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The Coulomb Gauge in Non-associative Gauge Theory

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Abstract

The aim of this paper is to extend existence results for the Coulomb gauge from standard gauge theory to a non-associative setting. Non-associative gauge theory is based on smooth loops, which are the non-associative analogs of Lie groups. The main components of the theory include a finite-dimensional smooth loop $\mathbb{L}$, its tangent algebra $\mathfrak{l}$, a finite-dimensional Lie group Ψ, that is the pseudoautomorphism group of $\mathbb{L}$, a smooth manifold M with a principal Ψ-bundle $\mathcal{P}$, and associated bundles $\mathcal{Q}$ and $\mathcal{A}$ with fibers $\mathbb{L}$ and $\mathfrak{l}$, respectively. A configuration in this theory is defined as a pair (s, ω), where s is a section of $\mathbb{Q}$ and ω is a connection on $\mathcal{P}$. The torsion $T^{(s,\omega)}$ is the key object in the theory, with a role similar to that of a connection in standard gauge theory. The original motivation for this study comes from G_2-geometry, and the questions of existence of G_2-structures with particular torsion types. In particular, given a fixed connection, we prove existence of configurations with divergence-free torsion, given a sufficiently small torsion in a Sobolev norm.

Keywords Gauge theory · Connections · Non-associativity · Coulomb gauge

Mathematics Subject Classification 53C07 · 57R57 · 53C10 · 20N05

1 Introduction

The main goal of this work is to extend results on existence of Coulomb gauge transformations from standard gauge theory to a *non-associative* setting. One of highly successful areas at the intersection of differential geometry, analysis, and mathematical physics is gauge theory, which, as it is well-known, is the study of connections on bundles with particular Lie groups as the structure groups. In [26], the author initiated a theory of smooth loops, which are non-associative analogs of Lie groups, and began the development of gauge theory based on loops, i.e. a non-associative gauge theory.

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The key example of a non-associative smooth loop is the loop of unit octonions. A non-associative gauge theory has the following components:

1. A finite-dimensional smooth (right) loop $\mathbb{L}$, i.e. a smooth manifold with a right multiplication diffeomorphism R_p defined for every $p \in \mathbb{L}$ and a distinguished identity element $1 \in \mathbb{L}$, with tangent algebra $\mathfrak{l}$ at identity.
2. A finite-dimensional Lie group Ψ that is the *pseudoautomorphism* group of $\mathbb{L}$, a non-associative generalization of an automorphism group.
3. A smooth manifold M with a principal Ψ-bundle $\mathcal{P}$, and associated bundles $\mathcal{Q}$ and $\mathcal{A}$ with fibers $\mathbb{L}$ and $\mathfrak{l}$, respectively. We also define an associated bundle $\mathcal{Q}'$ with fibers diffeomorphic to $\mathbb{L}$, but with a different action of Ψ (defined in detail in Sect. 2).
4. A configuration is defined by a pair (s, ω) , where s is a section of $\mathcal{Q}$ and ω is a connection on $\mathcal{P}$. Together they define the *torsion* $T^{(s,\omega)}$, which is an $\mathcal{A}$-valued 1-form on M. The torsion is then the key object in the theory, in the same way that a connection is the key object in standard gauge theory.
5. In addition to standard gauge transformations of ω by Ψ, we now also have transformations of s induced by loop multiplication. Both of these kinds of transformations induce transformations of the torsion.

The original motivation for studying non-associative gauge theories comes from G_2-geometry [24]. A G_2-structure on a 7-dimensional Riemannian manifold is a reduction of the structure group of the orthornormal frame bundle from $SO(7)$ to G_2, which is the automorphism group of the octonions. A further relationship between G_2-structures and the octonions is that unit norm sections of an octonion bundle parametrize G_2-structures that are associated with the same metric, also known as *isometric* G_2-structures. A defining characteristic of a G_2-structure is its torsion, and one of the unanswered questions in G_2-geometry is which torsion types are admissible within a fixed metric class. One of the main goals in the theory of G_2-structures is to obtain existence results for torsion-free G_2 -structures, similar to the Yau's Theorem [49], that settled the existence question for Calabi–Yau manifolds. While this goal is formulated in terms of G_2-structures, the real question is the existence of a Riemannian metric with holonomy group equal to G_2. The fact that for each metric there exists an entire family of compatible G_2-structures presents a degeneracy in this problem. Some of the existing approaches involve Laplacian flows of G_2-structures [4, 5, 7–9, 21, 23, 33, 38–40], with the hope of a flow eventually converging to a torsion-free G_2 -structure. As shown in [21], the Laplacian flow of a generic G_2-structure has a component that moves within a metric class, and that component is precisely given by $\operatorname{div} T$, where T is the torsion of the G_2-structure, expressed as a 2-tensor, so that taking the divergence makes sense. Laplacian flows have been more successful for *closed* G_2-structures, in which case $\operatorname{div} T$ automatically vanishes, and thus the degeneracy is resolved. More generally, however, this degeneracy is the source of non-parabolicity of Laplacian flows, such as in the case of *co-closed* G_2-structures [21]. Therefore, the condition $\operatorname{div} T = 0$ can be regarded as a *gauge-fixing condition*. Moreover, in [24], it was found that on a compact manifold, G_2 -structures with $\operatorname{div} T = 0$ are precisely the critical points of the L^2-norm of the torsion when restricted to a fixed metric class.

As shown in [24, 26], within the loop bundle framework, this is the precise analog of the Coulomb gauge condition.

Existence of G_2-structures with divergence-free torsion has been studied from different perspectives by several authors in [3, 12, 25, 41]. All these approaches relied on a flow of isometric G_2 -structures (or more generally, geometric structures in [41] and [13]), and have shown existence of a G_2-structure with divergence-free torsion as a long-term limit of the flow, given sufficiently small pointwise initial torsion or another quantity, the entropy.

The interpretation of isometric G_2-structures as an octonionic non-associative gauge theory allows to adapt some gauge theory techniques in this setting. Moreover, without much additional effort, more general loops can be considered, with potential wider-reaching applications.

In gauge theory there are a number of versions of local and global existence results for connections in the Coulomb gauge, depending on the desired regularity [11, 14–17, 47, 48]. In this paper we use the Quantitative Implicit Function Theorem for Banach Spaces, as given in [17], to prove the following main result.

Theorem A *Suppose $\mathbb{L}$ is a smooth compact loop with tangent algebra $\mathfrak{l}$ and pseudoautomorphism group Ψ. Let (M, g) be a closed, smooth Riemannian manifold of dimension $n \geq 2$, and let $\mathcal{P}$ be a Ψ-principal bundle over M and let $\mathcal{A}$ be the associated vector bundle to $\mathcal{P}$ with fibers isomorphic to $\mathfrak{l}$. Let ω be a smooth connection on $\mathcal{P}$. Also, suppose k is a non-negative integer and $r \geq 0$ such that $kr > n$. Then, there exist constants $\delta \in (0, 1]$ and $K \in (0, \infty)$, such that if $s \in \Gamma(\mathcal{Q})$ is a smooth defining section for which*

$$\left\| T^{(s,\omega)} \right\|_{W^{k-1,r}} < \delta,$$

then there exists a section $A \in W^{k,r}\left(\mathcal{Q}'\right)$, such that

$$\left(d^{\omega}\right)^{*} T^{(As,\omega)} = 0,$$

where d^{ω} is the exterior covariant derivative with respect to the connection ω, and

$$\left\| T^{(As,\omega)} \right\|_{W^{k-1,r}} < K \left\| T^{(s,\omega)} \right\|_{W^{k-1,r}} \left(1 + \left\| T^{(s,\omega)} \right\|_{W^{k-1,r}}^{k-1}\right). \tag{1.1}$$

If moreover, $(k-1)r \geq n$, then A is smooth.

Given a 7-manifold M with a G_2-structure φ, recall from [24], that an octonion bundle on M can be defined as $\mathbb{O}M \cong \Lambda^0 \oplus TM$ with a product induced from the G_2-structure. Then the set of unit sections of $\mathbb{O}M$, the unit octonion bundle $U\mathbb{O}M$ can be regarded as a loop bundle, which each section $V \in \Gamma(U\mathbb{O}M)$ defining an isometric G_2-structure φ_V with torsion $T^{(V)}$. Then, Theorem A, can be reformulated specifically for G_2-structures, to give the following result for existence of smooth G_2-structures with divergence-free torsion.

Theorem B *Suppose M is a closed 7-dimensional manifold with a smooth G_2-structure φ with torsion T with respect to the Levi-Civita connection ∇. Suppose $U\mathbb{O}M$ is the corresponding unit octonion bundle. Also, suppose k is a positive integer and r is a positive real number such that $kr > 7$. Then, there exist constants $\delta \in (0, 1]$ and $K \in (0, \infty)$, such that if T satisfies*

$$\|T\|_{W^{k,r}} < \delta,$$

then there exists a smooth section $V \in \Gamma(U\mathbb{O}M)$, such that

$$\operatorname{div} T^{(V)} = 0$$

and

$$\left\|T^{(V)}\right\|_{W^{k,r}} < K \|T\|_{W^{k,r}} \left(1 + \|T\|_{W^{k,r}}^k\right). \tag{1.2}$$

The results presented in this paper are of interest and importance in their own right, but perhaps even more crucially, they show that some well-known results and techniques from classical gauge theory can be reinterpreted and adapted in a non-associative setting. In particular, this may open the door to some analogues of Uhlenbeck compactness and a better understanding of the torsion of non-associative gauge theories. Furthermore, a non-associative version of Yang-Mills equations can be considered. Moreover, any such advances will give immediate results in G_2-geometry.

The structure of this paper is the following. In Sect. 2, we give an overview of smooth loops, extending [26]. We give the basic properties of a smooth loop $\mathbb{L}$, define the pseudoautomorphism group Ψ and the tangent algebra $\mathfrak{l}$ at identity. The algebra $\mathfrak{l}$ is a generalization of a Lie algebra, but due to the non-associativity of $\mathbb{L}$ does not satisfy the Jacobi identity. Similarly as for Lie algebras, there is a notion of an exponential map. There is however a family of brackets $[\cdot,\cdot]^{(s)}$ on $\mathfrak{l}$, defined for each point $s \in \mathbb{L}$. For later use, we also give estimates for the exponential and adjoint maps. In particular, we analyze solutions of the following initial value problem for $\mathfrak{l}$-valued maps $\eta(t)$:

$$\begin{cases} \frac{d\eta(t)}{dt} = [\xi, \eta(t)]^{(\exp_s(t\xi)s)} \\ \eta(0) = \eta_0 \end{cases}, \tag{1.3}$$

where $\xi \in \mathfrak{l}$ and $s \in \mathbb{L}$.

In Sect. 3, we switch attention to loop-valued maps. In particular, given a smooth manifold M, consider a map $s : M \longrightarrow \mathbb{L}$. Using this map, we may define products of $\mathbb{L}$-valued maps and brackets of $\mathfrak{l}$-valued maps. Then, using the right quotient, translating the differential ds to the tangent space at $1 \in \mathbb{L}$, we obtain an $\mathfrak{l}$-valued 1-form θ_s on M, which is the analogue of the Darboux derivative of Lie group-valued maps [45]. The differentials of various operations defined by s are then expressed in terms of θ_s. Suppose $A(t) = \exp_s(t\xi)$ for some $\mathfrak{l}$-valued map ξ. We show that $\theta_{A(t)s}$ satisfies a non-homogeneous version of (1.3).

Further, we define Sobolev spaces of maps from M, and show in Lemma 3.10 that, similarly as for Lie groups, $s \in W^{k,r}(M, \mathbb{L})$ if and only if $\theta_s \in$

$W^{k-1,r}(M, T^*M \otimes \mathfrak{l})$. Using the evolution equation satisfied by $\theta_{A(t)s}$ then allows us to obtain Sobolev space estimates of $\theta_{A(t)s}$ and other quantities that satisfy equations based on (1.3).

Theorem C *Let M be a compact Riemannian manifold and $\mathbb{L}$ is a smooth compact loop. Suppose k is a positive integer and r is a positive real number such that $kr > n = \dim M$. Let $s \in W^{k,r}(M, \mathbb{L})$ and $\xi \in W^{k,r}(M, \mathfrak{l})$, and suppose $A = \exp_s(\xi)$. Then,*

$$\|\theta_{As}\|_{W^{k-1,r}} \lesssim e^{Ck\|\xi\|_{C^0}} \left(\Theta^k + \Theta\right), \tag{1.4}$$

where $\Theta = \|\theta_s\|_{W^{k-1,r}} + \|\xi\|_{W^{k,r}}$, C is constant that depends only on $\mathbb{L}$, and $\lesssim$ denotes inequality up to a universal constant factor

Similarly, if $X = X(1)$, where $X(t)$ is 1 -parameter family of $\mathfrak{l}$-valued maps that satisfies

$$\begin{cases} \frac{dX(t)}{dt} = [\xi, X(t)]^{A(t)s} + Y \\ X(0) = X_0, \end{cases}$$

for $\mathfrak{l}$-valued maps X_0 and Y, then,

$$\|X\|_{W^{k,r}} \lesssim e^{C(k+1)\|\xi\|_{C^0}} \left(\|X_0\|_{W^{k,r}} + \|Y\|_{W^{k,r}}\right)\left(\Theta^{k+1} + \Theta\right). \tag{1.5}$$

In Sect. 4, we introduce a principal Ψ-bundle $\mathcal{P}$ over a compact manifold M, and then apply the above results to Ψ-equivariant maps from $\mathcal{P}$ to a loop $\mathbb{L}$ and other related spaces. This immediately then allows to consider sections of bundles over M that are associated to $\mathcal{P}$. In particular, suppose we have a connection ω on $\mathcal{P}$ and suppose s is a section of the associated loop bundle $\mathcal{Q}$, with fibers diffeomorphic to $\mathbb{L}$. It uniquely corresponds to a Ψ -equivariant map $\tilde{s} : \mathcal{P} \longrightarrow \mathbb{L}$, and thus we obtain the equivariant $\mathfrak{l}$-valued 1-form $\theta_{\tilde{s}}$ on $\mathcal{P}$. On the other hand, the connection ω defines a decomposition of $T\mathcal{P}$ into vertical and horizontal subspaces. Therefore, we may compose $\theta_{\tilde{s}}$ with the horizontal projection to obtain a *basic*, i.e. horizontal and equivariant $\mathfrak{l}$-valued 1-form on $\mathcal{P}$. This then corresponds to a section of a bundle over M, and gives us the *torsion* $T^{(s,\omega)}$ of the configuration (s, ω). Defining fiberwise loop multiplication, we see that all the possible configurations with a fixed ω may be obtained by multiplying s by some section A. Therefore, the *loop gauge transformations* are precisely the transformations $s \mapsto As$. Moreover, as it was already known previously, [24–27, 41], given appropriate algebraic conditions on the loop, the critical points of the functional $s \mapsto \int_M \left|T^{(s,\omega)}\right|^2 \mathrm{vol}$ are precisely the sections s for which $(d^\omega)^* T^{(s,\omega)} = 0$, which relates to the previous discussion on divergence-free torsion and the Coulomb gauge.

Considering the transformations of s of the form $s \mapsto \exp_s(\xi) s$, for $\mathfrak{l}$-valued sections ξ, and using the loop exponential map, as developed in Sect. 2, the quantity $T^{(\exp_s(t\xi)s,\omega)}$ is then shown to satisfy an ODE with the homogeneous part of the form (1.3). This allows to rewrite $T^{(\exp_s(t\xi)s,\omega)}$ in terms of the evolution operator for (1.3). The equation $(d^\omega)^* T^{(\exp_s(\xi)s,\omega)} = 0$ is then written as a second-order PDE for ξ.

This operator then defines a smooth functions between appropriate Banach spaces of sections of vector bundles, which allows to apply the Implicit Function Theorem from [17], to show existence of solutions for sufficiently small initial $T^{(s,\omega)}$ in an appropriate Sobolev norm, and thus prove Theorem A.

In Sect. 5, we carefully apply the general theory of smooth loops to the particular case of G_2-structures, and then Theorem B follows as an immediate corollary of Theorem A.

2 Smooth Loops

For a detailed introduction to smooth loops, the reader is referred to [26]. The reader can also refer to [29, 34, 43, 44, 46] for a discussion of these concepts.

Definition 2.1 A *loop* $\mathbb{L}$ is a set with a binary operation $p \cdot q$ with identity 1, and compatible left and right quotients $p\backslash q$ and p/q, respectively.

In particular, existence of quotients is equivalent to saying that for any $q \in \mathbb{L}$, the left and right product maps L_q and R_q are invertible maps. Restricting to the smooth category, we obtain the definition of a smooth loop.

Definition 2.2 A *smooth loop* is a smooth manifold $\mathbb{L}$ with a loop structure such that the left and right product maps are diffeomorphisms of $\mathbb{L}$.

Remark 2.3 In this paper we will not use the left quotient, so in fact everything that follows also holds true for smooth *right* loops, i.e. where only the right quotient is defined, but the left product is not necessarily invertible. However, for brevity, we will keep referring to *loops*, rather than *right loops*. It should be noted that the choice between left and right loops is arbitrary and is simply a matter of convention.

Example 2.4 Suppose G is a Lie group with a Lie subgroup H and consider the left quotient $K = {}_{H}\backslash G$. Suppose $\sigma : K \longrightarrow G$ is a section of G, regarded as a bundle over K. In particular, σ maps each right coset to a particular representative of that coset. Suppose $\sigma(H) = 1$. A product structure on K is then given by

$$(Ha) \circ (Hb) = Ha\,(\sigma(Hb)). \tag{2.1}$$

Consider the equation $(Hx) \circ (Ha) = Hb$, Since σ is a section, we can see right away that we have a unique solution $Hx = Hb\,(\sigma(Ha))^{-1}$. Thus, $(K, \circ)$ has *right* division, and is thus a *right loop* [34, 43]. To define left division, and hence to obtain a full loop structure, the section has to satisfy additional conditions [43, Proposition 1.6].

Example 2.5 Conversely, suppose $(K, \circ)$ is a (smooth) right loop. Let G be the group generated by the set of right translations of K and let H be the stabilizer subgroup of $1 \in K$ in G. Assume G has a right action on K. Define the section $\sigma : {}_{H}\backslash G \longrightarrow G$ given by $\sigma(Hg) = \left(R_{g^{-1}(1)}\right)^{-1} \in G$ for any $g \in G$. It is then easy to see [34, Sect. 2.B] that (2.1) recovers the right loop structure on K.

Remark 2.6 As Examples 2.4 and 2.5 above show, loops are always obtained as sections of groups. A partial classification of topological and smooth loops in given in [43], but a full classification is an open problem. The conditions for a smooth right loops are less restrictive, so they are expected to be more plentiful.

Example 2.7 A loop is a rather general algebraic structure, however sometimes there are several progressively stronger partial associativity assumptions. Here is a list of some of the ones that we will be referring to:

1. *Power-associativity* Any element $p \in \mathbb{L}$ generates a subgroup of $\mathbb{L}$. Power-associativity allows to unambiguously define integer powers p^n of elements.
2. *(Left)-alternative* For any $p, q \in \mathbb{L}$, $p \cdot pq = pp \cdot q$. Similarly we can define the right-alternative property (i.e. $q \cdot pp = qp \cdot p$). If $\mathbb{L}$ is both left-alternative and right-alternative, then it is said to be *alternative.*
3. *(Left)-power-alternative* For any $p, q \in \mathbb{L}$, $p^k \left(p^l q\right) = p^{k+l} q$, for any non-negative integers k, l. Similarly, we can define right-power-alternative loops.
4. *Moufang loop* A loop is a Moufang loop if it satisfies the identities (2.2) for any $p, q, r \in \mathbb{L}$. In particular, Moufang loops are alternative.

$$p\,(q \cdot pr) = (p \cdot qp)\,r. \tag{2.2a}$$

$$(pq \cdot r)\,q = p\,(qr \cdot q)\,. \tag{2.2b}$$

5. *Group*: clearly any associative loop is a group.

Definition 2.8 A *pseudoautomorphism* of a smooth loop $\mathbb{L}$ is a diffeomorphism $h : \mathbb{L} \longrightarrow \mathbb{L}$ for which there exists another diffeomorphism $h' : \mathbb{L} \longrightarrow \mathbb{L}$, known as the partial pseudoautomorphism corresponding to h, such that for any $p, q \in \mathbb{L}$,

$$h\,(pq) = h'\,(p)\,h\,(q)\,. \tag{2.3}$$

In particular, $h' = R^{-1}_{h(1)} \circ h$ and $h'\,(1) = 1$. The element $h\,(1) \in \mathbb{L}$ is the *companion* of h'. From (2.3), we also see the following property of h' with respect to quotients:

$$h'\,(p/q) = h\,(p)\,/h\,(q)\,. \tag{2.4}$$

It is easy to see that the sets of pseudoautomorphisms and partial pseudoautomorphisms are both groups. Denote the former by Ψ and the latter by Ψ'. We also see that the *automorphism* group of the loop $\mathbb{L}$ is the subgroup $H \subset \Psi$ which is the stabilizer of $1 \in \mathbb{L}$.

Remark 2.9 To avoid introducing additional notation, but at the risk of some ambiguity, we will use the same notation $\mathbb{L}$ to denote the underlying manifold, the loop, and the G-set with the full action of Ψ. However, since $\mathbb{L}$ also admits the action of Ψ', if a distinction between the G-sets is needed, we will use $\mathbb{L}'$ to denote the set $\mathbb{L}$ with the action of Ψ', Let $r \in \mathbb{L}$, then we may define a modified product $\circ_r$ on $\mathbb{L}$ via $p \circ_r q = (p \cdot qr)/r$, so that $\mathbb{L}$ equipped with product $\circ_r$ will be denoted by $(\mathbb{L}, \circ_r)$, the corresponding quotient will be denoted by $/_r$. We have the following properties [26].

Lemma 2.10 *Let $h \in \Psi$. Then, for any $p, q, r \in \mathbb{L}$,*

$$h'(p \circ_r q) = h'(p) \circ_{h(r)} h'(q) \qquad h'(p/_r q) = h'(p) /_{h(r)} h'(q). \tag{2.5}$$

Also, for any $A \in \mathbb{L}$,

$$(p \circ_r (q \circ_r A)) /_r A = p \circ_{Ar} q. \tag{2.6}$$

Lemma 2.11 *Suppose $A(t)$ and $B(t)$ are smooth curves in $\mathbb{L}$ with $A(t_0) = A_0$ and $B(t_0) = B_0$, then*

$$\left.\frac{d}{dt} A(t) / B(t)\right|_{t=t_0} = \left.\frac{d}{dt} A(t) / B_0\right|_{t=t_0} - \left.\frac{d}{dt} (A_0/B_0 \cdot B(t)) / B_0\right|_{t=t_0}. \tag{2.7}$$

Consider the tangent space $\mathfrak{l} := T_1\mathbb{L}$ at $1 \in \mathbb{L}$. For any $q \in \mathbb{L}$, the pushforward $(R_q)_*$ of the right translation map R_q defines a linear isomorphism from $\mathfrak{l}$ to $T_q\mathbb{L}$. In particular, let us denote the linear map $\left.(R_q)_*\right|_1 : \mathfrak{l} \longrightarrow T_q\mathbb{L}$ by ρ_q, and correspondingly, $\left.\left(R_q^{-1}\right)_*\right|_1 : T_q\mathbb{L} \longrightarrow \mathfrak{l}$ by ρ_q^{-1}. Similarly, for left multiplication, define $\lambda_q = \left.(L_q)_*\right|_1 : \mathfrak{l} \longrightarrow T_q\mathbb{L}$. On a smooth right loop, λ_q will not necessarily be invertible. The corresponding maps with respect to the product $\circ_r$ will be denoted by $R_q^{(r)}$, $\rho_q^{(r)}$, $\lambda_q^{(r)}$.

Definition 2.12 For any $\xi \in \mathfrak{l}$, define the *fundamental vector field* $\rho(\xi)$ for any $q \in \mathbb{L}$, $\rho(\xi)_q = \rho_q(\xi)$.

The above definition of the fundamental vector field is the analog of a right-invariant vector field in Lie theory. However, in the loop case, although this vector field is canonical in some sense, it is not *invariant* under right translations. We use fundamental vector fields to define the loop exponential map.

Definition 2.13 Suppose $\mathbb{L}$ is a smooth loop and suppose $q \in \mathbb{L}$. Then, given $\xi \in \mathfrak{l}$, for sufficiently small t, define

$$p(t) = \exp_q(t\xi)\, q. \tag{2.8}$$

to be the solution of the equation

$$\begin{cases} \frac{dp(t)}{dt} = \left.\rho(\xi)\right|_{p(t)} \\ p(0) = q \end{cases}. \tag{2.9}$$

Equivalently, $\tilde{p}(t) = \exp_q(t\xi)$ satisfies

$$\begin{cases} \frac{d\tilde{p}(t)}{dt} = \rho^{(q)}(\xi)\big|_{\tilde{p}(t)} \\ \tilde{p}(0) = 1 \end{cases}. \tag{2.10}$$

Remark 2.14 In general, the solution $\exp_q$ will only be defined in a neighborhood of $0 \in \mathfrak{l}$, however as shown in [36, 42], if the loop $\mathbb{L}$ is power-associative, so that powers of an element $p \in \mathbb{L}$ associate, then $p(nh) = p(h)^n$ can be defined unambiguously. We will show this from a different perspective further below. This can then be used to define the solution $p(t)$ for all t, and thus this extends $\exp_q$ to all of $\mathfrak{l}$.

Let us consider $d\exp_q$. From the definition of $\exp_q$, for any $\xi \in \mathfrak{l}$, have

$$d\exp_q\big|_0(\xi) = \frac{d}{dt}\exp_q(t\xi)\bigg|_{t=0} = \xi. \tag{2.11}$$

In particular, $\exp_s$ is smooth and since the identity map is a linear isomorphism, by the Inverse Function Theorem, we have the following.

Lemma 2.15 *For any* $q \in \mathbb{L}$, *the map* $\exp_q : \mathfrak{l} \longrightarrow \mathbb{L}$ *is a local diffeomorphism around* $0 \in \mathfrak{l}$.

Remark 2.16 To distinguish the different exponential maps on $\mathfrak{l}$, we will use a subscript to denote with respect to which element of $\mathbb{L}$ the exponential map is used. For any $q \in \mathbb{L}$, $\gamma(t) = \exp_q(t\xi)$ is a curve on $\mathbb{L}$ with $\gamma(0) = 1$ and $\gamma'(0) = \xi$, so when only infinitesimal behavior is needed, such as when computing differentials, we will drop the subscript since then it does not matter which particular exponential function is used.

On smooth loops, we can define an analog of the Lie group Maurer–Cartan form.

Definition 2.17 [26] The Maurer–Cartan form θ is an $\mathfrak{l}$-valued 1-form on $\mathbb{L}$, such that for any vector field X, and any $p \in \mathbb{L}$, $\theta(X)|_p = \rho_p^{-1}(X_p) \in \mathfrak{l}$. Equivalently, for any $\xi \in \mathfrak{l}$, $\theta(\rho(\xi)) = \xi$.

The loop Maurer–Cartan form allows us to define brackets on $\mathfrak{l}$. For each $p \in \mathbb{L}$ define the bracket $[\cdot,\cdot]^{(p)}$ given for any $\xi, \eta \in \mathfrak{l}$ by

$$[\xi, \eta]^{(p)} = -\theta([\rho(\xi), \rho(\eta)])|_p .$$

As shown in [26, Theorem 3.7], we can equivalently define

$$\begin{aligned} [\xi, \gamma]^{(p)} &= \frac{d}{dt}\left(\mathrm{Ad}^{(p)}_{\exp(t\xi)}\gamma\right)\bigg|_{t=0} \\ &= \frac{d^2}{dtd\tau}\exp(t\xi)\circ_p\exp(\tau\gamma)\bigg|_{t,\tau=0} \\ &\quad - \frac{d^2}{dtd\tau}\exp(\tau\gamma)\circ_p\exp(t\xi)\bigg|_{t,\tau=0}, \end{aligned} \tag{2.12}$$

where, for $p, q \in \mathbb{L}$, $\mathrm{Ad}_q^{(p)} : \mathfrak{l} \longrightarrow \mathfrak{l}$ is the differential at $1 \in \mathbb{L}$ of the conjugation map $r \mapsto \left(q \circ_p r\right) /_p q \in \mathbb{L}$.

Remark 2.18 In [26], the conjugation map $r \mapsto \left(q \circ_p r\right) /_p q$ was denoted by $\mathrm{Ad}_q^{(p)}$, and its differential as $\left(\mathrm{Ad}_q^{(p)}\right)_*$. However here we adopt notation that is more in line with standard usage in Lie theory.

Definition 2.19 The vector space $\mathfrak{l}$ equipped with the bracket $[\cdot,\cdot]^{(p)}$ is known as the *loop tangent algebra* $\mathfrak{l}^{(p)}$.

Since for each $p \in \mathbb{L}$ we have a bracket $[\cdot,\cdot]^{(p)}$ on $\mathfrak{l}$, we can define the corresponding bracket function.

Definition 2.20 The *bracket function* $b : \mathbb{L} \longrightarrow \mathfrak{l} \otimes \Lambda^2\mathfrak{l}^*$ is the map that takes $p \mapsto [\cdot,\cdot]^{(p)} \in \mathfrak{l} \otimes \Lambda^2\mathfrak{l}^*$, so that $b(\theta,\theta)$ is an $\mathfrak{l}$-valued 2-form on $\mathbb{L}$, i.e. $b(\theta,\theta) \in \Omega^2(\mathfrak{l})$.

Definition 2.21 For any $\eta, \gamma, \xi \in \mathfrak{l}$, and $p \in \mathbb{L}$, the *associator* $[\cdot,\cdot,\cdot]^{(p)}$ on $\mathfrak{l}^{(p)}$ is given by

$$\begin{aligned}\left[\eta,\gamma,\xi\right]^{(p)} &= \frac{d^3}{dt d\tau d\tau'} \exp(\tau\eta) \circ_p \left(\exp\left(\tau'\gamma\right) \circ_p \exp(t\xi)\right)\Big|_{t,\tau,\tau'=0} \\ &\quad - \frac{d^3}{dt d\tau d\tau'} \left(\exp(\tau\eta) \circ_p \exp\left(\tau'\gamma\right)\right) \circ_p \exp(t\xi)\Big|_{t,\tau,\tau'=0}.\end{aligned} \tag{2.13}$$

Moreover, define mixed associators between elements of $\mathbb{L}$ and $\mathfrak{l}$. An $(\mathbb{L},\mathbb{L},\mathfrak{l})$ -associator is defined for any $p, q \in \mathbb{L}$ and $\xi \in \mathfrak{l}$ as

$$[p,q,\xi]^{(s)} = \left(L_p^{(s)} \circ L_q^{(s)}\right)_* \xi - \left(L_{p\circ_s q}^{(s)}\right)_* \xi \in T_{p\circ_s q}\mathbb{L} \tag{2.14}$$

and an $(\mathbb{L},\mathfrak{l},\mathfrak{l})$-associator is defined for an $p \in \mathbb{L}$ and $\eta, \xi \in \mathfrak{l}$ as

$$\begin{aligned}[p,\eta,\xi]^{(s)} &= \frac{d}{dt d\tau} \left(p \circ_s \left(\exp(t\eta) \circ_s \exp(\tau\xi)\right)\right)\Big|_{t,\tau=0} \\ &\quad - \frac{d}{dt d\tau} \left(\left(p \circ_s \exp(t\eta)\right) \circ_s \exp(\tau\xi)\right)\Big|_{t,\tau=0},\end{aligned} \tag{2.15}$$

where we see that $[p,\eta,\xi]^{(s)} \in T_p\mathbb{L}$. Similarly, for other combinations. Also define the *left-alternating associator* $a : \mathbb{L} \longrightarrow \mathfrak{l} \otimes \Lambda^2\mathfrak{l}^* \otimes \mathfrak{l}^*$, given by

$$a_p(\eta,\gamma,\xi) = \left[\eta,\gamma,\xi\right]^{(p)} - \left[\gamma,\eta,\xi\right]^{(p)}. \tag{2.16}$$

Remark 2.22 From the definitions of the associators, it is easy to see that if $(\mathbb{L}, \circ_s)$ is power-associative, given $\xi \in \mathfrak{l}$, associators with any combinations of ξ and $\exp_s(t\xi)$, for any values of t, in the three entries, will vanish. For example,

$$[\xi,\xi,\xi]^{(s)}=0 \tag{2.17a}$$
$$[\xi,\xi,\exp_s(t\xi)]^{(s)}=0 \tag{2.17b}$$
$$\left[\xi,\exp_s(t\xi),\exp_s(\tau\xi)\right]^{(s)}=0, \tag{2.17c}$$

as well as any permutations.

Similarly, if $(\mathbb{L},\circ_s)$ is left-power-alternative then associators with any combination of ξ and $\exp_s(t\xi)$ in the first two entries will vanish, for example

$$[\xi,\xi,\eta]^{(s)}=0 \tag{2.18a}$$
$$[\xi,\exp_s(t\xi),\eta]^{(s)}=0 \tag{2.18b}$$
$$\left[\exp_s(t\xi),\exp_s(\tau\xi),\eta\right]^{(s)}=0, \tag{2.18c}$$

for any $\eta\in\mathfrak{l}$ and similarly with the third entry replaced by an element of $\mathbb{L}$.

From [26] we cite several useful properties of these brackets and associators.

Theorem 2.23 ([26, Theorem 3.20]) *Suppose $p,s\in\mathbb{L}$, and $\xi,\eta\in\mathfrak{l}$. Then the bracket $[\cdot,\cdot]^{(ps)}$ is related to $[\cdot,\cdot]^{(s)}$ via the expression*

$$[\xi,\eta]^{(ps)}=[\xi,\eta]^{(s)}+\left(\rho_p^{(s)}\right)^{-1}a_s(\xi,\eta,p). \tag{2.19}$$

Theorem 2.24 ([26, Theorem 3.10]) *The form θ satisfies*

$$d\theta=\frac{1}{2}b(\theta,\theta), \tag{2.20}$$

where wedge product of 1-forms is implied. Also, for any $\xi,\eta\in\mathfrak{l}$, we have

$$db(\xi,\eta)=a(\xi,\eta,\theta). \tag{2.21}$$

It follows that $\xi,\eta,\gamma\in\mathfrak{l}$, the generalized Jacobi identity is satisfied:

$$\mathrm{Jac}^{(s)}(\xi,\eta,\gamma)=a_s(\xi,\eta,\gamma)+a_s(\eta,\gamma,\xi)+a_s(\gamma,\xi,\eta), \tag{2.22}$$

where

$$\mathrm{Jac}^{(s)}(\xi,\eta,\gamma)=\left[\xi,\left[\eta,\gamma\right]^{(s)}\right]^{(s)}+\left[\eta,\left[\gamma,\xi\right]^{(s)}\right]^{(s)}+\left[\gamma,[\xi,\eta]^{(s)}\right]^{(s)}. \tag{2.23}$$

Remark 2.25 Equation (2.20) is the loop Maurer–Cartan equation. The key difference from the Maurer–Cartan equation on Lie groups is that on non-associative loops, $b(s)$ is non-constant on $\mathbb{L}$, unlike on Lie groups, where there is a unique bracket on the

Lie algebra, and hence $b(s)$ is constant. In particular, the non-constant b leads to a non-trivial associator (2.21) and may lead to the failure of the standard Jacobi identity to hold.

With respect to the action of Ψ, the bracket and the associator satisfy the following properties.

Lemma 2.26 *If $h \in \Psi$ and $q \in \mathbb{L}$, then, for any $\xi, \eta, \gamma \in \mathfrak{l}$,*

$$\begin{aligned} h'_*[\xi,\eta]^{(q)} &= \left[h'_*\xi, h'_*\eta\right]^{h(q)} \\ h'_*\left[\xi,\eta,\gamma\right]^{(q)} &= \left[h'_*\xi, h'_*\eta, h'_*\gamma\right]^{h(q)}. \end{aligned}$$

If $A(t)$ is a path on $\mathbb{L}$, with $A(t_0) = A_0$, and $\left.\frac{d}{dt}A(t)/A_0\right|_{t=t_0} = \xi \in \mathfrak{l}$, then for any $p, q \in \mathbb{L}$,

$$\left.\frac{d}{dt}p \circ_{A(t)} q\right|_{t=t_0} = [p,q,\xi]^{(A_0)} \in T_{p\circ_{A_0}q}\mathbb{L}. \tag{2.24}$$

Also, for any $\eta, \gamma \in \mathfrak{l}$,

$$\left.\frac{d}{dt}\left[\eta,\gamma\right]^{A(t)}\right|_{t=t_0} = a_{A_0}(\eta,\gamma,\xi) \tag{2.25}$$

Proof The first part is given in [26, Lemma 3.17]. To show (2.24), consider

$$\begin{aligned} \left.\frac{d}{dt}p \circ_{A(t)} q\right|_{t=t_0} &= \left.\frac{d}{dt}\left(p\left(qA(t)\right)\right)/A(t)\right|_{t=t_0} \\ &= \left.\frac{d}{dt}\left(p\left(qA(t)\right)\right)/A_0\right|_{t=t_0} - \left.\frac{d}{dt}\left(\left(p\circ_{A_0} q\right)\cdot A(t)\right)/A_0\right|_{t=t_0}, \end{aligned}$$

where we've used (2.7). Now,

$$\begin{aligned} \left(p\left(qA(t)\right)\right)/A_0 &= \left(p\left(q\left(A(t)/A_0\cdot A_0\right)/A_0\cdot A_0\right)\right)/A_0 \\ &= p\circ_{A_0}\left(q\circ_{A_0}\left(A(t)/A_0\right)\right) \\ \left(\left(p\circ_{A_0}q\right)\cdot A(t)\right)/A_0 &= \left(p\circ_{A_0}q\right)\circ_{A_0}\left(A(t)/A_0\right). \end{aligned}$$

Hence,

$$\begin{aligned} \left.\frac{d}{dt}p\circ_{(t)} q\right|_{t=t_0} = &\left.\frac{d}{dt}p\circ_{A_0}\left(q\circ_{A_0}\left(A(t)/A_0\right)\right)\right|_{t=0} \\ &- \left.\frac{d}{dt}\left(p\circ_{A_0}q\right)\circ_{A_0}\left(A(t)/A_0\right)\right|_{t=0}. \end{aligned}$$

To show (2.25), we could use (2.21), but more directly, we can obtain it from the definition, using (2.12):

$$[\xi,\gamma]^{(A(t))} = \frac{d^2}{d\tau d\tau'} \exp(\tau\eta) \circ_{A(t)} \exp(\tau'\gamma)\Big|_{\tau,\tau'=0} - \frac{d^2}{d\tau d\tau'} \exp(\tau'\gamma) \circ_{A(t)} \exp(\tau\eta)\Big|_{\tau,\tau'=0}. \tag{2.26}$$

Then, from (2.24),

$$\frac{d}{dt} \exp(\tau\eta) \circ_{A(t)} \exp(\tau'\gamma)\big|_{t=t_0} = \left[\exp(\tau\eta), \exp(\tau'\gamma), \xi\right]^{(A_0)}$$
$$\frac{d}{dt} \exp(\tau'\gamma) \circ_{A(t)} \exp(\tau\eta)\big|_{t=t_0} = \left[\exp(\tau'\gamma), \exp(\tau\eta), \xi\right]^{(A_0)}$$

and from the definition (2.13), we obtain (2.25). □

Let $\xi \in \mathfrak{l}$ and $s \in \mathbb{L}$. Also let $A(t) = \exp_s(t\xi)$ for t in some interval $I \subset \mathbb{R}$ that contains 0. Then consider a family $\eta(t) \in \mathfrak{l}$ that satisfies the following initial value problem:

$$\begin{cases} \frac{d\eta(t)}{dt} = [\xi, \eta(t)]^{(A(t)s)} \\ \eta(0) = \eta_0 \end{cases}. \tag{2.27}$$

In other words, this is linear first-order ODE $\dot{\eta} = \mathrm{ad}_\xi^{(A(t)s)}\eta$, so for all $t \in I$ there exists an evolution operator $U_\xi^{(s)}(t) \in GL(\mathfrak{l})$, with $U_\xi^{(s)}(0) = \mathrm{id}_{\mathfrak{l}}$, such that

$$\eta(t) = U_\xi^{(s)}(t)\,\eta_0. \tag{2.28}$$

From standard ODE theory, recall that if $\tau', \tau'' \in I$, then $U_\xi^{(s)}(\tau'')\,U_\xi^{(s)}(\tau')^{-1}$ is the evolution operator from τ' to τ'' and is given by:

$$U_\xi^{(s)}(\tau'')\,U_\xi^{(s)}(\tau')^{-1} = \mathrm{id}_{\mathfrak{l}} + \int_{\tau'}^{\tau''} \mathrm{ad}_\xi^{((\exp_s t_1\xi)s)}\,dt_1 + \sum_{n=2}^{\infty} \int_{\tau'}^{\tau''}\int_{\tau'}^{t_n} \dots \int_{\tau'}^{t_2} \mathrm{ad}_\xi^{((\exp_s t_n\xi)s)} \dots \mathrm{ad}_\xi^{((\exp_s t_1\xi)s)}\,dt_1 \dots dt_n. \tag{2.29}$$

The following properties of $U_\xi^{(s)}(t)$ follow immediately.

Lemma 2.27 *The evolution operator $U_\xi^{(s)}(t)$ satisfies the following properties:*

1. $U_{\tau\xi}^{(s)}(t) = U_\xi^{(s)}(\tau t)$, *for any t and τ, as long as $\exp_s(t\xi)$ and $\exp_s(\tau t\xi)$ are both defined.*

2. $U_{\xi}^{(s)}(t)\,\xi = \xi$.
3. *If $\mathbb{L}$ is compact, and $\mathfrak{l}$ is equipped with an inner product, then in a compatible operator norm, there exists a constant $C = \sup_{s\in\mathbb{L}} |b_s|$, such*

$$\left|U_{\xi}^{(s)}\left(\tau''\right)U_{\xi}^{(s)}\left(\tau'\right)^{-1} - \mathrm{id}_{\mathfrak{l}}\right| \le e^{C\left|\tau''-\tau'\right||\xi|} - 1. \tag{2.30}$$

Proof Item 1 follows from a change of variables in (2.27). For item 2, consider

$$X(t) = U_{\xi}^{(s)}(t)\,\xi - \xi.$$

Then,

$$\begin{aligned}\frac{dX(t)}{dt} = \frac{d\left(U_{\xi}^{(s)}(t)\,\xi\right)}{dt} &= \left[\xi, U_{\xi}^{(s)}(t)\,\xi\right]^{(A(t)s)} \\ &= [\xi, X(t)]^{(A(t)s)},\end{aligned}$$

since $[\xi,\xi]^{(A(t))} = 0$. Hence, $X(t) = U_{\xi}^{(s)}(t)\,X(0)$, but $X(0) = 0$, so $X(t) = 0$ for all t.

For the estimate, from (2.29), we obtain

$$\begin{aligned}\left|U_{\xi}^{(s)}\left(\tau''\right)U_{\xi}^{(s)}\left(\tau'\right)^{-1} - \mathrm{id}_{\mathfrak{l}}\right| &\le \exp\left(\int_{\tau'}^{\tau''}\left|\mathrm{ad}_{\xi}^{((\exp_s t\xi)s)}\right|dt\right) - 1 \\ &\le \exp\left(\left|\tau''-\tau'\right||\xi|\sup_{s\in\mathbb{L}}|b_s|\right) - 1\end{aligned}$$

Now, $s \mapsto |b_s|$ is a smooth real-valued map on a compact manifold, and is hence bounded. Therefore, there exists a constant $C = \sup_{s\in\mathbb{L}}|b_s|$ and hence $\sup_{t\in[0,1]}\left|b_{(\exp_s t\xi)s}\right| \le C$. Thus,

$$\left|U_{\xi}^{(s)}\left(\tau''\right)U_{\xi}^{(s)}\left(\tau'\right)^{-1} - \mathrm{id}_{\mathfrak{l}}\right| \le e^{C\left|\tau''-\tau'\right||\xi|} - 1.$$

□

Remark 2.28 Since $U_{\xi}^{(s)}(t) = U_{t\xi}^{(s)}(1)$, for brevity let us denote the operator $U_{\xi}^{(s)}(t)$ by $U_{t\xi}^{(s)}$.

If $\mathfrak{l}$ is a Lie algebra, then $\mathrm{ad}_{\xi}^{(A(t)s)} = \mathrm{ad}_{\xi}$ is independent of t, and then $U_{t\xi}^{(s)} = \exp\left(t\,\mathrm{ad}_{\xi}\right) = \mathrm{Ad}_{\exp t\xi}$. In the non-associative case, this is no longer true in general, but needs additional assumptions, as Theorem 2.29 below shows.

Theorem 2.29 *Let $s \in \mathbb{L}$, $\xi, \eta \in \mathfrak{l}$, and $A(t) = \exp_s(t\xi)$. Suppose $U_{t\xi}^{(s)}$ is the evolution operator for the equation (2.27) as in (2.28). Then,*

$$\operatorname{Ad}_{A(t)}^{(s)} \eta = U_{t\xi}^{(s)} \eta + U_{t\xi}^{(s)} \int_0^t \left(U_{\tau\xi}^{(s)}\right)^{-1} \left(\rho_{A(\tau)}^{(s)}\right)^{-1} \left([\xi, A(\tau), \eta]^{(s)}\right) d\tau. \quad (2.31)$$

Moreover,

1. *If $\mathbb{L}$ is compact, and $\mathfrak{l}$ is equipped with an inner product, then in a compatible operator norm, there exists a constant C that depends only on $\mathbb{L}$, such that,*

$$\left|\operatorname{Ad}_{A(t)}^{(s)} - U_{t\xi}^{(s)}\right| \leq C\left(e^{C|\xi|t} - 1\right). \quad (2.32)$$

2. *If $(\mathbb{L}, \circ_s)$ is left-power-alternative, then*

$$\operatorname{Ad}_{A(t)}^{(s)} = U_{t\xi}^{(s)}. \quad (2.33)$$

3. *If $(\mathbb{L}, \circ_s)$ is both left-power-alternative and right-power-alternative, then*

$$\operatorname{Ad}_{A(t)}^{(s)} = \exp\left(t \operatorname{ad}_{\xi}^{(s)}\right). \quad (2.34)$$

Proof Let $x(t) = \operatorname{Ad}_{A(t)}^{(s)} \eta$ and note that $x(0) = \eta$. Then, consider the derivative of $x(t)$. Let $B(t) = \exp_s(t\eta)$. From (2.10) we have

$$\frac{dA(t)}{dt} = \rho_{A(t)}^{(s)}(\eta) = \left.\frac{d}{d\tau} B(\tau) \circ_s A(t)\right|_{\tau=0}$$

Then, using (2.7) we have

$$\begin{aligned}
\frac{dx(t)}{dt} &= \left.\frac{d^2}{dt d\tau} (A(t) \circ_s (B(\tau))) /_s A(t)\right|_{\tau=0} \\
&= \left.\frac{d^2}{d\tau d\tau'} \left(A\left(\tau'\right) \circ_s A(t)\right) \circ_s (B(\tau)) /_s A(t)\right|_{\tau',\tau=0} \\
&\quad - \left.\frac{d^2}{d\tau d\tau'} (((A(t) \circ_s (B(\tau))) /_s A(t)) \circ_s \left(A\left(\tau'\right) \circ_s A(t)\right)) /_s A(t)\right|_{\tau',\tau=0} \\
&= -\left(\rho_{A(t)}^{(s)}\right)^{-1} [\xi, A(t), \eta]^{(s)} \\
&\quad + \left.\frac{d}{d\tau'} \left(A\left(\tau'\right) \circ_s \left(\operatorname{Ad}_{A(t)}^{(s)} \eta \circ_s A(t)\right)\right) /_s A(t)\right|_{\tau'=0} \\
&\quad - \left.\frac{d}{d\tau} (\operatorname{Ad}_{A(t)}^{(s)} \eta \circ_s \left(A\left(\tau'\right) \circ_s A(t)\right)) /_s A(t)\right|_{\tau=0}.
\end{aligned} \quad (2.35)$$

Using (2.6), the second term in (2.35) becomes

$$\frac{d}{d\tau'}\left(A\left(\tau'\right)\circ_s\left(\mathrm{Ad}^{(s)}_{A(t)}\,\eta\circ_s A\left(t\right)\right)\right)/_s A\left(t\right)\Big|_{\tau'=0}=\frac{d}{d\tau'}A\left(\tau'\right)\circ_{A(t)s}x\left(t\right)\Big|_{\tau'=0}$$

and similarly, the third term in (2.35) becomes

$$\frac{d}{d\tau}(\mathrm{Ad}^{(s)}_{A(t)}\,\eta\circ_s\left(A\left(\tau'\right)\circ_s A\left(t\right)\right))/_s A\left(t\right)\Big|_{\tau=0}=\frac{d}{d\tau'}x\left(t\right)\circ_{A(t)s}A\left(\tau'\right)\Big|_{\tau'=0}.$$

Using (2.12) we then conclude that

$$\frac{dx\left(t\right)}{dt}=\left[\xi,x\left(t\right)\right]^{(A(t)s)}-\left(\rho^{(s)}_{A(t)}\right)^{-1}\left[\xi,A\left(t\right),\eta\right]^{(s)}. \tag{2.36}$$

This is an inhomogeneous linear first order ODE. The homogeneous part is precisely (2.27), and hence we obtain precisely (2.31).

For the estimate, suppose $\mathbb{L}$ is compact. Then, using (2.31) and (2.30), we have

$$\begin{aligned}\left|\mathrm{Ad}^{(s)}_{A(t)}-U^{(s)}_{t\xi}\right|&\leq|\xi|\int_0^t\left|U^{(s)}_{t\xi}\left(U^{(s)}_{\tau\xi}\right)^{-1}\right|\left|\left(\rho^{(s)}_{A(\tau)}\right)^{-1}\left(\left[\cdot,A\left(\tau\right),\cdot\right]^{(s)}\right)\right|d\tau\\&\leq|\xi|\int_0^t e^{C|\xi|(t-\tau)}\left|\left(\rho^{(s)}_{A(\tau)}\right)^{-1}\left(\left[\cdot,A\left(\tau\right),\cdot\right]^{(s)}\right)\right|d\tau.\end{aligned}$$

However, $(A,s)\mapsto\left|\rho_A^{-1}\left(\left[\cdot,A,\cdot\right]^{(s)}\right)\right|$ is a real-valued function on a compact manifold, and hence there exists a constant C' which is the supremum of this function over $\mathbb{L}\times\mathbb{L}$. Hence,

$$\begin{aligned}\left|\mathrm{Ad}^{(s)}_{A(t)}-U^{(s)}_{t\xi}\right|&\leq C'|\xi|\int_0^t e^{C|\xi|(t-\tau)}d\tau\\&=\frac{C'}{C}\left(e^{C|\xi|t}-1\right).\end{aligned}$$

Renaming the constant C, we get (2.32).

Now if $(\mathbb{L},\circ_s)$ is left-power-alternative, the second term on the right hand side of (2.36) vanishes, since

$$\left[\xi,A\left(t\right),\eta\right]^{(s)}=\left[\xi,\exp_s\left(t\xi\right),\eta\right]^{(s)}=0.$$

Then, $\mathrm{Ad}^{(s)}_{A(t)}\,\eta$ satisfies the homogeneous equation, so the solution is just $\mathrm{Ad}^{(s)}_{A(t)}\,\eta=U^{(s)}_{t\xi}\eta$.

Suppose now $(\mathbb{L},\circ_s)$ is both left-power-alternative and right-power-alternative. From (2.19), we have

$$\left[\xi,x\left(t\right)\right]^{(A(t)s)}=\left[\xi,x\left(t\right)\right]^{(s)}+\left(\rho^{(s)}_{A(t)}\right)^{-1}a_s\left(\xi,x\left(t\right),A\left(t\right)\right).$$

If $(\mathbb{L}, \circ_s)$ is left-power-alternative, then first of all, the associator is skew-symmetric in the first two entries, so

$$a_s(\xi, x(t), A(t)) = -2[x(t), \xi, A(t)]^{(s)},$$

however due to right-power-alternativity, $\left[x(t), \xi, \exp_s(t\xi)\right]^{(s)} = 0$. Hence, in this case,

$$[\xi, x(t)]^{(A(t)s)} = [\xi, x(t)]^{(s)}.$$

Then, $x(t)$ satisfies the first order homogeneous ODE with constant coefficients:

$$\frac{dx(t)}{dt} = \mathrm{ad}_{\xi}^{(s)}(x(t)),$$

so $U_{\xi}(t) = \exp\left(t\,\mathrm{ad}_{\xi}^{(s)}\right)$ and hence the solution is now

$$x(t) = \exp\left(t\,\mathrm{ad}_{\xi}^{(s)}\right)\eta.$$

□

Corollary 2.30 *If* $(\mathbb{L}, \circ_s)$ *is power-associative, then*

1. $\mathrm{Ad}_{\exp_s(t\xi)}^{(s)}\xi = \xi$,
2. $\exp_s(2t\xi) = \exp_s(t\xi) \circ_s \exp_s(t\xi)$.

Proof From (2.31),

$$\mathrm{Ad}_{A(t)}^{(s)}\xi = U_{t\xi}^{(s)}\xi + U_{t\xi}^{(s)}\int_0^t \left(U_{t\xi}^{(s)}\right)^{-1}\rho_{A(\tau)}^{-1}\left([\xi, A(\tau), \xi]^{(s)}\right)d\tau. \tag{2.37}$$

However, by power-associativity (2.17),

$$[\xi, A(\tau), \xi]^{(s)} = \left[\xi, \exp_s(\tau\xi), \xi\right]^{(s)} = 0.$$

Hence, using Lemma 2.27, we obtain

$$\mathrm{Ad}_{A(t)}^{(s)}\xi = U_{t\xi}^{(s)}\xi = \xi.$$

For the second part, define

$$r(t) = \exp_s(t\xi) \circ \exp_s(t\xi).$$

Then, informally, we write

$$\begin{aligned}\frac{dr(t)}{dt} &= \exp_s(t\xi) \circ_s \left(\xi \circ_s \exp_s(t\xi)\right) \\ &\quad + \left(\xi \circ_s \exp_s(t\xi)\right) \circ_s \exp_s(t\xi).\end{aligned}$$

Now using power-associativity, we see that ξ associates with $\exp_s(t\xi)$ and since $\mathrm{Ad}^{(s)}_{\exp_s(t\xi)}\xi = \xi$, moreover ξ and $\exp_s(t\xi)$ commute. Hence, we can rewrite

$$\frac{dr(t)}{dt} = 2\xi \circ_s \left(\exp_s(t\xi) \circ_s \exp_s(t\xi)\right) = 2\xi \circ_s r(t).$$

The solution is thus

$$r(t) = \exp_s(2t\xi),$$

so by uniqueness of solutions we then have the needed equality. □

Remark 2.31 Corollary 2.30 thus shows that indeed, power-associativity allows to extend $\exp_s(t\xi)$ for all t. This is a slightly different proof of this fact compared to [36, 42]. The result from Corollary 2.30 also allows to conclude that if $(\mathbb{L}, \circ_s)$ is power-associative, then

$$\exp_s(t\xi) \circ_s \exp_s(\tau\xi) = \exp_s((t+\tau)\xi). \tag{2.38}$$

Theorem 2.32 *Suppose $\xi(t)$ is a path in $\mathfrak{l}$, with derivative $\dot{\xi}(t)$, then*

$$\frac{d}{dt}\left(\exp_s(\xi(t))\right) = \rho^{(s)}_{\exp_s(\xi(t))} U^{(s)}_{\xi(t)} \int_0^1 \left(U^{(s)}_{\tau\xi(t)}\right)^{-1} \dot{\xi}(t)\, d\tau.$$

Moreover,

1. *If $(\mathbb{L}, \circ_s)$ is left-power-alternative, then*

$$\frac{d}{dt}\left(\exp_s(\xi(t))\right) = \lambda^{(s)}_{\exp_s(\xi(t))} \int_0^1 \left(\mathrm{Ad}^{(s)}_{\exp_s \tau\xi(t)}\right)^{-1} \dot{\xi}(t)\, d\tau.$$

2. *If $(\mathbb{L}, \circ_s)$ is both left-power-alternative and right-power-alternative, then*

$$\frac{d}{dt}\left(\exp_s(\xi(t))\right) = \lambda^{(s)}_{\exp_s(\xi(t))} \int_0^1 \exp\left(-\tau\, \mathrm{ad}^{(s)}_{\xi(t)}\right) \dot{\xi}(t)\, d\tau.$$

Proof Using a similar approach as in the Lie group case, let

$$\Gamma(\tau, t) = \left(\rho^{(s)}_{\exp_s(\tau\xi(t))}\right)^{-1} \frac{\partial}{\partial t} \exp_s(\tau\xi(t)).$$

We can write this (somewhat informally) as

$$\Gamma(\tau,t)=\left(\frac{\partial}{\partial t}\exp_s(\tau\xi(t))\right)/_s\exp_s(\tau\xi(t)) \tag{2.39}$$

Note that $\Gamma(0,t)=0$. Then, consider

$$\begin{aligned}\frac{\partial\Gamma}{\partial\tau}&=\left(\frac{\partial}{\partial t}\left(\xi(t)\circ_s\exp_s(\tau\xi(t))\right)\right)/_s\exp_s(\tau\xi(t))\\&\quad-\left(\Gamma(\tau,t)\circ_s\left(\xi(t)\circ_s\exp_s(\tau\xi(t))\right)\right)/_s\exp_s(\tau\xi(t))\\&=\frac{\partial\xi(t)}{\partial t}+\left(\xi(t)\circ_s\frac{\partial}{\partial t}\exp_s(\tau\xi(t))\right)/_s\exp_s(\tau\xi(t))\\&\quad-\Gamma(\tau,t)\circ_{\exp_s(\tau\xi(t))s}\xi(t)\\&=\frac{\partial\xi(t)}{\partial t}+[\xi(t),\Gamma(\tau,t)]^{\exp_s(\tau\xi(t))s}.\end{aligned}$$

For each t, the homogeneous part of ODE is precisely (2.27), and since the initial condition is $\Gamma(0,t)=0$, we find that the solution of the inhomogeneous equation is

$$\Gamma(\tau,t)=U^{(s)}_{\tau\xi(t)}\int_0^\tau\left(U^{(s)}_{\tau'\xi(t)}\right)^{-1}\frac{\partial\xi(t)}{\partial t}d\tau'.$$

Setting $\tau=1$, we obtain

$$\left(\rho^{(s)}_{\exp_s(\xi(t))}\right)^{-1}\frac{\partial}{\partial t}\exp_s(\xi(t))=U^{(s)}_{\xi(t)}(1)\int_0^1U^{(s)}_{\xi(t)}(\tau')^{-1}\frac{\partial\xi(t)}{\partial t}d\tau'.$$

The special cases now follow immediately from (2.33) and (2.34). □

Corollary 2.33 *Let* $\xi,\eta\in\mathfrak{l}$, *then*

$$\left(\rho^{(s)}_{\exp_s\xi}\right)^{-1}d\exp_s\big|_\xi(\eta)=U^{(s)}_\xi(1)\int_0^1U^{(s)}_\xi(\tau')^{-1}\eta d\tau'. \tag{2.40}$$

Moreover, if $\mathbb{L}$ *is compact, given a norm on* $\mathfrak{l}$ *and a corresponding operator norm,*

$$\left|\left(\rho^{(s)}_{\exp_s\xi}\right)^{-1}d\exp_s\big|_\xi-\mathrm{id}_{\mathfrak{l}}\right|\le e^{C|\xi|}-1, \tag{2.41}$$

where $C>0$ *is a constant that depends on* $\mathbb{L}$.

Proof The expression (2.40) follows directly from Theorem 2.32. We thus have

$$\left(\rho^{(s)}_{\exp_s\xi}\right)^{-1}d\exp_s\big|_\xi-\mathrm{id}_{\mathfrak{l}}=\int_0^1\left(U^{(s)}_\xi\left(U^{(s)}_{\tau'\xi}\right)^{-1}-\mathrm{id}_l\right)d\tau'. \tag{2.42}$$

From (2.42) we then obtain (2.41). □

Let us now explore the dependence of $\exp_s$ on s. In particular, suppose we have a smooth 1-parameter family $s(\tau)$, with $s(0) = s \in \mathbb{L}$ and $\frac{d}{d\tau} s(\tau)\big|_{\tau=0} = \rho_s(\eta) \in T_s\mathbb{L}$, with $\eta \in \mathfrak{l}$. For each τ, $p(t,\tau) = \exp_{s(\tau)}(t\xi)$ satisfies

$$\begin{cases} \frac{dp(t,\tau)}{dt} = \rho^{(s(\tau))}(\xi)\big|_{p(t,\tau)} \\ p(0,\tau) = 1 \end{cases}. \tag{2.43}$$

Now for each t, $\frac{d}{d\tau} p(t,\tau)\big|_{\tau=0} \in T_{p(t,0)}\mathbb{L}$, so define

$$\begin{aligned} \sigma(t) &= \left(\rho^{(s)}_{p(t,0)}\right)^{-1} \frac{d}{d\tau} p(t,\tau)\bigg|_{\tau=0} \\ &= \frac{d}{d\tau}\left(p(t,\tau)/_s\, p(t,0)\right)\bigg|_{\tau=0}, \end{aligned} \tag{2.44}$$

so that in particular, $\sigma(t) \in \mathfrak{l}$.

Lemma 2.34 *Let* $A(t) = \exp_s(t\xi)$*, then the quantity* $\sigma(t)$ *is given by*

$$\sigma(t) = \left(U^{(s)}_{t\xi} - \mathrm{Ad}^{(s)}_{A(t)}\right)\eta. \tag{2.45}$$

In particular, if $(\mathbb{L}, \circ_s)$ *is left-power-alternative, then* $\sigma(t) = 0$ *for all* t.

Proof From (2.43), we have

$$\begin{aligned} \frac{dp(t,\tau)}{dt} &= \rho^{(s(\tau))}(\xi)\Big|_{p(t,\tau)} \\ &= \frac{d}{d\varepsilon} \exp_{s(\tau)}(\varepsilon\xi) \circ_{s(\tau)} p(t,\tau)\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} p(\varepsilon,\tau) \circ_{s(\tau)} p(t,\tau)\bigg|_{\varepsilon=0}, \end{aligned}$$

and therefore,

$$\begin{aligned} \frac{d\sigma(t)}{dt} &= \frac{d^2}{dt d\tau}\left(p(t,\tau)/_s\, p(t,0)\right)\bigg|_{\tau=0} \\ &= \frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,\tau) \circ_{s(\tau)} p(t,\tau)\right)/_s\, p(t,0)\bigg|_{\varepsilon,\tau=0} \\ &\quad - \frac{d^2}{d\varepsilon d\tau}\left(p(t,\tau)/_s\, p(t,0) \circ_s \left(p(\varepsilon,0) \circ_s p(t,0)\right)\right)/_s\, p(t,0)\bigg|_{\varepsilon,\tau=0}, \end{aligned} \tag{2.46}$$

where we used the derivative of the quotient formula (2.7) and also the fact that $p(0,\tau) = 1$ for all τ. Noting that for any τ, $\frac{d}{d\varepsilon} p(\varepsilon,\tau)\big|_{\varepsilon=0} = \xi$, consider the first term of (2.46):

$$\begin{aligned}
&\frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,\tau)\circ_{s(\tau)}p(t,\tau)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}\\
&\quad=\frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,0)\circ_{s(\tau)}p(t,\tau)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}\\
&\quad=\frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,0)\circ_{s(\tau)}p(t,0)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}\\
&\qquad+\frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,0)\circ_{s}p(t,\tau)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,0)\circ_{s(\tau)}p(t,0)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}\\
&=\left(\rho^{(s)}_{\exp_s(t\xi)}\right)^{-1}\left[\xi,\exp_s(t\xi),\eta\right]^{(s)}\\
&\quad\times\frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,0)\circ_{s}p(t,\tau)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}\\
&=\frac{d^2}{d\varepsilon d\tau}\left(p(\varepsilon,0)\circ_s\left(p(t,\tau)/_s p(t,0)\right)\circ_s p(t,0)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}\\
&=\frac{d^2}{d\varepsilon d\tau}p(\varepsilon,0)\circ_{p(t,0)s}\left(p(t,\tau)/_s p(t,0)\right)\Big|_{\varepsilon,\tau=0}.
\end{aligned}$$

The second term of (2.46) gives

$$\begin{aligned}
&\frac{d^2}{d\varepsilon d\tau}\left(p(t,\tau)/_s p(t,0)\circ_s\left(p(\varepsilon,0)\circ_s p(t,0)\right)\right)/_s p(t,0)\Big|_{\varepsilon,\tau=0}\\
&\quad=\frac{d^2}{d\varepsilon d\tau}\left(p(t,\tau)/_s p(t,0)\right)\circ_{p(t,0)s}p(\varepsilon,0)\Big|_{\varepsilon,\tau=0}.
\end{aligned}$$

Overall, (2.46) becomes

$$\begin{aligned}
\frac{d\sigma(t)}{dt}&=\frac{d^2}{d\varepsilon d\tau}p(\varepsilon,0)\circ_{p(t,0)s}\left(p(t,\tau)/_s p(t,0)\right)\Big|_{\varepsilon,\tau=0}\\
&\quad-\frac{d^2}{d\varepsilon d\tau}\left(p(t,\tau)/_s p(t,0)\right)\circ_{p(t,0)s}p(\varepsilon,0)\Big|_{\varepsilon,\tau=0}\\
&\quad+\left(\rho^{(s)}_{\exp_s(t\xi)}\right)^{-1}\left[\xi,\exp_s(t\xi),\eta\right]^{(s)}\\
&=\left[\xi,\sigma(t)\right]^{(\exp_s(t\xi)s)}+\left(\rho^{(s)}_{\exp_s(t\xi)}\right)^{-1}\left[\xi,\exp_s(t\xi),\eta\right]^{(s)}.
\end{aligned}$$

This is precisely the Eq. (2.36) satisfied by $-\operatorname{Ad}_{A(t)}^{(s)}\eta$, however with initial condition $\sigma(0)=0$. Therefore, the solution is

$$\sigma(t)=\left(U_{t\xi}^{(s)}-\operatorname{Ad}_{A(t)}^{(s)}\right)\eta.$$

If $(\mathbb{L},\circ_s)$ is left-power-associative, then from Theorem 2.29, $\operatorname{Ad}_{A(t)}^{(s)}=U_{t\xi}^{(s)}$, and thus $\sigma(t)=0$ for all t. □

We will assume that group Ψ of pseudoautomorphisms of $\mathbb{L}$ is a finite-dimensional Lie group, and suppose the Lie algebras of Ψ and $H_s=\operatorname{Aut}(\mathbb{L},\circ_s)$ are $\mathfrak{p}$ and $\mathfrak{h}_s$, respectively. In particular, $\mathfrak{h}_s$ is a Lie subalgebra of $\mathfrak{p}$. Also, we will assume that Ψ acts transitively on $\mathbb{L}$. The action of Ψ on $\mathbb{L}$ induces an action of the Lie algebra $\mathfrak{p}$ on $\mathfrak{l}$, which we will denote by $\cdot$.

Definition 2.35 Define the map $\varphi:\mathbb{L}\longrightarrow\mathfrak{l}\otimes\mathfrak{p}^*$ such that for each $s\in\mathbb{L}$ and $\gamma\in\mathfrak{p}$,

$$\varphi_s(\gamma)=\frac{d}{dt}\left(\exp_{\mathfrak{p}}(t\gamma)(s)\right)/s\Big|_{t=0}\in\mathfrak{l}, \tag{2.47}$$

where $\exp_{\mathfrak{p}}$ is the exponential map on the Lie algebra $\mathfrak{p}$.

Lemma 2.36 ([26, Theorem 3.25]) *The map φ as in (2.47) is equivariant with respect to corresponding actions of Ψ, in particular for $h\in\Psi$, $s\in\mathbb{L}$, $\gamma\in\mathfrak{p}$, we have*

$$\varphi_{h(s)}\left((\operatorname{Ad}_h)_*\gamma\right)=\left(h'\right)_*\varphi_s(\gamma). \tag{2.48}$$

Moreover, the image of φ_s is $\mathfrak{l}^{(s)}$ and the kernel is $\mathfrak{h}_s$, and hence, $\mathfrak{p}\cong\mathfrak{h}_s\oplus\mathfrak{l}^{(s)}$.

Lemma 2.37 ([26, Lemmas 3.33 and 3.35]) *Suppose $\xi\in\mathfrak{p}$ and $\eta,\gamma\in\mathfrak{l}$, then*

$$\xi\cdot[\eta,\gamma]^{(s)}=[\xi\cdot\eta,\gamma]^{(s)}+[\eta,\xi\cdot\gamma]^{(s)}+a_s(\eta,\gamma,\varphi_s(\xi)) \tag{2.49a}$$
$$\xi\cdot\varphi_s(\eta)=\eta\cdot\varphi_s(\xi)+\varphi_s\left([\xi,\eta]_{\mathfrak{p}}\right)+[\varphi_s(\xi),\varphi_s(\eta)]^{(s)}. \tag{2.49b}$$

Similarly as for Lie groups, we may define a Killing form $K^{(s)}$ on $\mathfrak{l}^{(s)}$. For $\xi,\eta\in\mathfrak{l}$, we have

$$K^{(s)}(\xi,\eta)=\operatorname{Tr}\left(\operatorname{ad}_{\xi}^{(s)}\circ\operatorname{ad}_{\eta}^{(s)}\right), \tag{2.50}$$

where $\circ$ is just composition of linear maps on $\mathfrak{l}$ and $\operatorname{ad}_{\xi}^{(s)}(\cdot)=[\xi,\cdot]^{(s)}$. Clearly $K^{(s)}$ is a symmetric bilinear form on $\mathfrak{l}$. In [26] it is shown that for $h\in\Psi$, and $\xi,\eta\in\mathfrak{l}$ it satisfies $K^{(h(s))}\left(h'_*\xi,h'_*\eta\right)=K^{(s)}(\xi,\eta)$.

General criteria for a loop algebra to admit a non-degenerate Killing form are currently not known, but it is known [37] that for a *semisimple Malcev* algebra, the

Killing form is non-degenerate. A *Malcev* algebra is the tangent algebra of a Moufang loop, and is an *alternative* algebra that also satisfies the following identity [36, 42]:

$$\left[\xi, \gamma, [\xi, \eta]^{(s)}\right]^{(s)} = \left[[\xi, \gamma, \eta]^{(s)}, \xi\right]^{(s)}. \tag{2.51}$$

Moreover, in this case, $K^{(s)}$ is $\mathfrak{p}$-invariant and $\mathrm{ad}^{(s)}$-invariant [26]. Suppose $s(t) = \exp_s(t\gamma)s$, then from (2.25), we see that generally,

$$\begin{aligned}\frac{dK^{(s(t))}}{dt}(\xi, \xi) &= \frac{d}{dt}\left(\mathrm{Tr}\left(\left[\xi, [\xi, \cdot]^{(s(t))}\right]^{(s(t))}\right)\right) \\ &= 2\,\mathrm{Tr}\left(\left[\xi, a_{s(t)}(\xi, \cdot, \gamma)\right]^{(s(t))}\right).\end{aligned} \tag{2.52}$$

In the special case of $\mathbb{L}$ being a Moufang loop, and thus every $\mathfrak{l}^{(s)}$ being a Malcev algebra, we have the following.

Lemma 2.38 *Suppose $\mathbb{L}$ is a Moufang loop. Then, $K^{(s)}$ is independent of s and for each $\gamma \in \mathfrak{l}$, then map $\mathrm{ad}_{\gamma}^{(s)}$ is skew-adjoint with respect to $K^{(s)}$.*

Proof If $\mathbb{L}$ is Moufang, then any $(\mathbb{L}, \circ_s)$ is also a Moufang loop, and hence for any s, $\mathfrak{l}^{(s)}$ is a Malcev algebra. Since a Malcev algebra is alternative, $a^{(s)}(\cdot,\cdot,\cdot) = 2[\cdot,\cdot,\cdot]^{(s)}$, and is moreover totally skew-symmetric. In particular, the Malcev identity (2.51) can be written as

$$a_s\left(\xi, \gamma, [\xi, \eta]^{(s)}\right) = \left[a_s(\xi, \gamma, \eta), \xi\right]^{(s)}. \tag{2.53}$$

In particular, taking the trace, we get

$$\begin{aligned}\mathrm{Tr}\, a_s\left(\xi, \gamma, [\xi, \cdot]^{(s)}\right) &= \mathrm{Tr}\left[a_s(\xi, \gamma, \cdot), \xi\right]^{(s)} \\ &= -\,\mathrm{Tr}\left[a_s(\xi, \gamma, [\xi, \cdot])\right]^{(s)} \\ &= 0.\end{aligned} \tag{2.54}$$

Then, (2.52) gives $\frac{dK^{(s(t))}}{dt}(\xi, \xi) = 0$. This shows that $K^{(s)}$ is constant on $\mathbb{L}$.

For the second part, from the generalized Jacobi identity (2.22), we obtain

$$\begin{aligned}K^{(s)}\left([\gamma, \eta]^{(s)}, \xi\right) = &-K^{(s)}\left(\eta, [\gamma, \xi]^{(s)}\right) \\ &+ \mathrm{Tr}\left[\eta, a_s(\cdot, \xi, \gamma) + a_s(\xi, \gamma, \cdot) + a_s(\gamma, \cdot, \xi)\right]^{(s)} \\ &+ \mathrm{Tr}\left[\xi, a_s(\cdot, \eta, \gamma) + a_s(\eta, \gamma, \cdot) + a_s(\gamma, \cdot, \eta)\right]^{(s)}.\end{aligned} \tag{2.55}$$

However, for an alternative algebra, this simplifies to

$$K^{(s)}\left(\left[\gamma,\eta\right]^{(s)},\xi\right)=-K^{(s)}\left(\eta,\left[\gamma,\xi\right]^{(s)}\right) \\ +3\operatorname{Tr}\left[\eta,a_s\left(\cdot,\xi,\gamma\right)\right]^{(s)}+3\operatorname{Tr}\left[\xi,a_s\left(\cdot,\eta,\gamma\right)\right]^{(s)}. \tag{2.56}$$

The second line is symmetric in ξ and η, so it is sufficient to consider the case $\xi=\eta$. Indeed, for $\xi=\eta$,using (2.54), this vanishes, so we get

$$K^{(s)}\left(\operatorname{ad}_{\gamma}^{(s)}\eta,\xi\right)=-K^{(s)}\left(\eta,\operatorname{ad}_{\gamma}^{(s)}\xi\right). \tag{2.57}$$

□

Remark 2.39 Note that in Lemma 2.38, we only used the trace of the Malcev identity. The non-degeneracy of the Killing form in a semi-simple Malcev algebra also hinges on the property (2.57), same as for semi-simple Lie algebras. This suggests that weaker conditions could be sufficient for these key properties.

3 Loop-Valued Maps

Let M be a smooth, n -dimensional manifold and let $s:M\longrightarrow\mathbb{L}$ be a smooth map. The map s can be used to define a product on $\mathbb{L}'$ -valued maps from M and a corresponding bracket on $\mathfrak{l}$-valued maps. Indeed, let $A,B:M\longrightarrow\mathbb{L}'$ and $\xi,\eta:M\longrightarrow\mathfrak{l}$ be smooth maps, then at each $x\in M$, define

$$As|_x=A_xs_x\in\mathbb{L} \tag{3.1a}$$

$$A\circ_s B|_x=A_x\circ_{s_x}B_x\in\mathbb{L} \tag{3.1b}$$

$$A/_sB|_x=A_x/_{s_x}B_x\in\mathbb{L} \tag{3.1c}$$

$$[\xi,\eta]^{(s)}\Big|_x=[\xi_x,\eta_x]^{(s_x)}\in\mathfrak{l}. \tag{3.1d}$$

In particular, the bracket $[\cdot,\cdot]^{(s)}$ defines the map $b_s:M\longrightarrow\Lambda^2\mathfrak{l}^*\otimes\mathfrak{l}$ (cf. Definition 2.20) We also have the corresponding associator $[\cdot,\cdot,\cdot]^{(s)}$ and the left-alternating associator map $a_s:M\longrightarrow\Lambda^2\mathfrak{l}^*\otimes\mathfrak{l}^*\otimes\mathfrak{l}$ (cf. Definition 2.21). Similarly, define the map $\varphi_s:M\longrightarrow\mathfrak{p}^*\otimes\mathfrak{l}$ (cf. Definition 2.35).

Then, similarly as for maps to Lie groups, we may define the (right) *Darboux derivative* θ_s of s, which is an $\mathfrak{l}$-valued 1-form on M given by the pull-back $s^*\theta$ of the Maurer–Cartan form on $\mathbb{L}$ [45]. In particular, at every $x\in M$,

$$(\theta_s)|_x=\rho_{s(x)}^{-1}\,ds|_x\,, \tag{3.2}$$

and for any vector $X\in T_xM$

$$(\theta_s)|_x(X)=\theta|_{s(x)}(ds|_x(X))\in\mathfrak{l}.$$

It is then clear that θ_s, being a pullback of θ, satisfies the loop Maurer–Cartan structural equation (2.20). In particular, for any vectors $X, Y \in T_x M$,

$$d\theta_s(X, Y) - [\theta_s(X), \theta_s(Y)]^{(s)} = 0. \tag{3.3}$$

We can then calculate the derivatives of these maps (3.1).

Theorem 3.1 ([26, Theorem 3.51]) *Let M be a smooth manifold and suppose $s \in C^\infty(M, \mathbb{L})$ and $A, B \in C^\infty(M, \mathbb{L}')$, then*

$$d(As) = \rho_s(dA) + \lambda_A(ds) \tag{3.4a}$$

$$d(A \circ_s B) = \rho_B^{(s)}(dA) + \lambda_A^{(s)}(dB) + [A, B, \theta_s]^{(s)} \tag{3.4b}$$

$$d(A/_s B) = \left(\rho_B^{(s)}\right)^{-1} dA - \left(\rho_B^{(s)}\right)^{-1}\left(\rho_{A/_s B}^{(s)} dB\right) \tag{3.4c}$$

$$- \left(\rho_B^{(s)}\right)^{-1}[A/_s B, B, \theta_s]^{(s)}, \tag{3.4d}$$

where as before, λ_A is pushforward of the left multiplication map L_A at identity. Suppose now $\xi, \eta \in C^\infty(M, \mathfrak{l})$, then

$$d[\xi, \eta]^{(s)} = [d\xi, \eta]^{(s)} + [\xi, d\eta]^{(s)} + a_s(\xi, \eta, \theta_s). \tag{3.5}$$

The $\mathfrak{l} \otimes \mathfrak{p}^*$-valued map $\varphi_s : M \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ satisfies

$$d\varphi_s = \mathrm{id}_{\mathfrak{p}} \cdot \theta_s - [\varphi_s, \theta_s]^{(s)}, \tag{3.6}$$

where $\mathrm{id}_{\mathfrak{p}}$ is the identity map of $\mathfrak{p}$ and $\cdot$ denotes the action of the Lie algebra $\mathfrak{p}$ on $\mathfrak{l}$.

The Killing form $K^{(s)} : M \longrightarrow \mathrm{Sym}^2\, \mathfrak{l}^*$ satisfies

$$dK^{(s)}(\xi, \xi) = 2\,\mathrm{Tr}\left([\xi, a_s(\xi, \cdot, \theta_s)]^{(s)}\right). \tag{3.7}$$

Given $A : M \longrightarrow \mathbb{L}'$ and $s : M \longrightarrow \mathbb{L}$, as shown in [26], we have the following expression for θ_{As}

$$\theta_{As} = \mathrm{Ad}_A^{(s)}\theta_s + \left(\rho_A^{(s)}\right)^{-1} dA. \tag{3.8}$$

Moreover, let us consider the evolution equation satisfied by $\theta_{A(t)s}$ for $A(t) = \exp_s(t\xi)$ for some $\xi : M \longrightarrow \mathfrak{l}$. This gives us the following.

Lemma 3.2 *Let $A(t) = \exp_s(t\xi)$, for $\xi \in C^\infty(M, \mathfrak{l})$, then*

$$\frac{d\theta_{A(t)s}}{dt} = \left[\xi, \theta_{A(t)s}\right]^{A(t)s} + d\xi \tag{3.9}$$

and hence

$$\theta_{A(t)s} = U_{t\xi}^{(s)}\theta_s + \left(U_{t\xi}^{(s)}\int_0^t \left(U_{\tau\xi}^{(s)}\right)^{-1} d\tau\right) d\xi, \tag{3.10}$$

where $U_{t\xi}^{(s)}$ *is the evolution operator for Eq.* (2.27)*, as given by* (2.28)*. Moreover, if* $\mathbb{L}$ *is compact and given a metric on* M *and an inner product on* $\mathfrak{l}$,

$$\left|\theta_{A(t)s}\right| \leq e^{Ct|\xi|}\left(|\theta_s| + t\,|d\xi|\right), \tag{3.11}$$

where C *is a constant that depends on* $\mathbb{L}$.

Proof We will write symbolically

$$\frac{dA}{dt} = \xi \circ_s A,$$

so

$$\begin{aligned}\frac{d\theta_{A(t)s}}{dt} &= \frac{d}{dt}\left(d\left(As\right)/As\right)\\ &= d\left(\frac{dA}{dt}s\right)/As\\ &\quad - \left(\left(d\left(As\right)/As\right)\cdot\left(\frac{dA}{dt}s\right)\right)/As\\ &= d\left(\left(\xi \circ_s A\right)s\right)/As\\ &\quad - \left(\theta_{As}\cdot\left(\left(\xi \circ_s A\right)s\right)\right)/As\\ &= d\left(\xi\left(As\right)\right)/As\\ &\quad + \left(\theta_{As}\cdot\left(\xi\left(As\right)\right)\right)/As\\ &= d\xi + \left(\xi \cdot d\left(As\right)\right)/As\\ &\quad -\theta_{As}\circ_{As}\xi\\ &= d\xi + \left[\xi, \theta_{As}\right]^{(As)}\end{aligned}$$

Solving this ODE, with $\theta_{A(0)s} = \theta_s$, we find (3.10). To obtain the estimate, we first have

$$\left|\theta_{A(t)s}\right| \leq \left|U_{t\xi}^{(s)}\right||\theta_s| + |d\xi|\int_0^t \left|U_{t\xi}^{(s)}\left(U_{\tau\xi}^{(s)}\right)^{-1}\right| d\tau,$$

but from (2.30), $\left|U_{t\xi}^{(s)}\right| \leq e^{Ct|\xi|}$ and $\left|U_{t\xi}^{(s)}\left(U_{\tau\xi}^{(s)}\right)^{-1}\right| \leq e^{C(t-\tau)|\xi|}$, so

$$\int_0^t \left|U_{t\xi}^{(s)}\left(U_{\tau\xi}^{(s)}\right)^{-1}\right| d\tau \leq te^{Ct|\xi|}, \tag{3.12}$$

since $t \geq 0$, and hence,

$$\left|\theta_{A(t)s}\right| \leq e^{Ct|\xi|}\left|\theta_s\right| + te^{Ct|\xi|}\left|d\xi\right|,$$

and thus indeed, we obtain (3.11). □

In a very similar fashion we obtain the same results for φ_s. Recall that $\mathfrak{p}$ is the Lie algebra of Ψ.

Lemma 3.3 *Let $A(t) = \exp_s(t\xi)$ and $\gamma \in \mathfrak{p}$, then*

$$\frac{d\varphi_{A(t)s}(\gamma)}{dt} = \left[\xi, \varphi_{A(t)s}(\gamma)\right]^{A(t)s} + \gamma \cdot \xi \tag{3.13}$$

and hence

$$\varphi_{A(t)s}(\gamma) = U_{t\xi}^{(s)}\varphi_s(\gamma) + \left(U_{t\xi}^{(s)}\int_0^t \left(U_{\tau\xi}^{(s)}\right)^{-1} d\tau\right)(\gamma \cdot \xi). \tag{3.14}$$

Moreover, if $\mathbb{L}$ is compact and given a metric on M and an inner product on $\mathfrak{l}$,

$$\left|\varphi_{A(t)s}(\gamma)\right| \leq e^{Ct|\xi|}\left(\left|\varphi_s(\gamma)\right| + t\left|\gamma \cdot \xi\right|\right). \tag{3.15}$$

Thus we see that three important quantities $\theta_{A(t)s}, U_{t\xi}^{(s)}, \varphi_{A(t)s}$ satisfy similar ODEs:

$$\begin{aligned}
\frac{d\theta_{A(t)s}}{dt} &= \left[\xi, \theta_{A(t)s}\right]^{A(t)s} + d\xi \\
\frac{dU_{t\xi}^{(s)}}{dt} &= \left[\xi, U_{t\xi}^{(s)}\right]^{A(t)s} \\
\frac{d\varphi_{A(t)s}}{dt} &= \left[\xi, \varphi_{A(t)s}\right]^{A(t)s} + \mathrm{id}\cdot\xi.
\end{aligned}$$

Suppose we have an affine connection ∇ on M, then by differentiating the above ODEs, we can obtain expressions for derivatives of θ_{As}, $U_{\xi}^{(s)}$, and φ_{As}. However, first we have a helpful technical lemma.

Lemma 3.4 *Suppose M is a smooth manifold and $\mathbb{L}$ a smooth loop, neither necessarily compact. Suppose $A : M \longrightarrow \mathbb{L}'$ and $s : M \longrightarrow \mathbb{L}$, and α is a k-linear map on $\mathfrak{l}$ that depends smoothly on A and s. Then, for $\xi_1, \ldots, \xi_k \in \mathfrak{l}$,*

$$\begin{aligned}
d\alpha(\xi_1, \ldots, \xi_k) = \alpha^{(1)}&(\xi_1, \ldots, \xi_k, \theta_s) \\
&+\alpha^{(2)}(\xi_1, \ldots, \xi_k, \theta_{As}),
\end{aligned} \tag{3.16}$$

where $\alpha^{(1)}$ and $\alpha^{(2)}$ are $(k+1)$-linear maps on $\mathfrak{l}$ that also depend smoothly on A and s.

In particular, given a metric on M and a norm on $\mathfrak{l}$, *we have the following pointwise bound*

$$\left|\nabla^k db_s\right| \leq f^{(k)}(s) \sum_{(i_1,\dots,i_{k+1})\in I} |\theta_s|^{i_1} |\nabla\theta_s|^{i_2} \dots \left|\nabla^k\theta_s\right|^{i_{k+1}} \tag{3.17}$$

where $I = \left\{(i_1,\dots,i_{k+1}) \in \mathbb{N}_0^{k+1},\ \text{such that } \sum_{m=1}^{k+1} m i_m = k+1\right\}$ *and* $f^{(k)} : \mathbb{L} \longrightarrow \mathbb{R}_+$ *is a continuous function for each* k.

Proof Since α depends on s and A, by chain rule we have

$$\begin{aligned} d\alpha &= \frac{\partial\alpha}{\partial s} \circ ds + \frac{\partial\alpha}{\partial A} \circ dA \\ &= \left(\frac{\partial\alpha}{\partial s} \circ \rho_s\right)\left(\rho_s^{-1} ds\right) + \left(\frac{\partial\alpha}{\partial A} \circ \rho_A^{(s)}\right)\left(\left(\rho_A^{(s)}\right)^{-1} dA\right) \\ &= \left(\frac{\partial\alpha}{\partial s} \circ \rho_s + \frac{\partial\alpha}{\partial A} \circ \rho_A^{(s)} \circ \mathrm{Ad}_A^{(s)}\right)\theta_s \\ &\quad + \left(\frac{\partial\alpha}{\partial A} \circ \rho_A^{(s)}\right)\theta_{As}, \end{aligned}$$

where we have used (3.8). So now we can set

$$\begin{aligned} \alpha^{(1)}(\xi_1,\dots,\xi_k,\xi_{k+1}) &= \left(\frac{\partial\alpha(\xi_1,\dots,\xi_k)}{\partial s} \circ \rho_s\right)\xi_{k+1} \\ &\quad + \left(\frac{\partial\alpha(\xi_1,\dots,\xi_k)}{\partial A} \circ \rho_A^{(s)} \circ \mathrm{Ad}_A^{(s)}\right)\xi_{k+1} \\ \alpha^{(2)}(\xi_1,\dots,\xi_k,\xi_{k+1}) &= \left(\frac{\partial\alpha(\xi_1,\dots,\xi_k)}{\partial A} \circ \rho_A^{(s)}\right)\xi_{k+1}. \end{aligned}$$

Thus, we obtain (3.16).

From (3.5), we know that for $\xi, \eta \in \mathfrak{l}$,

$$(db_s)(\xi,\eta) = a_s(\xi,\eta,\theta_s),$$

where the alternating associator a_s is a trilinear form on $\mathfrak{l}$. Hence, we have the following estimate

$$|db_s| \leq C\,|a_s|\,|\theta_s|,$$

where C is some universal constant. However a_s is smooth in s, so $|a_s|$ is in particular a continuous real-valued function on $\mathbb{L}$. Hence we can write

$$|db_s| \leq f^{(0)}(s)\,|\theta_s| \tag{3.18}$$

for some positive real-valued function f^0. Now, as we have just shown,

$$da_s(\xi_1, \xi_2, \xi_3, \xi_4) = a_s^{(1)}(\xi_1, \xi_2, \xi_3, \theta_s),$$

for some 4-linear form $a_s^{(1)}$. Therefore, for a vector field X_1 on M,

$$\begin{aligned}\left(\nabla_{X_1} db_s\right)(\xi, \eta) &= \left(d_{X_1} a_s\right)(\xi, \eta, \theta_s) + a_s\left(\xi, \eta, \nabla_{X_1}\theta_s\right)\\ &= a_s^{(1)}(\xi, \eta, \theta_s, \theta_s(X_1)) + a_s\left(\xi, \eta, \nabla_{X_1}\theta_s\right),\end{aligned}$$

so, we have the following estimate

$$\begin{aligned}|\nabla db| &\leq C\left(\left|a_s^{(1)}\right| |\theta_s|^2 + |a_s| |\nabla\theta_s|\right)\\ &\leq f^{(1)}(s)\left(|\theta_s|^2 + |\nabla\theta_s|\right),\end{aligned} \tag{3.19}$$

where $f^{(1)}(s) = C\max\left(\left|a_s^{(1)}\right|, |a_s|\right)$ is continuous.

Similarly, we obtain the expression for the second derivative of db_s, for a multilinear maps $a_s^{(2)}$ on $\mathfrak{l}$:

$$\begin{aligned}\left(\nabla_{X_2}\nabla_{X_1} db_s\right)(\xi, \eta) = a_s^{(2)}&(\xi, \eta, \theta_s, \theta_s(X_1), \theta_s(X_2)) + a_s^{(1)}\left(\xi, \eta, \nabla_{X_2}\theta_s, \theta_s(X_1)\right)\\ &+ a_s^{(1)}\left(\xi, \eta, \nabla_{X_1}\theta_s, \theta_s(X_2)\right) + a_s^{(1)}\left(\xi, \eta, \theta_s, \nabla_{X_2}\theta_s(X_1)\right)\\ &+ a_s\left(\xi, \eta, \nabla_{X_2}\nabla_{X_1}\theta_s\right),\end{aligned}$$

where X_1 and X_2 are vector fields on M. Hence,

$$\begin{aligned}\left|\nabla^2 db_s\right| &\leq C\left(\left|a_s^{(2)}\right| |\theta_s|^3 + 3\left|a_s^{(1)}\right| |\theta_s| |\nabla\theta_s| + |a_s|\left|\nabla^2\theta_s\right|\right)\\ &\leq f^{(2)}(s)\left(|\theta_s|^3 + |\theta_s| |\nabla\theta_s| + \left|\nabla^2\theta_s\right|\right),\end{aligned}$$

for a continuous function $f^{(2)}$.

Note that in these cases for $k = 0, 1, 2$, we can symbolically write

$$\nabla^k db_s = \sum_{(i_1,\ldots,i_{k+1})\in I} a_{i_1\ldots i_{k+1}}\left((\theta_s)^{i_1}, (\nabla\theta_s)^{i_2}, \ldots, \left(\nabla^k\theta_s\right)^{i_{k+1}}\right) \tag{3.20}$$

where $a_{i_1\ldots i_{k+1}}$ are multilinear maps that depend on s and $I = \Big\{(i_1, \ldots, i_{k+1}) \in \mathbb{N}_0^{k+1}$, such that $\sum_{m=1}^{k+1} m i_m = k+1\Big\}$. Proceeding by induction, consider

$$\begin{aligned}\nabla^{k+1} db_s &= \nabla\sum_{(i_1,\ldots,i_{k+1})\in I} a_{i_1\ldots i_{k+1}}\left((\theta_s)^{i_1}, (\nabla\theta_s)^{i_2}, \ldots, \left(\nabla^k\theta_s\right)^{i_{k+1}}\right)\\ &= \sum_{(i_1,\ldots,i_{k+1})\in I} a^{(0)}_{i_1\ldots i_{k+1}}\left((\theta_s)^{i_1+1}, (\nabla\theta_s)^{i_2}, \ldots, \left(\nabla^k\theta_s\right)^{i_{k+1}}\right)\end{aligned}$$

$$+\sum_{(i_1,\ldots,i_{k+1})\in I} a^{(1)}_{i_1\ldots i_{k+1}}\left((\theta_s)^{i_1-1},(\nabla\theta_s)^{i_2+1},\ldots,\left(\nabla^k\theta_s\right)^{i_{k+1}}\right)$$
$$+\cdots+\sum_{(i_1,\ldots,i_{k+1})\in I} a^{(k+1)}_{i_1\ldots i_{k+1}}\left((\theta_s)^{i_1},(\nabla\theta_s)^{i_2},\ldots,\left(\nabla^{k-1}\theta_s\right)^{i_k-1},\left(\nabla^k\theta_s\right)^{i_{k+1}+1}\right)$$
$$+\sum_{(i_1,\ldots,i_{k+1})\in I} a^{(k+2)}_{i_1\ldots i_{k+1}}\left((\theta_s)^{i_1},(\nabla\theta_s)^{i_2},\ldots,\left(\nabla^k\theta_s\right)^{i_{k+1}-1},\nabla^{k+1}\theta_s\right) \quad (3.21)$$

where $a^{(j)}_{i_1\ldots i_{k+1}}$ are new multilinear forms. The form $a^{(0)}_{i_1\ldots i_{k+1}}$ is obtained from $\nabla a_{i_1\ldots i_{k+1}}$, and adds another θ_s. Note that since $\sum_{m=1}^{k+1} mi_m = k+1$, replacing i_1 with i_1+1, increases this sum by 1.

The remaining terms in (3.21) are obtained from differentiating the derivatives of θ_s. Symbolically,

$$\nabla\left(\left(\nabla^j\theta\right)^{i_{j+1}}\right)\sim\left(\nabla^j\theta\right)^{i_{j+1}-1}\left(\nabla^{j+1}\theta\right),$$

so differentiation of each term decreases the power of $\nabla^j\theta$ by one, and adds another $\nabla^{j+1}\theta$. Again, in the sum $\sum_{m=1}^{k+1} mi_m = k+1$, replacing $ji_j+(j+1)\,i_{j+1}$ with $j\left(i_j-1\right)+(j+1)\left(i_{j+1}+1\right)$ increases the sum by 1.

Overall we can then rewrite (3.21) in the form

$$\nabla^{k+1}db_s=\sum_{(i_1,\ldots,i_{k+1},i_{k+2})\in I'} a_{i_1\ldots i_{k+2}}\left((\theta_s)^{i_1},(\nabla\theta_s)^{i_2},\ldots,\left(\nabla^{k+1}\theta_s\right)^{i_{k+2}}\right)$$

where $I'=\left\{(i_1,\ldots,i_{k+2})\in\mathbb{N}_0^{k+2},\ \text{such that } \sum_{m=1}^{k+2} mi_m=k+2\right\}$, hence proving the inductive step. The estimate then follows immediately. □

Lemma 3.5 *Suppose now M is a smooth manifold that is not necessarily compact, but* $\mathbb{L}$ *is a compact smooth loop. Then we have the following pointwise estimates*

$$\left|\nabla^j\theta_{A(t)s}\right|\le Ce^{C(j+1)t|\xi|}p_j(t),$$

where

$$p_j(t)=t\left|\nabla^j d\xi\right|+\sum_{J_j} t^{k_1+\cdots+k_j}\left|\theta_s\right|^{i_1}\left|\nabla\theta_s\right|^{i_2}\ldots\left|\nabla^j\theta_s\right|^{i_j}$$
$$\times\left|d\xi\right|^{k_1}\left|\nabla d\xi\right|^{k_2}\ldots\left|\nabla^{j-1}d\xi\right|^{k_j}, \quad (3.22)$$

with $J_j=\left\{\left(i_1,\ldots,i_j,k_1,\ldots,k_j\right)\in\mathbb{N}_0^{2j}:\sum_{m=1}^{j} mi_m+\sum_{m'=1}^{j} m'k_{m'}=j+1\right\}$.

In particular, for $k=1$ and $k=2$, we have

$$\left|\nabla\theta_{A(t)s}\right| \le Ce^{2Ct|\xi|}\left(|\theta_s|^2 + t\,|d\xi|\,|\theta_s| + t^2\,|d\xi|^2 + |\nabla\theta_s| + t\,|\nabla d\xi|\right) \tag{3.23}$$

$$\left|\nabla^2\theta_{A(t)s}\right| \le Ce^{3Ct|\xi|}\left(\left|\nabla^2\theta_s\right| + |\nabla\theta_s|\,|\theta_s| + |\theta_s|^3 + t\,|\nabla\theta_s|\,|d\xi| + t\,|\theta_s|^2\,|d\xi|\right.$$
$$\left. + t\,|\theta_s|\,|\nabla d\xi| + t^2\,|\theta_s|\,|d\xi|^2 + t^3\,|d\xi|^3 + t\left|\nabla^2 d\xi\right|\right). \tag{3.24}$$

Proof By differentiating (3.9), we see that the k-th covariant derivative of $\theta_{A(t)s}$ satisfies the following initial value problem

$$\frac{d\nabla^k\theta_{A(t)s}}{dt} = \sum_{k_1+k_2+k_3=k}\left(\nabla^{k_1}b_{A(t)s}\right)\left(\nabla^{k_2}\xi,\nabla^{k_3}\theta_{A(t)s}\right) + \nabla^k d\xi$$
$$\nabla^k\theta_{A(0)s} = \nabla^k\theta_s$$

for $k_1, k_2, k_3 \ge 0$. In particular,

$$\frac{d\nabla^k\theta_{A(t)s}}{dt} = \left[\xi,\nabla^k\theta_{A(t)s}\right]^{A(t)s} + \sum_{\substack{k_1+k_2+k_3=k\\ k_3<k}}\left(\nabla^{k_1}b_{A(t)s}\right)\left(\nabla^{k_2}\xi,\nabla^{k_3}\theta_{A(t)s}\right) + \nabla^k d\xi,$$

and thus the solution of the ODE is

$$\nabla^k\theta_{A(t)s} = U_{t\xi}^{(s)}\nabla^k\theta_s + \left(U_{t\xi}^{(s)}\int_0^t\left(U_{\tau\xi}^{(s)}\right)^{-1}d\tau\right)\nabla^k d\xi \tag{3.25}$$
$$+ U_{t\xi}^{(s)}\int_0^t\left(U_{\tau\xi}^{(s)}\right)^{-1}\sum_{\substack{k_1+k_2+k_3=k\\ k_3<k}}\left(\nabla^{k_1}b_{A(\tau)s}\right)\left(\nabla^{k_2}\xi,\nabla^{k_3}\theta_{A(\tau)s}\right)d\tau.$$

To estimate $\left|\nabla^k\theta_{A(t)s}\right|$, consider

$$\left|U_{t\xi}^{(s)}\nabla^k\theta_{A(t)s}\right| \le e^{Ct|\xi|}\left|\nabla^k\theta_s\right|$$
$$\left|\left(U_{t\xi}^{(s)}\int_0^t\left(U_{\tau\xi}^{(s)}\right)^{-1}d\tau\right)\nabla^k d\xi\right| \le te^{Ct|\xi|}\left|\nabla^k d\xi\right|$$

using (2.30) and (3.12), and moreover,

$$\left|\left(\nabla^{k_1}b_{A(\tau)s}\right)\left(\nabla^{k_2}\xi,\nabla^{k_3}\theta_{A(\tau)s}\right)\right| \le \left|\nabla^{k_1}b_{A(\tau)s}\right|\left|\nabla^{k_2}\xi\right|\left|\nabla^{k_3}\theta_{A(\tau)s}\right|.$$

From (3.17), and using the fact that $\mathbb{L}$ is compact, we know that

$$\left|\nabla^{k_1}b_{A(\tau)s}\right| \le C\sum_{(i_1,\ldots,i_{k_1})\in I_{k_1}}\left|\theta_{A(\tau)s}\right|^{i_1}\left|\nabla\theta_{A(\tau)s}\right|^{i_2}\ldots\left|\nabla^{k_1-1}\theta_{A(\tau)s}\right|^{i_{k_1}} \tag{3.26}$$

where $I_{k_1} = \left\{ \left(i_1, \ldots, i_{k_1}\right) \in \mathbb{N}^{k_1}, \text{ such that } \sum_{m=1}^{k_1} m i_m = k_1 \right\}$ and C is a constant that depends only on $\mathbb{L}$.

To proceed with an induction argument, we now need to complete the base step. First, from (3.11), we know that

$$\left|\theta_{A(t)s}\right| \leq e^{Ct|\xi|} \left(|\theta_s| + t\,|d\xi|\right).$$

Thus, for $k = 1$,

$$\begin{aligned}
\left|\nabla\theta_{A(t)s}\right| &\leq e^{Ct|\xi|} \left(|\nabla\theta_s| + t\,|\nabla d\xi|\right) \\
&\quad + Ce^{Ct|\xi|} \int_0^t e^{-C\tau|\xi|} \left(\left|b_{A(\tau)s}\right| |d\xi| \left|\theta_{A(\tau)s}\right| + \left|db_{A(\tau)s}\right| |\xi| \left|\theta_{A(\tau)s}\right|\right) d\tau \\
&\leq e^{Ct|\xi|} \left(|\nabla\theta_s| + t\,|\nabla d\xi|\right) \\
&\quad + Ce^{Ct|\xi|} \int_0^t e^{-C\tau|\xi|} \left(|d\xi| \left|\theta_{A(\tau)s}\right| + \left|\theta_{A(\tau)s}\right|^2 |\xi|\right) d\tau
\end{aligned}$$

where we used the estimate $\left|db_{A(\tau)s}\right| \leq C \left|\theta_{A(\tau)s}\right|$ (3.18). Moreover,

$$\begin{aligned}
\int_0^t e^{-C\tau|\xi|} |\nabla\xi| \left|\theta_{A(\tau)s}\right| d\tau &\leq \int_0^t |\nabla\xi| \left(|\theta_s| + \tau\,|d\xi|\right) d\tau \\
&\leq t\,|\nabla\xi|\,|\theta_s| + t^2\,|d\xi|^2 \\
\int_0^t e^{-C\tau|\xi|} \left|\theta_{A(\tau)s}\right|^2 |\xi|\, d\tau &\leq \int_0^t e^{C\tau|\xi|} \left(\left(|\theta_s| + \tau\,|d\xi|\right)^2 |\xi|\right) d\tau \\
&\leq \frac{1}{C} e^{Ct|\xi|} \left(|\theta_s| + t\,|d\xi|\right)^2.
\end{aligned}$$

Hence, for some new overall constant C, we have

$$\left|\nabla\theta_{A(t)s}\right| \leq Ce^{2Ct|\xi|} \left(|\theta_s|^2 + t\,|d\xi|\,|\theta_s| + t^2\,|d\xi|^2 + t\,|\nabla d\xi| + |\nabla\theta_s|\right). \quad (3.27)$$

Let us also complete the $k = 2$ case.

$$\begin{aligned}
\left|\nabla^2\theta_{A(t)s}\right| &\leq e^{Ct|\xi|} \left(\left|\nabla^2\theta_s\right| + t\left|\nabla^2 d\xi\right|\right) \\
&\quad + Ce^{Ct|\xi|} \int_0^t e^{-C\tau|\xi|} \left(\sum_{k_1+k_2+k_3=2} \left|\nabla^{k_1} b_{A(\tau)s}\right| \left|\nabla^{k_2}\xi\right| \left|\nabla^{k_3}\theta_{A(\tau)s}\right| \right) d\tau.
\end{aligned}$$

More explicitly,

$$\sum_{\substack{k_1+k_2+k_3=2\\k_3<2}} \left|\nabla^{k_1} b_{A(\tau)s}\right| \left|\nabla^{k_2}\xi\right| \left|\nabla^{k_3}\theta_{A(\tau)s}\right| = \left|\nabla d b_{A(\tau)s}\right| |\xi| \left|\theta_{A(\tau)s}\right| \\ + \left|d b_{A(\tau)s}\right| |d\xi| \left|\theta_{A(\tau)s}\right| \\ + \left|d b_{A(\tau)s}\right| |\xi| \left|\nabla\theta_{A(\tau)s}\right| \\ + \left|b_{A(\tau)s}\right| |\nabla d\xi| \left|\theta_{A(\tau)s}\right|.$$

Let

$$p_0(\tau) = |\theta_s| + \tau\, |d\xi| \\ p_1(\tau) = |\theta_s|^2 + \tau\, |d\xi|\, |\theta_s| + \tau^2 |d\xi|^2 + |\nabla\theta_s| + \tau\, |\nabla d\xi|$$

so that

$$\left|\theta_{A(\tau)s}\right| \le e^{C\tau|\xi|} p_0(\tau) \\ \left|\nabla\theta_{A(t)s}\right| \le C e^{2C\tau|\xi|} p_1(\tau).$$

Using also (3.18) and (3.19) we have

$$\left|\nabla d b_{A(\tau)s}\right| |\xi| \left|\theta_{A(\tau)s}\right| \le C e^{3C|\xi|\tau} |\xi|\, p_0 \left(p_0^2 + p_1\right) \\ \left|d b_{A(\tau)s}\right| |d\xi| \left|\theta_{A(\tau)s}\right| \le C e^{2C|\xi|\tau} p_0^2 |d\xi| \\ \left|d b_{A(\tau)s}\right| |\xi| \left|\nabla\theta_{A(\tau)s}\right| \le C e^{2C|\xi|\tau} |\xi|\, p_0 p_1 \\ \left|b_{A(\tau)s}\right| |\nabla d\xi| \left|\theta_{A(\tau)s}\right| \le C e^{C|\xi|\tau} |\nabla d\xi|\, p_0.$$

Since p_0 and p_1 are non-decreasing functions of τ, we can evaluate them at $\tau = t$. In particular,

$$e^{Ct|\xi|} \int_0^t e^{-C\tau|\xi|} \left|\nabla d b_{A(\tau)s}\right| |\xi| \left|\theta_{A(\tau)s}\right| d\tau \le C\, |\xi|\, p_0 \left(p_0^2 + p_1\right) e^{Ct|\xi|} \int_0^t e^{2C\tau|\xi|} d\tau \\ \le C e^{3C\tau|\xi|} p_0 \left(p_0^2 + p_1\right) \\ e^{Ct|\xi|} \int_0^t e^{-C\tau|\xi|} \left|d b_{A(\tau)s}\right| |d\xi| \left|\theta_{A(\tau)s}\right| d\tau \le Cte^{2|\xi|t} p_0^2 |d\xi| \le Cte^{3|\xi|t} p_0^2 |d\xi|,$$

and similarly for other terms. Overall, we find

$$\left|\nabla^2\theta_{A(t)s}\right| \le C e^{3Ct|\xi|} \left(p_0\left(p_0^2 + p_1\right) + t p_0^2 |d\xi| + p_0 p_1 + t\, |\nabla d\xi|\, p_0 + \left|\nabla^2\theta_s\right| \\ + t\left|\nabla^2 d\xi\right|\right) \\ \le C e^{3Ct|\xi|} \left(\left|\nabla^2\theta_s\right| + |\nabla\theta_s|\, |\theta_s| + |\theta_s|^3 + t\, |\nabla\theta_s|\, |d\xi| + t\, |\theta_s|^2 |d\xi| \\ + t\, |\theta_s|\, |\nabla d\xi| + t\left|\nabla^2 d\xi\right| + t^2 |\theta_s|\, |d\xi|^2 + t^3 |d\xi|^3\right).$$

Suppose for each $j < k$,

$$\left|\nabla^{j}\theta_{A(t)s}\right| \leq Ce^{(j+1)Ct|\xi|}p_{j}(t),$$

where

$$p_{j}(t) = t\left|\nabla^{j}d\xi\right| + \sum_{J_j} t^{k_1+\cdots+k_j}|\theta_s|^{i_1}|\nabla\theta_s|^{i_2}\ldots\left|\nabla^{j}\theta_s\right|^{i_{j+1}}$$
$$\times|d\xi|^{k_1}|\nabla d\xi|^{k_2}\ldots\left|\nabla^{j-1}d\xi\right|^{k_j},$$

where $J_j = \left\{\left(i_1,\ldots,i_{j+1},k_1,\ldots,k_{j+1}\right)\in\mathbb{N}^{2j+2} : \sum_{m=1}^{j+1} mi_m + \sum_{m=1}^{j} mk_m = j+1\right\}$.

Therefore,

$$\left|\sum_{\substack{k_1+k_2+k_3=k\\k_3<k}}\left(\nabla^{k_1}b_{A(\tau)s}\right)\left(\nabla^{k_2}\xi,\nabla^{k_3}\theta_{A(\tau)s}\right)\right|$$
$$\leq C\sum_{(i_0,i_1,\ldots,i_k)\in I'_k}\left|\nabla^{i_0}\xi\right|\left|\theta_{A(\tau)s}\right|^{i_1}\left|\nabla\theta_{A(\tau)s}\right|^{i_2}\ldots\left|\nabla^{k-1}\theta_{A(\tau)s}\right|^{i_k}$$
$$\leq C\sum_{(i_0,i_1,\ldots,i_k)\in I'_k}e^{C(k+1-i_0)\tau|\xi|}\left|\nabla^{i_0}\xi\right|p_0^{i_1}p_1^{i_2}\ldots p_{k_1-1}^{i_{k_1}}$$
$$\leq C\sum_{\substack{(i_0,i_1,\ldots,i_k)\in I'_k\\i_0>0}}e^{C(k+1-i_0)\tau|\xi|}\left|\nabla^{i_0}\xi\right|p_0^{i_1}p_1^{i_2}\ldots p_{k-1}^{i_k}$$
$$+C\sum_{(0,i_1,\ldots,i_k)\in I'_k}e^{C(k+1)\tau|\xi|}|\xi|p_0^{i_1}p_1^{i_2}\ldots p_{k_1-1}^{i_{k_1}},$$

where $I'_k = \left\{(i_0,i_1,\ldots,i_k)\in\mathbb{N}_0^{k+1},\ \text{such that}\ \sum_{m=1}^{k} mi_m + i_0 = k+1\right\}$.

Now, from (3.25), we find

$$\left|\nabla^{k}\theta_{A(t)s}\right| \leq Ce^{Ct|\xi|}\left|\nabla^{k}\theta_s\right| + te^{Ct|\xi|}\left|\nabla^{k}d\xi\right|$$
$$+Ce^{Ct|\xi|}\int_0^t\sum_{\substack{(i_1,\ldots,i_{k_1})\in I_{k_1}\\k_1+k_2=k,\ k_2>0}}e^{Ck_1\tau|\xi|}p_0^{i_1}p_1^{i_2}\ldots p_{k_1-1}^{i_{k_1}}\left|\nabla^{k_2}\xi\right|d\tau$$
$$+Ce^{Ct|\xi|}\int_0^t\sum_{(i_1,\ldots,i_k)\in I_k}e^{Ck\tau|\xi|}p_0^{i_1}p_1^{i_2}\ldots p_{k-1}^{i_k}|\xi|d\tau$$
$$\leq Ce^{Ct|\xi|}\left|\nabla^{k}\theta_s\right| + te^{Ct|\xi|}\left|\nabla^{k}d\xi\right|$$

$$+C \sum_{\substack{(i_1,\ldots,i_{k_1})\in I_{k_1}\\ k_1+k_2=k,\ k_2>0}} te^{C(k_1+1)t|\xi|} p_0^{i_1} p_1^{i_2} \ldots p_{k_1-1}^{i_{k_1}} \left|\nabla^{k_2}\xi\right|$$
$$+C \sum_{(i_1,\ldots,i_k)\in I_k} e^{C(k+1)t|\xi|} p_0^{i_1} p_1^{i_2} \ldots p_{k-1}^{i_k},$$

where we have bounded $p_i(\tau) \leq p_i(t)$, since these functions are non-decreasing. Also, in the first integral, we bounded $\int_0^t e^{Ck_1\tau|\xi|}d\tau \leq te^{Ck_1t|\xi|}$ and in the second integral, we used $\int_0^t e^{Ck\tau|\xi|}|\xi|\,d\tau \leq C'e^{Ck|\xi|}$ for some new constant C'. Further, we can bound

$$\begin{aligned}\left|\nabla^k\theta_{A(t)s}\right| &\leq Ce^{C(k+1)t|\xi|}\Bigg(\left|\nabla^k\theta_s\right| + \left|\nabla^k d\xi\right| \\ &\quad + \sum_{\substack{(i_1,\ldots,i_{k_1})\in I_{k_1}\\ k_1+k_2=k,\ k_2>0}} tp_0^{i_1} p_1^{i_2} \ldots p_{k_1-1}^{i_{k_1}} \left|\nabla^{k_2}\xi\right| \\ &\quad + \sum_{(i_1,\ldots,i_k)\in I_k} p_0^{i_1} p_1^{i_2} \ldots p_{k-1}^{i_k}\Bigg) \\ &\leq Ce^{C(k+1)t|\xi|} p_k(t).\end{aligned}$$

□

Remark 3.6 In the estimates that follow, to avoid introducing arbitrary constants, we will use the notation $\lesssim$ to denote inequality up to a constant factor, where the constant factor may depend on M and $\mathbb{L}$.

Corollary 3.7 *Under the same assumptions as Lemma 3.5, for $k > 0$, $b_{A(t)s}$ satisfies*

$$\left|\nabla^k b_{A(t)s}\right| \lesssim e^{Ckt|\xi|} p_{k-1}(t),$$

where $p_k(t)$ is given by (3.22).

Proof From (3.17),

$$\left|\nabla^k b_{As}\right| \lesssim \sum_{(i_1,\ldots,i_k)\in I_k} |\theta_{As}|^{i_1} |\nabla\theta_{As}|^{i_2} \ldots \left|\nabla^{k-1}\theta_{As}\right|^{i_k} \tag{3.28}$$

where $I_k = \left\{(i_1,\ldots,i_k) \in \mathbb{N}_0^k,\ \text{such that } \sum_{m=1}^k m i_m = k\right\}$. However, from Lemma 3.5,

$$\left|\nabla^j\theta_{A(t)s}\right| \lesssim e^{C(j+1)t|\xi|} p_j(t),$$

where

$$p_j(t) = t\left|\nabla^j d\xi\right| + \sum_{J_j} t^{k_1+\cdots+k_j} |\theta_s|^{i_1} |\nabla\theta_s|^{i_2}\ldots\left|\nabla^j\theta_s\right|^{i_{j+1}}$$
$$\times |d\xi|^{k_1}|\nabla d\xi|^{k_2}\ldots\left|\nabla^{j-1}d\xi\right|^{k_j},$$

with $J_j = \left\{\left(i_1,\ldots,i_{j+1},k_1,\ldots,k_j\right)\in\mathbb{N}_0^{2j+1} : \sum_{m=1}^{j+1} m i_m + \sum_{m'=1}^{j} m' k_{m'} = j+1\right\}$. So

$$\left|\nabla^k b_{A(t)s}\right| \lesssim e^{Ckt|\xi|}\sum_{(i_1,\ldots,i_k)\in I_k} p_0^{i_1}p_1^{i_2}\ldots p_{k-1}^{i_k}$$
$$\lesssim e^{Ckt|\xi|}p_{k-1}(t)$$

□

More generally, suppose we have 1-parameter family $X(t)$ of $\mathfrak{l}$-valued maps that satisfies

$$\begin{cases}\frac{dX(t)}{dt} = [\xi, X(t)]^{A(t)s} + Y \\ X(0) = X_0,\end{cases} \tag{3.29}$$

where Y is also an $\mathfrak{l}$-valued map. We know that

$$X(t) = U_{t\xi}^{(s)}X_0 + \left(U_{t\xi}^{(s)}\int_0^t\left(U_{\tau\xi}^{(s)}\right)^{-1}d\tau\right)Y, \tag{3.30}$$

and in particular,

$$|X(t)| \lesssim e^{tC|\xi|}\left(|X_0| + t|Y|\right). \tag{3.31}$$

Differentiating (3.29), we obtain estimates for higher derivatives of X.

Lemma 3.8 *Under the same assumptions as Lemma 3.5, suppose $X(t)$ is a 1-parameter family $X(t)$ of $\mathfrak{l}$-valued maps that satisfies (3.29). Then,*

$$\left|\nabla^k X(t)\right| \lesssim e^{(k+1)Ct|\xi|}\sum_{k'+k''=k} p_{k'-1}(t)\left(\left|\nabla^{k''}X_0\right| + t\left|\nabla^{k''}Y\right|\right),$$

with $p_{-1}(t) = 1$ and for $k \geq 0$, $p_k(t)$ is given by (3.22).

Proof Differentiating (3.29), for $k \geq 1$, we get

$$\frac{d\nabla^k X(t)}{dt} = \left[\xi, \nabla^k X(t)\right]^{A(t)s} + \sum_{\substack{k_1+k_2+k_3=k\\k_3<k}}\left(\nabla^{k_1}b_{A(t)s}\right)\left(\nabla^{k_2}\xi, \nabla^{k_3}X(t)\right) + \nabla^k Y,$$

and hence,

$$\nabla^k X(t) = U_{t\xi}^{(s)}\left(\nabla^k X_0\right) + \left(U_{t\xi}^{(s)}\int_0^t \left(U_{\tau\xi}^{(s)}\right)^{-1} d\tau\right)\nabla^k Y$$
$$+ \sum_{\substack{k_1+k_2+k_3=k \\ k_3<k}} \left(U_{t\xi}^{(s)}\int_0^t \left(U_{\tau\xi}^{(s)}\right)^{-1}\left(\nabla^{k_1} b_{A(\tau)s}\right)\left(\nabla^{k_2}\xi, \nabla^{k_3} X(\tau)\right) d\tau\right)$$

Let $q_0(t) = |X_0| + t|Y|$, so that

$$|X(t)| \lesssim e^{tC|\xi|} q_0(t).$$

Suppose for all $j < k$,

$$\left|\nabla^j X(t)\right| \lesssim e^{(j+1)Ct|\xi|} q_j(t),$$

where $q_j(t)$ is non-decreasing. Then,

$$\left|U_{t\xi}^{(s)}\int_0^t \left(U_{\tau\xi}^{(s)}\right)^{-1}\left(\nabla^{k_1} b_{A(\tau)s}\right)\left(\nabla^{k_2}\xi, \nabla^{k_3} X(\tau)\right) d\tau\right|$$
$$\lesssim e^{Ct|\xi|}\int_0^t e^{k_3 C\tau|\xi|}\left|\nabla^{k_1} b_{A(\tau)s}\right|\left|\nabla^{k_2}\xi\right| q_{k_3}(\tau)\, d\tau$$
$$\lesssim e^{Ct|\xi|}\int_0^t e^{(k_1+k_3)C\tau|\xi|} p_{k_1-1}\left|\nabla^{k_2}\xi\right| q_{k_3}(\tau)\, d\tau$$

For $k_2 = 0$,

$$e^{Ct|\xi|}\int_0^t e^{k_3 C\tau|\xi|}\left|\nabla^{k_1} b_{A(\tau)s}\right| |\xi|\, q_{k_3}(\tau)\, d\tau \lesssim e^{(k+1)Ct|\xi|} p_{k_1-1} q_{k_3}(t)$$

For $k_2 > 0$,

$$e^{Ct|\xi|}\int_0^t e^{k_3 C\tau|\xi|}\left|\nabla^{k_1} b_{A(\tau)s}\right|\left|\nabla^{k_2}\xi\right| q_{k_3}(\tau)\, d\tau \lesssim e^{(k_1+k_3+1)Ct|\xi|} p_{k_1-1}\left|\nabla^{k_2}\xi\right| q_{k_3}(t)$$
$$\lesssim e^{(k+1)Ct|\xi|} p_{k_1+k_2-1} q_{k_3}(t).$$

Thus,

$$\left|\nabla^k X(t)\right| \lesssim e^{(k+1)Ct|\xi|}\left(\left|\nabla^k X_0\right| + t\left|\nabla^k Y\right| + \sum_{k'+k''=k-1} p_{k'} q_{k''}\right)$$

Therefore, for $k \geq 1$,

$$q_k = \left|\nabla^k X_0\right| + t\left|\nabla^k Y\right| + \sum_{k'+k''=k-1} p_{k'} q_{k''}.$$

Setting $x_k = \left|\nabla^k X_0\right| + t\left|\nabla^k Y\right|$, it is then easy to see that

$$q_k = \sum_{L_k} p_0^{j_0} p_1^{j_1} \dots p_{k-1}^{j_{k-1}} x_l$$

where

$$L_k = \left\{ (j_0, \dots, j_{k-1}, x_l) \in \mathbb{N}_0^{k+2} : l + \sum_{m=0}^{j-1} (m+1)\, j_m = k \right\},$$

and thus

$$q_k \lesssim x_k + \sum_{k'+k''=k-1} p_{k'} x_{k''}.$$

Therefore,

$$\left|\nabla^k X(t)\right| \lesssim e^{(k+1)Ct|\xi|} \sum_{k'+k''=k} p_{k'-1} \left(\left|\nabla^{k''} X_0\right| + t\left|\nabla^{k''} Y\right| \right),$$

with $p_{-1} = 1$. □

We will need to be able to define loop-valued maps with Sobolev regularity. First, let us recall Sobolev spaces $W^{k,p}$ of functions between manifolds.

Lemma 3.9 *Suppose M is a compact n-dimensional manifold and suppose N is an l-dimensional manifold. Let k be a non-negative integer and $r \geq 0$ such that $kr > n$. Let $\Phi : N \longrightarrow \mathbb{R}^{2l}$ be a smooth embedding (by Whitney Embedding Theorem) and suppose $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for N. Suppose $f : M \longrightarrow N$ is a continuous map. Then, the following are equivalent:*

1. $f \in W^{k,r}(M, N)$
2. $\Phi \circ f \in W^{k,r}\left(M, \mathbb{R}^{2l}\right)$
3. $\phi_\alpha \circ f \in W^{k,r}\left(\phi_\alpha^{-1}(U_\alpha), \mathbb{R}^l\right)$ *for any chart* (U_α, ϕ_α)
4. $f^*\theta_G \in W^{k-1,r}(M, T^*M \otimes \mathfrak{g})$ *in case when if $N = G$, a compact Lie group, with Lie algebra $\mathfrak{g}$ and θ_G is the Maurer–Cartan form on G.*

In particular, conditions (2) and (3) are independent of the choice of the embedding Φ and the atlas (U_α, ϕ_α), respectively.

Note that the condition $kr > n$ is needed in Lemma 3.9 due to the Sobolev embedding $W^{k,r} \subset C^0$ for $kr > n$. We will prove a characterization of loop-valued $W^{k,p}$-maps in terms of the loop Maurer–Cartan form that is similar to item (4) in Lemma 3.9.

Lemma 3.10 *Suppose M is a compact n-dimensional manifold and suppose $\mathbb{L}$ is a smooth loop of dimension l (not necessarily compact) with tangent algebra $\mathfrak{l}$ and $\mathfrak{l}$-valued Maurer–Cartan form θ. Let k be a non-negative integer and $r \geq 0$ such that $kr > n$. Suppose $s : M \longrightarrow \mathbb{L}$ is a continuous map. Then, $s \in W^{k,r}(M, \mathbb{L})$ if and only if $\theta_s \in W^{k-1,r}(M, T^*M \otimes \mathfrak{l})$.*

Proof Suppose $s \in W^{k,r}(M, \mathbb{L})$. By Lemma 3.9, if $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for $\mathbb{L}$, then for each chart (U_α, ϕ_α), $\phi_\alpha \circ s \in W^{k,r}\left(\phi_\alpha^{-1}(U_\alpha), \mathbb{R}^l\right)$. Now, $\left\{s^{-1}(U_\alpha)\right\}$ is an open cover of M, but using compactness of M, let $\left\{s^{-1}(U_i)\right\}$ be an finite subcover, and suppose $\{u_i\}$ is a smooth partition of unity subordinate to this subcover. Then, we can write

$$\begin{aligned}\theta_s = \theta s_* &= \sum_i (u_i \theta) s_* \\ &= \sum_i \left((u_i\theta)\left(\phi_i^{-1}\right)_*\right)\left((\phi_i)_* s_*\right).\end{aligned}$$

For each i, $(\phi_i)_* s_* = (\phi_i \circ s)_* \in W^{k-1,r}\left(\phi_i^{-1}(U_i), \mathbb{R}^l\right)$. On the other hand $(u_i\theta)\left(\phi_i^{-1}\right)_*$ is a smooth function, and hence composition with it is a continuous map $W^{k-1,r}\left(\phi_i^{-1}(U_i), \mathbb{R}^l\right) \longrightarrow W^{k-1,r}\left(\phi_i^{-1}(U_i), T^*U_i \otimes \mathfrak{l}\right)$ (using [48, Lemma B.8]). Overall, we see that each term of this finite sum is bounded in the $W^{k-1,r}$ norm, and thus $\theta_s \in W^{k-1,r}(M, T^*M \otimes \mathfrak{g})$.

Conversely, suppose now $\theta_s \in W^{k-1,r}(M, T^*M \otimes \mathfrak{g})$. We will use item (2) in Lemma 3.9 to show that $s \in W^{k,r}(M, \mathbb{L})$. This adapts the proof of [48, Lemma B.5]. Let $\Phi : \mathbb{L} \longrightarrow \mathbb{R}^{2l}$ be a smooth embedding, so that $\Phi \circ s$ is continuous. In particular, $\Phi \circ s \in L^r\left(M, \mathbb{R}^{2l}\right)$. Now, let $x \in M$ and consider

$$\begin{aligned}d(\Phi \circ s)|_x &= d\Phi|_{s(x)}\, ds|_x \\ &= \left(d\Phi|_{s(x)}\, \rho_{s(x)}\right)\left(\rho_{s(x)}^{-1}\, ds|_x\right) \\ &= E(s(x))\,(\theta_s)_x,\end{aligned}$$

where, for each $p \in \mathbb{L}$, we have the linear map $E(p) = d\Phi|_p\, \rho_p : \mathfrak{l} \longrightarrow \mathbb{R}^{2l}$, and the map $p \mapsto E(p)$ is a smooth map from $\mathbb{L}$ to $\mathrm{Hom}\left(\mathfrak{l}, \mathbb{R}^{2l}\right)$. Thus, we can write

$$d(\Phi \circ s) = (E \circ s)\,\theta_s,$$

with $E \circ s$ being bounded in the operator norm, since s is continuous. Hence, there exists a constant $C > 0$, such that

$$\|d(\Phi \circ s)\|_{L^r} \leq C \|\theta_s\|_{L^r} \leq C \|\theta_s\|_{W^{k-1,r}}.$$

This shows that $\Phi \circ s \in W^{1,r}(M, \mathbb{R}^{2l})$. To show further that $\Phi \circ s \in W^{k,r}(M, \mathbb{R}^{2l})$, $k \geq 2$, similar estimates are obtained by considering higher derivatives. □

Theorem 3.11 *Let M be a compact Riemannian manifold, and $\mathbb{L}$ is a compact smooth loop. Let k be a non-negative integer and $r \geq 0$ such that $kr > n = \dim M$. Let $s \in W^{k,r}(M, \mathbb{L})$ and $\xi \in W^{k,r}(M, \mathfrak{l})$, and suppose $A = \exp_s(\xi)$. Then,*

$$\|\theta_{As}\|_{W^{k-1,r}} \lesssim e^{Ck\|\xi\|_{C^0}} \left(\Theta^k + \Theta\right), \tag{3.32}$$

where $\Theta = \|\theta_s\|_{W^{k-1,r}} + \|\xi\|_{W^{k,r}}$.

Similarly, if $X = X(1)$, where $X(t)$ is 1-parameter family of $\mathfrak{l}$-valued maps that satisfies (3.29). Then,

$$\|X\|_{W^{k,r}} \lesssim e^{C(k+1)\|\xi\|_{C^0}} \left(\|X_0\|_{W^{k,r}} + \|Y\|_{W^{k,r}}\right)\left(\Theta^{k+1} + \Theta\right)$$

Proof From Lemma 3.5, for each $j \leq k$, we have the pointwise estimate

$$\left|\nabla^{j-1}\theta_{As}\right| \lesssim e^{Cj|\xi|} p_{j-1},$$

where

$$p_{j-1} = \sum_{J_{j-1}} |\theta_s|^{i_1} |\nabla\theta_s|^{i_2} \cdots \left|\nabla^{j-1}\theta_s\right|^{i_j} |d\xi|^{k_1} |\nabla d\xi|^{k_2} \cdots \left|\nabla^{j-1} d\xi\right|^{k_j}, \tag{3.33}$$

with $J_{j-1} = \left\{(i_1, \ldots, i_j, k_1, \ldots, k_j) \in \mathbb{N}_0^{2j} : \sum_{m=1}^{j} m i_m + \sum_{m=1}^{j} m k_m = j\right\}$. Thus,

$$\left\|\nabla^{j-1}\theta_{As}\right\|_{L^r} \lesssim e^{Cj\|\xi\|_{C^0}} \left\|p_{j-1}\right\|_{L^r}.$$

Now, from Lemma A.1, if $\sum_{i'=1}^{k'} q_{i'} m_{i'} \leq k$, then

$$\left\|\prod_{i'=1}^{k'} A_{i'}^{m_{i'}}\right\|_{L^r} \lesssim \prod_{i'=1}^{k'} \|A_{i'}\|_{W^{k-q_{i'},r}}^{m_{i'}}. \tag{3.34}$$

We can apply this to (3.33), with the weight $q_i = i$ for each $\left|\nabla^i \theta_s\right|$ or $\left|\nabla^{i-1} d\xi\right|$ factor. Then,

$$\begin{aligned}\left\|p_{j-1}\right\|_{L^r} &\lesssim \sum_{J_{j-1}} \prod_{i'=1}^{j-1} \left\|\nabla^{i'-1}\theta_s\right\|_{W^{k-i',r}}^{i_{i'}} \prod_{i''=1}^{j} \left\|\nabla^{i''-1} d\xi\right\|_{W^{k-i'',r}}^{k_{i''}} \\ &= \sum_{J_{j-1}} \|\theta_s\|_{W^{k-1,r}}^{i_1} \cdots \left\|\nabla^{j-1}\theta_s\right\|_{W^{k-(j-1),r}}^{i_{j-1}} \|d\xi\|_{W^{k-1,r}}^{k_1} \cdots \left\|\nabla^{j-1} d\xi\right\|_{W^{k-j,r}}^{k_j}.\end{aligned}$$

Since for each $i' \leq k$, $\left\|\nabla^{i'-1}\theta_s\right\|_{W^{k-i',r}} \lesssim \|\theta_s\|_{W^{k-1,r}}$ and $\left\|\nabla^{i'-1} d\xi\right\|_{W^{k-i',r}} \lesssim \|\xi\|_{W^{k,r}}$, we obtain

$$\left\|p_{j-1}\right\|_{L^r} \lesssim \sum_{J_{j-1}} \|\theta_s\|_{W^{k-1,r}}^{i_1+\cdots+i_{j-1}} \|\xi\|_{W^{k,r}}^{k_1+\cdots+k_j}. \tag{3.35}$$

The right hand-side of (3.35) is thus a polynomial in $\|\theta_s\|_{W^{k-1,r}}$ and $\|\xi\|_{W^{k,r}}$, and from the definition of J_{j-1}, the degree of this polynomial is j and the lowest order terms are $\|\theta_s\|_{W^{k-1,r}}$ and $\|\xi\|_{W^{k,r}}$. Hence, we can write

$$\left\|\nabla^{j-1}\theta_{As}\right\|_{L^r} \lesssim e^{Cj\|\xi\|_{C^0}} \left(\Theta^j + \Theta\right)$$

where $\Theta = \|\theta_s\|_{W^{k-1,r}} + \|\xi\|_{W^{k,r}}$. In particular,

$$\|\theta_{As}\|_{L^r} \lesssim e^{C\|\xi\|_{C^0}}\Theta \lesssim e^{Cj\|\xi\|_{C^0}} \left(\Theta^j + \Theta\right).$$

Now, since

$$\|\theta_{As}\|_{W^{k-1,r}} \lesssim \|\theta_{As}\|_{L^r} + \left\|\nabla^{k-1}\theta_{As}\right\|_{L^r} \lesssim e^{Ck\|\xi\|_{C^0}} \left(\Theta^k + \Theta\right),$$

which gives us (3.32).

Now from Lemma 3.8, for each j,

$$\left|\nabla^j X\right| \lesssim e^{(j+1)C|\xi|} \sum_{j'+j''=j} p_{j'-1} \left(\left|\nabla^{j''} X_0\right| + \left|\nabla^{j''} Y\right|\right),$$

Hence,

$$\begin{aligned}\left\|\nabla^j X\right\|_{L^r} &\lesssim e^{(j+1)C\|\xi\|_{C^0}} \sum_{j'+j''=j} \left\|p_{j'-1}\left(\left|\nabla^{j''} X_0\right| + \left|\nabla^{j''} Y\right|\right)\right\|_{L^r} \\ &\lesssim e^{(j+1)C\|\xi\|_{C^0}} \sum_{J'_{j-1}} \left\| |\theta_s|^{i_1} |\nabla\theta_s|^{i_2} \cdots \left|\nabla^{j-1}\theta_s\right|^{i_{j-1}} |d\xi|^{k_1} |\nabla d\xi|^{k_2} \cdots \left|\nabla^{j-1} d\xi\right|^{k_j} \right. \\ &\quad \left. \times \left(\left|\nabla^{j''} X_0\right| + \left|\nabla^{j''} Y\right|\right)\right\|_{L^r}\end{aligned}$$

where

$$J'_{j-1} = \left\{ \left(i_1, \ldots, i_j, k_1, \ldots, k_j, j''\right) \in \mathbb{N}_0^{2j+1} : \sum_{m=1}^{j} m i_m + \sum_{m=1}^{j} m k_m + j'' = j \right\}.$$

Hence, from Lemma A.1,

$$\begin{aligned}\left\|\nabla^j X\right\|_{L^r} &\lesssim e^{(j+1)C\|\xi\|_{C^0}} \sum_{J'_{j-1}} \left(\left\|\nabla^{j''} X_0\right\|_{W^{k-j'',r}} + \left\|\nabla^{j''} Y\right\|_{W^{k-j'',r}} \right) \\ &\quad \times \prod_{i'=1}^{j-1} \left\|\nabla^{i'-1}\theta_s\right\|_{W^{k-i',r}}^{i_{i'}} \prod_{i''=1}^{j} \left\|\nabla^{i''-1} d\xi\right\|_{W^{k-i'',r}}^{k_{i''}} \\ &\lesssim e^{(j+1)C\|\xi\|_{C^0}} \left(\|X_0\|_{W^{k,r}} + \|Y\|_{W^{k,r}}\right) \sum_{J'_{j-1}} \|\theta_s\|_{W^{k-1,r}}^{i_1+\cdots+i_j} \|\xi\|_{W^{k,r}}^{k_1+\cdots+k_j} \\ &\lesssim e^{C(j+1)\|\xi\|_{C^0}} \left(\|X_0\|_{W^{k,r}} + \|Y\|_{W^{k,r}}\right) \left(\Theta^j + \Theta\right),\end{aligned}$$

similarly as before. Hence, we conclude that

$$\|X\|_{W^{k,r}} \lesssim e^{C(k+1)\|\xi\|_{C^0}} \left(\|X_0\|_{W^{k,r}} + \|Y\|_{W^{k,r}}\right) \left(\Theta^{k+1} + \Theta\right).$$

□

Corollary 3.12 *Under the same assumptions as Theorem 3.11, suppose* $A \in C^0\left(M, \mathbb{L}'\right)$. *Then,* $A \in W^{k,r}\left(M, \mathbb{L}'\right)$ *if and only if* $\theta_{As} \in W^{k-1,r}\left(M, T^*M \otimes \mathfrak{l}\right)$.

Proof The map $\mu : \mathbb{L}' \times \mathbb{L} \longrightarrow \mathbb{L}$ given by $(A, s) \mapsto As$ is smooth, hence the composition with μ is a continuous map from $W^{k,r}\left(M, \mathbb{L}' \times \mathbb{L}\right)$ to $W^{k,r}(M, \mathbb{L})$. If $A \in W^{k,r}\left(M, \mathbb{L}'\right)$, then since $s \in W^{k,r}(M, \mathbb{L}) \subset C^0(M, \mathbb{L})$, $As \in W^{k,r}(M, \mathbb{L})$, and hence from Lemma 3.10, $\theta_{As} \in W^{k-1,r}\left(M, T^*M \otimes \mathfrak{l}\right)$, and thus $\theta_A^{(s)} \in W^{k-1,r}\left(M, T^*M \otimes \mathfrak{l}\right)$.

Conversely, if $\theta_{As} \in W^{k-1,r}\left(M, T^*M \otimes \mathfrak{l}\right)$, then $As \in W^{k,r}(M, \mathbb{L})$. Since right division is a smooth map, and $s \in C^0(M, \mathbb{L})$, we conclude that $A \in W^{k,r}\left(M, \mathbb{L}'\right)$. □

4 Gauge Theory

Let $\mathbb{L}$ be a smooth loop with pseudoautomorphism group being the finite-dimensional Lie group Ψ. Similarly, let $\mathbb{L}'$ be the same underlying set as $\mathbb{L}$ but with the partial action of Ψ, as defined in Sect. 2. Also, let $\mathfrak{p}$ be the Lie algebra of Ψ. Let M be a smooth, finite-dimensional manifold with a Ψ-principal bundle $\pi : \mathcal{P} \longrightarrow M$.

Definition 4.1 Let $s : \mathcal{P} \longrightarrow \mathbb{L}$ be an equivariant map. In particular, given $p \in \mathcal{P}$, the equivalence class $\lfloor p, s_p \rfloor_\Psi$ defines a section of the associated bundle $\mathcal{Q} = \mathcal{P} \times_\Psi \mathbb{L}$, where $\lfloor p, s_p \rfloor_\Psi$ is the equivalence class with respect to the action of Ψ :

$$\left(p, s_p\right) \sim \left(ph, l_{h^{-1}}\left(s_p\right)\right) = \left(ph, s_{ph}\right) \text{ for any } h \in \Psi. \tag{4.1}$$

We will refer to s as *the defining map* or *defining section*.

We will define several associated bundles related to $\mathcal{P}$. As it is well-known, sections of associated bundles are equivalent to equivariant maps. With this in mind, we also give properties of equivariant maps that correspond to sections of these bundles. Let $h \in \Psi$ and, as before, denote by h' the partial action of h.

Bundle	**Equivariant map**	**Equivariance property**
$\mathcal{P}$	$k : \mathcal{P} \longrightarrow \Psi$	$k_{ph} = h^{-1} k_p$
$\mathcal{Q}' = \mathcal{P} \times_{\Psi'} \mathbb{L}'$	$q : \mathcal{P} \longrightarrow \mathbb{L}'$	$q_{ph} = \left(h'\right)^{-1} q_p$
$\mathcal{Q} = \mathcal{P} \times_\Psi \mathbb{L}$	$r : \mathcal{P} \longrightarrow \mathbb{L}$	$r_{ph} = h^{-1}\left(r_p\right)$
$\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$	$\eta : \mathcal{P} \longrightarrow \mathfrak{l}$	$\eta_{ph} = \left(h'\right)_*^{-1} \eta_p$
$\mathfrak{p}_{\mathcal{P}} = \mathcal{P} \times_{(\mathrm{Ad}_\xi)_*} \mathfrak{p}$	$\xi : \mathcal{P} \longrightarrow \mathfrak{p}$	$\xi_{ph} = \left(\mathrm{Ad}_h^{-1}\right)_* \xi_p$
$\mathrm{Ad}\left(\mathcal{P}\right) = \mathcal{P} \times_{\mathrm{Ad}_\Psi} \Psi$	$u : \mathcal{P} \longrightarrow \Psi$	$u_{ph} = h^{-1} u_p h$

(4.2)

Given equivariant maps $q, r : \mathcal{P} \longrightarrow \mathbb{L}'$, define an equivariant product using s, given for any $p \in \mathcal{P}$ by

$$q \circ_s r|_p = q_p \circ_{s_p} r_p. \tag{4.3}$$

Due to Lemma 2.10, the corresponding map $q \circ_s r : \mathcal{P} \longrightarrow \mathbb{L}'$ is equivariant, and hence $\circ_s$ induces a fiberwise product on sections of $\mathcal{Q}'$. Analogously, we define fiberwise quotients of sections of $\mathcal{Q}'$. Similarly, we define an equivariant bracket $[\cdot, \cdot]^{(s)}$ and the equivariant map φ_s (cf. Definition 2.35). Similarly, the Killing form $K^{(s)}$ (2.50) is then also equivariant.

Suppose the principal Ψ-bundle $\mathcal{P}$ has a principal Ehresmann connection given by the decomposition $T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP}$ and the corresponding vertical $\mathfrak{p}$-valued connection 1-form ω. Let S be a smooth manifold with an action of Ψ on it. Given an equivariant map $f : \mathcal{P} \longrightarrow S$, define

$$\nabla^\omega f := f_* \circ \mathrm{proj}_{\mathcal{H}} : T\mathcal{P} \longrightarrow \mathcal{HP} \longrightarrow TS. \tag{4.4}$$

This is then a horizontal map since it vanishes on any vertical vectors. The map $\nabla^\omega f$ is moreover still equivariant, and hence induces a covariant derivative on sections of the associated bundle $\mathcal{P} \times_\Psi S$.

If instead S is a vector space, then this reduces to the usual definition of the covariant derivative of a vector bundle-valued function and $\nabla^\omega f$ is a vector-bundle-valued 1-form. In a standard way we can then define the covariant exterior derivatives on vector-bundle-valued p-forms. This will be denoted by d^ω.

Following [26], let us define the torsion of the defining map s with respect to the connection ω.

Definition 4.2 The *torsion* $T^{(s,\omega)}$ of the defining map s with respect to ω is a horizontal $\mathfrak{l}$-valued 1-form on $\mathcal{P}$ given by $T^{(s,\omega)} = (s^*\theta) \circ \mathrm{proj}_{\mathcal{H}}$, where θ is Maurer–Cartan form of $\mathbb{L}$. Equivalently, at $p \in \mathcal{P}$, we have

$$T^{(s,\omega)}\Big|_p = \left(R_{s_p}^{-1}\right)_* \nabla^\omega s\big|_p \,. \tag{4.5}$$

Thus, $T^{(s,\omega)}$ is the horizontal component of $\theta_s = s^*\theta$. We also easily see that it is Ψ-equivariant. Thus, $T^{(s,\omega)}$ is a *basic* (i.e. horizontal and equivariant) $\mathfrak{l}$-valued 1-form on $\mathcal{P}$, and thus defines a 1-form on M with values in the associated vector bundle $\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$.

Recall that the curvature $F^{(\omega)} \in \Omega^2(\mathcal{P}, \mathfrak{p})$ of the connection ω on $\mathcal{P}$ is given by

$$F^{(\omega)} = d\omega \circ \mathrm{proj}_{\mathcal{H}} = d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{p}} \,, \tag{4.6}$$

where wedge product is implied. Given the defining map s, define $\hat{F}^{(s,\omega)} \in \Omega^2(\mathcal{P}, \mathfrak{l})$ to be the projection of the curvature $F^{(\omega)}$ to $\mathfrak{l}$ with respect to s, such that for any $X_p, Y_p \in T_p\mathcal{P}$,

$$\hat{F}^{(s,\omega)} = \varphi_s\left(F^{(\omega)}\right). \tag{4.7}$$

Theorem 4.3 ([26, Theorem 4.19]) *$\hat{F}^{(s,\omega)}$ and $T^{(s,\omega)}$ satisfy the following structure equation*

$$\hat{F}^{(s,\omega)} = d^\omega T^{(s,\omega)} - \frac{1}{2}\left[T^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)}, \tag{4.8}$$

where a wedge product between the 1*-forms $T^{(s,\omega)}$ is implied and d^ω is the covariant exterior derivative with respect to the connection ω of $T^{(s,\omega)} \in \Omega^1(\mathcal{A})$.*

In the case of an octonion bundle over a 7-dimensional manifold, this relationship between the torsion and a curvature component has been shown in [24].

As discussed earlier, equivariant horizontal forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M. So let us now switch perspective, and work in terms of sections of bundles. In particular, now we will consider s to be a smooth section of the bundle $\mathcal{Q}$, so that we will say $s \in \Gamma(\mathcal{Q})$, and will refer to it as the *defining section*. Similarly, we can also consider sections $A \in \Gamma\left(\mathcal{Q}'\right)$, which admit the partial action of Ψ. The product on elements of $\mathbb{L}'$ and $\mathbb{L}$, then carries over to sections of bundles, so that we have a product $\Gamma\left(\mathcal{Q}'\right) \times \Gamma(\mathcal{Q}) \longrightarrow \Gamma(\mathcal{Q})$.

The connection ω on $\mathcal{P}$ then induces connections on the associated bundles and correspondingly, covariant derivatives on sections of these bundles. The torsion $T^{(s,\omega)}$, as defined earlier, was a horizontal and equivariant 1-form on $\mathcal{P}$ with values in $\mathfrak{l}$, so it uniquely corresponds to a 1-form on M with values in the bundle $\mathcal{A}$, i.e., now we will consider $T^{(s,\omega)} \in \Omega^1(\mathcal{A})$.

In standard gauge theory, the key object is the connection, however, in the non-associative theory, in addition to the connection ω we also the defining section s. We then make the following definition.

Definition 4.4 A *non-associative gauge theory* is defined by the following objects:

1. A smooth loop $\mathbb{L}$ with a finite-dimensional pseudoautomorphism Lie group Ψ and tangent algebra $\mathfrak{l}$ at identity.
2. A smooth manifold M with a principal Ψ-bundle $\mathcal{P}$, and associated bundles $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{A}$, with fibers $\mathbb{L}$, $\mathbb{L}'$, and $\mathfrak{l}$, respectively
3. A *configuration* (s, ω), where $s \in \Gamma(\mathcal{Q})$ is a defining section and ω is a connection on $\mathcal{P}$. Each configuration carries torsion $T^{(s,\omega)} \in \Omega^1(\mathcal{A})$.

As we see, the key components are the loop $\mathbb{L}$, with its pseudoautomorphism group, and the corresponding principal bundle $\mathcal{P} \longrightarrow M$. Up to a choice of the configuration (s, ω), everything else follows uniquely. In particular, the associated bundles are unique because particular actions of Ψ are used to define them.

The group Ψ acts via standard gauge transformations on ω and also acts on the section s. These actions are related in the following way, as shown in [26],

$$T^{(s,h^*\omega)} = \left(h'_*\right)^{-1} T^{(h(s),\omega)}, \tag{4.9}$$

where h is a section of $\mathrm{Ad}\,\mathcal{P}$, so is fiberwise in Ψ. However, we will define loop gauge transformations in the following way.

Definition 4.5 A *loop gauge transformation* is a transformation of the defining section s by right multiplication by a section $A \in \Gamma\left(\mathcal{Q}'\right)$, such that $s \mapsto As$, and hence $T^{(s,\omega)} \mapsto T^{(As,\omega)}$.

With respect to a loop gauge transformation, the torsion and curvature $\hat{F}$ transform in the following way.

Lemma 4.6 ([26, Theorem 4.28]) *Suppose* $A \in \Gamma\left(\mathcal{Q}'\right)$ *and* $s \in \Gamma(\mathcal{Q})$. *Then,*

$$T^{(As,\omega)} = \left(\rho_A^{(s)}\right)^{-1} \nabla^{\omega} A + \mathrm{Ad}_A^{(s)}\, T^{(s,\omega)} \tag{4.10a}$$

$$\hat{F}^{(As,\omega)} = \left(\rho_A^{(s)}\right)^{-1} \left(F' \cdot A\right) + \mathrm{Ad}_A^{(s)}\, \hat{F}^{(s,\omega)}, \tag{4.10b}$$

where $F' \cdot A$ *denotes the infinitesimal action of* $\mathfrak{p}$ *on* $\mathbb{L}$.

Let us fix the connection ω, and suppose we have a path $\exp_s(t\xi)\, s \in \Gamma(\mathcal{Q})$. Then from Lemma 3.2, just by taking the horizontal projection, we immediately obtain that the corresponding one-parameter family of torsions satisfy an ODE similar to (3.9).

Lemma 4.7 *Suppose $T^{(s,\omega)}$ is the torsion with respect to a defining section $s \in \Gamma(\mathcal{Q})$ and a connection ω. Suppose $A_t = \exp_s(t\xi) \in \Gamma(\mathcal{Q}')$, then*

$$\frac{d}{dt} T^{(A_t s,\omega)} = \left[\xi, T^{(A_t s,\omega)}\right]^{(A_t s)} + d^{\omega}\xi \tag{4.11}$$

Using (4.10a) and (3.10), given $\xi \in \Gamma(\mathcal{A})$, we get

$$\begin{aligned} T^{((\exp_s \xi)s,\omega)} &= U_{\xi}^{(s)} T^{(s,\omega)} \\ &\quad + U_{\xi}^{(s)} \left(\int_0^1 U_{\xi}^{(s)}(\tau)^{-1} d\tau\right) d^{\omega}\xi. \end{aligned} \tag{4.12}$$

Now suppose the base manifold M is compact and Riemannian with a metric g and also that the loop $\mathbb{L}$ admits a non-degenerate Killing form on $\mathfrak{l}$. Then, define the functional

$$\mathcal{E}_{\omega}(s) = \int_M \left|T^{(s,\omega)}\right|^2_{(s)} \mathrm{vol}_g, \tag{4.13}$$

where $||_{(s)}$ is a combination of the metric g on M and the Killing form $K^{(s)}$ on sections of $\mathcal{A}$. Critical points then become analogues of the Coulomb gauge condition in gauge theory [10, 24–27, 41]

Theorem 4.8 *Suppose $\mathbb{L}$ is a semisimple Moufang loop, then the critical points of the functional (4.13) with respect to deformations of the defining section s are those for which*

$$\left(d^{\omega}\right)^* T^{(s,\omega)} = 0, \tag{4.14}$$

where $(d^{\omega})^*$ is the formal adjoint of d^{ω}.

Proof From Lemma 2.38, we know that for a Moufang loop, $K^{(s)}$ is actually independent of s, so we will drop the superscript and will just denote it by K. Moreover, it is invariant under $\mathrm{ad}^{(s)}$ for any s. Let $\langle\cdot,\cdot\rangle$ denote the combination of the metric g on M and the Killing form K on sections of $\mathcal{A}$.

Let us consider deformations of s. The semisimple condition implies K is non-degenerate. Consider a path $s_t = \exp_s(t\xi)\, s$ where $\xi \in \Gamma(\mathcal{A})$. Then,

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(s_t,\omega)\bigg|_{t=0} &= \frac{d}{dt}\int_M \left|T^{(s_t,\omega)}\right|^2_s \mathrm{vol}_g\bigg|_{t=0} \\ &= 2\int_M \left\langle T^{(s,\omega)}, \frac{d}{dt}T^{(s_t,\omega)}\bigg|_{t=0}\right\rangle \mathrm{vol}_g \\ &= 2\int_M \left\langle T^{(s,\omega)}, \left[\xi, T^{(s,\omega)}\right]^{(s)} + d^{\omega}\xi\right\rangle \mathrm{vol}_g, \end{aligned} \tag{4.15}$$

where we have used (4.11). The condition that K is invariant with respect to $[\cdot,\cdot]^{(s)}$ means that for any $\xi, \eta, \gamma \in \mathfrak{l}$,

$$K\left(\xi, [\eta, \gamma]^{(s)}\right) = -K\left([\eta, \xi]^{(s)}, \gamma\right).$$

Therefore, the first term in (4.15) becomes:

$$\begin{aligned}\left\langle T^{(s,\omega)}, \left[\xi, T^{(s,\omega)}\right]^{(s)}\right\rangle &= g^{ab} K\left(T_a^{(s,\omega)}, \left[\xi, T_b^{(s,\omega)}\right]^{(s)}\right) \\ &= -g^{ab} K\left(\left[T_a^{(s,\omega)}, T_b^{(s,\omega)}\right], \xi^{(s)}\right) \\ &= 0,\end{aligned}$$

and thus,

$$\left.\frac{d}{dt}\mathcal{E}(s_t, \omega)\right|_{t=0} = 2\int_M \left\langle \left(d^{\omega}\right)^* T^{(s,\omega)}, \xi\right\rangle \mathrm{vol}_g .$$

Thus critical points of $\mathcal{E}$ with respect to deformations of s satisfy

$$\left(d^{\omega}\right)^* T^{(s,\omega)} = 0. \tag{4.16}$$

□

Remark 4.9 In Theorem 4.8, we use the fact that the tangent algebra of a Moufang loop is a Malcev algebra, i.e. is alternative and satisfies the additional identity 2.51. Moreover, the semisimple condition implies that the Killing form is non-degenerate. As noted in Remark 2.39, the full Malcev algebra condition is likely to be too strong, and a weaker assumption may be sufficient to obtain these key properties and in fact obtain $(d^{\omega})^* T^{(s,\omega)} = 0$ as the equation for critical points. On the other hand, other techniques, such as introducing a different metric (such as the Killing-Ricci form on Lie triple systems [35]) or introducing modified connections may produce similar results in other settings.

To prove existence of transformations of s that lead to $(d^{\omega})^* T^{(s,\omega)} = 0$, we will adapt the procedures from [17], and in particular will apply the Banach Space Implicit Function Theorem (Theorem A.2). The relevant Banach spaces for us will be spaces of sections with appropriate regularity. The previously used notations Γ and Ω^k will always denote smooth sections and smooth bundle-valued forms, respectively. Given a smooth defining section $s \in \Gamma(\mathcal{Q})$ and a smooth connection ω, for any $k \in \mathbb{N}$ and $q \in [1, \infty]$, denote by $W^{k,q}_{(s,\omega)}\left(\Lambda^l T^*M \otimes \mathcal{A}\right)$ the Sobolev space of sections of $\Lambda^l T^*M \otimes \mathcal{A}$ with the norm given by

$$\|\chi\|_{W^{k,q}_{(s,\omega)}} = \left(\int_M |\chi|^q_{(s)} \mathrm{vol}_g\right)^{\frac{1}{q}} + \left(\int_M \left|\left(\nabla^{\omega}\right)^k \chi\right|^q_{(s)} \mathrm{vol}_g\right)^{\frac{1}{q}},$$

for $1 \leq q < \infty$ and

$$\|\chi\|_{W^{k,\infty}_{(s,\omega)}} = \operatorname{ess\,sup}_X |\chi|_{(s)} + \operatorname{ess\,sup}_X \left|\left(\nabla^{\omega}\right)^k \chi\right|_{(s)}.$$

Similarly we will denote $W^{0,r}_{(s,\omega)}$ as $L^r_{(s)}$. Differential operators on smooth sections extend naturally as bounded linear maps between Sobolev spaces, and will be denoted in the same way.

By Definition 4.2, the torsion of (s,ω) is just the horizontal component of θ_s, so we can immediately adapt the estimates from Sect. 3, we obtain the following estimates for torsion.

Lemma 4.10 *Suppose $\mathbb{L}$ is a smooth compact loop with tangent algebra $\mathfrak{l}$ and pseudoautomorphism group Ψ. Let (M,g) be a closed, smooth Riemannian manifold of dimension $n \geq 2$, and let $\mathcal{P}$ be a Ψ-principal bundle over M with connection ω and let $\mathcal{A}$ be the associated vector bundle to $\mathcal{P}$ with fibers isomorphic to $\mathfrak{l}$. Let ω be a smooth connection on $\mathcal{P}$ and let $s \in \Gamma(\mathcal{Q})$ be a smooth defining section. Also, suppose k is a non-negative integer and $r \geq 0$ such that $kr > n$. Let $\xi \in W^{k,r}_{(s,\omega)}(\mathcal{A})$, and suppose $A = \exp_s(\xi)$. Then,*

$$\left\|T^{(As,\omega)}\right\|_{W^{k-1,r}_{(s,\omega)}} \lesssim e^{Ck\|\xi_{(s,\omega)}\|_{C^0}}\left(\Theta^k + \Theta\right), \tag{4.17}$$

where $\Theta = \left\|T^{(s,\omega)}\right\|_{W^{k-1,r}_{(s,\omega)}} + \|\xi\|_{W^{k,r}_{(s,\omega)}}$.

Lemma 4.11 *Now suppose that $(k-1)r \geq n$. Given other hypotheses the same as in Lemma 4.10, if $\xi \in W^{k,r}_{(s,\omega)}(\mathcal{A})$, and given $A = \exp_s(\xi)$ such that*

$$\left(d^{\omega}\right)^* T^{(As,\omega)} = 0, \tag{4.18}$$

then in fact A is smooth.

Proof Using the Whitney Embedding Theorem, suppose $\mathbb{L}$ is smoothly embedded in some $\mathbb{R}^N$. We can define a loop product and quotient on the image of the embedding. Hence the bundles $\mathcal{Q}$ and $\mathcal{Q}'$ can be regarded as subbundles of a vector bundle over M. In particular, since s is smooth and $\exp_s : \mathfrak{l} \longrightarrow \mathbb{L}$ is also a smooth map, we find that since $kr > n$ and $\xi \in W^{k,r}(\mathcal{A})$, then $A = \exp_s(\xi) \in W^{k,r}\left(\mathcal{Q}'\right) \subset C^0\left(\mathcal{Q}'\right)$. In particular, A is a continuous section of a vector bundle. Using (4.10a), we have

$$\begin{aligned}
\left(d^{\omega}\right)^* T^{(As,\omega)} &= \left(d^{\omega}\right)^*\left(\left(\rho_A^{(s)}\right)^{-1} d^{\omega}A + \operatorname{Ad}_A^{(s)} T^{(s,\omega)}\right)\\
&= \left(\rho_A^{(s)}\right)^{-1}\left(d^{\omega}\right)^* d^{\omega}A - \left\langle d^{\omega}\left(\rho_A^{(s)}\right)^{-1}, d^{\omega}A\right\rangle_{TM}\\
&\quad + \left(d^{\omega}\right)^*\left(\operatorname{Ad}_A^{(s)} T^{(s,\omega)}\right),
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{TM}$ is the inner product on TM. Thus, we can rewrite (4.18) as

$$\left(d^{\omega}\right)^{*}d^{\omega}A=\rho_{A}^{(s)}\left\langle d^{\omega}\left(\rho_{A}^{(s)}\right)^{-1},d^{\omega}A\right\rangle_{TM}-\rho_{A}^{(s)}\left(\left(d^{\omega}\right)^{*}\left(\mathrm{Ad}_{A}^{(s)}T^{(s,\omega)}\right)\right) \tag{4.19}$$

$$=-\left\langle\left(d^{\omega}\rho_{A}^{(s)}\right)\left(\rho_{A}^{(s)}\right)^{-1},d^{\omega}A\right\rangle_{TM}+\rho_{A}^{(s)}\left\langle d^{\omega}\,\mathrm{Ad}_{A}^{(s)},T^{(s,\omega)}\right\rangle \tag{4.20}$$

$$-\rho_{A}^{(s)}\left(\mathrm{Ad}_{A}^{(s)}\left(d^{\omega}\right)^{*}T^{(s,\omega)}\right). \tag{4.21}$$

The operator $(d^{\omega})^{*}d^{\omega}$ on vector-bundle-valued 1-forms has a leading term of the form $-g^{ab}\partial_a\partial_b$ and is thus elliptic. From the hypothesis $(k-1)r\geq n$, must have $k\geq 2$, so $(d^{\omega})^{*}d^{\omega}$ extends to a bounded linear operator $W^{k,r}\longrightarrow W^{k-2,r}$.

Now since $A\in W^{k,r}\subset C^{0}$, and $T^{(s,\omega)}$ is smooth, for any $p>0$,

$$\left\|\left(d^{\omega}\right)^{*}d^{\omega}A\right\|_{L^{p}}\leq c\left(A\right)\left(\left\|\left|d^{\omega}A\right|^{2}\right\|_{L^{p}}+\left\|d^{\omega}A\right\|_{L^{p}}+\left\|\left(d^{\omega}\right)^{*}T^{(s,\omega)}\right\|_{L^{p}}\right) \tag{4.22}$$

Also, $\left\||d^{\omega}A|^{2}\right\|_{L^{p}}\leq\|d^{\omega}A\|_{L^{2p}}^{2}$. Since $\frac{k-1}{n}\geq\frac{1}{r}$, we see that $\frac{k-1}{n}>\frac{1}{r}-\frac{1}{q}$ for any $q>0$. By the Sobolev Embedding Theorem, this shows that

$$\left\|d^{\omega}A\right\|_{L^{q}}\lesssim\left\|d^{\omega}A\right\|_{W^{k-1,r}}\lesssim\|A\|_{W^{k,r}}.$$

Thus, (4.22) shows that $\left\|(d^{\omega})^{*}d^{\omega}A\right\|_{L^{p}}$ is bounded. By elliptic regularity, this implies that $A\in W^{2,p}$. In particular, if $p>n$, then $k>\frac{n}{p}+1$, and thus $A\in C^{1}$. Bootstrapping the elliptic regularity argument we then obtain the smoothness of A. In particular, note that this does not depend on the choice of embedding. □

Remark 4.12 The proof of Lemma 4.11 is an adaptation of the proof of [11, Proposition 2.3.4], where in particular the regularity of a gauge transformation to the Coulomb gauge was proved. In that case, $r=2$ and $n=4$, so the conditions $kr>n$ and $(k-1)r\geq n$ were equivalent since k is an integer. More generally, the condition that is needed for smoothness is somewhat stronger than the one needed for continuity.

Then, we have the main theorem.

Theorem 4.13 *Suppose $\mathbb{L}$ is a smooth compact loop with tangent algebra $\mathfrak{l}$ and pseudoautomorphism group Ψ. Let (M,g) be a closed, smooth Riemannian manifold of dimension $n\geq 2$, and let $\mathcal{P}$ be a Ψ-principal bundle over M with and let $\mathcal{A}$ be the associated vector bundle to $\mathcal{P}$ with fibers isomorphic to $\mathfrak{l}$. Let ω be a smooth connection on $\mathcal{P}$. Also, suppose k is a non-negative integer and $r\geq 0$ such that $kr>n$. Then, there exist constants $\delta\in(0,1]$ and $K\in(0,\infty)$, such that if $s\in\Gamma(\mathcal{Q})$ is a smooth defining section for which*

$$\left\|T^{(s,\omega)}\right\|_{W_{(s,\omega)}^{k-1,r}}<\delta,$$

then there exists a section $A \in W^{k,r}\left(\mathcal{Q}'\right)$, *such that*

$$\left(d^{\omega}\right)^{*} T^{(As,\omega)} = 0$$

and

$$\left\| T^{(As,\omega)} \right\|_{W^{k-1,r}_{(s,\omega)}} < K \left\| T^{(s,\omega)} \right\|_{W^{k-1,r}} \left(1 + \left\| T^{(s,\omega)} \right\|^{k-1}_{W^{k-1,r}}\right). \tag{4.23}$$

If moreover, $(k-1)\, r \geq n$, *then* A *is smooth.*

Proof Consider $\xi \in W^{k,r}_{(s,\omega)}(\mathcal{A})$ and $a \in W^{(k-1),r}_{(s,\omega)}(T^{*}M \otimes \mathcal{A})$. For now, let us drop the (s,ω) subscript in function spaces. Since k and r satisfy $kr > n$, by the Sobolev Embedding Theorem, $W^{k,r}$ embeds in C^{0}. Define the function

$$G : W^{(k-1),r}\left(T^{*}M \otimes \mathcal{A}\right) \times W^{k,r}(\mathcal{A}) \longrightarrow W^{(k-2),r}(\mathcal{A})$$

by

$$G(a,\xi) = \left(d^{\omega}\right)^{*}\left(U^{(s)}_{\xi} a + U^{(s)}_{\xi}\left(\int_{0}^{1} U^{(s)}_{\xi}(\tau)^{-1}\, d\tau\right) d^{\omega}\xi\right). \tag{4.24}$$

The assumption that $\xi \in C^{0}$, together with the smoothness of $U^{(s)}_{\xi}$ and the derivative maps, leads to the conclusion that G is a smooth map of Banach spaces. Note that using (4.12), we can write

$$G(a,\xi) = \left(d^{\omega}\right)^{*}\left(U^{(s)}_{\xi}\left(a - T^{(s,\omega)}\right) + T^{((\exp_{s}\xi)s,\omega)}\right) \tag{4.25}$$

Using the connection ω, let us define the bundle-valued Hodge Laplacian

$$\Delta^{(\omega)} = \left(d^{\omega}\right)^{*} d^{\omega} + \left(d^{\omega}\right)^{*} d^{\omega}.$$

On 0-forms it reduces to $\Delta^{(\omega)} = (d^{\omega})^{*} d^{\omega}$. It extends as an operator of Sobolev spaces as

$$\Delta^{(\omega)} : W^{k,r}\left(\Lambda^{l} T^{*}M \otimes \mathcal{A}\right) \longrightarrow W^{(k-2),r}\left(\Lambda^{l} T^{*}M \otimes \mathcal{A}\right),$$

and by standard elliptic theory is Fredholm with index 0 and a closed range

$$\left(\ker \Delta^{(\omega)}\right)^{\perp} \cap W^{(k-2),r}\left(\Lambda^{l} T^{*}M \otimes \mathcal{A}\right),$$

where $\perp$ denotes the L^{2}-orthogonal complement. Note that the kernel of $\Delta^{(\omega)}$ is independent of k.

To be able to apply the Implicit Function Theorem (Theorem A.2), in (4.24), let us constrain $\xi \in \left(\ker \Delta^{(\omega)}\right)^{\perp}$, and we also see that $\operatorname{Im} G \subset \left(\ker \Delta^{(\omega)}\right)^{\perp}$. This can be seen immediately. Suppose $\sigma = \left(d^{(\omega)}\right)^{*} \rho$ for some $\rho \in W^{(k-1),r}\left(T^{*}M \otimes \mathcal{A}\right)$ and $\gamma \in \ker \Delta^{(\omega)} \subset W^{(k-2),r}(\mathcal{A})$, then

$$\langle \sigma, \gamma\rangle_{L^2} = \left\langle \left(d^{(\omega)}\right)^{*} \rho, \gamma \right\rangle_{L^2} = \left\langle \rho, d^{(\omega)}\gamma \right\rangle_{L^2} = 0,$$

since on a compact manifold, $\gamma \in \ker \Delta^{(\omega)}$ if and only if $d^{(\omega)}\gamma = 0$. Hence the image of G is contained in $\left(\ker \Delta^{(\omega)}\right)^{\perp}$, which we'll denote for brevity by $K^{\perp}$, and so in fact,

$$G : W^{(k-1),r}\left(T^{*}M \otimes \mathcal{A}\right) \times \left(K^{\perp} \cap W^{k,r}(\mathcal{A})\right) \longrightarrow K^{\perp} \cap W^{(k-2),r}(\mathcal{A}).$$

Now let us consider the differential of G at $(a, \xi) = 0$ in the direction $(b, \eta) \in W^{(k-1),r}\left(T^{*}M \otimes \mathcal{A}\right) \times \left(K^{\perp} \cap W^{k,r}(\mathcal{A})\right)$:

$$\begin{aligned} DG|_{(0,0)}(b, \eta) &= \frac{d}{dt} \left(d^{\omega}\right)^{*} \left(U_{t\eta}^{(s)}(tb) + U_{t\eta}^{(s)} \left(\int_0^1 U_{t\eta}^{(s)}(\tau)^{-1} d\tau\right) d^{\omega}(t\eta)\right)\Bigg|_{t=0} \\ &= \left(d^{\omega}\right)^{*} b + \left(d^{\omega}\right)^{*} d^{\omega}\eta, \end{aligned}$$

since $U_0^{(s)} = \mathrm{id}_{\mathbb{L}}$. In particular, the partial derivative in the second direction is given by

$$\partial_2 G|_{(0,0)}(\eta) = \Delta^{(\omega)}\eta.$$

In Theorem A.2, let

$$\begin{aligned} X &= W^{(k-1),r}\left(T^{*}M \otimes \mathcal{A}\right) \\ Y &= K^{\perp} \cap W^{k,r}(\mathcal{A}) \\ Z &= K^{\perp} \cap W_{A_1}^{(k-2),r}(\mathcal{A}). \end{aligned}$$

Then, the map $\partial_2 G|_{(0,0)} : Y \longrightarrow Z$ is an isomorphism, and we define

$$N = \left\| \left(\partial_2 G|_{(0,0)}\right)^{-1} \right\|_{\operatorname{Hom}(Z,Y)}.$$

Let

$$\begin{aligned} U &= \left\{x \in W^{(k-1),r}\left(T^{*}M \otimes \mathcal{A}\right) : \ \|x\|_{W^{(k-1),r}} < \zeta\right\} \subset X \\ V &= \left\{y \in K^{\perp} \cap W^{k,r}(\mathcal{A}) : \ \|y\|_{W^{k,r}} < \zeta\right\} \subset Y, \end{aligned}$$

where $\zeta \in (0, 1]$ is small enough such that

$$\sup_{(x,y)\in U\times V} \left\| \partial_2 G|_{(x,y)} - \partial_2 G|_{(0,0)} \right\|_{\mathrm{Hom}(Y,Z)} \leq \frac{1}{2N}.$$

Also define the constant β as

$$\beta = \sup_{(x,y)\in U\times V} \left\| \partial_1 G|_{(x,y)} \right\|_{\mathrm{Hom}(X,Z)} < \infty.$$

Then, by the conclusion of Theorem A.2, there exist an open set $\tilde{U} \subset U$, given by

$$\tilde{U} = \left\{ x \in W^{(k-1),r} \left(T^*M \otimes \mathcal{A} \right) : \ \|x\|_{W^{(k-1),r}} < \delta \right\} \subset X,$$

where $\delta \in \left(0, \min\left\{\zeta, \frac{\zeta}{2\beta N}\right\}\right]$, and a unique smooth map

$$\xi : \tilde{U} \longrightarrow V,$$

such that $\xi(0) = 0$, and

$$\begin{aligned} G(a, \xi(a)) &= 0, \ \forall a \in \tilde{U} \\ D\xi|_a &= -\left(\partial_2 G|_{(a,\xi(a))} \right)^{-1} \partial_1 G|_{(a,\xi(a))} \in \mathrm{Hom}(X, Y), \ \forall a \in \tilde{U} \\ \|\xi(a_1) - \xi(a_2)\|_Y &\leq 2\beta N \|a_1 - a_2\|_{X,} \ \forall a_1, a_2 \in \tilde{U}. \end{aligned}$$

In particular, for any $a \in W^{(k-1),r}(T^*M \otimes \mathcal{A})$ with $\|a\|_{W^{(k-1),r}} < \delta$, there exists a section $A(a) = \exp_s(\xi(a))$ with $\xi \in W^{k,r}(\mathcal{A})$, for which

$$(d^\omega)^* \left(U_\xi^{(s)} \left(a - T^{(s,\omega)} \right) + T^{(As,\omega)} \right) = 0,$$

and

$$\|\xi(a)\|_{W^{k,r}} \leq 2\beta N \|a\|_{W^{(k-1),r}}.$$

Since s is smooth and $\exp_s : \mathfrak{l} \longrightarrow \mathbb{L}$ is a smooth map, this shows that $A \in W^{k,r}\left(\mathcal{Q}'\right)$.

Now suppose s and ω are such that $\left\| T^{(s,\omega)} \right\|_{W^{(k-1),r}} < \delta$, then setting $a = T^{(s,\omega)}$ gives $\xi_{(s,\omega)} = \xi\left(T^{(s,\omega)}\right)$, for which

$$\begin{aligned} (d^\omega)^* \left(T^{(As,\omega)} \right) &= 0 \\ \left\| \xi_{(s,\omega)} \right\|_{W^{k,r}} &< \zeta \\ \left\| \xi_{(s,\omega)} \right\|_{W^{k,r}} &\leq 2\beta N \left\| T^{(s,\omega)} \right\|_{W^{(k-1),r}}, \end{aligned}$$

where $A = \exp_s\left(\xi_{(s,\omega)}\right)$. From (4.17), we have

$$\left\|T^{(As,\omega)}\right\|_{W^{k-1,r}} \lesssim e^{Ck\|\xi_{(s,\omega)}\|_{C^0}}\left(\Theta^k+\Theta\right), \tag{4.26}$$

where $\Theta = \left(\left\|T^{(s,\omega)}\right\|_{W^{k-1,r}} + \left\|\xi_{(s,\omega)}\right\|_{W^{k,r}}\right)$. Now, using the estimate for ξ in terms of T, we get

$$\Theta \lesssim (1+2\beta N)\left(\left\|T^{(s,\omega)}\right\|_{W^{k-1,r}}\right),$$

and since $kr > n$, $\left\|\xi_{(s,\omega)}\right\|_{C^0} \lesssim \left\|\xi_{(s,\omega)}\right\|_{W^{k,r}} < \zeta$. Overall, combining the constants into a single constant K, we obtain

$$\left\|T^{(As,\omega)}\right\|_{W^{k-1,r}} < K\left\|T^{(s,\omega)}\right\|_{W^{k-1,r}}\left(1+\left\|T^{(s,\omega)}\right\|^{k-1}_{W^{k-1,r}}\right), \tag{4.27}$$

and hence (4.23).

If $(k-1)\,r \geq n$, then by Lemma 4.11, we see that A is smooth. □

5 G_2-Manifolds

The general picture considered in the previous sections can now be specialized to the case of manifolds with G_2-structure. The 14-dimensional group G_2 is the smallest of the five exceptional Lie groups and is defined as the automorphism group of the loop of unit octonions $U\mathbb{O}$. Let M be a compact 7-dimensional manifold with vanishing first and second Stiefel-Whitney classes, so that the manifold is both orientable and admit a spin structure. Then, as it is well-known [18, 19], M admits a G_2 -structure, that is a reduction of the structure group of the frame bundle to G_2. Since G_2 is a subgroup of $SO(7)$, the G_2 -structure can be extended uniquely to an $SO(7)$-structure, and thus defines a Riemannian metric g and orientation on M. Equivalently, given a Riemannian metric g, an $SO(7)$ -structure on M lifts to a spin structure, which is a principal Spin (7)-structure. Given the spin structure, we can then construct a spinor bundle $\mathcal{S}$ which will necessarily admit a nowhere vanishing section. Any such spinor section will then reduce the spin structure to a G_2-structure on M. Indeed, any unit spinor will hence define a G_2-structure that is compatible with the metric g.

Recall that Spin (7) has three low-dimensional real irreducible representations: 1-dimensional representation V_1, 7-dimensional "vector" representation V_7, and the 8-dimensional "spinor" representation S_7 [2]. The representations V_1 and V_7 descend to representations of $SO(7)$. Moreover, the Clifford product gives the map

$$V_7 \times S_7 \longrightarrow S_7. \tag{5.1}$$

Setting $V_8 = V_1 \oplus V_7$, we can then extend this map to $m : V_8 \times S_7 \longrightarrow S_7$. This product is non-degenerate, and fixing $\xi \in S_7$ allows to identify V_8 with S_7. Both spaces are then identified with the octonions and the product m then gives rise to octonion multiplication. The element ξ is identified with $1 \in \mathbb{O}$. The stabilizer of $\xi \in S_7$ under the action of Spin (7) is isomorphic to G_2. Note that V_8 here then corresponds to the irreducible "vector" representation of Spin (8) , while the two copies of S_7 are identified with the irreducible 8-dimensional chiral spinor representations $S_8^{\pm}$ of Spin (8) , and thus gives the normed triality of Spin (8) [2]. Since the map m preserves norms, it restricts to unit spheres in V_8 and S_7, which we will denote by $U\mathbb{O}'$ and $U\mathbb{O}$, respectively, because they correspond to $\mathbb{L}'$ and $\mathbb{L}$ in the general theory in Sect. 2. Clearly, $U\mathbb{O}$ is a compact smooth loop. The tangent space at 1 to $U\mathbb{O}$ is then isomorphic to $\mathbb{R}^7 \cong \operatorname{Im} \mathbb{O}$. We thus have the following identification of objects.

Object	**Loops**	**Octonions**
Pseudoautomorphism group	Ψ	Spin (7)
Partial pseudoautomorphism group	Ψ'	$SO\,(7)$
Automorphism group	H	G_2
Lie algebra of Ψ	$\mathfrak{p}$	$\mathfrak{so}\,(7)$
Loop with full action of Ψ	$\mathbb{L}$	$U\mathbb{O} \subset S_7$
Loop with partial action of Ψ	$\mathbb{L}'$	$U\mathbb{O}' \subset V_8$
Tangent algebra	$\mathfrak{l}$	$\operatorname{Im} \mathbb{O} \cong V_7 \cong \mathbb{R}^7$

Therefore, on the manifold M as above, the spin structure corresponds to a principal Ψ-bundle in the general theory, the unit spinor bundle $U\mathcal{S}$ corresponds to the bundle $\mathcal{Q}$ and the unit subbundle $U\mathbb{O}M$ of $\mathbb{O}M \cong \Lambda^0 \oplus TM$ corresponds to $\mathcal{Q}'$. This is precisely the octonion bundle introduced in [24]. Hence, we have the following dictionary relating objects in the general loop bundle theory and G_2-geometry.

Loop bundles	**G_2-geometry**
$\mathcal{P}$	Spin structure: principal Spin (7)-bundle over M
$\mathcal{Q}' = \mathcal{P} \times_{\Psi'} \mathbb{L}'$	Unit octonion bundle $U\mathbb{O}M$
$\mathcal{Q} = \mathcal{P} \times_{\Psi} \mathbb{L}$	Unit spinor bundle $U\mathcal{S}$
$\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$	Bundle of imaginary octonions: TM
$\mathfrak{p}_{\mathcal{P}} = \mathcal{P} \times_{(\mathrm{Ad}_\xi)_*} \mathfrak{p}$	$\mathfrak{so}\,(7)$-bundle over $M \cong \Lambda^2 T^* M$
$\mathrm{Ad}\,(\mathcal{P}) = \mathcal{P} \times_{\mathrm{Ad}_\Psi} \Psi$	$SO\,(7)$ gauge transformations

G_2-structures can also be described using differential forms since G_2 is alternatively defined as the subgroup of $GL\,(7, \mathbb{R})$ that preserves a particular 3-form φ_0 [30].

Definition 5.1 Let $\left(e^1, e^2, \ldots, e^7\right)$ be the standard basis for $\left(\mathbb{R}^7\right)^*$, and denote $e^i \wedge e^j \wedge e^k$ by e^{ijk}. Then define φ_0 to be the 3-form on $\mathbb{R}^7$ given by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{5.2}$$

Then G_2 is defined as the subgroup of $GL\,(7, \mathbb{R})$ that preserves φ_0.

It turns out that there is a 1-1 correspondence between G_2 -structures on a 7-manifold and smooth 3-forms φ for which the 7-form-valued bilinear form B_φ as defined by (5.3) is positive definite (for more details, see [6] and the arXiv version of [28]).

$$B_{\varphi}\left(u,v\right)=\frac{1}{6}\left(u\lrcorner\varphi\right)\wedge\left(v\lrcorner\varphi\right)\wedge\varphi. \tag{5.3}$$

Here the symbol $\lrcorner$ denotes contraction of a vector with the differential form: $(u\lrcorner\varphi)_{mn}=u^{a}\varphi_{amn}$.

A smooth 3-form φ is said to be *positive* if B_{φ} is the tensor product of a positive-definite bilinear form and a nowhere-vanishing 7-form. In this case, it defines a unique Riemannian metric g_{φ} and volume form vol_{φ} such that for vectors u and v, the following holds

$$g_{\varphi}\left(u,v\right)\mathrm{vol}_{\varphi}=\frac{1}{6}\left(u\lrcorner\varphi\right)\wedge\left(v\lrcorner\varphi\right)\wedge\varphi. \tag{5.4}$$

An equivalent way of defining a positive 3-form φ, is to say that at every point, φ is in the $GL\left(7,\mathbb{R}\right)$-orbit of φ_0. It can be easily checked that the metric (5.4) for $\varphi=\varphi_0$ is in fact precisely the standard Euclidean metric g_0 on $\mathbb{R}^7$. Therefore, every φ that is in the $GL\left(7,\mathbb{R}\right)$-orbit of φ_0 has an *associated* Riemannian metric g that is in the $GL\left(7,\mathbb{R}\right)$-orbit of g_0. The only difference is that the stabilizer of g_0 (along with orientation) in this orbit is the group $SO\left(7\right)$, whereas the stabilizer of φ_0 is $G_2\subset SO\left(7\right)$. This shows that positive 3-forms forms that correspond to the same metric, i.e., are *isometric*, are parametrized by $SO\left(7\right)/G_2\cong\mathbb{RP}^7\cong S^7/\mathbb{Z}_2$. Therefore, on a Riemannian manifold, metric-compatible G_2-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or alternatively, by sections of an S^7-bundle, with antipodal points identified. The precise parametrization of isometric G_2-structures is given in Theorem 5.2.

Theorem 5.2 ([7]) *Let M be a 7-dimensional smooth manifold. Suppose φ is a positive 3-form on M with associated Riemannian metric g. Then, any positive 3-form $\tilde{\varphi}$ for which g is also the associated metric, is given by the following expression:*

$$\tilde{\varphi}=\sigma_{A}\left(\varphi\right)=\left(a^{2}-\left|\alpha\right|^{2}\right)\varphi-2a\alpha\lrcorner\left(*\varphi\right)+2\alpha\wedge\left(\alpha\lrcorner\varphi\right) \tag{5.5}$$

where $A=(a,\alpha)$ is a pair with a a scalar function on M and α a vector field such that

$$a^{2}+\left|\alpha\right|^{2}=1 \tag{5.6}$$

The pair $A=(a,\alpha)$ can in fact be also interpreted as a *unit octonion* section, where a is the real part, and α is the imaginary part. The relationship between octonion bundles and G_2 -structures was developed in detail in [24]. In particular, sections of a unit octonion bundle over M parametrize G_2 -structures that are associated to the same metric.

Definition 5.3 The *octonion bundle* $\mathbb{O}M$ on M is the rank 8 real vector bundle given by

$$\mathbb{O}M \cong \Lambda^0 \oplus TM \tag{5.7}$$

where $\Lambda^0 \cong M \times \mathbb{R}$ is a trivial line bundle. At each point $p \in M$, $\mathbb{O}_p M \cong \mathbb{R} \oplus T_p M$.

The definition (5.7) gives a natural decomposition of octonions on M into real and imaginary parts. We may write $A = (\operatorname{Re} A, \operatorname{Im} A)$ or $A = \begin{pmatrix} \operatorname{Re} A \\ \operatorname{Im} A \end{pmatrix}$. Since $\mathbb{O}M$ is defined as a tensor bundle, the Riemannian metric g on M induces a metric on $\mathbb{O}M$. Let $A = (a, \alpha) \in \Gamma(\mathbb{O}M)$. Then,

$$|A|^2 = a^2 + |\alpha|_g^2 \tag{5.8}$$

The metric allows to define the subbundle $U\mathbb{O}M$ of octonions of unit norm and allows allows to define a vector cross product on TM.

Definition 5.4 Given the G_2-structure φ on M, we define a *vector cross product with respect to* φ on M. Let α and β be two vector fields, then define

$$\left\langle \alpha \times_\varphi \beta, \gamma \right\rangle = \varphi(\alpha, \beta, \gamma) \tag{5.9}$$

for any vector field γ [20, 31].

Using the inner product and the cross product, we can now define the *octonion product* on $\mathbb{O}M$.

Definition 5.5 Let $A, B \in \Gamma(\mathbb{O}M)$. Suppose $A = (a, \alpha)$ and $B = (b, \beta)$. Given the vector cross product (5.9) on M, we define the *octonion product* $A \circ_\varphi B$ *with respect to* φ as follows:

$$A \circ_\varphi B = \begin{pmatrix} ab - \langle \alpha, \beta \rangle \\ a\beta + b\alpha + \alpha \times_\varphi \beta \end{pmatrix} \tag{5.10}$$

If there is no ambiguity as to which G_2-structure is being used to define the octonion product, we will simply write AB to denote it. In particular, $|AB| = |A|\,|B|$.

Given a G_2-structure φ with an associated metric g, we may use the metric to define the Levi-Civita connection ∇. The *intrinsic torsion* of a G_2-structure is then defined by $\nabla\varphi$. Following [22, 32], we can write

$$\nabla_a \varphi_{bcd} = -2T_a{}^e \psi_{ebcd} \tag{5.11}$$

where T_{ab} is the *full torsion tensor*. Similarly, we can also write

$$\nabla_a \psi_{bcde} = 8T_{a[b}\varphi_{cde]} \tag{5.12}$$

We can also invert (5.11) to get an explicit expression for T

$$T_a{}^m = -\frac{1}{48}\left(\nabla_a \varphi_{bcd}\right)\psi^{mbcd}. \tag{5.13}$$

This 2-tensor fully defines $\nabla\varphi$ [22].

Remark 5.6 The torsion tensor T as defined here is actually corresponds to $-T$ in [24], $-\frac{1}{2}T$ in [22] and $\frac{1}{2}T$ in [32]. Even though this requires extra care when translating various results, it will turn out to be more convenient.

Given a unit norm spinor section $\xi \in \Gamma(\mathcal{S})$, a G_2-structure 3-form φ_ξ is defined in the following way:

$$\varphi_\xi(\alpha, \beta, \gamma) = -\langle \xi, \alpha \cdot (\beta \cdot (\gamma \cdot \xi)) \rangle_S, \tag{5.14}$$

where $\cdot$ denotes Clifford multiplication, α, β, γ are arbitrary vector fields and $\langle \cdot, \cdot \rangle_S$ is the inner product on the spinor bundle. The Levi-Civita connection lifts to the spinor bundle $\mathcal{S}$, giving the spinorial covariant derivative ∇^S. Then, the torsion $T^{(\xi)}$ of φ_ξ is given by [1, Definition 4.2 and Lemma 4.3]

$$\nabla^S_X \xi = T^{(\xi)}_X \cdot \xi, \tag{5.15}$$

Note that in [1], the torsion endomorphism is denoted by S.

As discussed above, in the special case of G_2-structures on 7 -manifolds, the correspondence with the general loop bundle picture is the following: the Ψ-principal bundle is the Spin (7)-principal bundle that is the spin structure and the loop bundle $\mathcal{Q}$ is the unit spinor bundle $U\mathcal{S}$. Then, fixing a section $\xi \in \Gamma(U\mathcal{S})$, we can rewrite the general Definition 4.2 of the torsion $T^{(\xi,\nabla)}$ of the section ξ with respect to the Levi-Civita connection ∇ precisely in the form (5.15), showing that the torsion $T^{(\xi)}$ of the G_2-structure φ_ξ is precisely equal to $T^{(\xi,\nabla)}$.

The covariant exterior derivative d^ω in the general case is then equal to the Levi-Civita covariant exterior derivative d^∇, and in particular, the adjoint $(d^\omega)^*$, which appears in Theorem 4.13, is then equal to $\left(d^\nabla\right)^*$. On vector-valued 1-forms, such as the torsion, it is easy to see that in components, $\left(\left(d^\nabla\right)^* T\right)^a = -\nabla^b T_b{}^a = -(\operatorname{div} T)^a$. Therefore, the condition $(d^\omega)^* T = 0$ is equivalent to $\operatorname{div} T = 0$ in this case.

Similarly, given a unit octonion section $A \in \Gamma(U\mathbb{O}M)$, $A \cdot \xi$ is again a unit spinor which defines a G_2 -structure $\varphi_{A\cdot\xi}$. Considering both A and ξ as octonions in $U\mathbb{O}'$ and $U\mathbb{O}$, respectively, this is just octonion multiplication $A\xi$, and $\varphi_{A\cdot\xi} = \varphi_{A\xi} = \sigma_A(\varphi_\xi)$. Therefore, all isometric G_2-structures are given by $\varphi_{A\xi}$ for some unit octonion section A. The curvature component $\hat{F}$ is then equal to a particular component of the Riemann curvature tensor. These relationships are explored in detail in [24].

Thus we can reformulate Theorem 4.13 for G_2-structures.

Theorem 5.7 *Suppose M is a closed 7-dimensional manifold with a smooth G_2-structure φ with torsion T with respect to the Levi-Civita connection ∇. Also, suppose k is a positive integer and p is a positive real number such that $kp > 7$. Then, there exist constants $\delta \in (0, 1]$ and $K \in (0, \infty)$, such that if T satisfies*

$$\|T\|_{W^{k,p}} < \delta,$$

then there exists a smooth section $V \in \Gamma(U\mathbb{O}M)$, *such that*

$$\operatorname{div} T^{(V)} = 0$$

and

$$\left\| T^{(V)} \right\|_{W^{k,p}} < K \|T\|_{W^{k,p}} \left(1 + \|T\|_{W^{k,p}}^k\right). \tag{5.16}$$

Remark 5.8 If we choose $p = 2$ to work with Hilbert spaces, then for a smooth section V, we need $k \geq 4$, so the condition on T is to be sufficiently small in the $W^{4,2}$-norm.

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Data Availability Not available.

Appendix

Lemma A.1 *Let* k, k', n *be positive integers and* $kp > n$, *for a positive real number* p, *and let* $A_1, \ldots, A_{k'}$ *be real-valued functions on a compact* n*-dimensional Riemannian manifold* M. *Also, suppose* $m_1, \ldots, m_{k'}$ *are non-negative integers and* $q_1, \ldots, q_{k'}$ *are positive integers such that* $\sum_{j=1}^{k''} q_j m_j \leq k$, *then*

$$\left\| \prod_{j=1}^{k'} A_j^{m_j} \right\|_{L^p} \lesssim \prod_{j=1}^{k'} \|A_j\|_{W^{k-q_j,p}}^{m_j}. \tag{A.1}$$

Proof Let $k'' = \sum_{j=1}^{k''} q_j m_j \leq k$. Then suppose $p_j = \frac{pk''}{q_j m_j}$ for all j for which $m_j > 0$, so that $\frac{1}{p_j} = \frac{q_j m_j}{pk''}$, and hence $\sum_{j=1}^{k'} \frac{1}{p_j} = \frac{1}{p}$. Thus, from Hölder's inequality, we have

$$\left\| \prod_{j=1}^{k'} A_j^{m_j} \right\|_{L^p} \lesssim \prod_{j=1}^{k'} \|A_j\|_{L^{m_j p_j}}^{m_j}.$$

Now note that using the definition of p_j, $\frac{q_j}{k''} = \frac{p}{p_j m_j} \leq 1$, and hence

$$\begin{aligned} \frac{k - q_j}{n} &= \frac{k}{n}\left(1 - \frac{q_j}{k}\right) \\ &\geq \frac{k}{n}\left(1 - \frac{q_j}{k''}\right) \\ &= \frac{k}{n}\left(1 - \frac{p}{p_j m_j}\right). \end{aligned}$$

Since by assumption, $\frac{k}{n} > \frac{1}{p}$, we obtain

$$\frac{k-q_j}{n} > \frac{1}{p} - \frac{1}{p_j m_j}.$$

Using a version of the Sobolev Embedding Theorem, this shows that indeed,

$$\|A_j\|_{L^{m_j p_j}} \lesssim \|A_j\|_{W^{k-q_j,p}},$$

and (A.1) follows. □

Theorem A.2 (Banach space quantitative implicit function theorem [17, Theorem F.1]) *Let $k \geq 1$ be an integer or ∞, and let X, Y, Z be real Banach spaces. Suppose $U \subset X$ and $V \subset Y$ are open neighborhoods of points $x_0 \in X$ and $y_0 \in Y$ and $f : U \times V \longrightarrow Z$ is a C^k map such that $f(x_0, y_0) = 0$ and the partial derivative of f at (x_0, y_0) with respect to the second variable, $\partial_2 f|_{(x_0,y_0)} \in Hom(Y, Z)$ is an isomorphism of Banach spaces. Define*

$$N = \left\| \left(\partial_2 f|_{(x_0,y_0)}\right)^{-1} \right\|_{\mathrm{Hom}(Z,Y)}.$$

Let $\zeta \in (0, 1]$ be small enough such that the open ball $B_\zeta(x_0) \subset U$ and $B_\zeta(y_0) \subset V$, and assume

$$\sup_{(x,y)\in B_\zeta(x_0)\times B_\zeta(y_0)} \left\| \partial_2 f|_{(x,y)} - \partial_2 f|_{(x_0,y_0)} \right\|_{\mathrm{Hom}(Y,Z)} \leq \frac{1}{2N}$$
$$\beta = \sup_{(x,y)\in B_\zeta(x_0)\times B_\zeta(y_0)} \left\| \partial_1 f|_{(x,y)} \right\|_{\mathrm{Hom}(X,Z)} < \infty.$$

Then there exist a constant $\delta \in \left(0, \min\left\{\zeta, \frac{\zeta}{2\beta N}\right\}\right]$ and unique C^k map $g : B_\delta(x_0) \longrightarrow B_\zeta(y_0)$ such that $y_0 = g(x_0)$ and

$$f(g(x), x) = 0, \ \forall x \in B_\delta(x_0)$$
$$Dg|_x = -\left(\partial_2 f|_{(x,g(x))}\right)^{-1} \partial_1 f|_{(x,g(x))} \in \mathrm{Hom}(X, Y), \ \forall x \in B_\delta(x_0)$$
$$\|g(x_1) - g(x_2)\|_Y \leq 2\beta N \|x_1 - x_2\|_X, \ \forall x_1, x_2 \in B_\delta(x_0).$$

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