NONCLASSICAL MINIMIZING SURFACES WITH SMOOTH BOUNDARY

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Abstract

We construct a Riemannian metric g on $\mathbb{R}^4$ (arbitrarily close to the euclidean one) and a smooth simple closed curve $\Gamma \subset \mathbb{R}^4$ such that the unique area minimizing surface spanned by Γ has infinite topology. Furthermore the metric is almost Kähler and the area minimizing surface is calibrated.

1. Introduction

Consider a smooth closed simple curve Γ in $\mathbb{R}^n$. The existence of oriented surfaces which bound Γ and minimize the area can be approached in two different ways. Following the classical work of Douglas and Rado we can fix an abstract connected smooth surface Σ_g of genus g whose boundary $\partial \Sigma_g$ consists of a single connected component and look at smooth maps $\Phi : \Sigma_g \to \mathbb{R}^n$ with the property that the restriction of Φ to $\partial \Sigma_g$ is an homeomorphism onto Γ. We then consider the infimum $A_g(\Gamma)$ over all such Φ and all Riemannian metrics h on Σ of

$$\int_{\Sigma_g} |\nabla \Phi|^2 \, \mathrm{dvol}_h \, .$$

If $A_g(\Gamma) < A_{g-1}(\Gamma)$, then there is a minimizer (Φ, h) and the image of h is an immersed surface of genus g, with possible branch points, see [**17**, **27**, **8**] and also [**25**, **29**]. The second, more intrinsic, approach was pioneered later by De Giorgi, in the codimension 1 case [**9**], and by Federer and Fleming in higher codimension [**19**]. They look at a suitable measure-theoretic generalization of smooth oriented surfaces, called integral currents T, whose generalized boundary is given by $[\![\Gamma]\!]$ and minimize a suitable generalization of the area, called mass. In this framework a minimizer always exist and competitors do not have any topological restriction.

A basic question is whether the Federer-Fleming solution T coincides with the Douglas-Rado solutions for some genus g. This is true if the curve Γ is sufficiently regular ($C^{k,\alpha}$ for $k + \alpha > 2$, because combining De Giorgi's interior regularity theorem [**10**] with Hardt and Simon's

boundary regularity theorem [24], we know that every minimizer is an embedded C^2 surface up to the boundary Γ, in particular it has finite genus g_0. As corollaries, any conformal parametrization Φ of T gives a minimizer in the sense of Douglas and Rado, while $A_g(\Gamma) = A_{g_0}(\Gamma)$ for every $g > g_0$. If we instead merely assume that Γ has finite length, Fleming showed in [20] that it is possible to have $A_{g+1}(\Gamma) < A_g(\Gamma)$ for g arbitrarily large, implying in particular that every integral current minimizer has infinite topology, see also [3] for related phenomena.

In higher codimension, namely for $n \geq 4$, it is known that the minimizer T is in general not regular, neither in the interior nor at the boundary. Concerning the interior regularity, it has been shown by Chang in [7] that T is smooth in $\mathbb{R}^n \setminus \Gamma$ up to a discrete set of singular branch points and self-intersections (we in fact refer to [15, 12, 13, 14] for a complete proof, as Chang needs a suitable modification of the techniques of Almgren's monumental monograph [2] to start his argument, and the former has been given in full details in [13]). As a corollary we know therefore that for any point $p \notin \Gamma$ there is a neighborhood U in which T is the union of finitely many topological disks. Nonetheless it is still an open problem whether "globally" such solutions T have finite topology. So far this can be only concluded if Γ is of class $C^{k,\alpha}$ for $k + \alpha > 2$ and lies in the boundary of a uniformly convex open set, because Allard's boundary regularity theorem [1] rules out boundary singularities.

In the general case, however, very little is known about the boundary regularity of area minimizing integral currents. The first result has been established by the authors and A. Massaccesi in the recent work [11], which shows that, if Γ is of class $C^{k,\alpha}$ for $k + \alpha > 3$, then the set of regular boundary points is open and dense in Γ. On the other hand the same paper gives a smooth simple closed curve Γ in $\mathbb{R}^4$ bounding a (unique) minimizer T which has infinitely many singularities. Such T is, however, still an immersed disk, which has a countable number of self-intersections accumulating towards a boundary branch point: it is, in particular, a Douglas-Rado solution with genus $g = 0$.

In his work [30] White conjectures that the Federer-Fleming solution has finite genus if Γ is real analytic. If White's conjecture were true, then the main theorem in [30] would imply that, for real analytic Γ, the set of boundary and interior singular points is finite and it would also exclude the presence of branch points at the boundary: the (finitely many) singular boundary points would all arise as self intersections.

As already mentioned, the example in [11] shows that the latter conclusion would certainly be false for smooth Γ in $\mathbb{R}^4$. In this note we show that, if we perturb the Euclidean metric in an appropriate way, the same curve bounds a unique area minimizing integral current with

infinite topology. In particular, if we look at White's conjecture in Riemannian manifolds, real analyticity is a necessary assumption to exclude infinite topology of the Federer-Fleming solution. Our precise theorem is the following, where we denote by δ the standard Euclidean metric.

Theorem 1.1. *For every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there is a smooth metric g on $\mathbb{R}^4$, a smooth oriented curve Γ in the unit ball $\mathbf{B}_1$ passing through the origin and a smooth oriented surface Σ in $\mathbf{B}_1 \setminus \{0\}$ such that:*

(a) *$g = \delta$ on $\mathbb{R}^4 \setminus \mathbf{B}_1$ and $\|g - \delta\|_{C^N} < \varepsilon$;*
(b) *$[\![\Sigma]\!]$ is the unique area minimizing integral current in the Riemannian manifold $(\mathbb{R}^4, g)$ which bounds $[\![\Gamma]\!]$;*
(c) *Σ has infinite topology.*

In our example Σ has (only) one singularity at the origin. The latter is a boundary singular point and Σ displays a sequence of interior necks accumulating to it. A simple modification of our proof gives the existence of an area-minimizing current which bounds a smooth curve in a smooth Riemannian manifold and has an infinite number of interior branch points accumulating to the boundary. For the precise statement see Theorem 6.1 below. For the proofs of both Theorem 1.1 and Theorem 6.1 it is essential that we are allowed to perturb the Euclidean metric. In particular the question whether such examples can exist in some Euclidean space remains open.

As pointed out, the question of whether the Federer-Fleming solution coincides with a Douglas-Rado solution is closely related to the regularity theory for area minimizers. We therefore close this introduction with a brief (and certainly not exhaustive) review of what is known for the Douglas-Rado solution. Interior branch points can be excluded in codimension 1, i.e. for surfaces in $\mathbb{R}^3$, see [**26**, **4**, **5**, **23**] and the discussion in [**16**, Section 6.4]. In higher codimension both interior branch points and self intersections are possible (primary examples are holomorphic curves in $\mathbb{C}^k = \mathbb{R}^{2k}$). Concerning boundary branch points, it is well known that they can exist in higher codimension if the boundary curve is just C^k. The example of [**11**] mentioned above shows that they can exist even if it is C^∞, while the aforementioned paper of White [**30**] excludes their existence when Γ is real analytic. In fact the same conclusion was drawn much earlier in codimension 1 by a classical paper of Gulliver and Lesley, [**22**].

In codimension 1 the existence of boundary branch points for the Douglas Rado solution is still an open question and it is probably the most important one in the field, we refer again to the discussion in [**16**, Section 6.4] for a detailed account of the known results. In [**21**] Gulliver provides an interesting example of a C^∞ curve in $\mathbb{R}^3$ which bounds a minimal disk with one boundary branch point, however it is not known

wether this surface is a Douglas-Rado solution. We note in passing that Gulliver's proof gives as well a Douglas-Rado disk-type solution (in fact a Federer Fleming solution) in $\mathbb{R}^6$ spanning a C^∞ curve and with a boundary branch point.

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2. Preliminaries

The Riemannian manifold $(\mathbb{R}^4, g)$ of Theorem 1.1 has in fact a very special geometric structure, since it is an almost Kähler manifold.

Definition 2.1. An almost complex structure on a 4-dimensional smooth manifold M is given by a smooth $(1,1)$ tensor J with the property that $J^2 = -\operatorname{Id}$. The structure is almost Kähler if there is a smooth Riemannian metric g with the properties that:

(i) J is isometric, namely $g(JV, JW) = g(V, W)$ for every vector fields V and W;

(ii) The 2-form defined by $\omega(V, W) := -g(V, JW)$ is closed.

ω will be called the almost Kähler form associated to the almost Kähler structure.

Theorem 1.1 will then be a corollary of the following

Theorem 2.2. *For every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there is a smooth metric g on $\mathbb{R}^4$, a smooth oriented curve Γ in the unit ball $\mathbf{B}_1$ passing through the origin and a smooth oriented surface Σ in $\mathbf{B}_1 \setminus \{0\}$ such that:*

(a) *$g = \delta$ on $\mathbb{R}^4 \setminus \mathbf{B}_1$ and $\|g - \delta\|_{C^N} < \varepsilon$;*

(b1) *there is an almost complex structure J for which the conditions (i) and (ii) of Definition 2.1 hold;*

(b2) *$[\![\Sigma]\!]$ bounds $[\![\Gamma]\!]$ and the pull-back of the corresponding ω on Σ is the volume form with respect to the metric g;*

(c) *Σ has infinite topology.*

Property (b2) is usually referred to as ω calibrating the surface Σ. It is a classical elementary, yet powerful, remark of Federer that the conditions (b1)-(b2) imply, by an inequality of Wirtinger, the minimality of the current $[\![\Sigma]\!]$, cf. [**18**]. Wirtinger's theorem shows that

$$\omega(V, W) \leq 1$$

whenever

$$|V \wedge W|_g := \sqrt{g(V, V)g(W, W) - g(V, W)^2} \leq 1$$

and that the equality holds if and only if $W = JV$. In the language of geometric measure theory Wirtinger's inequality implies that the comass (relative to the metric g) of the form ω is 1. Moreover, we infer from the second part of Wirtinger's Theorem (the characterization of the equality case) that ω is pulled back to the standard volume form on Σ if and only if there is a positively oriented tangent frame of the tangent bundle to Σ of the form $\{V, JV\}$. Consider now any current (not necessarily integral!) T which bounds $[\![\Gamma]\!]$. Since ω is closed and $\mathbb{R}^4$ has trivial topology, ω has a primitive α. We then must have

$$T(\omega) = T(d\alpha) = \int_\Gamma \alpha = \int_\Sigma d\alpha = \int_\Sigma \omega = \mathbf{M}([\![\Sigma]\!]) \,.$$

On the other hand Wirtinger's inequality implies that the comass of ω in the metric g is 1 and thus the mass of T is necessarily larger than $T(\omega)$.

This shows that $[\![\Sigma]\!]$ is area minimizing. In order to conclude that it is the unique minimizer, we must appeal to the boundary regularity theory developed in [11]. First of all observe that, by [11, Theorem 2.1] the interior regular set $\Lambda := \mathrm{Reg}_i(T)$ of the current T is connected, it is an orientable submanifold of $\mathbb{R}^4$ and (up to a change of orientation) $T = [\![\Lambda]\!]$. Moreover, by [11, Theorem 1.6] there is at least one point $p \in \Gamma \setminus \{0\}$ and a neighborhood U of p such that $\Lambda \cap U$ is a smooth oriented surface with smooth oriented boundary $\Gamma \cap U$. By the argument above we must have $T(\omega) = \mathbf{M}(T)$ and this implies, by Wirtinger's Theorem, that the tangent planes to Λ are invariant under the action of J. The same holds for the tangent planes to Σ. In particular, the tangents to Σ and Λ must coincide at every point $q \in \Gamma \cap U$ and they must have the same orientation. Since both are smooth minimal surfaces in U, the unique continuation for elliptic systems implies that they coincide in a neighborhood of q. Again, thanks to the unique continuation principle and the connectedness of Λ we conclude that Λ is in fact a subset of Σ. However, since they have the same area, this implies that $[\![\Sigma]\!] = T$.

3. Proof of Theorem 2.2: Part I

In this section we slightly modify the construction given in [11, Section 2.3] to achieve a smooth curve Γ in $\mathbb{R}^4$ and an integral current T in $\mathbb{R}^4$ such that

(i) T bounds $[\![\Gamma]\!]$ and is area minimizing in $(\mathbb{R}^4, \delta)$ (i.e. with respect to the Euclidean metric), in fact T is induced by an holomorphic subvariety in $\mathbb{R}^4 \setminus \Gamma$;

(ii) T is regular at $\Gamma \setminus \{0\}$;

(iii) 0 is an accumulation point for the interior singular set of T, denoted by $\mathrm{Sing}_i(T)$;

(iv) At each $p \in \mathrm{Sing}_i(T)$ there is a neighborhood U such that T in U consists of two holomorphic curves intersecting transversally at p.

First of all consider the complex plane with an infinite slit

$$\mathbb{K} := \{z \in \mathbb{C}\} \setminus \{z \in \mathbb{R} : z \leq 0\}.$$

We consider the usual inverse arctan : $\mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ on the real axis of the trigonometric function tan and we fix a determination of the complex logarithm on $\mathbb{K}$ which coincides with

$$\mathrm{Log}\, z = \log|z| + i \arctan \frac{\mathrm{Im}\, z}{\mathrm{Re}\, z}.$$

on the open half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. Correspondingly we define the functions $z^{-\alpha} = \exp(-\alpha \mathrm{Log}\, z)$ for $\alpha \in (0, 1/2)$ and

$$f_k(z) = \exp(-z^{-\alpha}) \sin \left(\mathrm{Log}\, z + \frac{3 - 2k}{6} \pi i \right) \qquad \text{for } k = 0, 1, 2, 3.$$

Observe that:

(i) If we extend each f_k to the origin as 0, then f_k is a smooth function over any wedge

$$\mathbb{K}_a := \{z : -\mathrm{Re}\, z \leq a|\mathrm{Im}\, z|\}$$

with a positive.

(ii) Since $\exp(-z^{-\alpha})$ does not vanish on $\overline{\mathbb{H}} \setminus \{0\}$ (recall that $\alpha \in (0, 1/2)$), the zero set Z_k of f_k in $\overline{\mathbb{H}} \setminus \{0\}$ is given by

$$Z_k = \left\{ z \in \overline{\mathbb{H}} : \mathrm{Log}\, z + \frac{3 - 2k}{6} \pi i \in \pi \mathbb{Z} \right\},$$

namely by

$$(3.1) \qquad Z_k = \left\{ \exp \left(n\pi + i \frac{2k - 3}{6} \pi \right) : n \in \mathbb{Z} \right\}.$$

Consider next the function

$$g(z) = \prod_{k=0}^{3} f_k(z).$$

We then conclude that g is holomorphic on $\mathbb{K}$, it is C^∞ on $\mathbb{K}_a$ for every $a > 0$ and its zero set in $\overline{\mathbb{H}}$, which we denote by Z, is given by

$$Z = \{0\} \cup \bigcup_{k=0}^{3} Z_k.$$

Define now the map $G : \mathbb{K} \to \mathbb{C}^2$ by $G(z) = (z^3, g(z))$. We consider a smooth simple curve $\gamma \subset \mathbb{K}_1$ which in a neighborhood of the origin is tangent to the imaginary axis and we let $D \subset \mathbb{K}_1$ be the open disk bounded by γ. Following the arguments of [11, Section 2.3] it is not difficult to see that γ can be chosen so that:

(A) $\{g = 0\} \cap \gamma = \{0\}$;

(B) $\{g = 0\} \cap D \subset \overline{\mathbb{H}}$, hence $\{g = 0\} \cap D \subset Z$ and, for each $k \in \{0, \ldots, 3\}$, it contains all sufficiently small elements of Z_k, namely there is a positive constant c_0 such that $\{z \in Z_k : |z| \leq c_0\} \subset D$.

The current $T := G_\sharp \llbracket D \rrbracket$ is integer rectifiable, it has multiplicity one (in particular it coincides with $\llbracket G(D) \rrbracket$) and

$$\partial T = G_\sharp \partial \llbracket D \rrbracket = G_\sharp \llbracket \gamma \rrbracket \ .$$

Observe that $G(D)$ is an holomorphic curve of $\mathbb{C}^2$, which carries a natural orientation. If $\llbracket G(D) \rrbracket$ denotes the corresponding integer rectifiable current, we then can follow the argument in [**11**, Section 2.3] to show that $T = \llbracket G(D) \rrbracket$ and Federer's classical argument implies that T is area minimizing for the standard Euclidean metric.

The arguments given in [**11**, Section 2.3] show that $G_\sharp \llbracket \gamma \rrbracket = \llbracket G(\gamma) \rrbracket$ and $G(\gamma) \subset \mathbb{C}^2 = \mathbb{R}^4$ is a smooth embedded curve. The same arguments also show that $G(D)$ is a smooth immersed surface, that it is embedded outside the discrete set $G(Z)$ and that at each point $q \in G(Z \cap D)$ it consists of two holomorphic graphs intersecting transversally.

4. Proof of Theorem 2.2: Part II

In order to conclude the proof of Theorem 1.1 the idea is to modify the example of the previous section and substitute the self-intersection of each singular point $q \in G(Z \cap D)$ with a neck. In order for the new surface to be area minimizing we will then perturb the Euclidean metric and the standard complex structure to a nearby metric and a nearby almost Kähler structure. More precisely, order the points $\{p_k\}_{k \in \mathbb{N}}$ of the discrete set $G(Z \cap D)$. Fix sufficiently small balls $\mathbf{B}_{100r_k}(p_k)$ so that they are all disjoint and do not intersect the boundary curve Γ. Recall that $G(D) \cap \mathbf{B}_{100r_k}(p_k)$ consist of two holomorphic disks intersecting transversally at p_k. In particular, we can assume that the two tangents to these disks are given by π_1 and π_2, where π_1 and π_2 are two distinct affine complex planes, namely

$$(4.1) \qquad \pi_1 = p_k + \{(z, w) : a_1 z + b_1 w = 0\}$$

$$(4.2) \qquad \pi_2 = p_k + \{(z, w) : a_2 z + b_2 w = 0\}$$

for two different points $[a_1, b_1], [a_2, b_2] \in \mathbb{CP}^1$. The idea is to choose a sufficiently small $\eta_k > 0$, substitute the surface $G(D)$ inside $\mathbf{B}_{r_k}(p_k)$ with the holomorphic subvariety

$$\Lambda_k := \{p_k + (z, w) : (a_1 z + b_1 w)(a_2 z + b_2 w) = \eta_k\} \, ,$$

and glue it back to the original surface $G(D)$ in the annulus

$$\mathbf{B}_{100r_k}(p_k) \setminus \overline{\mathbf{B}}_{r_k}(p_k) \, .$$

If η_k and r_k are sufficiently small, we can assume that

$$\Lambda_k \cap \mathbf{B}_{100r_k}(p_k) \setminus \overline{\mathbf{B}}_{r_k}(p_k) \quad \text{and} G(D) \cap \mathbf{B}_{100r_k}(p_k) \setminus \overline{\mathbf{B}}_{r_k}(p_k)$$

consist each of two annuli, respectively Λ_k^1, Λ_k^2 and Σ_k^1, Σ_k^2, where Λ_k^i is close to Σ_k^i. Moreover, again by assuming that η_k and r_k are sufficiently small, each Λ_k^i and Σ_k^i are graphs of holomorphic functions over the plane $p_k + \pi_i$. We now wish to glue the surfaces Σ_k^1 and Λ_k^1 and Σ_k^2 and Λ_k^2 and modify the Euclidean metric and the standard Kähler structure in the annulus $\mathbf{B}_{100r_k}(p_k) \setminus \overline{\mathbf{B}}_{r_k}(p_k)$ to a nearby Riemannian metric with a corresponding almost Kähler structure, so that the glued surface is calibrated by the associated almost Kähler form. Both the new metric and the corresponding almost Kähler form will coincide with the Euclidean metric and the standard Kähler form outside of a neighborhood of the glued surface. By assuming r_k and η_k very small, we can reduce to perform such glueing in neighborhoods of the planar annuli $(p_k + \pi_1) \cap \mathbf{B}_{100r_k}(p_k) \setminus \overline{\mathbf{B}}_{r_k}(p_k)$ and $(p_k + \pi_2) \cap \mathbf{B}_{100r_k}(p_k) \setminus \overline{\mathbf{B}}_{r_k}(p_k)$, which are disjoint. In particular we can assume that we glue the two pairs of surfaces and we modify the metric and the Kähler form in two separate regions. A schematic picture summarizing our discussion is given in the picture below.
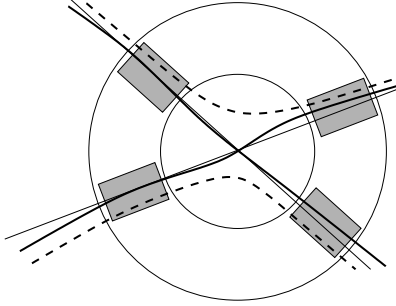

Figure 1. A schematic picture of the procedure outlined above. The figure contains cross sections of the corresponding objects with the real affine plane $p_k + \mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}$. In particular, $G(D)$ is pictured by the thick continuous curves, which in p_k are tangent to the union of two crossing complex lines $p_k + \pi_1$ and $p_k + \pi_2$. The dashed lines represent the hyperbola Λ_k. The surface Σ will coincide with the dashed lines in the inner ball, with the thick lines outside the outer ball and with a smooth interpolation between the two surfaces in the annular region. The interpolation will take place in the shadowed region, where both Λ_k and $G(D)$ are graphical over the corresponding portion of $p_k + \pi_i$.

The corresponding metric g_k will coincide with the euclidean one outside of the annulus and will have the property that, if we set $\bar{k} := \max\{k, N\}$, then

$$(4.3) \qquad \|g_k - \delta\|_{C^{\bar{k}}} < \varepsilon 2^{-\bar{k}-1} \,,$$

where ε is the constant of Theorem 2.2. The latter estimate will be achieved by choosing η_k appropriately small, so that the graphs Λ_k^i almost coincide with the graphs Σ_k^i.

The surface Σ and the metric g of Theorem 2.2 will then be defined as follows:

- Outside of $\bigcup_k \mathbf{B}_{100r_k}(p_k)$ Σ coincides with $G(D)$ and the metric g is the Euclidean metric.
- Inside each $\overline{\mathbf{B}_{r_k}(p_k)}$ Σ coincides with the holomorphic submanifold Λ_k and the metric g is the Euclidean metric.
- In the annulus $\mathbf{B}_{100r_k}(p_k) \setminus \overline{\mathbf{B}_{r_k}(p_k)}$ Σ is the glued surface and g is the metric g_k described above.

The existence of the (local) glued surface and of the metric g_k is thus the key point and is guaranteed by the glueing proposition below (after appropriate rescaling). In the rest of the note we use the following notation:

- $D_r(p) \subset \mathbb{C}$ is the disk centered at $p \in \mathbb{C}$ of radius r; p will be omitted if it is the origin.
- ω_0 is the Kähler form on $\mathbb{R}^4 = \mathbb{C}^2$ and δ is the Euclidean metric on $\mathbb{R}^4$.
- J_0 is the standard complex structure on $\mathbb{R}^4$, namely $J_0(a, b, c, d) = (-b, a, -d, c)$.
- Norms on functions, tensors, etc. are computed with respect to the Euclidean metric.

Proposition 4.4 (Glueing). *For every $\eta > 0, N \in \mathbb{N}$ there is $\varepsilon > 0$ with the following property. Assume that $f, h : D_{10} \setminus \overline{D}_1 \to \mathbb{C}$ are two holomorphic maps with*

$$\|f\|_{C^{N+2}} + \|h\|_{C^{N+2}} \leq \varepsilon \,.$$

Then there are

  (i) *a metric $g \in C^\infty$ with $\|g - \delta\|_{C^N} \leq \eta$ and $g = \delta$ outside $(D_9 \setminus \overline{D}_1) \times D_{2\eta}$,*

  (ii) *an almost Kähler structure J compatible with g such that $\|J - J_0\|_{C^N} \leq \eta$ and $J = J_0$ outside $(D_9 \setminus \overline{D}_1) \times D_{2\eta}$,*

  (iii) *an associated almost Kähler form ω with $\|\omega - \omega_0\|_{C^N} \leq \eta$ and $\omega = \omega_0$ outside $(D_9 \setminus \overline{D}_1) \times D_{2\eta}$*

  (iii) *and a function $\zeta : D_{10} \setminus \overline{D}_1 \to D_\eta$*

such that

  (a) *$\zeta = (\operatorname{Re} f, \operatorname{Im} f)$ on $D_2 \setminus \overline{D_1}$ and $\zeta = (\operatorname{Re} h, \operatorname{Im} h)$ on $D_{10} \setminus \overline{D_9}$;*

(b) ω *calibrates the graph of* ζ.

5. Proof of the glueing proposition

Before coming to the proof, let us recall some known facts from symplectic geometry. First of all, a 2-form α on $\mathbb{R}^{2n}$ is called nondegenerate if for every point $p \in \mathbb{R}^{2n}$ the corresponding skew-symmetric bilinear map $\alpha_p : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is nondegenerate, namely

$$(5.1) \qquad \forall u \in \mathbb{R}^{2n} \neq 0 \; \exists v \in \mathbb{R}^{2n} \text{ with } \alpha_p(u,v) \neq 0.$$

Given a skew-symmetric form we can define $A_p : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as

$$(5.2) \qquad \omega_p(v,w) = -\delta(v, A_p w)$$

where we recall that δ is the Euclidean metric. The nondegeneracy condition (5.1) is equivalent to $\ker A_p = \{0\}$. Note that if ω_0 is the standard Kähler form of $\mathbb{R}^{2n}$, then $A_p = J_0$ and that

$$\|\alpha - \omega_0\|_{C^N} \leq \eta \Rightarrow \|A_p - J_0\|_{C^N} \leq C\eta$$

In particular, any 2-form which is sufficiently close to ω in the C^0 norm is necessarily nondegenerate.

We start with the following particular version of the Poincaré Lemma. Since we have not been able to find a precise reference, we give the explicit argument.

Lemma 5.3. *Assume $U \subset \mathbb{R}^4$ is a star-shaped domain with respect to the origin and let β be a closed 2-form, with the property that the pull back of β on $\{X_3 = X_4 = 0\}$ vanishes. Then there is a primitive α with the properties that*

- *α vanishes identically on $\{X_3 = X_4 = 0\}$;*
- *$\|\alpha\|_{C^N} \leq C\|\beta\|_{C^{N+1}}$, where the constant C depends only on the diameter of U.*

Proof. First of all recall the standard formula for the primitive of a form given by integration along rays (cf. [**28**, Theorem 4.1]). Namely, if

$$\bar{\beta} = \sum_{i<j} \bar{\beta}_{ij} \, dX_i \wedge dX_j \,,$$

then a primitive $\bar{\alpha}$ can be computed using the formula

$$(5.4) \qquad \bar{\alpha}(X) = \sum_i \sum_j (-1)^{j-1} \int_0^1 t\bar{\beta}_{ij}(tX) \, dt \, X_j dX_i$$

with the convention that $\bar{\beta}_{ij} = -\bar{\beta}_{ji}$ if $i > j$. Using the latter expression we obviously have $\|\bar{\alpha}\|_{C^N} \leq C\|\bar{\beta}\|_{C^N}$. Moreover, if $\bar{\beta}$ vanishes identically on $\{X_3 = X_4 = 0\}$ then clearly $\bar{\alpha}$ vanishes identically on $\{X_3 = X_4 = 0\}$.

Given a general closed 2-form β, we then look for a 1-form ϑ which vanishes on $\{X_3 = X_4 = 0\}$ and with the property that $\bar{\beta} := \beta - d\vartheta$ vanishes on $\{X_3 = X_4 = 0\}$. The resulting α will then be found as $\bar{\alpha} + \vartheta$, where $\bar{\alpha}$ is the primitive of $\bar{\beta}$ given in the formula (5.4). In order to find ϑ we first write β in the form

$$\beta = f\, dX_1 \wedge dX_2 + \underbrace{(a_1 dX_1 + a_2\, dX_2)}_{=:\lambda} \wedge dX_3$$
$$+ \underbrace{(b_1\, dX_1 + b_2\, dX_2 + b_3\, dX_3)}_{=:\mu} \wedge dX_4\,.$$

By assumption f equals 0 on $\{X_3 = X_4 = 0\}$. Let us set

$$\vartheta = -X_3\mu - X_4\lambda\,,$$

so that

$$\beta - d\vartheta = f\, dX_1 \wedge dX_2 + X_3 d\mu + X_4 d\lambda\,.$$

Since f vanishes on $\{X_3 = X_4 = 0\}$ we then get the desired property that $\beta - d\vartheta$ vanishes on it as well.   q.e.d.

Proof of the Glueing Proposition. We will focus on the construction of the triple, whereas the estimates are a simple consequence of the algorithm.

Step 1: Definition of ζ and a new system of coordinates: First we smoothly extend f inside D_1 and we then define ζ as

$$\zeta = (\operatorname{Re} f, \operatorname{Im} f)\varphi + (\operatorname{Re} h, \operatorname{Im} h)(1 - \varphi)\,.$$

where $\varphi \in C_c^\infty(D_5)$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on D_4. In particular

$$\zeta = f \qquad \text{on } D_4 \qquad \text{and} \qquad \zeta = h \qquad \text{outside } D_5.$$

We now choose a system of coordinates $X := (X_1, \ldots, X_4)$ such that $\|X - \operatorname{Id}\|_{C^{N+1}} \le C\varepsilon$,

(5.5) $\Sigma = \operatorname{graph}(\zeta) = \{X_3 = X_4 = 0\}$

and

(5.6) $T_p\Sigma = \operatorname{Ker} dX_3 \cap \operatorname{Ker} dX_4 \qquad T_p\Sigma^\perp = \operatorname{Ker} dX_1 \cap \operatorname{Ker} dX_2$

Note that this can be done by, for instance, taking normal coordinates around Σ, provided ε is chosen sufficiently small.

More precisely, we first choose two vector fields ξ, τ along Σ such that:

- $|\xi_p| = |\tau_p| = 1$ and $\xi_p \perp \tau_p$ (in the euclidean metric);
- $T_p\Sigma^\perp = \operatorname{span}(\xi_p, \tau_p)$.

We set

$$Y(x_1, x_2, x_3, x_4)$$
$$= (x_1, x_2, \zeta_1(x_1, x_2), \zeta_2(x_1, x_2)) + x_3\xi(x_{,1}, x_2) + x_4\tau(x_1, x_2)\,,$$

where

$$\zeta = (\zeta_1, \zeta_2) \,,$$
$$p = (x_1, x_2, \zeta_1(x_1, x_2), \zeta_2(x_1, x_2)) \in \Sigma$$
$$\xi(x_{,1}, x_2) = \xi_p$$
$$\tau(x_1, x_2) = \tau_p \,.$$

In order to get vector fields ζ and τ whose derivatives are under control, a standard procedure is to take the standard vector fields $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$, project them orthogonally onto $T_p^\perp \Sigma$ and apply the Gram-Schmidt orthogonalization procedure to them. Simple computations give that

$$\|\xi - e_3\|_{C^{N+1}} + \|\tau - e_4\|_{C^{N+1}} \leq C\|\zeta\|_{C^{N+2}} \,.$$

Note in particular that, if ε is chosen sufficiently small, Y is a diffeomorphism onto its image and that the latter contains $D_8 \times D_8$. Letting $X = Y^{-1}$ it is immediate to check that (5.5) is satisfied and thus also the first equality in (5.6). To check the second one simply note, by the very definition of X,

$$\{X_1 = c_1, X_2 = c_2\} = p + T_p\Sigma^\perp \qquad X(p) = (c_1, c_2, 0, 0).$$

From now on, with a slight abuse of notation, we will denote by $D_r \times D_s$ the product of disks in the X system of coordinates, that is

$$D_r \times D_s = \{X_1^2 + X_2^2 < r^2 \,, X_3^2 + X_4^2 < s^2\}$$

and we will work in the domain $D_8 \times D_8$. Given that $\|DX - \mathrm{Id}\|_{C^0} + \|DY - \mathrm{Id}\|_{C^0} \leq \varepsilon$ and assuming, without loss of generality, that X and Y keep the origin fixed, such sets are comparable to the corresponding products $D_\rho^e \times D_\sigma^e$ in the euclidean system of coordinates, namely

$$D_{C^{-1}r}^e \times D_{C^{-1}s}^e \subset D_r \times D_s \subset D_{Cr}^e \times D_{Cs}^e$$

where the constant C approaches 1 as $\varepsilon \to 0$.

Step 2: Construction of the 2 form: We take $\sigma \ll \eta$ and, provided $\varepsilon \ll \sigma$, we claim the existence of a 2-form ω on $D_8 \times D_8$ such that

(a) ω is closed (and hence exact);
(b) The pull back of ω and ω_0 are the same on Σ.
(c) For all $p \in \Sigma \cap \big((D_7 \setminus \overline{D_3}) \times D_8 \big)$

(5.7) $\omega_p = \omega_p(v, w) = 0 \qquad$ for all $v \in T_p\Sigma$ and all $w \in T_p\Sigma^\perp$

(d) $\omega = \omega_0$ outside of $D_7 \setminus \overline{D_3} \times D_{2\sigma}$
(e) $\|\omega - \omega_0\|_{C^N} \leq \eta$.

To construct the form we observe that, on Σ,

$$i_\Sigma^\sharp \omega_0 = a(X_1, X_2)dX_1 \wedge dX_2$$

for a suitable smooth function a. Extending a constant in the X_3, X_4 coordinates we can write

$$\omega_0 = a(X_1, X_2)dX_1 \wedge dX_2 + \overline{\omega}$$

where $\bar{\omega}$ is pulled back to 0 on Σ. Note that $d\bar{\omega} = 0$ since ω is closed. Moreover

(5.8) $\qquad \|a - 1\|_{C^{N+1}(D_8)} + \|\overline{\omega} - dX_3 \wedge dX_4\|_{C^{N+1}(D_8 \times D_8)} \leq o_\varepsilon(1)$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. We define

(5.9) $\qquad\qquad \beta = a(X_1, X_2)dX_1 \wedge dX_2 + dX_3 \wedge dX_4.$

We now apply Lemma 5.3 to find a primitive ϑ of $\omega_0 - \beta = \overline{\omega} - dX_3 \wedge dX_4$ which equals 0 on Σ. We also let φ be a smooth cut-off function such that

$$\begin{cases} \varphi \equiv 0 & \text{on } (D_6 \setminus D_4) \times D_\sigma. \\ \varphi \equiv 1 & \text{outside } (D_7 \setminus \overline{D}_3) \times D_{2\sigma} \end{cases}$$

Note in particular that, provided $\varepsilon \ll \sigma$,

(5.10) $\qquad\qquad \Sigma \cap \{\varphi \neq 0\} \subset \operatorname{graph} f \cup \operatorname{graph} h\,.$

We define

$$\omega = \beta + d(\varphi\vartheta) = \beta + \varphi(\omega_0 - \beta) + d\varphi \wedge \vartheta.$$

Clearly ω satisfies (a) and (d). Property (e) follows by choosing $\varepsilon \ll \sigma$ as a consequence of the construction of Lemma 5.3 and of (5.8). Moreover since ϑ vanishes on Σ and the pull-backs of β and ω_0 on Σ are the same, also (b) is satisfied. To check (c) we note that due to (5.6) and the definition of β we have that $\beta_p(v, w) = 0$ for all p in the domain of X and $v \in T_p\Sigma, w \in T_p\Sigma^\perp$. In particular (5.7) is satisfied on $\{\varphi = 0\}$. Since f, h are holomorphic outside D_2, by (5.10), for $p \in \{\varphi \neq 0\} \cap (D_8 \setminus \overline{D}_2) \times D_8$ the spaces $T_p\Sigma$ and $T_p\Sigma^\perp$ are perpendicular complex lines. Hence ω_0 satisfies (5.7) there, since

$$\omega\big|_\Sigma = (1 - \varphi)\beta + \varphi\omega_0,$$

ω satisfies (5.7) as well and (c) is verified.

Step 3: Definition of the almost complex structure and of the metric: To conclude the proof it will be enough to construct a metric g and a compatible almost complex structure J. Here we follow a method used in [**6**]. Let A_p be the skew-symmetric matrix defined in (5.2). In particular $Q_p = -A_p^2 = A_p A_p^t$ defines a positive definite quadratic form and thus it admits a (positive definite) square root. We set

$$g_p = \big(-A_p^2\big)^{-\frac{1}{2}} \qquad \text{and} \qquad J_p = g_p^{-1}A_p.$$

Note that g_p and J_p equal, respectively, the euclidean metric δ and the usual complex structure J_0 where $\omega = \omega_0$. Furthermore, since g_p commutes with A_p, one immediately verifies that

$$J_p^2 = -\operatorname{Id} \qquad g_p(J_p v, J_p w) = g_p(v, w) \qquad \omega_p(v, w) = -g_p(v, J_p w)$$

so that the triple (g, J, ω) defines an almost Kähler structure on $D_8 \times D_8$ which coincides with the canonical one where $\omega = \omega_0$. We are thus left to prove that ω calibrates Σ on $(D_{10} \setminus \overline{D_1}) \times D_{10}$. This is clear in the region where $\omega = \omega_0$, because in that region Σ equals either the graph of f or that of h and these are holomorphic outside D_1. Hence it is enough to verify that Σ is calibrated in $\big((D_7 \setminus \overline{D_3}) \times D_8\big)$. To this end note that, if $p \in \Sigma \cap \big((D_7 \setminus \overline{D_3}) \times D_8\big)$, by (5.7) and the definition of A_p,

$$0 = \omega_p(v, w) = -\delta(v, A_p w) = \delta(A_p v, w) \qquad \text{for all } v \in T_p\Sigma \ , \ w \in T_p\Sigma^\perp.$$

In particular A_p maps $T_p\Sigma$ into itself (and $T_p\Sigma^\perp$ into itself as well). The same is true then for J_p and thus, if $g_p(v, v) = 1$, $(v, J_p v)$ is a g_p-orthonormal frame of $T_p\Sigma$. This implies that ω is pulled back on Σ to the g-volume form and concludes the proof.   q.e.d.

6. Branching singularities

A simple modifications of the ideas outlined above proves the following

Theorem 6.1. *For every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there is a smooth metric g on $\mathbb{R}^4$, a smooth oriented curve Γ in the unit ball $\mathbf{B}_1$ passing through the origin and a smooth oriented surface Σ in $\mathbf{B}_1 \setminus \{0\}$ such that:*

(a) *$g = \delta$ on $\mathbb{R}^4 \setminus \mathbf{B}_1$ and $\|g - \delta\|_{C^N} < \varepsilon$;*

(b) *$[\![\Sigma]\!]$ is the unique area minimizing integral current in the Riemannian manifold $(\mathbb{R}^4, g)$ which bounds $[\![\Gamma]\!]$;*

(c') *There is an finite number of branching singularities $p_k \in \Sigma \setminus \Gamma$ accumulating to the only boundary singular point 0.*

The idea of the proof is to produce the analogous to Theorem 2.2 where the conclusion (c) therein is substituted by the conclusion (c') above. Here we sketch the necessary modifications to the arguments given for Theorem 2.2.

We start by constructing an example of an holomorphic subvariety inducing an area minimizing current T as in Section 3 where the property (iv) is however replaced by

(v) *At each $p \in \operatorname{Sing}_i(T)$ there is a neighborhood U such that T in U consists of *four* holomorphic curves intersecting transversally at p.*

More precisely there are four distinct elements

$$[a_1, b_1], [a_2, b_2], [a_3, b_3], [a_4, b_4] \in \mathbb{CP}^1$$

such that the tangent cone to T at p is given by the union of four corresponding complex lines:

$$\left\{ (z,w) : \prod_{i=1}^{4}(a_i z + b_i w) = 0 \right\}.$$

In order to achieve such object we construct a similar function g as in Section 3, by defining

$$f_k(z) = \exp(-z^\alpha) \sin\left(\operatorname{Log} z + \frac{7-2k}{14}\pi i \right) \qquad \text{for } k = 0, 1 \ldots, 7$$

and

$$g(z) = \prod_{k=0}^{7} f_k(z).$$

We then proceed as in Section 3 to define the zero sets Z_k of f_k on $\overline{\mathbb{H}} \backslash \{0\}$, the set $Z = \{0\} \cup \bigcup_k Z_k$, the curve γ and the corresponding disk D, where we require the properties analogous to (A) and (B) therein. We finally define the map $G(z) := (z^7, g(z))$ and the current T is thus given by $[\![G(D)]\!]$.

Next, proceeding as in Section 4, in a sufficiently small ball of radius r_k centered at $p_k \in \operatorname{Sing}_i(T)$ we wish to replace $G(D)$ with another holomorphic subvariety, which has a branching singularity at p_k. Since $G(D)$ is, at small scale, very close to the cone

$$C_k := \bigcup_{i=1}^{4} \underbrace{\{p_k + (z,w) : (a_i z + b_i w) = 0\}}_{=:\pi_{k,i}},$$

the idea is to choose

$$\Lambda_k := \left\{ p_k + (z,w) : \prod_{i=1}^{4}(a_i z + b_i w) = \eta_k(z^3 - w^2) \right\},$$

where η_k is again a very small parameter. Choosing r_k and η_k sufficiently small, we can ensure that $G(D) \cap \mathbf{B}_{100 r_k} \setminus \overline{\mathbf{B}}_{r_k}(p_k)$ and $\Lambda_k \cap \mathbf{B}_{100 r_k} \setminus \overline{\mathbf{B}}_{r_k}(p_k)$ consist each of four annuli which are graphs over corresponding annular regions of the four distinct complex lines $\pi_{k,i}$, $i = 1, \ldots, 4$. We can obviously engineer such graphs to be arbitrarily close to the corresponding planes, and hence to fall, after appropriating rescaling under the assumption of the glueing Proposition 4.4. Hence the construction of Σ and of the almost Kähler structure (g, J, ω) follows the same arguments.

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