Additive Schwarz Preconditioners for C^0 Interior Penalty Methods for a State Constrained Elliptic Distributed Optimal Control Problem

Susanne C. Brenner, Li-Yeng Sung, and Kening Wang

1 Introduction

Let Ω be a bounded convex polygon in $\mathbb{R}^2$, $f \in L_2(\Omega)$, and $\beta > 0$ be a constant. We consider the following elliptic optimal control problem: Find $(y,u) \in H_0^1(\Omega)\times L_2(\Omega)$ that minimize the functional

$$J(y,u) = \frac{1}{2}\int_\Omega (y-f)^2 + \frac{\beta}{2}\int_\Omega u^2\,dx$$

subject to

$$-\Delta y = u \;\text{ in }\; \Omega, \qquad y = 0 \;\text{ on }\; \partial\Omega,$$

and $y \le \psi$ in Ω, where $\psi \in W^{3,p}(\Omega)$ for $p > 2$, and $\psi > 0$ on $\partial\Omega$.

By elliptic regularity (cf. [6]), we can reformulate the model problem as follows: Find $y \in K$ such that

$$y = \underset{v\in K}{\operatorname{argmin}}\left[\frac{1}{2}a(v,v) - (f,v)\right], \tag{1}$$

where $K = \{v \in H^2(\Omega)\cap H_0^1(\Omega) : v \le \psi \;\text{ in }\; \Omega\}$,

$$a(v,w) = \beta\int_\Omega \nabla^2 v : \nabla^2 w\,dx + \int_\Omega vw\,dx \quad \text{and} \quad (f,v) = \int_\Omega fv\,dx.$$

Here $\nabla^2 v : \nabla^2 w = \sum_{i,j=1}^2 \frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 w}{\partial x_i \partial x_j}$ is the inner product of the Hessian matrices of v and w. Once y is calculated, then u can be determined by $u = -\Delta y$.

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A quadratic C^0 interior penalty method for the minimization problem (1) was analyzed in [4]. The goal of this paper is to apply the ideas in [3] for an obstacle problem of clamped Kirchhoff plates to develop and analyze additive Schwarz preconditioners for the discrete problem in [4].

2 The C^0 Interior Penalty Method

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω consisting of convex quadrilaterals, and let $V_h \subset H_0^1(\Omega)$ be the standard Q_k finite element space (the space of polynomials of degree $\leq k$ in each variable) associated with $\mathcal{T}_h$.

The discrete problem of the optimal control problem (1) resulting from the C^0 interior penalty method is to find

$$y_h = \operatorname*{argmin}_{v \in K_h} \left[\frac{1}{2} a_h(v,v) - (f,v) \right], \tag{2}$$

where

$$K_h = \{v \in V_h : v(p) \leq \psi(p), \quad \forall\, p \in \mathcal{N}_h\},$$

$$a_h(v,w) = \beta \Big[\sum_{D \in \mathcal{T}_h} \int_D \nabla^2 v : \nabla^2 w \, dx + \sum_{e \in \mathcal{E}_h^i} \frac{\eta}{|e|} \int_e \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] ds$$
$$+ \sum_{e \in \mathcal{E}_h^i} \int_e \left(\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] \right) ds \Big] + \sum_{D \in \mathcal{T}_h} \int_D vw \, dx,$$

$\mathcal{N}_h$ is the set of nodes in Ω associated with V_h, $\mathcal{E}_h^i$ is the set of edges in $\mathcal{T}_h$ that are interior to Ω, $\eta > 0$ is a sufficiently large penalty parameter, and the jump $[\![\cdot]\!]$ and the average $\{\!\{\cdot\}\!\}$ are defined as follows. Let e be an interior edge shared by two elements, D_- and D_+, and n_e be the unit normal vector pointing from D_- to D_+, we define

$$\left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} = \frac{1}{2} \left(\frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2} \right).$$

Note that $a_h(\cdot,\cdot)$ is a consistent bilinear form for the biharmonic equation with the boundary conditions of simply supported plates (cf. [4]).

It follows from the standard theory that the discrete problem (2) has a unique solution $y_h \in K_h$ characterized by the discrete variational inequality

$$a_h(y_h, v_h - y_h) \geq (f, v_h - y_h) \qquad \forall\, v_h \in K_h. \tag{3}$$

Moreover, there exists a positive constant C independent of h such that (cf. [4])

$$\|y - y_h\|_h \leq Ch^\alpha,$$

where $\|\cdot\|_h$ is the mesh-dependent energy norm defined by

$$\|v\|_h^2 = \beta\Big(\sum_{D\in\mathcal{T}_h} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h^i} \frac{1}{|e|}\|\,[[\partial v/\partial n]]\,\|^2_{L_2(e)}\Big) + \|v\|^2_{L_2(\Omega)},$$

h is the mesh size of the triangulation, and $\alpha \in (0,1]$ is the index of elliptic regularity that is determined by the interior angles of Ω.

3 The Primal-Dual Active Set Algorithm

By introducing a Lagrange multiplier $\lambda_h : \mathcal{N}_h \to \mathbb{R}$, the discrete variational inequality (3) is equivalent to

$$a_h(y_h, v) - (f, v) = -\sum_{p\in\mathcal{N}_h} \lambda_h(p)v(p) \qquad \forall\, v \in V_h, \tag{4}$$

$$y_h(p) - \psi(p) \geq 0,\ \ \lambda_h(p) \geq 0 \ \text{ and }\ (y_h(p) - \psi(p))\lambda_h(p) = 0 \quad \forall\, p \in \mathcal{N}_h \tag{5}$$

Moreover, the optimality conditions (5) can be written concisely as

$$\lambda_h(p) = \max(0, \lambda_h(p) + c(y_h(p) - \psi(p))) \qquad \forall\, p \in \mathcal{N}_h, \tag{6}$$

where c is a large positive number. The system (4) and (6) can then be solved by a primal-dual active set (PDAS) algorithm (cf. [7, 8]).

Given the $k-$th approximation (y_k, λ_k), the $(k+1)-$st iteration of the PDAS algorithm is to find (y_{k+1}, λ_{k+1}) such that

$$a_h(y_{k+1}, v) - (f, v) = -\sum_{p\in\mathcal{N}_h} \lambda_{k+1}(p)v(p) \qquad \forall\, v \in V_h, \tag{7a}$$

$$y_{k+1}(p) = \psi(p) \qquad \forall\, p \in \mathcal{A}_k, \tag{7b}$$

$$\lambda_{k+1}(p) = 0 \qquad \forall\, p \in \mathcal{I}_k, \tag{7c}$$

where $\mathcal{A}_k = \{p \in \mathcal{N}_h : \lambda_k(p) + c(y_k(p) - \psi(p)) > 0\}$ is the active set determined by (y_k, λ_k), and $\mathcal{I}_k = \mathcal{N}_h \backslash \mathcal{A}_k$ is the inactive set. The iteration terminates when $\mathcal{A}_{k+1} = \mathcal{A}_k$. Given a sufficiently accurate initial guess, the PDAS algorithm converges superlinearly to the unique solution of (3) (cf. [7]).

From (7b) and (7c), we can reduce (7a) to an auxiliary system that only involves the unknowns of $y_{k+1}(p)$ for $p \in \mathcal{I}_k$. But even so, for small h, the reduced auxiliary system is still large, sparse, and ill-conditioned. To solve such systems more efficiently, we can apply the preconditioned conjugate gradient method.

Let $\widetilde{\mathcal{N}}_h$ be a subset of $\mathcal{N}_h$. We define $\widetilde{T}_h : V_h \to V_h$, the truncation operator, by

$$(\widetilde{T}_h v)(p) = \begin{cases} v(p) & \text{if } \ p \in \widetilde{\mathcal{N}}_h, \\ 0 & \text{if } \ p \in \mathcal{N}_h \backslash \widetilde{\mathcal{N}}_h. \end{cases}$$

Then $\widetilde{T}_h$ is a projection from V_h onto $\widetilde{V}_h = \widetilde{T}_h V_h$. Moreover, let $\widetilde{A}_h : \widetilde{V}_h \to \widetilde{V}_h'$ be defined by

$$\langle \widetilde{A}_h v, w \rangle = a_h(v, w) \qquad \forall\, v, w \in \widetilde{V}_h,$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $\widetilde{V}_h' \times \widetilde{V}_h$.

In the context of solving (3), the set $\widetilde{\mathcal{N}}_h$ represents the inactive set that appears in an iteration of the PDAS algorithm and $\widetilde{A}_h$ represents the stiffness matrix for the corresponding auxiliary system. Our goal is to develop preconditioners for $\widetilde{A}_h$ whose performance is independent of $\widetilde{\mathcal{N}}_h$.

4 A One-Level Additive Schwarz Preconditioner

Let Ω_j, $1 \le j \le J$, be overlapping subdomains of Ω such that $\Omega = \bigcup_{j=1}^J \Omega_j$, diam $\Omega_j \approx H$, and the boundaries of Ω_j are aligned with $\mathcal{T}_h$. We assume that there exist non-negative $\theta_j \in C^\infty(\bar{\Omega})$ for $1 \le j \le J$ such that

$$\begin{aligned} \theta_j &= 0 \qquad\qquad \text{on } \ \Omega \setminus \Omega_j, \\ \sum_{j=1}^J \theta_j &= 1 \qquad\qquad \text{on } \ \bar{\Omega}, \\ \|\nabla \theta_j\|_{L_\infty(\Omega)} \le \frac{C_\dagger}{\delta}, &\qquad \|\nabla^2 \theta_j\|_{L_\infty(\Omega)} \le \frac{C_\dagger}{\delta^2}, \end{aligned}$$

where $\nabla^2 \theta_j$ is the Hessian of θ_j, $\delta > 0$ measures the overlap among subdomains, and $C_\dagger$ is a positive constant independent of h, H, and J. Moreover, we assume that

$$\text{any point in } \ \Omega \ \text{ can belong to at most } \ N_c \ \text{ many subdomains,}$$

where the positive integer N_c is independent of h, H, J and δ.

Let $\widetilde{V}_j$ be the subspace of $\widetilde{V}_h$ whose members vanish at all nodes outside Ω_j, and let $\widetilde{A}_j : \widetilde{V}_j \to \widetilde{V}_j'$ be defined by

$$\langle \widetilde{A}_j v, w \rangle = a_{h,j}(v, w) \qquad \forall\, v, w \in \widetilde{V}_j,$$

where

$$\begin{aligned} a_{h,j}(v, w) = \beta \Big[\sum_{D \in \mathcal{T}_h} \int_D \nabla^2 v : \nabla^2 w \, dx + \sum_{\substack{e \in \mathcal{E}_h^i \\ e \subset \overline{\Omega}_j \setminus \partial\Omega}} \frac{\eta}{|e|} \int_e \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] ds \\ + \sum_{\substack{e \in \mathcal{E}_h^i \\ e \subset \overline{\Omega}_j \setminus \partial\Omega}} \int_e \left(\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] + \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \right) ds \Big] + \sum_{D \in \mathcal{T}_h} \int_D v w \, dx. \end{aligned}$$

The one-level additive Schwarz preconditioner $B_{OL} : \widetilde{V}_h' \to \widetilde{V}_h$ is then defined by

$$B_{OL} = \sum_{j=1}^{J} \widetilde{I}_j \widetilde{A}_j^{-1} \widetilde{I}_j^t,$$

where $\widetilde{I}_j : \widetilde{V}_j \longrightarrow \widetilde{V}_h$ $(1 \le j \le J)$ is the natural injection operator, and $\widetilde{I}_j^t : \widetilde{V}_h' \to \widetilde{V}_j'$ is the transpose of $\widetilde{I}_j$.

With similar arguments as in [3], we can obtain the following result.

Theorem 1 *It holds that*

$$\kappa(B_{OL}\widetilde{A}_h) = \frac{\lambda_{\max}(B_{OL}\widetilde{A}_h)}{\lambda_{\min}(B_{OL}\widetilde{A}_h)} \le C_1 \delta^{-4},$$

where the positive constant C_1 is independent of H, h, j, δ and $\widetilde{N}_h$.

Remark 1 The condition number estimate given in Theorem 1 is identical to the one for the plate bending problem without obstacles, which indicates that the obstacle is invisible to the one-level additive Schwarz preconditioner.

5 A Two-level Additive Schwarz Preconditioner

A two-level additive Schwarz preconditioner contains not only subdomain solves, but also a coarse grid solve. Let $\mathcal{T}_H$ be a coarse quasi-uniform triangulation for Ω whose mesh size is comparable to the diameters of the subdomains Ω_j, $1 \le j \le J$, and $V_H \subset H_0^1(\Omega)$ be the Q_k finite element space associated with $\mathcal{T}_H$.

Since the Q_{k+2} Bogner-Fox-Schmit (BFS) tensor product element is a C^1 relative of the Q_k tensor product element (cf. [2]), we define $W_H \subset H^2(\Omega) \cap H_0^1(\Omega)$ to be the Q_{k+2} BFS finite element space associated with $\mathcal{T}_H$. The two spaces V_H and W_H can be connected by an enriching operator E_H which is constructed by the averaging technique (cf. [2, 3]).

Now we define $I_0 : V_H \longrightarrow V_h$ by

$$I_0 = \Pi_h \circ E_H$$

where $\Pi_h : C^0(\bar{\Omega}) \to V_h$ is the nodal interpolation operator.

Let $\widetilde{V}_0 \subset \widetilde{V}_h$ be defined by

$$\widetilde{V}_0 = \widetilde{T}_h \, I_0 \, V_H,$$

and let the operator $\widetilde{A}_0 : \widetilde{V}_0 \to \widetilde{V}_0'$ be defined by

$$\langle \widetilde{A}_0 v, w \rangle = a_h(v, w) \qquad \forall\, v, w \in \widetilde{V}_0.$$

Then the two-level additive Schwarz preconditioner $B_{TL} : \widetilde{V}_h' \to \widetilde{V}_h$ is given by

$$B_{TL} = \sum_{j=0}^{J} \widetilde{I}_j \, \widetilde{A}_j^{-1} \, \widetilde{I}_j^t,$$

where $\widetilde{I}_j : \widetilde{V}_j \to \widetilde{V}_h$ $(0 \leq j \leq J)$ is the natural injection operator, and $\widetilde{I}_j^t$ is the transpose of $\widetilde{I}_j$.

Following the arguments in [3], we can obtain an estimate on the condition number of $B_{TL}\widetilde{A}_h$.

Theorem 2 *It holds that*

$$\kappa(B_{TL}\widetilde{A}_h) \leq C_2 \min\Big((H/h)^4, \delta^{-4}\Big), \tag{8}$$

where C_2 is a positive constant independent of H, h, j, δ and $\widetilde{N}_h$.

Remark 2 When the obstacle is present, it is necessary to include the truncation operator in the construction of $\widetilde{V}_0$. Therefore, the condition number estimate (8) for the two-level additive Schwarz preconditioner is different from the one for the plate bending problem without obstacles (cf. [5]) which takes the form

$$\kappa(B_{TL}A_h) \leq C_* \Big(1 + (H/\delta)^4\Big).$$

6 Numerical Results

We consider the obstacle problem (cf. [1]) with $\Omega = (-0.5, 0.5)^2, \beta = 0.1, \psi = 0.01$, and $f = 10(\sin(2\pi(x_1 + 0.5)) + (x_2 + 0.5))$. We discretize the model problem by the C^0 interior penalty method that is based on a rectangular mesh, and choose V_h to be the standard Q_2 finite element space with the mesh size $h = 2^{-\ell}$, where ℓ is the refinement level. The resulting discrete variational inequalities are solved by the PDAS algorithm, in which we choose the constant c to be 10^8. The initial guess for the PDAS algorithm is taken to be the solution at the previous level or zero when $\ell = 1$.

The graphs of the numerical solution y_h and the discrete active set $\mathcal{A}_k$ at refinement level 7 are given in Figure 1.

For comparison, we first calculate the condition number of the un-preconditioned auxiliary system $\widetilde{A}_h$ in each iteration of the PDAS algorithm and then take the average. The average condition numbers and numbers of iterations of the PDAS algorithm for various levels are presented in Table 1.

We apply the one-level and two-level additive Schwarz preconditioners to the auxiliary system in each iteration of the PDAS algorithm. The average condition numbers of both preconditioned auxiliary systems for 4, 16, 64, and 256 subdomains with small overlap, $\delta = h$, are reported in Table 2 and Table 3 respectively. Comparing with the condition numbers of the unpreconditioned auxiliary systems in Table 1, both one-level and two-level algorithms show dramatical improvements.

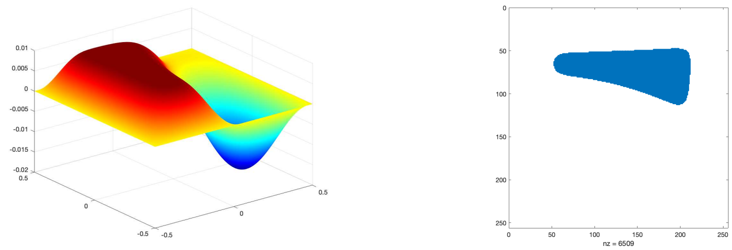

Fig. 1: The numerical solution y_h (left) and the discrete active set $\mathcal{A}_k$ (right) at refinement level 7

	$\kappa(\widetilde{A}_h)$	PDAS Iterations
$\ell = 1$	1.7604×10^1	5
$\ell = 2$	2.2085×10^2	10
$\ell = 3$	4.3057×10^3	5
$\ell = 4$	6.7740×10^4	8
$\ell = 5$	1.0849×10^6	12
$\ell = 6$	1.8038×10^7	15

Table 1: Average condition number of $\widetilde{A}_h$, and number of iterations of the PDAS algorithm

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	5.8672×10^0	—	—	—
$\ell = 3$	1.9350×10^1	5.1410×10^1	—	—
$\ell = 4$	9.9423×10^1	2.4134×10^2	6.6698×10^2	—
$\ell = 5$	6.9235×10^2	1.7965×10^3	3.4752×10^3	1.0282×10^4
$\ell = 6$	5.6185×10^3	1.4676×10^4	2.8898×10^4	5.6312×10^4

Table 2: Average condition number of $B_{OL}\widetilde{A}_h$ with small overlap

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	5.4489×10^0	—	—	—
$\ell = 3$	8.1290×10^0	1.2913×10^1	—	—
$\ell = 4$	3.6660×10^1	1.8647×10^1	3.4614×10^1	—
$\ell = 5$	2.1670×10^2	4.0108×10^1	4.6832×10^1	7.9579×10^1
$\ell = 6$	1.5552×10^3	2.4043×10^2	5.5854×10^1	1.0981×10^2

Table 3: Average condition number of $B_{TL}\widetilde{A}_h$ with small overlap

Moreover, similar simulations for generous overlap $\delta = H$ are also performed. The average condition numbers of the one-level and two level additive Schwarz preconditioned auxiliary systems for various number of subdomains are presented in Tables 4 and 5 .

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	1.0000×10^0	—	—	—
$\ell = 3$	1.0000×10^0	1.1796×10^1	—	—
$\ell = 4$	1.0000×10^0	1.2828×10^1	1.1154×10^2	—
$\ell = 5$	1.0000×10^0	1.3457×10^1	1.1315×10^2	1.5925×10^3
$\ell = 6$	1.0000×10^0	1.4041×10^1	1.1760×10^2	1.6453×10^3

Table 4: Average condition number of $B_{OL}\widetilde{A}_h$ with generous overlap

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	1.2500×10^0	—	—	—
$\ell = 3$	1.2500×10^0	7.8441×10^0	—	—
$\ell = 4$	1.2500×10^0	9.1917×10^0	2.4105×10^1	—
$\ell = 5$	1.2500×10^0	9.9897×10^0	2.5678×10^1	5.8649×10^1
$\ell = 6$	1.2500×10^0	1.0569×10^1	2.6729×10^1	6.3733×10^1

Table 5: Average condition number of $B_{TL}\widetilde{A}_h$ with generous overlap

7 Conclusion

We present additive Schwarz preconditioners for the auxiliary systems that appear in a primal-dual active set algorithm for solving a state constrained elliptic distributed optimal control problem discretized by a C^0 interior penalty method. Both the one-level and two-level preconditioners improve the condition numbers of the auxiliary systems significantly.

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