



A categorical Connes' $\chi(M)$

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Abstract

Popa introduced the tensor category $\tilde{\chi}(M)$ of approximately inner, centrally trivial bimodules of a II_1 factor M , generalizing Connes' $\chi(M)$. We extend Popa's notions to define the W^* -tensor category $\text{End}_{\text{loc}}(\mathcal{C})$ of local endofunctors on a W^* -category $\mathcal{C}$. We construct a unitary braiding on $\text{End}_{\text{loc}}(\mathcal{C})$, giving a new construction of a braided tensor category associated to an arbitrary W^* -category. For the W^* -category of finite modules over a II_1 factor, this yields a unitary braiding on Popa's $\tilde{\chi}(M)$, which extends Jones' κ invariant for $\chi(M)$. Given a finite depth inclusion $M_0 \subseteq M_1$ of non-Gamma II_1 factors, we show that the braided unitary tensor category $\tilde{\chi}(M_\infty)$ is equivalent to the Drinfeld center of the standard invariant, where M_∞ is the inductive limit of the associated Jones tower. This implies that for any pair of finite depth non-Gamma subfactors $N_0 \subseteq N_1$ and $M_0 \subseteq M_1$, if the standard invariants are not Morita equivalent, then the inductive limit factors N_∞ and M_∞ are not isomorphic.

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1 Introduction

Tensor categories have come to play an important role in noncommutative analysis, arising as categories of bimodules of C^* and von Neumann algebras and as representation categories of compact quantum groups. In subfactor theory, the standard invariant of a finite index II_1 subfactor [40, 71] can be described by a unitary tensor category (a.k.a. a semisimple rigid C^* -tensor category), together with a chosen unitary Frobenius algebra object internal to this category [29, 44, 51, 57]. In operator algebraic approaches to quantum field theories (AQFT) and topologically ordered spin systems, braided tensor categories arise in the DHR theory of superselection sectors of nets of von Neumann algebras [17, 22, 30, 50, 63]. Here, the presence of a braiding yields an incredibly rich structure theory which does not have an obvious analog in the purely ‘noncommutative’ world of ordinary subfactors.

There are, however, less widely recognized instances of braided tensor categories arising in the theory of II_1 factors. In [16], Connes introduced an invariant $\chi(M)$ of a II_1 factor M , which is the abelian subgroup of $\text{Out}(M)$ consisting of the image of approximately inner and centrally trivial automorphisms. In [38], Jones defines a quadratic form κ on the group $\chi(M)$. These invariants can be used to distinguish various II_1 factors from group von Neumann algebras. An abelian group together with a quadratic form defines (uniquely up to braided equivalence) a braided 2-group [21], which linearizes to a braided unitary tensor category. Generalizations of these constructions in the relative context were studied by Kawahigashi [48, 49] and utilized in the study of orbit equivalence of group actions by Ioana [35].

In his groundbreaking work [68, 72], Popa introduced notions of approximately inner and centrally free for finite index subfactors, which played a key role in his classification result for subfactors in terms of their standard invariants. In [68, Def. 2.5], Popa also considers a definition of a centrally trivial subfactor (as an ‘opposite’ to his notion of centrally free), and in [68, Rem. 2.7], he discusses how this definition and his definition of approximately inner subfactor have a natural generalization to bimodules. He introduces the unitary tensor category $\tilde{\chi}(M)$ of dualizable approximately inner and centrally trivial bimodules of a II_1 factor, generalizing Connes’ $\chi(M)$. Popa asks

whether this $\tilde{\chi}(M)$ is a 'commutative' tensor category, i.e., does it admit a braiding? In this article, we answer Popa's question positively.

Theorem A *Let M be a II_1 factor. Then $\tilde{\chi}(M)$ admits a unitary braiding (see Eq. (5)). Furthermore, if N is another II_1 factor stably isomorphic to M , then $\tilde{\chi}(M) \cong \tilde{\chi}(N)$ as braided unitary tensor categories.*

Notably, $\tilde{\chi}(M)$ recovers Connes' $\chi(M)$ as the equivalence classes of invertible bimodules in $\tilde{\chi}(M)$ whose left and right von Neumann dimension are equal (to one), and our braiding on $\tilde{\chi}(M)$ recovers Jones' κ (see Example 4.11 for more details). We may thus think of $\tilde{\chi}(M)$ as a unitary braided categorical extension of the braided 2-group $(\chi(M), \kappa)$.

The existence of a braiding on $\tilde{\chi}(M)$ is surprising from a categorical viewpoint. Indeed, von Neumann algebras form a 2-category whose 1-morphisms are bimodules and whose 2-morphisms are intertwiners. The unitary tensor category $\tilde{\chi}(M)$ is a full subcategory of $\text{End}(M) \cong \text{Bim}(M)$ in this 2-category. Braidings arise formally in the context of 3-categories by looking at endomorphisms of some identity 1-morphism. This is the algebraic structure underlying conformal nets, which produces unitary modular tensor categories in the rational setting [4, 5, 50]. The presence of a braiding, together with its behavior under local extensions in Theorem B below, suggests that von Neumann algebras may be objects in a yet to be discovered 3-category.

As expected from experience with Connes' χ , using [16, 68, 74], it is straightforward to show that $\tilde{\chi}(R)$, $\tilde{\chi}(N)$ and $\tilde{\chi}(R \otimes N)$ are trivial where R denotes the hyperfinite II_1 factor and N is any non-Gamma II_1 factor. In order to leverage these facts to compute some non-trivial examples, we prove the following theorem, which is similar in spirit to Connes' short exact sequence [16] (c.f. [7, Prop. 6.4] for a parallel result in the conformal net context).

Theorem B *Let $N \subseteq M$ be a finite index II_1 subfactor such that $Q := {}_N L^2 M_N \in \tilde{\chi}(N)$ is a commutative Q -system. Then $\tilde{\chi}(M) \cong \tilde{\chi}(N)_Q^{\text{loc}}$ as braided unitary tensor categories.*

In [74], Popa studied Connes' χ in the context of inductive limits of Jones towers of finite depth finite index non-Gamma II_1 subfactors $N \subseteq M$. He shows $\chi(M_\infty) = 1$ for a large class of non-Gamma inclusions $N \subseteq M$ for which M_∞ is McDuff but not isomorphic to $R \otimes N$ for N non-Gamma, resolving a question of Connes. Building off Popa's techniques, in this paper we will directly compute $\tilde{\chi}(M_\infty)$ as a unitary braided tensor category.

To state our results, recall the standard invariant of a finite depth finite index II_1 subfactor $N \subseteq M$ consists of the indecomposable 2×2 multifusion category $\mathcal{C}(N \subseteq M)$ of N - N , N - M , M - M , and M - N bimodules generated by $L^2 M$, together with the choice of generating object ${}_N L^2 M_M$. We define Morita equivalence of two standard invariants $\mathcal{C}(N_1 \subseteq N_2)$ and $\mathcal{C}(M_1 \subseteq M_2)$ as Morita equivalence of the underlying multifusion categories [19, Sect. 7.12]. The Drinfeld center $\mathcal{Z}(\mathcal{C}(N \subseteq M))$ is a braided unitary fusion category, and indecomposable multifusion categories are Morita equivalent if and only if their Drinfeld centers are equivalent as braided fusion categories [19, Sect. 8.5]. We have the following theorem.

Theorem C *Let $N \subseteq M$ be a finite depth finite index inclusion of non-Gamma II_1 factors, and let M_∞ denote the inductive limit factor of the Jones tower. Then $\tilde{\chi}(M_\infty) \cong \mathcal{Z}(\mathcal{C}(N \subseteq M))$ as braided unitary tensor categories.*

We get the following immediate corollary.

Corollary D *If $N_1 \subseteq N_2$ and $M_1 \subseteq M_2$ are finite depth inclusions of non-Gamma II_1 factors with $\mathcal{C}(N_1 \subseteq N_2)$ not Morita equivalent to $\mathcal{C}(M_1 \subseteq M_2)$, then the II_1 factors N_∞ and M_∞ are not stably isomorphic.*

This corollary shows that remarkably, the (stable) isomorphism class of the inductive limit II_1 factor M_∞ remembers the standard invariant of the initial finite depth subfactor $N \subseteq M$ up to Morita equivalence. In fact, our result complements the rigidity result of Popa, which states that the II_1 factor M_∞ remembers the inclusion $N \subseteq M$ up to weak equivalence [75, Def. 3.1.4 and Cor. 3.6]. As another consequence, our computation for $\tilde{\chi}(M_\infty)$ also computes the ordinary $\chi(M_\infty)$ and κ invariants as the braided subcategory of invertible objects in $\mathcal{Z}(\mathcal{C}(N \subseteq M))$. By Popa's result [74, Thm. 4.2], this is isomorphic to the relative $\chi(M' \cap M_\infty \subseteq N' \cap M_\infty)$ studied by Kawahigashi [48]. The latter inclusion is a finite index hyperfinite II_1 subfactor with standard invariant equivalent to $\mathcal{C}(N \subseteq M)$.

There are many examples of finite depth non-Gamma inclusions. Popa and Shlyakhtenko showed that every subfactor standard invariant can be realized as an inclusion of II_1 factors isomorphic to $L\mathbb{F}_\infty$ [78]. In [24], Guionnet–Jones–Shlyakhtenko provide an alternative realization of finite depth standard invariants as inclusions of interpolated free group factors [25] in their diagrammatic reproof of Popa's celebrated subfactor reconstruction theorem [69]. As every indecomposable unitary multifusion category is Morita equivalent to any of its unitary fusion category diagonal summands, we obtain the following corollary.

Corollary E *For $\mathcal{C}$ a unitary fusion category, its Drinfeld center $\mathcal{Z}(\mathcal{C})$ is realized as $\tilde{\chi}(M)$ for some McDuff II_1 factor M .*

We remark that $\tilde{\chi}(M)$ is a distinct construction from the Drinfeld center $\mathcal{Z}(\text{Bim}(M))$. Indeed, whenever M has full fundamental group $\mathbb{R}_{>0}$, there is an $\mathbb{R}_{>0}$ grading on $\text{Bim}(M)$ from the modular distortion, i.e., the square root of the ratio of the left and right von Neumann dimensions [2, 67]. This grading gives a canonical copy of $\text{Hilb}_{\text{fd}}(\mathbb{R})$ in $\mathcal{Z}(\text{Bim}(M))$ which forgets to the trivial bimodule [28, Sect. 3B]. Hence for R or for the examples M_∞ from Theorem C, $\tilde{\chi}(M)$ is not $\mathcal{Z}(\text{Bim}(M))$.

While we have stated our results above for $\tilde{\chi}(M)$, our analysis actually occurs in a much more general categorical setting. We define the notions of approximately inner and centrally trivial for endofunctors on an arbitrary W^* -category with separable preduals (see Sect. 3.2). Functors which are both approximately inner and centrally trivial are said to be *local*. We construct a unitary braiding on this category (without any dualizability assumptions), and prove all the axioms are satisfied here. Thus we get a new construction of a canonical braided W^* -tensor category $\text{End}_{\text{loc}}(\mathcal{C})$ from an arbitrary W^* -category $\mathcal{C}$.

When $\mathcal{C} = \text{Mod}_{\text{fpp}}(M)$, the finitely generated projective modules of a separable finite von Neumann algebra, under the well known equivalence between

$\text{End}(\text{Mod}_{\text{fgp}}(M))$ and $\text{Bim}_{\text{fgp}}(M)$, our definitions of approximately inner and centrally trivial agree with Popa's from [68]. We use this equivalence to express the braiding as a bimodule intertwiner in Eq. (5). Aside from the greater generality, one of the reasons we use the functor language for our constructions and proofs is that the category of endofunctors is strict, making some commuting diagrams significantly simpler. In addition, in the categorically oriented functor approach, the concept of approximately commuting diagram significantly reduces the complexity of proofs and their verification.

Outline

In Sect. 2, we recall the notion of W^* -category, and we pay special attention to the W^* 2-category of von Neumann algebras, bimodules, and intertwiners. In Sect. 2.3, we discuss the canonical σ -strong* topology on the hom spaces of a W^* -category, which is essential to our construction.

In Sect. 3, we introduce the notion of an approximate natural transformation between endofunctors of a W^* -category, which we use to define the notions of approximately inner and centrally trivial for endofunctors. We show that the local endofunctors which are both approximately inner and centrally trivial admit a canonical unitary braiding in Sect. 3.3.

In Sect. 4, we translate our construction into the language of dualizable bimodules over a II_1 factor M , and we calculate many examples of $\tilde{\chi}(M)$ in Sect. 4.3. In Sect. 5, we prove Theorem B, and in Sect. 6, we prove Theorem C. To prove these theorems, we make heavy use of the Q-system realization machinery developed in [13, 44] and the graphical calculus for unitary tensor categories.

2 Preliminaries

We assume the reader is relatively familiar with the basics of von Neumann algebras, in particular II_1 factors, where our main references include [1, 8, 41, 85]. Most von Neumann algebras that appear in this article are assumed to be *separable* (their preduals are separable), with the exception of ultraproducts in Sect. 4.3 below.

We also assume the reader is relatively familiar with the basics of tensor categories and 2-categories, where our main references include [19, 34, 47]. Of particular importance is the graphical string diagrammatic calculus for 2-categories and tensor categories [34, Sect. 1.1.1 and 8.1.2]. For a 2-category $\mathcal{C}$, objects are represented by 2D shaded regions, 1-morphisms are represented by labelled 1D strands read from left-to-right, and 2-morphisms are represented by labelled 0D coupons which are read bottom-to-top. 1-composition is read left-to-right similar to the relative tensor product of bimodules, and 2-composition is read bottom-to-top.

These string diagrams are formally dual to pasting diagrams, and typically associators and unitors are suppressed whenever possible. As a tensor category is equivalent to a 2-category with one object, the graphical calculus for tensor categories has no shadings for regions; objects are represented by labelled 1D strings, and morphisms

are represented by labelled 0D coupons read bottom-to-top. Tensor product is read left-to-right, and composition of morphisms is read bottom-to-top. Our 2-categories and tensor categories are C^*/W^* (see Sects. 2.1 and 2.2 below), and we represent the $\dagger$ -operation by vertical reflection of diagrams.

2.1 C^*/W^* -categories

We begin with the basics of C^* and W^* -categories. The latter were first introduced in [26].

Definition 2.1 A C^* -category is a $\mathbb{C}$ -linear category $\mathcal{C}$ such that:

- for each pair of objects $a, b \in \mathcal{C}$, there is a conjugate linear involution $\dagger : \mathcal{C}(a \rightarrow b) \rightarrow \mathcal{C}(b \rightarrow a)$, satisfying $(f \cdot g)^\dagger = g^\dagger \cdot f^\dagger$,
- for each pair of objects $a, b \in \mathcal{C}$, there is a Banach norm on $\mathcal{C}(a \rightarrow b)$ satisfying $\|f\|^2 = \|f^\dagger \cdot f\| = \|f \cdot f^\dagger\|$ for all $f \in \mathcal{C}(a \rightarrow b)$, and
- for all $f \in \mathcal{C}(a \rightarrow b)$, $f^\dagger \cdot f$ is a positive element in the C^* -algebra $\mathcal{C}(a \rightarrow a)$. That is, there is a $g \in \mathcal{C}(a \rightarrow a)$ such that $f^\dagger \cdot f = g^\dagger \cdot g$.

A W^* -category is a C^* -category such that every hom space $\mathcal{C}(a \rightarrow b)$ admits a predual Banach space. We call a W^* -category *separable* if all such preduals are separable Banach spaces.

Assumption 2.2 In this article, we assume all C^*/W^* -categories are *unitarily Cauchy complete*, meaning they admit all finite orthogonal direct sums and are orthogonal projection complete. There is a formal construction to complete any C^*/W^* -category which satisfies a universal property; we refer the reader to [27, Sect. 3.1.1] for more details.

Remark 2.3 Every unitarily Cauchy complete C^*/W^* -category admits a canonical left Hilb_{fd} -module category structure. That is, for each $c \in \mathcal{C}$ and finite dimensional Hilbert space H , there is an object $H \triangleright c \in \mathcal{C}$, unique up to canonical unitary isomorphism. Moreover, there is a canonical unitary associator $H \triangleright (K \triangleright a) \cong (H \otimes K) \triangleright a$. In the sequel, we will assume our Hilb_{fd} -module category structure is *strictly unital*, i.e., $\mathbb{C} \triangleright c = c$ for all $c \in \mathcal{C}$.

Definition 2.4 A $\dagger$ -functor between C^* -categories is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F(f^\dagger) = F(f)^\dagger$ for all morphisms f in $\mathcal{C}$. Two C^* -categories are (unitarily) *equivalent* if there are $\dagger$ -functors each way whose appropriate composites are unitarily naturally isomorphic to the appropriate identity $\dagger$ -functors.

For W^* -categories, one restricts to the *normal* $\dagger$ -functors which are weak*-continuous on hom spaces. Equivalence is defined similarly as before, but restricting to normal $\dagger$ -functors.

Example 2.5 The most important example of a W^* -category for our article is the finitely generated projective right modules for a II_1 factor M . There are two dagger equivalent such categories that one can work with:

- Hilbert spaces H equipped with a normal right M -action such that the von Neumann dimension $\dim(H_M)$ is finite, or

- finitely generated projective right Hilbert W^* -modules (see Sect. 2.2 below for more details).

To see the equivalence, the map from the first to the second is taking bounded vectors (the $\xi \in H$ such that $\widehat{m} \mapsto \xi m$ extends to a bounded map $L^2 M \rightarrow H$), and the map from the second to the first is $-\otimes_M L^2 M$ (the inner product is given by $\langle \eta \otimes \widehat{m}, \xi \otimes \widehat{n} \rangle := \langle \langle \xi | \eta \rangle_M \widehat{m}, \widehat{n} \rangle_{L^2 M}$).

We will use the second definition above for the convenience that we may state many results for all $\xi \in X_M$ rather than for all bounded vectors. However, one can work with the first definition provided that one restricts to bounded vectors when appropriate.

Example 2.6 For a separable C^* -algebra A , $\text{Rep}(A)$ is the W^* -category of (non-degenerate) $*$ -representations of A on separable Hilbert spaces. This category is relevant in the operator algebraic study of quantum statistical mechanics.

Definition 2.7 Given a Hilb_{fd} -module C^* category $\mathcal{C}$, a finite dimensional Hilbert space H , and any $\dagger$ -functor $F \in \text{End}(\mathcal{C})$, we have a canonical *braid-like* unitary natural isomorphism

$$\sigma_{F,H} : F(H \triangleright -) \rightarrow H \triangleright F(-)$$

defined as follows. For an orthonormal basis $\{e_i\}$, we may identify its elements as bounded operators $e_i \in B(\mathbb{C}, H)$ defined by $1 \mapsto e_i$. Then $e_i^\dagger \in B(H, \mathbb{C})$ is given by $e_j^\dagger e_i = \delta_{i,j}$. We then define $\sigma_{F,H}$ in components by

$$\sigma_{F,H}^a := \sum_i (e_i \triangleright F(1_a)) \cdot F(e_i^\dagger \triangleright 1_a),$$

which does not depend on the choice of orthonormal basis of H .

Unitarity is straightforward to verify. To show naturality, let $f \in \mathcal{C}(a \rightarrow b)$. Then

$$\begin{aligned} (1_H \triangleright F(f)) \cdot \sigma_{F,H}^a &= \sum_i (e_i \triangleright F(f)) \cdot F(e_i^\dagger \triangleright 1_a) \\ &= \sum_i (e_i \triangleright 1_b) \cdot F(f) \cdot F(e_i^\dagger \triangleright 1_a) \\ &= \sum_i (e_i \triangleright 1_b) \cdot F(e_i^\dagger \triangleright f) = \sigma_{F,H}^b \cdot F(1_H \triangleright f). \end{aligned}$$

The family σ also satisfies the following monoidality conditions (where we have suppressed the module category associator).

Proposition 2.8 (1) For any $F \in \text{End}(\mathcal{C})$, $\sigma_{F, H \otimes K}^a = (1_H \triangleright \sigma_{F,K}^a) \cdot \sigma_{F,H}^{K \triangleright a}$.
(2) For any $G \in \text{End}(\mathcal{C})$ we have $\sigma_{F \circ G, H}^a = \sigma_{F,H}^{G(a)} \cdot F(\sigma_{G,H}^a)$.

Proof. Let $\{f_j\}$ be an orthonormal basis for K , which we identify with bounded operators $f_j \in B(\mathbb{C}, K)$. Then

$$\begin{aligned} (1_H \triangleright \sigma_{F,K}^a) \cdot \sigma_{F,H}^{K \triangleright a} &= \sum_{i,j} (1_H \triangleright f_j \triangleright F(1_a)) \cdot (1_H \triangleright F(f_j^\dagger \triangleright 1_a)) \\ &\quad \cdot (e_i \triangleright F(1_{K \triangleright a}) \cdot F(e_i^\dagger \triangleright 1_{K \triangleright a})) \\ &= \sum_{i,j} (1_H \triangleright f_j \triangleright F(1_a)) \cdot (e_i \triangleright F(1_a)) \cdot F(f_j^\dagger \triangleright 1_a) \\ &\quad \cdot F(e_i^\dagger \triangleright 1_{K \triangleright a}) = \sigma_{F,H \otimes K}^a \end{aligned}$$

and

$$\begin{aligned} \sigma_{F,H}^{G(a)} \cdot F(\sigma_{G,H}^a) &= \sum_i (e_i \triangleright FG(1_a)) \cdot F(e_i^\dagger \triangleright G(1_a)) \cdot F(e_i \triangleright G(1_a)) \cdot FG(e_i^\dagger \triangleright 1_a) \\ &= \sum_i (e_i \triangleright FG(1_a)) \cdot FG(e_i^\dagger \triangleright 1_a) = \sigma_{F \circ G, H}^a. \quad \square \end{aligned}$$

2.2 Modules and correspondences of W^* -algebras

We now recall the definition of the W^* 2-category $W^*\text{Alg}_{\text{fpg}}$ of finitely generated projective right W^* -correspondences, after which we formally define the finitely generated projective right modules $\text{Mod}_{\text{fpg}}(M)$. Our exposition follows [13, Sect. 2.2], which was adapted from [9, Sect. 8]. Other references include [65, 81].

Definition 2.9 The W^* 2-category $W^*\text{Alg}_{\text{fpg}}$ is given as follows.

- objects are von Neumann algebras
- 1-morphisms are finitely generated projective right W^* -correspondences. In more detail, given von Neumann algebras A, B , a 1-morphism ${}_A X_B$ is a Banach space equipped with a right B -action and a right B -valued inner product satisfying
 - $\langle \eta | \xi_1 + \xi_2 b \rangle_B = \langle \eta | \xi_1 \rangle_B + \langle \eta | \xi_2 \rangle_B b$,
 - $\langle \eta_1 + \eta_2 b | \xi \rangle_B = \langle \eta_1 | \xi \rangle_B + b^* \langle \eta_2 | \xi \rangle_B$,
 - $\langle \eta | \xi \rangle_B^* = \langle \xi | \eta \rangle_B$, and
 - $\langle \xi | \xi \rangle_B \geq 0$ with equality if and only if $\xi = 0$.

By the Cauchy–Schwarz inequality, $\|\langle \xi | \xi \rangle_B\|_B$ defines a norm on X , which is required to be complete. Moreover, we require the left A -action to be by adjointable operators.

The finitely generated projective condition says that as a right B -module, X_B is unitarily isomorphic to pB^n for some (adjointable) orthogonal projection $p \in \text{End}_{-B}(B^n)$.

The W^* condition amounts to requiring that:

- ${}_A X_B$ has a predual,
- the B -valued inner product $\langle \cdot | \cdot \rangle_B$ is separately weak*-continuous, and

- the left A -action $A \rightarrow \text{End}(X_B)$ is normal.

Composition of 1-morphisms is the relative tensor product.

- 2-morphisms ${}_A X_B \Rightarrow {}_A Y_B$ are the adjointable right B -linear operators that commute with the left A -action.

Definition 2.10 For a von Neumann algebra M , we define $\text{Mod}_{\text{fgp}}(M) := W^*\text{Alg}_{\text{fgp}}(\mathbb{C} \rightarrow M)$.

Given $X_M \in \text{Mod}_{\text{fgp}}(M)$, a finite X_M -basis is a finite subset $\{\beta\} \subseteq X$ such that $\sum_{\beta} \beta \langle \beta | \xi \rangle_M = \xi$ for all $\xi \in X$ [69, Sect. 1.1.3], [10, Sect. 3.1.1]. As we only work with finitely generated projective modules in this article, all X_M -bases will be finite, so we omit the word ‘finite’ without confusion. We call such a basis *orthogonal* if $\langle \beta | \beta' \rangle_M$ is equal to $\delta_{\beta=\beta'}$ times an orthogonal projection in M . Given an X_M -basis, one can always obtain an orthogonal X_M -basis using the Gram-Schmidt orthogonalization procedure [1, Lem. 8.5.2].

Remark 2.11 It is well known that the W^* -tensor category $\text{End}(\text{Mod}_{\text{fgp}}(M))$ of normal $\dagger$ -endofunctors is dagger equivalent to $\text{Bim}_{\text{fgp}}(M)^{\text{mp}}$, the monoidal opposite of $\text{Bim}_{\text{fgp}}(M)$. That is, every normal $\dagger$ -endofunctor of $\text{Mod}_{\text{fgp}}(M)$ is of the form $-\boxtimes_M X$ for some $X \in \text{Mod}_{\text{fgp}}(M)$. For example, this equivalence is explained in [33, Sect. 3.2] for infinite von Neumann algebras using the fact that $\text{End}(\text{Mod}_{\text{fgp}}(M))$ is unitarily equivalent to the orthogonal projection completion of the W^* -tensor category $\text{End}(M)$.

Remark 2.12 There is another way that the category of $\text{End}(\text{Mod}_{\text{fgp}}(N))$ is used in practice, particularly among the II_1 factor community. This stems from the fact that $\text{Mod}_{\text{fgp}}(N)$ is equivalent to the unitary Cauchy completion of N thought of as a W^* -category with one object. Objects in the completion are pairs (n, p) , where $n \in \mathbb{N}$ and $p \in M_n(N)$ is a projection. Morphisms $(n, p) \rightarrow (m, q)$ are elements of $qM_{m \times n}(N)p$. By the universal property of Cauchy completion, an endofunctor is determined by where it sends $(1, 1_N)$ together with its action on $\text{End}((1, 1_N)) \cong N$. In other words, an endofunctor in $\text{End}(\text{Mod}_{\text{fgp}}(N))$ is completely determined up to unitary natural isomorphism by a (unital) homomorphism $N \rightarrow pM_n(N)p$ for some projection $p \in M_n(N)$, called a *cofinite morphism* of N in [84]. Furthermore, a natural transformation is uniquely determined by its $(1, 1_N)$ -component.

2.3 The σ -strong* topology on a W^* -category

Let $\mathcal{C}$ be a separable W^* -category, which has a canonical weak* topology on each hom space.

Definition 2.13 For each $a, b \in \mathcal{C}$, the σ -strong* topology τ on $\mathcal{C}(a \rightarrow b)$ is defined as follows: $f_i \rightarrow 0$ σ -strong* if and only if $f_i^\dagger f_i \rightarrow 0$ and $f_i f_i^\dagger \rightarrow 0$ weak* (σ -weakly).

Facts 2.14 The σ -strong* topology τ on the hom spaces of $\mathcal{C}$ satisfies the following properties:

- ($\tau 1$) composition is jointly τ -continuous on norm bounded subsets (if $\{a_n\}, \{b_n\}$ are uniformly norm bounded, and $a_n \rightarrow a$ and $b_n \rightarrow b$ then $a_n b_n \rightarrow ab$).

- (τ_2) $\dagger$ is τ -continuous on norm bounded subsets of morphism spaces.
 (τ_3) τ restricted to the unit ball of any morphism space is completely metrizable.

The following proposition basically follows from [8, III.2.2.2].

Proposition 2.15 *Suppose M, N are von Neumann algebras and $\Phi : M \rightarrow N$ is a unital $*$ -homomorphism. Then Φ is normal if and only if it is σ -strong* continuous on the unit ball of M .*

Proof If Φ is σ -strong* continuous on bounded sets, then for any increasing bounded net (x_i) in M with $x_i \nearrow x$, $x_i \rightarrow x$ in the σ -strong* topology. Hence $\Phi(x_i) \nearrow \Phi(x)$, and Φ is normal.

The converse argument is similar to [8, III.2.2.2]. If $x_i \rightarrow 0$ σ -strong*, then $x_i x_i^* \rightarrow 0$ and $x_i^* x_i \rightarrow 0$ σ -weakly. Hence $\Phi(x_i)^* \Phi(x_i) \rightarrow 0$ and $\Phi(x_i) \Phi(x_i)^* \rightarrow 0$ σ -weakly, which implies $\Phi(x_i) \rightarrow 0$ σ -strong*. $\square$

As all W^* -categories were assumed to admit finite orthogonal direct sums, we have the following immediate corollary.

Corollary 2.16 *Suppose $\mathcal{C}, \mathcal{D}$ are W^* -categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ is a $\dagger$ -functor. Then F is normal if and only if F is σ -strong* continuous on norm bounded subsets of hom spaces in $\mathcal{C}$.*

Example 2.17 Let (M, tr_M) be a finite von Neumann algebra equipped with a faithful tracial state τ . In this case, the σ -strong* topology on the unit ball of M is exactly the $\|\cdot\|_2$ -topology, where $\|x\|_2^2 := \text{tr}_M(x^*x)$ [41, Prop. 9.1.1]. In fact, we may also describe the entire σ -strong* topology τ on $\text{Mod}_{\text{fpp}}(M)$ on norm bounded sets as coming from a $\|\cdot\|_2$ -norm induced by canonical commutant traces.

In more detail, for $X_M \in \text{Mod}_{\text{fpp}}(M)$, the canonical *commutant trace* [69, Sect. 1.1.3(c)], [10, Def. 3.1.4] on the finite von Neumann algebra $\text{End}(X_M)$ is given by

$$\text{Tr}_X(f) := \sum_b \text{tr}_M(\langle b|xb \rangle_M^X)$$

where $\{b\}$ is any finite X_M -basis. Observe that Tr_X is independent of the choice of basis. When M is a II_1 factor, $\text{Tr}(\text{id}_X)$ equals the right von Neumann dimension of $X \otimes_M L^2 M$.

Observe that the maps $(\text{Tr}_X)_{X \in \text{Mod}_{\text{fpp}}(M)}$ endow $\text{Mod}_{\text{fpp}}(M)$ with a *unitary categorical trace* in the spirit of [83, Def. 3.7] (see also [27, Def. 3.59]). Indeed, for $f \in \text{Hom}(X_M \rightarrow Y_M)$, we choose a finite X_M -basis $\{b\}$ and a finite Y_M -basis $\{c\}$, and we calculate

$$\begin{aligned}
\mathrm{Tr}_X(f^\dagger f) &= \sum_b \mathrm{tr}_M(\langle b|f^\dagger f b\rangle_M^X) = \sum_b \mathrm{tr}_M(\langle f b|f b\rangle_M^Y) \\
&= \sum_{b,c} \mathrm{tr}_M(\langle f b|c \langle c|f b\rangle_M^Y \rangle_M^Y) \\
&= \sum_{b,c} \mathrm{tr}_M(\langle f b|c \rangle_M^Y \langle c|f b\rangle_M^Y) = \sum_{b,c} \mathrm{tr}_M(\langle f^\dagger c|b \rangle_M^X \langle b|f^\dagger c \rangle_M^X) \\
&= \sum_{b,c} \mathrm{tr}_M(\langle b|b|f^\dagger c \rangle_M^X |f^\dagger c \rangle_M^X) \\
&= \sum_c \mathrm{tr}_M(f^\dagger c |f^\dagger c \rangle_M^X) = \sum_c \mathrm{tr}_M(c |f f^\dagger c \rangle_M^X) = \mathrm{Tr}_Y(f f^\dagger).
\end{aligned}$$

Using this categorical trace, for $f \in \mathrm{Hom}(X_M \rightarrow Y_M)$, we define $\|f\|_2 := \mathrm{Tr}_X(f^\dagger \circ f)^{1/2}$. Observe that $\|\cdot\|_2$ on $\mathrm{Hom}(X_M \rightarrow Y_M)$ is exactly the restriction of $\|\cdot\|_2$ on $\mathrm{End}(X_M \oplus Y_M)$, which is again defined using the canonical commutant trace $\mathrm{Tr}_{X \oplus Y}$. Thus the σ -strong* topology τ on $\mathrm{Mod}_{\mathrm{fgp}}(M)$ exactly corresponds to the $\|\cdot\|_2$ -topology on norm-bounded sets.

The following remark will be used later in Sect. 5.2.

Remark 2.18 Suppose (M, tr_M) is a finite von Neumann algebra equipped with a normal faithful trace tr_M . Suppose $X_M \in \mathrm{Mod}_{\mathrm{fgp}}(M)$ and $N \subseteq (M, \mathrm{tr}_M)$ is strongly Markov inclusion [42, Def. 2.8], i.e., there is a finite M_N -basis $\{c\}$ which satisfies $[M : N] := \sum_c c c^* \in [1, \infty)$. (This definition was based on [3] and [69, Sect. 1.1.3 and 1.1.4].) Here, the right N -valued inner product on M_N is given by $\langle a|b \rangle_N = E_N(a^* b)$ where $E_N : M \rightarrow N$ is the unique trace-preserving conditional expectation.

We now compare Tr_{X_M} and $\mathrm{Tr}_{X \boxtimes_M M_N}$. If $\{b\}$ is a basis for X_M and $\{c\}$ is a basis for M_N , then $\{b \boxtimes c\}$ is a basis for $X \boxtimes_M M_N$. Thus for $f \in \mathrm{End}(X_M)$,

$$\begin{aligned}
\mathrm{Tr}_{X \boxtimes_M M_N}(f \boxtimes \mathrm{id}_{M_N}) &= \sum_{b,c} \mathrm{tr}_N(\langle b \boxtimes c|f b \boxtimes c \rangle_N^{X \boxtimes_M M_N}) \\
&= \sum_{b,c} \mathrm{tr}_N(E_N(c^* \langle b|f b \rangle_M^X c)) \\
&= \sum_{b,c} \mathrm{tr}_M(c^* \langle b|f b \rangle_M^X c) = \sum_{b,c} \mathrm{tr}_M(\langle b|f b \rangle_M^X c c^*) \\
&= [M : N] \sum_b \mathrm{tr}_M(\langle b|f b \rangle_M^X) = [M : N] \mathrm{Tr}_X(f).
\end{aligned}$$

This says the (faithful) restriction functor $- \boxtimes M_N : \mathrm{Mod}_{\mathrm{fgp}}(M) \rightarrow \mathrm{Mod}_{\mathrm{fgp}}(N)$ is a continuous embedding (a homeomorphism onto its image) of hom spaces with respect to the $\tau_M - \tau_N$ topologies. This means that $f_n \rightarrow 0$ in τ_M if and only if $f_n \boxtimes \mathrm{id}_{M_N} \rightarrow 0$ in τ_N .

3 Approximate natural transformations and local endofunctors

In this section, given a W^* -category, we define its canonical braided W^* -tensor category of local endofunctors which are both approximately inner and centrally trivial. To define these notions, we first introduce the concept of an approximate natural transformation.

3.1 Approximate natural transformations

Suppose $\mathcal{C}$ is a separable W^* -category, and recall $\text{End}(\mathcal{C})$ denotes the normal $\dagger$ -endofunctors of $\mathcal{C}$. We define $\ell^\infty(\mathbb{N}, \mathcal{C})$ as the W^* -category with the same objects as $\mathcal{C}$ and whose morphisms are uniformly norm-bounded sequences of morphisms in $\mathcal{C}$. The composition and $\dagger$ in $\ell^\infty(\mathbb{N}, \mathcal{C})$ are defined pointwise.

Definition 3.1 For each $a, b \in \mathcal{C}$, we define

$$\begin{aligned}\mathcal{I}(a \rightarrow b) &:= \left\{ f = (f_n) \in \ell^\infty(\mathbb{N}, \mathcal{C}(a \rightarrow b)) \mid f_n \rightarrow_\tau 0 \right\} \\ \mathcal{C}^\infty(a \rightarrow b) &:= \left\{ f \in \ell^\infty(\mathbb{N}, \mathcal{C}(a \rightarrow b)) \mid \forall g \in \mathcal{I}(b \rightarrow c), g \cdot f \in \mathcal{I}(a \rightarrow c) \text{ and } \forall h \in \mathcal{I}(d \rightarrow a), f \cdot h \in \mathcal{I}(d \rightarrow b) \right\}.\end{aligned}$$

We view $\mathcal{C}^\infty$ as the *idealizer* of $\mathcal{I}$ in $\ell^\infty(\mathbb{N}, \mathcal{C})$. We call $f \in \mathcal{C}^\infty(a \rightarrow b)$ an *approximate morphism*, and we say two morphisms $f, g \in \mathcal{C}^\infty(a \rightarrow b)$ are *equivalent* or *approximately equal* if $f - g \in \mathcal{I}(a \rightarrow b)$. Observe that $\mathcal{C}^\infty$ is a category under pointwise composition of approximate morphisms. By $(\tau 1)$ and $(\tau 2)$, $\mathcal{I}(a \rightarrow b) \subseteq \mathcal{C}^\infty(a \rightarrow b)$ for all a, b , and $\mathcal{I}$ defines a $\dagger$ -closed ideal in $\mathcal{C}^\infty$.

We define a $\dagger$ -category $\tilde{\mathcal{C}}$ with the same objects as $\mathcal{C}$, and hom spaces $\tilde{\mathcal{C}}(a \rightarrow b) := \mathcal{C}^\infty(a \rightarrow b) / \mathcal{I}(a \rightarrow b)$. For $f \in \mathcal{C}^\infty(a \rightarrow b)$, we write $\tilde{f}$ for its image in $\tilde{\mathcal{C}}(a \rightarrow b)$. Observe we can view $\mathcal{C}$ as a $\dagger$ -subcategory of $\tilde{\mathcal{C}}$ by mapping $f \in \mathcal{C}(a \rightarrow b)$ to the image of the constant sequence $(\tilde{f}) \in \tilde{\mathcal{C}}(a \rightarrow b)$. In what follows, we identify $f \in \mathcal{C}(a \rightarrow b)$ with $(f)_{n \in \mathbb{N}} \in \mathcal{C}^\infty(a \rightarrow b)$ and $(\tilde{f})_{n \in \mathbb{N}} \in \tilde{\mathcal{C}}(a \rightarrow b)$.

Definition 3.2 A diagram in $\mathcal{C}^\infty$ is said to *approximately commute* if the corresponding diagram in the quotient $\tilde{\mathcal{C}} = \mathcal{C}^\infty / \mathcal{I}$ actually commutes. That is, given $a, b, c, d \in \mathcal{C}$ and morphisms $f \in \mathcal{C}^\infty(a \rightarrow b)$, $g \in \mathcal{C}^\infty(b \rightarrow d)$, $h \in \mathcal{C}^\infty(a \rightarrow c)$, and $k \in \mathcal{C}^\infty(c \rightarrow d)$, the diagram

$$\begin{array}{ccc} a & \xrightarrow{f_n} & b \\ h_n \downarrow & & \downarrow g_n \\ c & \xrightarrow{k_n} & d \end{array}$$

approximately commutes if $g_n \cdot f_n - k_n \cdot h_n \rightarrow_\tau 0$.

For a normal $\dagger$ -endofunctor $F \in \text{End}(\mathcal{C})$, for any $f = (f_n) \in \mathcal{C}^\infty(a \rightarrow b)$, $(F(f_n)) \in \mathcal{C}^\infty(F(a) \rightarrow F(b))$, so F descends to a $\dagger$ -endofunctor on $\tilde{\mathcal{C}}$.

Definition 3.3 Given two functors $F, G \in \text{End}(\mathcal{C})$ an *approximate natural transformation* is a family $\{\eta^a \in \mathcal{C}^\infty(F(a) \rightarrow G(a))\}_{a \in \mathcal{C}}$ such that for every $f \in \mathcal{C}(a \rightarrow b)$, $\eta^b \cdot f = f \cdot \eta^a$ in $\tilde{\mathcal{C}}$. In other words, the following diagram approximately commutes:

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_n^a} & G(a) \\ F(f) \downarrow & & \downarrow G(f) \\ F(b) & \xrightarrow{\eta_n^b} & G(b). \end{array}$$

Clearly every natural transformation $\eta : F \Rightarrow G$ gives an approximate natural transformation.

Warning 3.4 The collection of normal $\dagger$ -endofunctors of $\mathcal{C}$ and approximate natural transformations between them (up to $\mathcal{I}$) clearly forms a $\dagger$ -category containing $\text{End}(\mathcal{C})$ as a (non-full) subcategory. It is tempting to think that this should also form a tensor category, with tensor product being composition of endofunctors as usual. However, the horizontal composition of two approximate natural transformations is not well-defined in general. This is a fundamental point: the category of endofunctors and approximately natural transformations is *not* a tensor category in general, as endomorphisms of the ‘unit’ in this category (see Definition 3.7) is not a commutative algebra.

Definition 3.5 We call $v \in \mathcal{C}^\infty(a \rightarrow b)$ an *approximate isometry* if its image $\tilde{v} \in \tilde{\mathcal{C}}(a \rightarrow b)$ is an isometry. For $\dagger$ -endofunctors $F, G \in \text{End}(\mathcal{C})$, an approximate natural transformation $v : F \Rightarrow G$ is called an *approximately isometric natural transformation* if $v^a \in \mathcal{C}^\infty(F(a) \rightarrow G(a))$ is an approximate isometry for each $a \in \mathcal{C}$.

Remark 3.6 (Arrow flipping) Suppose $f \in \mathcal{C}^\infty(a \rightarrow c)$, $v \in \mathcal{C}^\infty(a \rightarrow b)$, and $w \in \mathcal{C}^\infty(b \rightarrow c)$ such that the diagram

$$\begin{array}{ccc} a & \xrightarrow{f_n} & c \\ & \searrow v_n & \nearrow w_n \\ & b & \end{array}$$

approximately commutes.

- If w is an approximate isometry, then $\begin{array}{ccc} a & \xrightarrow{f_n} & c \\ & \searrow v_n & \nearrow w_n^\dagger \\ & b & \end{array}$ approximately commutes.
- If v is an approximate coisometry, then $\begin{array}{ccc} a & \xrightarrow{f_n} & c \\ & \nwarrow v_n^\dagger & \nearrow w_n \\ & b & \end{array}$ approximately commutes.

Indeed, these remarks follow since $\tilde{v}$, respectively $\tilde{w}$, is an actual isometry, respectively coisometry, in $\tilde{\mathcal{C}}$.

Definition 3.7 An approximately natural transformation from the identity functor to itself is called a *central sequence*.

Lemma 3.8 Suppose $\mathcal{C}$ is a separable W^* -category which is not necessarily unitarily Cauchy complete. Every central sequence of $\mathcal{C}$ has a canonical extension to the unitary Cauchy completion of $\mathcal{C}$. Moreover, this extension gives a bijective correspondence between the equivalence classes of central sequences of $\mathcal{C}$ and the equivalence classes of central sequences of the unitary Cauchy completion of $\mathcal{C}$.

Proof We proceed in 2 steps; first, we show the result for the orthogonal direct sum completion $\text{Add}(\mathcal{C})$, and second, we show the result for the orthogonal projection completion $\text{Proj}(\mathcal{C})$.

Step 1: We claim a central sequence η of $\mathcal{C}$ gives a central sequence of $\text{Add}(\mathcal{C})$ by

$\text{Add}(\eta)_{ii} := \pi_i^\dagger \eta^{c_i} \pi_i$ for $\bigoplus_i c_i \in \text{Add}(\mathcal{C})$, where $\pi_j : \bigoplus c_i \rightarrow c_j$ denotes the canonical projections satisfying $\sum_i \pi_i^\dagger \pi_i = \text{id}_{\bigoplus_i c_i}$ and $\pi_i \circ \pi_j^\dagger = \delta_{i=j} \text{id}_{c_i}$. Let $f^n = (f_{ji}^n) \in \text{Add}(\mathcal{C})^\infty(\bigoplus_i c_i \rightarrow \bigoplus_j d_j)$, with each $(f_{ji}^n) \in \mathcal{C}^\infty(c_i \rightarrow d_j)$ a bounded sequence of morphisms. Then clearly $(f^n) \in \mathcal{I}(\bigoplus_i c_i \rightarrow \bigoplus_j d_j)$ if and only if each $(f_{ji}^n) \in \mathcal{I}(c_i \rightarrow d_j)$. Therefore since $\eta \in \mathcal{C}^\infty$, we have $\text{Add}(\eta) \in \text{Add}(\mathcal{C})^\infty$.

To see that $\text{Add}(\eta)$ defines a natural transformation of the identity functor, let $f = (f_{ji}) \in \text{Add}(\mathcal{C})(\bigoplus_i c_i \rightarrow \bigoplus_j d_j)$ as above. Then in $\tilde{\mathcal{C}}$, $\text{Add}(\eta)f = (\eta_n^{d_j} f_{ji}) = (f_{ji} \eta_n^{c_i}) = f \text{Add}(\eta)$. Moreover, the assignment $\eta \mapsto \text{Add}(\eta)$ clearly preserves equivalence of central sequences.

Conversely, given a central sequence η of $\text{Add}(\mathcal{C})$, we automatically get a central sequence of $\mathcal{C}$ by considering the canonical embedding $\mathcal{C} \hookrightarrow \text{Add}(\mathcal{C})$. Moreover, for $\bigoplus_i c_i \in \text{Add}(\mathcal{C})$, the off-diagonal terms of $\eta^{\bigoplus_i c_i}$ go to zero σ -strong* as $\eta^{\bigoplus_i c_i}$ approximately commutes with the π_j , and the diagonal term corresponding to c_i must be approximately equivalent to η^{c_i} .

Step 2: A central sequence η of $\mathcal{C}$ gives a central sequence of $\text{Proj}(\mathcal{C})$ by defining

$\text{Proj}(\eta)^{(c,p)} := p \eta^c p$ for $(c, p) \in \text{Proj}(\mathcal{C})$. Given $(a, p), (b, q) \in \text{Proj}(\mathcal{C})$, $\mathcal{I}((a, p) \rightarrow (b, q)) = q \mathcal{I}(a \rightarrow b) p$, so $\text{Proj}(\eta) \in \text{Proj}(\mathcal{C})^\infty$. Moreover, this construction preserves equivalence of central sequences.

Conversely, given a central sequence η of $\text{Proj}(\mathcal{C})$, we automatically get a central sequence of $\mathcal{C}$ by considering the canonical embedding $\mathcal{C} \hookrightarrow \text{Proj}(\mathcal{C})$. Clearly starting with a central sequence in $\mathcal{C}$, extending to $\text{Proj}(\mathcal{C})$ as above, and restricting back to $\mathcal{C}$ yields the same central sequence. In the other direction, let η be a central sequence in $\text{Proj}(\mathcal{C})$. We need to show that in $\widehat{\text{Proj}(\mathcal{C})}$, $p \eta^{(c, \text{id}_c)} p = \eta^{(c,p)}$. But note we can view an orthogonal projection $p \in \text{End}_{\mathcal{C}}(c)$ as a morphism in $\text{Proj}(\mathcal{C})((c, p) \rightarrow (c, \text{id}_c))$. Then in $\widehat{\text{Proj}(\mathcal{C})}$, $\eta^{(c,p)} = \eta^{(c,p)} p = (\eta^{(c,p)} p) p = p \eta^{(c, \text{id}_c)} p$ as desired.

□

Example 3.9 For a separable von Neumann algebra M , central sequences in the sense of Definition 3.7 for the W^* -category $\text{Mod}_{\text{fgp}}(M)$ exactly agree with the usual notion of central sequences of M . Indeed, consider the category with one object M_M whose endomorphisms is M acting by left multiplication. An approximate natural transformation of the identity functor is exactly a sequence (x_n) such that $x_n m - m x_n \rightarrow 0$ in the σ -strong* topology for all $m \in M$. Now since $\text{Mod}_{\text{fgp}}(M)$ is the unitary Cauchy completion of this one object category, the claim follows by Lemma 3.8.

The next lemma will be important in the next subsection.

Lemma 3.10 Suppose H, K are finite dimensional Hilbert spaces with orthonormal bases $\{e_i\}, \{f_j\}$ respectively. For a collection of maps $(\eta^a \in C^\infty(H \triangleright a \rightarrow K \triangleright a))_{a \in \mathcal{C}}$, we define $\eta_{i,j}^a := (f_j^\dagger \triangleright 1_a) \eta^a (e_i \triangleright 1_a) \in C^\infty(a \rightarrow a)$. Then $\eta : H \triangleright - \Rightarrow K \triangleright -$ defines an approximate natural transformation if and only if $\eta_{i,j}$ is a central sequence for each i, j .

Proof Note that $\eta^a = \sum_{i,j} (f_j \triangleright 1_a) \eta_{i,j}^a (e_i^\dagger \triangleright 1_a)$. Then for $g \in \mathcal{C}(a \rightarrow b)$,

$$\begin{aligned} g \eta_{i,j}^a - \eta_{i,j}^b g &= g (f_j^\dagger \triangleright 1_a) \eta^a (e_i \triangleright 1_a) - (f_j^\dagger \triangleright 1_b) \eta^b (e_i \triangleright 1_b) g \\ &= (f_j^\dagger \triangleright 1_a) (1_K \triangleright g) \eta^a (e_i \triangleright 1_a) - (f_j^\dagger \triangleright 1_b) \eta^b (1_H \triangleright g) (e_i \triangleright 1_b) \\ &= (f_j^\dagger \triangleright 1_b) \left((1_K \triangleright g) \eta^a - \eta^b (1_H \triangleright g) \right) (e_i \triangleright 1_a), \end{aligned}$$

and

$$(1_K \triangleright g) \eta^a - \eta^b (1_H \triangleright g) = \sum_{i,j} (f_j \triangleright 1_b) \left(g \eta_{i,j}^a - \eta_{i,j}^b g \right) (e_i^\dagger \triangleright 1_a).$$

Thus $(1_K \triangleright g) \eta^a - \eta^b (1_H \triangleright g) = 0$ in $\tilde{\mathcal{C}}$ if and only if $g \eta_{i,j}^a - \eta_{i,j}^b g = 0$ in $\tilde{\mathcal{C}}$ for all i, j . $\square$

3.2 Centrally trivial and approximately inner endofunctors

For the rest of this section, $\mathcal{C}$ is a fixed separable W^* -category.

Definition 3.11 A functor $F \in \text{End}(\mathcal{C})$ is called *centrally trivial* if for all finite dimensional Hilbert spaces and all approximate natural transformations $\eta : H \triangleright - \Rightarrow K \triangleright -$, the following diagram approximately commutes.

$$\begin{array}{ccc} F(H \triangleright a) & \xrightarrow{F(\eta_n^a)} & F(K \triangleright a) \\ \sigma_{F,H}^a \downarrow & & \downarrow \sigma_{F,K}^a \\ H \triangleright F(a) & \xrightarrow{\eta_n^{F(a)}} & K \triangleright F(a) \end{array}$$

Proposition 3.12 A functor F is centrally trivial if and only if for all central sequences η , $F(\eta^a) = \eta^{F(a)}$ in $\tilde{\mathcal{C}}$.

Proof It is clear that $F(\eta^a) = \eta^{F(a)}$ in $\tilde{\mathcal{C}}$ for all central sequences η if F is centrally trivial. Conversely, for each approximate natural transformation $\eta : H \triangleright - \Rightarrow K \triangleright -$, $\eta_{i,j}$ defined in Lemma 3.10 is a central sequence for each i, j . Then by Definition 2.7,

$$\begin{aligned} \sigma_{F,K}^a F(\eta^a) (\sigma_{F,H}^a)^\dagger &= \sum_j (f_j \triangleright F(1_a)) F(f_j^\dagger \triangleright 1_a) F(\eta^a) \sum_i F(e_i \triangleright 1_a) (e_i^\dagger \triangleright F(1_a)) \\ &= \sum_{i,j} (f_j \triangleright F(1_a)) F(\eta_{i,j}^a) (e_i^\dagger \triangleright F(1_a)) \\ &= \sum_{i,j} (f_j \triangleright F(1_a)) \eta_{i,j}^{F(a)} (e_i^\dagger \triangleright F(1_a)) = \eta^{F(a)}, \end{aligned}$$

which implies that F is centrally trivial. $\square$

Definition 3.13 We denote the full subcategory of $\text{End}(\mathcal{C})$ of centrally trivial endofunctors by $\text{End}_{\text{ct}}(\mathcal{C})$.

Proposition 3.14 $\text{End}_{\text{ct}}(\mathcal{C})$ is a replete unitarily Cauchy complete W^* -tensor subcategory of $\text{End}(\mathcal{C})$.

Proof Suppose G, G' are centrally trivial. For each central sequence η , by Proposition 3.12, we have

$$G(G'(\eta^a)) = G(\eta^{G'(a)}) = \eta^{G(G'(a))},$$

which implies $G \circ G'$ is centrally trivial.

Now suppose G is centrally trivial and $v : F \Rightarrow G$ is an isometric natural transformation for some other endofunctor $F \in \text{End}(\mathcal{C})$. That $F(\eta^a) = \eta^{F(a)}$ for all central sequences η follows from the following approximately commuting diagram

$$\begin{array}{ccccc} & & F(\eta_n^a) & & \\ & \swarrow & \text{ } & \searrow & \\ F(a) & \xrightarrow{v^a} & G(a) & \xrightarrow{G(\eta_n^a)} & G(a) & \xrightarrow{(v^a)^\dagger} & F(a) \\ & \nwarrow & \eta_n^{G(a)} & \nearrow & \eta_n^{F(a)} & \nwarrow & \\ & & \eta_n^{F(a)} & & \end{array} \quad (1)$$

where we have used Remark 3.6 to flip the arrow on the right. Considering the case when v is unitary shows that $\text{End}_{\text{ct}}(\mathcal{C})$ is replete.

As $\mathcal{C}$ admits orthogonal direct sums, so does $\text{End}(\mathcal{C})$. Suppose G, G' are centrally trivial, and let η be a central sequence. Then

$$(G \oplus G')(\eta^a) = G(\eta^a) \oplus G'(\eta^a) = \eta^{G(a)} \oplus \eta^{G'(a)}.$$

By the proof of Step 1 of Lemma 3.8, $\eta^{G(a)} \oplus \eta^{G'(a)} = \eta^{(G \oplus G')(a)}$ in $\tilde{\mathcal{C}}$, and thus $G \oplus G'$ is centrally trivial.

As $\mathcal{C}$ is orthogonal projection complete, so is $\text{End}(\mathcal{C})$. Suppose G is centrally trivial and $\pi : G \Rightarrow G$ is an orthogonal projection natural transformation. Then π orthogonally splits in $\text{End}(\mathcal{C})$, so there is an $F \in \text{End}(\mathcal{C})$ and an isometry natural transformation $v : F \Rightarrow G$. But then $F \in \text{End}_{\text{ct}}(\mathcal{C})$ by (1) above. $\square$

Example 3.15 For a finite dimensional Hilbert space H , the functor $H \triangleright -$ is centrally trivial. Note that the identity functor $\text{id}_{\mathcal{C}}$ is centrally trivial, and thus so is $\bigoplus_i \text{id}_{\mathcal{C}}$. Since the functor $H \triangleright -$ is equivalent to $\bigoplus_i \text{id}_{\mathcal{C}}$, by Proposition 3.14, $H \triangleright -$ is centrally trivial.

Definition 3.16 A functor $F \in \text{End}(\mathcal{C})$ is called *approximately inner* if there exists a finite dimensional Hilbert space H and an approximate isometry natural transformation $v : F \Rightarrow H \triangleright -$ in $\text{End}(\tilde{\mathcal{C}})$, i.e.,

$$\begin{array}{ccc} F(a) & \xrightarrow{v_n^a} & H \triangleright a \\ F(f) \downarrow & & \downarrow 1_H \triangleright f \\ F(b) & \xrightarrow{v_n^b} & H \triangleright b \end{array}$$

approximately commutes. The pair (v, H) is called an *approximating sequence* for F .

Definition 3.17 We denote the full subcategory of $\text{End}(\mathcal{C})$ of approximately inner endofunctors by $\text{End}_{\text{ai}}(\mathcal{C})$.

Proposition 3.18 $\text{End}_{\text{ai}}(\mathcal{C})$ is a replete unitarily Cauchy complete W^* -tensor subcategory of $\text{End}(\mathcal{C})$.

Proof Suppose F and F' are approximately inner, with approximating sequences (v, H) and (w, K) respectively. We claim $(\alpha_{H,K,-}^{-1} \cdot v^{K \triangleright -} \cdot F(w), H \otimes K)$ is an approximating sequence for $F \circ F'$, where the unitary natural transformation $\alpha_{H,K,-} : (H \otimes K) \triangleright - \Rightarrow H \triangleright (K \triangleright -)$ is the left module associator. For $f \in \mathcal{C}(a \rightarrow b)$, consider the diagram

$$\begin{array}{ccccccc} F(F'(a)) & \xrightarrow{F(w_n^a)} & F(K \triangleright a) & \xrightarrow{v_n^{K \triangleright a}} & H \triangleright (K \triangleright a) & \xrightarrow{\alpha_{H,K,a}^\dagger} & (H \otimes K) \triangleright a \\ \downarrow F(F'(f)) & & \downarrow F(1_K \triangleright f) & & \downarrow 1_H \triangleright (1_K \triangleright f) & & \downarrow 1_{H \otimes K} \triangleright f \\ F(F'(b)) & \xrightarrow{F(w_n^b)} & F(K \triangleright b) & \xrightarrow{v_n^{K \triangleright b}} & H \triangleright (K \triangleright b) & \xrightarrow{\alpha_{H,K,b}^\dagger} & (H \otimes K) \triangleright b \end{array}$$

The left square commutes because F' is approximately inner and F is τ -continuous on bounded subsets. The middle square commutes because F is approximately inner. The right square commutes because $\alpha_{H,K,-}^\dagger$ is natural.

Now suppose F is approximately inner with approximating sequence (v, H) and $u : G \Rightarrow F$ is a isometric natural transformation for some other endofunctor $G \in$

$\text{End}(\mathcal{C})$. It is easy to see $(v \cdot u, H)$ is an approximating sequence for G from the fact that $v \cdot u$ is again approximately isometric. The case when u is unitary shows that $\text{End}_{\text{ai}}(\mathcal{C})$ is replete. The case when u is an isometry shows that since $\mathcal{C}$ is orthogonal projection complete, then so is $\text{End}_{\text{ai}}(\mathcal{C})$.

Finally, as $\mathcal{C}$ is orthogonal direct sum complete, the orthogonal direct sum of functors is defined. If F and F' are approximately inner with approximating sequence (v, H) and (w, K) respectively, then $(v \oplus w, H \oplus K)$ is an approximating sequence for $F \oplus F'$, so $F \oplus F' \in \text{End}_{\text{ai}}(\mathcal{C})$. $\square$

3.3 Relative braiding between centrally trivial and approximately inner endofunctors

The goal of this section is to show that the subcategories $\text{End}_{\text{ct}}(\mathcal{C})$ and $\text{End}_{\text{ai}}(\mathcal{C})$ of $\text{End}(\mathcal{C})$ ‘commute’ with each other in the sense of Definition 3.24 below.

Proposition 3.19 *Suppose $F \in \text{End}_{\text{ai}}(\mathcal{C})$ with approximating sequence (v, H) and $G \in \text{End}_{\text{ct}}(\mathcal{C})$. Then $(v^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v^a) \in \tilde{\mathcal{C}}(G(F(a)) \rightarrow F(G(a)))$ is independent of the choice of approximating sequence (v, H) for F .*

Proof Suppose (v, H) and (w, K) are approximating sequences for the endofunctor $F \in \text{End}_{\text{ai}}(\mathcal{C})$. Observe that $vw^\dagger : K \triangleright - \Rightarrow H \triangleright -$ is approximately natural. Now since G is centrally trivial, by Definition 3.11,

$$\begin{array}{ccccc}
 & & G(F(a)) & & \\
 & \nearrow^{G((v_n^a)^\dagger)} & & \searrow^{G(w_n^a)} & \\
 G(H \triangleright a) & & & & G(K \triangleright a) \\
 \sigma_{G,H}^a \downarrow & & & & \downarrow \sigma_{G,K}^a \\
 H \triangleright G(a) & \xrightarrow{(v_n^{G(a)})^\dagger} & F(G(a)) & \xrightarrow{w_n^{G(a)}} & K \triangleright G(a)
 \end{array}$$

approximately commutes for each object $a \in \mathcal{C}$. Then by Remark 3.6,

$$\begin{array}{ccccc}
 & & G(F(a)) & & \\
 & \nearrow^{G(v_n^a)} & & \searrow^{G(w_n^a)} & \\
 G(H \triangleright a) & & & & G(K \triangleright a) \\
 \sigma_{G,H}^a \downarrow & & & & \downarrow \sigma_{G,K}^a \\
 H \triangleright G(a) & \xrightarrow{(v_n^{G(a)})^\dagger} & F(G(a)) & \xleftarrow{(w_n^{G(a)})^\dagger} & K \triangleright G(a)
 \end{array}$$

approximately commutes. Therefore, $(v^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v^a) \in \tilde{\mathcal{C}}(G(F(a)) \rightarrow F(G(a)))$ is independent of the choice of approximating sequences (v, H) for F . $\square$

The proof of the following lemma is standard and left to the reader.

Lemma 3.20 *Suppose (x_n) is a sequence in a metric space (X, d) which satisfies the following property:*

- For all functions $k : \mathbb{N} \rightarrow \mathbb{N}$ such that $n < k_n \leq k_{n+1}$, $d(x_n, x_{k_n}) \rightarrow 0$. (Note here that (x_{k_n}) is not quite a subsequence as terms can repeat.)

Then (x_n) is Cauchy.

Theorem 3.21 Suppose $F \in \text{End}_{\text{ai}}(\mathcal{C})$ and $G \in \text{End}_{\text{ct}}(\mathcal{C})$. For each $a \in \mathcal{C}$, there exists a unique morphism in $\mathcal{C}$, $u_{G,F}^a \in \mathcal{C}(G(F(a)) \rightarrow F(G(a)))$, such that for all approximate sequences (v, H) , the following diagram approximately commutes.

$$\begin{array}{ccc} G(F(a)) & \xrightarrow{u_{G,F}^a} & F(G(a)) \\ G(v_n^a) \downarrow & & \uparrow (v_n^{G(a)})^\dagger \\ G(H \triangleright a) & \xrightarrow{\sigma_{G,H}^a} & H \triangleright G(a) \end{array}$$

Proof Suppose (v, H) is an approximating sequence for F . Then for any function $k : \mathbb{N} \rightarrow \mathbb{N}$ such that $n < k_n \leq k_{n+1}$, $(v' := (v_{k_n})_n, H)$ is also an approximating sequence for F . By Proposition 3.19,

$$(v^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v^a) = (v'^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v'^a)$$

in $\tilde{\mathcal{C}}(G(F(a)) \rightarrow F(G(a)))$. By $[(\tau 3)]$, the τ topology on any bounded subspace of $\mathcal{C}(G(F(a)) \rightarrow F(G(a)))$ is completely metrizable. Then by Lemma 3.20 and the definition of $\tilde{\mathcal{C}}$, the bounded sequence

$$(v^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v^a) = \left((v_n^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v_n^a) \right)_n$$

is a Cauchy sequence, and we denote $u_{G,F}^a$ to be its unique limit. Again, by Proposition 3.19, $u_{G,F}^a$ does not depend on the choice of the approximating sequence for F . $\square$

Proposition 3.22 For $F \in \text{End}_{\text{ai}}(\mathcal{C})$ and $G \in \text{End}_{\text{ct}}(\mathcal{C})$, $u_{G,F}^a$ is unitary.

Proof By Proposition 3.19, for an approximating sequence (v, H) for F , the following diagram approximately commutes.

$$\begin{array}{ccc} G(H \triangleright a) & \xrightarrow{G(v_n^a \cdot (v_n^a)^\dagger)} & G(H \triangleright a) \\ \sigma_{G,H}^a \downarrow & & \downarrow \sigma_{G,H}^a \\ H \triangleright G(a) & \xrightarrow{v_n^{G(a)} \cdot (v_n^{G(a)})^\dagger} & H \triangleright G(a) \end{array}$$

Since $\sigma_{G,H}^a$ is unitary, then in $\tilde{\mathcal{C}}$, we have

$$\begin{aligned}
 u_{G,F}^a \cdot (u_{G,F}^a)^\dagger &= \left((v^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v^a) \right) \cdot \left(G(v^a)^\dagger \cdot (\sigma_{G,H}^a)^\dagger \cdot v^{G(a)} \right) \\
 &= (v^{G(a)})^\dagger \cdot \left(\sigma_{G,H}^a \cdot G(v^a) \cdot G(v^a)^\dagger \cdot (\sigma_{G,H}^a)^\dagger \right) \cdot v^{G(a)} \\
 &= (v^{G(a)})^\dagger \cdot \left(v^{G(a)} \cdot (v^{G(a)})^\dagger \right) \cdot v^{G(a)} \\
 &= \text{id}_{F(G(a))}, \\
 (u_{G,F}^a)^\dagger \cdot u_{G,F}^a &= \left(G(v^a)^\dagger \cdot (\sigma_{G,H}^a)^\dagger \cdot v^{G(a)} \right) \cdot \left((v^{G(a)})^\dagger \cdot \sigma_{G,H}^a \cdot G(v^a) \right) \\
 &= G(v^a)^\dagger \cdot \left((\sigma_{G,H}^a)^\dagger \cdot v^{G(a)} \cdot (v^{G(a)})^\dagger \cdot \sigma_{G,H}^a \right) \cdot G(v^a) \\
 &= G(v^a)^\dagger \cdot \left(G(v^a) \cdot G(v^a)^\dagger \right) \cdot G(v^a) \\
 &= \text{id}_{G(F(a))}.
 \end{aligned}$$

Since the inclusion $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is faithful, we are finished. $\square$

Proposition 3.23 For $F \in \text{End}_{\text{ai}}(\mathcal{C})$ and $G \in \text{End}_{\text{ct}}(\mathcal{C})$, the family $u_{G,F} : \{u_{G,F}^a\}_{a \in \mathcal{C}}$ is a natural transformation $G \circ F \Rightarrow F \circ G$.

Proof For $f \in \mathcal{C}(a \rightarrow b)$, it suffices to prove the following diagram approximately commutes.

$$\begin{array}{ccccc}
 G(F(a)) & \xrightarrow{u_{G,F}^a} & F(G(a)) & & \\
 \downarrow G(F(f)) & \searrow G(v_n^a) & \nearrow (v_n^{G(a)})^\dagger & & \downarrow F(G(f)) \\
 & G(H \triangleright a) & \xrightarrow{\sigma_{G,H}^a} & H \triangleright G(a) & \\
 & \downarrow G(1_H \triangleright f) & & \downarrow 1_H \triangleright G(f) & \\
 & G(H \triangleright b) & \xrightarrow{\sigma_{G,H}^b} & H \triangleright G(b) & \\
 \uparrow G(v_n^b) & & & & \uparrow (v_n^{G(b)})^\dagger \\
 G(F(b)) & \xrightarrow{u_{G,F}^b} & F(G(b)) & &
 \end{array}$$

The top and bottom squares commute by definition of $u_{G,F}^a$ by Theorem 3.21. The left square commutes because F is approximately inner and G is τ -continuous on bounded sets. The right square commutes by approximate naturality. The middle square commutes by naturality of σ . $\square$

In the next definition, we use strict monoidal categories simply because the case we care about is strict, but one obtains the general definition by inserting coheretors where appropriate.

Definition 3.24 Let $\mathcal{C}$ be a (strict) C^* -tensor category, and $\mathcal{A}, \mathcal{B}$ full, replete $\dagger$ -tensor subcategories of $\mathcal{C}$. A *centralizing structure* on the pair $(\mathcal{A}, \mathcal{B})$ is a family of unitary isomorphisms $u_{a,b} : a \otimes b \rightarrow b \otimes a$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$ satisfying the following conditions:

- (1) (Natural) For $a, a' \in \mathcal{A}$ and $f \in \mathcal{C}(a \rightarrow a')$ and $b, b' \in \mathcal{B}$, $g \in \mathcal{C}(b \rightarrow b')$,
 $(g \otimes f) \circ u_{a,b} = u_{a',b'} \circ (f \otimes g)$.
- (2) (Braid relation 1) For $a \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $(1_a \otimes u_{a,b'}) \circ (u_{a,b} \otimes 1_{b'}) = u_{a,b \otimes b'}$.
- (3) (Braid relation 2) For $a, a' \in \mathcal{A}$ and $b \in \mathcal{B}$, $(u_{a,b} \otimes 1_{a'}) \circ (1_a \otimes u_{a',b}) = u_{a \otimes a',b}$.

The goal of this section is to construct a *centralizing structure* for the pair of full, replete subcategories $(\text{End}_{\text{ct}}(\mathcal{C}), \text{End}_{\text{ai}}(\mathcal{C}))$ inside $\text{End}(\mathcal{C})$.

Proposition 3.25 For $F \in \text{End}_{\text{ai}}(\mathcal{C})$ and $G \in \text{End}_{\text{ct}}(\mathcal{C})$, $u_{G,F}$ satisfies condition (1) in Definition 3.24.

Proof Suppose $G, G' \in \text{End}_{\text{ct}}(\mathcal{C})$ and $F, F' \in \text{End}_{\text{ai}}(\mathcal{C})$ with approximating sequences (v, H) and (w, K) respectively. Let $\eta \in \text{End}(\mathcal{C})(F \Rightarrow F')$ be a natural transformation. We shall show $u_{G,F'}^a \cdot G(\eta^a) = \eta^{G(a)} \cdot u_{G,F}^a$. It suffices to prove the following diagram approximately commutes.

$$\begin{array}{ccccc}
 G(F(a)) & \xrightarrow{u_{G,F}^a} & F(G(a)) & & \\
 \downarrow G(\eta^a) & \searrow G(v_n^a) & \nearrow (v_n^{G(a)})^\dagger & & \downarrow \eta^{G(a)} \\
 & G(H \triangleright a) & \xrightarrow{\sigma_{G,H}^a} & H \triangleright G(a) & \\
 & \downarrow G((w_n \cdot \eta \cdot v_n^\dagger)^a) & & \downarrow (w_n \cdot \eta \cdot v_n^\dagger)^{G(a)} & \\
 & G(K \triangleright a) & \xrightarrow{\sigma_{G,K}^a} & K \triangleright G(a) & \\
 & \uparrow G(w_n^a) & & \uparrow (w_n^{G(a)})^\dagger & \\
 G(F'(a)) & \xrightarrow{u_{G,F'}^a} & F'(G(a)) & &
 \end{array}$$

The left/right squares commute since v, w are approximately isometric. The top/bottom squares commute by the definition of $u_{G,F}$. The middle square commutes because G is centrally trivial.

Let $\psi \in \text{End}(\mathcal{C})(G \Rightarrow G')$. We shall show $u_{G',F}^a \cdot \psi^{F(a)} = F(\psi^a) \cdot u_{G,F}^a$. It suffices to prove the following diagram approximately commutes.

$$\begin{array}{ccccc}
 G(F(a)) & \xrightarrow{u_{G,F}^a} & F(G(a)) & & \\
 \downarrow \psi^{F(a)} & \searrow G(v_n^a) & \nearrow (v_n^{G(a)})^\dagger & & \downarrow \psi^{G(a)} \\
 & G(H \triangleright a) & \xrightarrow{\sigma_{G,H}^a} & H \triangleright G(a) & \\
 & \downarrow \psi^{H \triangleright a} & & \downarrow 1_{H \triangleright \psi^a} & \\
 & G'(H \triangleright a) & \xrightarrow{\sigma_{G',H}^a} & H \triangleright G'(a) & \\
 & \uparrow G'(v_n^a) & & \uparrow (v_n^{G'(a)})^\dagger & \\
 G'(F(a)) & \xrightarrow{u_{G',F}^a} & F(G'(a)) & &
 \end{array}$$

The left square commutes by the naturality of ψ . The right square commutes because F is approximately inner. The top/bottom squares commute by the definition of $u_{G,F}$. The middle square commutes by the naturality of σ . $\square$

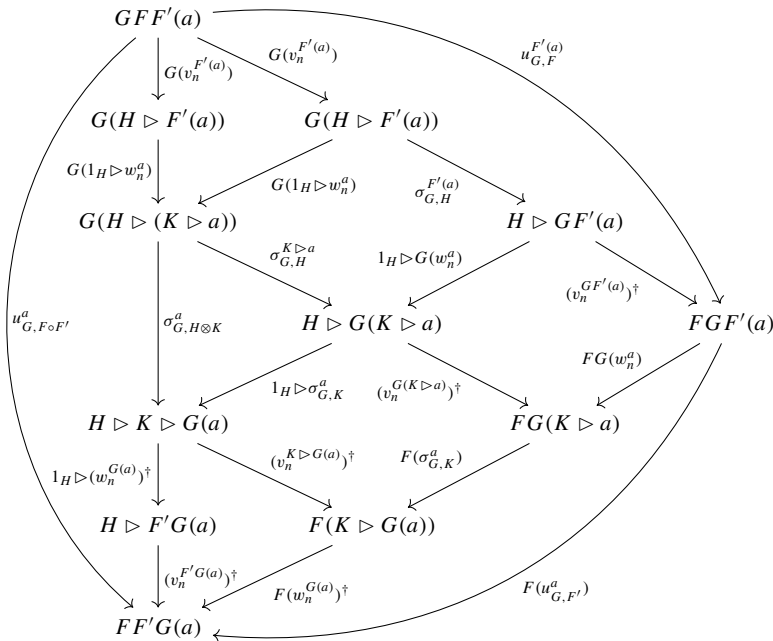
Proposition 3.26 For $F \in \text{End}_{\text{ai}}(\mathcal{C})$ and $G, G' \in \text{End}_{\text{ct}}(\mathcal{C})$, $u_{G,F}$ satisfies condition (2) of Definition 3.24.

Proof It suffices to show that for an approximating sequence (v, H) for F , the following diagram approximately commutes.

For any choice of approximating sequence, the outer three cells approximately commute by the definition of u from Theorem 3.21. The upper triangle approximately commutes as v is approximately isometric. The middle triangle approximately commutes by Proposition 2.8(2). The lower triangle is trivial. Finally, the remaining square approximately commutes because $vv^\dagger : H \triangleright - \Rightarrow H \triangleright -$ is approximately natural and G' is centrally trivial (see also Proposition 3.22). $\square$

Proposition 3.27 For $F, F' \in \text{End}_{\text{ai}}(\mathcal{C})$ and $G \in \text{End}_{\text{ct}}(\mathcal{C})$, $u_{G,F}$ satisfies condition (3) of Definition 3.24.

Proof We must prove for each fixed $a \in \mathcal{C}$, $F(u_{G,F'}^a)u_{G,F}^{F'(a)} = u_{G,F \circ F'}^a$. To do so, we carefully choose approximating sequences (v, H) and (w, K) for F and F' respectively such that the following diagram approximately commutes.



For any choices of approximating sequences, the outer three cells approximately commute by the definition of u from Theorem 3.21. The upper left square is trivial. The adjacent square to its lower right approximately commutes by the naturality of $\sigma_{G,H}$. The left middle triangle approximately commutes by Proposition 2.8(1), and the square to the lower right of this triangle approximately commutes by approximate naturality of v . This leaves us to consider the two remaining squares

$$\begin{array}{ccc}
 H \triangleright K \triangleright G(a) \xrightarrow{(v_n^{K \triangleright G(a)})^\dagger} F(K \triangleright G(a)) & & H \triangleright GF'(a) \xrightarrow{(v_n^{GF'(a)})^\dagger} FGF'(a) \\
 \downarrow 1 \triangleright (w_n^{G(a)})^\dagger & \downarrow F(w_n^{G(a)})^\dagger & \downarrow 1_H \triangleright G(w_n^a) \\
 H \triangleright F'G(a) \xrightarrow{(v_n^{F'G(a)})^\dagger} FF'G(a) & \text{and} & H \triangleright G(K \triangleright a) \xrightarrow{(v_n^{G(K \triangleright a)})^\dagger} FG(K \triangleright a)
 \end{array}$$

These squares may not approximately commute for an arbitrary choice of approximating sequences, but we can get around this by replacing v with a subsequence, as any approximating sequence for F can be used. Indeed, for $b, c \in \mathcal{C}$, let $d_{b \rightarrow c}$ denote a metric inducing the τ -topology on bounded subsets of $\mathcal{C}(b \rightarrow c)$. Since $(w_n^{G(a)})^\dagger$ and $G(w_n^a)$ are morphisms in $\mathcal{C}$ for each fixed n , using approximate naturality of $v^\dagger$, we may inductively choose $1 \leq k_{n-1} < k_n$ so that both

$$d_{H \triangleright K \triangleright G(a) \rightarrow F'G(a)} \left(F(w_n^{G(a)})^\dagger \cdot (v_{k_n}^{K \triangleright G(a)})^\dagger, (v_{k_n}^{F'G(a)})^\dagger \cdot (1_H \triangleright (w_n^{G(a)})^\dagger) \right) < \frac{1}{n} \quad \text{and}$$

$$d_{H \triangleright GF'(a) \rightarrow FG(K \triangleright a)} \left(FG(w_n^a) \cdot (v_{k_n}^{GF'(a)})^\dagger, (v_{k_n}^{G(K \triangleright a)})^\dagger \cdot (1_H \triangleright G(w_n^a)) \right) < \frac{1}{n}$$

simultaneously. Thus replacing (v_n^c) with $(v_{k_n}^c)$ for all $c \in \mathcal{C}$, the previous arguments still hold, as $((v_{k_n}^c)_c, H)$ is still an approximating sequence for F , and these two squares approximately commute for our fixed $a \in \mathcal{C}$. Since we only need to verify condition (2) of Definition 3.24 for one $a \in \mathcal{C}$ at a time, the result follows. $\square$

Definition 3.28 We define the category $\text{End}_{\text{loc}}(\mathcal{C})$ of *local endofunctors*, to be the full W^* -monoidal subcategory of $\text{End}(\mathcal{C})$ whose objects are normal $\dagger$ -endofunctors which are both approximately inner and centrally trivial. By Propositions 3.14 and 3.18, $\text{End}_{\text{loc}}(\mathcal{C})$ is replete and unitarily Cauchy complete. The family of unitary natural transformations $u_{G,F} : G \circ F \Rightarrow F \circ G$ equips $\text{End}_{\text{loc}}(\mathcal{C})$ with the structure of a *braided W^* -tensor category*.

4 $\tilde{\chi}(M)$ for finite von Neumann algebras via bimodules

In this section, we give the main application of the construction of the last section to give a definition of $\tilde{\chi}(M)$ for a II_1 -factor M in terms of bimodules.

Given a W^* -tensor category $\mathcal{C}$, the *dualizable part*, denoted $\mathcal{C}_{\text{dualizable}}$, is the full tensor subcategory whose objects have two sided duals. Observe that if $\text{End}_{\mathcal{C}}(1_{\mathcal{C}})$ is finite dimensional, then $\mathcal{C}_{\text{dualizable}}$ is a rigid C^*/W^* tensor category. If moreover $\mathcal{C}_{\text{dualizable}}$ is semisimple (equivalently orthogonal projection complete), it is called a *unitary multitensor category* [67]; it is called a *unitary tensor category* if $\text{End}_{\mathcal{C}}(1_{\mathcal{C}})$ is one dimensional.

Definition 4.1 Given a von Neumann algebra M , we denote the braided unitary tensor category $\tilde{\chi}(M) := \text{End}_{\text{loc}}(\text{Mod}_{\text{fpp}}(M))_{\text{dualizable}}$.

Identifying $\text{End}(\text{Mod}_{\text{fpp}}(M)) \simeq \text{Bim}_{\text{fpp}}(M)^{\text{mp}}$ as in Remark 2.11, we call a bimodule $X \in \text{Bim}_{\text{fpp}}(M)$ approximately inner (respectively centrally trivial) if the functor $- \boxtimes_M X$ is approximately inner (respectively centrally trivial). We see the underlying unitary tensor category of $\tilde{\chi}(M)$ agrees with the definition of $\tilde{\chi}(M)$ from [68, Rem. 2.7]. We may thus think of $\tilde{\chi}(M)$ as the dualizable approximately inner and centrally trivial bimodules of M whose conjugate bimodule is also approximately inner and centrally trivial.

Since $\text{End}_{\text{loc}}(\text{Mod}_{\text{fpp}}(M))$ is braided, we get a monoidal equivalence from $\tilde{\chi}(M)$ to its monoidal opposite $\tilde{\chi}(M)^{\text{mp}}$, which allows us to bypass this opposite issue. We address this in detail in Remark 4.9 below and the discussion thereafter.

Note the dualizable objects in $\text{Bim}_{\text{fpp}}(M)$ are precisely the bifinite Hilbert bimodules of M , and the dual object is the conjugate bimodule. It is easy to see that H is centrally trivial if and only if $\overline{H}$ is centrally trivial, but for approximately inner, any such relationship is not obvious. Thus it may be possible for a centrally trivial bifinite bimodule to be approximately inner, but its conjugate bimodule may not be

approximately inner. We do not have an example, but we cannot rule this out at this time.

4.1 Module and bimodule realization

In order to translate the results of Sect. 3, we use the graphical calculus for $\text{Mod}_{\text{fgp}}(M)$ as a right $\text{Bim}_{\text{fgp}}(M)$ -module W^* -category. One way to do this is to use the realization graphical calculus from [13] based on [31, Sect. 4.1]. We only introduce the part of the graphical calculus that we need in this section, and we introduce the rest of the graphical calculus for Q -system realization in Sect. 5.1 below.

In the 2D graphical calculus for $W^*\text{Alg}$, von Neumann algebras are denoted by shaded regions, bimodules are denoted by 1D strands, and intertwiners are denoted by 0D coupons. For the rest of this section, let M be a II_1 factor. Of particular importance is the right M -module M_M . The missing label on the left hand side is inferred to be $\mathbb{C}$, which is always represented by the empty shading. That is, we identify $\text{Mod}_{\text{fgp}}(M) = W^*\text{Alg}(\mathbb{C} \rightarrow M)$ in the 2D graphical calculus. We denote $\mathbb{C}M_M$ by a dashed line which is shaded by M on the right hand side.

$$\text{shaded square} = M \quad \text{dashed square} = \mathbb{C} \quad \text{dashed square with shaded right side} = \mathbb{C}M_M.$$

A bounded, adjointable intertwiner $f : Y_M \rightarrow Z_M$ is denoted graphically by

$$\text{coupon } f \text{ in a square} : Y_M \rightarrow Z_M; \quad \text{dashed square with shaded right side} = Y_M \quad \text{and} \quad \text{dashed square with shaded left side} = Z_M.$$

Construction 4.2 [13, Const. 4.1] Given $Y_M \in \text{Mod}_{\text{fgp}}(M)$, the map $x \mapsto L_y$ where $L_y(m) := ym$ gives a canonical isomorphism $Y_M \cong |Y|_M := \text{Hom}(M_M \rightarrow Y_M)$ such that $\langle x|y \rangle_M = L_x^\dagger L_y$. In Sects. 4.2 and 5.2 below, we make heavy use of this identification.

$$\text{coupon } y \text{ in a square} \in |Y| := \text{Hom}(M_M \rightarrow Y_M).$$

The right M -action is given by identifying $M = \text{End}(M_M)$ and stacking coupons:

$$\text{stacked coupons } y \text{ and } a \text{ in a square} = y \triangleleft a.$$

The condition that $\{c_j\}$ is a Y_M -basis can be written graphically as

$$\sum_j \left(\begin{array}{c} \boxed{c_j} \\ \vdots \\ \boxed{c_j^\dagger} \end{array} \right) = \text{vertical line}.$$

If $X \in \text{Bim}_{\text{fpp}}(M)$, then we define the realization $|X|$ slightly differently:

$$\left(\begin{array}{c} \boxed{x} \end{array} \right) \in |X| := \text{Hom}(M_M \rightarrow M \boxtimes_M X_M).$$

While this definition is canonically isomorphic to the previous definition via the canonical unitor $M \boxtimes_M X \cong X$, this definition offers the advantage of depicting both the left and the right M -actions graphically by

$$\left(\begin{array}{c} \boxed{a} \\ \vdots \\ \boxed{x} \end{array} \right) = a \triangleright x \quad \text{and} \quad \left(\begin{array}{c} \boxed{x} \\ \vdots \\ \boxed{b} \end{array} \right) = x \triangleleft b.$$

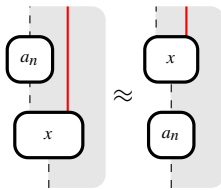
4.2 Centrally trivial and approximately inner bimodules

In this section we clarify the equivalence between our definitions of approximately inner and centrally trivial given in terms of endofunctors, and Popa's original definitions as translated to bimodules, which are much more natural from the point of view of a single von Neumann algebra [68, Rem. 2.7]. For this section, let M be a finite separable von Neumann algebra with faithful normal trace tr_M .

Notation 4.3 For norm bounded sequences $(f_n)_n, (g_n)_n \subseteq \text{Hom}(X_M \rightarrow Y_M)$, we write $f_n \approx g_n$ if $\lim_n \|f_n - g_n\|_2 = 0$. For $f \in \text{Hom}(X_M \rightarrow Y_M)$, we write $f \approx f_n$ if $\lim_n \|f - f_n\|_2 = 0$. As a consequence, $f \approx g$ if and only if $f = g$.

We remark that since composition is jointly τ -continuous on norm-bounded subsets of hom spaces, we may use $f_n \approx g_n$ as a *local relation* amongst morphisms in $\text{Mod}_{\text{fpp}}(M)$. If we precompose with an appropriate h , we still have $f_n \circ h \approx g_n \circ h$, and similarly for composing on the other side, or both sides simultaneously. Similarly, since $\boxtimes$ is separately normal in each variable, tensoring with a fixed k is separately τ -continuous on norm-bounded subsets. Thus we still have $f_n \boxtimes k \approx g_n \boxtimes k$, and similarly for tensoring on the other side, or both sides simultaneously.

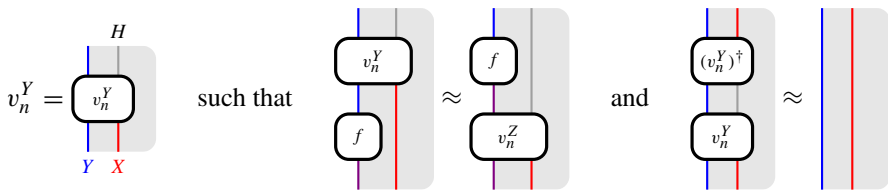
Proposition 4.4 X is centrally trivial over M if and only if for all central sequences $(a_n)_n \subseteq M$ and for all $x \in X$, $\|a_n x - x a_n\|_2 \rightarrow 0$, i.e.,



$$\forall x \in X, \quad \forall \text{ central sequences } (a_n) \subseteq M.$$

Proof Recall a central sequence of $\text{Mod}_{\text{fpp}}(M)$ is a natural transformation of the identity functor. By Example 3.9, equivalence classes of these central sequences agree with the usual equivalence classes of central sequences of M . The result now follows directly from Proposition 3.12, which translates into the displayed condition in the statement of the proposition. $\square$

Now by definition, the functor $-\boxtimes_M X$ is approximately inner if there exists an approximately natural isometry $v_n^Y : Y \boxtimes_M X \rightarrow Y \otimes H$, i.e., we have



for all intertwiners $f \in \text{Hom}(Z_M \rightarrow Y_M)$.

Definition 4.5 For $Y \in \text{Mod}_{\text{fpp}}(M)$, an *approximate Y_M -basis* is a sequence $\{b_i^{(n)}\}_{i=1}^{m(n)} \subseteq Y$ such that $\sup_n m(n) < \infty$, $\sup_{i,n} \|\langle b_i^{(n)} | b_i^{(n)} \rangle_M^Y\| < \infty$, and

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{i=1}^m b_i^{(n)} \langle b_i^{(n)} | x \rangle_M^Y \right\|_2 = 0 \quad \forall x \in Y.$$

For $X \in \text{Bim}_{\text{fpp}}(X)$, an *approximately inner X_M -basis* is an approximate X_M -basis such that

$$\|ab_i^{(n)} - b_i^{(n)}a\|_2 \rightarrow 0 \quad \forall a \in M. \quad (2)$$

Proposition 4.6 A bimodule X is approximately inner over M if and only if there exists an approximately inner X_M -basis.

Proof Suppose X is approximately inner. By Definition 3.16, there exists a finite dimensional Hilbert space H and an approximate natural isometry $v = (v_n) : - \boxtimes_M$

$X \rightarrow H \triangleright -$. Let $\{e_i\}$ be an orthonormal basis of H , and define $b_i^{(n)}$ as follows.

$$\begin{array}{c} \boxed{b_i^{(n)}} \\ \hline \end{array} := \begin{array}{c} \boxed{(v_n^M)^\dagger} \\ \hline \bullet e_i \end{array}.$$

Observe that for all $x \in X$, we have

$$\sum_i b_i^{(n)} \langle b_i^{(n)} | x \rangle_M^X = \sum_i \begin{array}{c} \boxed{b_i^{(n)}} \\ \hline \boxed{(b_i^{(n)})^\dagger} \\ \hline \boxed{x} \end{array} = \sum_i \begin{array}{c} \boxed{(v_n^M)^\dagger} \\ \hline \bullet e_i \bullet e_i^* \\ \hline \boxed{v_n^M} \\ \hline \boxed{x} \end{array} = \sum_i \begin{array}{c} \boxed{(v_n^M)^\dagger} \\ \hline \boxed{v_n^M} \\ \hline \boxed{x} \end{array} = (v_n^M)^\dagger (v_n^M(x)).$$

The condition (2) follows immediately by approximate naturality of v on the M -component.

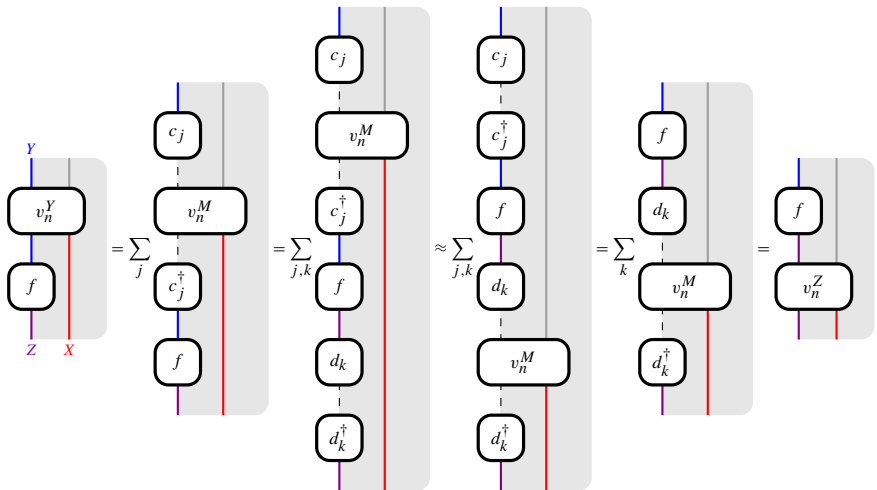
Conversely, starting with an approximately inner X_M -basis, we define

$$\begin{array}{c} H \\ \hline \boxed{v_n^M} \\ \hline \end{array} := \sum_i \begin{array}{c} \bullet e_i \\ \hline \boxed{v_n^M} \\ \hline \bullet e_i^* \end{array} = \sum_i \begin{array}{c} \bullet e_i \\ \hline \boxed{(b_i^{(n)})^\dagger} \\ \hline \end{array},$$

and we define each v_n^Y for $Y \in \text{Mod}_{\text{fip}}(M)$ in terms of v_n^M and a Y_M -basis $\{c_j\}$:

$$\begin{array}{c} \boxed{v_n^Y} \\ \hline \textcolor{blue}{Y} \quad \textcolor{red}{X} \end{array} := \sum_j \begin{array}{c} \boxed{c_j} \\ \hline \boxed{v_n^M} \\ \hline \boxed{c_j^\dagger} \end{array}.$$

We have v_n^Y is norm-bounded as $\sum_j \|c_j\|_2^2 < \infty$. To see the approximate naturality of $\{v_n^Y\}_{Y \in \text{Mod}_{\text{fgp}}(M)}$, for $f \in \text{Hom}(Z_M \rightarrow Y_M)$,



We see v_n^Y is independent of the choice of Y_M -basis by taking $f = \text{id}_Y$ above. $\square$

In [68, Def. 1.1], Popa gave a definition of approximate innerness for a finite index II_1 subfactor $N \subseteq M$. We can also view ${}_N M_N \in \text{Bim}_{\text{fgp}}(N)$ with $\langle x|y \rangle_N := E_N(x^*y)$. We will show in Proposition 4.8 below that $N \subseteq M$ is approximately inner in the sense of [68, Def. 1.1] if and only if ${}_N M_N$ is approximately inner.

We quickly recall the notion of ultraproduct for II_1 (sub)factors following [1, Sect. 5.4]. Let ω be a non-principal ultrafilter on $\mathbb{N}$. For a II_1 factor N , define $N^\omega = \ell^\infty(\mathbb{N}, M)/\mathcal{I}$, where $\mathcal{I}$ is the ideal of sequences which converge to 0 in $\|\cdot\|_2$ along ω . Then N^ω is a II_1 factor, with trace given by taking the limit along ω .

Now consider a finite index II_1 subfactor $N \subseteq M$. Then $N^\omega \subseteq M^\omega$ is another II_1 subfactor with the same Jones index and trace preserving conditional expectation extending $E : M \rightarrow N$. Using this expectation, we can view M^ω as an N^ω bimodule. We can also consider the inclusion $N' \cap N^\omega \subseteq N' \cap M^\omega$, and E restricts to the trace preserving conditional expectation $E : N' \cap M^\omega \rightarrow N' \cap N^\omega$. We recall the following definition due to Popa.

Definition 4.7 [68, Def. 1.1 and Prop. 1.2] A finite index II_1 subfactor $N \subseteq M$ is called *approximately inner* if the inclusion $N' \cap N^\omega \subseteq_E N' \cap M^\omega$ is $[M : N]^{-1}$ -Markov, i.e., there is a (finite) Pimsner–Popa basis $\{b\}$ for $N' \cap M^\omega$ over $N' \cap N^\omega$ which satisfies $\sum_b bb^* = [M : N]$.

Proposition 4.8 A finite index II_1 subfactor $N \subseteq M$ is approximately inner if and only if ${}_N M_N \in \text{Bim}_{\text{fgp}}(N)$ is approximately inner.

Proof It is easy to see that for a finite index subfactor $N \subseteq M$, if ${}_N M_N \in \text{Bim}_{\text{fgp}}(N)$ is approximately inner, then an approximate M_N -basis gives an honest Pimsner–Popa

basis for M^ω over N^ω , and the extra commutativity condition (2) means this Pimsner–Popa basis lies in $N' \cap M^\omega$. Thus we have a Pimsner–Popa basis for $N' \cap M^\omega$ over $N' \cap N^\omega$ which lifts to a Pimsner–Popa basis for M^ω over N^ω , which implies $N' \cap N^\omega \subseteq_E N' \cap M^\omega$ is $[M : N]^{-1}$ -Markov.

Conversely, suppose $N \subseteq M$ is approximately inner. Let $\{b_i^{(n)}\}_{i=1}^k$ be a representative of the Pimsner–Popa basis for $N' \cap N^\omega \subseteq_E N' \cap M^\omega$ which lifts to a Pimsner–Popa basis for $N^\omega \subseteq_E M^\omega$, which exists by [72, Proposition 2.2 (3)]. Note that for every $x \in M$ and $y \in N$,

$$\left\| x - \sum_i b_i^{(n)} E((b_i^{(n)})^* x) \right\|_2 \longrightarrow_\omega 0 \quad \text{and} \quad \| y b_j^{(n)} - b_j^{(n)} y \|_2 \longrightarrow_\omega 0 \quad \forall j.$$

Let $F \subset (N)_1$ and $G \subset (M)_1$ be countable σ -strong* dense subsets. (Since M, N are II_1 factors, we are really working with the $\|\cdot\|_2$ -topology.) Write $F = \bigcup_l F_l$ and $G = \bigcup_l G_l$ where the F_l and G_l increasing sequences of finite subsets. For each l , we have

$$\sum_{x \in G_l} \left\| x - \sum_i b_i^{(n)} E((b_i^{(n)})^* x) \right\|_2 + \sum_j \sum_{y \in F_l} \| y b_j^{(n)} - b_j^{(n)} y \|_2 \longrightarrow_\omega 0.$$

Therefore there exists a subsequence of the $b_i^{(n_k)}$ such that

$$\sum_{x \in G_l} \left\| x - \sum_i b_i^{(n_k)} E((b_i^{(n_k)})^* x) \right\|_2 + \sum_j \sum_{y \in F_l} \| y b_j^{(n_k)} - b_j^{(n_k)} y \|_2 \longrightarrow 0.$$

In particular, we can choose $c_i^{(l)} := b_i^{(n_k)}$ where n_k is sufficiently large so that the above sum is less than 2^{-l} .

We claim is the sequence $\{c_i^{(l)}\}$ is an approximately inner $L^2 M_N$ -basis. Clearly the collection $\{c_i^{(l)}\}$ satisfies the conditions from Definition 4.5 for $x \in G$ while it satisfies (2) for $y \in F$. The result follows since G and F are σ -strong* dense in $(M)_1$ and $(N)_1$ respectively. $\square$

Remark 4.9 (cf. [19, Ex. 8.1.9]) For a monoidal category $\mathcal{C}$, the *monoidal opposite* $\mathcal{C}^{\text{mp}}$ is the same category with the opposite monoidal product given by $a \otimes_{\text{mp}} b := b \otimes_{\mathcal{C}} a$. It is equipped with the inverse associator:

$$\alpha_{a,b,c}^{\text{mp}} : (a \otimes_{\text{mp}} b) \otimes_{\text{mp}} c := c \otimes (b \otimes a) \xrightarrow{\alpha_{c,b,a}^{-1}} (c \otimes b) \otimes a =: a \otimes_{\text{mp}} (b \otimes_{\text{mp}} c).$$

Now suppose $\mathcal{C}$ is a braided monoidal category. Observe that the braiding endows the linear equivalence $\mathcal{C} \rightarrow \mathcal{C}^{\text{mp}}$ with a monoidal structure

$$\mu_{a,b} : \text{id}(a \otimes b) = a \otimes b \xrightarrow{\beta_{a,b}} b \otimes a = \text{id}(a) \otimes_{\text{mp}} \text{id}(b)$$

giving an equivalence of monoidal categories $\mathcal{C} \simeq \mathcal{C}^{\text{mp}}$. Now transporting β to $\mathcal{C}^{\text{mp}}$ along this monoidal equivalence endows $\mathcal{C}^{\text{mp}}$ with the braiding

$$\beta_{a,b}^{\text{mp}} := \mu_{b,a} \circ \text{id}(\beta_{a,b}) \circ \mu_{a,b}^{-1} : a \otimes_{\text{mp}} b = b \otimes a \xrightarrow{\beta_{b,a} \circ \beta_{a,b} \circ \beta_{a,b}^{-1} = \beta_{b,a}} a \otimes b = b \otimes_{\text{mp}} a$$

such that $(\text{id}, \mu) : (\mathcal{C}, \alpha, \beta) \rightarrow (\mathcal{C}^{\text{mp}}, \alpha^{\text{mp}}, \beta^{\text{mp}})$ is a braided monoidal equivalence. (Observe that if we chose our monoidal structure for id to be $\mu_{a,b} := \beta_{b,a}^{-1}$, we would still obtain $\beta_{a,b}^{\text{mp}} = \beta_{b,a}$ as the transported braiding on $\mathcal{C}^{\text{mp}}$.)

Using the above remark, we now translate the definition of the unitary braiding into the language of bimodules. For $X, Y \in \text{Bim}_{\text{fgp}}(M)$, we write $u_{X,Y} = u_{-\boxtimes_M Y, -\boxtimes_M X}^{L^2 M}$. Choosing approximately inner X_M, Y_M -bases $\{b_i^{(n)}\}_i, \{c_j^{(n)}\}_j$ and ordinary X_M, Y_M -bases $\{b_i\}_i, \{c_j\}_j$ respectively, unpacking Theorem 3.21 and Definition 2.7 gives the following formulas.

$$u_{X,Y} = \text{crossing} \approx \text{box } (v_n^Y)^\dagger, v_n^M \approx \sum_i \text{box } (v_n^Y)^\dagger, v_n^M \text{ with } e_i \approx \sum_{i,j} \text{box } (v_n^M)^\dagger, v_n^M \text{ with } e_i, c_j^\dagger \approx \sum_{i,j} \text{box } (b_i^{(n)})^\dagger, (b_i^{(0)})^\dagger \text{ with } c_j^\dagger \quad (3)$$

$$u_{Y,X}^\dagger = \text{crossing} \approx \text{box } (w_n^M)^\dagger, w_n^X \approx \sum_j \text{box } (w_n^M)^\dagger, w_n^X \text{ with } f_j \approx \sum_{i,j} \text{box } (w_n^M)^\dagger, w_n^M \text{ with } f_j, b_i \approx \sum_{i,j} \text{box } (c_j^{(n)})^\dagger, (c_j^{(0)})^\dagger \text{ with } b_i, b_i^\dagger \quad (4)$$

That is, the braidings $u_{X,Y}, u_{Y,X}^\dagger$ can be expressed as the following $\|\cdot\|_2$ -limits of M -finite rank operators:

$$u_{X,Y} = \lim_{\tau} \sum_{i,j} \left| c_j \boxtimes b_i^{(n)} \right\rangle \left\langle b_i^{(n)} \boxtimes c_j \right| \quad \text{and} \quad u_{Y,X}^{\dagger} = \lim_{\tau} \sum_{i,j} \left| c_j^{(n)} \boxtimes b_i \right\rangle \left\langle b_i \boxtimes c_j^{(n)} \right|. \quad (5)$$

The content of Sect. 3 then translates into the facts that:

- The limit $u_{X,Y}$ exists and is independent of the choice of (approximately inner) X_M, Y_M -bases (Theorem 3.21 and Proposition 3.23),
- $u_{X,Y}$ is unitary (Proposition 3.22),
- $u_{X,Y}$ is natural in X and Y (Proposition 3.25), and
- $u_{X,Y}$ satisfies the braid relations (Propositions 3.27, 3.26).

4.3 Examples

We now compute many examples of $\tilde{\chi}(M)$ for various von Neumann algebras M . In this section, in order to easily make contact with other results in the literature, we work with the Hilbert space version of $\text{Mod}_{\text{fgp}}(M)$, i.e., Hilbert spaces with a normal right M -action such that $\dim(H_M) < \infty$ (see Example 2.5 for the equivalence with $W^*\text{Alg}_{\text{fgp}}(\mathbb{C} \rightarrow M)$).

Now we recall two different notions of central sequence in the von Neumann algebra literature which agree for finite von Neumann algebras but are subtly distinct in general. The standard notion to use outside the finite setting is *centralizing sequence*, introduced by Connes in [14, 15] in terms of the predual. Equivalently (e.g., see [32, Lem. 1.8]), we say a norm bounded sequence $\{x_i\} \subseteq M$ is centralizing if for all $\eta \in L^2 M$, $\|x_i \eta - \eta x_i\|_2 \rightarrow 0$. This is in contrast to *central sequences*, which satisfy $x_i m - m x_i \rightarrow 0$, in the strong*-topology (the latter notion is compatible with our use of the term central sequence). Note every centralizing sequence is central, but in general the converse is not true. For finite von Neumann algebras, however, these two notions agree. In both settings, we say a sequence is *trivial* if there exists a scalar λ such that $x_i - \lambda 1_M \rightarrow 0$ in the strong*-topology.

For our formalism, the notion of central sequence is the correct one; however, we will occasionally need to make use of results that are stated in terms of centralizing sequences.

Example 4.10 ($L^\infty(X, \mu)$) For the abelian von Neumann algebra $L^\infty(X, \mu)$ over a finite measure space (X, μ) , constant sequences are centrally trivial. Thus any centrally trivial right-finite correspondence is simply a finitely generated projective module, made into a bimodule by defining the right action to be the left action. All of these correspondences are inner, hence approximately inner. This braided category is equivalent to the symmetric monoidal category of finitely generated projective modules, which is equivalent to the category of finite dimensional measurable Hilbert bundles over (X, μ) [18].

Example 4.11 (Connes' $\chi(M)$ [16] and Jones' κ [38]) Let M be a separable finite von Neumann algebra with faithful normal trace tr_M . We show that Connes' $\chi(M)$ embeds as a multiplicative subgroup of the monoid of equivalence classes of invertible objects in $\tilde{\chi}(M)$.

For $\alpha \in \text{Aut}(M)$, we define the corresponding M – M bimodule as ${}_{\alpha}L^2M$. Denoting the image of 1_M in ${}_{\alpha}L^2M$ by Ω , the left action is given by $a \triangleright x\Omega := \alpha(a)x\Omega$, the right action is given by $x\Omega \triangleleft b = xb\Omega$, and the M -valued inner product is given by $\langle x\Omega | x'\Omega \rangle_M^{L^2M} := x^*x'$. Moreover, the map $\alpha \mapsto {}_{\alpha}L^2M$ descends to a group isomorphism from $\text{Out}(M)$ onto the group of unitary equivalence classes of invertible M – M bimodules H such that $\dim({}_MH) = \dim(H_M) = 1$.

Recall that an automorphism α is approximately inner if there exists a sequence of unitaries $u_n \in U(M)$ such that $\|\alpha(x)u_n - u_nx\|_2 \rightarrow 0$ for all $x \in M$. According to Proposition 4.6, to show ${}_{\alpha}L^2M$ is approximately inner, it suffices to show $\{u_n\Omega\}$ is an approximate Pimsner–Popa basis centralized by M . It is clear that $u_n \triangleright \langle u_n\Omega | x\Omega \rangle_M^{L^2M} = x$ for all $x \in M$, and for all $a \in M$,

$$\|a \triangleright u_n\Omega - u_n\Omega \triangleleft a\|_2 = \|\alpha(a)u_n\Omega - u_na\Omega\|_2 \rightarrow 0.$$

Therefore, ${}_{\alpha}L^2M$ is approximately inner. By Proposition 4.4, the bimodule ${}_{\alpha}L^2M$ is centrally trivial if α is centrally trivial.

Now assume M is a II_1 factor. The subcategory of $\tilde{\chi}(M)$ spanned by the image of $\chi(M)$ is a pointed braided unitary tensor category. The entire braided tensor structure here is uniquely determined by the quadratic form κ on $\chi(M)$ determined by $u := u_{{}_{\alpha}L^2M, {}_{\alpha}L^2M} = \kappa(\alpha)1_{{}_{\alpha}L^2M \boxtimes_M {}_{\alpha}L^2M}$. Since

$$u = \|\cdot\|_2 - \lim_n |\Omega \boxtimes u_n\Omega\rangle \langle u_n\Omega \boxtimes \Omega|,$$

for an arbitrary bounded vector $\Omega \boxtimes m\Omega$, we see

$$\kappa(\alpha)(\Omega \boxtimes m\Omega) = u(\Omega \boxtimes m\Omega) = \Omega \boxtimes \|\cdot\|_2 - \lim_n u_n^* \alpha(u_n)m\Omega.$$

Thus this κ is precisely Jones' quadratic form κ on $\chi(M)$ [38, Def. 2.4].

Further references for Connes' $\chi(M)$ and Jones' κ include [12, 20, 37].

Example 4.12 ($\tilde{\chi}(R)$ is trivial) The following proposition specialized to the case $S = \mathbb{C}$ shows that the only centrally trivial bimodules of R are inner, and thus $\tilde{\chi}(R)$ is trivial. This extends Connes' result that $\chi(R)$ is trivial [16].

Proposition 4.13 *Let $M = R \otimes S$ where R is the hyperfinite II_1 factor and S is any factor. Let H be a separable M – M Hilbert bimodule such that for all $\xi \in H$ and all central sequences $x = (x_n)_{n \in \mathbb{N}} \subseteq R$, $\|(x_n \otimes 1_S)\xi - \xi(x_n \otimes 1_S)\|_2 \rightarrow 0$. Then $H \cong L^2R \otimes K$ for some S – S bimodule K .*

Proof Let $\xi \in H$ with $\|\xi\| = 1$. Represent $R = \bigotimes_{i=1}^{\infty} M_2(\mathbb{C})$, and denote $R_n = \bigotimes_{i=1}^n M_2(\mathbb{C}) \subseteq R$. We claim there is some n_0 so that for all unitaries $u \in R_{n_0}' \cap R$, $\|(u \otimes 1_S)\xi - \xi(u \otimes 1_S)\| < \frac{1}{2}$. Otherwise, we could find a sequence of unitaries $u_n \in R_n' \cap R$ with $\|(u_n \otimes 1_S)\xi - \xi(u_n \otimes 1_S)\| \geq \frac{1}{2}$. However by construction the sequence $(u_n)_{n \in \mathbb{N}} \subseteq R$ is central, contradicting the hypothesis.

Choose such an n_0 , and consider the weakly compact convex subset

$$\overline{\text{co}}^w \left\{ (u \otimes 1_S) \xi (u^* \otimes 1_S) \mid u \in R'_{n_0} \cap R \right\} \subseteq H.$$

This has a unique element of minimal norm, ξ_0 . By hypothesis

$$2\|\xi\|^2 - 2\operatorname{Re}\langle (u \otimes 1_S) \xi (u^* \otimes 1_S), \xi \rangle = \|(u \otimes 1_S) \xi (u^* \otimes 1_S) - \xi\|^2 < \frac{1}{4}$$

and thus

$$\frac{7}{8} < \operatorname{Re}\langle (u \otimes 1_S) \xi (u^* \otimes 1_S), \xi \rangle.$$

Therefore $\xi_0 \neq 0$. Since $\|(u \otimes 1_S) \xi_0 (u^* \otimes 1_S)\| = \|\xi_0\|$, uniqueness of ξ_0 implies $(u \otimes 1_S) \xi_0 = \xi_0 (u \otimes 1_S)$ for all unitaries (hence all elements) $u \in R'_n \cap R$. Note that $R_n \xi_0 R_n$ is a cyclic bimodule over the finite dimensional full matrix algebra R_n , and thus contains a non-zero R_n central vector ξ_1 (which is evidently $R'_n \cap R$ -central). Thus since $R = \langle R_n, R'_n \cap R \rangle$, we see that ξ_1 is R -central and thus R -bounded by [66, Lem. 3.20]. This means the $R - R$ bimodule $H_1 := \overline{(R \otimes 1_S) \xi_1 (R \otimes 1_S)}^{\|\cdot\|}$ is canonically isomorphic to $L^2 R$ as an $R - R$ bimodule via the map defined by $1_R \rightarrow \xi_1$. Choosing a vector $\xi' \in H_1^\perp$, we can repeat this procedure, obtaining a decomposition as R -bimodules $H \cong L^2 R \otimes K$ where K is a separable multiplicity space. But note $K \cong \Omega_R \otimes K$ is the space of $R \otimes 1_S$ central vectors hence is closed under the left and right actions of $1_R \otimes S$. $\square$

The following proposition is well known to experts. We record it here for completeness and convenience of the reader. The concise proof included below was suggested by a helpful referee.

Proposition 4.14 *Let $N \subseteq M$ be a finite index II_1 subfactor. Then $M = N \vee (N' \cap M)$ if and only if $L^2 M \cong \bigoplus_i L^2 N$ as $N - N$ bimodules.*

Proof Suppose $M = N \vee (N' \cap M)$. By [39], $N' \cap M$ is finite dimensional, and thus isomorphic to a multimatrix algebra. The only way that $N \vee (N' \cap M)$ can be a factor is if $N' \cap M$ is a full matrix algebra, i.e., isomorphic to $M_k(\mathbb{C})$ for some $k \in \mathbb{N}$. Since $N' \cap M$ is finite dimensional, the algebraic tensor product $N \otimes (N' \cap M)$ is also the spatial tensor product, and the canonical map $N \otimes (N' \cap M) \rightarrow N \vee (N' \cap M) = M$ is surjective. But since $N \otimes (N' \cap M) \cong N \otimes M_k(\mathbb{C})$ is a II_1 factor, the canonical map is also injective, and thus $M \cong N \otimes M_k(\mathbb{C})$, implying the result.

The converse is obvious. $\square$

Example 4.15 ($\tilde{\chi}(N)$ is trivial for N non-Gamma) Let N be a non-Gamma II_1 factor, and $H \in \tilde{\chi}(N)$ irreducible. Setting $X = L^2 N \oplus H$ and $M := |X \boxtimes_N \bar{X}|$, we get a finite index II_1 subfactor $N \subseteq M$ by Example 5.3 which is approximately inner by Proposition 4.8. By [72, Proposition 2.6 (iv)], $M = N \vee (N' \cap M)$, so by Proposition 4.14, ${}_N L^2 M_N \cong \bigoplus_i {}_N L^2 N_N$. On the other hand, $L^2 M \cong X \boxtimes_N \bar{X} \cong$

$L^2N \oplus H \oplus \overline{H} \oplus (H \boxtimes_N \overline{H})$ contains H as an irreducible summand, and thus we have $H \cong L^2N$ as an N - N bimodule.

Example 4.16 ($\tilde{\chi}(R \otimes N)$ is trivial for N non-Gamma) Let $H \in \tilde{\chi}(R \otimes N)$ be irreducible, and consider $X = L^2(R \otimes N) \oplus H$. Then since H is centrally trivial, so is X . By Lemma 4.13, $X \cong L^2(R) \otimes K$ where K is an N - N bifinite bimodule. In particular, setting $M := |K \boxtimes_N \overline{K}|$, $N \subseteq M$ is a finite index II_1 subfactor by Example 5.3. Furthermore, since N is non-Gamma, M is non-Gamma by [76, Prop. 1.11].

Now since H is approximately inner, the $R \otimes N - R \otimes N$ bimodule $X \boxtimes_{R \otimes N} \overline{X} \cong L^2(R \otimes M)$ is approximately inner, hence $R \otimes N \subseteq R \otimes M$ is a finite index approximately inner subfactor. But note this subfactor is simply $R \otimes (N \subseteq M)$. By Proposition 4.8 (using Definition 4.7), the inclusion $(R \otimes N)' \cap (R \otimes N)^\omega \subseteq (R \otimes N)' \cap (R \otimes M)^\omega$ is $[M : N]^{-1}$ -Markov, and has a finite Pimsner–Popa basis lifting to a Pimsner–Popa basis of $(R \otimes N)^\omega \subseteq (R \otimes M)^\omega$ (where ω is a non-principal ultrafilter).

Furthermore, since $R \otimes N = (R \otimes 1) \vee (1 \otimes N)$, we have the equality

$$(R \otimes N)' \cap (R \otimes M)^\omega = ((1 \otimes N)' \cap (R \otimes M)^\omega) \cap ((R \otimes 1)' \cap (R \otimes M)^\omega). \quad (6)$$

By [74, Prop. 3.2(1)], the inclusion $1 \otimes N \subseteq R \otimes M$ has spectral gap, and thus

$$(1 \otimes N)' \cap (R \otimes M)^\omega = ((1 \otimes N)' \cap (R \otimes M))^\omega = (R \otimes (N' \cap M))^\omega. \quad (7)$$

Since $(R \otimes 1)^\omega \subseteq (R \otimes M)^\omega$, combining (6) and (7), we have

$$\begin{aligned} (R \otimes N)' \cap (R \otimes M)^\omega &= (R \otimes (N' \cap M))^\omega \cap ((R \otimes 1)' \cap (R \otimes M)^\omega) \\ &= (R \otimes 1)' \cap (R \otimes (N' \cap M))^\omega. \end{aligned}$$

Thus there is a finite Pimsner–Popa basis $\{\tilde{m}_i\}$ for $(R \otimes N)^\omega \subseteq (R \otimes M)^\omega$ with $\tilde{m}_i = \{b_i^{(k)}\}_{k \in \mathbb{N}}$ where each $b_i^{(k)} \in R \otimes (N' \cap M)$.

In particular, for any $m \in M$ and any $\varepsilon > 0$, there exists finitely many elements $r_i \in R$, $k_i \in N' \cap M$, and $n_i \in N$ such that $\|(1_R \otimes m) - \sum_i r_i \otimes k_i n_i\|_2 < \varepsilon$. But then applying the trace preserving conditional expectation $E = \text{tr}_R \otimes \text{id}_M : R \otimes M \rightarrow M$, we have

$$\begin{aligned} \left\| m - \sum_i \text{tr}_R(r_i) k_i n_i \right\|_2 &= \left\| E((1_R \otimes m) - \sum_i r_i \otimes k_i n_i) \right\|_2 \\ &\leq \left\| (1_R \otimes m) - \sum_i r_i \otimes k_i n_i \right\|_2 < \varepsilon. \end{aligned}$$

Therefore $N \vee (N' \cap M) = M$. By Proposition 4.14, this implies that ${}_N L^2 M_N \cong \bigoplus_i {}_N L^2 N_N$. But since $|X \boxtimes_{R \otimes N} \overline{X}|$ is isomorphic to $L^2 R \otimes |K \boxtimes_{R \otimes N} \overline{K}|$ as N - N bimodules, we have that $X \boxtimes_{R \otimes N} \overline{X} \cong \bigoplus_i L^2(R \otimes N)$ as $R \otimes N - R \otimes N$ bimodules. But recall $X \boxtimes_{R \otimes N} \overline{X} \cong L^2(R \otimes N) \oplus H \oplus \overline{H} \oplus (H \boxtimes_{R \otimes N} \overline{H})$, which contains H as an irreducible summand. Thus H is trivial.

5 Local extension

In this section, we prove Theorem B, i.e., $\tilde{\chi}(|Q|) = \tilde{\chi}(M)_Q^{\text{loc}}$ for a commutative Q-system $Q \in \tilde{\chi}(M)$, where $|Q|$ is the realization of Q defined in Sect. 5.1 below. This result appears as Theorem 5.16.

5.1 Q-system realization

Q-systems are unitary versions of Frobenius algebra objects which were originally introduced by Longo in [52] to describe the canonical endomorphism for type III subfactors [51]. In this section, we define the *realization* procedure [13, 44] (based on [55] and [31, Sect. 4.1]) which given a Q-system Q over a II_1 factor M , recovers a von Neumann algebra $|Q|$ containing M and a conditional expectation $E_M : |Q| \rightarrow M$ with finite Pimsner–Popa index. This story works in much broader generality, but we restrict to II_1 factors here for ease of exposition. As in Sect. 4.1, for this section, M is a II_1 factor, and we denote $\mathbb{C}$, M , ${}_{\mathbb{C}}M_M$ as before:

$$\text{shaded circle} = M \quad \text{dashed circle} = \mathbb{C} \quad \text{dashed circle with vertical line} = {}_{\mathbb{C}}M_M.$$

Definition 5.1 Given a II_1 factor M , a *Q-system* in $\text{Bim}_{\text{fgp}}(M)$ is a triple (Q, m, i) where $Q \in \text{Bim}(M)$ is bimodule, and $m : Q \boxtimes_M Q \rightarrow Q$ and $i : M \rightarrow Q$ are bounded maps that satisfy certain relations best described graphically. Representing M by a shaded region and M by a strand, m is a trivalent vertex, and i is a univalent vertex; adjoints are represented by vertical reflection.

$$\text{shaded circle} = M \quad \text{vertical line} = {}_M Q_M. \quad \text{trivalent vertex (top)} = m \quad \text{trivalent vertex (bottom)} = m^\dagger \quad \text{univalent vertex (top)} = i \quad \text{univalent vertex (bottom)} = i^\dagger.$$

The axioms that m, i must satisfy are associativity, unitality, the Frobenius relations, and unitary separability ($mm^\dagger = \text{id}_Q$). We refer the reader to [13, Sect. 3.1] for a full discussion with many helpful diagrams. We call a Q-system $Q \in \text{Bim}_{\text{fgp}}(M)$ *connected* if $\text{Hom}_{M-M}(M \rightarrow Q) = \mathbb{C}i$.

Definition 5.2 [13, Sect. 4.1] For a Q-system $(Q, m, i) \in \text{Bim}(M)$, its *realization* $|Q|$ is the unital $*$ -algebra with underlying vector space is $\text{Hom}_{\mathbb{C}-M}({}_{\mathbb{C}}M_M \rightarrow {}_{\mathbb{C}}M \boxtimes_M Q_M)$, whose elements are denoted by

$$\text{diagram of a box labeled } q \text{ with a vertical line through it} \in |Q| := \text{Hom}_{\mathbb{C}-M}({}_{\mathbb{C}}M_M \rightarrow {}_{\mathbb{C}}M \boxtimes_M Q_M).$$

The multiplication, unit, and adjoint, respectively, of $|Q|$ are given by

$$q_1 \cdot q_2 := \text{diagram with } q_1 \text{ and } q_2 \text{ boxes connected by a line}, \quad 1_{|Q|} := \text{diagram with a single dot}, \quad q^* := \text{diagram with } q^\dagger \text{ box and a line}$$

Identifying $M = \text{End}(M_M)$, the inclusion $M \hookrightarrow |Q|$ is given by

$$\text{End}({}_{\mathbb{C}} M_M) \ni \text{diagram with } m \text{ box} \mapsto \text{diagram with } m \text{ box and a dot} \in |Q|$$

By [13, Rem. 4.4 and Prop. 4.6], $|Q|$ is a finite (Pimsner–Popa) index von Neumann algebra over M . Moreover, $|Q|$ is a II_1 factor if and only if $\text{End}_{Q-Q}(Q) = \mathbb{C} \text{id}_Q$. In this case, the unique trace-preserving conditional expectation is given by

$$E_M(q) = \frac{1}{[|Q| : M]} \cdot \text{diagram with } q \text{ box and a dot}$$

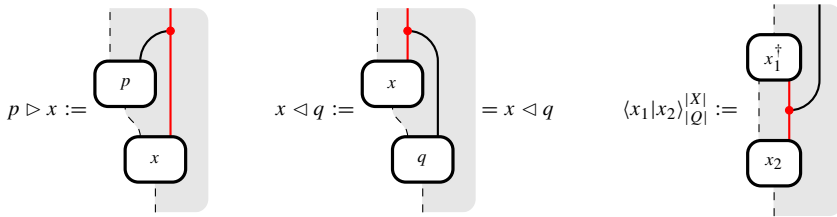
Example 5.3 (Canonical Q-systems) Suppose ${}_N H_P \in W^* \text{Alg}_{\text{fgp}}(N \rightarrow P)$. Then $H \boxtimes_P \overline{H}$ is a Q-system in $\text{Bim}_{\text{fgp}}(N)$. By [82, Prop. 3.1], there is a canonical isomorphism $H \boxtimes_P \overline{H} \cong L^2(P^{\text{op}})' \cap B(H)$, and thus $|H \boxtimes_P \overline{H}| \cong (P^{\text{op}})' =: M$. Observe that the relative commutant of $N \subseteq M$ is exactly given by $N' \cap M = N' \cap (P^{\text{op}})' = \text{End}_{N-P}(H)$.

Remark 5.4 Let N be a II_1 factor. If Q is a connected Q-system in $\text{Bim}_{\text{fgp}}(N)$, then $N \subseteq |Q|$ is a finite index irreducible II_1 subfactor. By the W^* version of [13, Prop. 4.16], $|Q|$ is also a connected Q-system in $\text{Bim}_{\text{fgp}}(N)$, and $Q \cong |Q|$ as Q-systems. Hence the connected Q-systems in $\text{Bim}_{\text{fgp}}(N)$ are exactly II_1 factors M containing N such that $N \subseteq M$ is finite index and irreducible.

Definition 5.5 [13, Sect. 4.2] Suppose $P, Q \in \text{Bim}_{\text{fgp}}(M)$ are two Q-systems and X is a $P - Q$ bimodule. The *realization* $|X| := \text{Hom}_{\mathbb{C}-M}({}_{\mathbb{C}} M_M \rightarrow {}_{\mathbb{C}} M \boxtimes_M X_M)$ of X is a $|P| - |Q|$ bimodule whose elements are denoted by

$$\text{diagram with } x \text{ box} \in |X| := \text{Hom}_{\mathbb{C}-M}({}_{\mathbb{C}} M_M \rightarrow {}_{\mathbb{C}} M \boxtimes_M X_M).$$

with left $|P|$ and right $|Q|$ -actions given respectively by



Clearly $|X|$ has a predual as it is a hom space in a W^* -category. By [13, Lem. 2.3 and Prop. 2.4], the right $|Q|$ -valued inner product is separately weak*-continuous, and the left action of $|P|$ on X is normal. Hence $|X| \in W^*\text{Alg}_{\text{fgp}}(|P| \rightarrow |Q|)$ is a W^* -correspondence.

Notation 5.6 Given another Q -system $R \in \text{Bim}_{\text{fgp}}(M)$ and a $Q - R$ bimodule Y , there is a notion of the relative tensor product of X with Y over Q , denoted $X \otimes_Q Y$. We refer the reader to [13, Sect. 3.2] for the detailed definition. We have two canonical projectors which we denote graphically as follows:

$$\begin{array}{|c|} \hline \text{red line} \\ \hline \end{array} : X \boxtimes_M Y \rightarrow X \otimes_Q Y \quad \text{and} \quad \begin{array}{|c|} \hline \text{blue line} \\ \hline \end{array} : |X \boxtimes_M Y| \rightarrow |X| \boxtimes_{|Q|} |Y|; \quad \text{grey box} = |Q|. \quad (8)$$

For the second diagram, we omit the external shadings, which may denote either a left/right M -action, or a left $|P|$ and right $|R|$ -action depending on context.

Remark 5.7 By the W^* version of [13, Thm. A], realization $|\cdot|$ gives a dagger 2-equivalence from the Q -system completion of $W^*\text{Alg}$ to $W^*\text{Alg}$. Thus $|\cdot|$ gives an equivalence from the unitary tensor category of $Q - Q$ bimodules in $\text{Bim}_{\text{fgp}}(M)$ with the tensor product $\otimes_Q$ to $\text{Bim}_{\text{fgp}}(|Q|)$ with the Connes fusion relative tensor product $\boxtimes_Q$. Moreover, the canonical tensorator $\mu_{X,Y} : |X| \boxtimes_{|Q|} |Y| \rightarrow |X \otimes_Q Y|$ fits into a commuting diagram with the canonical projectors (8):

$$\begin{array}{ccc} \begin{array}{|c|} \hline \text{red line} \\ \hline \end{array} & \searrow & |X| \boxtimes_{|Q|} |Y| \\ |X \boxtimes_M Y| & & \downarrow \mu_{X,Y} \\ \begin{array}{|c|} \hline \text{blue line} \\ \hline \end{array} & \searrow & |X \otimes_Q Y| \end{array} \quad (9)$$

5.2 Local extension

We now turn to the proof that $\tilde{\chi}(|Q|) = \tilde{\chi}(M)_Q^{\text{loc}}$.

Definition 5.8 A Q -system Q in a unitary braided tensor category $\mathcal{C}$ is called *commutative* if

$$\boxed{\text{crossing}} = \boxed{\text{crossing}} = \boxed{\text{crossing}}.$$

Suppose $Q \in \mathcal{C}$ is a commutative connected Q -system and let $X \in \mathcal{C}$ be a right Q -module. We say X is *local* if

$$\boxed{\text{crossing}} = \boxed{\text{crossing}}.$$

We can turn a local Q -module X_Q into a Q - Q bimodule by defining the left action map by

$$\boxed{\text{crossing}} := \boxed{\text{crossing}} = \boxed{\text{crossing}}.$$

It is well known that under the bimodule relative tensor product $\otimes_Q$, the collection of local Q -modules $\mathcal{C}_Q^{\text{loc}}$ is a unitary braided tensor category, where the braiding is inherited from $\mathcal{C}$.

Our goal now is to prove $\tilde{\chi}(|Q|) \cong \tilde{\chi}(M)_Q^{\text{loc}}$ as braided unitary tensor categories.

Lemma 5.9 If $(a_n)_n \subseteq M$ is a central sequence, then $(a_n)_n$ is also a central sequence in $|Q|$.

Proof Since Q is centrally trivial, for all $q \in |Q|$, $\|a_n q - q a_n\|_2 \rightarrow 0$ by Proposition 4.4. $\square$

Proposition 5.10 A Q -system $Q \in \tilde{\chi}(M)$ is commutative if and only if for every approximate inner Q_M -basis $\{q_i^{(n)}\}$, $(q_i^{(n)})_n$ is a central sequence in $|Q|$ for each fixed i .

Proof Suppose $\{q_j\}_j$ is a $\mathcal{Q}_M$ -basis and $\{q_i^{(n)}\}_i$ is an approximate $\mathcal{Q}_M$ -basis. By (3) and (4),

$$\begin{array}{c} \text{Diagram 1: A box with a crossing is approximately equal to a sum over } i, j \text{ of a box containing } q_j, q_i^{(n)}, q_j^\dagger, (q_i^{(n)})^\dagger. \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 2: A box with a crossing is approximately equal to a sum over } i, j \text{ of a box containing } q_j^{(n)}, q_i, (q_j^{(n)})^\dagger, q_i^\dagger. \end{array} \quad (10)$$

First, if each $q_i^{(n)}$ is a central sequence, it is straightforward to see that $\mathcal{Q}$ is commutative using (10).

Conversely, if $\mathcal{Q}$ is commutative, to show each $q_i^{(n)}$ is a central sequence, since $\|q_i^{(n)}a - aq_i^{(n)}\|_2 \rightarrow 0$ for each $a \in M$, it suffices to prove $\|q_i^{(n)}q_j - q_jq_i^{(n)}\|_2 \rightarrow 0$ for each j . For each i, j and n , define

$$x_n^{i,j} := q_i^{(n)}q_j - q_jq_i^{(n)} = \begin{array}{c} \text{Diagram 3: Box with } q_i^{(n)}, q_j \end{array} - \begin{array}{c} \text{Diagram 4: Box with } q_j, q_i^{(n)} \end{array} \in \text{Hom}(M_M \rightarrow \mathcal{Q}_M).$$

Observe that in $\text{End}(\mathcal{Q}_M)$, then again by (10) we have

$$\sum_{i,j} x_n^{i,j} (x_n^{i,j})^\dagger = \sum_{i,j} \left(\begin{array}{c} \text{Diagram 5: Box with } q_i^{(n)}, q_j, q_j^\dagger, (q_i^{(n)})^\dagger \end{array} - \begin{array}{c} \text{Diagram 6: Box with } q_i^{(n)}, q_j, (q_i^{(n)})^\dagger, q_j^\dagger \end{array} - \begin{array}{c} \text{Diagram 7: Box with } q_j, q_i^{(n)}, q_j^\dagger, (q_i^{(n)})^\dagger \end{array} + \begin{array}{c} \text{Diagram 8: Box with } q_j, q_i^{(n)}, (q_i^{(n)})^\dagger, q_j^\dagger \end{array} \right)$$

☐

$$\begin{aligned} X_M \otimes_M L^2 M & \quad \text{with} \quad (x \otimes \Omega_M, y \otimes \Omega_M) := (\langle y|x \rangle_M^X \Omega_M, \Omega_M)_{L^2 M} = \text{tr}_M(\langle y|x \rangle_M^X) \\ X_{|Q|} \otimes_{|Q|} L^2 |Q| & \quad \text{with} \quad (x \otimes \Omega_{|Q|}, y \otimes \Omega_{|Q|}) := (\langle y|x \rangle_{|Q|}^X \Omega_{|Q|}, \Omega_{|Q|})_{L^2 |Q|} = \text{tr}_{|Q|}(\langle y|x \rangle_{|Q|}^X) \\ & \quad = \text{tr}_M \circ E_M(\langle y|x \rangle_{|Q|}^X) = \frac{1}{||Q|| : M|} \text{tr}_M(\langle y|x \rangle_M^X). \end{aligned}$$

$X_{|Q|} \otimes_{|Q|} L^2|Q|_{|Q|}$. One calculates that

$$\begin{aligned} \dim(X \otimes_M L^2 M_M) &= \sum_i \langle b_i \otimes \Omega_M, b_i \otimes \Omega_M \rangle = \sum_i \operatorname{tr}_M(\langle b_i | b_i \rangle_M), \\ \dim(X \otimes_{|Q|} L^2|Q|_{|Q|}) &= \sum_i \langle b_i \otimes \Omega_{|Q|}, b_i \otimes \Omega_{|Q|} \rangle = \frac{1}{[|Q| : M]} \sum_i \operatorname{tr}_M(\langle b_i | b_i \rangle_M) \\ &= \frac{1}{[|Q| : M]} \dim(X \otimes_M L^2 M_M), \end{aligned}$$

as expected.

It is important to remember that the subfactor is not $M \subset |Q|$, but rather $\iota(M) \subset |Q|$. As we have normalized our Q-system (Q, m, i) so that $i^\dagger i = d_Q$, the M -valued inner product on $Q_M \cong |Q|_M$ will not agree with the $\iota(M)$ -valued inner product on $|Q|$ defined in the usual subfactor way [6, p18] under the isomorphism $M \cong \iota(M)$. Indeed,

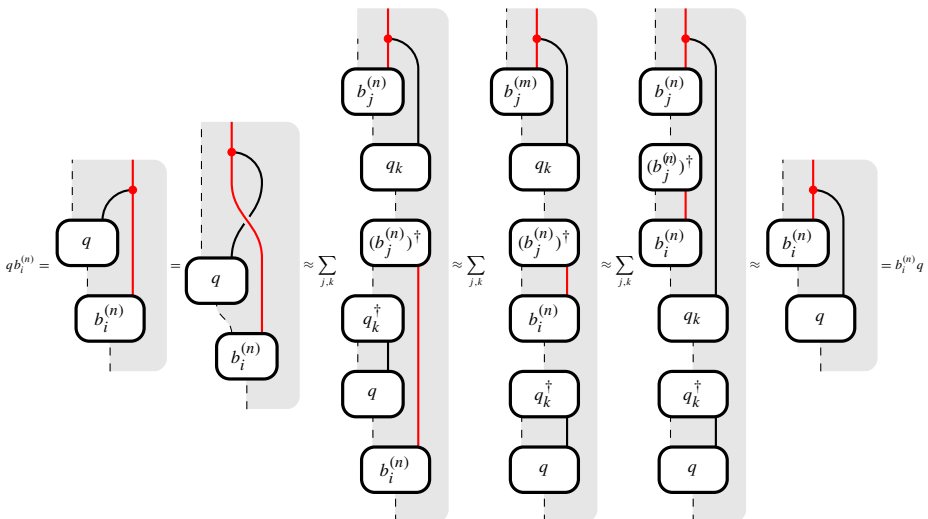
$$\langle x|y \rangle_{\iota(M)} = E_M(\langle x|y \rangle_{|Q|}) = E_M(x^* y) = d_Q^{-1} \langle x|y \rangle_M.$$

This is perfectly fine, since rescaling an M -valued inner product cannot change the von Neumann dimension after applying $- \otimes_M L^2 M$, as the rescaled M -valued inner product will also rescale a right M -basis.

Proposition 5.13 *Realization $|\cdot|$ takes every bimodule in $\tilde{\chi}(M)_Q^{\text{loc}}$ into $\tilde{\chi}(|Q|)$.*

Proof Suppose $X \in \tilde{\chi}(M)_Q^{\text{loc}}$. We show $|Q||X|_{|Q|}$ is approximately inner and centrally trivial.

Since X is approximately inner over M , by Proposition 4.6, there is an approximately inner X_M -basis $\{b_i^{(n)}\} \subseteq X$. By Lemma 5.11, $\{b_i^{(n)}\}$ is an approximate $|X|_{|Q|}$ -basis. Now we show (2), i.e., $\|qb_i^{(n)} - b_i^{(n)}q\|_2 \rightarrow 0$ for all $q \in |Q|$.



The second equality uses the hypothesis that X is a local bimodule. The third $\approx$ uses (4). The fourth $\approx$ holds because $\langle q_k | q \rangle_M^Q \in M$ and $\|[x, b_i^{(n)}]\|_2 \rightarrow 0$ for $x \in M$. The fifth $\approx$ holds because $\langle b_j^{(n)} | b_i^{(n)} \rangle_M^X$ is a central sequence in M for each i, j and Q is centrally trivial over M . The sixth $\approx$ holds from the definition of (approximate) basis for X_M and Q_M . We conclude that $|X|$ is approximately inner over $|Q|$.

Since X is centrally trivial over M , by Proposition 4.4, for all central sequences $(a_n)_n \subseteq M$ and all $x \in X$, $\|a_n x - x a_n\|_2 \rightarrow 0$. If $(q_n)_n \subseteq |Q|$ is a central sequence, then

$$\begin{aligned}
 q_n x &= \text{Diagram 1} = \text{Diagram 2} \approx \sum_{j,k} \text{Diagram 3} \approx \sum_{j,k} \text{Diagram 4} \approx \sum_{j,k} \text{Diagram 5} \approx \text{Diagram 6} = x q_n
 \end{aligned}$$

The second equality uses the hypothesis that X is local. The third $\approx$ uses (3). The fourth $\approx$ holds because $\langle q_k^{(n)} | q_n \rangle_M^Q$ is a central sequence in M , and X is centrally trivial. The fifth $\approx$ holds because $\langle b_j | x \rangle_M^X \in M$ and $\|[a, q_k^{(n)}]\|_2 \rightarrow 0$ for $a \in M$. The sixth $\approx$ holds from the definition of (approximate) basis for X_M and Q_M . We conclude that $|X|$ is centrally trivial over $|Q|$. $\square$

The following proposition is straightforward; we omit its proof.

Proposition 5.14 (1) *If $\{b_i\}_i$ is an $|X|_{|Q|}$ -basis and $\{q_j\}_j$ is a Q_M -basis, then*

$$\left\{ \text{Diagram of } c_{i,j} := \text{Diagram of } b_i \text{ and } q_j \right\}_{i,j}$$

is an $|X|_M$ -basis.

- (2) If $\{b_i^{(n)}\}$ is an approximate $X|_Q$ -basis and $\{q_j^{(n)}\}$ is an approximately inner Q_M -basis, then

$$\left\{ \begin{array}{c} \text{Diagram showing } c_{i,j}^{(n)} \text{ and } b_i^{(n)}, q_j^{(n)} \text{ in boxes with arrows and a red dot} \\ \vdots \\ \text{Diagram showing } c_{i,j}^{(n)} \text{ and } b_i^{(n)}, q_j^{(n)} \text{ in boxes with arrows and a red dot} \end{array} \right\}_{i,j}$$

is an approximate $|X|_M$ -basis. Moreover, if $\{b_i^{(n)}\}_i$ is approximately inner, so is $\{c_{i,j}^{(n)}\}$.

Since X_M is canonically isomorphic to $|X|_M$, we may view (1) as an X_M -basis and (2) as an approximate(ly inner) X_M -basis.

Proposition 5.15 Every bimodule in $\tilde{\chi}(|Q|)$ is unitarily isomorphic to a realization of a bimodule in $\tilde{\chi}(M)_Q^{\text{loc}}$.

Proof By Remark 5.7, it suffices to consider a Q – Q bimodule ${}_Q X_Q$ in $\text{Bim}_{\text{fgp}}(M)$ such that $|X| \in \tilde{\chi}(|Q|)$. In order to show $X \in \tilde{\chi}(M)_Q^{\text{loc}}$, must prove X is centrally trivial and approximately inner over M , and X is a local Q – Q bimodule.

By Lemma 5.9, any central sequence $(a_n) \subseteq M$ is also a central sequence in $|Q|$. By Proposition 4.4, X is centrally trivial over $|Q|$, so X is centrally trivial over M . By Propositions 5.14 and 4.6, we have X is approximately inner over M . Therefore, as an M – M bimodule, $X \in \tilde{\chi}(M)$.

Since X is already a Q – Q bimodule, it remains to show X is local. Let $\{b_i\}, \{q_j\}$ be $X|_Q, Q_M$ -bases and let $\{b_i^{(n)}\}, \{q_j^{(n)}\}$ be approximately inner $X|_Q, Q_M$ -bases respectively. Defining $\{c_{i,j}\}$ and $\{c_{i,j}^{(n)}\}$ as in Proposition 5.14 gives an X_M -basis and an approximately inner X_M -basis respectively.

For the over braiding,

The first $\approx$ uses (3), the second equality is the definition of $c_{i,j}$, the third equality uses associativity of the Q – Q bimodule actions, the fourth $\approx$ uses that X is centrally trivial over $|Q|$, the fifth $\approx$ uses (10), and the sixth equation follows from

For the under braiding,

$$\begin{aligned}
 & \approx \sum_{i,j,k} \text{Diagram 1} = \sum_{i,j,k} \text{Diagram 2} = \sum_{i,j,k} \text{Diagram 3} = \sum_{i,j,k} \text{Diagram 4} \approx \sum_{i,j,k} \text{Diagram 5} \\
 & \approx \sum_i \text{Diagram 6} = \sum_i \text{Diagram 7} \approx \text{Diagram 8}
 \end{aligned}$$

The first $\approx$ uses (4), the second equality is the definition of $c_{i,j}^{(n)}$, the third equality uses associativity of the Q - Q bimodule actions, the fourth $\approx$ uses X is centrally trivial over $|Q|$ so $\|[b_i^{(n)}, q_k]\|_2 \rightarrow 0$, the fifth $\approx$ uses (10), and the sixth equality uses (11) again. $\square$

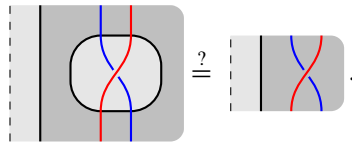
Theorem 5.16 *Realization gives a braided unitary equivalence $\tilde{\chi}(M)_Q^{\text{loc}} \rightarrow \tilde{\chi}(|Q|)$.*

Proof. By Remark 5.7, realization $|\cdot|$ gives a unitary tensor equivalence from Q - Q bimodules in $\text{Bim}_{\text{fgp}}(M)$ to $\text{Bim}_{\text{fgp}}(|Q|)$. By Proposition 5.13, for $X \in \tilde{\chi}(M)_Q^{\text{loc}}$, $|X| \in \tilde{\chi}(|Q|)$, and by Proposition 5.15, every bimodule in $\tilde{\chi}(|Q|)$ arises in this way. Since $\tilde{\chi}(M)_Q^{\text{loc}}$ is a full subcategory of the Q - Q bimodules in $\text{Bim}_{\text{fgp}}(M)$, $\tilde{\chi}(|Q|)$ is a full subcategory of $\text{Bim}_{\text{fgp}}(|Q|)$, and realization $|\cdot|$ is fully faithful, it restricts to a unitary tensor equivalence $\tilde{\chi}(M)_Q^{\text{loc}} \rightarrow \tilde{\chi}(|Q|)$.

It remains to verify that $|\cdot| : \tilde{\chi}(M)_Q^{\text{loc}} \rightarrow \tilde{\chi}(|Q|)$ is braided, i.e., the following diagram commutes.

$$\begin{array}{ccccc}
 |X| \boxtimes_{|Q|} |Y| & \xrightarrow{u_{X,Y}^{|Q|}} & |Y| \boxtimes_{|Q|} |X| \\
 \downarrow \mu_{X,Y} & \searrow & \swarrow & \downarrow \mu_{Y,X} \\
 |X| \otimes_Q |Y| & \hookrightarrow & |X| \boxtimes_M |Y| & \xrightarrow{|u_{X,Y}^M|} & |Y| \boxtimes_M |X| & \twoheadrightarrow & |Y| \otimes_Q |X|
 \end{array}$$

The two triangles on either side commute by (9), so it remains to prove the inner square commutes. Graphically denoting the II_1 factor $|Q|$ and the canonical projector $|X| \boxtimes_M |Y| \rightarrow |X| \otimes_{|Q|} |Y|$ as in (8), as realization is fully faithful, this is the condition that



Let $\{c_k\}$ be a $Y_{|Q|}$ -basis, and let $\{b_i^{(n)}\}$ be an approximately inner $X_{|Q|}$ -basis. Let $\{q_l\}$ be a Q_M -basis and $\{q_j^{(n)}\}$ be an approximately inner Q_M -basis. According to Proposition 5.14, $\{c_k q_l\}$ is a Y_M basis and $\{b_i^{(n)} q_j^{(n)}\}$ is an approximately inner X_M basis. Then

The diagram illustrates a sequence of transformations between string diagrams, representing a proof. The diagrams are arranged in two rows. The top row starts with a small diagram on the left, followed by a sequence of diagrams connected by $\approx$ and $=$ symbols. The bottom row continues the sequence, ending with a small diagram on the right. The diagrams use blue and red strands, boxes labeled with algebraic elements like c_k , q_l , $b_i^{(n)}$, $q_l^{(n)}$, $c_k^\dagger$, $q_l^\dagger$, and $b_i^{(n)\dagger}$, and summation symbols with indices i, j, k, l .

The first $\approx$ uses (3) for $X \boxtimes_M Y$, and the second equality uses associativity of the bimodule actions. The third $\approx$ uses X is approximately inner over $|Q|$, Y is centrally trivial over $|Q|$ and $(q_l^{(n)})$ is a central sequence in $|Q|$, and the fourth $\approx$ uses (10). The fifth equality uses an argument similar to (11). The sixth equality is just isotopy, and the final $\approx$ uses (3) for $X \boxtimes_{|Q|} Y$.

Finally, since

The diagram shows two equations. The first equation is: a vertical line with a dashed line to its left is equal to a vertical line with a dashed line to its left and a crossing, which is equal to a sum over j of a box labeled q_j and a box labeled $q_j^\dagger$. The second equation is: a crossing of two lines is equal to a crossing of two lines with a different orientation.

we have

The diagram shows two equations. The first equation is: a crossing of two lines is equal to a sum over j of a box labeled q_j and a box labeled $q_j^\dagger$. The second equation is: a crossing of two lines is equal to a crossing of two lines with a different orientation.

6 Calculation of $\tilde{\chi}(M_\infty)$ for a non-Gamma finite depth II_1 subfactor

In this section, we calculate $\tilde{\chi}(M_\infty)$ for the inductive limit II_1 factor obtained from iterating Jones' basic construction for a finite depth finite index non-Gamma II_1 subfactor $N \subseteq M$. These examples are motivated by [74].

Suppose $N \subseteq M$ a finite depth, finite index II_1 subfactor, and let $\mathcal{C} = {}_N\mathcal{C}_N$ denote the unitary fusion category of N - N bimodules generated by ${}_NM_N$. The results of [79, Sects. 3 and 4] give a bijective correspondence between equivalence classes of (bifinite) bimodules of M_∞ which restrict to $R \otimes N$ -bimodules of the form $\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}$ and objects of the Drinfeld center $\mathcal{Z}(\mathcal{C})$. The main goal of this section is to extend this bijection to a fully faithful unitary tensor functor $\Phi : \mathcal{Z}(\mathcal{C}) \rightarrow \text{Bim}_{\text{fgp}}(M_\infty)$ such that when N is non-Gamma, Φ takes values in $\tilde{\chi}(M_\infty)$ and is a braided unitary equivalence. To do so, we rely on the Q-system realization language from [13] together with the coend realization viewpoint of [44].

We begin this section with some basics on unitary fusion categories and subfactor standard invariants.

6.1 Basics on unitary fusion categories and subfactor standard invariants

A *unitary fusion category* is a unitary tensor category with only finitely many isomorphism classes of simple objects. A unitary fusion category $\mathcal{C}$ has three commuting involutions $\dagger, \vee, \bar{}$, and the composite of any two is the third. Here, $\vee$ is the unique unitary dual functor [67, 86] giving the canonical unitary spherical structure of $\mathcal{C}$ [54], and we may define $\bar{} := \vee \dagger = \dagger \vee$.

Definition 6.1 The *Drinfeld center* of a unitary fusion category $\mathcal{C}$ is $\mathcal{Z}(\mathcal{C}) = \text{End}_{\mathcal{C}-\mathcal{C}}(\mathcal{C})$, the Morita dual of $\mathcal{C}^{\text{mp}} \boxtimes \mathcal{C}$ acting on $\mathcal{C}$ by $(a^{\text{mp}} \boxtimes b) \triangleright c := b \otimes c \otimes a$, where $\mathcal{C}^{\text{mp}}$ is the monoidal opposite of $\mathcal{C}$ from Remark 4.9. Note that the unitary dual functor $\vee$ gives a unitary tensor equivalence $\mathcal{C}^{\text{mp}} \rightarrow \mathcal{C}^{\text{op}}$, the opposite fusion category with the opposite arrows, but the same tensor product. It is useful in the subsections below to identify $\mathcal{Z}(\mathcal{C})$ with $\text{End}_{\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}}(\mathcal{C})$ with the action $(a^{\text{op}} \boxtimes b) \triangleright c := b \otimes c \otimes \bar{a}$.

Now $\mathcal{Z}(\mathcal{C})$ has another description in terms of pairs (z, σ_z) of an object $z \in \mathcal{C}$ equipped with a half-braiding σ_z , where $\mathcal{Z}(\mathcal{C})$ acts on $\mathcal{C}$ via the forgetful functor $(z, \sigma_z) \mapsto z$ [19, Sects. 7.13 and 8.5]. Our convention for the half-braiding σ_z is that the strands for objects in $\mathcal{C}$ pass over the z -strand:

$$\sigma_{c,z} := \begin{array}{c} \text{blue strand} \\ \text{red strand } c \\ \text{blue strand } z \end{array}$$

Thus the braiding $(z, \sigma_z) \otimes (w, \rho_w) \rightarrow (w, \rho_w) \otimes (z, \sigma_z)$ in $\mathcal{Z}(\mathcal{C})$ is given by $\rho_{z,w}$.

Definition 6.2 There are many equivalent notions of the standard invariant for a finite index II_1 subfactor $N \subseteq M$. For this article, the standard invariant will mean the 2×2 unitary multitensor category $\mathcal{C}(N \subseteq M)$ of N - N , N - M , M - N , and M - M bimodules generated by $L^2 M$ under $\boxtimes, \oplus$, and $\subseteq$, with generating object ${}_N L^2 M_M$.

$$\mathcal{C}(N \subseteq M) = \begin{pmatrix} {}_N \mathcal{C}_N & {}_N \mathcal{C}_M \\ {}_M \mathcal{C}_N & {}_M \mathcal{C}_M \end{pmatrix}$$

Observe that $\mathcal{C}(N \subseteq M)$ is multifusion if and only if $N \subseteq M$ is finite depth. In this case, the corners ${}_N \mathcal{C}_N$ and ${}_M \mathcal{C}_M$ of N - N and M - M bimodules generated by $L^2 M$ respectively are unitary fusion categories which are Morita equivalent, and thus share the same Drinfeld center $\mathcal{Z}(\mathcal{C})$.

Remark 6.3 Suppose $\mathcal{C}$ and $\mathcal{D}$ are two unitary fusion categories and ${}_C \mathcal{M}_{\mathcal{D}}$ is an indecomposable unitary $\mathcal{C} - \mathcal{D}$ bimodule category witnessing a Morita equivalence. Using the internal hom [64] (see also [61, Appendix A]), we can form a 2×2 unitary multifusion category by

$$\begin{pmatrix} \mathcal{C} & \mathcal{M} \\ {}_{\mathcal{M}^{\text{op}}} & \mathcal{D} \end{pmatrix}. \quad (12)$$

For a simple $X \in \mathcal{M}$, we get two Q-systems $X \otimes \bar{X} = \text{End}_{\mathcal{C}}(X) \in \mathcal{C}$ and $\bar{X} \otimes X = \text{End}_{\mathcal{D}}(X) \in \mathcal{D}$. The map $\text{Ad}(X) : d \mapsto X \otimes d \otimes \bar{X}$ gives a unitary tensor equivalence

between $\mathcal{D}$ and $X \otimes \overline{X} - X \otimes \overline{X}$ bimodules in $\mathcal{C}$. A similar result holds on the other side.

Suppose now we have a fully faithful unitary tensor functor $F : \mathcal{C} \rightarrow \text{Bim}_{\text{fgp}}(N)$ for a II_1 factor N . Then the realization $M := |X \otimes \overline{X}|$ is a II_1 factor containing N , and the standard invariant of $N \subseteq M$ is unitarily equivalent to the 2×2 unitary multifusion category (12) with generator $|X|$ as an N - M bimodule. By Remark 5.7, we get a fully faithful unitary tensor functor $G : \mathcal{D} \rightarrow \text{Bim}_{\text{fgp}}(M)$ from realization as

$$\mathcal{D} \xrightarrow{\text{Ad}(X)} \text{Bim}_{\mathcal{C}}(X \otimes \overline{X}) \xrightarrow{|\cdot|} \text{Bim}_{\text{fgp}}(M).$$

We now give an important example of Remark 6.3 which will be used in this section below.

Example 6.4 Let $\mathcal{C}$ be a unitary fusion category, and consider the $\mathcal{C} \boxtimes \mathcal{C}^{\text{mp}} - \mathcal{Z}(\mathcal{C})$ Morita equivalence bimodule $\mathcal{C}$. One calculates that $\underline{\text{End}}_{\mathcal{C}^{\text{mp}} \boxtimes \mathcal{C}}(1_{\mathcal{C}}) = \bigoplus_{c \in \text{Irr}(\mathcal{C})} \overline{c}^{\text{mp}} \boxtimes c$, which we call the *symmetric enveloping Q-system* after [70, 73]. This algebra object first appeared in [53], and its realization was later shown to be equivalent to the symmetric enveloping algebra/asymptotic inclusion by [56]. The infinite version of this algebra object plays a very important role for the study of analytic properties of infinite unitary tensor categories [80]. Identifying $\mathcal{C}^{\text{mp}} \cong \mathcal{C}^{\text{op}}$ via $\vee$, which will be useful in the sequel, the symmetric enveloping Q-system is given by $S := \bigoplus_{c \in \text{Irr}(\mathcal{C})} c^{\text{op}} \boxtimes c$. By Remark 6.3, $\mathcal{Z}(\mathcal{C}) \cong \text{Bim}_{\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}}(S)$. On the other hand, one calculates that $\underline{\text{End}}_{\mathcal{Z}(\mathcal{C})}(1_{\mathcal{C}}) = I(1_{\mathcal{C}})$, where $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ is the adjoint of the forgetful functor.

6.2 Q-system realization as a coend

Suppose $\mathcal{C}$ is a unitary fusion category and $G : \mathcal{C} \rightarrow \text{Bim}_{\text{fgp}}(N)$ is a unitary tensor functor, where N is a II_1 factor. Given a Q-system $Q \in \mathcal{C}$, the realization $|G(Q)|$ is a II_1 multifactor (finite direct sum of II_1 factors) which is a factor if and only if Q is simple as a Q - Q bimodule in $\mathcal{C}$.

By the Yoneda lemma, we have a canonical isomorphism of vector spaces

$$|G(Q)| := \text{Hom}(N_N \rightarrow N \boxtimes_N G(Q)_N) \cong \bigoplus_{c \in \mathcal{C}} \mathcal{C}(c \rightarrow Q) \otimes_{\mathbb{C}} G(c).$$

We graphically represent elements of this tensor product by

$$\sum_{c \in \text{Irr}(\mathcal{C})} \begin{array}{c} \text{---} Q \\ | \\ \text{---} c \\ \text{---} c \\ | \\ \text{---} c \end{array} := \sum_{c \in \text{Irr}(\mathcal{C})} \begin{array}{c} \text{---} Q \\ | \\ \text{---} c \\ | \\ \text{---} c \end{array} \otimes_{\mathbb{C}} \begin{array}{c} c \\ | \\ \text{---} c \\ | \\ \text{---} c \end{array} = \sum_{c \in \text{Irr}(\mathcal{C})} f_c \otimes_{\mathbb{C}} \xi_c.$$

Here, the orange line represents the functor $G^\circ := \text{Forget} \circ G$ viewed as a W^* -algebra object in $\text{Fun}(\mathcal{C} \rightarrow \text{Vect})$ [44, Prop. 2.18], where $\text{Forget} : \text{Bim}_{\text{fgp}}(N) \rightarrow \text{Vect}$ is the forgetful functor. The shaded half of the diagram is read top to bottom, and the tensorator $G_{a,b}^2$ is denoted by appending an orange trivalent vertex below.

Under this isomorphism of vector spaces, the multiplication and $*$ -structure are given by

$$(f_a \otimes \xi_a)(g_b \otimes \eta_b) = \sum_{\substack{c \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}} \text{Diagram} \quad (f_a \otimes \xi_a)^* = \text{Diagram}^*,$$

The diagrams are string diagrams representing the multiplication and adjoint operation. The first diagram shows a composition of two boxes f_a and g_b with inputs a and b respectively, and outputs Q . Below them, a box ξ_a and η_b have inputs a and b respectively, and output c . An orange line connects the output c of the bottom boxes to the input c of the top boxes. The second diagram shows a box f_a with input Q and output $\bar{a}$, and a box ξ_a with input $\bar{a}$ and output a . An orange line connects the output a of the bottom box to the input a of the top box.

and the unit is given by

$$1 = \text{Diagram} ; \quad \Omega_N = 1 \in {}_N N_N.$$

The diagram shows a box Ω_N with input Q and output a . An orange line connects the output a of the box to the input a of the box.

Example 6.5 Suppose now $N \subseteq M$ is a finite depth, finite index II_1 subfactor. The algebra M considered as an N - N bimodule ${}_N M_N \in {}_N \mathcal{C}_N$ is the canonical Q-system $X \boxtimes_M \bar{X}$ corresponding to the generator $X := {}_N M_M \in {}_N \mathcal{C}_M$ as discussed in Example 5.3. The realization $|{}_N M_N|$ is canonically $*$ -isomorphic to M :

$$\begin{aligned} |{}_N M_N| &= \text{Hom}(N_N \rightarrow N \boxtimes_N M_N) = \text{Hom}(N_N \rightarrow N \boxtimes_N M \boxtimes_M M_N) \\ &\cong \text{Hom}(M_M \rightarrow M_M) = M. \end{aligned}$$

Now consider the Jones tower obtained by iterating Jones' basic construction defined inductively by $M_{n+1} := \text{End}((M_n)_{M_{n-1}})$ [23, 39]

$$M_0 = N \subseteq M = M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots.$$

The II_1 factor M_n is $*$ -isomorphic to the realization of the Q-system $(X \boxtimes_M \bar{X})^{\boxtimes n} \cong X^{\text{alt} \boxtimes n} \boxtimes \bar{X}^{\text{alt} \boxtimes n}$ which has multiplication and unit given by

$$\text{Diagram} \quad \text{Diagram} \quad \left| \quad n = \text{id}_{X^{\text{alt} \boxtimes n}}; \quad X^{\text{alt} \boxtimes n} := \underbrace{X \boxtimes_M \bar{X} \boxtimes_N \cdots \boxtimes_N \bar{X}}_{n \text{ tensorands}}.$$

The first diagram shows a box with inputs n and n and output n . The second diagram shows a box with inputs n and n and output n . The third diagram shows a box with input n and output n .

Indeed,

$$\text{Hom}(N_N \rightarrow N \boxtimes_N X^{\text{alt} \boxtimes n} \boxtimes \overline{X^{\text{alt} \boxtimes n}}) \cong \text{End}(N \boxtimes_N X^{\text{alt} \boxtimes n}_{M \text{ or } N}) \cong M_n$$

by the multistep Jones basic construction [6, 77]. Another way to see this is to use Remark 6.3; for example, the map $\text{Ad}(X)$ takes the basic construction $\langle M, N \rangle = \overline{X} \boxtimes_N X$ to $(X \boxtimes_M \overline{X})^{\boxtimes 2}$ with the multiplication as claimed.

As a coend realization, we have a canonical Frobenius reciprocity isomorphism

$$M_n = \bigoplus_{c \in \text{Irr}(C)} \mathcal{C}(c \rightarrow X^{\text{alt} \boxtimes n} \boxtimes \overline{X^{\text{alt} \boxtimes n}}) \otimes_{\mathbb{C}} G(c) \cong \bigoplus_{c \in \text{Irr}(C)} \mathcal{C}(c \boxtimes X^{\text{alt} \boxtimes n} \rightarrow X^{\text{alt} \boxtimes n}) \otimes_{\mathbb{C}} G(c). \quad (13)$$

Under this isomorphism, in the coend realization diagrammatic calculus, we can draw the $X^{\text{alt} \boxtimes n}$ horizontally, where the horizontal line should be viewed as slightly tilted going from the bottom right to the top left, as indicated by the cyan arrows below.

$$k \text{ --- } \boxed{f} \text{ --- } k \in \mathcal{C}(c \boxtimes X^{\text{alt} \boxtimes n} \rightarrow X^{\text{alt} \boxtimes n}) \quad (14)$$

The multiplication, $*$, and unit in the realization $|X^{\text{alt} \boxtimes n} \boxtimes \overline{X^{\text{alt} \boxtimes n}}| \cong M_n$ are now represented respectively by

$$\sum_{\substack{a, b, c \in \text{Irr}(C) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}} \text{Diagram 1}, \quad \text{Diagram 2}, \quad \text{and} \quad \text{Diagram 3} = \text{Diagram 4}.$$

Here, the $X^{\text{alt} \boxtimes n}$ horizontal strand is read bottom to top, i.e.,

$$\text{--- } n \text{ ---} = \text{--- } X^{\text{alt} \boxtimes n} \text{ ---} = \begin{array}{c} \text{--- } X^? \text{ ---} \\ \vdots \\ \text{--- } X \text{ ---} \\ \text{--- } \overline{X} \text{ ---} \\ \text{--- } X \text{ ---} \end{array}.$$

6.3 The inductive limit factor as a realization

We now give a graphical representation of the inductive limit II_1 factor M_∞ from the realized Jones tower from Example 6.5. We begin with a short remark about inductive limits in the category of tracial von Neumann algebras and trace-preserving unital $*$ -homomorphisms, followed by a brief review of construction of the hyperfinite II_1 subfactor $R \subseteq P$ with the opposite standard invariant as our finite depth, finite index II_1 subfactor $N \subseteq M$.

Remark 6.6 Consider the category whose objects are pairs (M, tr) where M is a separable von Neumann algebra and tr is a faithful normal tracial state, and whose morphisms are trace-preserving unital $*$ -homomorphisms (which are automatically normal by [8, III.2.2.2] and [41, Prop. 9.1.1]). It is well known that this category admits inductive limits. We briefly recall the construction for completeness and convenience of the reader.

For an increasing sequence of tracial von Neumann algebras (M_n, tr_n) , we get a tracial von Neumann algebra $\varinjlim M_n$ by taking the bicommutant of $\bigcup_n M_n$ on its GNS Hilbert space with respect to the limit trace, which is equipped with the faithful tracial state $\varinjlim \text{tr} := \langle \cdot, \Omega \rangle_{L^2(\bigcup_n M_n)}$. For any tracial von Neumann algebra (N, tr) and trace-preserving maps $\varphi_n : M_n \rightarrow N$ compatible with the inclusions, we get a unique trace-preserving map $\varphi : \bigcup_n M_n \rightarrow N$, which extends uniquely to a trace-preserving map $\varinjlim M_n \rightarrow N$.

Indeed, for a fixed $x \in \varinjlim M_n$, let $\xi \in L^2 N$ be the image of $x\Omega$ under the induced map of GNS spaces $L^2(\bigcup_n M_n) \rightarrow L^2 N$ given by the extension of $m\Omega \mapsto \varphi(m)\Omega$. Since there is a bounded sequence $(x_k) \subseteq \bigcup_n M_n$ with $x_k \rightarrow x$ in $\|\cdot\|_2$, we see

$$\begin{aligned} \|L_\xi n\Omega\|_2 &= \|\xi n\|_2 = \|Jn^* J\xi\|_2 = \lim \|Jn^* J\varphi(x_k)\Omega\|_2 = \lim \|\varphi(x_k)n\Omega\|_2 \\ &\leq \limsup \|\varphi(x_k)\| \cdot \|n\Omega\|_2 = \limsup \|x_k\| \cdot \|n\Omega\|_2. \end{aligned}$$

Thus ξ is N -bounded, and necessarily of the form $\varphi(x)\Omega$ for some $\varphi(x) \in N$. Since multiplication is jointly SOT-continuous on bounded subsets and $*$ is SOT-continuous on bounded subsets of $\varinjlim M_n$, it is easily verified that the above definition of $\varphi(x)$ extends φ to a unital $*$ -homomorphism $\varinjlim M_n \rightarrow N$.

For the rest of this section, we fix a finite depth finite index II_1 subfactor $N \subseteq M$. Recall that its standard invariant can also be described as the subfactor planar algebra $\mathcal{P}_\bullet$ whose box spaces are given by

$$\mathcal{P}_{k,+} := \text{End}(X^{\text{alt}} \boxtimes^k) \quad \text{and} \quad \mathcal{P}_{k,-} := \text{End}(\overline{X}^{\text{alt}} \boxtimes^k).$$

These finite dimensional von Neumann algebras are equipped with the canonical traces which agree with the categorical traces on $\mathcal{C}(N \subseteq M)$.

We now rapidly recall how to construct a hyperfinite II_1 subfactor with the opposite standard invariant [62, 69]. For a detailed diagrammatic exposition (in the multifactor setting), see [2, Sect. 5.1]. Since $N \subseteq M$ is finite depth, the inductive limit tracial

von Neumann algebras $\varinjlim \mathcal{P}_{k,\pm}$ are hyperfinite II_1 factor. We have a trace-preserving injection $\mathcal{P}_{n,+} \hookrightarrow \mathcal{P}_{n+1,-}$ by adding an $\overline{X}$ strand to the left, giving a II_1 subfactor

$$R := \varinjlim \mathcal{P}_{n,+} \subseteq \varinjlim \mathcal{P}_{n,-} =: P.$$

It is well known that by the Ocneanu Compactness Theorem [46, Thm. 5.7.6], the inclusion $R^{\text{op}} \subseteq P^{\text{op}}$ has the same standard invariant as $N \subseteq M$ [62, 69], and thus $R \subseteq P$ has the opposite standard invariant.

Letting $\mathcal{C} := {}_N \mathcal{C}_N$ be the unitary fusion category generated by ${}_N M_N$, the above construction gives a fully faithful unitary tensor functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{Bim}_{\text{fgp}}(R)$. By [6, Prop. 3.2], simple objects $c \in \text{Irr}(\mathcal{C})$ correspond to minimal projections $p \in M'_0 \cap M_{2n}$ under the correspondence

$$M'_0 \cap M_{2n} \ni p \mapsto pM_n \in \text{Bim}_{\text{fgp}}(N).$$

Now consider the Jones tower

$$R_0 = R \subseteq P = R_1 \subseteq R_2 \subseteq R_3 \subseteq \cdots$$

of our hyperfinite II_1 subfactor $R \subseteq P$. Simple objects $c^{\text{op}} \in \text{Irr}(\mathcal{C}^{\text{op}})$ correspond to the opposite projections $p^{\text{op}} \in R'_0 \cap R_{2n} \cong (M'_0 \cap M_{2n})^{\text{op}}$, which corresponds to the bimodule $p^{\text{op}}R_n$. As R_n is also isomorphic to $\varinjlim \mathcal{P}_{k,\pm}$ (here, $\pm$ depends on the parity of n), we can realize the bimodule $p^{\text{op}}R_n$ graphically as an inductive limit:

$$F(c^{\text{op}}) = p^{\text{op}}R_n \cong \varinjlim \mathcal{C}(c \boxtimes X^{\text{alt} \boxtimes k} \rightarrow X^{\text{alt} \boxtimes k}) \quad (15)$$

under the isometric right R_n -inner product preserving inclusions

$$k \text{ --- } \boxed{f} \text{ --- } k \xrightarrow{\quad} k \text{ --- } \overline{\boxed{f}} \text{ --- } k$$

$\quad \quad \quad c \quad \quad \quad c$

Indeed, $p^{\text{op}}R_n \cong p^{\text{op}}R_n f_n \subseteq p^{\text{op}}R_{2n} f_n$, where f_n is the multistep Jones projection [6, 77]. In diagrams, for $\xi \in \mathcal{C}(X^{\text{alt} \boxtimes k+n} \rightarrow X^{\text{alt} \boxtimes k+n}) \subseteq R_n \hookrightarrow R_{2n}$, we have

$$f_n = \frac{1}{[M:N]^{n/2}} \left(\begin{array}{c} n \\ \text{---} \end{array} \right) \left(\begin{array}{c} n \\ \text{---} \end{array} \right) \Rightarrow p^{\text{op}} \xi f_n = \frac{1}{[M:N]^{n/2}} \left(\begin{array}{c} k \\ \text{---} \end{array} \right) \left(\begin{array}{c} n \\ \text{---} \end{array} \right) \left(\begin{array}{c} n \\ \text{---} \end{array} \right)$$

It is now visibly evident how to implement the isomorphism (15).

Now, since we have two subfactors $N \subseteq M$ and $R \subseteq P$ with opposite standard invariants, we get two fully faithful unitary tensor functors $F : \mathcal{C}^{\text{op}} \rightarrow \text{Bim}_{\text{fgp}}(R)$ and $G : \mathcal{C} \rightarrow \text{Bim}_{\text{fgp}}(N)$. Consider the *symmetric enveloping Q-system* $S := \bigoplus_{c \in \text{Irr}(\mathcal{C})} c^{\text{op}} \boxtimes c \in \mathcal{C}^{\text{op}} \boxtimes \mathcal{C}$, which is simple as an $S-S$ bimodule, giving the realized II_1 factor $|(F \boxtimes G)(S)|$.

Proposition 6.7 *The realization $|(F \boxtimes G)(S)|$ is $*$ -isomorphic to the inductive limit II_1 factor $M_\infty = \varinjlim M_n$.*

Proof. Observe that by the Yoneda Lemma, we have canonical isomorphisms

$$\begin{aligned} |(F \boxtimes G)(S)| &\cong \bigoplus_{a,b \in \text{Irr}(\mathcal{C})} (\mathcal{C}^{\text{op}} \boxtimes \mathcal{C})(a^{\text{op}} \boxtimes b \rightarrow S) \otimes_{\mathbb{C}} (F \boxtimes G)(a^{\text{op}} \boxtimes b) \\ &= \bigoplus_{c \in \text{Irr}(\mathcal{C})} (\mathcal{C}^{\text{op}} \boxtimes \mathcal{C})(c^{\text{op}} \boxtimes c \rightarrow S) \otimes_{\mathbb{C}} (F \boxtimes G)(c^{\text{op}} \boxtimes c) \\ &\cong \bigoplus_{c \in \text{Irr}(\mathcal{C})} (F \boxtimes G)(c^{\text{op}} \boxtimes c) \\ &= \bigoplus_{c \in \text{Irr}(\mathcal{C})} F(c^{\text{op}}) \otimes_{\mathbb{C}} G(c) \end{aligned}$$

Thus $|(F \boxtimes G)(S)|$ is $*$ -isomorphic to the more general coend realization of F° and G° from [44, Sect. 4] (see also [43, Ex. 5.33]), where as above, $\circ$ denotes taking the underlying vector space. Using this identification,

$$\bigoplus_{c \in \text{Irr}(\mathcal{C})} F(c^{\text{op}}) \otimes_{\mathbb{C}} G(c) \cong \varinjlim \bigoplus_{c \in \text{Irr}(\mathcal{C})} \mathcal{C}(c \boxtimes X^{\text{alt} \boxtimes k} \rightarrow X^{\text{alt} \boxtimes k}) \otimes_{\mathbb{C}} G(c) \stackrel{(13)}{\cong} \varinjlim M_n. \quad \square$$

6.4 Embedding $\mathcal{Z}(\mathcal{C})$ into $\tilde{\chi}(M_\infty)$ when $N \subseteq M$ is finite depth

As in Sect. 6.3 above, for this section, we fix a finite depth, finite index II_1 subfactor $N \subseteq M$, and let $\mathcal{C} = {}_N\mathcal{C}_N$ denote the unitary fusion category of N – N bimodules generated by ${}_NM_N$. In this section, we extend this bijection to a unitary tensor functor $\Phi : \mathcal{Z}(\mathcal{C}) \rightarrow \text{Bim}_{\text{fgp}}(M_\infty)$ such that when N is non-Gamma, Φ takes values in $\tilde{\chi}(M_\infty)$ as is braided. To do so, we use the description of the inductive limit II_1 factor $M_\infty = |(F \boxtimes G)(S)|$ obtained from iterating the Jones basic construction afforded by Proposition 6.7 for the fully faithful unitary tensor functors $F : \mathcal{C}^{\text{op}} \rightarrow \text{Bim}_{\text{fgp}}(R)$ from (15) and $G : \mathcal{C} \rightarrow \text{Bim}_{\text{fgp}}(N)$ associated to the subfactor $N \subset M$.

First, applying [13, Cor. C] in the W^* setting to the fully faithful unitary tensor functor $F \boxtimes G : \mathcal{C}^{\text{op}} \boxtimes \mathcal{C} \rightarrow \text{Bim}_{\text{fgp}}(R \otimes N)$, bimodule realization gives a fully faithful tensor functor from $\text{Bim}_{\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}}(S) \rightarrow \text{Bim}_{\text{fgp}}(M_\infty)$. (See also Remark 5.7 above.) Explicitly, on an S – S bimodule $X = \bigoplus_{a,b \in \text{Irr}(\mathcal{C})} X_{ab} \otimes (a^{\text{op}} \boxtimes b)$, we have

$$\begin{aligned} |X| &= \text{Hom} \left((R \otimes N)_{R \otimes N} \rightarrow \bigoplus_{a,b \in \text{Irr}(\mathcal{C})} X_{ab} \otimes (F(a^{\text{op}}) \otimes G(b))_{R \otimes N} \right) \\ &\cong \bigoplus_{a,b \in \text{Irr}(\mathcal{C})} X_{ab} \otimes F(a^{\text{op}}) \otimes G(b). \end{aligned}$$

Using the well-known equivalence $\text{Bim}_{\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}}(S) \cong \mathcal{Z}(\mathcal{C})$ from Example 6.4, we get the following proposition.

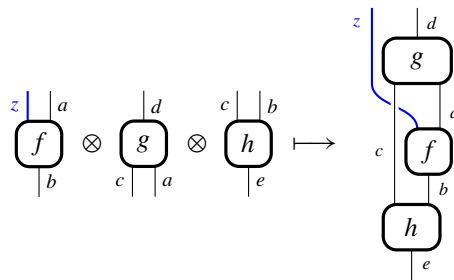
Proposition 6.8 *Bimodule realization gives a fully faithful unitary tensor functor $\Phi : \mathcal{Z}(\mathcal{C}) \rightarrow \text{Bim}_{\text{fgp}}(M_\infty)$.*

We now want an explicit model for $\Phi(z, \sigma_z)$ for each object $(z, \sigma_z) \in \mathcal{Z}(\mathcal{C})$. To do so, we give an explicit description of the $S - S$ bimodule $X_z \in \text{Bim}_{\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}}(S)$ under the unitary tensor equivalence.

Definition 6.9 Given $(z, \sigma_z) \in \mathcal{Z}(\mathcal{C})$, we define $X_z := \bigoplus_{a,b \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes (a^{\text{op}} \boxtimes b)$. The left S -module structure of X_z is given as follows. First, we observe

$$\begin{aligned} S \otimes_{\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}} X_z &= \bigoplus_{a,b,c \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes ((c^{\text{op}} \otimes a^{\text{op}}) \boxtimes (c \otimes b)) \\ &= \bigoplus_{a,b,c,d,e \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes \mathcal{C}^{\text{op}}(d^{\text{op}} \rightarrow c^{\text{op}} \otimes a^{\text{op}}) \\ &\quad \otimes \mathcal{C}(e \rightarrow c \otimes b) \otimes (d^{\text{op}} \boxtimes e) \\ &= \bigoplus_{a,b,c,d,e \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes \mathcal{C}(c \otimes a \rightarrow d) \otimes \mathcal{C}(e \rightarrow c \otimes b) \\ &\quad \otimes (d^{\text{op}} \boxtimes e) \end{aligned}$$

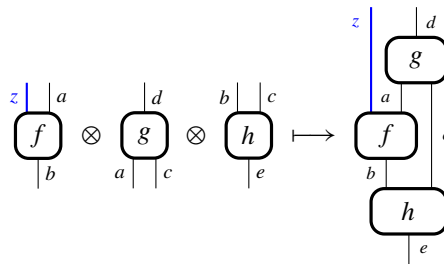
The left action map is given component-wise by



The right S -module structure is defined similarly. Observe

$$\begin{aligned} X_z \otimes_{\mathcal{C}^{\text{op}} \boxtimes \mathcal{C}} S &= \bigoplus_{a,b,c \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes (a^{\text{op}} \otimes c^{\text{op}}) \boxtimes (b \otimes c) \\ &= \bigoplus_{a,b,c,d,e \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes \mathcal{C}(a \otimes c \rightarrow d) \otimes \mathcal{C}(e \rightarrow b \otimes c) \\ &\quad \otimes (d^{\text{op}} \boxtimes e), \end{aligned}$$

and the right action map is given component-wise by



We now describe the realization

$$|X_z| = \varinjlim_{a, b \in \text{Irr}(\mathcal{C})} \bigoplus_{a, b \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes_{\mathbb{C}} \mathcal{C}(a \boxtimes X^{\text{alt} \boxtimes n} \rightarrow X^{\text{alt} \boxtimes n}) \otimes_{\mathbb{C}} G(b). \quad (16)$$

First, note that by semisimplicity and Frobenius reciprocity, we can alternatively describe the first two tensorands in the direct sum for $|X_z|$ as

$$\left. \begin{aligned} \bigoplus_{a \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \rightarrow z \otimes a) \otimes_{\mathbb{C}} \mathcal{C}(a \boxtimes X^{\text{alt} \boxtimes n} \rightarrow X^{\text{alt} \boxtimes n}) \\ \cong \bigoplus_{a \in \text{Irr}(\mathcal{C})} \mathcal{C}(\bar{z} \boxtimes b \rightarrow a) \otimes_{\mathbb{C}} \mathcal{C}(a \rightarrow X^{\text{alt} \boxtimes n} \boxtimes \overline{X^{\text{alt} \boxtimes n}}) \\ \cong \mathcal{C}(\bar{z} \boxtimes b \rightarrow X^{\text{alt} \boxtimes n} \boxtimes \overline{X^{\text{alt} \boxtimes n}}) \\ \cong \mathcal{C}(b \boxtimes X^{\text{alt} \boxtimes n} \rightarrow z \boxtimes X^{\text{alt} \boxtimes n}) \end{aligned} \right\} \quad (17)$$

We may thus graphically represent elements in a dense subspace of $|X_z|$ as

$$\bigoplus_{b \in \text{Irr}(\mathcal{C})} \left(\begin{array}{c} \text{Diagram with box } f_b \text{ and } \xi_b \text{ connected by a horizontal line, with inputs } n, b \text{ and outputs } n, b. \end{array} \right) \in \varinjlim_{b \in \text{Irr}(\mathcal{C})} \bigoplus_{b \in \text{Irr}(\mathcal{C})} \mathcal{C}(b \boxtimes X^{\text{alt} \boxtimes n} \rightarrow z \boxtimes X^{\text{alt} \boxtimes n}) \otimes_{\mathbb{C}} G(b)$$

where as in (14), the horizontal line in the top half of the diagram should be viewed as slightly tilted going from the bottom right to the top left, as indicated by the cyan arrows above. The right M_{∞} -action on $|X_z|$ is the obvious diagrammatic one, and the left one is similar, but uses the half-braiding for z :

$$(f_a \otimes \xi_a) \triangleright (g_b \otimes \eta_b) := \sum_{\substack{c \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}} \text{Diagram}.$$

The M_∞ -valued inner product of $|X_z|$ is given by

$$\langle f_a \otimes \xi_a | g_b \otimes \eta_b \rangle_{M_\infty}^{|X_z|} := \sum_{\substack{c \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}} \text{Diagram}.$$

By the definition of the realization (16) for $|X_z|$ and $|X_w|$ and the right and left S -action on X_z and X_w respectively, the tensorator $\Phi_{z,w}^2$ is given on $|X_z| \boxtimes_{M_\infty} |X_w|$ by

$$\text{Diagram} \xrightarrow{\Phi_{z,w}^2} \sum_{\substack{e, f \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes c \rightarrow e) \\ \beta \in \text{ONB}(b \otimes d \rightarrow f)}} \text{Diagram} \xleftrightarrow{(17)} \sum_{f, \beta} \text{Diagram}.$$

under the semisimplicity isomorphism (17).

We now show that the image of the unitary tensor functor $\Phi : \mathcal{Z}(\mathcal{C}) \rightarrow \text{Bim}_{\text{fgp}}(M_\infty)$ lies in $\tilde{\chi}(M_\infty)$ when N is non-Gamma.

Lemma 6.10 *Since $N \subseteq M$ is finite depth, there is a $k > 0$ such that every $c \in \text{Irr}(\mathcal{C})$ is isomorphic to a summand of $(X \boxtimes \bar{X})^{\boxtimes k} = X^{\text{alt} \boxtimes 2k}$. There is a finite subset $\{e_i\}_{i=1}^m \subseteq \mathcal{C}^{\text{op}}(X^{\text{alt} \boxtimes 2k} \rightarrow z \boxtimes X^{\text{alt} \boxtimes 2k})$ such that*

$$\text{id}_{z \boxtimes X^{\text{alt} \boxtimes 2k}} = \frac{2k}{z} = \sum_{i=1}^m \frac{2k}{z} \text{Diagram} \frac{2k}{z}. \quad (18)$$

Proof For each $c \in \text{Irr}(\mathcal{C})$, we pick

- a finite family of isometries $\{t_c^i : c \rightarrow X^{\text{alt} \boxtimes 2k}\}_{i=1}^{m_c}$ such that $(t_c^i)^\dagger \cdot t_c^i = 1_c$ and $\sum_c \sum_{i=1}^{m_c} t_c^i \cdot (t_c^i)^\dagger = 1_{X^{\text{alt} \boxtimes 2k}}$, and
- an orthonormal basis $\{\alpha_c\} \subseteq \mathcal{C}(c \rightarrow z \boxtimes X^{\text{alt} \boxtimes 2k})$ under the isometry inner product, i.e., $\alpha_c^\dagger \cdot \alpha_{c'} = \delta_{c=c'} \text{id}_c$.

Then we have (reading diagrams right to left)

$$\frac{2k}{z} = \sum_{c \in \text{Irr}(\mathcal{C})} \sum_{\alpha_c} \frac{2k}{z} \xrightarrow{\alpha_c} c \xrightarrow{\alpha_c^\dagger} z = \sum_{c \in \text{Irr}(\mathcal{C})} \sum_{\alpha_c} \sum_{i=1}^{m_c} \frac{2k}{z} \xrightarrow{\alpha_c} c \xrightarrow{(t_c^i)^\dagger} 2k \xrightarrow{t_c^i} c \xrightarrow{\alpha_c^\dagger} z.$$

So we define our set $\{e_i\}$ to be $\bigcup_{c \in \text{Irr}(\mathcal{C})} \{\alpha_c \cdot (t_c^i)^\dagger\}_{i=1}^{m_c}$. $\square$

Proposition 6.11 Let $\{e_i\}$ be as in (18) above. For $n \geq 0$, define subsets $\{b_i\}$ and $\{b_i^{(n)}\}$ of $|X_z|$ by

$$b_i := \begin{array}{c} \begin{array}{c} 2k \\ z \end{array} \xrightarrow{e_i} \begin{array}{c} 2k \\ 1_c \\ 1_c \end{array} \\ \Omega_N \end{array} \quad b_i^{(n)} := \begin{array}{c} \begin{array}{c} 2k \\ 2n \\ z \end{array} \xrightarrow{e_i} \begin{array}{c} 2k \\ 2n \\ 1_c \\ 1_c \end{array} \\ \Omega_N \end{array}. \quad (19)$$

Then $\{b_i\}$ is an $|X_z|$ -basis and $\{b_i^{(n)}\}$ is an approximately inner $|X_z|_{M_\infty}$ -basis. This implies $|X_z|$ is approximately inner.

Proof. The first claim is immediate from (18). Similarly, $\{b_i^{(n)}\}$ is an $|X_z|_{M_\infty}$ -basis for every fixed n , and moreover, $[b_i^{(n)}, a] = 0$ for all $a \in M_{2n} \subseteq M_\infty$. Since $M_\infty = \varinjlim M_n$, for $a \in M_\infty$, $\|a - E_{M_n}(a)\|_2 \rightarrow 0$ (e.g., see [2, Lem. B.7]). Then for all $a \in M_\infty$,

$$\begin{aligned} \|ab_i^{(n)} - b_i^{(n)}a\|_2 &\leq \|(a - E_{M_n}(a))b_i^{(n)}\|_2 + \|E_{M_n}(a)b_i^{(n)} - b_i^{(n)}E_{M_n}(a)\|_2 \\ &\quad + \|b_i^{(n)}(E_{M_n}(a) - a)\|_2 \\ &\leq 2\|b_i^{(n)}\|_2\|E_{M_n}(a) - a\|_2 \rightarrow 0. \end{aligned} \quad \square$$

Assumption 6.12 For the remainder of this section, we now assume the II_1 factor N in our finite index finite depth subfactor $N \subseteq M$ is non-Gamma. This implies M is also non-Gamma by [76, Prop. 1.11].

Lemma 6.13 Let $\{b_i\}$ be as in (19). For each central sequence $(a_n) \subseteq N_\infty$, $\|a_nb_i - b_ia_n\|_2 \rightarrow 0$.

Proof. Since $N \subseteq M$ is finite depth and N is non-Gamma, by [74, Prop. 3.2(3)], every Jones basic construction M_n has spectral gap in M_∞ for every n . This implies that $\|a_n - E_{M'_{2k+1} \cap M_\infty}(a_n)\|_2 \rightarrow 0$. Since $[E_{M'_{2k+1} \cap M_\infty}(a_n), b_i] = 0$, we have

$$\begin{aligned} \|a_n b_i - b_i a_n\|_2 &\leq \|(a_n - E_{M'_{2k+1} \cap M_\infty}(a_n)) b_i\|_2 \\ &\quad + \|E_{M'_{2k+1} \cap M_\infty}(a_n) b_i - b_i E_{M'_{2k+1} \cap M_\infty}(a_n)\|_2 \\ &\quad + \|b_i (E_{M'_{2k+1} \cap M_\infty}(a_n) - a_n)\|_2 \\ &\leq \|b_i\|_2 \cdot \|a_n - E_{M'_{2k+1} \cap M_\infty}(a_n)\|_2 \\ &\quad + \|a_n - E_{M'_{2k+1} \cap M_\infty}(a_n)\|_2 \cdot \|b_i\|_2 \longrightarrow 0. \end{aligned} \quad \square$$

Proposition 6.14 $|X_z|$ is centrally trivial.

Proof. By Proposition 4.4, we must show that for each central sequence $(a_n) \subseteq M_\infty$, $\|a_n x - x a_n\|_2 \rightarrow 0$, for every $x \in |X_z|$. Suppose (a_n) is such a central sequence. Let $\{b_i\}$ be the $|X_z|_{M_\infty}$ -basis as in (19), and let $K > 0$ such that $\|b_i\|_2 \leq K$ for all i . By Lemma 6.13,

$$\begin{aligned} \|a_n x - x a_n\|_2 &= \left\| a_n \sum_i b_i \langle b_i | x \rangle_{M_\infty}^{X_z} - \sum_i b_i \langle b_i | x \rangle_{M_\infty}^{X_z} a_n \right\|_2 \\ &\leq \left\| \sum_i (a_n b_i - b_i a_n) \langle b_i | x \rangle_{M_\infty}^{X_z} \right\|_2 + \left\| \sum_i b_i (a_n \langle b_i | x \rangle_{M_\infty}^{X_z} - \langle b_i | x \rangle_{M_\infty}^{X_z} a_n) \right\|_2 \\ &\leq K \|x\|_2 \cdot \sum_i \|a_n b_i - b_i a_n\|_2 + K \sum_i \left\| a_n \langle b_i | x \rangle_{M_\infty}^{X_z} - \langle b_i | x \rangle_{M_\infty}^{X_z} a_n \right\|_2 \\ &\longrightarrow 0. \end{aligned} \quad \square$$

Combining the statements of Propositions 6.11 and 6.14, our unitary tensor functor $\Phi : \mathcal{Z}(\mathcal{C}) \rightarrow \text{Bim}_{\text{fpp}}(M_\infty)$ lands in $\tilde{\chi}(M_\infty)$.

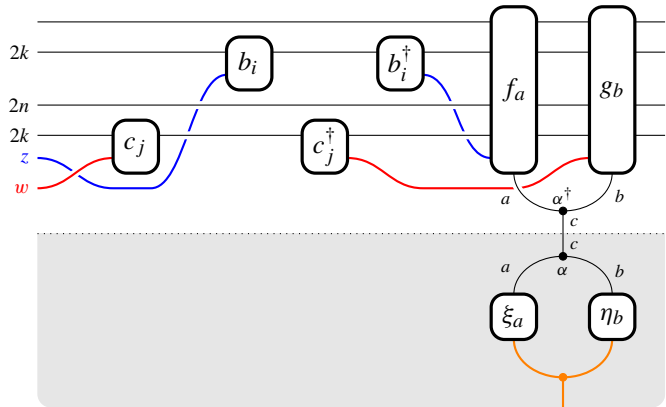
Proposition 6.15 The unitary tensor functor $\Phi : \mathcal{Z}(\mathcal{C}) \rightarrow \tilde{\chi}(M_\infty)$ is braided, i.e., for $z, w \in \mathcal{Z}(\mathcal{C})$, the following diagram commutes

$$\begin{array}{ccc} |X_z| \boxtimes_{M_\infty} |X_w| & \xrightarrow{u_{|X_z|, |X_w|}} & |X_w| \boxtimes_{M_\infty} |X_z| \\ \Phi_{z, w}^2 \downarrow & & \downarrow \Phi_{w, z}^2 \\ |X_{z \otimes w}| & \xrightarrow{\Phi(\sigma_{z, w})} & |X_{w \otimes z}| \end{array} \quad (20)$$

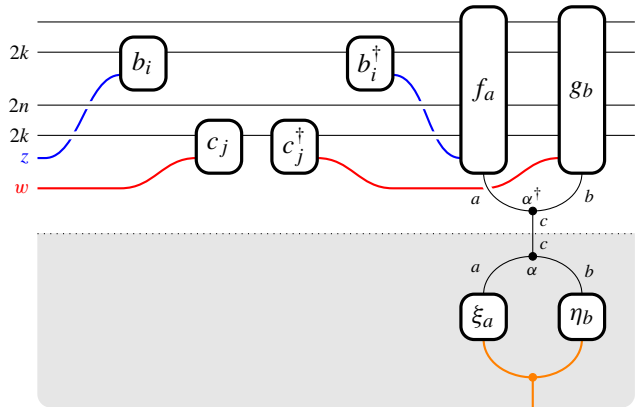
Proof Let $\{b_i^{(n)}\} \subseteq |X_z|$ and $\{c_j\} \subseteq |X_w|$ be approximately inner $|X_z|_{M_\infty}$ -basis and $|X_w|_{M_\infty}$ -basis respectively as in (19). By (5), $u_{|X_z|, |X_w|} = \lim_n \sum_{i, j} |c_j \boxtimes b_i^{(n)}\rangle \langle b_i^{(n)} \boxtimes c_j|$. For $x = (f_a \otimes \xi_a) \in |X_z|$ and $y = (g_b \otimes \eta_b) \in |X_w|$ with $f_a \in \mathcal{C}(X^{\text{alt} \boxtimes 2t} \rightarrow a \boxtimes X^{\text{alt} \boxtimes 2t})$ and $g_b \in \mathcal{C}(X^{\text{alt} \boxtimes 2t} \rightarrow b \boxtimes X^{\text{alt} \boxtimes 2t})$ for t sufficiently large, we have

$$\sum_{i,j} \Phi_{w,z}^2(|c_j \boxtimes b_i^{(n)}\rangle \langle b_i^{(n)} \boxtimes c_j|x \boxtimes y\rangle_{M_\infty}^{|X_z| \boxtimes_{M_\infty} |X_w|})$$

$$= \sum_{i,j} \sum_{\substack{c \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}}$$



$$= \sum_{i,j} \sum_{\substack{c \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}}$$



$$\begin{aligned}
 &= \sum_{\substack{c \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}} \\
 &\quad \begin{array}{c} \text{Diagram 1: A box labeled } \alpha^\dagger \text{ with inputs } a \text{ (red) and } b \text{ (black) and output } c \text{ (black). Above it are boxes } f_a \text{ and } g_b \text{ with inputs } 2t \text{ (black) and } z \text{ (blue) and } w \text{ (red). Below it is a shaded box containing boxes } \xi_a \text{ and } \eta_b \text{ with inputs } a \text{ (black) and } b \text{ (black) and output } c \text{ (black).} \end{array} \\
 &= \sum_{\substack{c \in \text{Irr}(\mathcal{C}) \\ \alpha \in \text{ONB}(a \otimes b \rightarrow c)}} \\
 &\quad \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but with a dashed box around the } \alpha^\dagger \text{ box and a wavy line connecting the } z \text{ and } w \text{ inputs.} \end{array} \\
 &= \Phi(\sigma_{z,w}) \cdot \Phi_{z,w}^2(x \boxtimes y).
 \end{aligned}$$

Thus (20) commutes on a dense subspace of $|X_z| \boxtimes_{M_\infty} |X_w|$, and the result follows. $\square$

Corollary 6.16 *Let $\mathcal{C}$ be a braided fusion category, then there exists a II_1 factor M with a braided fully faithful monoidal functor $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C}) \rightarrow \tilde{\chi}(M)$.*

6.5 Calculation of $\tilde{\chi}(M_\infty)$ when $N \subseteq M$ is non-Gamma and finite depth

As in Sect. 6.3 above, we suppose $N \subseteq M$ is a fixed finite depth finite index II_1 subfactor. We also continue Assumption 6.12 that N is non-Gamma. We now prove our main result, which uses a technical result on centralizers in braided tensor categories in Sect. 6.6 below.

Theorem 6.17 *Let $N \subseteq M$ be a finite depth finite index inclusion of non-Gamma II_1 factors. Let M_∞ denote the inductive limit II_1 factor from the Jones tower, and let $\mathcal{C} = {}_N\mathcal{C}_N$ be the even part of the standard invariant $\mathcal{C}(N \subseteq M)$. Then $\tilde{\chi}(M_\infty) \cong \mathcal{Z}(\mathcal{C})$.*

Proof Consider our construction of $\Phi : \mathcal{Z}(\mathcal{C}) \rightarrow \tilde{\chi}(M_\infty)$. Let $L := I(1_{\mathcal{C}}) \in \mathcal{Z}(\mathcal{C})$ be the canonical Lagrangian algebra, where $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ is adjoint to the forgetful functor. By Example 6.4, the Q-systems $L \in \mathcal{Z}(\mathcal{C})$ and $S \in \mathcal{C}^{\text{op}} \boxtimes \mathcal{C}$ are related as in Remark 6.3, i.e., $S = X \otimes \bar{X}$ and $L = \bar{X} \otimes X$ for $X = 1_{\mathcal{C}}$ in the Morita equivalence $\mathcal{C}^{\text{op}} \boxtimes \mathcal{C} - \mathcal{Z}(\mathcal{C})$ bimodule category $\mathcal{C}$. This result first appeared in [36]; see also [58]. Thus we can identify $|\Phi(L)|$ with the basic construction of $R \otimes N \subseteq M_\infty$ by the discussion in Example 6.5. But this implies that $|\Phi(L)|$ is Morita equivalent to $R \otimes N$, and in particular, by Example 4.16, $\tilde{\chi}(|\Phi(L)|) = \tilde{\chi}(R \otimes N)$ is trivial.

By Proposition 5.16, $\tilde{\chi}(M_\infty)_{\Phi(L)}^{\text{loc}} \cong \tilde{\chi}(|\Phi(L)|)$. This implies no non-trivial simple object in $\tilde{\chi}(M_\infty)$ centralizes $\Phi(\mathcal{Z}(\mathcal{C}))$, since the free module functor $x \mapsto x \otimes L$ for $x \in \Phi(\mathcal{Z}(\mathcal{C}))' \subseteq \tilde{\chi}(M_\infty)$ is fully faithful. Thus $\Phi(\mathcal{Z}(\mathcal{C}))' \subseteq \tilde{\chi}(M_\infty)$ is trivial. Since $\mathcal{Z}(\mathcal{C})$ is non-degenerately braided by [58], by Proposition 6.21 in the next subsection, $\Phi(\mathcal{Z}(\mathcal{C})) = \tilde{\chi}(M_\infty)$. $\square$

Remark 6.18 Kawahigashi studied a relative version of Connes' $\chi(M)$ and Jones κ invariant for finite index II_1 subfactors $N \subseteq M$ [48, 49]. In particular, Kawahigashi provides bounds and computations for relative χ for finite depth finite index subfactors of the hyperfinite II_1 factor. For a given finite depth *hyperfinite* subfactor $N \subseteq M$, there exists a non-Gamma inclusion $A \subseteq B$ with the same standard invariant [78]. By [74, Thm. 4.2], $\chi(A_\infty) \cong \chi(N \subseteq M)$. By Theorem 6.17 and Example 4.11, $\chi(N \subseteq M) \cong \text{Inv}(\mathcal{Z}(\mathcal{C}(N \subseteq M)))$, the group of isomorphism classes of invertible objects of the Drinfeld center of $\mathcal{C}(N \subseteq M)$.

Remark 6.19 Suppose $N \subseteq M$ is a finite index inclusion of non-Gamma II_1 factors with A_{2n} Jones–Temperley–Lieb standard invariant. Then $\mathcal{Z}(\mathcal{C}(N \subseteq M))$ is a unitary modular tensor category with no non-trivial invertible objects. This distinguishes the corresponding M_∞ factors pairwise, but they all have the same trivial Connes' χ invariant. Popa considers these examples of II_1 factors with trivial χ which are not *s-McDuff* in [74], answering a question of Connes in the negative. (Recall a II_1 factor is *s-McDuff* if it is of the form $R \otimes N$ for N non-Gamma). This leads us to ask the natural extension of Connes question.

Question 6.20 *If M is McDuff and $\tilde{\chi}(M)$ is trivial, is M s-McDuff?*

6.6 A technical result on centralizers in braided tensor categories

The goal of this section is to prove the following technical result for braided unitary tensor categories. We expect this result holds in the greater generality of semisimple ribbon tensor categories; see the paragraph before [59, Prop. 2.5] for more details in this direction.

Proposition 6.21 *Suppose $\mathcal{C}$ is an arbitrary braided unitary tensor category and $\mathcal{D} \subsetneq \mathcal{C}$ is a non-degenerately braided proper fusion subcategory. There exists $a \in \text{Irr}(\mathcal{C}) \setminus \text{Irr}(\mathcal{D})$ which centralizes $\mathcal{D}$.*

Let $\mathcal{C}$ be a braided unitary tensor category. Let $\text{tr}_{\mathcal{C}}$ denote the (unnormalized) categorical trace corresponding to the unique unitary spherical structure from minimal solutions to the conjugate equations [54]. (The normalization is $\text{tr}_{\mathcal{C}}(\text{id}_c) = \dim(c)$ for each $c \in \mathcal{C}$.) For each $a \in \text{Irr}(\mathcal{C})$ define a function $\gamma_a : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{C}$ by

$$\gamma_a(b) := \frac{1}{d_b} \text{tr}_{\mathcal{C}}(\sigma_{b,a} \sigma_{a,b}).$$

Then $\gamma_a(b)$ extends linearly to a *character* on the fusion ring, i.e.

$$\gamma_a(b)\gamma_a(c) = \sum_d N_{bc}^d \gamma_a(d).$$

Furthermore, characters are related by the following equation:

$$\frac{\gamma_a(c)\gamma_b(c)}{d_c} = \sum_e \frac{d_e}{d_a d_b} N_{ab}^c \gamma_e(c). \quad (21)$$

Suppose $\mathcal{D}$ is a non-degenerately braided full fusion subcategory of $\mathcal{C}$ (which is thus modular by unitarity). Then by non-degeneracy, $\{\gamma_a\}_{a \in \text{Irr}(\mathcal{D})}$ forms a complete set of characters of $K_0(\mathcal{D})$. But any $b \in \text{Irr}(\mathcal{C})$ also defines a character γ_b , and thus $\gamma_b|_{\text{Irr}(\mathcal{D})} = \gamma_{f(b)}$ for some uniquely defined $f(b) \in \text{Irr}(\mathcal{D})$. Thus we have a function $f : \text{Irr}(\mathcal{C}) \rightarrow \text{Irr}(\mathcal{D})$.

Now we define the *fusion hypergroup* of a semisimple unitary tensor category $\mathcal{C}$ as the fusion algebra $K_0(\mathcal{C})$ with the distinguished basis $\lambda_a = \frac{[a]}{d_a}$. We then have

$$\lambda_a \lambda_b = \sum_c M_{ab}^c \lambda_c \quad \text{where} \quad M_{ab}^c := \frac{d_c}{d_a d_b} N_{ab}^c.$$

Lemma 6.22 *The assignment $f(\lambda_a) := \lambda_{f(a)}$ extends to a homomorphism of fusion algebras $K_0(\mathcal{C}) \rightarrow K_0(\mathcal{D})$.*

Proof For $x \in \text{Irr}(\mathcal{D})$ and $a, b \in \text{Irr}(\mathcal{C})$, we compute $d_x^{-1} \gamma_a(x) \gamma_b(x)$ in two ways. First, we can apply (21) and then swap γ_c with $\gamma_{f(c)}$, or we can swap γ_a, γ_b with $\gamma_{f(a)}, \gamma_{f(b)}$ respectively and then apply (21). Equating these two computations gives the equality

$$\sum_{c \in \text{Irr}(\mathcal{C})} \frac{d_c}{d_a d_b} N_{ab}^c \gamma_{f(c)}(x) = \sum_{y \in \text{Irr}(\mathcal{D})} \frac{d_y}{d_{f(a)} d_{f(b)}} N_{f(a) f(b)}^y \gamma_y(x),$$

which implies

$$\sum_{y \in \text{Irr}(\mathcal{D})} \left(\sum_{c \in f^{-1}(y)} M_{ab}^c - M_{f(a) f(b)}^y \right) \gamma_y(x) = 0. \quad (22)$$

Since (22) holds for all $x \in \text{Irr}(\mathcal{D})$, we have

$$\sum_{y \in \text{Irr}(\mathcal{D})} \left(\sum_{c \in f^{-1}(y)} M_{ab}^c - M_{f(a) f(b)}^y \right) \gamma_y = 0,$$

which is an equation in the space of functions on $\text{Irr}(\mathcal{D})$. But since $\{\gamma_y\}_{y \in \text{Irr}(\mathcal{D})}$ is a complete set of characters for the fusion algebra $K_0(\mathcal{D})$, it forms a basis for the space $\text{Fun}(\text{Irr}(\mathcal{D}) \rightarrow \mathbb{C})$ (where we identify $\text{Fun}(\text{Irr}(\mathcal{D}) \rightarrow \mathbb{C})$ with the dual space $K_0(\mathcal{D})^\vee$), and is thus linearly independent. This immediately implies

$$\sum_{c \in f^{-1}(y)} M_{ab}^c = M_{f(a) f(b)}^y \quad \forall y \in \text{Irr}(\mathcal{D}).$$

We then see that

$$\begin{aligned} f(\lambda_a)f(\lambda_b) &= \lambda_{f(a)}\lambda_{f(b)} = \sum_y M_{f(a)f(b)}^y \lambda_y = \sum_y \left(\sum_{c \in f^{-1}(y)} M_{ab}^c \right) \lambda_y \\ &= \sum_c M_{ab}^c \lambda_{f(c)} = f(\lambda_a \lambda_b) \end{aligned}$$

as claimed. $\square$

Proof of Proposition 6.21 We prove the contrapositive. That is, we will show that if $\mathcal{C}$ is a braided unitary tensor category and $\mathcal{D} \subseteq \mathcal{C}$ is a non-degenerately braided fusion subcategory, then the absence of a nontrivial centralizing simple object for $\mathcal{D}$ in $\text{Irr}(\mathcal{C})$ implies $\mathcal{C} = \mathcal{D}$.

First, by [59, Prop. 2.5], $c \in \text{Irr}(\mathcal{C})$ centralizes $\mathcal{D}$ if and only if $\gamma_c|_{\mathcal{D}} = \gamma_{f(c)} = \gamma_{1_{\mathcal{D}}}$. Suppose that the only $c \in \text{Irr}(\mathcal{C})$ for which $f(c) = 1_{\mathcal{D}}$ is $c = 1_{\mathcal{C}}$. Let $\tau_{\mathcal{C}}$ be the functional on $K_0(\mathcal{C})$ which picks off the coefficient of the identity object, and similarly define $\tau_{\mathcal{D}}$ on $K_0(\mathcal{D})$. Note that $\tau_{\mathcal{C}}, \tau_{\mathcal{D}}$ are positive definite on the $*$ -algebras $K_0(\mathcal{C}), K_0(\mathcal{D})$ respectively. For any $\eta \in K_0(\mathcal{C})$, our hypothesis implies $\tau(\eta) = \tau(f(\eta))$, where we have extended f linearly. But then $f : K_0(\mathcal{C}) \rightarrow K_0(\mathcal{D})$ is injective, since if $f(\eta) = 0$, then $0 = f(\eta^*)f(\eta) = f(\eta^*\eta)$, and thus $0 = \tau_{\mathcal{D}}(f(\eta^*\eta)) = \tau_{\mathcal{C}}(\eta^*\eta)$, which implies $\eta = 0$. In particular,

$$\text{rank}(\mathcal{C}) = \dim(K_0(\mathcal{C})) \leq \dim(K_0(\mathcal{D})) = \text{rank}(\mathcal{D}).$$

But as $\mathcal{D} \subseteq \mathcal{C}$, we must have $\mathcal{D} = \mathcal{C}$. $\square$

7 Open problems

We end our article by advertising the following list of open problems related to $\tilde{\chi}(M)$. In this section, we assume M is a separable II_1 factor. The first question below was asked by Popa when he introduced $\tilde{\chi}(M)$ and remains open.

Question 7.1 [68, Rem. 2.7] *If ${}_M X_M \in \text{Bim}_{\text{fpp}}(M)$ is a dualizable (bifinite) M – M bimodule which is approximately inner and centrally trivial, is its conjugate $\overline{{}_M X_M}$ also approximately inner and centrally trivial?*

Our local extension result Theorem B is similar in spirit to Connes' exact sequence to compute $\tilde{\chi}(M)$, but not an exact analog. Popa discusses the possible existence of a categorical analog of Connes' short exact sequence.

Question 7.2 [68, Rem. 2.7] *If $N \subseteq M$ is a finite index II_1 subfactor, is there a categorical analog of Connes' short exact sequence to compute $\tilde{\chi}(M)$ in terms of N and the categorical data of the standard invariant?*

The next two problems were suggested by Yasuyuki Kawahigashi based on the analogy with conformal nets afforded by the local extension result Theorem B.

Question 7.3 (Kawahigashi) *Do the $SU(N)_k$ modular tensor categories from the WZW models arise as $\tilde{\chi}(M)$ for some finite von Neumann algebra M ?*

Question 7.4 (Kawahigashi) *Is there a quantity measuring the size of $\tilde{\chi}(M)$, analogous to the μ -index $\sum_X d_X^2$ for a conformal net?*

Based on Kawahigashi's questions above, we ask the following.

Question 7.5 *Is the braiding on $\tilde{\chi}(M)$ always non-degenerate?*

The next question was alluded to in the introduction after Theorem A based on the existence of the braiding on $\tilde{\chi}(M)$ and the local extension result Theorem B.

Question 7.6 *Is there a 3-category whose objects are Π_1 factors M such that $\text{End}(\text{id}_M) \cong \tilde{\chi}(M)$?*

At this time, it may be more tractable to develop more robust evidence for the existence of such a 3-category. In particular, for conformal nets with a group action, the fixed point DHR braided tensor category can be related to the original braided tensor category by the categorical process of gauging [11, 60]. This can be understood as arising from the 3-categorical structure of conformal nets [5] by the techniques developed in [45].

Question 7.7 *Suppose G is a finite group acting outerly on a Π_1 factor M such that the action is neither approximately inner nor centrally trivial. Is the braided tensor category $\tilde{\chi}(M^G)$ the gauging of $\tilde{\chi}(M)$?*

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Declarations

Conflict of interest On behalf of all authors, David Penneys states that there is no conflict of interest.

Availability of data and material. Not applicable

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