$$

Notice that $\mathcal{T}_3$ is a special case for $\mathcal{T}_1$ and $\mathcal{T}_2$ when we set $\tau_a = \tau_b$.

In the bootstrap method, C_i^* can be sampled from the local KM estimator or the censored quantile regression model. We include both options in the comparisons.

The simulation uses 1000 Monte Carlo data sets and the bootstrap size is $B = 500$ throughout. Tables B1 and B2 in the Appendix summarize the numerical results, and they show that all the tests under consideration here maintain the nominal significance level quite well. Figures 2 and 3 visually compare the power results.

As shown in Figure 2, the tests have similar power regardless of whether we use the local KM or the censored quantile regression to sample C_i^*. The performances of $\mathcal{T}_1$ and $\mathcal{T}_2$ are comparable in Case (i). However, $\mathcal{T}_2$ has greater power than $\mathcal{T}_1$ in Case (ii), because by design, the signs of

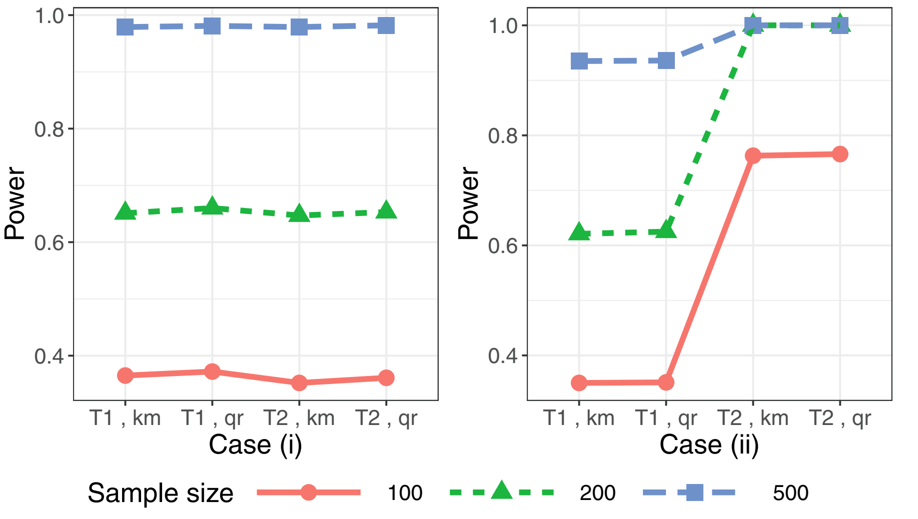

FIGURE 2: Comparison of the empirical power between $\mathcal{T}_1$ and $\mathcal{T}_2$. The chosen quantile region is $[0.5, 0.85]$ for Case (i) and $[0.1, 0.7]$ for Case (ii). (T1,km) and (T2,km) denote the test statistic $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, with C_i^* sampled from the local KM estimator. Similarly, (T1,qr) and (T2,qr) denote the two test statistics with C_i^* sampled from the censored quantile regression model.

$\beta_2(\tau)$ and $\beta_3(\tau)$ change from negative to positive as τ increases in this case. The test $\mathcal{T}_1$ targets the average effect over τ, so there is cancellation in the average.

From Figure 3, we see that the power of $\mathcal{T}_3$ varies greatly with the choice of the quantile level τ. The choice of quantile region does have some impact on the power of $\mathcal{T}_2$ but much less so. Also note that in Case (ii), even though the magnitude of the effect is the same at τ and $1-\tau$, the structure of the model implies that the differences in the lower tail are much easier to detect than those in the upper tail.

Overall, our empirical results suggest that $\mathcal{T}_2$ is a preferred test for regional quantile regression.

4. AN EXAMPLE: NATURAL MORTALITY IN BIGHORN SHEEP

In this section, we apply our proposed method to test the effect of early environment conditions on the natural mortality of adult bighorn sheep using the data analyzed in Douhard et al. (2019). This data set is available for download from the Dryad Digital Repository: https://doi.org/10.5061/dryad.6bm4228. The data set contains the survival times of 351 bighorn sheep born at Ram Mountain in Alberta, Canada, from 1973 to 2010. Other covariates included in the data are sex, adult environment conditions, and an indicator of whether cougar predation exists nearby. The early and adult environment conditions are measured as the 3-year average of the average mass of the 15-month-old yearlings. Because we are interested in the natural mortality rate, the lifetimes of sheep that were shot by hunters are considered as censored. In the data, the lifetimes of 19 of 191 female sheep are censored and 53 of 160 male sheep are censored.

Douhard et al. (2019) use the logarithm of survival time as the response and sex, cougar predation, early environment conditions, adult environment conditions, and the interaction between sex and the early environment conditions as predictors. Results from the Cox proportional hazards model used in Douhard et al. (2019) show that female sheep with a better early environment tend to live longer (P-value $= 0.0042$). But this phenomenon is not observed for male sheep (P-value $= 0.1747$), and the interaction effect between sex and early environment is not significant either (P-value $= 0.4341$). This seemingly contradictory result may be due to the

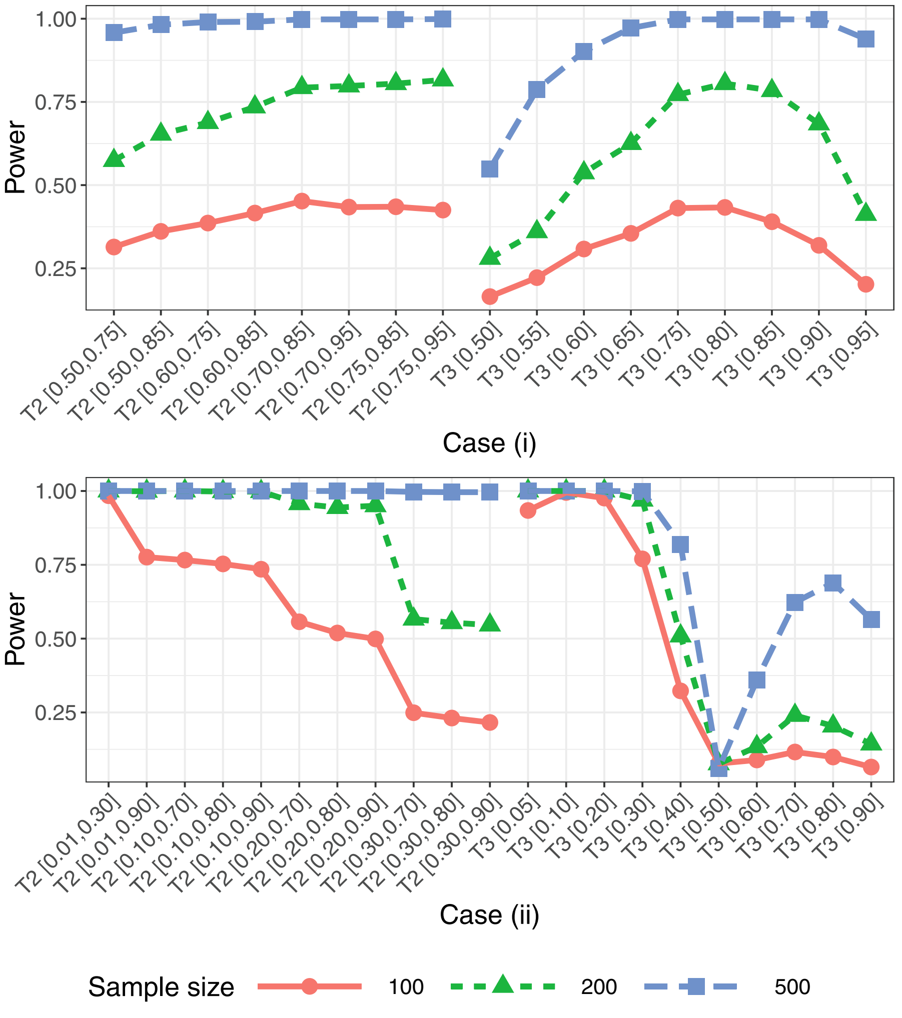

FIGURE 3: Comparison of the empirical power between $\mathcal{T}_2$ and $\mathcal{T}_3$ with different quantile regions/levels. T2 $[\tau_a, \tau_b]$ denotes the test statistic $\mathcal{T}_2$ over τ in $[\tau_a, \tau_b]$ and T3 $[\tau]$ denotes the test statistic $\mathcal{T}_3$ at τ. The value C_i^* is sampled from the censored quantile regression model.

lack of statistical power to detect the early environment effect on male sheep or the interaction effect between sex and early environment.

The Cox proportional hazards model assumes that the effect of a covariate on the hazard ratio is a constant, which precludes many forms of heterogeneity. Alternatively, we fit the model with censored quantile regression with the same covariates. Figure 4 shows the estimated coefficients and the 95% pointwise confidence band built with the paired bootstrap. We notice that the estimated coefficient for cougar predation is positive at the lower tail and is close to 0 for the upper tail. This type of heterogeneity cannot be observed under the Cox proportional hazards model.

From Figure 4, we see that the effect of early environment conditions for female sheep is significant for a large range of τ. But the effect of early environment conditions for male sheep is only significant for values of τ around 0.2. It is very difficult to detect the early environment effect on male sheep if one only looks at a single value of τ because it is hard to know which τ to look at beforehand and a multiplicity adjustment would be needed if one conducts tests at several quantile levels individually.

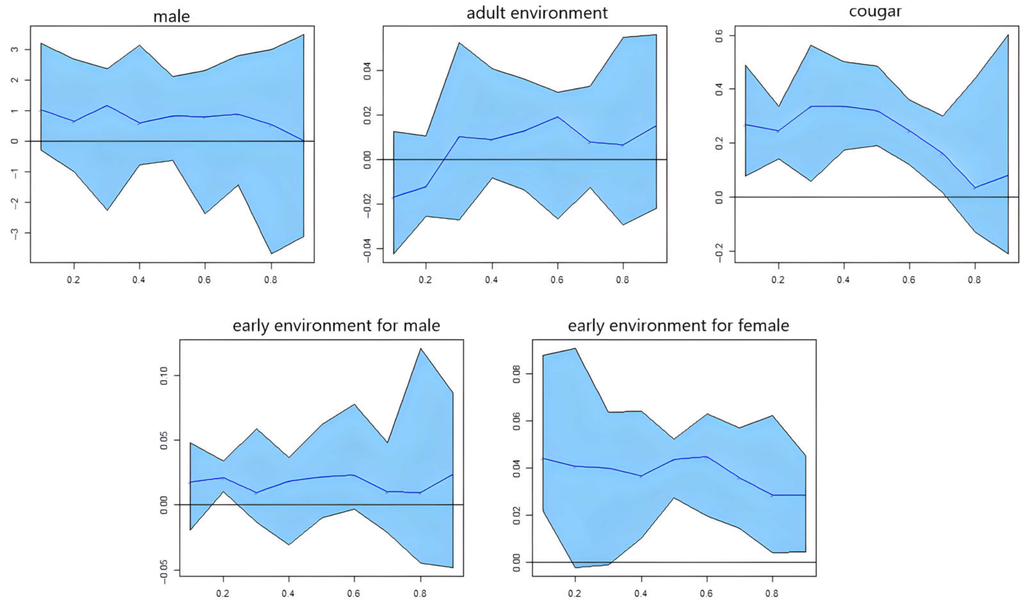

FIGURE 4: Pointwise confidence bands for the censored quantile regression coefficients. The horizontal axes are the quantile level τ.

In the next step, we conduct the proposed rank-based test. Because our aim is to test the overall effect of early environment conditions on male and female sheep, we choose $[\tau_a, \tau_b]$ to cover a large quantile region. With the test $\mathcal{T}_2$ and $[\tau_a, \tau_b] = [0.01, 0.8]$, we find that the interaction between early environment and sex is significant (P-value $= 0.016$). Furthermore, we detect that male sheep with better early environments tend to live longer (P-value $= 0.020$), and that the same holds for female sheep (P-value < 0.001). According to our analysis, good early environment conditions have a positive effect on the survival time for both male and female sheep, and the effect on female sheep is greater than on male sheep. We are able to arrive at the same conclusion if $\mathcal{T}_1$ is used or if we set $[\tau_a, \tau_b]$ to be other quantile regions such as $[0.1, 0.8]$ or $[0.1, 0.7]$, indicating the robustness of the proposed test to the choice of the quantile regions.

5. CONCLUDING REMARKS

In the present article, we propose rank-based tests for censored quantile regression inference with a new construction of regression rank scores for censored outcomes. We show that the rank-based tests can be used naturally for regional quantile regression inference to detect the impact of certain covariates over a quantile region, and that such tests often enjoy better stability than statistical tests targeted at a single quantile level. We develop a model-based bootstrap method that can effectively estimate the reference distributions of the rank-based tests under the global null hypothesis.

To further enhance the power of the rank-based tests, it is possible to use the weighted rank-score statistics with adaptively selected weights to reflect the heterogeneity of the conditional distributions across different values of the covariates. There has been some recent work in identifying the optimally efficient quantile regression estimation under the regional or global linear quantile models, for example, see Chen et al. (2017). It is an interesting open problem as to how optimally weighted rank-based tests can be implemented for censored quantile regression inference.

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APPENDIX A: PROOFS

In this section, we present the proofs of Theorem 1, Theorem 2, and Proposition 2. To simplify the notations, we write $F(\cdot|\boldsymbol{x}_i)$, $f(\cdot|\boldsymbol{x}_i)$, $G(\cdot|\boldsymbol{x}_i)$, and $g(\cdot|\boldsymbol{x}_i)$ as $F_i(\cdot)$, $f_i(\cdot)$, $G_i(\cdot)$, and $g_i(\cdot)$, respectively.

Proof of Theorem 1. Let $\boldsymbol{d}_i$ be an l-dimensional vector with bounded support. Write

$$\Psi(\boldsymbol{w}(t),\boldsymbol{b}) = \sum_i \boldsymbol{d}_i \left(w_i(t)\psi\left(Y_i - \boldsymbol{x}_{i1}^T\boldsymbol{b}, t\right) + \left(1 - w_i(t)\right)\psi\left(Y_\infty - \boldsymbol{x}_{i1}^T\boldsymbol{b}, t\right)\right),$$

where $\psi(u,t) = t - \mathbb{I}(u < 0)$.

Notice that for any l-dimensional vector $\boldsymbol{v}$ and a compact set $\mathcal{B} \in \mathbb{R}^p$, the class of functions

$$\mathcal{G} = \left\{\boldsymbol{v}^T\boldsymbol{d}_i\left(w_i(t)\psi\left(Y_i - \boldsymbol{x}_{i1}^T\boldsymbol{b}, t\right) + (1 - w_i(t))\psi\left(Y_\infty - \boldsymbol{x}_{i1}^T\boldsymbol{b}, t\right)\right), \boldsymbol{b} \in \mathcal{B}\right\}$$

is a VC subgraph class with $\mathbb{E}(g^2) < \infty$ for $g \in \mathcal{G}$. Without loss of generality assume $\boldsymbol{x}_{i1}^T\boldsymbol{b}_1 \le \boldsymbol{x}_{i1}^T\boldsymbol{b}_2$,

$$\begin{aligned}\mathbb{E}\big(g(\boldsymbol{b}_1) - g(\boldsymbol{b}_2)\big)^2 &\le \mathbb{E}\left(\left(\boldsymbol{v}^T\boldsymbol{d}_i\right)^2\mathbb{I}\left(\boldsymbol{x}_{i1}^T\boldsymbol{b}_1 \le Y_i \le \boldsymbol{x}_{i1}^T\boldsymbol{b}_2\right)\right) + \mathbb{E}\left(\left(\boldsymbol{v}^T\boldsymbol{d}_i\right)^2\mathbb{I}\left(\boldsymbol{x}_{i1}^T\boldsymbol{b}_1 \le Y_\infty \le \boldsymbol{x}_{i1}^T\boldsymbol{b}_2\right)\right)\\ &\le K\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|,\end{aligned}$$

where K is a large constant. Because $\|\boldsymbol{\beta}_1(t) - \tilde{\boldsymbol{\beta}}_1(t)\|$ is $O_p(n^{-1/2})$ uniformly for $t \in [t_0, t_M]$ by (C9), we have

$$\Psi\left(\boldsymbol{w}(t), \tilde{\boldsymbol{\beta}}_1(t)\right) - \Psi\left(\boldsymbol{w}(t), \boldsymbol{\beta}_1(t)\right) - \mathbb{E}\left(\Psi\left(\boldsymbol{w}(t), \tilde{\boldsymbol{\beta}}_1(t)\right) - \Psi\left(\boldsymbol{w}(t), \boldsymbol{\beta}_1(t)\right)\right) = o_p(n^{1/2}),$$

uniformly for $t \in [t_0, t_M]$. Calculating the expectation term in the above equation,

$$\begin{aligned}&\mathbb{E}\left(\Psi\left(\boldsymbol{w}(t), \tilde{\boldsymbol{\beta}}_1(t)\right) - \Psi\left(\boldsymbol{w}(t), \boldsymbol{\beta}_1(t)\right)\right)\\ &\quad= \sum_i \boldsymbol{d}_i\mathbb{E}\left(\left(\mathbb{I}\left(T_i \le \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\right) - \mathbb{I}\left(T_i \le \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\right)\mathbb{I}(T_i \le C_i)\right.\\ &\qquad\left.+ w_i(t)\left(\mathbb{I}\left(C_i \le \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\right) - \mathbb{I}\left(C_i \le \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\right)\mathbb{I}(T_i > C_i)\right)\\ &\quad= \sum_i \boldsymbol{d}_i\mathbb{E}\Big(\left(\mathbb{I}\left(T_i \le \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\right) - \mathbb{I}\left(T_i \le \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\right)\mathbb{I}\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t) \le C_i\right)\\ &\qquad+ w_i(t)\Big(\mathbb{I}\left(C_i \le \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\right) - \mathbb{I}\left(C_i \le \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\Big)\mathbb{I}\big(T_i > \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\big)\Big) + O_p(1)\\ &\quad= \sum_i \boldsymbol{d}_i\mathbb{E}\Big(\big(\mathbb{I}(T_i \le \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\big) - \mathbb{I}\big(T_i \le \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\big)\big)\mathbb{I}\big(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t) \le C_i\big)\Big) + O_p(1)\\ &\quad= \sum_i \boldsymbol{d}_i\mathbb{E}\big(\mathbb{I}(T_i \le \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\big) - \mathbb{I}\big(T_i \le \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\big)\big)\mathbb{E}\big(\mathbb{I}\big(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t) \le C_i\big)\big) + O_p(1).\end{aligned} \tag{A1}$$

In the above derivation, the second equality follows because the probability that both T_i and C_i are between $\boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)$ and $\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)$ is of order n^{-1} for $\|\tilde{\boldsymbol{\beta}}_1(t)-\boldsymbol{\beta}_1(t)\| = O(n^{-1/2})$. The third equality follows because the probability that C_i is between $\boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)$ and $\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)$ is of order $n^{-1/2}$ and $w_i(t)$ is $O_p(n^{-1/2})$ for such terms. By the Taylor expansion,

$$\mathbb{E}\left(\mathbb{I}\left(T_i \le \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\right) - \mathbb{I}\left(T_i \le \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\right) = \boldsymbol{x}_{i1}^T f_i\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\left(\tilde{\boldsymbol{\beta}}_1(t)-\boldsymbol{\beta}_1(t)\right) + O(n^{-1}).$$

Therefore

$$\Psi\left(\boldsymbol{w}(t),\tilde{\boldsymbol{\beta}}_1(t)\right) - \Psi(\boldsymbol{w}(t),\boldsymbol{\beta}_1(t)) - \boldsymbol{D}^T\boldsymbol{V}(t)\boldsymbol{X}_1\left(\tilde{\boldsymbol{\beta}}_1(t)-\boldsymbol{\beta}_1(t)\right) = o_p(n^{1/2}),$$

where $\boldsymbol{V}(t)$ is a diagonal matrix with $V_{ii}(t) = f_i\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\left(1-G_i\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\right)$ and $\boldsymbol{D}$ is an $n\times p$ matrix with $\boldsymbol{d}_i^T$ as the ith row. Let $\boldsymbol{CI}_t := \{i : \Delta_i = 0 \text{ and } t > \hat{\tau}_i\}$. Because $\hat{w}_i(t) \ne w_i(t)$ only for $i \in \boldsymbol{CI}_t$,

$$\begin{aligned}\Psi\left(\hat{\boldsymbol{w}}(t),\tilde{\boldsymbol{\beta}}_1(t)\right) = &\sum_{i\in CI_t}\boldsymbol{d}_i\left(\hat{w}_i(t)-w_i(t)\right)\mathbb{I}\left(C_i < \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\right) + \Psi\left(\boldsymbol{w}(t),\boldsymbol{\beta}_1(t)\right)\\ &+\boldsymbol{D}^T\boldsymbol{V}(t)\boldsymbol{X}_1\left(\tilde{\boldsymbol{\beta}}_1(t)-\boldsymbol{\beta}_1(t)\right) + o_p(n^{1/2}),\end{aligned} \tag{A2}$$

uniformly for $t\in\mathcal{S}$. Define $a_i^c(t) := 1 - w_i(t)\mathbb{I}\left(Y_i - \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t) < 0\right)$. Notice that $\Psi\left(\hat{\boldsymbol{w}}(t),\tilde{\boldsymbol{\beta}}_1(t)\right) = \sum_i\left(\hat{a}_i^c(t)-1+t\right)$ and $\Psi\left(\boldsymbol{w}(t),\boldsymbol{\beta}_1(t)\right) = \sum_i\left(a_i^c(t)-1+t\right)$, Equation (A2) gives us

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_i\boldsymbol{d}_i\left(\hat{a}_i^c(t)-1+t\right) = &\frac{1}{\sqrt{n}}\sum_i\boldsymbol{d}_i\left(a_i^c(t)-1+t\right) + \frac{1}{n}\boldsymbol{D}^T\boldsymbol{V}(t)\boldsymbol{X}_1\sqrt{n}\left(\tilde{\boldsymbol{\beta}}_1(t)-\boldsymbol{\beta}_1(t)\right)\\ &+\frac{1}{n}\sum_i\boldsymbol{d}_i\frac{\sqrt{n}(\hat{\tau}_i-\tau_i)}{(1-\tau_i)^2}\mathbb{I}\left(Y_i > C_i\right)\mathbb{I}\left(\boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\ge C_i\right) + o_p(1).\end{aligned} \tag{A3}$$

Notice that the first two terms also appear in the derivation of quantile regression without censoring in similar forms, while the third term is the error incurred by estimating $w_i(t)$ with $\hat{w}_i(t)$. Following Theorem 3.1 from the study by Portnoy & Lin (2010), the third term $\boldsymbol{DT}_{n,d}(t) := \frac{1}{n}\sum_i\boldsymbol{d}_i\frac{\sqrt{n}(\hat{\tau}_i-\tau_i)}{(1-\tau_i)^2}\mathbb{I}(Y_i > C_i)\mathbb{I}\left(\boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\ge C_i\right)$ converges to

$$\boldsymbol{DT}_d(t) = \int_0^t\boldsymbol{\Gamma}_d(u)\boldsymbol{B}_n(u)du,$$

where

$$\boldsymbol{\Gamma}_d(t) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\frac{g_i\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)}{(1-t)\left(1-G_i\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right)\right)}\boldsymbol{d}_i\boldsymbol{x}_{i1}^T.$$

Set $\boldsymbol{d}_i = \boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2}$, Equation (A3) becomes

$$\frac{1}{\sqrt{n}}\sum_i\left(\boldsymbol{x}_{i2}-\hat{\boldsymbol{x}}_{i2}\right)\hat{a}_i^c(t) = \boldsymbol{W}_{n,x_{i2}-\hat{x}_{i2}}(t) + \boldsymbol{K}_{x_{i2}-\hat{x}_{i2}}(t)\boldsymbol{B}_n(t) + \boldsymbol{DT}_{n,x_{i2}-\hat{x}_{i2}}(t) + o_p(1), \tag{A4}$$

where $\boldsymbol{W}_{n,d}(t) = \frac{1}{\sqrt{n}}\sum_i\boldsymbol{d}_i\left(a_i^c(t)-1+t\right)$ and $\boldsymbol{K}_d(t) = \lim_n\frac{1}{n}\boldsymbol{D}^T\boldsymbol{V}(t)\boldsymbol{X}_1$.

By (C9), $\boldsymbol{B}_n(t)$ converges to a Gaussian process $\boldsymbol{B}(t)$ for $t \in [t_0, t_M]$. It is easy to see that the $\boldsymbol{W}_{n,x_{i2}-\hat{x}_{i2}}(t)$ converges to a zero-mean Gaussian process $\boldsymbol{W}_{x_{i2}-\hat{x}_{i2}}(t)$. Thus, under the null hypothesis, $\frac{1}{\sqrt{n}}\sum_i \left(\boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2}\right) \hat{a}_i^c(t)$ converges to a zero-mean Gaussian process. Then the desired result follows naturally. ■

Write $\tau_i^* = \inf_\tau \left\{\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1(\tau) \geq C_i^*\right\}$ and $\hat{\tau}_i^* = \inf_\tau \left\{\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1^*(\hat{\tau}_i) \geq C_i^*\right\}$, and define

$$\hat{w}_i^*(\tau) = \begin{cases} \frac{\tau-\hat{\tau}_i^*}{1-\hat{\tau}_i^*}, & \Delta_i^* = 0, \tau \geq \hat{\tau}_i^*, \\ 1, & \Delta_i^* = 0, \tau < \hat{\tau}_i, \\ 1, & \Delta_i^* = 1, \end{cases} \tag{A5}$$

Let $w_i^*(\tau)$ be defined as (A5) where $\hat{\tau}_i^*$ is replaced by τ_i^*.

Lemma 1. *Using Portnoy's censored quantile regression estimator, and under regularity Conditions (C1)-(C8), given the sample $\left\{Y_i, \Delta_i, \boldsymbol{x}_i\right\}_{i=1}^n$, for $k = 1, \ldots, M$,*

$$\sup_k \|\hat{\boldsymbol{\beta}}_1^*(t_k) - \hat{\boldsymbol{\beta}}_1(t_k)\| = O_{p^*}(n^{-1/2}) \tag{A6}$$

with probability going to 1 in the original space.

Proof. We restrict our analysis to the set $\mathcal{D}$ where $\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1(\tau)$ is monotone in τ. As shown in the study by Portnoy & Lin (2010), $\mathbb{P}(\mathcal{D}) \to 1$. For $\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1(\tau)$ monotone, T_i^* is generated from a valid quantile process $\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1(\tau)$ and many arguments in Portnoy & Lin (2010) can be carried over to the bootstrap space. Following Portnoy & Lin (2010), we shall show by induction that for $k = 1, \ldots, M$,

$$\sum_{i \in CI_k} |\hat{\tau}_i^* - \tau_i^*| \leq c_{k,n}, \tag{A7}$$

$$\|\hat{\boldsymbol{\beta}}_1^*(t_k) - \hat{\boldsymbol{\beta}}_1(t_k)\| \leq 2r_1 n^{-1} c_{k,n}, \tag{A8}$$

where $c_{k,n} = R_n\left(1 + 2r_1 r_2 E_n^* \delta_n\right)^{k-1}$, $R_n = \sup_k \|\Psi^*\left(\boldsymbol{w}^*(t_k), \hat{\boldsymbol{\beta}}(t_k)\right)\|$, $E_n^* = \max\left(\frac{\tilde{E}_n}{2r_1 r_2}, 1 + \tilde{E}_n \delta_n\right)$ with $\tilde{E}_n$ a random bound and $n^{-1/2} \ll \delta_n \ll n^{-1/4}$ is the grid size, and r_1 and r_2 are the two constants given in Equations (A12) and (A15) below.

When $k = 1$, by (C8), there is no censoring for $t \leq t_1$ with $o_p(1)$, thus $\sum_{i \in CI_{t_1}} |\hat{\tau}_i^* - \tau_i^*| = 0$. Since there is no censoring at the t_1-quantile, $\|\hat{\boldsymbol{\beta}}_1^*(t_1) - \hat{\boldsymbol{\beta}}_1(t_1)\| \leq 2r_1 n^{-1} c_{1,n}$ is given by Theorem 3.1 of the study by Sun & He (2021), where the root-n consistency of $\hat{\boldsymbol{\beta}}_1^*(t)$ for the model-based bootstrap without censoring is proved.

Assume Equations (A7) and (A8) are satisfied when $k = l$. At the t_{l+1} level,

$$\begin{aligned} \sum_{i=1}^n |\hat{w}_i^*(t_{l+1}) - w_i^*(t_{l+1})| &= \sum_{i \in CI_l} |\hat{w}_i^*(t_{l+1}) - w_i^*(t_{l+1})| + \sum_{\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_l)} |\hat{w}_i^*(t_{l+1}) - w_i^*(t_{l+1})| \\ &= \sum_{i \in CI_l} \frac{\left(1 - t_{l+1} |\hat{\tau}_i^* - \tau_i^*|\right)}{(1 - \hat{\tau}_i^*)} + \sqrt{n} E_n \delta_n \end{aligned}$$

$$\leq \sum_{i \in CI_l} \frac{1-\epsilon}{\epsilon^2} |\hat{\tau}_i^* - \tau_i^*| + \sqrt{n} E_n \delta_n$$
$$\leq \frac{1-\epsilon}{\epsilon^2} c_{l,n} \left(1 + \tilde{E}_n \delta_n\right),$$

where E_n and $\tilde{E}_n$ are two random bounds. By Lemma 4.1 of the study by He & Shao (1996), we have $\left\{\boldsymbol{\theta} : \|\boldsymbol{\theta} - \hat{\boldsymbol{\beta}}_1(t_{l+1})\| \leq K n^{-1/2}\right\}$,

$$\Psi^* \left(\boldsymbol{w}^*(t_{l+1}), \boldsymbol{\theta}\right) - \Psi^* \left(\boldsymbol{w}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1(t_{l+1})\right) - \mathbb{E}^* \left(\Psi^* \left(\boldsymbol{w}^*(t_{l+1}), \boldsymbol{\theta}\right) - \Psi^* \left(\boldsymbol{w}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1(t_{l+1})\right)\right) = O_p^* \left(n^{1/4} \log n\right). \quad \text{(A9)}$$

Similar to the derivation of Equation (A1),

$$\begin{aligned}
&\mathbb{E}^* \left(\Psi_{l+1}^* \left(\boldsymbol{w}(t_{l+1}), \boldsymbol{\theta}\right) - \Psi_{l+1}^* \left(\boldsymbol{w}(t_{l+1}), \hat{\boldsymbol{\beta}}_1(t_{l+1})\right)\right) \\
&= \sum_i \boldsymbol{d}_i \underbrace{\mathbb{E}^* \left(\mathbb{I}\left(T_i^* \leq \boldsymbol{x}_{i1}^T \boldsymbol{\theta}\right) - \mathbb{I}\left(T_i^* \leq \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1})\right)\right)}_{\text{I}} \underbrace{\mathbb{E}^* \left(\mathbb{I}\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_k) \leq C_i^*\right)\right)}_{\text{II}} + O_{p^*}(1).
\end{aligned}$$

We now calculate the expectation of I and II. According to the bootstrap algorithm, $T_i^* = \boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1(u_i)$ for $2\epsilon < u_i < \min(1-\epsilon, \tau_U)$. When $u_i < 2\epsilon$ or $u_i > 1-\epsilon$, the probability that T_i^* is between $\boldsymbol{x}_{i1}^T \boldsymbol{\theta}$ and $\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1})$ tends to 0 by the asymptotic monotonicity of $\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1(\cdot)$. Let $\boldsymbol{\eta} = \tilde{\boldsymbol{\beta}}_1(u_i) - \boldsymbol{\beta}_1(u_i)$, we have for $2\epsilon < u_i < \min(1-\epsilon, \tau_U)$,

$$\begin{aligned}
\text{I} &= \mathbb{E}^* \left(\mathbb{I}(\boldsymbol{x}_{i1}^T \boldsymbol{\theta} - \boldsymbol{x}_{i1}^T \boldsymbol{\eta} < \boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(u_i) < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1}) - \boldsymbol{x}_{i1}^T \boldsymbol{\eta})\right) \\
&= \int_{\boldsymbol{x}_{i1}^T \theta - \boldsymbol{x}_{i1}^T \boldsymbol{\eta}}^{\boldsymbol{x}_{i1}^T \hat{\beta}_1(t_{l+1}) - \boldsymbol{x}_{i1}^T \boldsymbol{\eta}} f_i(c) dc \\
&= \int_{\boldsymbol{x}_{i1}^T \theta - \boldsymbol{x}_{i1}^T \boldsymbol{\eta}}^{\boldsymbol{x}_{i1}^T \hat{\beta}_1(t_{l+1}) - \boldsymbol{x}_{i1}^T \boldsymbol{\eta}} f_i\left(\boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_{l+1})\right) + O\left(c - \boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_{l+1})\right) dc \\
&= f_i\left(\boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_{l+1})\right) \left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1}) - \boldsymbol{x}_{i1}^T \boldsymbol{\theta}\right) + O_p(n^{-1}).
\end{aligned}$$

By the bootstrap design $C_i^* = \hat{G}^{-1}(v_i | \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2})$ for $v_i < \tau_{V_i}$, where τ_{V_i} is the largest value, $G^{-1}(\cdot | \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2})$ is identifiable. Notice that $G^{-1}(\tau_{V_i} | \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}) \geq \boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_{l+1})$ because otherwise $G^{-1}(\cdot | \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2})$ would be identifiable at τ_{V_i}. If $v_i \geq \tau_{V_i}$ then $\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1}) \leq C_i^*$ since we impute a very large value for C_i^*. Thus

$$\mathbb{P}^*\left(C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1})\right) = \mathbb{P}^* \left(v_i < \hat{G}_i\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1})\right)\right) = \hat{G}_i\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1})\right).$$

By Theorem 2.1 of Gonzalez-Manteiga & Cadarso-Suarez (1994),

$$\sup_t \sup_{\boldsymbol{x}} |\hat{G}(t|\boldsymbol{x}) - G(t|\boldsymbol{x})| = O_p((\log n)^{1/2} n^{-1/4-\gamma_0/2}),$$

where $0 < \gamma_0 < 1/4$. Thus

$$\text{II} = 1 - \hat{G}_i\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1})\right) = 1 - G_i\left(\boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_{l+1})\right) + O_p(n^{-1/4} \log n),$$

and

$$\mathbb{E}^* \left(\Psi^*\left(\boldsymbol{w}^*(t_{l+1}), \boldsymbol{\theta}\right) - \Psi^*\left(\boldsymbol{w}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1(t_{l+1})\right)\right)$$
$$= \sum_i \boldsymbol{d}_i f_i\left(\boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_{l+1})\right) \left(1 - G_i\left(\boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_{l+1})\right)\right) \left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1(t_{l+1}) - \boldsymbol{x}_{i1}^T \boldsymbol{\theta}\right) + O_p(n^{1/4}\log n) + O_p^*(1). \tag{A10}$$

Then Equation (A9) becomes

$$\Psi^*\left(\hat{\boldsymbol{w}}^*(t_{l+1}), \boldsymbol{\theta}\right) = \sum_{i \in CI_l} \boldsymbol{d}_i\left(\hat{w}_i^*(t_{l+1}) - w_i^*(t_{l+1})\right)\mathbb{I}\left(C_i^* < \boldsymbol{x}_{i1}^T\boldsymbol{\theta}\right) + \Psi^*\left(\boldsymbol{w}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1(t_{l+1})\right)$$
$$+ \boldsymbol{D}^T \boldsymbol{V}(t)\boldsymbol{X}_1\left(\boldsymbol{\theta} - \hat{\boldsymbol{\beta}}_1(t_{l+1})\right) + O_p(n^{1/4}\log n) + O_p^*(1). \tag{A11}$$

Set $\boldsymbol{\theta} = \hat{\boldsymbol{\beta}}_1^*(t_{l+1})$ and $\boldsymbol{d}_i = \boldsymbol{x}_{i1}$ in the above equation. (This is possible because if $\|\hat{\boldsymbol{\beta}}_1^*(t_{l+1}) - \hat{\boldsymbol{\beta}}_1(t_{l+1})\| \geq Kn^{-1/2}$ for K is large, the gradient condition cannot be satisfied by Equation (A11).) Then,

$$\|\hat{\boldsymbol{\beta}}_1^*(t_{l+1}) - \hat{\boldsymbol{\beta}}_1(t_{l+1})\| = \Bigg\| \left(\boldsymbol{X}_1^T \boldsymbol{V}(t_{l+1})\boldsymbol{X}_1\right)^{-1} \Bigg(\sum_{i \in CI_l} \boldsymbol{x}_{i1}\left(\hat{w}_i^*(t_{l+1}) - w_i^*(t_{l+1})\right)\mathbb{I}\left(C_i^* < \boldsymbol{x}_{i1}^T\hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right)$$
$$- \Psi^*\left(\hat{\boldsymbol{w}}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right) + \Psi^*\left(\boldsymbol{w}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1(t_{l+1})\right) + O_{p^*}(n^{1/4}\log n)$$
$$+ O_p(n^{1/4}\log n)\Bigg)\Bigg\|.$$

By (C3) and (C4), there exists an $a > 0$ such that the largest eigenvalue of $\left(\boldsymbol{X}_1^T \boldsymbol{V}(t)\boldsymbol{X}_1\right)^{-1} \leq an^{-1}$. Let

$$r_1 = an^{-1}\frac{1-\epsilon}{\epsilon^2}, \tag{A12}$$

$$\|\hat{\boldsymbol{\beta}}_1^*(t_{l+1}) - \hat{\boldsymbol{\beta}}_1(t_{l+1})\| \leq an^{-1}\Bigg(\sum_{i \in CI_l}\left(\hat{w}_i^*(t_{l+1}) - w_i^*(t_{l+1})\right) + \|\Psi_{l+1}^*\left(\hat{\boldsymbol{w}}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right)\|$$
$$+ \|\Psi_{l+1}^*\left(\boldsymbol{w}^*(t_{l+1}), \hat{\boldsymbol{\beta}}_1(t_{l+1})\right)\| + O_{p^*}(n^{1/4}\log n) + O_p(n^{1/4}\log n))$$
$$\leq an^{-1}\left(\frac{1-\epsilon}{\epsilon^2}c_{l,n}\left(1 + \tilde{E}_n\delta_n\right) + R_n\right)$$
$$\leq r_1 n^{-1} c_{l,n}\left(1 + \tilde{E}\delta_n\right) + r_1 n^{-1} c_{l,n}$$
$$\leq 2r_1 n^{-1} c_{l+1,n}.$$

Therefore, Equation (A8) holds. To verify Equation (A7), consider

$$\sum_{i \in CI_{l+1}} |\hat{\tau}_i^* - \tau_i^*| \leq \sum_{i \in CI_l} |\hat{\tau}_i^* - \tau_i^*| + \sum_i |\hat{\tau}_i^* - \tau_i^*|\mathbb{I}\left(\boldsymbol{x}_{i1}^T\hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T\hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right). \tag{A13}$$

We aim to bound the last term in the previous equation. Let $j = j(i)$ such that $t_j \le \hat{\tau}_i^* \le t_{j+1}$, since both $\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1^* \left(\hat{\tau}_i^*\right)$ and $\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1 \left(\tau_i^*\right)$ are equal to $\boldsymbol{C}_i^*$,

$$0 = \boldsymbol{x}_{i1}^T \left(\tilde{\boldsymbol{\beta}}_1^* \left(\hat{\tau}_i^*\right) - \tilde{\boldsymbol{\beta}}_1 \left(\hat{\tau}_i^*\right)\right) + \boldsymbol{x}_{i1}^T \left(\tilde{\boldsymbol{\beta}}_1 \left(\hat{\tau}_i^*\right) - \tilde{\boldsymbol{\beta}}_1 \left(\tau_i^*\right)\right).$$

Define $\hat{\alpha}_i^*$ such that $\tilde{\boldsymbol{\beta}}_1^* \left(\hat{\tau}_i^*\right) = \hat{\boldsymbol{\beta}}_1^*(t_j) + \hat{\alpha}_i^* \left(\hat{\boldsymbol{\beta}}_1^*(t_{j+1}) - \hat{\boldsymbol{\beta}}_1^*(t_j)\right)$,

$$\boldsymbol{x}_{i1}^T \left(\tilde{\boldsymbol{\beta}}_1^* \left(\hat{\tau}_i^*\right) - \tilde{\boldsymbol{\beta}}_1 \left(\hat{\tau}_i^*\right)\right) = \hat{\alpha}_i^* \boldsymbol{x}_{i1}^T \left(\hat{\boldsymbol{\beta}}_1^*(t_j) - \hat{\boldsymbol{\beta}}_1(t_j)\right) + \left(1 - \hat{\alpha}_i^*\right) \boldsymbol{x}_{i1}^T \left(\hat{\boldsymbol{\beta}}_1^*(t_{j+1}) - \hat{\boldsymbol{\beta}}_1(t_{j+1})\right).$$

Let $h_i(\cdot)$ be the right derivative of $\boldsymbol{x}_{i1}^T \tilde{\boldsymbol{\beta}}_1(\cdot)$, by the Taylor expansion,

$$\boldsymbol{x}_{i1}^T \left(\tilde{\boldsymbol{\beta}}_1 \left(\hat{\tau}_i^*\right) - \tilde{\boldsymbol{\beta}}_1 \left(\tau_i^*\right)\right) = \left(\hat{\tau}_i^* - \hat{\tau}_i\right) h_i(t_j) + O\left(\delta_n^2\right).$$

Thus, we have

$$\sqrt{n}\left(\hat{\tau}_i^* - \tau_i^*\right) = h_i(t_j) \boldsymbol{x}_{i1}^T \boldsymbol{B}_{i,j}^* + O\left(\delta_n^2\right), \tag{A14}$$

where

$$\boldsymbol{B}_{i,j}^* = \sqrt{n}\left(\hat{\alpha}_i^* \left(\hat{\boldsymbol{\beta}}_1^*(t_j) - \hat{\boldsymbol{\beta}}_1(t_j)\right) + \left(1 - \hat{\alpha}_i^*\right)\left(\hat{\boldsymbol{\beta}}_1^*(t_{j+1}) - \hat{\boldsymbol{\beta}}_1(t_{j+1})\right)\right).$$

Set r_2 to be a constant that satisfies

$$r_2 \ge \|\boldsymbol{x}_{i1}\| g_i \left(\boldsymbol{x}_{i1}^T \boldsymbol{\beta}_1(t_l)\right) \quad \text{for any } \ i, l. \tag{A15}$$

By Equation (A14), Equation (A13) becomes

$$\begin{aligned}
\sum_{i \in C I_{l+1}} |\hat{\tau}_i^* - \tau_i^*| &\le c_{l,n} + \sum_i \left(n^{-1/2} h_i(t_j) \boldsymbol{B}_{i,l}^* \mathbb{I}\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right)\right) + O_{p^*}\left(n\delta_n^2\right) \\
&\le c_{l,n} + \sum_i \left(r_2 n^{-1/2} \boldsymbol{x}_{i1}^T \boldsymbol{B}_{i,l}^* \delta_n\right) \\
&\le c_{l,n} + 2 r_1 r_2 c_{l,n} \left(1 + \tilde{E}_n \delta_n\right) \delta_n \\
&\le c_{l+1,n}.
\end{aligned}$$

In the second line of the above derivation, we replace $\mathbb{I}(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_{l+1}))$ with its expectation, which is of order δ_n. The error incurred by this replacement is dominated by $c_{l,n}$ because

$$\begin{aligned}
&\mathbb{E}^*\left(\sum_i \left(\mathbb{I}\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right) - \mathbb{E}^*\left(\mathbb{I}\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right)\right)\right)\right)^2 \\
&= \sum_i \mathbb{E}^*\left(\mathbb{I}\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right) - \mathbb{E}^*\left(\mathbb{I}\left(\boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_l) < C_i^* < \boldsymbol{x}_{i1}^T \hat{\boldsymbol{\beta}}_1^*(t_{l+1})\right)\right)\right)^2 \\
&= O(n\delta_n).
\end{aligned}$$

■

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Lemma 2. *Let* $\boldsymbol{DT}^*_{n,d}(t) := \frac{1}{n}\sum_i \boldsymbol{d}_i \frac{\sqrt{n}(\hat{\tau}^*_i - \tau^*_i)}{(1-\tau^*_i)^2} \mathbb{I}\left(Y^*_i > C^*_i\right) \mathbb{I}\left(\boldsymbol{x}^T_{i1}\hat{\boldsymbol{\beta}}^*_1(t) \geq C^*_i\right)$. *Assume regularity Conditions (C1)–(C7) and (C9), given the sample* $\left\{Y_i, \Delta_i, \boldsymbol{x}_i\right\}^n_{i=1}$, $\boldsymbol{DT}^*_{n,d}(t)$ *converges to*

$$\boldsymbol{DT}^*_d(t) := \int_0^t \boldsymbol{\Gamma}_d(u)\boldsymbol{B}^*_n(u)du,$$

uniformly for $t \in [t_1, t_M]$, *with probability going to 1 in the original space.*

Proof. The lemma can be proved by adjusting the arguments that study $\boldsymbol{DT}_{n,d}$ in Theorem 3.1 in Portnoy & Lin (2010) to the bootstrap space. ■

Proof of Proposition 2. By Lemma 1 and arguments similar to those in the proof of Theorem 1,

$$\Psi\left(\boldsymbol{w}^*(t), \tilde{\boldsymbol{\beta}}^*_1(t)\right) - \Psi\left(\boldsymbol{w}^*(t), \tilde{\boldsymbol{\beta}}_1(t)\right) - \mathbb{E}^*\left(\Psi\left(\boldsymbol{w}^*(t), \tilde{\boldsymbol{\beta}}^*_1(t)\right) - \Psi\left(\boldsymbol{w}^*(t), \tilde{\boldsymbol{\beta}}_1(t)\right)\right) = o_{p^*}\left(n^{1/2}\right),$$

uniformly for $t \in [t_1, t_M]$. By Equation (A10),

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_i \boldsymbol{d}_i\left(\hat{a}^{c*}_i(t) - 1 + t\right) = \frac{1}{\sqrt{n}}\sum_i \boldsymbol{d}_i\left(a^{c*}_i(t) - 1 + t\right) + \frac{1}{n}\boldsymbol{D}^T\boldsymbol{V}(t)\boldsymbol{X}_1\sqrt{n}\left(\hat{\boldsymbol{\beta}}^*_1(t) - \hat{\boldsymbol{\beta}}_1(t)\right) \\ + \frac{1}{n}\sum_i \boldsymbol{d}_i \frac{\sqrt{n}\left(\hat{\tau}^*_i - \tau^*_i\right)}{\left(1-\tau^*_i\right)^2}\mathbb{I}\left(Y^*_i > C^*_i\right)\mathbb{I}\left(\boldsymbol{x}^T_{i1}\hat{\boldsymbol{\beta}}^*_1(t) \geq C^*_i\right) + o_{p^*}(1),\end{aligned} \tag{A16}$$

where $\hat{a}^{c*}_i(t) = 1 - \hat{w}^*_i(t)\mathbb{I}\left(Y^*_i - \boldsymbol{x}^T_{i1}\tilde{\boldsymbol{\beta}}^*_1(t) < 0\right)$ and $a^{c*}_i(t) = 1 - w^*_i(t)\mathbb{I}\left(Y^*_i - \boldsymbol{x}^T_{i1}\tilde{\boldsymbol{\beta}}_1(t) < 0\right)$.

Set $\boldsymbol{d}_i = \boldsymbol{x}_{i1}$, we have $\frac{1}{\sqrt{n}}\sum_i \boldsymbol{x}_{i1}\left(\hat{a}^{c*}_i(t) - 1 + t\right) = o(1)$, and Equation (A16) becomes

$$\boldsymbol{K}_{\boldsymbol{x}_{i1}}(t)\boldsymbol{B}^*_n(t) = \int_0^t \boldsymbol{\Gamma}_{\boldsymbol{x}_{i1}}(u)\boldsymbol{B}^*_n(u)du + \boldsymbol{W}^*_{n,\boldsymbol{x}_{i1}}(t) + o_{p^*}(1), \tag{A17}$$

where $\boldsymbol{W}^*_{n,d_i}(t) = \frac{1}{\sqrt{n}}\sum \boldsymbol{d}_i\left(a^{c*}_i(t) - 1 + t\right)$.

Set $\boldsymbol{d}_i = \boldsymbol{x}_{i1}$ in Equation (A3), we have

$$\boldsymbol{K}_{\boldsymbol{x}_{i1}}(t)\boldsymbol{B}_n(t) = \int_0^t \boldsymbol{\Gamma}_{\boldsymbol{x}_1}(u)\boldsymbol{B}_n(u)du + \boldsymbol{W}_{n,\boldsymbol{x}_1}(t) + o_p(1). \tag{A18}$$

Solving $\boldsymbol{B}_n(t)$ in Equation (A18) by Theorem 10 in Gill & Johansen (1990), we have

$$\boldsymbol{K}_{\boldsymbol{x}_{i1}}(t)\boldsymbol{B}_n(t) = \boldsymbol{W}_{n,\boldsymbol{x}_{i1}}(t) + \int_0^t \mathcal{I}(s,t)\boldsymbol{\Gamma}_{\boldsymbol{x}_{i1}}(s)\boldsymbol{K}^{-1}_{\boldsymbol{x}_{i1}}(s)\boldsymbol{W}_{n,\boldsymbol{x}_{i1}}(s)ds + o_p(1), \tag{A19}$$

where $\mathcal{I}(s,t) = \boldsymbol{\Pi}_{u\in(s,t]}\left(I_p + d\boldsymbol{\Gamma}_{\boldsymbol{x}_{i1}}(u)\boldsymbol{K}^{-1}_{\boldsymbol{x}_{i1}}(u)\right)$. Solving $\boldsymbol{B}^*_n(t)$ in Equation (A17),

$$\boldsymbol{K}_{\boldsymbol{x}_{i1}}(t)\boldsymbol{B}^*_n(t) = \boldsymbol{W}^*_{n,\boldsymbol{x}_{i1}}(t) + \int_0^t \mathcal{I}(s,t)\boldsymbol{\Gamma}_{\boldsymbol{x}_{i1}}(s)\boldsymbol{K}^{-1}_{\boldsymbol{x}_{i1}}(s)\boldsymbol{W}^*_{n,\boldsymbol{x}_{i1}}(s)ds + o^*_p(1). \tag{A20}$$

We only need to compare the limiting distribution of $\boldsymbol{W}_{n,\boldsymbol{x}_{i1}}(t)$ and $\boldsymbol{W}^*_{n,\boldsymbol{x}_{i1}}(t)$. By calculation, $\mathbb{E}\left(a_i^c(t)\right) = \mathbb{P}\left(T_i > \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t)\right) = 1-t$, and $\mathbb{E}^*\left(a_i^{c*}(t)\right) = \mathbb{P}\left(T_i^* > \boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t)\right) = 1-t$ with probability going to 1. Now consider the second moment, for $t_1 \le t_2$,

$$\begin{aligned}
&\mathbb{E}\left(\left(1-a_i^c(t_1)\right)\left(1-a_i^c(t_2)\right)\right)\\
&\quad = \mathbb{P}\left(T_i < C_i, T_i < \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t_1)\right) + w_i(t_1)w_i(t_2)\mathbb{P}\left(T_i > C_i, C_i < \boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t_1)\right)\\
&\quad = t_1 - \frac{\left((1-t_2)G_i\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(t_2)\right)\right)\left(\int_0^{t_1} G_i(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(u))du\right)}{\int_0^{t_2} G_i\left(\boldsymbol{x}_{i1}^T\boldsymbol{\beta}_1(u)\right)du}.
\end{aligned}$$

Repeating the same calculation in the bootstrap space,

$$\begin{aligned}
\mathbb{E}^*\left(\left(1-a_i^{c*}(t_1)\right)\left(1-a_i^{c*}(t_2)\right)\right) &= t_1 - \frac{\left((1-t_2)\hat{G}_i\left(\boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(t_2)\right)\right)\left(\int_0^{t_1} \hat{G}_i(\boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(u))du\right)}{\int_0^{t_2} G_i\left(\boldsymbol{x}_{i1}^T\tilde{\boldsymbol{\beta}}_1(u)\right)du}\\
&= \mathbb{E}\left(\left(1-a_i^c(t_1)\right)\left(1-a_i^c(t_2)\right)\right) + o_p(1).
\end{aligned}$$

Thus, $\boldsymbol{W}_{n,\boldsymbol{x}_{i1}}(t)$ and $\boldsymbol{W}^*_{n,\boldsymbol{x}_{i1}}(t)$ converge to the same Gaussian process. Therefore $\boldsymbol{B}_n(t)$ and $\boldsymbol{B}_n^*(t)$ converge to the same Gaussian process by Equations (A19) and (A20). ■

Proof of Theorem 2. Set $\boldsymbol{d}_i = \boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2}$ in Equation (A16),

$$\frac{1}{\sqrt{n}}\sum_i \left(\boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2}\right)\hat{a}_i^{c*}(t) = \boldsymbol{W}^*_{n,\boldsymbol{x}_{i2}-\hat{\boldsymbol{x}}_{i2}}(t) + \boldsymbol{K}_{\boldsymbol{x}_{i2}-\hat{\boldsymbol{x}}_{i2}}(t)\boldsymbol{B}_n^*(t) + \boldsymbol{D}\boldsymbol{T}^*_{n,x_{i2}-\hat{x}_{i2}}(t) + o_p^*(1). \tag{A21}$$

By the assumption that $\boldsymbol{B}_n^*(t)$ is bootstrap consistent, given the data, $\frac{1}{\sqrt{n}}\sum_i\left(\boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2}\right)\hat{a}_i^{c*}(t)$ converges to the same process as $\frac{1}{\sqrt{n}}\sum_i\left(\boldsymbol{x}_{i2} - \hat{\boldsymbol{x}}_{i2}\right)\hat{a}_i^{c}(t)$. Then it follows immediately that the conditional distribution of $\mathcal{T}_1^*/\mathcal{T}_2^*$ will converge to the same limiting distribution as $\mathcal{T}_1/\mathcal{T}_2$. ■

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APPENDIX B: SIMULATION RESULTS

TABLE B1: Comparison of the empirical Type I error rate and power under Case (i).

	$n = 100$		$n = 200$		$n = 500$	
	Type I error	Power	Type I error	Power	Type I error	Power
$\mathcal{T}_1^{\text{km}}[0.50, 0.85]$	0.046	0.365	0.047	0.651	0.057	0.979
$\mathcal{T}_1^{\text{km}}[0.75, 0.85]$	0.056	0.438	0.029	0.798	0.057	0.998
$\mathcal{T}_2^{\text{km}}[0.50, 0.85]$	0.049	0.352	0.048	0.647	0.054	0.979
$\mathcal{T}_2^{\text{km}}[0.75, 0.85]$	0.056	0.433	0.029	0.797	0.055	0.998
$\mathcal{T}_1^{\text{qr}}[0.50, 0.85]$	0.057	0.372	0.048	0.660	0.059	0.981
$\mathcal{T}_1^{\text{qr}}[0.75, 0.85]$	0.061	0.440	0.037	0.809	0.056	0.998
$\mathcal{T}_2^{\text{qr}}[0.50, 0.85]$	0.061	0.361	0.047	0.653	0.056	0.982
$\mathcal{T}_2^{\text{qr}}[0.75, 0.85]$	0.062	0.435	0.036	0.805	0.057	0.998
$\mathcal{T}_3^{\text{qr}}[0.50]$	0.044	0.165	0.055	0.280	0.054	0.549
$\mathcal{T}_3^{\text{qr}}[0.60]$	0.056	0.308	0.051	0.537	0.052	0.902
$\mathcal{T}_3^{\text{qr}}[0.75]$	0.059	0.431	0.041	0.772	0.057	0.998
$\mathcal{T}_3^{\text{qr}}[0.85]$	0.064	0.390	0.046	0.784	0.058	0.998

Note: $\mathcal{T}_1^{\text{km}}[\tau_a, \tau_b]$ denotes the test statistic $\mathcal{T}_1$ over τ in $[\tau_a, \tau_b]$ with C_i^* sampled from the local KM estimator. Similarly, $\mathcal{T}_2^{\text{qr}}[\tau_a, \tau_b]$ denotes the test statistic $\mathcal{T}_2$ over τ in $[\tau_a, \tau_b]$ with C_i^* sampled from the censored quantile regression model. The term $\mathcal{T}_3^{\text{qr}}[\tau]$ denotes the test statistic $\mathcal{T}_3$ at τ. All tests use the nominal significance level of 0.05. The standard errors for the Type I errors are no higher than 0.007.

TABLE B2: Comparison of the empirical Type I error rate and power under Case (ii).

	$n = 100$		$n = 200$		$n = 500$	
	Type I error	Power	Type I error	Power	Type I error	Power
$\mathcal{T}_1^{\mathrm{km}}[0.10, 0.70]$	0.046	0.350	0.057	0.621	0.049	0.935
$\mathcal{T}_1^{\mathrm{km}}[0.20, 0.80]$	0.042	0.158	0.058	0.273	0.053	0.502
$\mathcal{T}_2^{\mathrm{km}}[0.10, 0.70]$	0.044	0.763	0.063	1.000	0.051	1.000
$\mathcal{T}_2^{\mathrm{km}}[0.20, 0.80]$	0.038	0.514	0.064	0.939	0.056	1.000
$\mathcal{T}_1^{\mathrm{qr}}[0.10, 0.70]$	0.045	0.351	0.063	0.625	0.051	0.936
$\mathcal{T}_1^{\mathrm{qr}}[0.20, 0.80]$	0.045	0.108	0.056	0.274	0.057	0.497
$\mathcal{T}_2^{\mathrm{qr}}[0.10, 0.70]$	0.046	0.766	0.064	1.000	0.053	1.000
$\mathcal{T}_2^{\mathrm{qr}}[0.20, 0.80]$	0.045	0.519	0.058	0.944	0.055	1.000
$\mathcal{T}_3^{\mathrm{qr}}[0.10]$	0.034	0.995	0.05	1.000	0.047	1.000
$\mathcal{T}_3^{\mathrm{qr}}[0.30]$	0.040	0.77	0.06	0.969	0.056	0.999
$\mathcal{T}_3^{\mathrm{qr}}[0.70]$	0.039	0.116	0.045	0.240	0.064	0.623
$\mathcal{T}_3^{\mathrm{qr}}[0.90]$	0.051	0.065	0.055	0.143	0.051	0.565

Note: See the Note of Table B1 for more details.

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