A 3-categorical perspective on G-crossed braided categories

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September 2, 2020

Abstract

A braided monoidal category may be considered a 3-category with one object and one 1-morphism. In this paper, we show that, more generally, 3-categories with one object and 1-morphisms given by elements of a group G correspond to G-crossed braided categories, certain mathematical structures which have emerged as important invariants of low-dimensional quantum eld theories. More precisely, we show that the 4-category of 3-categories C equipped with a 3-functor $BG \to C$ which is essentially surjective on objects and 1-morphisms is equivalent to the 2-category of G-crossed braided categories. This provides a uniform approach to various constructions of G-crossed braided categories.

1 Introduction

G-crossed braided categories [EGNO15, §8.24] (see also §4.1) have emerged as important mathematical structures describing symmetry enriched invariants of quantum eld theories in low dimensions. In particular, G-crossed braided categories arise from global symmetries in (1+1)D chiral conformal eld theory ([Kir02, Kir01, Müg05]) and (2+1)D topological phases of matter [BBCW19], and as invariants of three-dimensional homotopy quantum eld theories [Tur10, SW18]. They are a central object of study in the theory of G-extensions of fusion categories [ENO10, GNN09, CGPW16]. In this article we describe a higher categorical approach to G-crossed braided categories, which unies these perspectives.

When G is trivial, a G-crossed braided category is exactly a braided monoidal category. It is well-known that braided monoidal categories are 'the same as' 3-categories[1] with exactly one object and one 1-morphism [BD95, Table 21] and [CG11]. This is an instance of the Delooping Hypothesis [BS10, §5.6 and Hypothesis 22] relating k-fold degenerate (n+k)-categories with k-fold monoidal n-categories. However, twice degenerate 3-categories, 3-functors, transformations, modications, and perturbations form a 4-category, whereas braided monoidal categories, braided monoidal functors, and monoidal natural transformations only form a 2-category. This discrepancy can be resolved by viewing '2-fold degeneracy' as a structure on a 3-category rather than a property, namely the structure of a 1-surjective pointing[2] [BS10, Sec 5.6]. Explicitly, the Delooping Hypothesis may then be understood as asserting that the 4-category of 3-categories equipped with 1-surjective pointings and pointing-preserving higher morphisms between them is in fact a 2-category (all hom 2-categories between 2-morphisms are contractible) and is equivalent to the 2-category of braided monoidal categories.

Rather than pointing by something contractible (i.e., a point), we can also study 'pointings' by other categories. In this article, we show that 1-surjective G-pointed 3-categories, i.e 3-categories equipped with a 1-surjective 3-functor from a group G viewed as a 1-category BG with one object, are 'the same as' G-crossed braided categories.

Theorem A. The 4-category[3] $3\mathrm{Cat}_G$ of 1-surjective G-pointed 3-categories and pointing-preserving higher morphisms (see Denition 3.2) is equivalent to the 2-category GCrsBrd of G-crossed braided categories. In particular, every hom 2-category between parallel 2-morphisms in $3\mathrm{Cat}_G$ is contractible.

[1] In this article, by a 3-category we mean an algebraic tricategory in the sense of [Gur13, Def 4.1], and by functor, transformation, modication, and perturbation, we mean the corresponding notions of trihomomorphism, tritransformation, trimodication, and perturbation of [Gur13, Def 4.10, 4.16, 4.18, 4.21].

[2] A functor between n-categories $G \to C$ is k-surjective if it is essentially surjective on objects and on j-morphisms for all $1 \le j \le k$. A k-surjective pointing on an n-category C is a k-surjective functor $* \to C$.

[3] We never actually work with a 4-category, as all our results can be stated and proven at the level of 2-categories. See Remark 3.3 for more details.

We prove Theorem A as follows. First, we show in Theorem 3.4 and Corollary 3.5 that $3\mathrm{Cat}_G$ is 2-truncated by showing it is equivalent to the strict sub-2-category $3\mathrm{Cat}_G^{st}$ of strict G-pointed 3-categories, whose objects are Gray-categories with precisely one object, whose sets of 1-morphisms is exactly G, and composition of 1-morphisms is the group multiplication. Then in Theorem 4.1, we construct a strict 2-equivalence between $3\mathrm{Cat}_G^{st}$ and the strict 2-category $\mathrm{GCrsBrd}^{st}$ of strict G-crossed braided categories. Finally, by [Gal17], every G-crossed braided category is equivalent to a strict one (see Denition 4.5 for more details), so that GCrsBrd is equivalent to its full 2-subcategory $\mathrm{GCrsBrd}^{st}$. In summary, we construct the following zig-zag of strict equivalences, where the hooked arrows denote inclusions of full subcategories.

$$3\mathrm{Cat}_G \xleftarrow[3.4]{\text{Thm.}} 3\mathrm{Cat}_G^{st} \xrightarrow[4.1]{\text{Thm.}} \mathrm{GCrsBrd}^{st} \xhookrightarrow[\text{[Gal17]}]{} \mathrm{GCrsBrd} \qquad (1)$$

For the trivial group G = feg, Theorem A specializes to the Delooping Hypothesis for twice degenerate 3-categories (also see [CG11], which uses so called 'iconic natural transformations' rather than pointings).

Corollary B. The 4-category[3] $3\mathrm{Cat}_{feg}$ of 1-surjective pointed 3-categories is equivalent to the 2-category of braided monoidal categories.

Our main theorem was inspired by, and is closely related to, the following two results: Passing from a G-pointed 3-category to the associated G-crossed braided category generalizes a result of [BGM19] which constructs G-crossed braided categories from group actions on 2-categories; see Example 1.11 for more details. A version of the construction of a G-pointed 3-category from a G-crossed braided category is discussed in [Cui19], and we use this construction in Section 4.3 to prove essential surjectivity of the 2-functor $3\mathrm{Cat}_G^{st}$! $\mathrm{GCrsBrd}^{st}$.

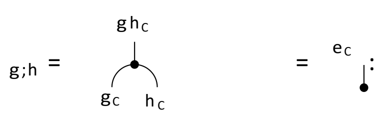
1.1 G-crossed braided categories from G-pointed 3-categories

In the proof of Theorem A, we construct the equivalence $3\mathrm{Cat}_G$ GCrsBrd by passing through appropriate strictications, resulting in the zig-zag (1) of strict equivalences. For the reader's convenience, we now sketch a direct construction of a G-crossed braided category (dened in x4 below) from a G-pointed 3-category, without passing through strictications.

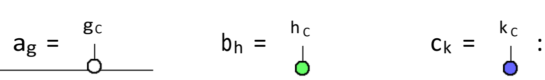
For a group G, we denote by BG the delooping of G, i.e., G considered as a 1-category with one object. Let C be a 3-category equipped with a 3-functor[4] : BG ! C.

To construct the G-crossed braided category, we will make use of the graphical calculus of Gray-categories (outlined in Section 2.2 below) and hence assume that C has been strictied to a Gray-category.[5] Unpacking the (weak) 3-functor into the data (; ; ; !; ;) as described in Appendix A, the G-crossed braided category C may be constructed as follows. Strictifying the situation slightly, we may assume that C has only one object, i.e. is a Gray-monoid, a monoid object in Gray viewed as a monoidal 2-category, and that the underlying 2-functor of is strict (the unitor and compositor data ^{1}; ^{2} of is trivial).

We write g_C := (g) 2 C, and we dene C_g := $C(1_C$! $g_C)$ for each g 2 G. We denote the tensorator $_{g;h}$ 2 $C(g_C$ h_C ! $gh_C)$ and unitor 2 $C(1_C$! $e_C)$ of by trivalent and univalent vertices respectively



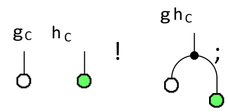
We denote 1-morphisms a_g 2 $C(1_C$! $g_C)$ by shaded disks as follows:



[4]Since a k-category may be viewed as an n-category for n k with only identity r-morphism for n r > k, it makes sense to talk about an n-functor from a k-category to an n-category.

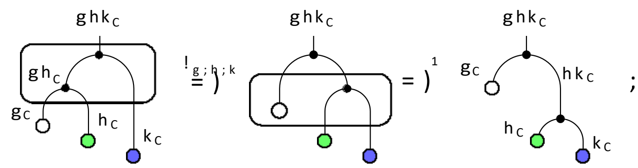
[5]In fact, [Gut19] justies working with this graphical calculus even in the context of weak 3-categories.

For $g, h \in G$, we define a tensor product $(a_g, b_h) \mapsto a_g \otimes b_h$ by



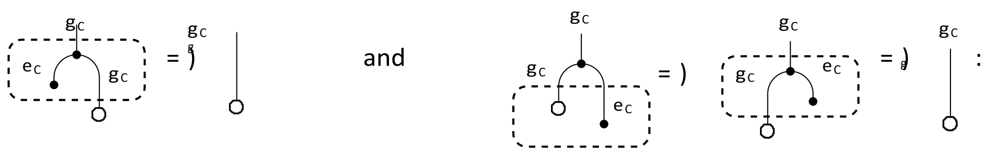
(2)

and we define the associator $_{gh,k}$ ($_{g,h}$ C_k)) $_{g,hk}$ (C_g $_{h,k}$) by

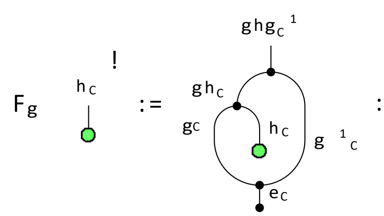


(3)

where denotes the interchanger in C (see §2 below). We define the unit object $1_C :=$ $\in C_e$. Unitors $_{e,g}$ (i)) id_{C_g} and $_{g,e}$ (i)) id_{C_g} are given respectively by

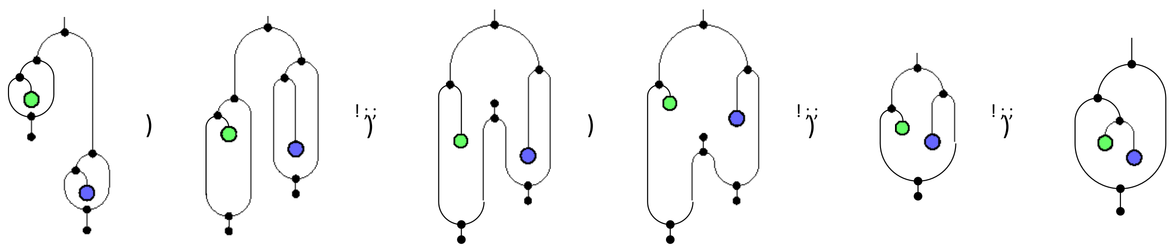


We define a G-action $F_g : C_h \to C_{ghg^{-1}}$ by



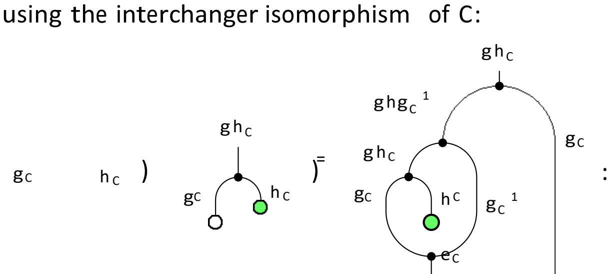
(4)

The functors F_g come equipped with natural isomorphisms $^g_{h,k}$: $(F_g \quad F_g)$) F_g $ghg^{-1};gkg^{-1}$ built from the coherence isomorphisms ^{g} ! ; ; and interchangers between two black nodes and between a black node and a shaded disk. For example, $_{b_h;c_k}$ is given by



(5)

The tensorator $_{g;h} : F_g F_h$) F_{gh} and the unit map $_h : id_C \to F_e|_{C_h}$ are defined similarly. The G-crossed braiding natural isomorphisms $_{g;h}$: $a_g \otimes b_h \to F_g(b_h) \otimes a_g$ are also defined similarly using the interchanger isomorphism of C:



(6)

1.2 The delooping hypothesis

Recall $(n+k)$-categories form an $(n+k+1)$-category, whereas k-fold monoidal n-categories only form an $(n+1)$-category. Thus one should not think of 'k-fold degeneracy' as a property of an $(n+k)$-category C but rather as additional structure, namely the structure of a $(k-1)$-surjective pointing, and require all morphisms and higher morphisms between these categories to preserve pointings [BS10, Sec 5.6]. Explicitly, the Delooping Hypothesis may then be understood as asserting that $(k-1)$-connected pointed $(n+k)$-categories and pointing-preserving higher morphisms form an $(n+1)$-category which is equivalent to the $(n+1)$-category of k-fold monoidal n-categories. This is an instance of a more general higher categorical principle.

Definition 1.1. We call a functor $F : C \to D$ of n-categories k-surjective[6] if it is essentially surjective on objects and parallel r-morphisms for $r \leq k$. By convention, any functor is (-1)-surjective.

Hypothesis 1.2. Let G be a n-category. The full $(n+1)$-subcategory of the under-$(n+1)$-category $nCat_{G/}$ on the k-surjective functors out of G is an $(n-k)$-category, i.e., all hom $(k+1)$-categories between parallel $(n-k)$-morphisms are contractible.

Remark 1.3. We expect hypothesis 1.2 is a direct consequence of more common assumptions on the $(n+1)$-category of n-categories: Namely, following [BS10, §5.5], we say that a functor $F : C \to D$ between n-categories is j-monic[7] if it is essentially surjective on k-morphisms for all $k > j$ (including $k = n+1$, where we interpret surjectivity to mean faithfulness on n-morphisms). By [BS10, Hypothesis 17], the (weak) fibers of such a j-monic functor are expected to be (possibly poset-enriched) $(j-1)$-categories.[8] Dually, a functor $G : C \to D$ between n-categories is j-epic if for every n-category E, the pre-composition functor $nCat(p, E) : nCat(D \to E) \to nCat(C \to E)$ is $(n-1-j)$-monic. In particular, any j-surjective functor in the sense of Definition 1.1 is j-epic.[9] Combining these observations, given a k-surjective functor $p : G \to C$ and an n-category E, the pre-composition functor $nCat(p, E) : nCat(C \to E) \to nCat(G \to E)$ is $(n-1-k)$-monic. Hence, its fiber at a $g \in nCat(G \to E)$ is a (possibly poset-enriched) $(n-1-k)$-category. But the fiber of the pre-composition functor $nCat(p, E)$ at $g : G \to E$ is the hom-category $nCat_{G/}(g, p)$ of the under-category of n-categories under G. Therefore, the full subcategory of $nCat_{G/}$ on the k-surjective functors is a (possibly poset-enriched) $(n-k)$-category. Moreover, essential (0-)surjectivity of $p : G \to C$ (also cf. Footnote 9) should imply that the pre-composition functor $nCat(p, E) : nCat(C \to E) \to nCat(G \to E)$ is n-conservative[10], and hence that the enriching posets of the (weak) fibers of $nCat(p, E)$ are honest sets.

Example 1.4 (k-fold monoidal n-categories). In the case where $G = *$ is the terminal category, Hypothesis 1.2 asserts that $(k-1)$-surjective (k-fold degenerate) pointed $(n+k)$-categories form an $(n+1)$-category. The Delooping Hypothesis [BS10, §5.6 and Hypothesis 22] identifies this $(n+1)$-category with the $(n+1)$-category of k-fold monoidal n-categories.

An important consequence of Hypothesis 1.2 is that it allows us to study certain higher-categorical objects, namely k-surjective functors and their higher transformations, using lower-categorical machinery. In many instances, there exist concrete descriptions of the resulting low-dimensional categories which have been developed and appear in mathematics and physics independently.

As a concrete example, it is easier to describe and work with the 1-category of monoids and monoid homomorphisms than its unpointed variant, the 2-category of categories, functors, and natural transformations.

[6] This notion of k-surjectivity does not coincide with the one used in [BS10], where a functor is said to be k-surjective if it is essentially surjective on k-morphisms.

[7] Many of the definitions and statements in this remark are extensively developed in the setting of $(\infty, 1)$-categories [Lur09a, Sec 5], and in particular in the $(n+1, 1)$-category of n-categories. However, we are not able to use these $(\infty, 1)$-notions and statements for our purposes, as we are working in the $(n+1, n+1)$-category of n-categories. For example, our j-monomorphisms do not coincide with the $(\infty, 1)$-categorical j-monomorphisms (in this context also known as $(j-1)$-truncated morphisms) as the latter only fulfill essential surjectivity conditions with respect to invertible cells.

[8] A functor between n-groupoids is j-monic if and only if its fibers are $(j-1)$-categories. For functors between general n-categories, j-monomorphisms have truncated fibers but the converse is not necessarily true.

[9] More generally, j-surjective functors are expected to correspond to 'strong j-epimorphisms' [BS10, Hypothesis 21], that is, functors that have the left lifting property with respect to j-monomorphisms. Since the $(n+1)$-category of n-categories has finite limits, any such 'strong j-epimorphism' is in particular a j-epimorphism; see [BS10, Sec 5.5].

[10] An n-functor $F : C \to D$ is n-conservative if it reflects n-isomorphisms, i.e., for every n-morphism $\alpha : f \Rightarrow g$ in D for which $F(\alpha)$ is an isomorphism, it follows that α is an isomorphism.

Similarly, it is easier to describe and work with the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations than its upointed variant, the 3-category of 2-categories, 2-functors, 2-transformations, and 2-modications. Similar examples are shown in Figure 1.

	G	n + k = 0	n + k = 1	n + k = 2	n + k = 3
k = 1	;	0-category	1-category	2-category	3-category
k = 0		point	monoid	monoidal category	monoidal 2-category
k = 1	BG		normal subgroup of G	G-crossed monoid	G-crossed braided category

Figure 1: (n + k)-categories equipped with k-surjective functors from G form an n-category

In this article, we focus on 1-surjective functors from the delooping BG of G, i.e., the 1-category with one object and endomorphisms G.

Hypothesis 1.5 (G-crossed delooping). For n 1, the (n + 3)-category of 1-surjective functors from BG into (n + 2)-categories is equivalent to the (n + 1)-category of G-crossed braided n-categories.

While we do not present a general denition of G-crossed braided n-category here, this hypothesis is a desideratum for any such denition (such as for example via Müller and Woike's 'little bundles' op-erad [MW19]). Observe that the k = 1 version of the delooping hypothesis follows as a consequence for the trivial group G = feg.

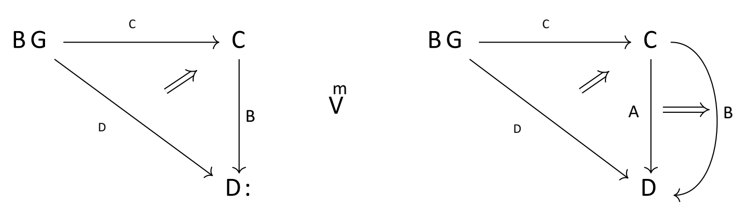
In the following, as a warm-up to our main theorem, we discuss the low-dimensional versions (n = 0 and n = 1) of Hypothesis 1.5 appearing in the last row of Figure 1.

Example 1.6 (G-crossed monoids as G-pointed 2-categories). The 3-category $2Cat_G$ of 2-categories C equipped with 1-surjective 2-functors BG ! C is equivalent to the 1-category of G-crossed monoids, or 'G-crossed braided 0-categories', dened below. Explicitly, the 2-category $2Cat_G$ has

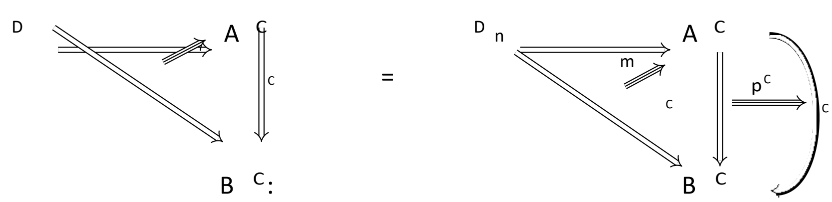
objects $(C; ^C)$ where C is a 2-category and ^{C} : BG ! C is a 1-surjective 2-functor,

1-morphisms (A;) : $(C; ^C)$! $(D; ^D)$ where A : C ! D is a 2-functor and : ^{D}) A ^{C} is an invertible 2-transformation,

2-morphisms (; m) : (A;)) (B;) where : A) B is a 2-transformation



3-morphisms p : (; m) V (; n) where p : V is a 2-modication such that



On the other hand, a natural decategorication of a G-crossed braided monoidal category is a G-graded monoid $M = q_{g2G}M_g$ together with a group homomorphism ^{M} : G ! Aut(M) such that the following axioms are satised:

$_g{}^M(m_h)$ 2 $M_{ghg}{}_1$ for all g 2 G and m_h 2 M_h, and

$m_g \; n_h = {}_g\,(n_h^M)\; m_g$ for all $m \; 2 \; M_g$ and $n_h \; 2 \; M_h$.

We call such a pair $(M; {}^M)$ a G-crossed monoid, or a 'G-crossed braided 0-category'. Morphisms $(M; {}^M)$! $(N; {}^N)$ are G-graded monoid homomorphisms that intertwine the G-actions.

To see that $2Cat_G$ is equivalent to the category of G-crossed monoids, we mirror our proof of Theorem A. One rst shows that $2Cat_G$ is equivalent to the 1-category $2Cat^{st}_G$ with

- objects strict monoidal categories C whose set of objects is $fg_C g_{g2G}$ with $1_C = e_C$ and tensor product given by the group multiplication, and
- morphisms $A : C \; ! \; D$ are strict monoidal functors such that $A(g_C) = g_D$ for all $g \; 2 \; G$.

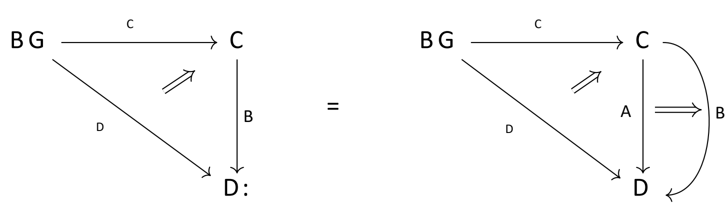
The equivalence from $2Cat^{st}_G$ to G-crossed monoids is given by taking hom from 1_C. We set $M_g := C(1_C \; ! \; g_C)$, and the multiplication on $M := q_{g2G} M_g$ is in C. The G-action ${}^M : G \; ! \; Aut(M)$ is given by conjugation:

$$ {}^M_g(m_h) := id_{g_C} \; {}^C \; m_h \; {}^C \; id_{g^{\;1}} \; 2 \; M_{ghg^{\;1}} = C(1_C \; ! \; ghg^{\;1}): $$

One then veries the G-crossed braiding axiom by a G-graded version of Eckmann-Hilton. A 1-morphism $A \; 2 \; 2Cat^{st}_G(C \; ! \; D)$ yields a G-graded monoid homomorphism by restricting to $M_g = C(1_C \; ! \; g_C)$. This monoid homomorphism is compatible with the G-actions by strictness of A. Finally, one veries this construction is an equivalence of categories.

Example 1.7 (Normal subgroups as G-pointed 1-categories). The 2-category Cat_G of 1-categories C equipped with 1-surjective functors $BG \; ! \; C$ is equivalent to the set of normal subgroups of G (which we may think of as the '0-category of G-crossed braided (1)-categories', see below). Explicitly, Cat_G has

- objects $(C; {}^C)$ where C is a category and ${}^C : BG \; ! \; C$ is a 1-surjective functor,
- 1-morphisms $(A;) : (C; {}^C) \; ! \; (D; {}^D)$ where $A : C \; ! \; D$ is a functor and $: {}^D \;) \; A \; {}^C$ is a natural isomorphism, and
- 2-morphisms $: (A;) \;) \; (B;)$ are natural transformations $: A \;) \; B$ such that



It is straightforward to verify that this 2-category is equivalent to a set. Moreover, up to equivalence, the data of a 1-surjective functor ${}^C : BG \; ! \; C$ is equivalent to the data of a normal subgroup of G, obtained as the kernel of the surjective group homomorphism $G \; ! \; Aut_C({}^C())$. Hence, the 2-category Cat_G is equivalent to the set of normal subgroups of G.

Employing 'categorical negative thinking' as in [BS10, x2], we may in fact think of a normal subgroup of G as a 'G-crossed braided (1)-category', and hence of the set of normal subgroups as 'the 0-category of G-crossed braided (1)-category' as it appears in Hypothesis 1.5: Since a (1)-category may be thought of as a truth value [BS10, x2], one may dene a G-graded (1)-category to be a monoid homomorphism G ! Bool = $(fT; Fg; \wedge)$, where Bool denotes the Booleans which one may think of as the commutative monoid (symmetric monoidal 0-category) of (1)-categories. Indeed, by taking the kernel, such 'G-graded (1)-categories' correspond to normal subgroups of G. This correspondence may be seen a further decategoried analogue of our construction. Indeed, given $(C; {}^C)$, the corresponding monoid homomorphism G ! Bool is exactly given by $g \; ! \; C(id_{c()} \; ! \; {}^C(g))$, where the latter is the Boolean which is true if $id_{c()} = (g)$ and false otherwise.

Example 1.8 (Shaded monoidal algebras). In [GMP+18, Defn. 3.18 and 3.26], the authors dene the notion of a shaded monoidal algebra, which is an operadic approach to 2-categories with a chosen set of objects and a set of generating 1-morphisms. The statements of [GMP+18, Thm. 3.21 and Cor. 3.23] can be understood as examples of Hypothesis 1.2. Indeed, equipping a 2-category with a set of objects and a generating set of 1-morphisms is equivalent to pointing by the free category on a graph . Hence the 3-category of 1-surjective -pointed 2-categories is equivalent to the 1-category of -shaded monoidal algebras.

Remark 1.9 (Planar algebras). Expanding on Example 1.8, Jones' planar algebras [Jon99] reect the philosophy of Hypothesis 1.2. A 2-shaded planar algebra may be understood as a pivotal 2-category C with precisely two objects 'unshaded' and 'shaded' together with a generating dualizable 1-morphism between them with loop modulus . This choice of generating 1-morphism may be understood as equipping C with a 1-surjective pivotal functor ^{C} : $TLJ()\to C$, where $TLJ()$ is the free spherical 2-category on a dualizble 1-morphism with quantum dimension . By (a pivotal version of) Hypothesis 1.2, such pivotal 2-categories and functors preserving this 'TLJ-pointing' actually form a 1-category, which is equivalent to the 1-category of 2-shaded planar algebras and planar algebra homomorphisms.

Another instance of this philosophy appears in [HPT16] which shows the 2-category ModTens of pointed module tensor categories over a braided pivotal category V (dened in [HPT16, x3.1]) is 1-truncated [HPT16, Lem. 3.6]. By [HPT16, Thm. A], ModTens is equivalent to the 1-category of anchored planar algebras in V.

1.3 Examples

Our main theorem asserts an equivalence between 1-surjective functors $BG\to C$ and G-crossed braided categories. Starting with an arbitrary 3-functor $:BG\to C$ we may factor it through a 1-surjective functor $^0 : BG\to C^0$ (where C^0 is the subcategory of C with objects and 1-morphisms in the essential image of , and all 2- and 3-morphisms between them) and apply our construction from x1.1 to obtain a G-crossed braided category. Most examples discussed below arise in this way.

Example 1.10 (Delooped braided monoidal categories). Let B be a braided monoidal category, and denote the corresponding 3-category with one object and one 1-morphism by B^2B. Observe that every weak 3-functor $BG\to B^2B$ is automatically 1-surjective. Such 3-functors $BG\to B^2B$ factor through the maximal sub-3-groupoid B^2B of B^2B, delooping the braided monoidal groupoid B of invertible objects and morphisms in B. Assuming the homotopy hypothesis for algebraic trigroupoids,[11] such functors correspond to homotopy classes of maps from the classifying space BG to the 1-connected homotopy 3-type B^2B.

Such 1-connected 3-types are completely determined by the abelian group $_2(B^2B) = \mathrm{Inv}(B)$ of isomorphism classes of invertible objects of B, the abelian group $_3(B^2B) = \mathrm{Aut}(1_B)$ of automorphisms of the tensor unit 1_B of B, and the k{invariant $q \in H^4(K(\mathrm{Inv}(B);2);\mathrm{Aut}(1_B)) = \mathrm{Quad}(\mathrm{Inv}(B);\mathrm{Aut}(1_B))$, the group of quadratic functions on Inv(B) valued in $\mathrm{Aut}(1_B)$ [EM54], which is explicitly given by the quadratic function

$$q:\mathrm{Inv}(B)\to\mathrm{Aut}(1_B)\qquad\text{given by}\qquad q(b) := \mathrm{ev}_b\ {}_{b;b}\ {}^{B}{}_{1}\ \mathrm{coev}_b:$$

Here, $\mathrm{ev}_b : b^{1}$ $b\to I$ and $\mathrm{coev}_b : I\to b$ b^{1} denote a choice of pairing between b and b^{1} and $_{b;b^{1}} : b$ $b^{1}\to b^{1}$ b denotes the braiding.

By [Mac52, EM54], the group $\mathrm{Quad}(\mathrm{Inv}(B);\mathrm{Aut}(1_B))$ is further isomorphic to the group $H^3_{ab}(\mathrm{Inv}(B);\mathrm{Aut}(1_B))$ of abelian 3-cocycles $(;)$, consisting of pairs of a group 3-cocycle $:\mathrm{Inv}(B)^3\to\mathrm{Aut}(1_B)$ and a certain '-twisted-bilinear' form $:\mathrm{Inv}(B)^2\to\mathrm{Aut}(1_B)$. We refer the reader to [Bra20, (1.2) and x11] for more details.

By the obstruction theory for homotopy classes of maps into such Postnikov towers (cf. [ENO10, Theorem 1.3]), it follows that, up to natural isomorphism, 3-functors $BG\to B^2B$ correspond to the following data:

- a 2-cocycle $\in Z^2(G;\mathrm{Inv}(B))$, up to coboundary;
- a 3-cochain $! \in C^3(G;\mathrm{Aut}(1_B))$ such that $d! = (;)$, where $(;) \in Z^4(G;\mathrm{Aut}(1_B))$ is the 4-cocycle in the image of the Pontryagin-Whitehead morphism[12] $(;) : H^2(G;\mathrm{Inv}(B))\to$

[11] The article [Lac11] constructs a model category structure on the category of Gray-categories and Gray-functors which restrict to a model structure on Gray-groupoids. Even though it is shown that the corresponding homotopy category of Gray-groupoids localized at the Gray-equivalences is equivalent to the category of homotopy 3-types and homotopy classes of continuous maps, to the best of our knowledge, it has not yet been shown that this category is also equivalent to the 1-category whose objects are Gray-groupoids (or algebraic trigroupoids) and whose morphisms are natural equivalence classes of weak 3-functors.

[12] Under the isomorphism $H^4(K(\mathrm{Inv}(B);2);\mathrm{Aut}(1_B)) = H^3_{ab}(\mathrm{Inv}(B);\mathrm{Aut}(1_B))$, the abelian 3-cocycle $(;)$ corresponds to a map $(;) : K(\mathrm{Inv}(B);2)\to K(\mathrm{Aut}(1_B);4)$. From this perspective, the Pontryagin-Whitehead morphism $(;) : H^2(G;\mathrm{Inv}(B))\to H^4(G;\mathrm{Aut}_B(1_B))$ is simply given by postcomposing a class $! : BG\to K(\mathrm{Inv}(B);2)$ with $(;)$.

$H^4(G; \mathrm{Aut}_B(1_B))$ for the k-invariant $(;) \, 2 \, H^4(K(\mathrm{Inv}(B); 2); \mathrm{Aut}(1_B))$. An explicit expression for the 4-cocycle $(;) \, 2 \, Z^4(G; \mathrm{Aut}(1_B))$ is given by

$$(;)(g; h; k; ') = {}_{k;';g;h} \, {}_{ghk;';gh;k} \, {}_{;g;h} \, {}^{1}_{ghk;';g;hk} \, {}_{h;k} \, {}_{g;hk'} \, {}_{;hk;';h;k} \, {}^{1} \; {}_{g;hk'} \, {}_{;h;k'} \, {}_{;k;'} \, {}_{gh;k'} \, {}_{;g;h} \, {}^{1}_{;k;'} \, {}_{gh;k'} \, {}_{;k;'} \, {}_{;g;h} \qquad (7)$$

This explicit expression can also be obtained, up to conventions, by taking the trivial G-action in [CGPW16, Eq. (5.6)].

In fact, after strictifying B to a strict braided monoidal category, so that B^2B is a Gray-category, this cohomological data may be directly read o from the components of the weak 3-functor $: BG \, ! \, B^2B$, using notation from Appendix A, as follows: We may assume the underlying 2-functor of is strictly unital, i.e., ${}^1_g = \mathrm{id}_1$ for all $g \, 2 \, G$. By (F-I).ii, this implies ${}^2_{g;h} = \mathrm{id}_{1_B}$ for all $g; h \, 2 \, G$. We write ${}_{g;h} := {}^2_{g;h} \, 2 \, \mathrm{Inv}(B)$. By (F-II).iii, $\mathrm{id}_{;\mathrm{id}_g \, h} = \mathrm{id} \; 2 \; \mathrm{End}({}_{g;h})$. Using the isomorphism $!_{g;h;k} : {}_{gh;k} \; {}_{g;h} \; ! \; {}_{g;hk} \; {}_{h;k}$; descends to a 2-cocycle in $Z^2(G; \mathrm{Inv}(B))$. To translate ! into a 3-cochain in $C^3(G; \mathrm{Aut}_B(1_B))$, we let C be a skeletalization of B. In C, we may identify all automorphism spaces of C with $\mathrm{Aut}(1_B)$, and hence recover the associator in C as an element of $Z^3(\mathrm{Inv}(C); \mathrm{Aut}(1_B))$, and descend the isomorphisms $!_{g;h;k} : {}_{gh;k} \; {}_{g;h} \; ! \; {}_{g;hk} \; {}_{h;k}$ to a 3-cochain ! in $C^3(G; \mathrm{Aut}_B(1_B))$. Unpacking[13] (F-1) leads to $d! = (;)$.

We can now explicitly describe the G-crossed braided category resulting from our construction from this cohomological data by interpreting the diagrams (2), (3), (4), (5), (6).

- All g-graded components are B,
- the monoidal structure is given by interpreting (2): $a_g \; b_h := {}_{g;h} \; a_g \; b_h$, with associator given by interpreting (3):

$$\begin{array}{l} {}_{gh;k} \; {}_{g;h} \; a_g \; b_h \; c_k \xrightarrow{!_{g;h;k} \; \mathrm{id}} {}_{g;hk} \; {}_{h;k} \; a_g \; b_h \; c_k \xrightarrow{\mathrm{id} \; {}^{1}_{a_g;\; h;k} \; \mathrm{id}} {}_{g;hk} \; a_g \; {}_{h;k} \; b_h \; c_k : \end{array}$$

- the G-action is given by interpreting (4): $F_g(b_h) := {}_{gh;g^{1}} \; {}^{1}_{g;h} \; b_h \; {}_{g;g^{1}}$, with tensorator ^{g} given by interpreting (5).
- the G-crossed braiding is given by interpreting (6).

One can view the resulting G-crossed extension as a twisting of the trivial extension by a 2-cocycle [ENO10, Pf. of Thm. 1.3]. When B is fusion, this is a G-crossed zesting of the trivial G-crossed extension B Vec(G) of B [DGP+20].

Example 1.11 (Generalized relative center construction). The article [BGM19] shows that every (weak) G-action on a 2-category may be strictied to a strict G-action on a strict 2-category, encoded by a group homomorphism $: G \, ! \, \mathrm{Aut}^{st}(B)$, where $\mathrm{Aut}^{st}(B)$ is the group of strict 2-equivalences of B which admit strict inverses. From such a strict G-action, the authors then construct a G-crossed braided monoidal cate-gory $Z_G(B)$ whose g-graded component is the category of pseudonatural transformations and modications $\mathrm{PseudoNat}(\mathrm{id}_B \;\; (g))$: Since $(e) = \mathrm{id}_B$, the trivial graded component is the Drinfeld center $Z(B)$. This construction generalizes the construction of the relative center $Z_C(D)$ of a G-extension D of a fusion category C;

by [GNN09], $Z_C(D)$ is a G-crossed braided fusion category whose trivial graded component is $Z(C)$.

Our construction of a G-crossed braided monoidal category from a G-pointed 3-category may be understood as a generalization of [BGM19] from G-actions on 2-categories, encoded by 3-functors BG ! 2Cat from BG into the 3-category of 2-categories, to arbitrary 3-functors BG ! C. In particular, we show in Section 3.2 that we may strictify a 1-surjective weak 3-functor BG ! C to a Gray-functor BG ! C^0 into a Gray-category C^0 equivalent to C, and construct a G-crossed braided category from this data.

[13] Unpacking (F-1) in C introduces six additional associator terms, one for every vertex of the hexagon commutative diagram. As these terms correspond to the two dierent ways to associate each of the vertex 1-cells in (F-1), the associators alternate and [1] around the diagram. The resulting 12 sided commutative diagram exactly reproduces, up to conventions, a simplication of [CGPW16, Fig. 1] where the G-action is trivial. Five of these 12 terms give d!, while the other 7 terms give (7). Since the diagram commutes, we have d! = (;) as desired.

Example 1.12 (G-crossed extension theory for braided fusion categories). Let C be a braided fusion category, and consider the monoidal 2-category Mod(C) of finite semisimple module categories [Gre10, DR18]. Given a monoidal 2-functor $: G \to \mathrm{Mod}(C)$, our construction produces the G-crossed braided fusion category

$$\bigoplus_{g \in G} \mathrm{Hom}(C_C \to (g)_C) = \bigoplus_{g \in G} (g)$$

which is a G-crossed braided extension of the e-graded piece $\mathrm{End}_{\mathrm{Mod}(C)}(C_C) = C$. This G-crossed braided category is equivalent to the G-crossed extension constructed in [ENO10] (which moreover gives an alternate proof that faithful G-crossed extensions of braided fusion categories are in fact classified by monoidal 2-functors $G \to \mathrm{Mod}(C)$).

Example 1.13 (Permutation crossed extensions). Let C be a symmetric monoidal 3-category, and let A be an object of C. Then there exists a monoidal 2-functor $: S_n \to \mathrm{End}(A^n)$, where denotes the symmetric monoidal product in C. Our construction produces a S_n-crossed braided category whose trivially graded piece is $\mathrm{End}(\mathrm{id}_{A^n})$. For example, if A is an object in the 3-category of fusion categories [DSPS13, Hau17, JFS17], there is an equivalence

$$\mathrm{End}(\mathrm{id}_{A^n}) = Z(A^n) = Z(A)^n;$$

where $Z(A)$ is the Drinfeld center of A, and the resulting S_n-crossed braided category is what is known as a permutation crossed extension of $Z(A)^n$. More generally, the article [GJ19] shows that such permutation crossed extensions of C^n exist for any modular tensor category.

Example 1.14 (Conformal nets). Consider the symmetric monoidal 3-category[14] of coordinate free conformal nets CN defined in [DH12, BDH15, BDH17, BDH19, BDH18]. A 3-functor $BG \to CN$ amounts to a conformal net $A \in CN$ together with a generalized action of G on the net A by invertible topological defects. Applied to such a 3-functor, our construction produces a G-crossed braided category whose trivial graded component is the braided category $\mathrm{End}_{CN}(1_A) = \mathrm{Rep}(A)$ of (super-selection) sectors [BDH15, Sec 1.B] of A. We expect this generalizes a construction of Müger [Müg05], which produces a G-crossed braided category from the action of global symmetries on a coordinatized conformal net. However, it is difficult to compare these two G-crossed braided categories, since it is not obvious how to construct a symmetric monoidal 3-category of coordinatized conformal nets.

Example 1.15 (Topological phases). The collection of (2+1)D gapped topological phases is expected to form a 3-category [GJF19b, GJF19a]. Given a global, onsite symmetry, there is an associated G-crossed braided category of twist defects [BBCW19]. Our construction can be understood as a direct generalization of this heuristic. Indeed, our pictures and arguments can be viewed as a more mathematically precise version of the arguments and structure given in the physical context (e.g., see [BBCW19, Fig. 7]).

Example 1.16 (Homotopy quantum field theory). Homotopy quantum field theories are topological field theories on bordisms equipped with a map to a fixed target space. If this target space is the classifying space BG of a finite group G, such field theories are also known as G-equivariant field theories. Following the cobordism hypothesis [BD95, Lur09b], such a fully extended (framed) 3-dimensional G-equivariant topological field theory valued in a fully dualizable symmetric monoidal 3-category corresponds to a 3-functor $BG \to C$ (i.e. a fully dualizable object A in C equipped with an 'internal G-action', given by a monoidal 2-functor $X : G \to \mathrm{End}_C(A)$). It therefore follows from Theorem A that to any such field theory, there is an associated G-crossed braided category.

In particular, if Fus is the 3-category of fusion categories introduced in [DSPS13], we expect the G-crossed braided category constructed via Theorem A from a fully extended G-equivariant three-dimensional field theory valued in Fus to coincide with the G-crossed braided category constructed in [SW18] by evaluating the field theory on (G-structured) circles. In particular, if G is trivial, this recovers the construction of the Drinfeld center of a fusion category A as $\mathrm{FusCat}({}_A A_A \quad {}_A A_A)$.

[14] The notion of tricategory used in [DH12, BDH18], namely an internal bicategory in Cat, is expected, but not proven to be equivalent to the notion of algebraic tricategory [Gur13] used in the present article.

1.4 Outline

Section 2 contains basic denitions and a brief introduction to the graphical calculus of Gray-monoids used throughout.

Section 3 proves various strictication results for 1-surjective pointed 3-categories (x3.1) and higher morphisms between them (x3.2, x3.3, x3.4, x3.5) and shows that $3\mathrm{Cat}_G$ (Denition 3.2) is equivalent to its strict sub-2-category $3\mathrm{Cat}_G^{st}$ (Corollary 3.5).

Section 4 denes the 2-category GCrsBrd of G-crossed braided categories (x4.1) and its equivalent full sub-2-category $\mathrm{GCrsBrd}^{st}$, constructs the strict 2-functor $3\mathrm{Cat}_G^{st}$! $\mathrm{GCrsBrd}^{st}$ (x4.2) and proves that it is an equivalence (x4.3).

Section 5 discusses how various properties and structures on a 1-surjective G-pointed 3-category, such as linearity and rigidity, may be translated across the equivalence of Theorem A to the resulting G-crossed braided category.

Appendix A unpacks the denitions of (weak) 3-functors, transformations, modications and perturbations between Gray-monoids in terms of the graphical calculus.

Appendices B and C contain most of the coherence proofs from Sections 3 and 4, respectively.

Acknowledgements The authors would like to thank Shawn Cui, Nick Gurski, Niles Johnson, and Andre Henriques for helpful conversations.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140, while the authors DP and DR were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester. CJ was supported by NSF DMS grant 1901082. DP was supported by NSF DMS grant 1654159. DR is grateful for the nancial support and hospitality of the Max Planck Institute for Mathematics where part of this work was carried out.

2 Background on 3-categories and monoidal 2-categories

In this article, by a 3-category we mean an algebraic tricategory in the sense of [Gur13, Def 4.1], and by functor, transformation, modication, and perturbation, we mean the corresponding notions of trihomomorphism, tritransformation, trimodication, and perturbation of [Gur13, Def 4.10, 4.16, 4.18, 4.21]. We include Appendix A below which unpacks the full denitions of these notions for Gray-monoids using the graphical calculus discussed in x2.2 below. When we consider stricter notions of categories or functors we will always use appropriate adjectives such as 'Gray' or 'strict'.

Remark 2.1. In this article, we use the term invertible as a property, i.e., the existence of a coherent inverse. Indeed, by [Gur12], every invertible 1-morphism (biequivalence) in a 3-category is part of a biadjoint biequivalence, and every invertible 2-morphism is part of an adjoint equivalence. Moreover, there is a contractible space of choices for these coherent inverses. Whenever we need to make such choices, we will refer back to this remark.

2.1 Gray-categories and Gray-monoids

In this section, we give a terse denition of Gray-category and Gray-monoid, and a brief discussion on the diagrammatic calculus for Gray-monoids. We refer the reader to [Gur06] for a more detailed treatment of Gray-categories and to [BMS12, x2.6] or [Bar14] for a more detailed treatment of the graphical calculus.

Denition 2.2. The symmetric monoidal category Gray is the 1-category of strict 2-categories and strict 2-functors equipped with the Gray monoidal structure [Gur06, x5]. A Gray-category is a category enriched in Gray in the sense of [Kel05]. A Gray-monoid is a monoid object in Gray. Given a Gray-monoid C, its delooping BC is the Gray-category with one object and endomorphisms C.

We now unpack the notion of Gray-monoid from Denition 2.2.

Notation 2.3. Given a Gray-monoid C, we refer to its objects, 1-morphisms, and 2-morphisms as 0-cells, 1-cells, and 2-cells respectively in order to distinguish these basic components of C from morphisms in an ambient category in which C lives.

The remarks and warning below are adapted directly from [DR18].

Remark 2.4. Unpacking Denition 2.2, a Gray-monoid consists of the following data:

(D1) a strict 2-category C, where composition of 1-morphisms is denoted by and composition of 2-morphisms is denoted by ;

(D2) an identity 0-cell 1_C 2 C;

(D3) strict left and right tensor product 2-functors L_a = a
and R_a =
a for each object a 2 C:

$$L_a = a$$
$$: C ! \quad C R_a =$$
$$a : C ! \quad C;$$

(D4) an interchanger 2-isomorphism $_{x;y}$ for each pair of 1-cells x : a ! b and g : c ! d:

$$_{x;y} : (x$$
$$id_d) \; (id_a$$
$$y) \;) \quad (id_b$$
$$y) \; (x$$
$$id_c)$$

subject to the following conditions:

(C1) left and right tensor product agree: for all objects a; b 2 C, $L_a b$ = $R_b a$ = a
b;

(C2) tensor product is strictly unital and associative:

$$L_{1_C} = id_C = R_{1_C}$$
$$L_a L_b = L_a$$
$$_b R_b R_a = R_a$$
$$_b L_a R_b = R_b L_a;$$

(C3) the interchanger respects identities, i.e., for a 0-cell A 2 C and a 1-cell f : C ! D,

$$_{f;id_A} = id_f$$
$$_{A \; id_A;f} = id_{A} \; _f$$

(C4) the interchanger respects composition, i.e., for x : a ! a^0, x^0 : a^0 ! a^{00}, y : b ! b^0 and y^0 : b^0 ! b^{00},

$$_{x^0 x;y} = (_{x^0;y} \; (x$$
$$id_b)) \; ((x^0$$
$$id_{b^0}) \; _{x;y}) \; _{x;y^0 y} = ((id_{a^0}$$
$$y^0) \; _{x;y}) \; (_{x;y^0} \; (id_a$$
$$y))$$

(C5) the interchanger is natural, i.e., for 1-cells x; x^0 : a ! a^0; y; y^0 : b ! b^0 and 2-cells : x) x^0, : y) y^0,

$$_{x^0;y} \; (($$
$$id_{b^0}) \; (id_a$$
$$y)) = ((id_{a^0}$$
$$y) \; ($$
$$id_b)) \; _{x;y \; x;y^0} \; ((x$$
$$id_{b^0}) \; (id_a$$

)) = ((id$_{a^0}$

) (x

id$_b$)) x;y

(C6) the interchanger respects tensor product, i.e., for x : a ! a^0, y : b ! b^0 and z : c ! c^0,

id$_a$

y;z = id$_a$

y;z x

id$_b$;z = x;id$_b$

z x;y

id$_c$ = x;y

id$_c$

A Gray-monoid is called linear if the underlying 2-category is linear and for all objects a the functors a

and
a are linear.

Warning 2.5 (Horizontal composition of 1-morphisms). We warn the reader that the tensor product in a Gray-monoid does not provide a unique definition of the tensor product of two 1-cells. Given $x : a \to b$ and $y : c \to d$, we define

$$x \otimes y := (x \otimes \mathrm{id}_d)\,(\mathrm{id}_a \otimes y); \tag{8}$$

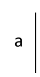
this convention is known as nudging [GPS95, §4.5]. We use a similar nudging convention for the tensor product of 2-cells. With this convention, the data of a Gray-monoid C as described in Definition 2.4 gives rise to an (opcubical cf [Gur13, §8]) algebraic tricategory BC [Gur13, Thm. 8.12].

Remark 2.6 (Strictification for monoidal 2-categories). By the strictification for tricategories from [GPS95] or [Gur13, Cor. 9.16], every (linear) weakly monoidal weak 2-category admits a monoidal 2-equivalence to a (linear) Gray-monoid of the form in Definition 2.4.

2.2 Graphical calculus for Gray-monoids

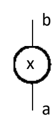
Gray-categories admit a graphical calculus of surfaces, lines, and vertices in three-dimensional space. We refer the reader to [BMS12, §2.6] for a rigorous discussion. Here, we will only ever work in a two-dimensional projection of this graphical calculus for Gray-monoids. Our exposition below follows [Bar14].

The 0-cells of our strict 2-category C (D1) are denoted by strands in the plane



and the identity 0-cell 1_C (D2) is denoted by the empty strand. The 1-cells are denoted by coupons between labelled strands

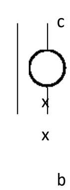
$x : a \to b$



The composition of 1-cells is denoted by vertical stacking of such diagrams.

The strict tensor product $\otimes$ is denoted by horizontal juxtaposition. For example, the tensor product functors L_a and R_a (D3) are denoted by placing a strand labelled by a to the left or right respectively.

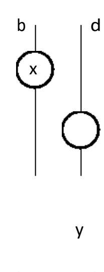
$L_a(x : b \to c) := \mathrm{id}_a \otimes x = $



$R_a(x : b \to c) := x \otimes \mathrm{id}_a = $

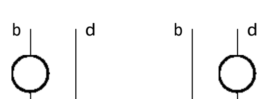
Given $x : a \to b$ and $y : c \to d$, we define their tensor product using the nudging convention from Warning 2.5.

$x \otimes y := (x \otimes \mathrm{id}_d)(\mathrm{id}_a \otimes y) = $



Observe that no two coupons ever share the same vertical height.

The 2-cells are inherently 3-dimensional, and can be thought of as 'movies' between our 2-dimensional string diagrams. Rather than drawing 2-cells, we denote them by arrows $\Rrightarrow$ between diagrams corresponding to their source and target 1-cells. For example, the interchanger ${}_{x;y}$ from (D4) is simply denoted by



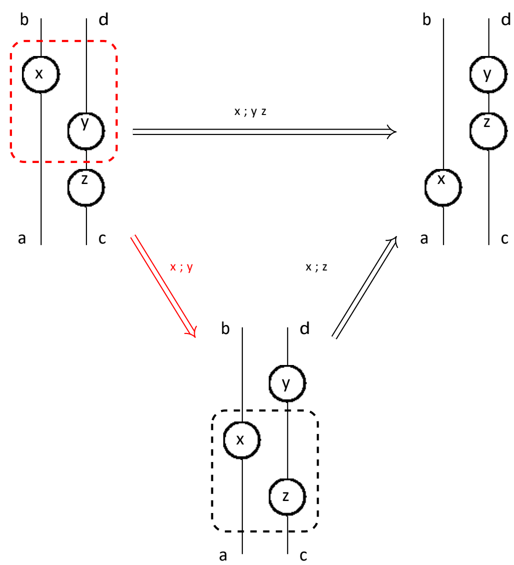
x y

x ; y =) : y

x

a c a c

Notation 2.7. When working with Gray-monoids, one often needs to whisker 2-cells between 1-cells, and the notation can quickly become cumbersome. Instead, we use the convention of a dashed box when we apply a 2-cell locally to a 1-cell, and we simply label the whiskered 2-cell by the name of the locally applied 2-cell. Later on, we will draw commutative diagrams whose vertices are 1-cells. When we want to apply two 2-cells locally in dierent places to the same 1-cell, we will use two dashed boxes with dierent colors, usually red and blue. When one of these two 2-cells is applied to the entire diagram, we do not use a dashed box, and we only use one dashed box of another color, usually red. As an explicit example, the second equation in (C4) in string diagrams is given by:



For the convenience of the reader, we have included Appendix A which unpacks the notions of 3-functor, transformation, modication, and perturbation for Gray-monoids using this graphical calculus.

3 Strictifying G-pointed 3-categories

Let G be a group. We recall from §1.1 that BG denoted the delooping of G, i.e., G considered as a 1-category with one object. As discussed at the beginning of §2, the terms n-category and n-functor for $n \geq 3$ will always mean weak n-categories and weak n-functors. Observe that since a k-category may be viewed as an n-category for $n \geq k$ with only identity higher morphisms, we may talk about an n-functor from a k-category to an n-category. Recall from Remark 2.1 that we use the adjective invertible for (bi)adjoint (bi)equivalences.

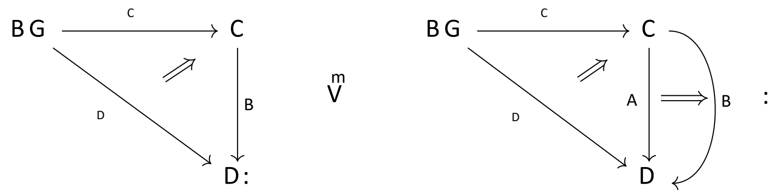
Denition 3.1. A 3-functor $A : C \to D$ is 1-surjective if it is essentially surjective on objects and if for every pair of objects $c_1; c_2$ of C, the 2-functors $A_{c_1;c_2} : C(c_1 \to c_2) \to D(A(c_1) \to A(c_2))$ are essentially surjective on objects.

Denition 3.2. Let G be a group. We dene the 4-category[15] $3Cat_G$ of G-pointed 3-categories to be the full sub-4-category of the under-category $3Cat_{BG=}$ on the 1-surjective 3-functors $BG \to C$. Explicitly, this 4-category can be described as follows:

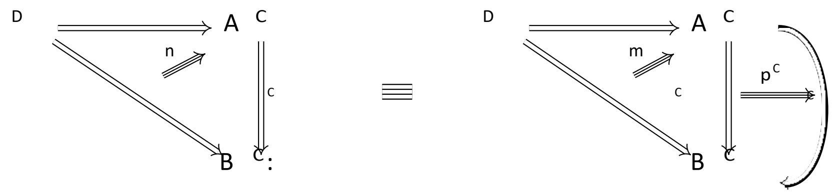
- objects are 3-categories C equipped with a 1-surjective 3-functor $^C : BG \to C$.
- 1-morphisms $(A;) : (C; ^C) \to (D; ^D)$ are pairs where $A : C \to D$ is a 3-functor and $: ^D \Rightarrow A\, ^C$ is an invertible natural transformation;

[15] All results in this section can be stated and proved at the level of various 2-categories of $(k-1)$-morphisms, k-morphisms and equivalence classes of $(k+1)$-morphisms of $3Cat_G$; we therefore will not show that $3Cat_G$ forms a 4-category — and in fact will not even choose any denition of 4-category. We only use the conceptual idea of a 4-category as an underlying organizational principle for our results.

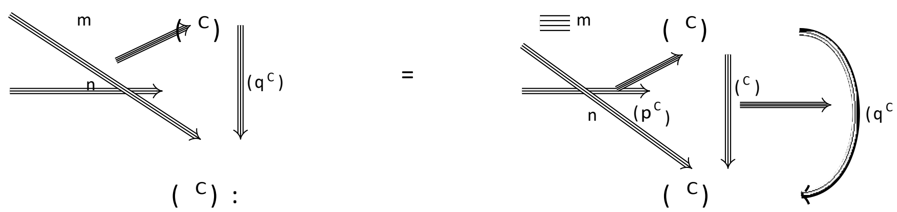
2-morphisms (; m) : (A;)) (B;) are pairs where : A) B is a natural transformation and m is an invertible modication

 (9)

3-morphisms (p;) : (; m) V (; n) are modications p : V together with an invertible perturba-tion :

 (10)

4-morphisms : (p;) ≡ (q;) are perturbations : p q satisfying

 (11)

Remark 3.3. As stated, Denition 3.2 and Theorem 3.4 below assume the existence of a (weak) 4-category 3Cat of algebraic tricategories, trifunctors, tritransformations, modications, and perturbations which has the appropriate homotopy bicategories between parallel k-morphisms. Assuming the existence of such a 4-category 3Cat, we may dene $3Cat_G$ as a certain full sub-4-category of the under-category as in Denition 3.2. In Theorem 3.4 we show, working a bicategory at a time, that this 4-category $3Cat_G$ is equivalent to a sub-4-category $3Cat_G^{pt}$ with only identity 3- and 4-morphisms, and is hence equivalent to a bicategory. After having established Theorem 3.4, we will from then on only work with this bicategory $3Cat_G^{pt}$.

Unfortunately, to the best of our knowledge, such a 4-category 3Cat has not yet been constructed in any of the established models of weak 4-category. However, none of the results in this article truly depend on the specics of 4-categories, and 4-categories only appear as a convenient conceptual organizing tool.

The reader uncomfortable with this sort of model-independent argument may unpack the statement of our main Theorem A to assert the following:

(1) For a pair of parallel '1-morphisms' as in Denition 3.2, the bicategory of 2-morphisms, 3-morphisms and 4-morphisms between them is equivalent to a set.

(2) The bicategory of objects, 1-morphisms and 2-morphisms up to invertible 3-morphisms of Denition 3.2 is equivalent to the 2-category of G-crossed braided categories.

Theorem 3.4. The 4-category $3Cat_G$ is equivalent to the 4-subcategory $3Cat_G^{pt}$ where

objects $(BC; {}^C)$ are those objects of $3Cat_G$ for which the 3-category is a Gray-category with one object, and hence given by BC for some Gray-monoid C, and for which ^{C} : BG ! BC is a strictly 1-bijective Gray-functor. Equivalently, an object is a Gray-monoid C whose set of 0-cells is $fg_C := {}^C(g)g_{g2G}$ and composition of 0-cells given by group multiplication.

1-morphisms $(A;) : (BC; {}^C) ! (BD; {}^D)$ satisfy:

– $A(g_C) = g_D$ for all $g \in G$,
– the adjoint equivalence :
$_D$ (A A)) A ^{A} ^{1} ^{e}
$_C$ satises $_{g_C;h_C}$ = id_{gh_D} : g_D
h_D) gh_D, – the adjoint equivalence ^{A} = (^{A}; $_1$) : I_D) A I_C satises ^{A} = id_{e_D}, and ^{A} = A^1,
– the associators and unitors $!^A$; $‘^A$; r^A are identities,
– = e_D and $_g$ = id_{g_D}, and $_{id_g}$ = A_g, ^{1}
– ^{1} = $id_{id_{e_D}}$ and $_{g;h}^2$ = $id_{id_{gh_D}}$ for all $g;h \in G$.

2-morphisms (; m) : (A;)) (B;) satisfy = e_D, $_{c\,g}$ = id_g $_e$ ^{1} = id_{id} $_{e_D}$ and $^2_{g_C;h_C}$ = $id_{id_{gh_D}}$ and m = e_D, m_g = $id_{id_{g_D}}$. That is, m is the identity modication.

3-morphisms (p;) : (; m) V (; n) satisfy p = id_e , p_g = id_{id} $_{g_D}$, and = id_{id} $_{e_D}$. That is, there are only identity 3-morphisms.

4-morphisms : (p;) ≡ (q;) satisfy = id_{id} $_{e_D}$. That is, the only 4-endomorphism of an identity 3-morphism is the identity.

Proof. In §3.1, 3.2, 3.3, 3.4, 3.5 below we show that every object, 1-morphism, 2-morphism, 3-morphism, and 4-morphism respectively in $3Cat_G$ is equivalent to one of the desired form in $3Cat_G^{pt}$. All proofs in these further subsections amount to checking the appropriate coherences for 3-functors, 3-natural transformations, 3-modications, and 3-perturbations outlined in Appendix A and are deferred to Appendix B. We signify where the reader may nd the deferred proof of a statement by including a small box with a link to the appropriate appendix after the statement. □

Since the only 3- and 4-morphisms of $3Cat_G^{pt}$ are identities, it is evident that $3Cat_G^{pt}$ | and hence by Theorem 3.4 also $3Cat_G$ | is 2-truncated and actually denes a 2-category. In the following corollary, we give a streamlined description of this 2-category without the redundant data.

Corollary 3.5. The 4-category $3Cat_G^{pt}$ is isomorphic to the strict 2-category $3Cat_G^{st}$, dened as follows:

An object is a Gray-monoid C whose set of 0-cells is G (below, we will denote the elements of G seen as 0-cells in C by g_C) and composition of 0-cells is given by group multiplication.

A 1-morphism A : C → D is a 3-functor A : BC → BD such that

– $A(g_C) = g_D$ for all $g \in G$,
– the adjoint equivalence :
$_D$ (A A)) A ^{1} ^{1} ^{1}
$_C$ satises $_{g;h}$ = id_{gh} : g_D
h_D) gh_D, – the adjoint equivalence ^{A} = (^{A}; ^{A}) : I_D) A I_C satises ^{A} = id_{e_D}, and ^{A} = A_e,
– the associators and unitors $!^A$; $‘^A$; r^A are identities.

A 2-morphism : A) B is a natural transformation such that = e_D, $_g$ = id_{g_D}, ^{1} = id_{id_e} $_D$ and $^2_{g;h}$ = $id_{id_{gh_D}}$ for all $g;h \in G$.

Composition of 1- and 2-morphisms is the usual composition of 3-functors and natural transformations [Gur13].

Proof. The natural transformation , the modications m and p and the perturbations and in the state-ment of Theorem 3.4 are completely determined by the imposed conditions on their coecients. Moreover, the so dened coecients always assemble into natural transformations, modications, and perturbations, respectively, between the respective morphisms described in Corollary 3.5.

We now show that $3Cat_G^{st}$ is indeed a strict 2-category. Suppose we have two composable 1-morphisms $(A; A^1; A^2; ^A; ^A) \in 3Cat_G^{st}(D \to E)$ and $(B; B^1; B^2; ^B; ^B) \in 3Cat_G^{st}(C \to D)$. Then the formulas for the components for the composite $(A\ B; (A\ B)^2; (A\ B)^2; ^{AB}; ^{AB})$ are given by

$$(A\ B)_g^1 = A(B_g)^1\ A_g^1 \qquad \forall g \in G$$
$$(A\ B)^2_{x;y} = A(B^2_{x;y})\ A^2_{B(x);B(y)} \qquad \forall x \in C(h_C \to k_C); \forall y \in C(g_C \to h_C)$$
$$^{AB}_{x;y} = A(_{x;y})^B\ ^A_{B(x);B(y)} \qquad \forall x \in C(g_C \to k_C); \forall y \in C(h_C \to ‘_C)$$
$$^{AB}_1 = A(_1)^B_1;\ ^A$$

which are easily seen to be strictly associative and strictly unital. It is also straightforward to see that composition of 2-morphisms is strictly associative and strictly unital as well. □

3.1 Strictifying objects

In the following section, we prove the 'object part' of Theorem 3.4 and show that every object = ^{C} : BG ! C of the 4-category 3Cat$_G$ is equivalent to a strictly 1-bijective Gray-functor ^{0}: BG ! BC^0, where C^0 is a Gray-monoid whose set of 0-cells is G with composition the group multiplication. The following lemma is a direct consequence of Gurski's strictication of 3-categories [Gur13, Cor. 9.15].

Lemma 3.6. Any 1-surjective 3-functor : BG ! C is equivalent, in 3Cat$_G$, to a 1-surjective 3-functor ^{0} : BG ! BC^0 where C^0 is a Gray-monoid.

Proof. By [Gur13, Cor. 9.15], there is a Gray-category C$^0_0$ and a 3-equivalence C ! C$^0_0$. By 1-surjectivity of , it follows that the composite BG ! C ! C$^0_0$ factors through the full endomorphism Gray-monoid C^0 of C$^0_0$ on the single object in the image of the composite, resulting in a 3-functor ^{0}: BG ! BC^0 which is equivalent to : BG ! C in 3Cat$_G$. □

To further strictify : BG ! BC, we use the following direct consequence of a theorem of Buhne [Buh14]. Recall that a 3-functor F : A ! B between Gray-categories A and B is locally strict if the 2-functors F$_{a;b}$: A(a ! b) ! B(F(a) ! F(b)) are strict.

Proposition 3.7. Given Gray-monoids G; C and a locally strict 3-functor : BG ! BC, there exists a Gray-monoid C^0, an equivalence A : BC ! BC^0, a Gray-functor ^{0} : BG ! BC^0 and a natural isomorphism ^{0}) A .

Proof. By [Buh14, Thm. 8], every locally strict 3-functor from a (small) Gray-category into a cocomplete Gray-category is equivalent to a Gray-functor. Here, cocomplete is used in the sense of enriched category theory [Kel05, x3.2].

Given two Gray-categories A; B, we denote by [A; B] the Gray-category of Gray-functors A ! B. Consider the Gray-enriched Yoneda embedding y : BC ! [BC^{op}; Gray], where the target is cocomplete as Gray is cocomplete [Kel05, x3.3]. The composite

BG ! BC^{y} ! [(BC)^{op}; Gray]

is a composite of a locally strict 3-functor with a Gray-functor and hence itself locally strict. Therefore, there is a Gray-functor ^{0}: BG ! [(BC)^{op}; Gray] which is equivalent to the composite.

Now we dene BC^0 to be the full sub-Gray-category of [(BC)^{op}; Gray] on the object ^{0}() and dene ^{0}: BG ! BC^0 as the codomain-restriction of ^{0} to BC^0. Finally, observe that both the Gray-Yoneda embedding y : BC ! [(BC)^{op}; Gray] and the inclusion BC^0 ! [(BC)^{op}; Gray] are fully faithful[16] Gray-functors which map the single objects of BC and BC^0 to equivalent objects. Hence, there is an equivalence A : BC ! BC^0 and a natural isomorphism : ^{0}) A . □

Remark 3.8. In general, we cannot get rid of the local strictness assumption on by the example given in [Buh15, Ex 2.2].

Theorem 3.9 (Strictifying objects). Every object (C;) 2 3Cat $_G$ is equivalent to an object (BC^0; ^{0}) of the subcategory 3Cat$^{pt}_G$ where ^{0} : BG ! BC^0 is a strictly 1-bijective Gray-functor into a Gray-monoid C^0 whose set of 0-cells is G with composition the group multiplication.

Proof. Since BG is a 1-category, it follows from [Buh15, Cor 2.6] that every 3-functor BG ! C is equivalent to a locally strict 3-functor. Applying Proposition 3.7, we obtain a Gray-monoid D and a Gray-functor ^{D} : BG ! BD such that (BD; ^{D}) is equivalent to (C;) in 3Cat$_G$.

Let D^0 be the full 2-subcategory of D whose objects are exactly those in the image of ^{D}. Since ^{D} is a Gray-functor, D^0 is a Gray-submonoid of D, which comes equipped with the corestricted Gray-functor $^{D^0}$: BG ! BD^0 which is strictly 1-surjective, i.e., onto Ob(D^0). Since ^{D} is 1-surjective, (BD; ^{D}) is equivalent in 3Cat$_G$ to (BD^0; $^{D^0}$).

Since $^{D^0}$: BG ! BD^0 is a strictly 1-surjective Gray-functor, there is in particular a surjective homomorphism : G ! Ob(D^0). We dene a Gray-monoid C^0 as follows. The 0-cells of C^0 are the elements of G,

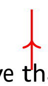
[16] Here, by a fully faithful Gray-functor we mean a Gray-functor F : A ! B whose induced 2-functors F$_{a;b}$: A(a ! b) ! B(F(a) ! F(b)) are isomorphisms in Gray.

and hom categories are given by $\mathrm{Hom}_{C^0}(g \to h) := \mathrm{Hom}_{D^0}((g) \to (h))$. Since is a homomorphism, C^0 inherits a Gray-monoid structure from D^0 together with an obvious strictly 1-bijective Gray-homomorphism $^0 : BG \to BC^0$. Since is surjective, $(BC^0; ^0)$ is equivalent to $(BD^0; ^{D^0})$ in $3\mathrm{Cat}_G$. □

3.2 Strictifying 1-morphisms

Given objects $(BC; ^C)$ and $(BD; ^D)$ in $3\mathrm{Cat}^{pt}_G$ comprised of Gray-monoids C and D and a strictly 1-bijective Gray-functor $(A;) \in 3\mathrm{Cat}_G \quad (BC; ^C) \to (BD; ^D)$, we construct a 1-morphism $(B;) \in 3\mathrm{Cat}^{pt}_G$ and an equivalence $(B;) \Rightarrow (A;)$. As $C; D$ are Gray-monoids, we make heavy use of the graphical calculus discussed in §2.2.

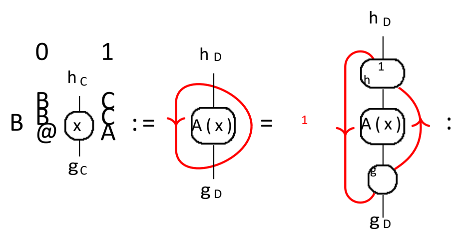
Recall that the 3-functor A consists of the data from Definition A.1. The invertible natural transformation $: ^D \Rightarrow A \quad ^C$ is comprised of the data from Definition A.2. We depict by an oriented red strand:



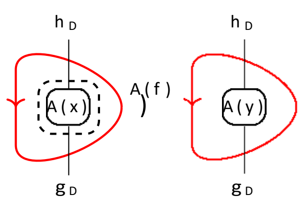
:

By the third unitality bullet point in (T-II), we have that $_{\mathrm{id}_g} = A^1_g$ since ^{C} is strict. By Remark 2.1, there is a contractible choice of ways to extend the invertible 0-cell to a biadjoint biequivalence (BB); we do so arbitrarily.

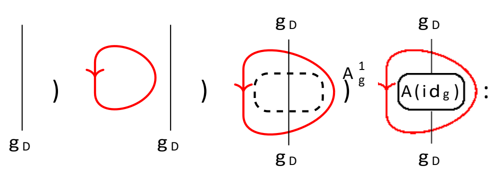
We now define $B : BC \to BD$ as follows. First, $B(g_C) := g_D$ for all $g \in G$. Given $x \in C(g_C \to h_C)$, we define



Given $x; y \in C(g_C \to h_C)$ and $f \in C(x \Rightarrow y)$, we define $B(f)$ to be the following 2-cell in D:

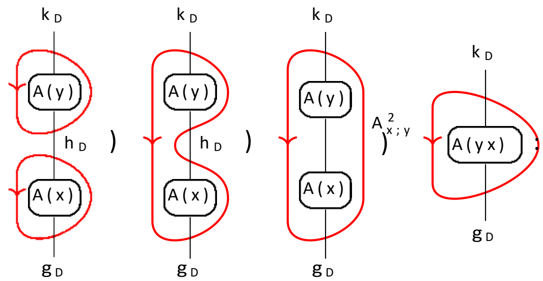


For $g \in G$, we define $B_g^1 \in D(\mathrm{id}_{g_D} \Rightarrow B(\mathrm{id}_{g_C}))$ to be the composite



(12)

For x 2 C(g_C ! h_C) and y 2 C(h_C ! k_C), we dene $B^2_{x;y}$ 2 D(B(y) B(x)) B(y x)) to be the composite

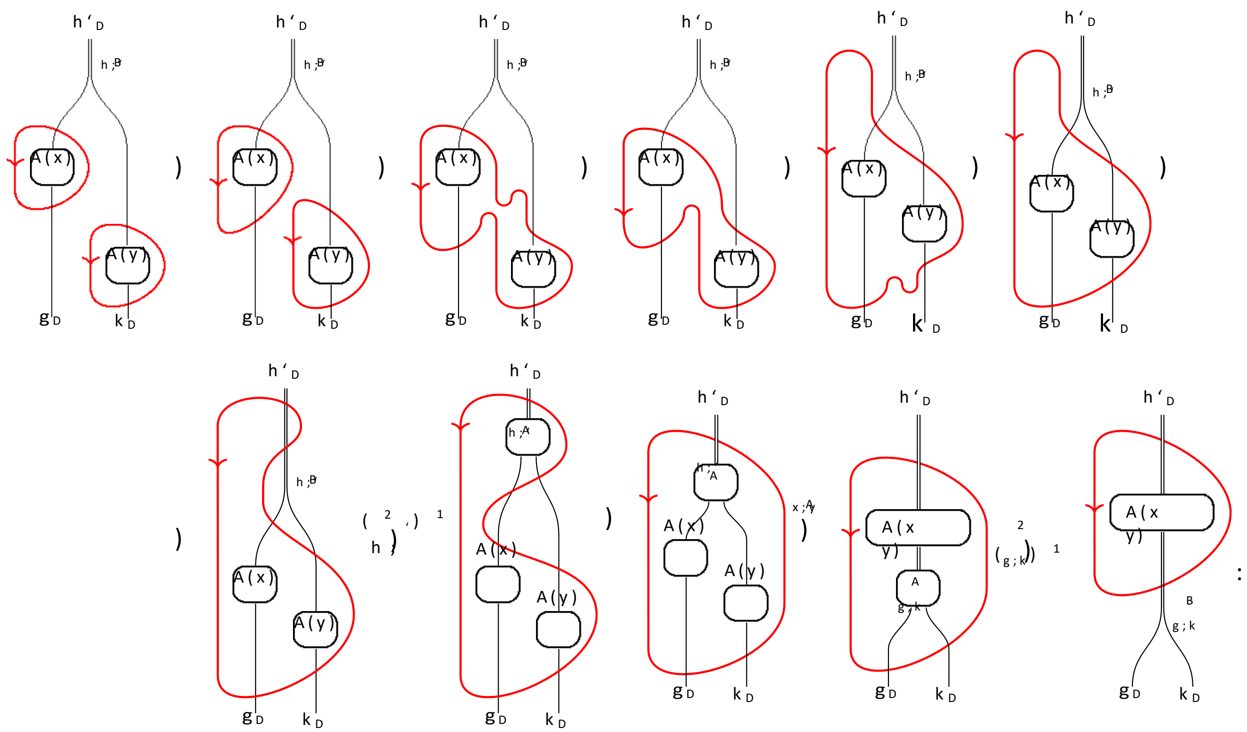


(13)

Lemma 3.10. The data (B; B^1; B^2) : C ! D denes a 2-functor. xB.1

We now endow B with the structure of a weak 3-functor BC ! BD.

Construction 3.11. We dene an adjoint equivalence ^{B} : $_D$ ^{B} (B B)) B $_C$ as follows. First, we dene $^B_{g;h}$ 2 D(g_D h_D ! gh_D) to be the identity. Next, for x 2 C(g_C ! h_C) and y 2 C(k_C ! '$_C$), we dene the natural isomorphism $_{x;y}$ 2 D($_{g;'}$ (B(x) B(y))) B(x y) $_{g;k}$) to be the composite



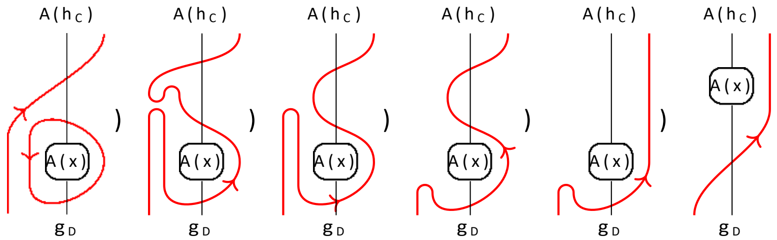
:

We dene an adjoint equivalence ^{B} = (^{B}; ^{B}) : 1_{I_D}) B I_C by ^{B} = id_e, and ^{B} := B^1_1 2 D(id_{e_D}) B(id_{e_C})) from (12). Finally, we dene the associator !^{B} and unitors '^{B}; r^B to be identities.

Lemma 3.12. The data (^{B}; ^{B}; !^{B}; '^{B}; r^B) endows B : BC ! BD with the structure of a weak 3-functor. xB.1

Lemma 3.13. The data = (:= e_D; $_g$:= id_g ; $_{id_g}$:= B^1_g; ^{1} := id_{id} ; ^{2} := id_{id}) : ^{D}) B ^{C} denes a natural isomorphism. xB.1

We now dene for $x \in C(g_C \to h_C)$ the 2-cell $_x$ given by



(14)

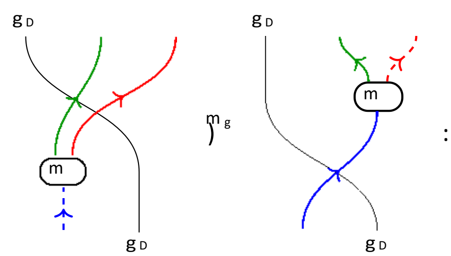
Theorem 3.14. The 1-morphisms (A;); (B;) 2 $3Cat_G((BC; {}^C) \to (BD; {}^D))$ are equivalent via the 2-morphism (; id) : (B;)) (A;) where = (:= ; $_g$:= $_{g;x}$; ^{1} := ^{1}; $^2_{g;h}$:= $^2_{g;h}$) : B) A is the natural isomorphism where $_x$ is given in (14) above. B.1

Remark 3.15. Working a bit harder, we can actually make (B;) strictly unital, i.e., $B(id_{g_C}) = id_{g_D}$ and $B^1_g = id_{g_D}$ for all $g \in G$. This has the following advantages: becomes trivial, $^B_{id_e;x_C} = id_{B(x)}$ for all $x \in C(g_C \to h_C)$ by (F-V), $^D = B\ ^C$ on the nose, and : ^{D}) B ^{C} is the identity transformation. Unfortunately, this would complicate our denition of the coherence data for B considerably, and it would further obfuscate the reasons why certain commuting diagrams commute in the sequel. Moreover, it has not yet been shown in the literature that every G-crossed braided functor is equivalent to a strictly unital one, although this would follow as a corollary of our main theorem. We are thus content to work with our (B;) with completely determined by B.

3.3 Strictifying 2-morphisms

Suppose $(BC; {}^C); (BD; {}^D) \in 3Cat^{pt}_G$ and $(A;); (B;) : (BC; {}^C) \to (BD; {}^D)$ are two 1-morphisms in $3Cat^{pt}_G$. Since (A;); (B;) are 1-morphisms in $3Cat^{pt}_G$, $A(g_C) = g_D = B(g_C)$ for all $g \in G$, and $= e_D =$ and $_g = id_{g_D} = {}_g$. Suppose (; m) :2 $3Cat_G((A;) \Rightarrow (B;))$. We prove that (; m) is equivalent to a 2-morphism (; id) 2 $3Cat^{pt}_G((A;) \Rightarrow (B;))$.

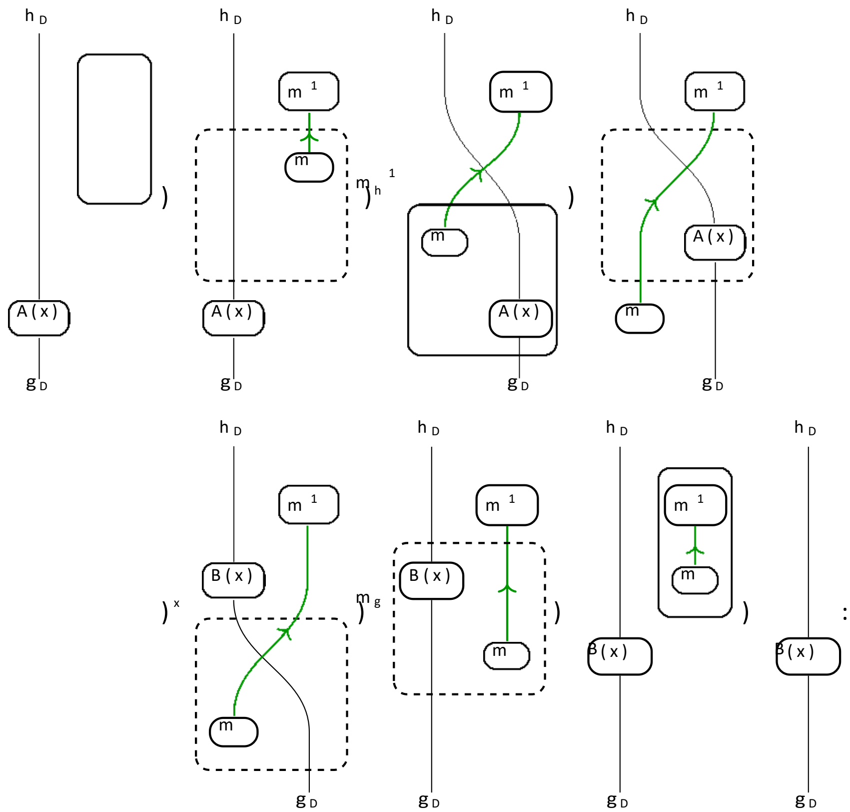
As in Dention A.2, we denote the 0-cell by an oriented green strand. The modication m = (m; m_g) as in Denition A.3 consists of an invertible 1-cell m :) together with coherent invertible 2-cells



(15)

Observe that since $= e_D =$ and $_g = id_g = {}_g$, we may completely omit the dashed lines in (15). As in Remark 2.1, we extend the invertible 1-cell $m \in D$ to an adjoint equivalence arbitrarily.

For $x \; 2 \; C(g_C \; ! \;\; h_C)$, we dene an invertible 2-cell $_x$ as the following composite:



(16)

We dene the unit map as in (T-III) by $^1 := \mathrm{id}_{\mathrm{id}_{e_D}}$ and the monoidal map as in (T-IV) by $_{g;h} \overset{2}{:=} \mathrm{id}_{\mathrm{id}_{gh_D}}$.

Lemma 3.16. The data $:= (= e; _g = \mathrm{id}_g; _x; ^1 := \mathrm{id}_{\mathrm{id}_e} \;\; ; _{g;h} := \mathrm{id}_{\mathrm{id}_{gh}} \;\;)$ together with the identity modication denes a 2-morphism $(; \mathrm{id}) \; 2 \; 3\mathrm{Cat}^{pt}_G((A;) \;\; (B;))$. xB.2

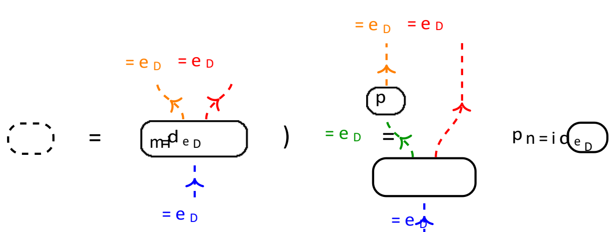
Observe now that by the strictness properties of and , $m : e_D$) . Erasing the dotted lines from (15) for m_g, we see that the same data as $m = (m; m_g)$ actually denes an invertible modication V !

Theorem 3.17. The 2-morphisms $(; m); (; \mathrm{id}) \; 2 \; 3\mathrm{Cat}_G((A;) \;\; (B;))$ are equivalent via the 3-morphism $(m; \mathrm{id}) \; 2 \; 3\mathrm{Cat}_G((; \mathrm{id}) \; V \;\; (; \mathrm{id}))$. xB.2

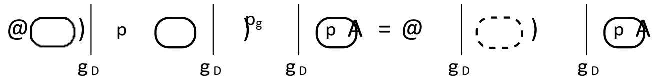
3.4 Strictifying 3-morphisms

Suppose now that $(; m = \mathrm{id}); (; n = \mathrm{id}) : (A;) \;\; (B;)$ are two 2-morphisms in $3\mathrm{Cat}^{pt}_G$ and $(p;) : (; \mathrm{id}) \; V \;\; (; \mathrm{id})$ is a 3-morphism in $3\mathrm{Cat}_G$.

First, since $(; \mathrm{id}); (; \mathrm{id})$ are 2-morphisms in $3\mathrm{Cat}^{pt}_G$, we have that $= e_D =$ and $_g = \mathrm{id}_{g_D} = _g$ for all $g \; 2 \; G$, and the modications are identities. This means the perturbation is a 2-cell



satisfying (P-1) in Denition A.4. We may thus view as an invertible 2-cell $\mathrm{id}_{\mathrm{id}_{e_D}}$) p, under which (P-1) becomes



$$8 g \in G. \quad (17)$$

Lemma 3.18. Any 3-morphism in $3\mathrm{Cat}_G$ between 2-morphisms in the subcategory $3\mathrm{Cat}_G^{pt}$ is an endomorphism. xB.3

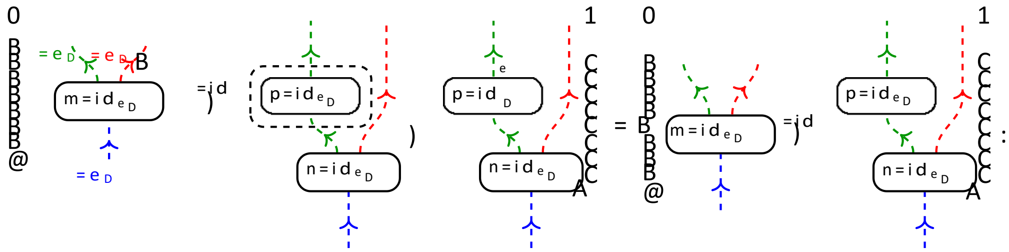
Theorem 3.19. Any 3-morphism in $3\mathrm{Cat}_G$ between 2-morphisms in the subcategory $3\mathrm{Cat}_G^{pt}$ is isomorphic to the identity 3-morphism.

Proof. First, by Lemma 3.18, every 3-morphism is a 3-endomorphism. Suppose (; id) is a 2-morphism in $3\mathrm{Cat}_G^{pt}$ and (p;) is a 3-endomorphism of (; id). As above, we may view as an invertible 2-morphism id_{e_D}) p that satises (17). This is exactly saying that is a perturbation $\mathrm{id}_{(;\mathrm{id})}$ ≡ (p;). □

3.5 Strictifying 4-morphisms

Theorem 3.20. The only 4-endomorphism in $3\mathrm{Cat}_G$ of an identity 3-morphism in the subcategory $3\mathrm{Cat}_G^{pt}$ is the identity.

Proof. Suppose is a 4-endomorphism of an identity 3-morphism (p = id; = id) in $3\mathrm{Cat}_G^{pt}$. Then satises the criterion (11), which in diagrams is



We conclude that = id. □

4 G-crossed braided categories

In x4.1 below, we dene the strict 2-category GCrsBrd of G-crossed braided categories. By [Gal17], GCrsBrd is equivalent to the full 2-subcategory $\mathrm{GCrsBrd}^{st}$ of strict G-crossed braided categories. In this section, we prove our second main theorem.

Theorem 4.1. The 2-category $3\mathrm{Cat}_G^{st}$ is equivalent to $\mathrm{GCrsBrd}^{st}$.

Proof. In x4.2 below, we construct a strict 2-functor $3\mathrm{Cat}_G^{st} \to \mathrm{GCrsBrd}^{st}$. In x4.3 below, we show this 2-functor is an equivalence. Indeed, the 2-functor is essentially surjective on objects by [Cui19] explained at the beginning of x4.3, essentially surjective on 1-morphisms by Theorem 4.21, and fully faithful on 2-morphisms by Theorem 4.22. We defer all further proofs in this section to Appendix C. □

We thus have the following zig-zag of strict equivalences denoted and an isomorphism =, where the hooked arrows denote inclusions of full subcategories.

$$3\mathrm{Cat}_G \xleftarrow{\text{Thm. } 3.4} 3\mathrm{Cat}_G^{pt} \xleftrightarrow[\text{Cor. } 3.5]{=} 3\mathrm{Cat}_G^{st} \xrightarrow{\text{Thm. } 4.1} \mathrm{GCrsBrd}^{st} \xhookrightarrow{[\text{Gal17}]} \mathrm{GCrsBrd}$$

4.1 Denitions

Let G be a group. We now give a denition of a (possibly non-additive) G-crossed braided category. Below, we give a denition in terms of the component categories C_g. When each component C_g is linear and the tensor product functors and G-action functors are linear, $C := \bigoplus_{g \in G} C_g$ is an ordinary G-crossed braided monoidal category in the sense of [EGNO15, x8.24] (except possibly not rigid nor fusion).

Denition 4.2. A G-crossed braided category C consists of the following data:

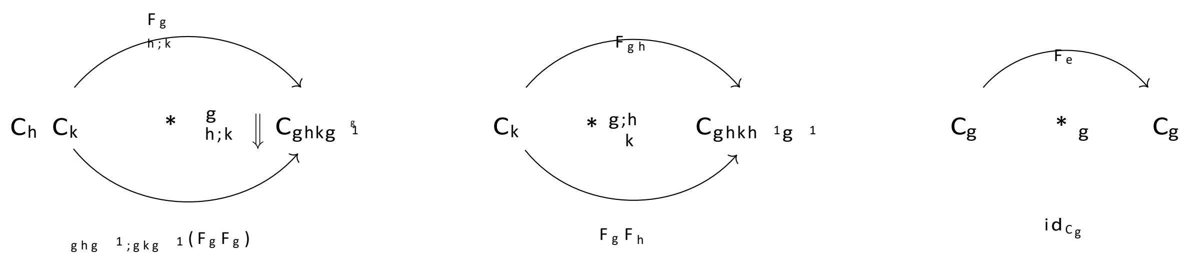
- a collection of categories $(C_g)_{g2G}$;
- a family of bifunctors $_{g;h} : C_g \; C_h \; ! \; C_{gh}$;
- an associator natural isomorphism $_{g;h;k}$: $_{gh;k}$ ($_{g;h}$ id_{C_k})) $_{g;hk}$ (id_{C_g} $_{h;k}$);
- a unit object 1_C 2 C_e;
- unitor natural isomorphisms : $_{e;g}$ (1_C)) id_{C_g} and : $_{g;e}$ (1_C)) id_{C_g}.

Using the convention

$$a_g \; b_h := {}_{g;h}(a_g \; b_h) \qquad 8\, a_g \; 2 \; C_g \text{ and } b_h \; 2 \; C_h;$$

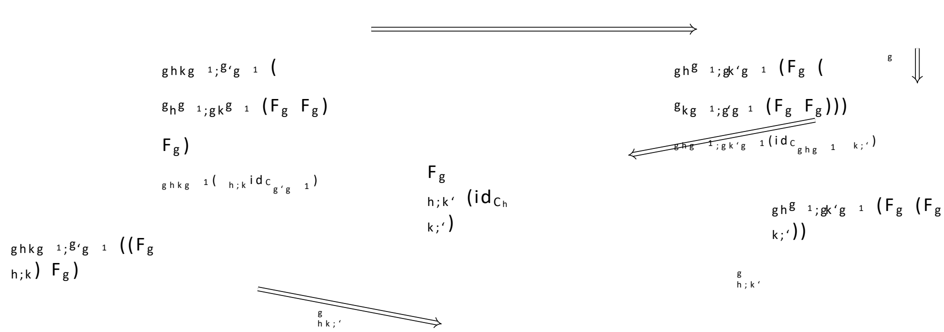
this data should satisfy the obvious pentagon and triangle axioms of a monoidal category.

Moreover, C is equipped with a G-action $F_g : C_h \; ! \; C_{ghg^{-1}}$ together with an isomorphism $i_g : 1_C \; ! \; F_g(1_C)$ and natural isomorphisms ^{g}, $^{g;h}$, and $_g$



which satisfy the following associativity and unitality conditions where we suppress whiskering:

(1) (associativity) The following diagram commutes:



(2) (unitality) For every a_h 2 C_h, the following diagram commutes:

e;ghg 1(i g id C h g g 1) $F_g(1_C)$

$F_g(a_h)$

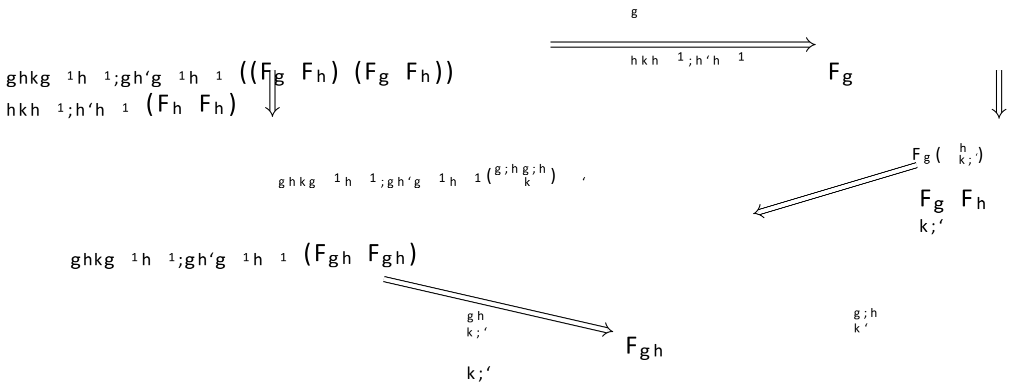
$F_g(a_h)$ g e;h

$F_g(a_h) \xleftarrow[F(a_h)]{g} F_g(1_C$

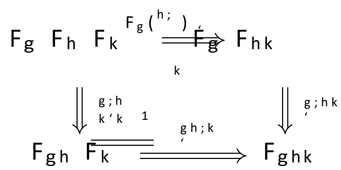
$a_h)$

as does a similar diagram where 1_C appears on the right with .

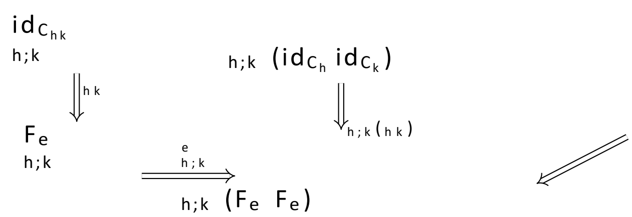
(1) (monoidality) The following diagram commutes:



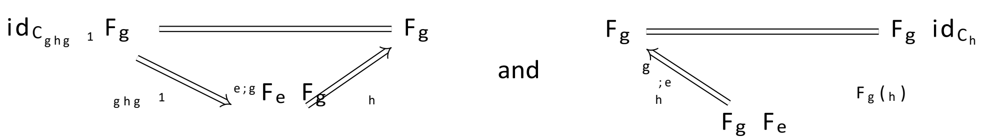
(2) (associativity) The following diagram commutes:



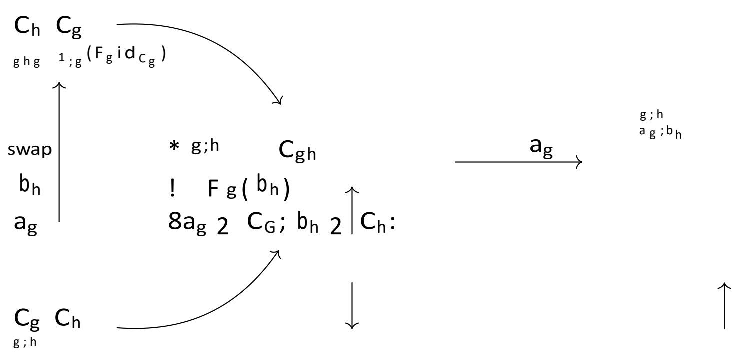
(1) (monoidality) The following diagram commutes:



(2) (unitality) The following diagrams commute:



Finally, we have the G-crossed braiding natural isomorphism



The G-action and G-crossed braiding are subject to the following coherence axioms taken from [EGNO15]. For all $a_g \in C_g$, $b_h \in C_h$, and $c_k \in C_k$, the following diagrams commute, where suppress all labels.

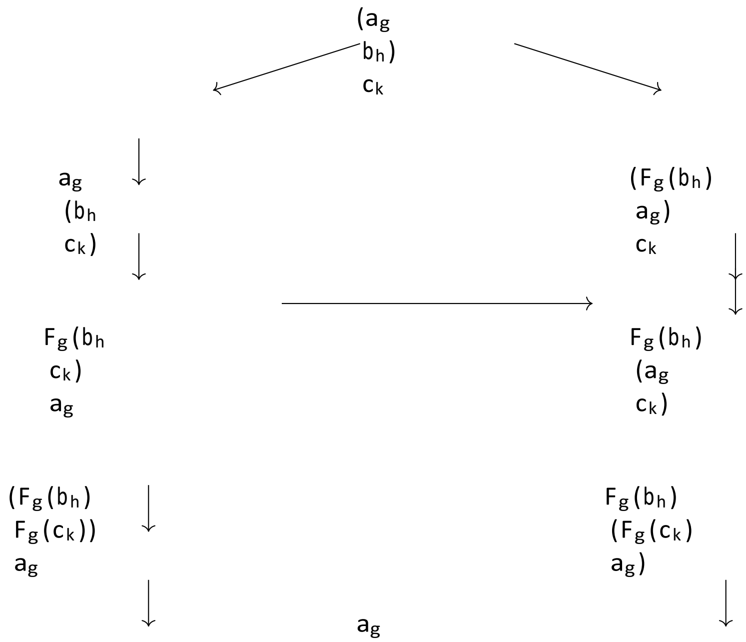
$F_{ghg\ _1}F_g(c_k)$
$F_g(b_h)$

$F_g(F_h(c_k)$
$b_h)$

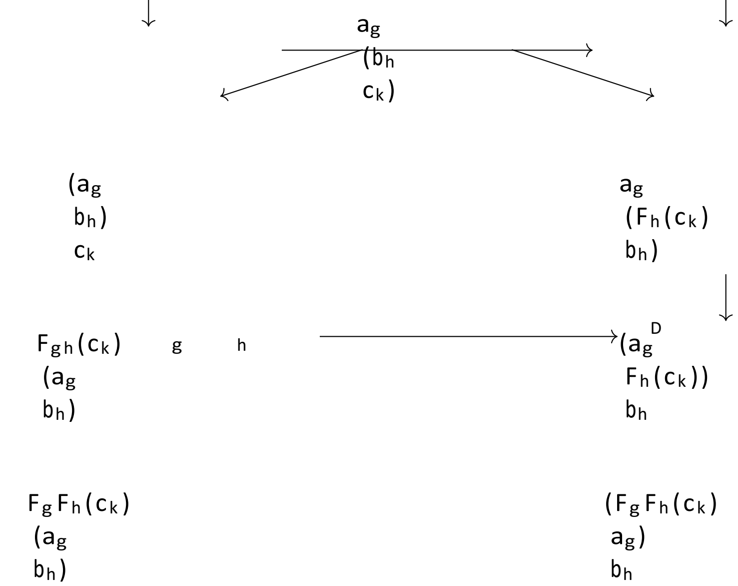
$\downarrow$

$F_{gh}(c_k)$
$F_g(b_h)$ (1)

$\longrightarrow F_g F_h(c_k)$
$F_g(b_h)$

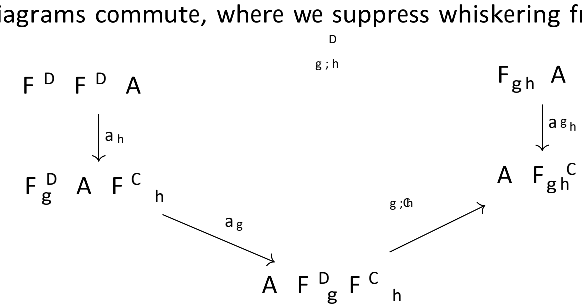


(2)

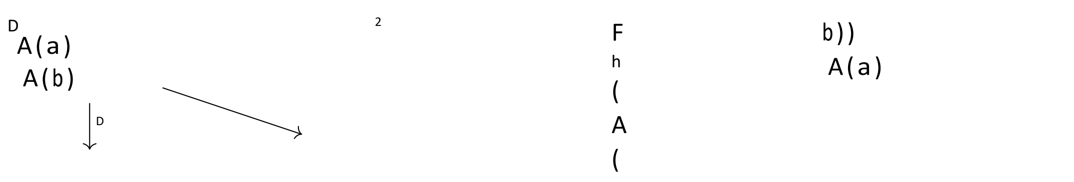


(3)

Definition 4.3. Given two G-crossed braided categories C and D, a G-crossed braided functor $(A, a) : C \to D$ consists of a family of functors $(A_g : C_g \to D_g)_{g\in G}$ together with a unitor isomorphism $A^1 : 1_D \to A(1_C)$ and a tensorator natural isomorphism $A^2_{a_g, b_h} : A(a_g) \otimes A(b_h) \to A(a_g \otimes b_h)$ for all $a_g \in C_g$ and $b_h \in C_h$ satisfying the obvious coherences. The monoidal functor $A = (A_g; A^1; A^2)$ comes equipped with a family $a = \{a_g : F^D_g \circ A \Rightarrow A \circ F^C_g\}_{g\in G}$ of monoidal natural isomorphisms such that for all $g, h \in G$, the following diagrams commute, where we suppress whiskering from the notation.



(1)



$A_{a;b}$ $\longrightarrow$ $A(a \otimes b)$

$A_{F^C(b);a}$ $\quad A(c^2)$ $\downarrow$

$A(F^C(b) \otimes a)$ $\nearrow$

$(\,^2)$

$A(F^C(b)) \otimes A(a)$

Definition 4.4. If $(A; a), (B; b) : \mathcal{C} \to \mathcal{D}$ are G-crossed braided functors, a G-crossed braided natural transformation $h : (A; a) \Rightarrow (B; b)$ is a monoidal natural transformation $h : A \Rightarrow B$ such that for all $g \in G$, the following diagram commutes.

$$\begin{array}{ccc} F_g^{\mathcal{D}} \circ A & \xrightarrow{F_g^{\mathcal{D}}(h_{(\cdot)})} & F_g^{\mathcal{D}} \circ B \\ \downarrow{a_g} & & \downarrow{b_g} \\ A \circ F_g^{\mathcal{C}} & \xrightarrow{h_{F_g^{\mathcal{C}}(\cdot)}} & B \circ F_g^{\mathcal{C}} \end{array} \qquad (18)$$

It is straightforward to verify that G-crossed braided categories, functors, and natural transformations assemble into a strict 2-category called GCrsBrd with familiar composition formulas similar to those from the strict 2-category of monoidal categories. (See the proof of Proposition 4.13 in Appendix C for full details.)

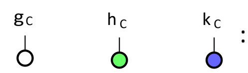
Denition 4.5 (Adapted from [Gal17, p.6]). A G-crossed braided category is called strict if ; ; are all identities, and all i_g, ^{g}, $^{g;h}$, and ^{g} are identities. Observe this implies that F_e is the identity as well.

By the main theorem of [Gal17], every G-crossed braided category is equivalent (via a G-crossed braided functor which is an equivalence of categories) to a strict G-crossed braided category. In particular, the 2-category GCrsBrd is equivalent to the full subcategory $\mathrm{GCrsBrd}^{st}$ of strict G-crossed braided categories.

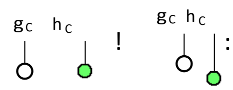
4.2 A strict 2-functor $3\mathrm{Cat}_G^{st}$ to $\mathrm{GCrsBrd}^{st}$

In this section, we construct a strict 2-functor $3\mathrm{Cat}_G^{st}$! $\mathrm{GCrsBrd}^{st}$. We begin by explaining how to obtain a strict G-crossed braided category C from an object C 2 $3\mathrm{Cat}_G^{st}$, i.e., C is a Gray-monoid with 0-cells $fg_C g_{g2G}$ with 0-composition the group multiplication.

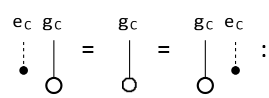
Construction 4.6. For each g 2 G, we dene the category $C_g := C(1_C$! $g_C)$. We denote 1-cells in C_g by small disks. For better readability, we distinguish dierent 1-cells in a given diagram by dierent shadings of the corresponding disks. We will use the shorthand notation that white, green, and blue shaded disks correspond to 1-cells into g_C; h_C; and k_C, respectively:



We dene the bifunctor
$_{g;h}$: C_g C_h ! C_{gh} by
:

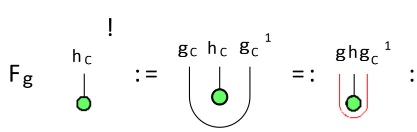


The associator
$_{gh;k}$ (
$_{g;h}$ C_k) ^{g})
$_{g;hk}$ (C_g
$_{h;k}$) is the identity. The unit object 1_C := id_e 2 C_e, which we denote by a univalent vertex attached to a dashed string. The unitors
$_{e;g}$ (i)) id_C and
$_{g;e}$ (i)) id_{C_g} are also identities



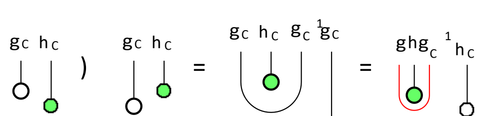
Clearly the associators and unitors satisfy the obvious pentagon and triangle axioms of a G-crossed braided category.

Construction 4.7 (G-action). We dene a G-action F_g : C_h ! $C_{ghg^{-1}}$ by



On the right hand side, we abbreviate this 'cup' action by a single g-labelled red cup drawn under the respective node. The functors F_g are strict tensor functors, i.e., the tensorators ^{g} :
$_{ghg^{-1};gkg^{-1}}$ $(F_g F_g)$) F_g
$_{h;k}$ are identity natural isomorphisms. The tensorator $_{g;h}$: F_g F_h) F_{gh} and the unit map $_h$: id_C ! F_e are also both identities. It is straightforward to see that these identity natural isomorphisms
^{g}, $^{g;h}$, and $_h$ satisfy (1), (2), (1), (2), (1), (2).

Construction 4.8 (G-crossed braiding). The G-crossed braiding natural isomorphisms $^{g;h}$ are given by interchangers in C:



Theorem 4.9. The data $(\mathcal{C}; \ {}_{g;h}; F_g; {}^{g;h})$ from Constructions 4.6, 4.7, and 4.8 forms a strict G-crossed braided category. §C.1

Now suppose that $\mathcal{C}; \mathcal{D} \in 3\mathrm{Cat}^{st}_G$ and $A \in 3\mathrm{Cat}^{st}_G(\mathcal{C} \to \mathcal{D})$ This means $A(g_{\mathcal{C}}) = g_{\mathcal{D}}$ on the nose for all $g \in G$, the adjoint equivalence ^{A} : ^{A} ${}_{\mathcal{D}}$ $(A \ A) \Rightarrow A$ ${}_{\mathcal{C}}$ satisfies ${}_{g;h} = \mathrm{id}_{gh} \in \mathcal{D}(g_{\mathcal{D}}$ ^{1} ${}_{\mathcal{D}}$ ^{1} ${}_{e}$ ${}_{\mathcal{D}}$ ${}_{\mathcal{C}}$ $h_{\mathcal{D}} \to gh_{\mathcal{D}})$, the adjoint equivalence ${}^A : ({}^A; {}^A) : I_{\mathcal{D}} \Rightarrow A \ I_{\mathcal{C}}$ satisfies ${}^A = \mathrm{id}_e$, and ${}^A := A^1 \in \mathcal{D}(\mathrm{id}_e \ ) \ B(\mathrm{id}_e \ ))$, and the associators and unitors $!; `; r$ are identities.

Let C and D be the strict G-crossed braided categories obtained from $\mathcal{C}$ and $\mathcal{D}$ respectively from Theorem 4.9. We now define a G-crossed braided functor $(A; a) : C \to D$.

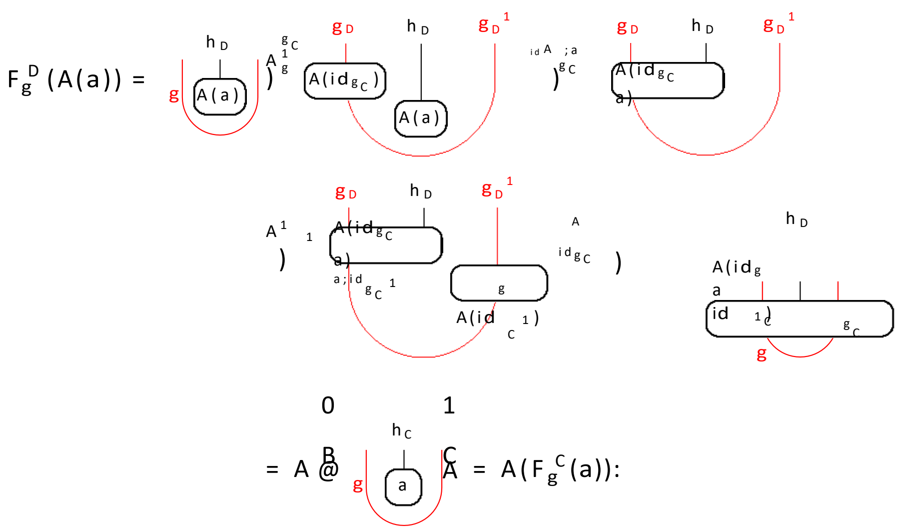
Construction 4.10. First, for $a \in C_g := \mathcal{C}(e_{\mathcal{C}} \to g_{\mathcal{C}})$, we define $A(a) := A(a) \in \mathcal{D}(e_{\mathcal{D}} \to g_{\mathcal{D}}) = D_g$. For $x \in C_g(a \to b)$, we define $A(x) := A(x) \in D_g(A(a) \to A(b))$. It is straightforward to verify A is a functor. We now endow A with a tensorator. For $a \in C_g$ and $b \in C_h$, we define $A^2_{a;b} \in D(A(a) \ A(b) \to A(a \ b))$ to be

$${}^A_{a;b} : (A(a) \ \mathrm{id}_h) \ (\mathrm{id}_e \ A(b)) \Rightarrow A((a \ \mathrm{id}_h) \ (\mathrm{id}_e \ b)):$$

We define the unitor by $A^1 := A^1_e \in \mathcal{D}(1_{\mathcal{D}} \to A(1_{\mathcal{C}})) = \mathcal{D}(\mathrm{id}_{e_{\mathcal{D}}} \to A(\mathrm{id}_{e_{\mathcal{C}}}))$.

Lemma 4.11. The data $(A; A^1; A^2) : C \to D$ is a G-graded monoidal functor. §C.1

We now construct the compatibility a between the G-actions on C and D. For $a \in C_h = \mathcal{C}(1_{\mathcal{C}} \to h_{\mathcal{C}})$, we define $a^a_g : F^D_g(A(a)) \Rightarrow A(F^C_g(a))$ using the tensorator ^{A}:



(19)

Theorem 4.12. The data $(A; A^1; A^2; a)$ is a G-crossed braided monoidal functor. §C.1

Proposition 4.13. The map $(A; {}^A; {}^A) \mapsto (A; a)$ strictly preserves identity 1-morphisms and composition of 1-morphisms. §C.1

Suppose $\mathcal{C}; \mathcal{D} \in 3\mathrm{Cat}^{st}_G$, $A; B \in 3\mathrm{Cat}^{st}_G(\mathcal{C} \to \mathcal{D})$, and $\ \in 3\mathrm{Cat}^{st}_G(A \Rightarrow B)$. This means that $\ = e_{\mathcal{D}}$ and ${}_g = \mathrm{id}_{g_{\mathcal{D}}}$ for all $g \in G$. Let $C; D$ be the G-crossed braided categories obtained from $\mathcal{C}; \mathcal{D}$ respectively from Theorem 4.9. Let $(A; a); (B; b) : C \to D$ be the G-crossed braided functors obtained from $A; B$ respectively from Theorem 4.12,

Construction 4.14. We define $h : (A; a) \Rightarrow (B; b)$ by $h_a := {}_a \in \mathcal{D}(A(a) \Rightarrow B(a))$ for $a \in C_g = \mathcal{C}(1_{\mathcal{C}} \to g_{\mathcal{C}})$.

Theorem 4.15. The data h defines a G-crossed braided natural transformation $(A; a) \Rightarrow (B; b)$. §C.1

Theorem 4.16. The map $\mathcal{C} \mapsto (C; \ {}_{g;h}; F_g; {}^{g;h})$, $(A; {}^A; {}^A) \mapsto (A; a)$, $\ \mapsto h$ is a strict 2-functor $3\mathrm{Cat}^{st}_G \to \mathrm{GCrsBrd}$.

Proof. By Proposition 4.13, we saw that this candidate 2-functor strictly preserves identity 1-morphisms and composition of 1-morphisms. It remains to prove that the map ! h preserves identities and 2-composition. This is immediate from Construction 4.14 as ( )a = a a as 3Cat^{s}t is strict. $_{G}$ □

4.3 The 2-functor is an equivalence

We now show our 2-functor $3\mathrm{Cat}_G^{st} \to G\mathrm{CrsBrd}^{st}$ constructed in §4.2 is an equivalence.

Essential surjectivity on objects. We begin by showing essential surjective, applying the techniques from [Cui19]. Suppose C is a strict G-crossed braided category. We define a Gray-monoid $\mathcal{C} \in 3\mathrm{Cat}_G^{st}$ as follows. The 0-cells of $\mathcal{C}$ are simply the elements of G, and 0-composition is the group multiplication. For $g, h \in G$, we define the hom category $\mathcal{C}(g \to h) := C_{hg^{-1}}$. These hom categories comprise a strict 2-category by defining vertical composition by the tensor product in C, i.e., if $a \in \mathcal{C}(g \to h) = C_{hg^{-1}}$ and $b \in \mathcal{C}(h \to k) = C_{kh^{-1}}$, we define

$$b \circ_{\mathcal{C}} a := b \otimes_C a = \mu_{kh^{-1}, hg^{-1}}(b \otimes a).$$

It is straightforward to verify that $\mathcal{C}$ is a strict 2-category by strictness of the associator and unitor of C.

We now endow $\mathcal{C}$ with a monoidal product and interchanger. We define the monoidal product with identity 1-morphisms as follows. Given $a \in \mathcal{C}(g \to h) = C_{hg^{-1}}$ and $k \in G$, we set

$$a \otimes \mathrm{id}_k := a \in C_{hg^{-1}} = C_{hkk^{-1}g^{-1}} = \mathcal{C}(gk \to hk);$$

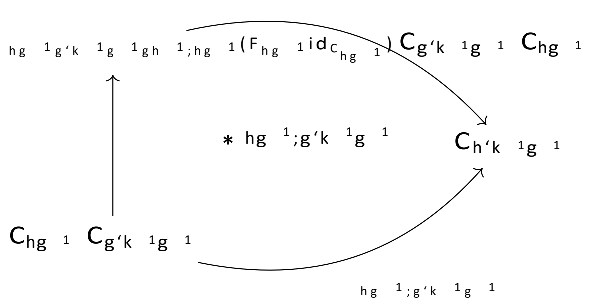
i.e., tensoring on the right with id_k does nothing. Tensoring on the left, however, implements the G-action:

$$\mathrm{id}_k \otimes a := F_k(a) \in C(kgh^{-1}k^{-1}) = \mathcal{C}(hg \to kh).$$

The interchanger ϕ is given by the G-crossed braiding. In more detail, given $a \in \mathcal{C}(g \to h) = C_{hg^{-1}}$ and $b \in \mathcal{C}(k \to \ell) = C_{\ell k^{-1}}$, we define

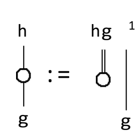
$$\phi_{a,b} := \beta^{hg^{-1}, g\ell k^{-1}\ell^{-1}}_{a,b} \in \mathcal{C}((a \otimes \mathrm{id}_\ell) \circ (\mathrm{id}_g \otimes b) \Rightarrow (\mathrm{id}_h \otimes b) \circ (a \otimes \mathrm{id}_k))$$

Indeed, since C is strict, $F_{hg^{-1}} = F_h F_{g^{-1}}$ on the nose, and $\beta^{hg^{-1}, g\ell k^{-1}g^{-1}}$ is a natural isomorphism

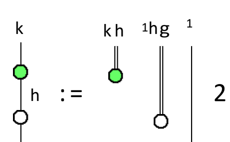


.

Notation 4.17. In the graphical calculus, one can think of a 1-cell in $\mathcal{C}(g \to h)$ as a 1-cell in $\mathcal{C}(1_C \to hg^{-1})$ with a g-strand on the right hand side, which does nothing.



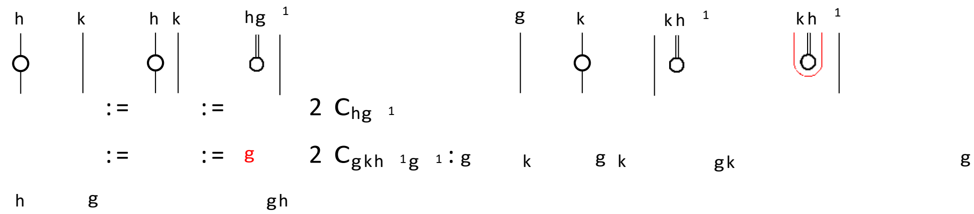
Vertical composition is then given by



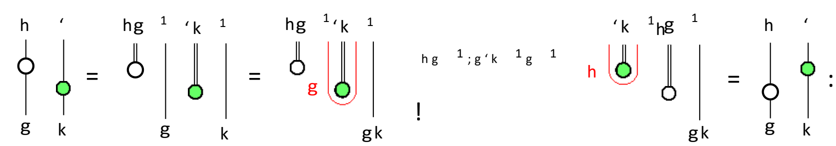
$$_{kh^{-1};hg^{-1}}(C_{kh^{-1}} \; C_{hg^{-1}}) = C_{kg^{-1}}:$$

g g

Tensoring by an identity strand on the right adds a strand on the right which does nothing, whereas tensoring by an identity on the left implements the G-action.[17]



That the interchanger is given by the G-crossed braiding can now be represented graphically by



One then checks that C dened as above is a Gray-monoid and thus denes an object of $3Cat^{st}_G$. Indeed, the verication is entirely similar to [Cui19] (see also [DR18, Construction 2.1.23]). Moreover, applying our construction from Theorem 4.9 to the so dened C, recovers the G-crossed braided category C on the nose. Hence, the strict 2-functor $3Cat^{st}_G \to GCrsBrd^{st}$ is in fact strictly surjective on objects.

Essential surjectivity on 1-morphisms. Let C; D be the G-crossed braided categories obtained from C; D 2 $3Cat^{st}_G$ respectively from Theorem 4.9, and suppose (A; a) : C ! D is a G-crossed braided functor. We now construct an A 2 $3Cat^{st}_G$(C ! D) which maps to (A; a) under Construction 4.10 and (19).

Construction 4.18. First, we must have $A(g_C) = g_D$ for all g 2 G. Recall that we have an isomorphism of categories $C(g_C \to h_C) = C(1_C \to hg^{-1})$ given by the strict 2-functor R_{g_C} $^{-1}$ = g ^{1}. For a 1-cell x 2 $C(g_C \to h_C)$, we dene A(x) := A(x g ^{1}) g g_D, and similarly for 2-cells f 2 C(x) y). We dene the unitor

$A^1_g := A_e$
g_D 2 $D(id_{g_D}$) $A(id_{g_C}) = A(id_{e_C})$
g_D);

and for x 2 $C(g_C \to h_C)$ and y 2 $C(h_C \to k_C)$, the compositor $A^2_{y;x}$ as the composite

A(y) A(x) = A(A(y
h_C ^{1}) C C
h_D) (A(x
g_C ^{1})
g_D)
= (A(y
h_C ^{1}) C C C
hg_D ^{1} C C
g_D) (A(x
g_C ^{1})
g_D) = ((A(y
h_C ^{1})
hg_D ^{1}) (A(x
g_C ^{1}))
g_D
= ((A(y
h_C ^{1})
A(x
g_C ^{1}))
g_D

! (A((y h ^{1}) (x g ^{1})) g_D = (A ((y h 1 h g 1)

(x
g 1))
g_D = (A((y
g 1) (x
g 1))
g_D
= (A((y x)
g 1)
g_D
= A(y x):

C

g_D strict
Nudging (8)

Nudging (8)

g_C^{1} strict

Lemma 4.19. The data $(A; A^1; A^2)$ denes a 2-functor C ! D such that $A(g_C) = g_D$ for all g 2 G. xC.2

Construction 4.20. The adjoint equivalence ^{A} :

A

$_D$ (A A)) A

$_C$ is dened as follows. First, $_{g;h}$:= id_{gh_D} 2 D(g_D

h_D) gh_D). For x 2 C(g_C ! h_C) and y 2 C(k_C ! '$_C$), we dene the natural

[17] This graphical calculus is analogous to diagrams for endomorphisms of a von Neumann algebra or a Cuntz C-algebra [Izu17, x2] where adding a strand labelled by an endomorphisms of a von Neumann algebra on the right does nothing, and adding a strand on the left implements the action of that endomorphism.

isomorphism $_{x;y}^{A}$ 2 D(A(x)
A(y)) A(x
y)) by the composite

A(x)
A(y) = A(x
$g_C{}^{1}$))
g_D
A(y
$k_C{}^{1}$))
k_D

= A(x D
$g_C{}^{1}$)) D
F_g (A(y C C
$k_C{}^{1}$)))
g_D
k_D = A(x
g ^{1}))
F_g (A(y
k ^{1})))
gk_D
! A(x g C
$g_C{}^{1}$))
A(F ^{C}(y
k ^{1})))
gk_D
$^{A\ 2}$! A(x C
$g_C{}^{1}$
F_g (y
$k_C{}^{1}$))
gk_D
= A(x g g
F ^{C} $_{1}$F ^{C}(y C
$k_C{}^{1}$)
$g_C{}^{1}$)
gk_D = A(x
F_e (y
$k_C{}^{1}$)
$g_C{}^{1}$)
gk_D
= A(x
y
$k_C{}^{1}$
$g_C{}^{1}$)
gk_D = A(x
y
$(gk)_C{}^{1}$)
gk_D
= A(x
y):

The adjoint equivalence ^{A} = ($^{!}$; ^{A})$_{1}$: I_D) A I_C is dened by ^{A} := id_e , ^{A} := A_1^1. The associator ! ^{A} and the unitors $\ell^A; r^A$ are all dened to be identities.

Theorem 4.21. The data $(A; {}^A; {}^A)$ denes a 1-morphism in $3Cat^{st}(C_G \to D)$. x C.2

Finally, we observe that the G-crossed braided functor constructed from A in Theorem 4.12 is exactly (A; a) by construction. Indeed, the strict 2-functors e^C and e^D are the identity on the nose. Hence, $3Cat^{st}_G \to GCrsBrd^{st}$ is in fact surjective on 1-morphisms on the nose.

Fully faithfulness on 2-morphisms. For $C; D \in 3Cat^{st}_G$, $A; B \in 3Cat^{st}_G(C \to D)$, and $\in 3Cat^{st}_G(A \Rightarrow B)$, let $C; D$ be the G-crossed braided categories obtained from $C; D$ respectively from Theorem 4.9, and let $(A; a); (B; b) : C \to D$ be the G-crossed braided functors obtained from $A; B$ respectively from Theorem 4.12. In Construction 4.14 we dened $h : (A; a) \Rightarrow (B; b)$ by $h_a := {}_a \in D(A(a) \Rightarrow B(a))$ for $a \in C_g = C(1_C \to g_C)$.

Theorem 4.22. The map $\mapsto h$ is a bijection $3Cat^{st}_G(A \Rightarrow B) \to GCrsBrd^{st}(A \Rightarrow B)$. x C.2

5 Induced properties and structures

Theorem A constructs an equivalence between G-crossed braided categories and 1-surjective G-pointed 3-categories. In this section, we investigate how various additional structures and properties of 3-categories, such as linearity and dualizability translate into the corresponding properties of G-crossed braided categories. Let ${}^C : BG \to C$ be a 1-surjective G-pointed 3-category and let $\{C_g\}_{g \in G}$ be the corresponding G-crossed braided category constructed via Theorem A.

The rst result below is immediate.

Proposition 5.1 (Linearity). If C is a linear 3-category, then $C := \bigoplus_{g \in G} C_g$ is a G-crossed braided category in these sense of [EGNO15, x8.24]. □

Following the conventions in [DSPS13, Defs. 2.1.1, 2.1.2, 2.1.4], given a 1-morphism $f : c \to d$ in a 2-category, we write $(f^L : d \to c; ev_f : f^L \otimes f \Rightarrow id_c; coev_f : id_d \Rightarrow f \otimes f^L)$ for the left adjoint of f and $(f^R : d \to c; {}_f ev : f \otimes f^R \Rightarrow id_d; {}_f coev : id_c \Rightarrow f^R \otimes f)$ for its right adjoint. Given an object x in a monoidal category M, we write $(x^-; ev_x : x^- \otimes x \to 1_M; coev_x : 1_M \to x \otimes x^-)$ for the right dual of x, and $({}^-x; {}_x ev : x \otimes {}^-x \to 1_M; {}_x coev : 1_M \to {}^-x \otimes x)$ for the left dual of x.

Recall that a braided monoidal category has right duals if and only if it has left duals.

Lemma 5.2. A G-crossed braided monoidal category $C = \bigoplus_{g \in G} C_g$ has right duals if and only if it has left duals.

Proof. We prove that having right duals implies having left duals; the other direction is analogous. Suppose $x \in C_g$ has a right dual $(x^- \in C_{g^{-1}}; ev_x : x^- \otimes x \to 1_C; coev_x : 1_C \to x \otimes x^-)$. Then, ${}^-x := g^{-1}(x^-)$ is a left dual with the following evaluation and coevaluation morphism:

$$ {}_x ev := ev_x \circ (\chi_{g;g^{-1}}^{x} \otimes id_{x^-}) \circ (\chi_{x;g^{-1}(x^-)}) : x \otimes g^{-1}(x^-) \to 1_C $$

$$ {}_x coev := \chi_{g^{-1}(x^-);x} \circ \chi_{x;x^-} \circ g^{-1}(coev_x) : 1_C \to {}^-x \otimes x $$

That these maps satisfy the zig-zag/snake equations is straightforward. □

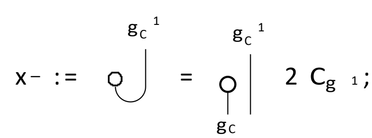
Remark 5.3. Similar to Lemma 5.2, every 2-morphism between invertible 1-morphisms (or more generally, between fully dualizable 1-morphisms) in a 3-category has a right adjoint if and only if it has a left adjoint [Reu19, Prop. A.2].

Proposition 5.4 (Rigidity). Suppose C is linear so that Proposition 5.1 holds. If every 2-morphism in $C(\mathcal{C}(e) \to \mathcal{C}(g))$ has either a right or a left adjoint (and thus necessarily both by Remark 5.3) for all $g \in G$, then the G-crossed braided linear monoidal category C is rigid.

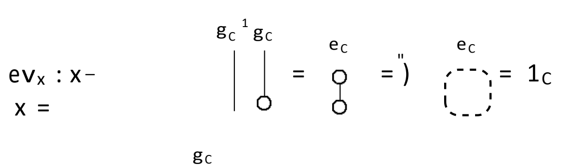
Proof. As the statement and the assumptions in this proposition are invariant under equivalences in $3Cat_G$ and GCrsBrd respectively, we may assume that C is an object of $3Cat_G^{st}$, and hence the delooping BA of a Gray-monoid A whose set of 0-cells is $\{g_A\}_{g \in G}$ with 0-composition the group multiplication, and that C is the strict G-crossed braided category obtained from Constructions 4.6–4.8.

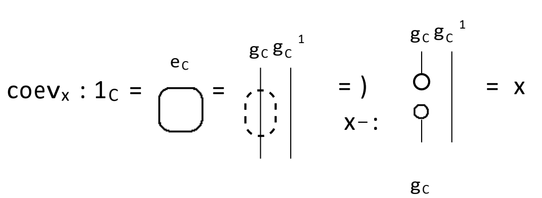
By Lemma 5.2 it suces to prove that for every $g \in G$, every object $x \in C_g$ (given by a 1-cell $x : 1_A \to g_A$ in the strict 2-category A) has either a right dual or a left dual. We assume the underlying 1-cell $x : 1_A \to g_A$ has a left adjoint $x^L : g_A \to 1_A$ in the 2-category A and prove that the corresponding object $x \in C_g$ has a right dual in the monoidal category C. We use the shorthand notation

Setting

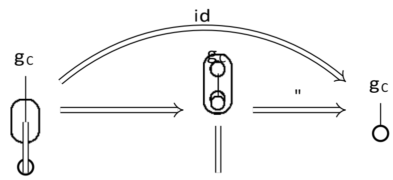


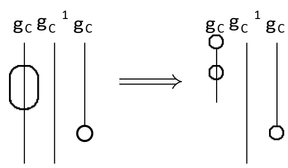
it is a direct consequence of the adjunction between x and x^L (here denoted $\varepsilon : x^L \otimes x \Rightarrow id_{e_C}$ and $\eta : id_{g_C} \Rightarrow x \otimes x^L$) that the following evaluation and coevaluation morphisms exhibit x^- as a right dual of x:





We explicitly prove the relation $(\mathrm{id}_x$ $\mathrm{ev}_x)$ $(\mathrm{coev}_x$ $\mathrm{id}_x) = \mathrm{id}_x$; the other relation is left to the reader.





In the diagram above, the composite $(\mathrm{id}_x$ $\mathrm{ev}_x)$ $(\mathrm{coev}_x$ $\mathrm{id}_x)$ is the path going down and then to the right. The square commutes as both maps are identical. The triangle commutes by the adjunction. □

Remark 5.5. There is a version of Proposition 5.4 that holds in the non-linear setting; one must be careful to dene the correct notion of duals.

The following propostion is also immediate.

Proposition 5.6 (Multifusion). Suppose C is as in the hypotheses of Proposition 5.4 so that C is rigid linear monoidal. If each 2-morphism category $C(^C(e) \,!\, ^C(g))$ is semisimple, then C is multifusion. If moreover the 2-morphism $\mathrm{id}_{C(e)} : {}^C(e) \,!\, {}^C(e)$ is simple, then C is fusion. □

Since the fusion 2-categories of [DR18] satisfy the hypotheses of Proposition 5.6, we get the following corollary.

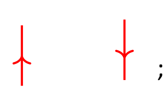
Corollary 5.7. If C is a fusion 2-category in the sense of [DR18] and : G ! C is a monoidal 2-functor which is essentially surjective on objects, then C is a G-crossed braided fusion category. □

Remark 5.8 (Unitarity). We dene a dagger structure on a Gray-monoid C in terms of the unpacked Denition 2.4 above. We require the strict 2-category C to be a dagger 2-category, all 2-functors to be dagger 2-functors, and all isomorphisms to be unitary. Similarly, one can dene the notion of a C or W Gray-monoid. Given a dagger Gray-monoid C and an appropriately compatible G action on C (all actions are by dagger functors and all isomorphisms are unitary), we expect our construction will yield a G-crossed braided dagger category. We expect analogous results in the C and W settings. However, the notion of dagger Gray-monoid is not compatible with weak equivalences and Gray-ication. These notions merit further study.

A Functors and higher morphisms between Gray-monoids

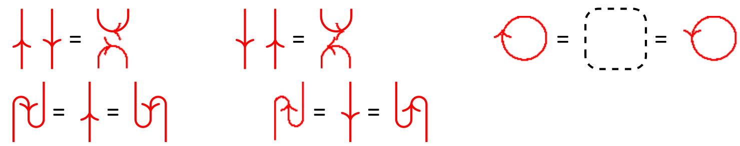
In this section, we unpack the denitions of trihomomorphism, tritransformation, trimodiciation, and perturbation of [Gur13, Def 4.10, 4.16, 4.18, 4.21] between two Gray-monoids in terms of the graphical calculus.

We remind the reader that as in Notation 2.3, given a Gray-monoid C, we refer to its objects, 1-morphisms, and 2-morphisms as 0-cells, 1-cells, and 2-cells respectively in order to distinguish these basic components of C from morphisms in an ambient category in which C lives. The notion of adjoint equivalence in a 2-category is well-known, so we will not unpack it further. A biadjoint biequivalence [Gur12] 0-cell in a Gray-monoid consists of 0-cells ; [1] which we depict in the graphical calculus as oriented red strands:



and cup and cap 1-morphisms

together with 2-isomorphisms



(BB)

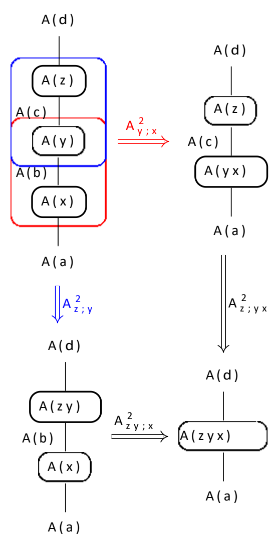
fullling certain coherence conditions; see [Gur12, Def. 2.1, Rem. 2.2, Def. 2.3].

A.1 3-functors between Gray-monoids

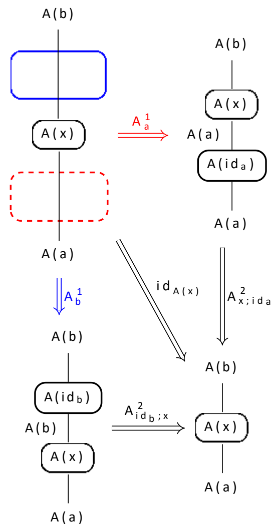
Denition A.1. Suppose C; D are Gray-monoids. A 3-functor A : BC ! BD consists of:

(F-I) A 2-functor $(A; A^1; A^2)$: C ! D. That is, a function on globular sets A, an invertible 2-cell A^1_c : $id_{A(c)}$) A(id_c), and an invertible 2-cell $A^2_{y;x}$: A(y) A(x)) A(y x), which satisfy the following coherence conditions:

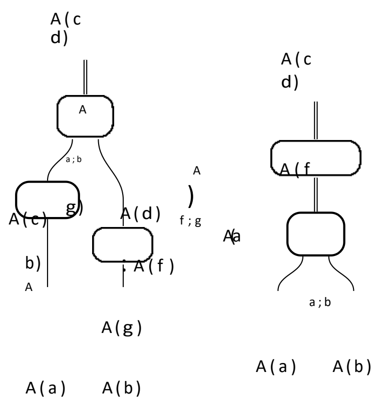
(F-I).i For all x 2 C(a ! b), y 2 C(b ! c), and z 2 C(c ! d), the following diagram commutes:



(F-I).ii For all x 2 C(a ! b), the following two triangles commute:

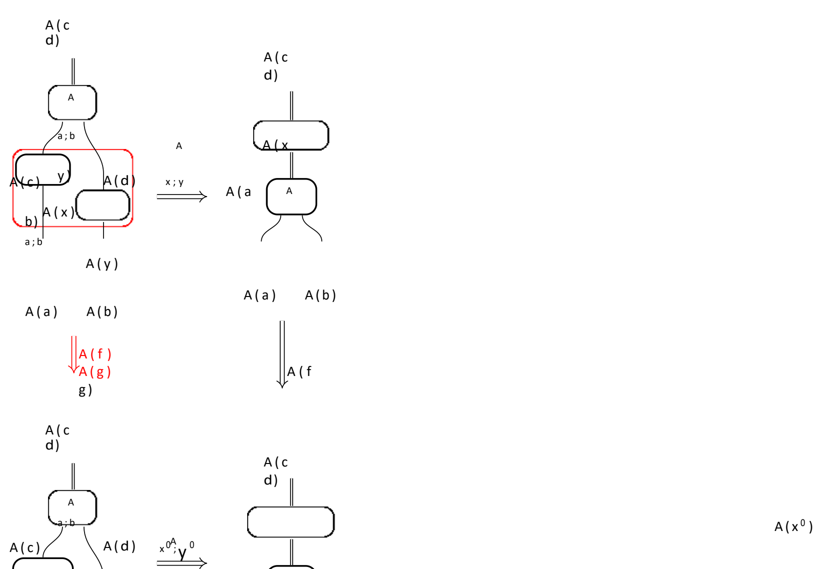


(F-II) An adjoint equivalence ^{A} : $_D$ (A A)) A $_C$ in the 2-category of 2-functors C C ! D. Explicitly, this is given by, for each pair of 0-cells (a; b) 2 C C, an adjoint equivalence 1-cell $^A_{a;b}$: A(a) A(b) ! A(a b) and for each pair of 1-cells (x; y) : (a; b) ! (c; d), an invertible 2-cell



That ^{A} is a 2-transformation means we have the following cohrences.

(F-II).i For all x; x^0 : a ! c and y; y^0 : b ! d and all f : x) x^0 and g : y) y, the following square commutes:



(y^0)

$A(x^0$
$y^0)$

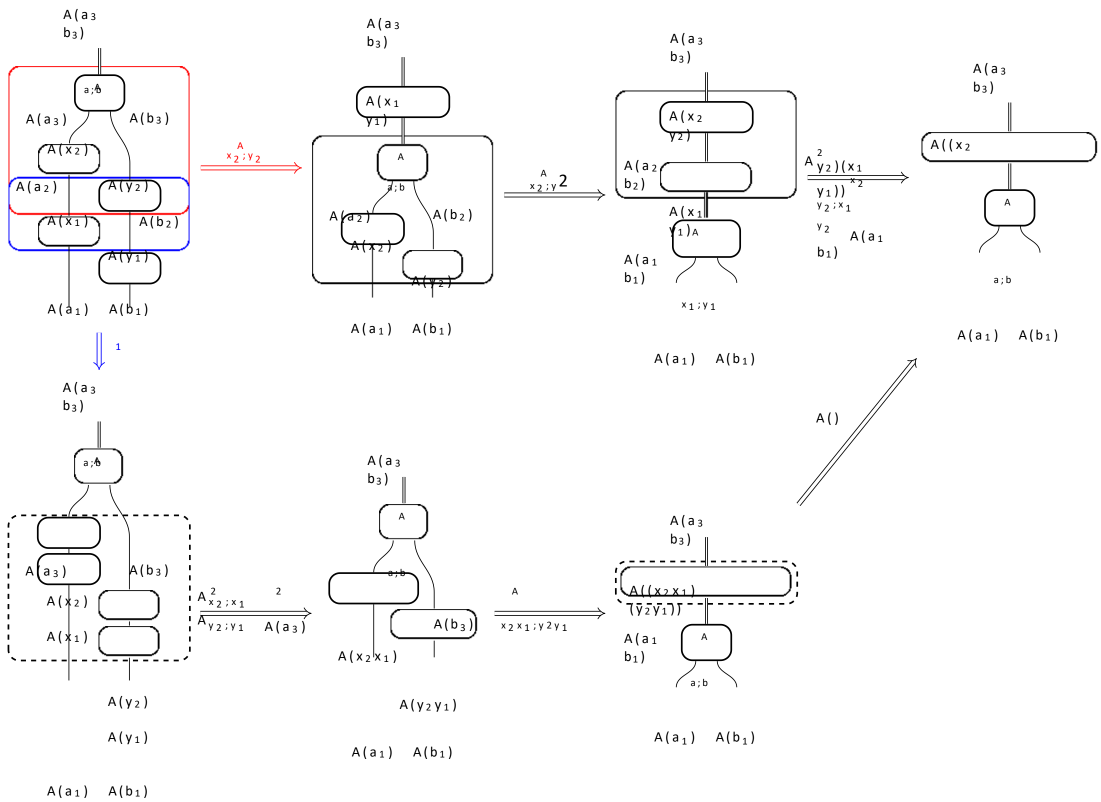
A(a
b)

a;b

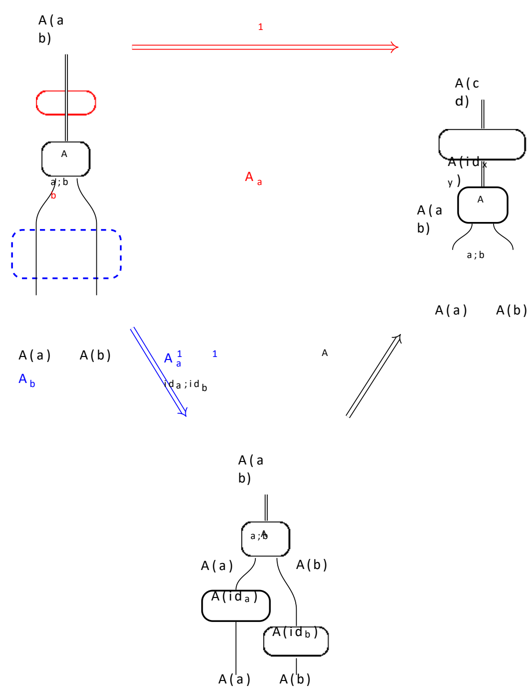
A(a) A(b)

A(a) A(b)

(F-II).ii For all 1-cells x_1 2 $C(a_1$! $a_2)$, x_2 2 $C(a_2$! $a_3)$, y_1 2 $C(b_1$! $b_2)$, and y_2 2 $C(b_2$! $b_3)$,

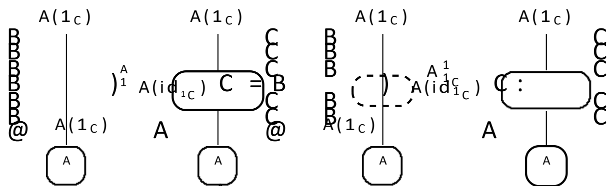


(F-II).iii For all 0-cells a; b 2 C, the following diagram commutes:



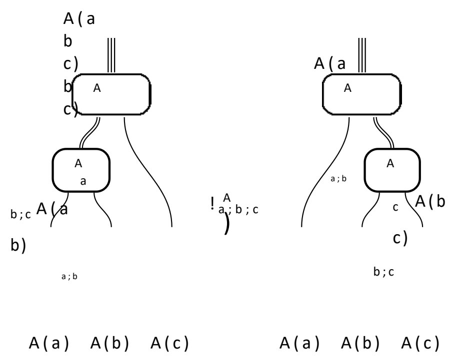
(F-III) An adjoint equivalence ^{A} : I_D) A I_C (in the 2-category of 2-functors ! D) where I_C : ! C is the inclusion of the trivial 2-category into C which picks out 1_C; id_{1_C}; $id_{id_{1_C}}$, and similarly for D. Explicitly,

this is given by an adjoint equivalence 1-cell ${}^{A}_{1} : 1_D \; ! \; A(1_C)$ and an invertible 2-cell 0

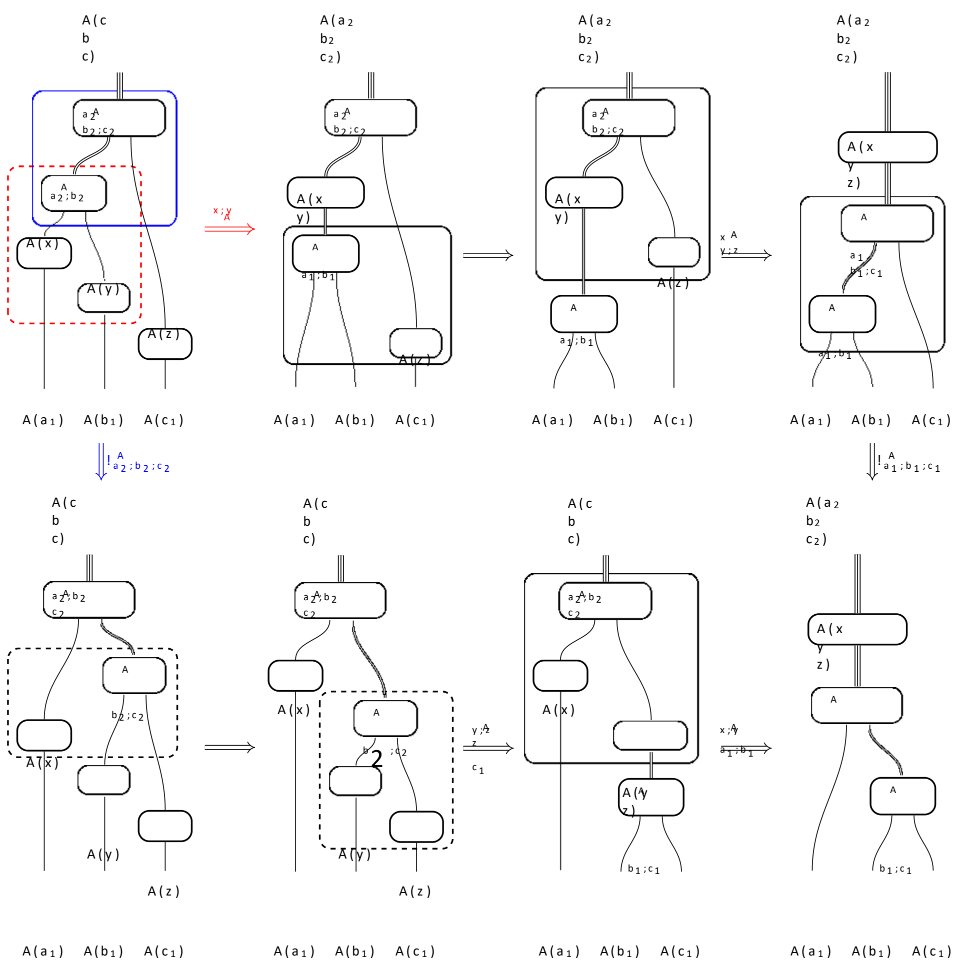


That ^{A} is a 2-transformation implies that ${}^{A}_{1}$ equals the map on the right hand side above, which is a whiskering with A^1_e. This means ${}^{A}_{1}$ is automatically natural and compatible with A^2.

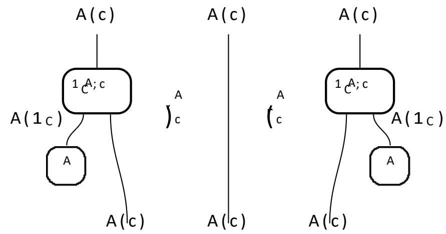
(F-IV) An invertible associator 2-modication $!^A$. Explicitly, this is given by, for each a; b; c 2 C, an invertible 2-cell



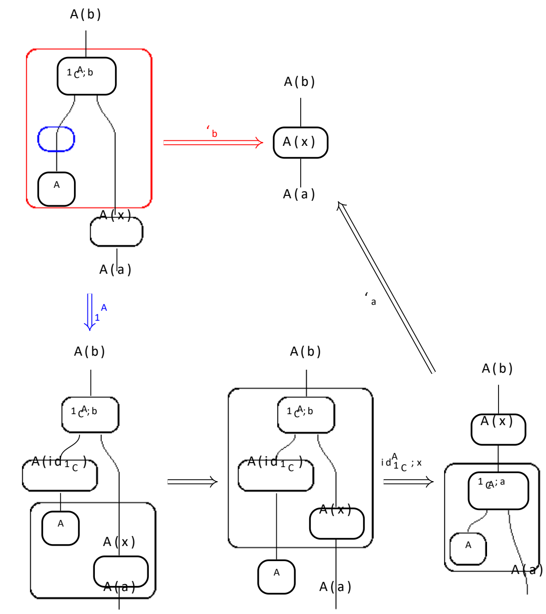
and the fact that ! is a 2-modication means that for all x 2 $C(a_1 \; ! \; a_2)$, y 2 $C(b_1 \; ! \; b_2)$, and z 2 $C(c_1 \; ! \; c_2)$,



(F-V) invertible unitor 2-modications ‘^{A} and r^A, i.e., for each c 2 C, invertible 2-cells



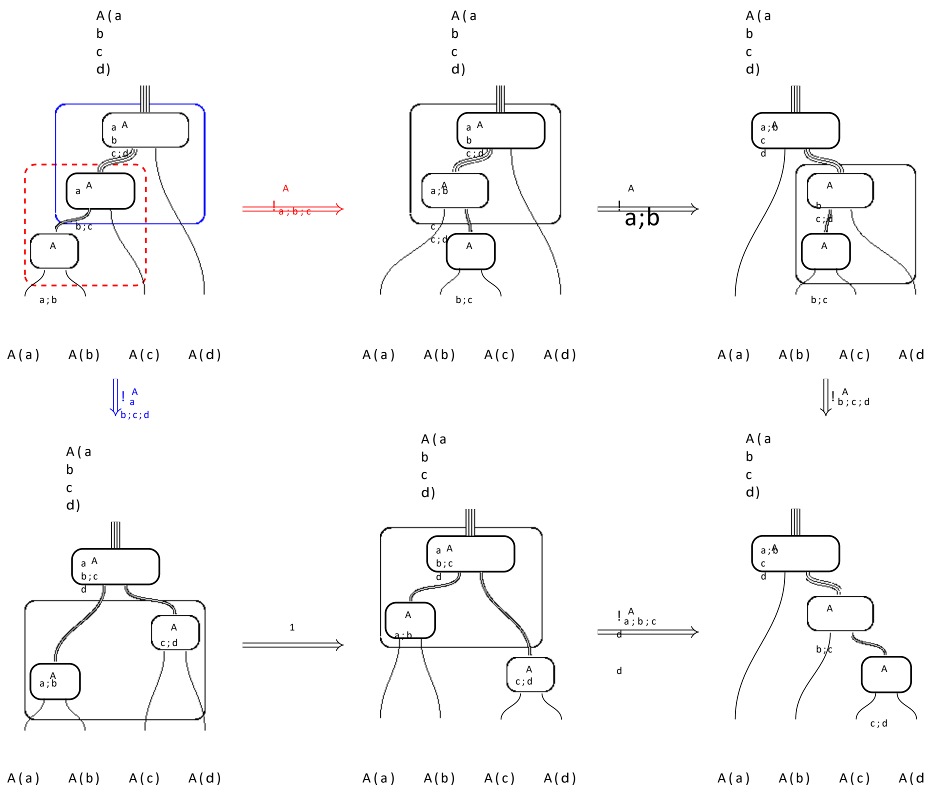
The fact that ‘ and r are 2-modications means that for all x 2 C(a ! b), the following diagram commutes:



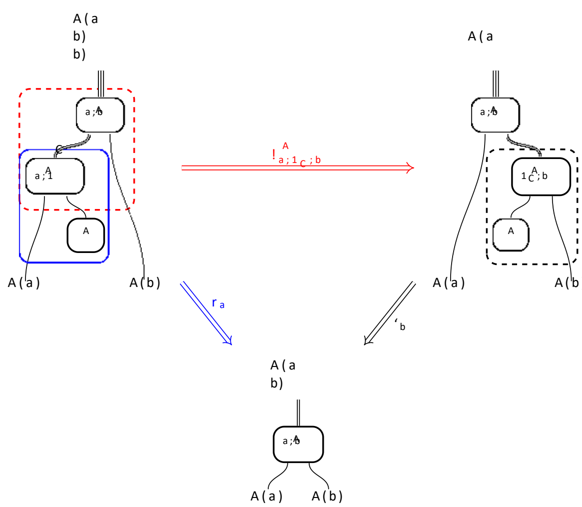
and a similar condition for r.

This data is subject to the additional two coherence conditions c.f. [Gur13, Def. 4.10]:

(F-1) For all a; b; c; d 2 C, the following diagram commutes:



(F-2) For all a; b; c 2 C, the following diagram commutes:



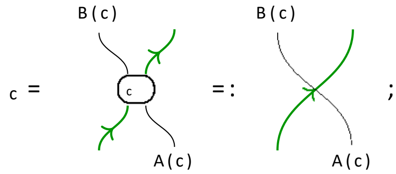
A.2 Transformations between functors of Gray-monoids

Denition A.2. Suppose C; D are Gray-monoids and A; B : BC ! BD are 3-functors. A transformation : A) B consists of:

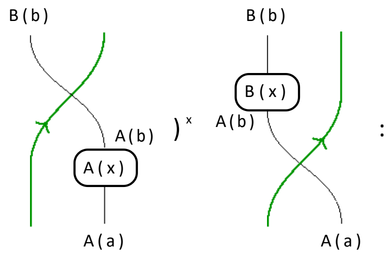
(T-I) An object 2 D, which we depict by an oriented green strand: 

(T-II) An adjoint equivalence : D(id;) A) D(; id) B in the 2-category of 2-functors C ! D. Explicitly, this is given by, for each c 2 C, an adjoint equivalence 1-cell $_c$: A(c)) B(c)

which we depict as a crossing

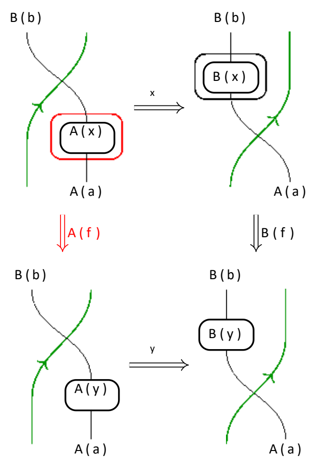


together with, for each x 2 C(a ! b), invertible 2-cells

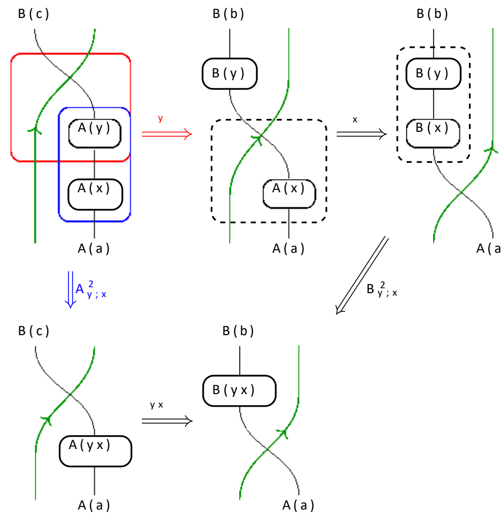


The fact that is a 2-natural transformation means that:

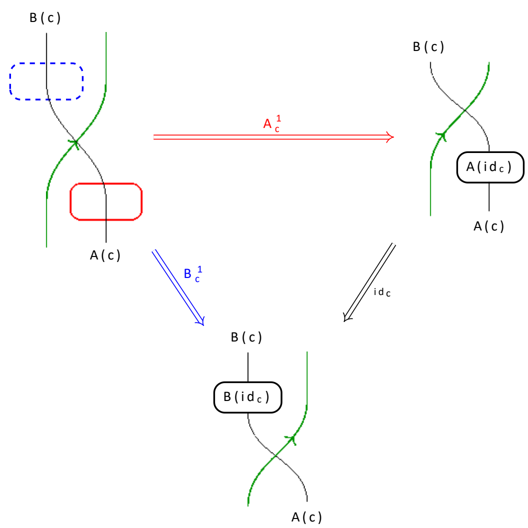
(T-II).i For every x; y 2 C(a ! b) and f 2 C(x) y), the following diagram commutes:



(T-II).ii For every $x \in C(a \to b)$ and $y \in C(b \to c)$, the following diagram commutes:

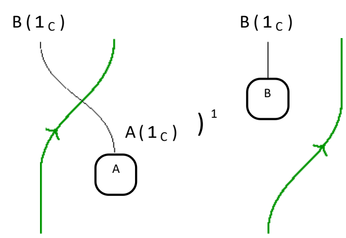


(T-II).iii For every $c \in C$, the following diagram commutes:



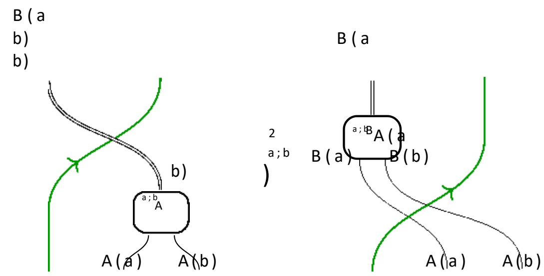
Observe this immediately implies that $_{\mathrm{id}_c} = B^{1}{}_{c}(A^{1})\,{}_{c}^{1}$ for all $c \in C$.

(T-III) A unit coherence invertible 2-modication

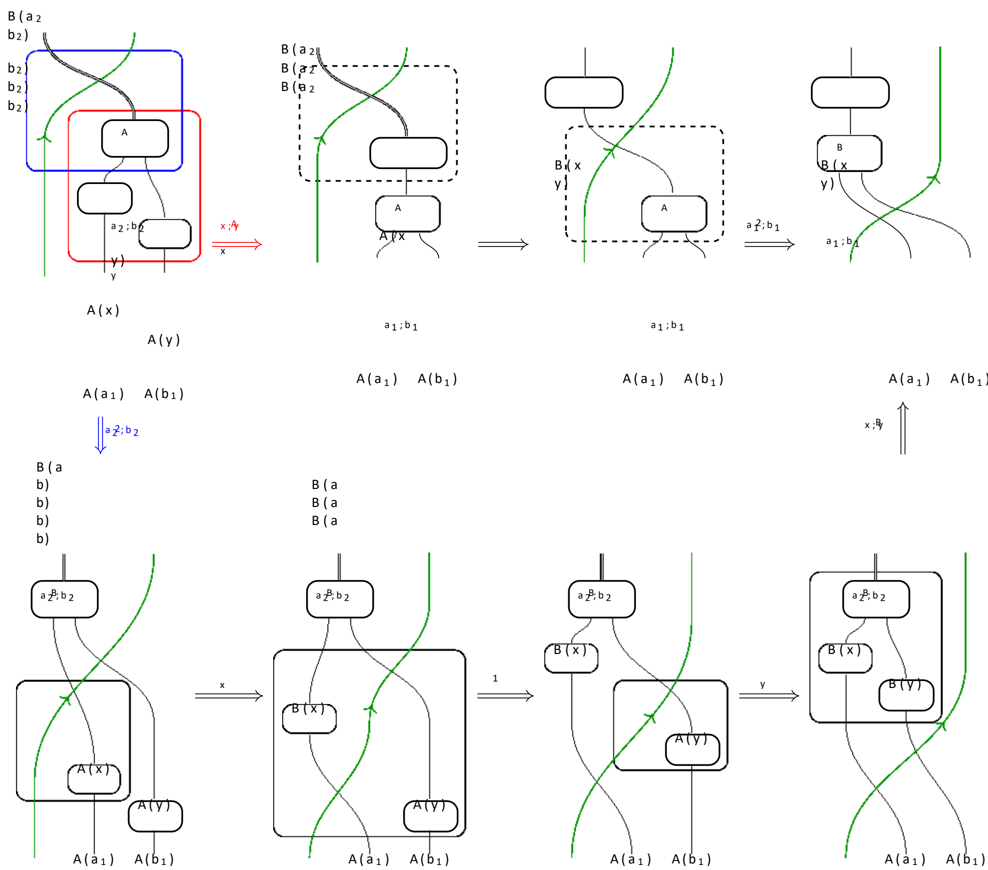


The 2-modication criterion for ^{1} is automatically satised by (F-III) (which says $^A = {}_1A^1$ and ${}_1 = {}^B_1{}^1_C$) and (T-II).iii above.

(T-IV) For every $a;b$ 2 C, a monoidality coherence invertible 2-modication

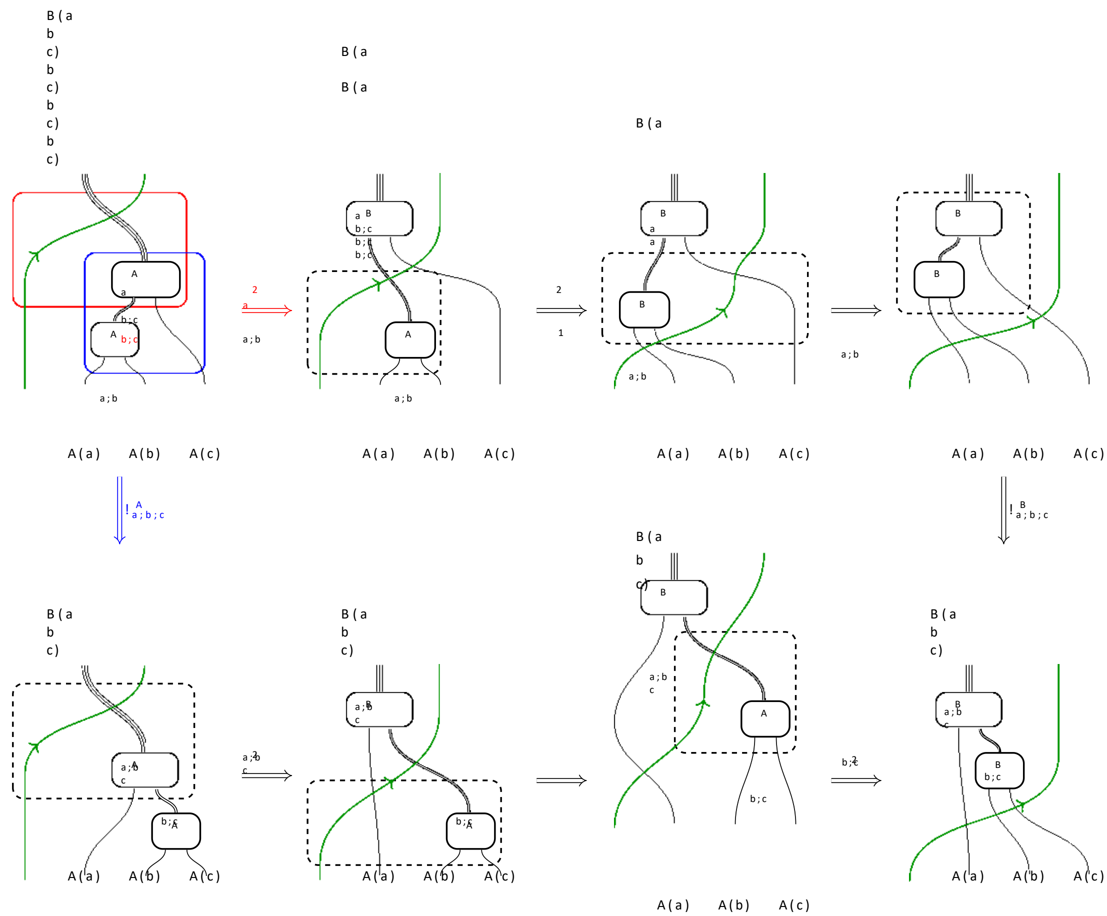


That ^{2} is a 2-modication means that for all x 2 $C(a_1 ! \;\; a_2)$ and y 2 $C(b_1 ! \;\; b_2)$,

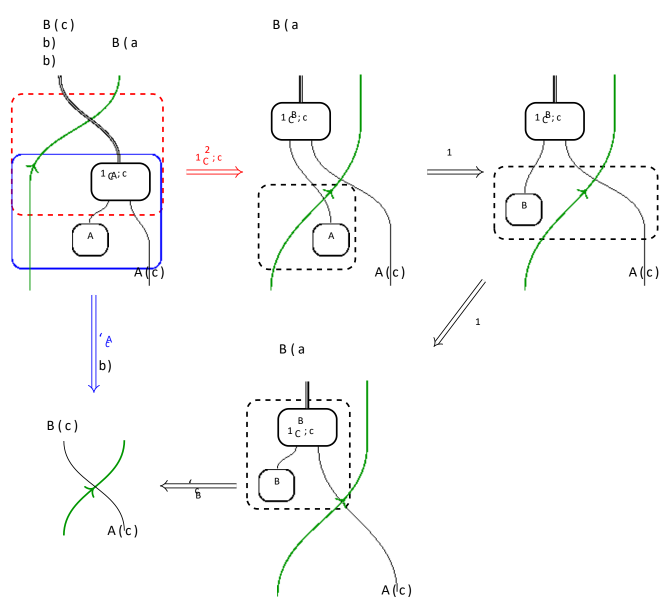


This data is subject to the following additional three coherences c.f. [Gur13, Def. 4.16]

(T-1) For all a; b; c 2 C, the following diagram commutes:



(T-2) For all c 2 C, the following diagram commutes:

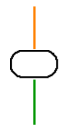


and a similar coherence equation holds for r_c^A as well.

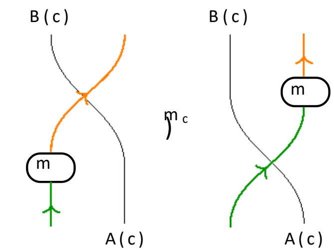
Observe that (T-2) completely determines ^{1} in terms of lower data. This means that one needs only verify the existence of some ^{1} satisfying (T-2) to verify (T-III) above.

A.3 Modications between transformations of Gray-monoids

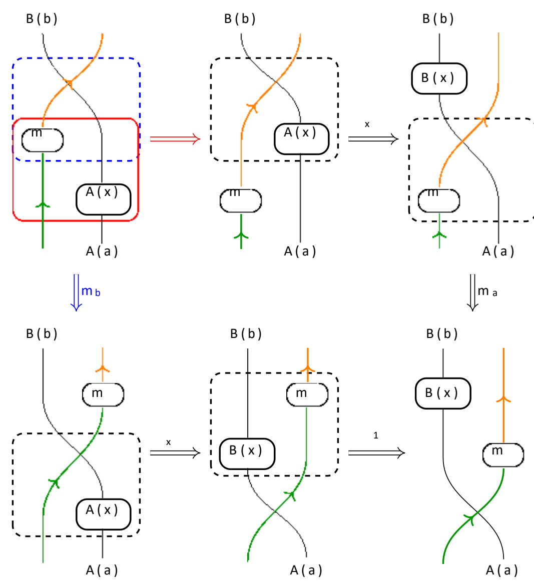
Denition A.3. Suppose C; D are Gray-monoids, A; B : BC ! BD are 3-functors, and ; : A) B are transformations. A modication m : V consists of:

(M-I) a 1-cell m : ! depicted m 

(M-II) For each c 2 C, an invertible 2-modication

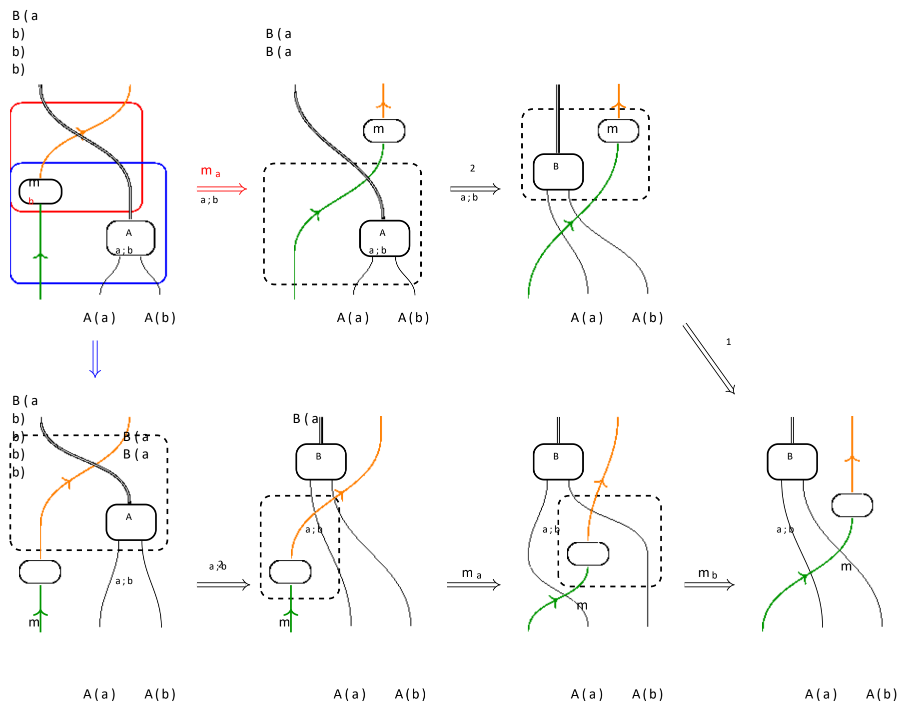


Explicitly, this means that m_c satises the following coherence condition for all x : a ! b:

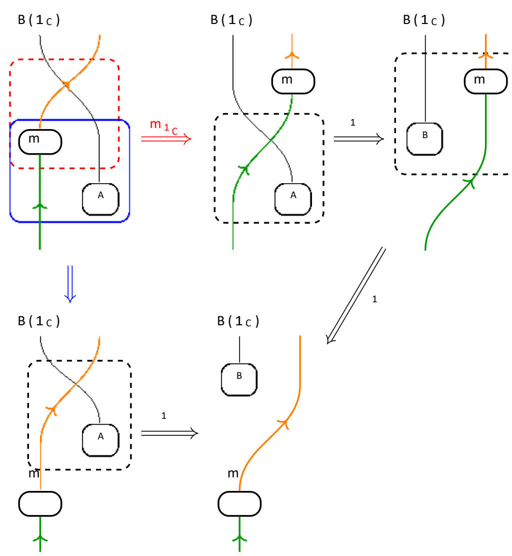


This data is subject to the following two conditions c.f. [Gur13, Def. 4.18]:

(M-1) For all a; b 2 C,



(M-2) The following diagram commutes:

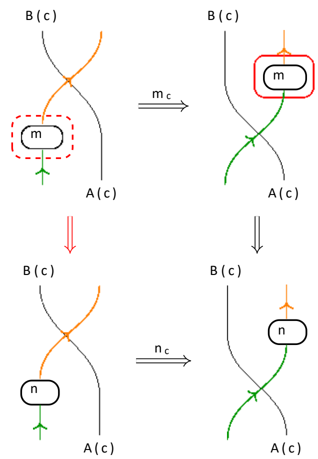


Observe this coherence completely determines m_{1_C} in terms of [1] and [1].

A.4 Perturbations between modications of Gray-monoids

Denition A.4. Suppose C; D are Gray-monoids, A; B : BC ! BD are 3-functors, ; : A) B are transformations, and m; n : V are modications. A perturbation : m ≡ n consists of a 2-cell : m) n satisfying the following coherence condition c.f. [Gur13, Def. 4.21]:

(P-1) For each c 2 C, the following square commutes:



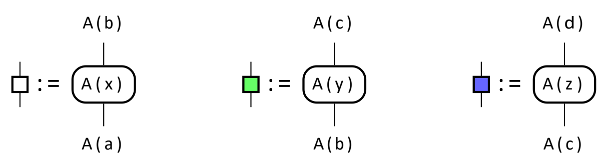
B Coherence proofs for strictication

This appendix contains all proofs from x3 which amount to checking/using various coherence conditions from Appendix A. As most of the proofs in this section are similar, we provide full detail for one part of each coherence proof below, and we explain the components of the proof in other parts whose details are left to the reader. To make the commutative diagrams more readable, we suppress all whiskering notation, including Notation 2.7.

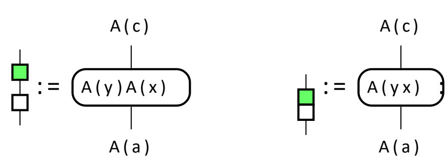
B.1 Coherence proofs for Strictifying 1-morphisms x3.2

We remind the reader that (BC; C); (BD; D) are two objects in 3Cat$^{pt}_{G}$, so that C; D are Gray-monoids, and (A;) 2 3Cat$_G$((BC; C) ! (BD; D)). We specied data for (B;) : (BC; C) ! (BD; D) above in x3.2, together with data for (; id) : (B;)) (A;).

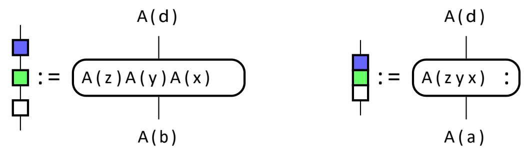
Notation B.1. In this section, we will use a shorthand notation for 1-cells in D for proofs using commutative diagrams. For x 2 C(a ! b), y 2 C(b ! c), and z 2 C(c ! d), we will denote their corresponding image in D under A as a small shaded square, e.g.,



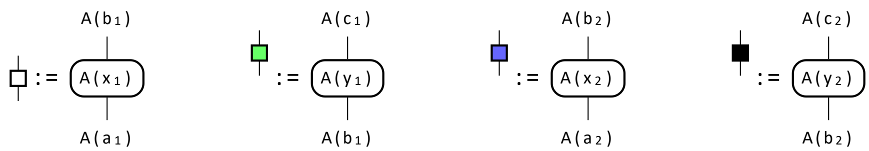
While the 1-composition in D is stacking of diagrams, we denote A applied to a 1-composite in C by vertically joining the shaded squares:



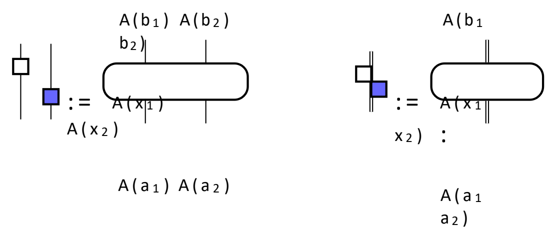
Since 1-composition in C and D are both strict, we denote a triple 1-composite by stacking three boxes, and we denote A applied to a triple 1-composite by vertically joining three boxes:



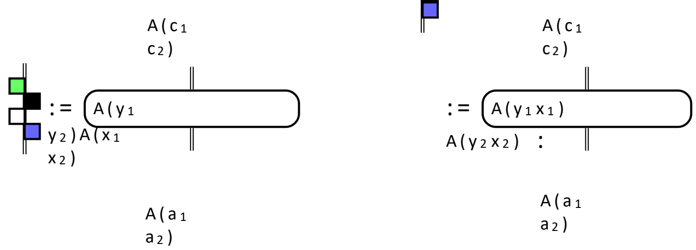
For the composite of 1-cells, we use the nudging convention as in (8). We denote A applied to a composite of 1-cells in C by joining the shaded boxes along corners. For the following example, given x_1 2 $C(a_1$! $b_1)$, y_1 2 $C(b_1$! $c_1)$, x_2 2 $C(a_2$! $b_2)$, and y_2 2 $C(b_2$! $c_2)$, we write



We then write

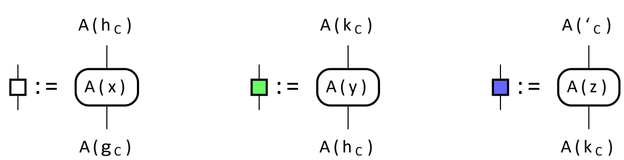


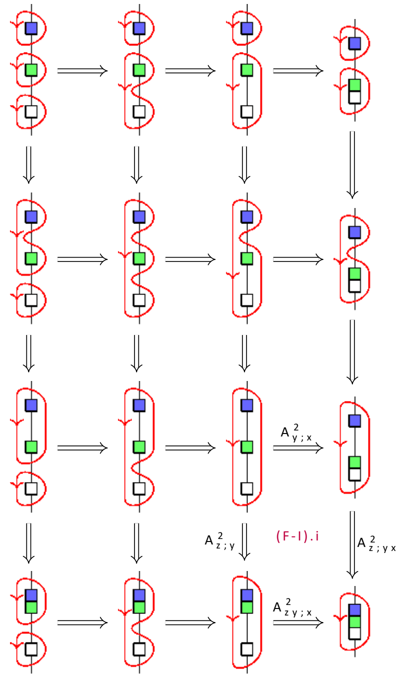
In this notation, we would write the following diagram for  A applied to the following composites:



Proof of Lem. 3.10: $(B; B^1; B^2)$ is a 2-functor. We must check (F-I) for B. We provide a complete proof for (F-I).i, and leave most of (F-I).ii as an exercise for the reader.

(F-I).i For x 2 $C(g_C$! $h_C)$, y 2 $C(h_C$! $k_C)$, and z 2 $C(k_C$! $`_C)$, we use the following shorthand as in Notation B.1:





Every square except for the bottom right square commutes by functoriality of 1-cell composition in a Gray-monoid, that is, applying two 2-cells locally to non-overlapping regions in a 1-cell commutes. The bottom right square commutes by (F-I).i applied to the underlying 2-functor of A.

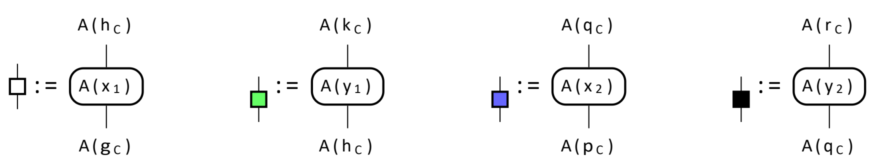
(F-I).ii Follows from the properties of the adjoint equivalence (see Remark B.2 below) together with (F-I).ii for the underlying 2-functor of A. □

Remark B.2. In subsequent proofs, we will freely combine squares that commute by functoriality of 1-cell composition when the involved 2-cells applied locally are part of the biadjoint biequivalence (BB) or the adjoint equivalences $_g$. We will then simply state this larger face commutes by the properties of the adjoint equivalence , i.e., the properties of the biadjoint biequivalence (BB) and the properties of the adjoint equivalences $_g$.

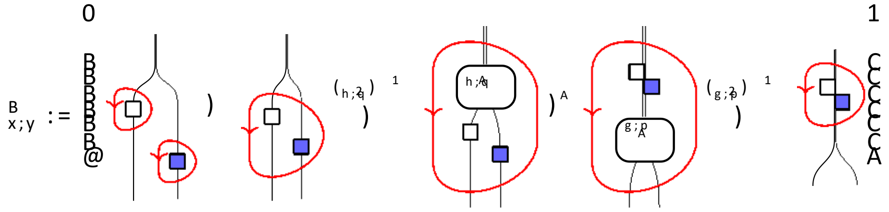
Proof of Lem. 3.12: (B; ^{B}; ^{B}; !; ‘; r) is a weak 3-functor BC ! BD.

(F-II).i Every component which makes up ^{B}, in Construction 3.11, especially ^{A}, is natural.

(F-II).ii For x_1 2 C(a_1 ! b_1), y_1 2 C(b_1 ! c_1), x_2 2 C(a_2 ! b_2), and y_2 2 C(b_2 ! c_2), we use the following shorthand as in Notation B.1:

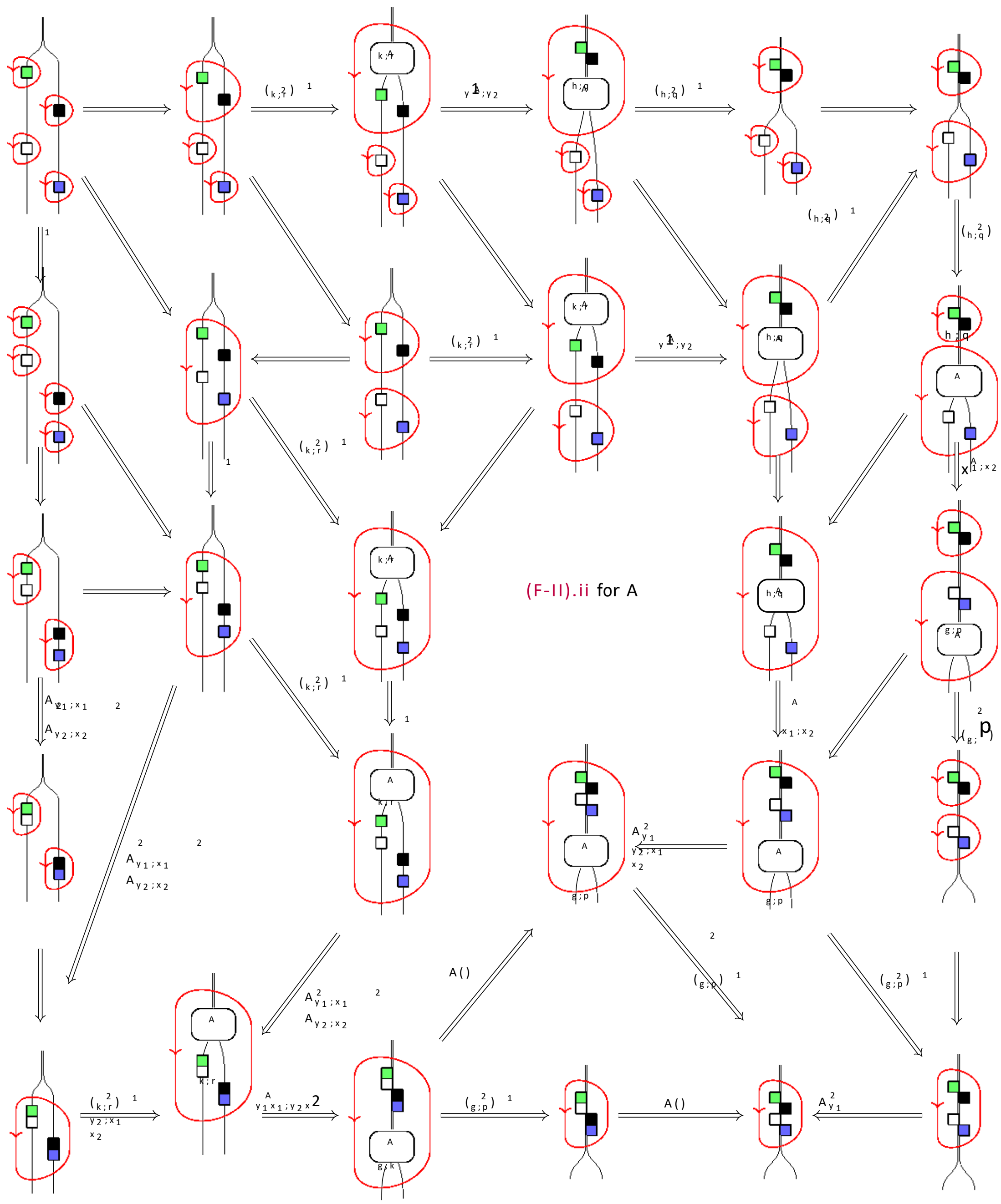


For the following diagram to t on one page, we compress the denition of $^B_{x;y}$ from Construction 3.11 into four steps



(20)

we suppress as many interchangers as possible, and we combine commuting squares involving only and the $_g$ as in Remark B.2.



Non-labelled faces either commute by functoriality of 1-cell composition , axioms (C4) and (C5) of the interchanger, or Remark B.2.

(F-II).iii This follows by Remark B.2 and functoriality of 1-cell composition , together with (F-II).iii applied to A.

(F-III) This part is automatic as $^B{}_1 := B^1{}_e$

(F-IV) This follows by Remark B.2 and functoriality of 1-cell composition , together with (F-IV) applied to A and two instances of (T-1) for the transformation : ^{D}) A ^{C}.

(F-V) This follows by Remark B.2 and functoriality of 1-cell composition , together with (F-V) applied to A and two instances of (T-2) for the transformation $: ^{D} \Rightarrow A^{C}$.

(F-1) Every map is the identity map.

(F-2) Every map is the identity map. □

Proof of Lem. 3.13: $(; _{g}; _{\mathrm{id}_g}; ^{1}; ^{2})$ is a 2-natural transformation $^{D} \Rightarrow B^{C}$. (T-II).i This condition is immediate as the only 1-cells and 2-cells in BG are identities.

(T-II).ii This step amounts to checking $B^{2}_{\mathrm{id}_g;\mathrm{id}_g} (B^{1}_{g} B_{g})^{1} = B^{1}_{g}$. Using Remark B.2 and functoriality of 1-cell composition , this reduces to the identity $A^{2}_{\mathrm{id}_g;\mathrm{id}_g} (A^{1}_{g} A^{1}_{g}) = A^{1}_{g}$.

(T-II).iii This condition is immediate as $(^{D})^{1}_{g} = \mathrm{id}_{\mathrm{id}_{g_D}}$ and $^{1}_{g} = B^{1}_{g}$.

(T-III) This condition is automatically satised.

(T-IV) This condition is immediate as $^{D}_{g;h} = \mathrm{id}_{gh_D} = ^{CB}_{g;h}$ and $_{g} = \mathrm{id}_g$ for all $g \in G$. (T-1) Every map is the identity map.

(T-2) Every map is the identity map. □

Proof of Thm. 3.14: $(; \mathrm{id}) : (B;) \Rightarrow (A;)$ is an invertible 2-morphism in $3\mathrm{Cat}_G$.

It suces to prove that denes a 2-transformation $: B \Rightarrow A$, as it clearly invertible.

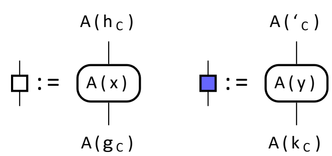
(T-II).i Every component which makes up $_x$ in (14) is natural in x.

(T-II).ii This follows by Remark B.2.

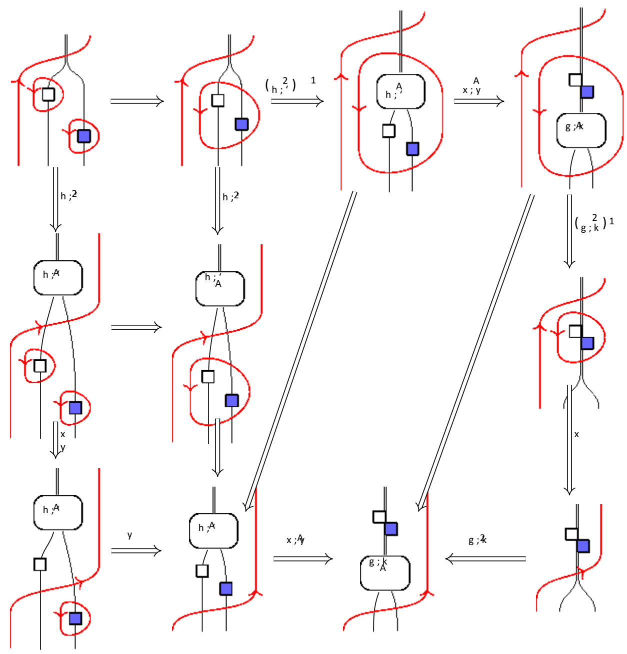
(T-II).iii This follows by Remark B.2.

(T-III) This condition is automatically satised.

(T-IV) For $x \in C(g_C \to h_C)$ and $y \in C(k_C \to \ell_C)$, we use the following shorthand as in Notation B.1:



For the following diagram to t on one page, we compress the denition of $^{B}_{x;y}$ from Construction 3.11 into four steps as in (20), we suppress as many interchangers as possible, and we combine commuting squares involving only and the $_g$ as in Remark B.2.



Every square here commutes by properties of the biadjoint biequivalence (BB), the adjoint equivalences $_g$, and functoriality of 1-cell composition .

(T-1) Since $!^{B}_{g;h;k}$ is the identity, it is equal to $!^{D}_{g;h;k}$ Since (T-1) holds for $: {}^{D}) \quad A^{C}$, we conclude (T-1) holds for $: B) \quad A$.

(T-2) Since $'^{B}_{g}; r^{B}_{g}$ are identities, they are equal to $'^{D}_{g}; r^{D}_{g}$ respectively. Since (T-2) holds for $: {}^{D}) \quad A^{C}$, we conclude (T-2) holds for $: B) \quad A$. □

B.2 Coherence proofs for Strictifying 2-morphisms x3.3

We remind the reader that $(BC; {}^{C}); (BD; {}^{D})$ are two objects in $3Cat^{pt}$, $(A;); (B;) \; 2 \; 3Cat^{pt}((BC; {}^{C}_{G}) ! \; (BD; {}^{D}))$, and $(; m) \; 2 \; 3Cat_G((A;)) \quad (B;))$. We specied data for $(; id) : (A;))_{G} \; (B;)$ above in x3.3.

Proof of Lem. 3.16: $(; id) : (A;)) \quad (B;)$ is a 2-morphism in $3Cat_G$. It suces to check that $: A) \quad B$ is a 2-natural transformation.

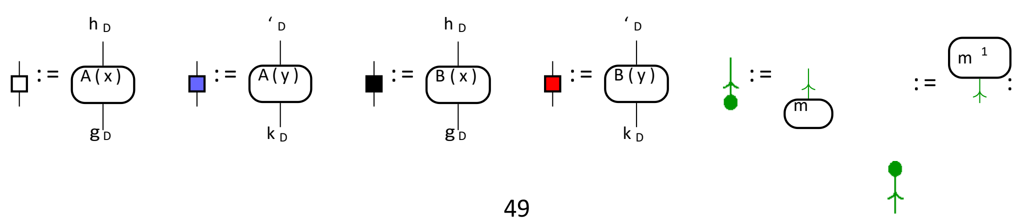
(T-II).i Every component which makes up $_x$ in (16) is natural in x.

(T-II).ii This follows by Remark B.2 and functoriality of 1-cell composition , together with (T-II).ii applied to $: {}^{D}) \quad A^{C}$.

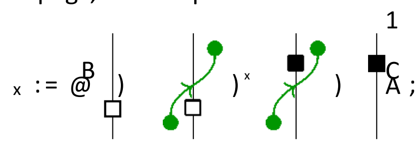
(T-II).iii This follows by Remark B.2 and functoriality of 1-cell composition , together with (T-II).iii applied to $: {}^{D}) \quad A^{C}$.

(T-III) This condition is automatically satised.

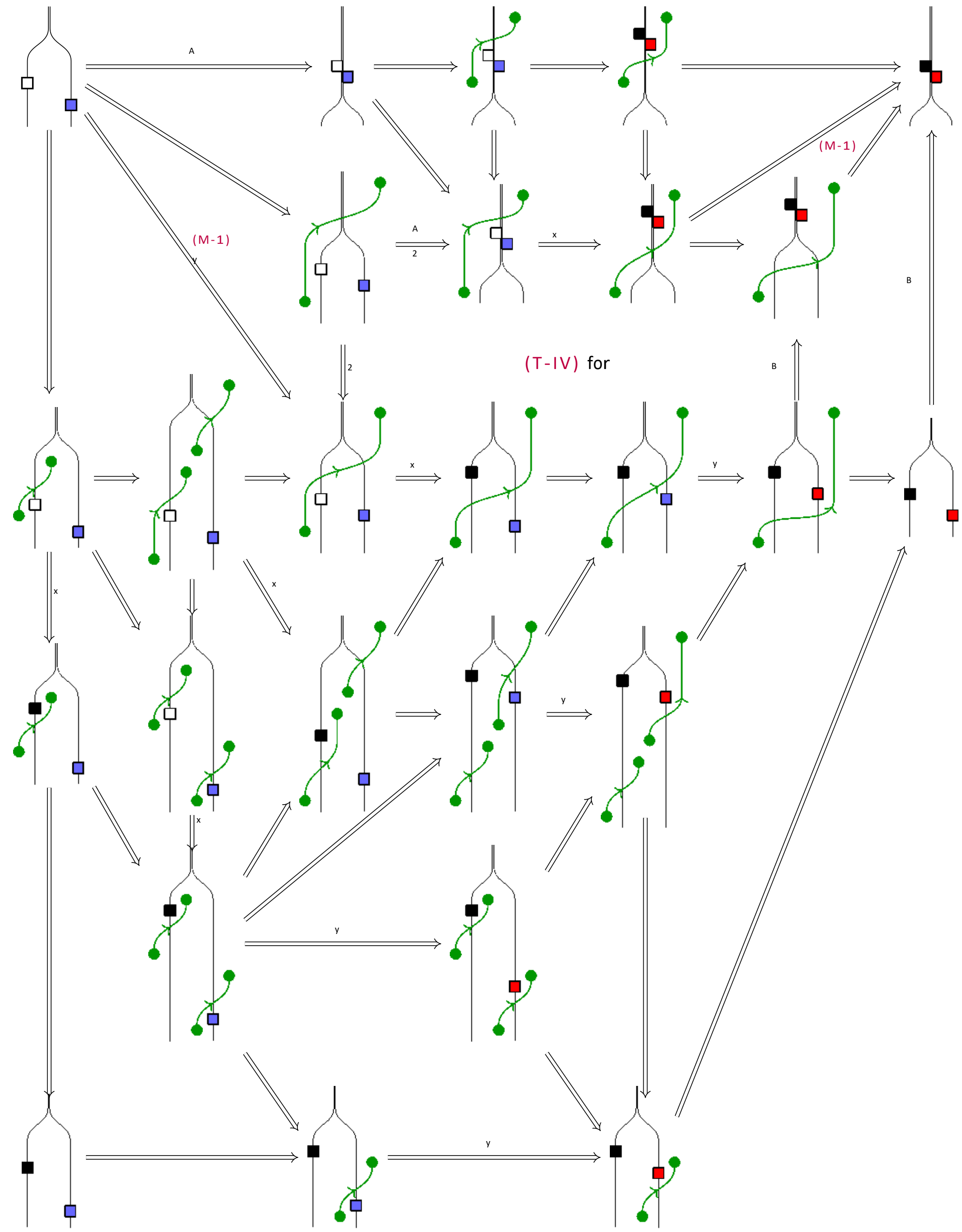
(T-IV) For $x \; 2 \; C(g_C ! \quad h_C)$ and $y \; 2 \; C(k_C ! \quad '_C)$, we use the following shorthand as in Notation B.1:



For the following diagram to t on one page, we compress the denition of from (16) into three steps 0



and we combine and suppress as many interchangers as possible, simply writing .

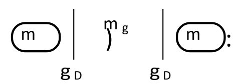


Non-labelled faces commute by either functoriality of 1-cell composition or by properties of a (bi)adjoint (bi)equivalence.

(T-1) Every map is the identity map.

(T-2) Every map is the identity map. □

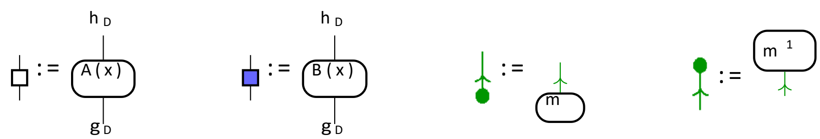
We remind the reader that by the strictness properties for ; as components of 1-morphisms in $3\mathrm{Cat}^{pt}_G$, m : e$_D$) , and m_g is an invertible 2-cell

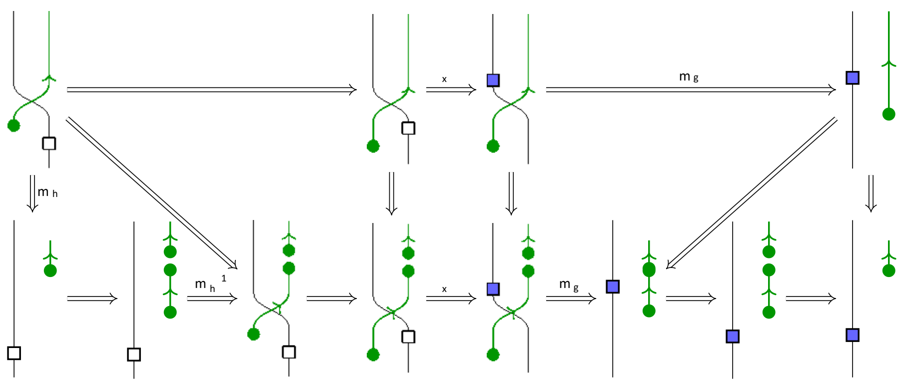


(15)

Proof of Thm. 3.17: (m; id) : (; id) V (; m) is an invertible 3-morphism in $3\mathrm{Cat}_G$. It suces to check that m : V is an invertible 3-modication.

(M-II) This condition corresponds for m : V corresponds to the outside of the following commutative diagram, where we use the following shorthand notation for x 2 C(g_C ! h_C) and m; m ^{1}:





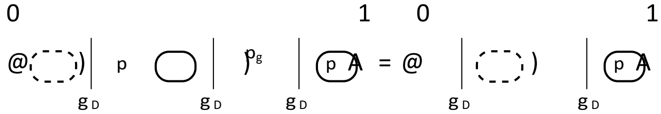
All inner faces in the above diagram are squares which commute by functoriality of 1-cell composition .

(M-1) This is exactly (M-1) applied to m viewed as a modication m : V (id$_c$) as in (9) above

(M-2) By the strictness properties of (A;) and (B;), (M-2) for the modication m : V (id $_c$) as in (9) above tells us that m_{1_C} = ^{1} on the nose. This exactly gives the coherence (M-2) for . □

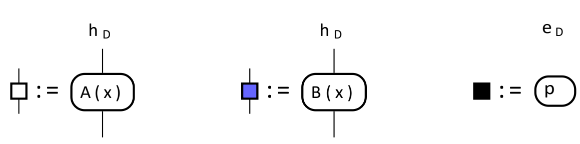
B.3 Coherence proofs for Strictifying 3-morphisms x3.4

We remind the reader that in this section, (; m = id); (; n = id) : (A;)) (B;) are two 2-morphisms in $3\mathrm{Cat}^{pt}_G$ and (p;) : (; id) V (; id) is a 3-morphism in $3\mathrm{Cat}_G$. This means is an invertible 2-cell $\mathrm{id}_{\mathrm{id}_{e_D}}$) p satisfying the coherence

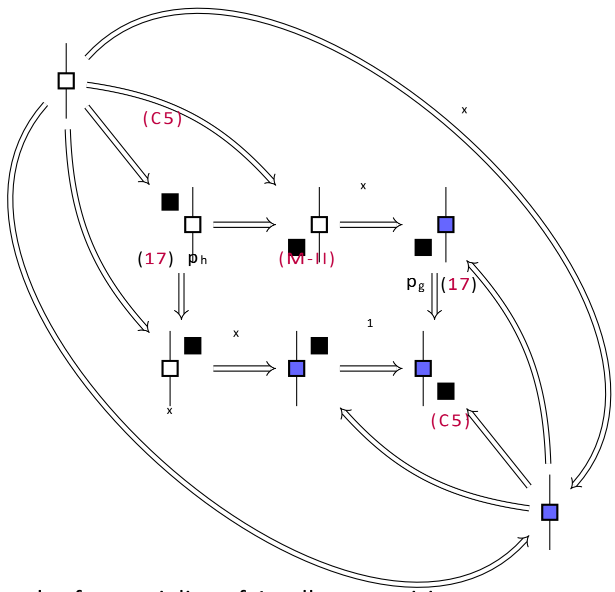


8 g 2 G: (17)

Proof of Lem. 3.18: $_x$ = $_x$ for all x 2 C(g_C ! h_C). For x 2 C(g_C ! h_C), we use the following shorthand as in Notation B.1.



The outside of the following commutative diagram is a bigon with one arrow $_x$ and one arrow $_x$; hence $_x$ = $_x$:



The unlabeled faces commute by functoriality of 1-cell composition . □

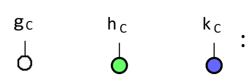
C Coherence proofs for G-crossed braided categories

This appendix contains all proofs from x4 which amount to checking/using various coherence conditions using the properties listed in Appendix A. To make the commutative diagrams more readable, we suppress all whiskering notation, including Notation 2.7.

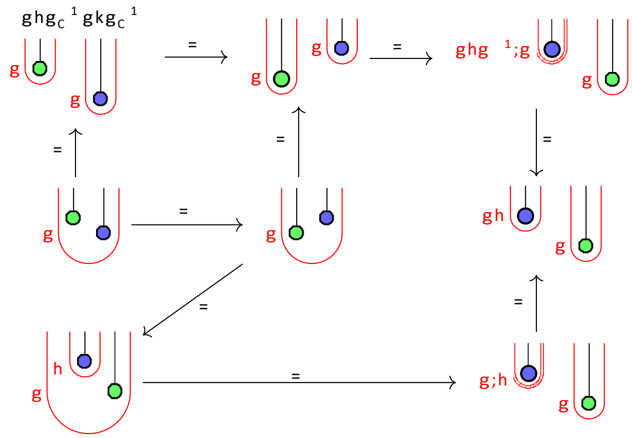
C.1 Coherence proofs for the 2-functor $3Cat_G^{st}$ to $GCrsBrd^{st}$ from x4.2

We now supply the proofs for statements in x4.2. We remind the reader that (C; $_{g;h}$; F_g; $_{g;h}$) is the data constructed from C 2 $3Cat_G^{st}$ in Constructions 4.6, 4.7, and 4.8.

Proof of Thm. 4.9: (C; $_{g;h}$; F_g; $^{g;h}$) forms a strict G-crossed braided category. We remind the reader that we use the shorthand notation that white, green, and blue shaded disks correspond to 1-morphisms into g_C; h_C; and k_C, respectively:



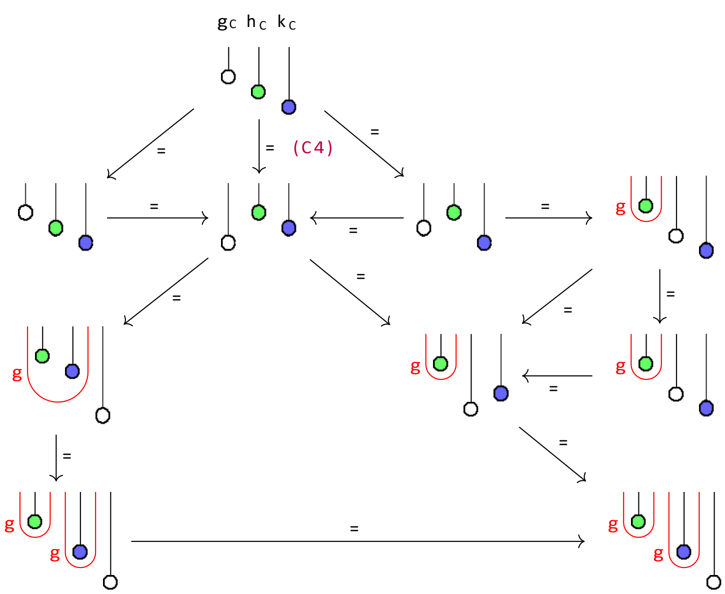
It remains to check the commutativity of (1), (2), and (3). We treat (1) in detail. Going around the outside of the diagram below corresponds to (1). The large face consists of only equalities, so it manifestly commutes.



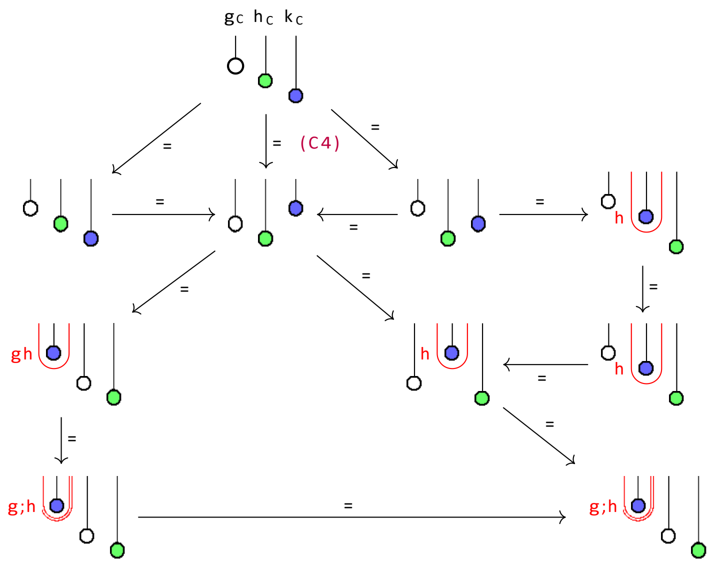
(1)

The top left square commutes as the only two non-trivial maps are the same interchanger.

The equations (2) and (3) are similar. In the two diagrams below, the outside 7 diagrams are the vertices in the heptagons (2) and (3) respectively. There is only one non-trivial face in each the two diagrams below corresponding to these two coherences, and this face commutes by the axiom (C4) of the interchanger in a Gray-monoid.



(2)



(3)

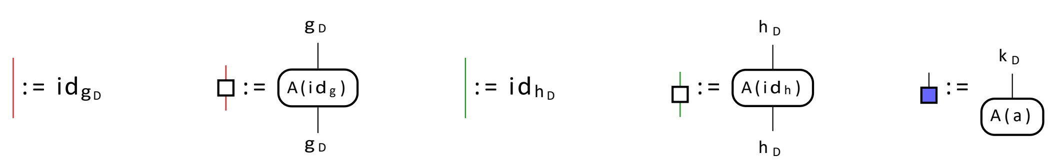
This completes the proof. □

For C; D 2 3Cat$_G^{st}$, A; B 2 3Cat$_G^{st}$(C ! D), and 2 3Cat$_G^{st}$(A) B), let C; D be the G-crossed braided categories obtained from C; D respectively from Theorem 4.9. In Construction 4.10 and (19), we dened the data (A; a) : C ! D.

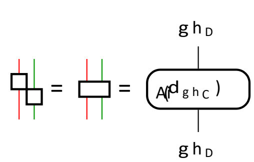
Proof of Lem. 4.11: (A; A^1; A^2) : C ! D is a G-graded monoidal functor. That each A$_g$ is a functor follows immediately from the fact that A is a functor. The data A^2 satises associativity by property (F-IV) of (A; ^{A}; ^{A}), and the data A^2 and A^1 satises unitality by property (F-V) of (A; ^{A}; ^{A}). (Observe that in (F-IV) and (F-V), all instances of , !^{A}, '^{A}, and r^A are identities, so these reduce to the usual associativity and unitality conditions for a monoidal functor.) □

Proof of Thm. 4.12: (A; A^1; A^2; a) : C ! D is a G-crossed braided monoidal functor. Naturality of a follows by naturality of A^1 and (F-II).i of ^{A}. It remains to prove the coherences (1) and (2).

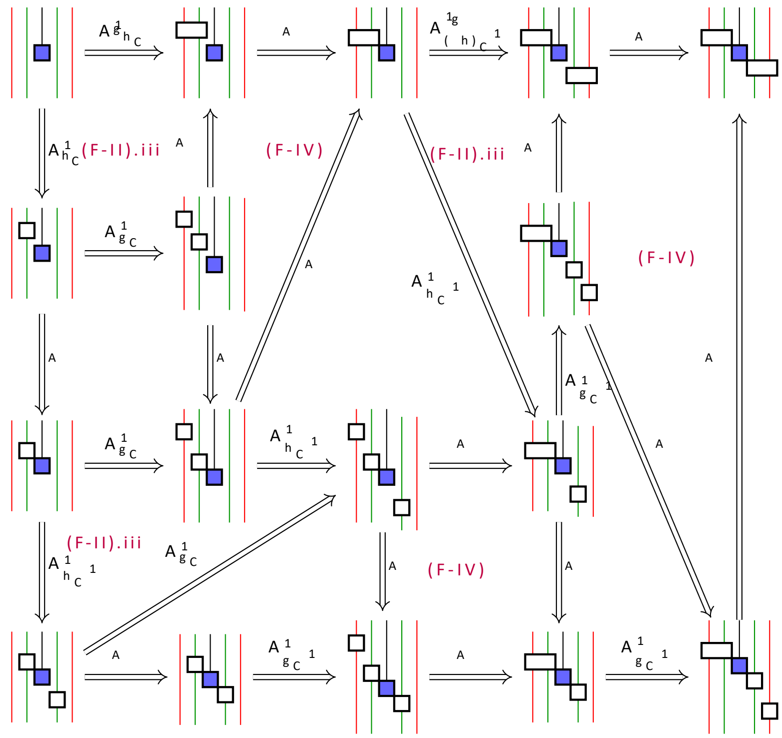
(1) Observe that since C and D are strict, the coherence condition (1) is actually a triangle. For a 2 C$_k$, we use the shorthand notation a small shaded box for A(a). For g; h 2 G and a 2 C$_k$ = C(1$_C$! k$_C$), we use the following shorthand as in Notation B.1:



Observe that since C is Gray, we have an equality id$_{g_C}$ id$_{h_C}$ = id$_{gh_C}$:

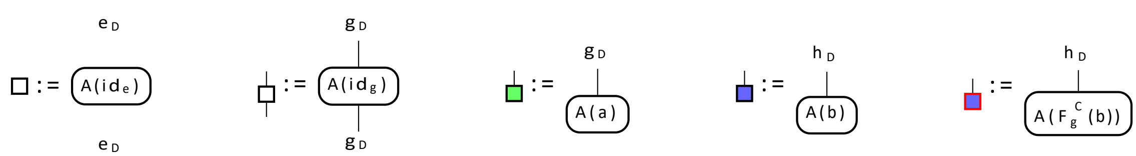


Expanding (19), we see that (1) follows from the following commuting diagram. (Recall that the cups on the bottom in (19) are really identity maps, and do not need to be drawn.)

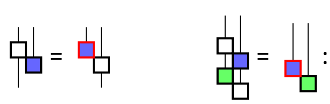


Each square above is labelled by the property for A which makes it commute. Unlabelled squares commute by functoriality of 1-cell composition .

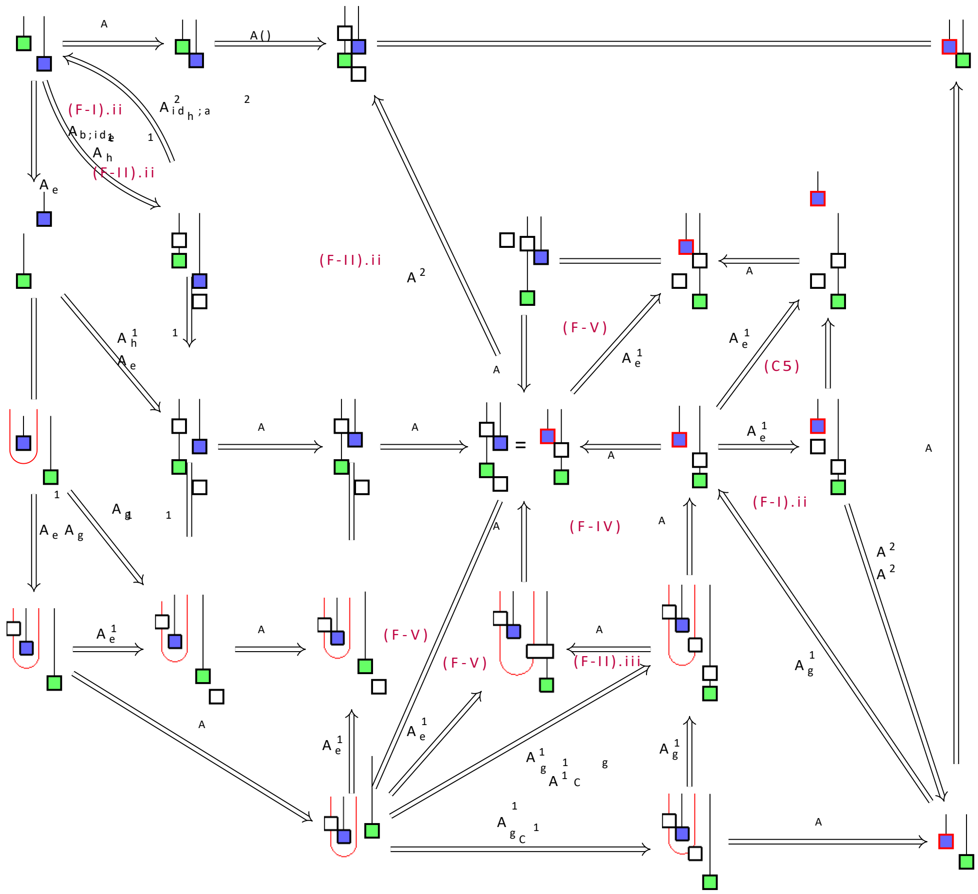
(2) For $g, h \in G$, $a \in \mathcal{C}_g = \mathcal{C}(1_{\mathcal{C}} \to g_{\mathcal{C}})$, and $b \in \mathcal{C}_h = \mathcal{C}(1_{\mathcal{C}} \to h_{\mathcal{C}})$, we use the following shorthand as in Notation B.1:



Recall that by Construction 4.8 of the G-crossed braiding in C, we have the identities

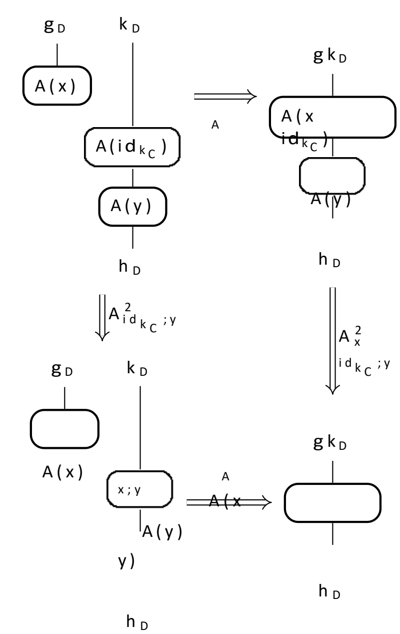


Going around the outside of the diagram below corresponds to (2).



Again, each square above is labelled by the property for A which makes it commute. □

Remark C.1. By an argument similar to the right half of the commutative diagram in the proof of (2) above, for a functor A 2 3Cat$^{\mathrm{st}}_{G}$, x 2 C(1$_C$! g$_C$), and y 2 C(h)C ! k$_C$), the following square commutes:



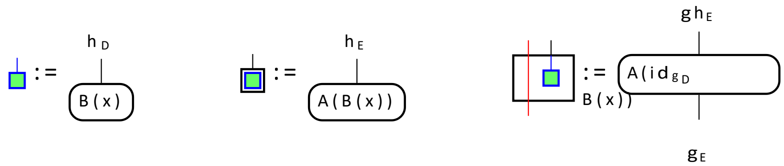
(21)

Proof of Prop. 4.13: (A; ^{A}; ^{A}) ! (A; a) is strict.

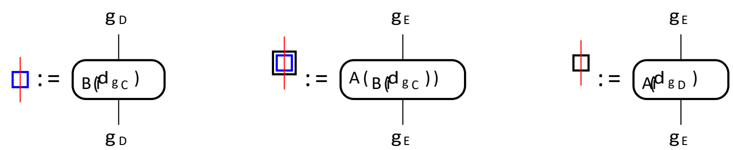
It is straightforward to see that if $(A; A^1; A^2; {}^A; {}^A) \in 3\mathrm{Cat}^{st}_G(C \to C)$ is the identity 3-functor, then so is $(A; A^1; A^2; a) \in \mathrm{GCrsBrd}^{st}(C \to C)$. Suppose now we have two composable 1-morphisms $(A; A^1; A^2; {}^A; {}^A) \in 3\mathrm{Cat}^{st}_G(D \to E)$ and $(B; B^1; B^2; {}^B; {}^B) \in 3\mathrm{Cat}^{st}_G(C \to D)$. We now calculate the composition formu-las for the composite G-crossed braided monoidal functor $(A \quad B; (A \quad B)^1; (A \quad B)^2; a \quad b)$ associated to $(A\,B; (A\,B)^2; (A\,B)^2; {}^{AB}; {}^{AB})$. The unitor $(A\,B)^1$ and tensorator $(A\,B)^2$ are straightforward:

$$(A \quad B)^1 = (A \quad B)^1_e = A(B^1)_e A^1 = {}_e A(B^1) \quad B^1 \quad {}_e$$
$$(A \quad B)^2_{x;y} = {}_{x;y}^{A\,B} = A({}_{x;y})\ {}^B_{B(x);B(y)}{}^A = A(B_{x;y})\ A^2_{B(x);B(y)}{}^2.$$

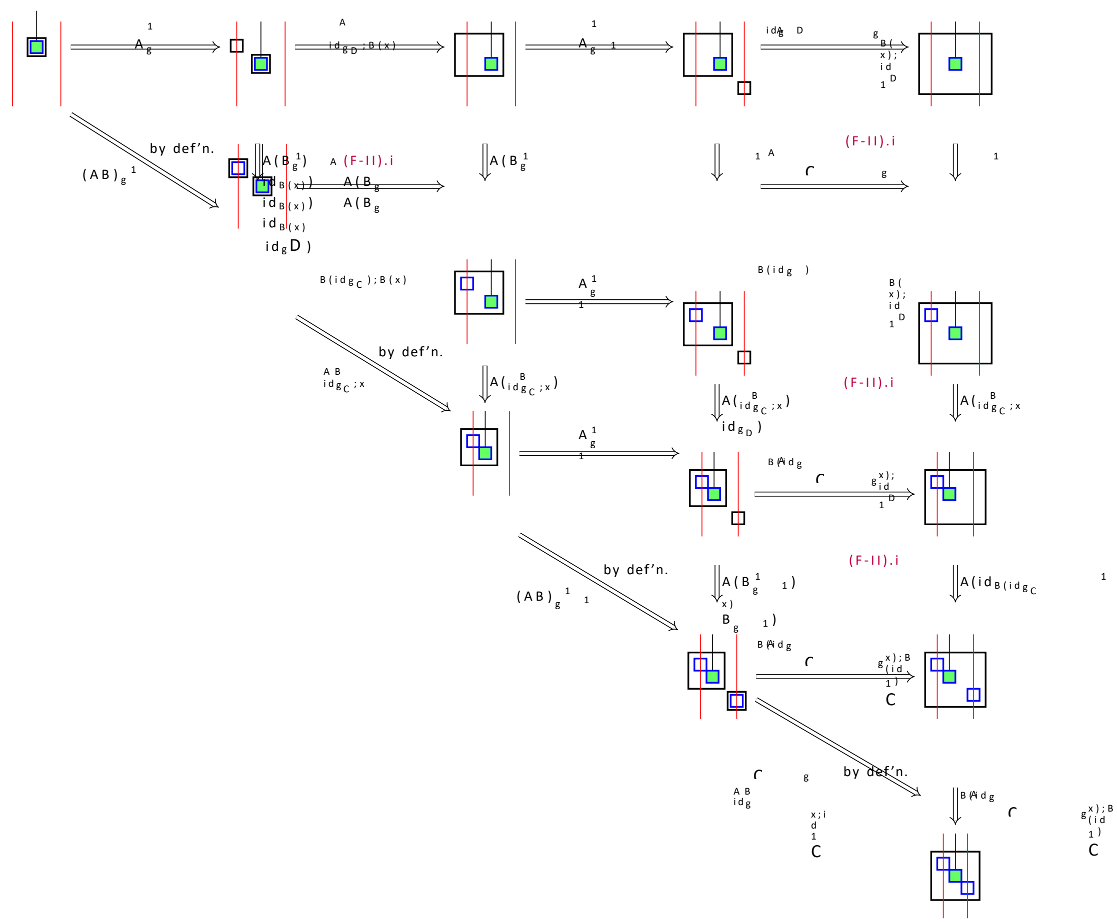
To compute $(a \quad b)_x$ for $x \in C(1_C \to g_C)$, we use the following shorthand as in Notation B.1, where black rectangles and strings corresponds to 1-cells in E after applying A, and blue rectangles and strands corresponds to 1-cells in D after applying B. We also draw red strands to denote id_g in both D and E. For example:



We draw unshaded boxes on red strands to denote $B(\mathrm{id}_{g_C}); B(\mathrm{id}_{g_D}{}^1, A(\mathrm{id}_{g_D}); A(\mathrm{id}_{g_D}{}^1.$



The composite along the diagonal in the commuting diagram below is the denition of $(a \quad b)_x$. Each face without a label above commutes by functoriality of 1-cell composition .



As the above diagram commutes, $(a\ b)_x = A(a_{F^D_g(B(x))})\ a_{B(x)}$.

Finally, we observe this data agrees with the composite data for the data for the composite of the G-crossed braided monoidal functors $(A; A^1; A^2; a)$ and $(B; B^1; B^2; b)$ in GCrsBrd. □

For C; D 2 $3Cat_G^{st}$, A; B 2 $3Cat_G^{st}$(C ! D), and 2 $3Cat_G^{st}$(A) B), let C; D be the G-crossed braided categories obtained from C; D respectively from Theorem 4.9, and let (A; a); (B; b) : C ! D be the G-crossed braided functors obtained from A; B respectively from Theorem 4.12. In Construction 4.14 we dened h : (A; a)) (B; b) by h_a := $_a$ 2 D(A(a)) B(a)) for a 2 C_g = C(1_C ! g_C).

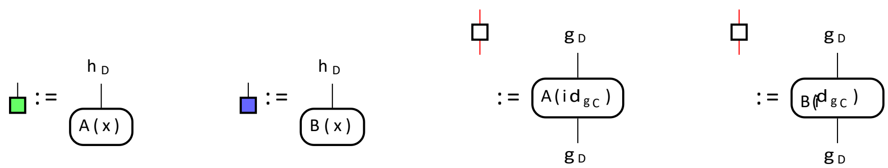
Proof of Thm. 4.15: h : (A; a)) (B; b) is a G-crossed braided monoidal transformation.

Naturality: This is immediate by the denition h_x := $_x$ for x 2 C_g = C(1_C ! g_C) and (T-II).i.

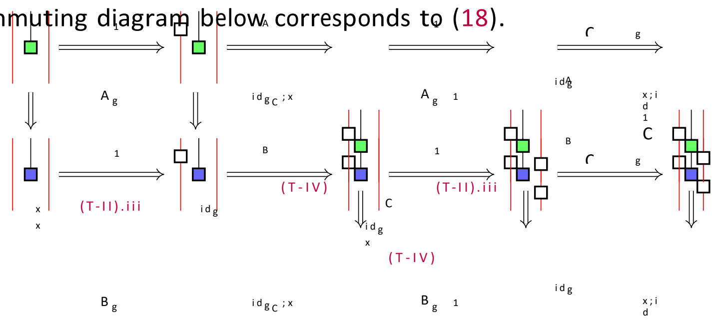
Unitality: By (T-II).iii. h_{1_C} = id_{e_C} = $B^1{}_e(A^1)_e{}^1$ = B^1 (A^1) ^{1}.

Monoidality: That $B^2_{x;y}$ (h_x 2
h_y) = h_x
$_y$ $A_{x;y}$ follows immediately by (T-IV).

(18) For x 2 C_h = C(1_C ! h_C), we use the following shorthand as in Notation B.1. We also draw red strands to denote id_g in D, and we draw unshaded boxes on red strands to denote each of A(id_{g_C}); A(id_{g_C} ^{1} and B(id_{g_C}); B(id_{g_D} ^{1}, For example:



The outside of the commuting diagram below corresponds to (18).



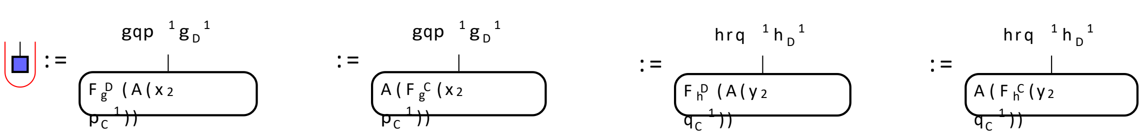
This completes the proof. □

C.2 Coherence proofs for the equivalence x4.3

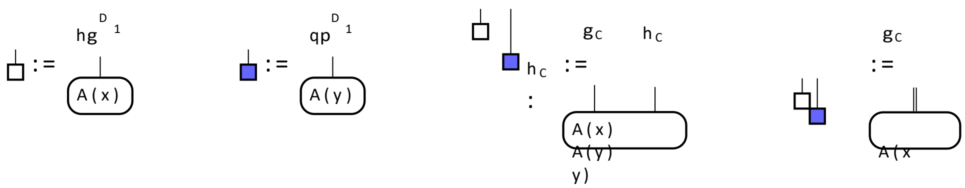
In this section, we supply the proofs from x4.3 which prove that the strict 2-functor $3Cat_G^{st}$! $GCrsBrd^{st}$ from Theorem 4.16 is an equivalence. We begin by expanding on Notation B.1.

Notation C.2. In this section, we use an expanded shorthand notation for 1-cells in D and D for proofs using commutative diagrams. For x_1 2 C(g_C ! h_C), x_2 2 C(h_C ! k_C), y_1 2 C(p_C ! q_C), and y_2 2 C(q_C ! r_C), we will denote the image under A after tensoring with the identity of the source object using small shaded squares with one strand coming out of the top, e.g.,

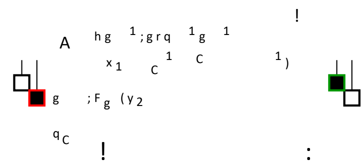
We denote the G-actions F_g^D and F_h^D as in Construction 4.8 by a red strand underneath the 1-morphism in C, where red corresponds to g and green corresponds to h. We denote A applied to the G-actions F_g^C and F^C by outlining the shaded square with red or green respectively, e.g.,



We use a similar convention for the composite of 1-cells as in Notation B.1. For example, if $x \in C_g = C(1_C \to g_C)$ and $y \in C_h = C(1_C \to h_C)$, we write

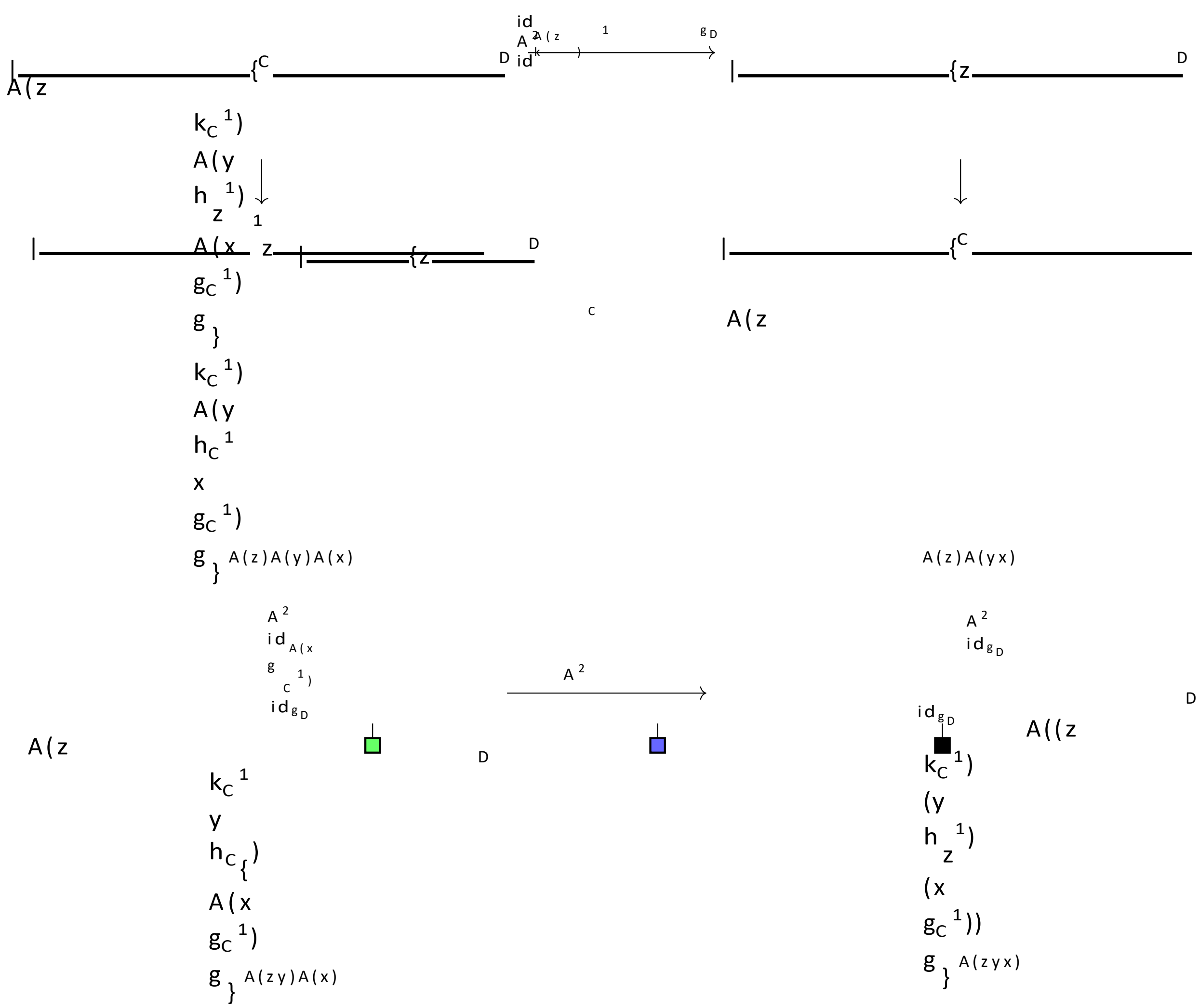


This means that by Construction 4.8 of the G-crossed braiding in C, we have that



Proof of Lem. 4.19: $(A; A^1; A^2) : C \to D$ is a 2-functor.

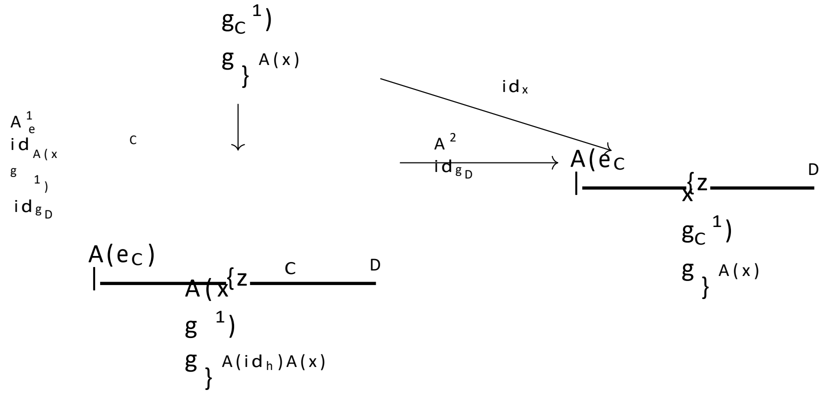
(F-I).i For $x \in C(g_C \to h_C)$, $y \in C(h_C \to k_C)$, and $z \in C(k_C \to \ell_C)$, using the nudging convention (8), the square for A^2 is exactly



which commutes by strictness of C; D and associativity of A^2.

(F-I).ii For $x \in C(g_C \to h_C)$, using the nudging convention (8), the lower triangle for A^1 and A^2 is exactly

e_D A(x



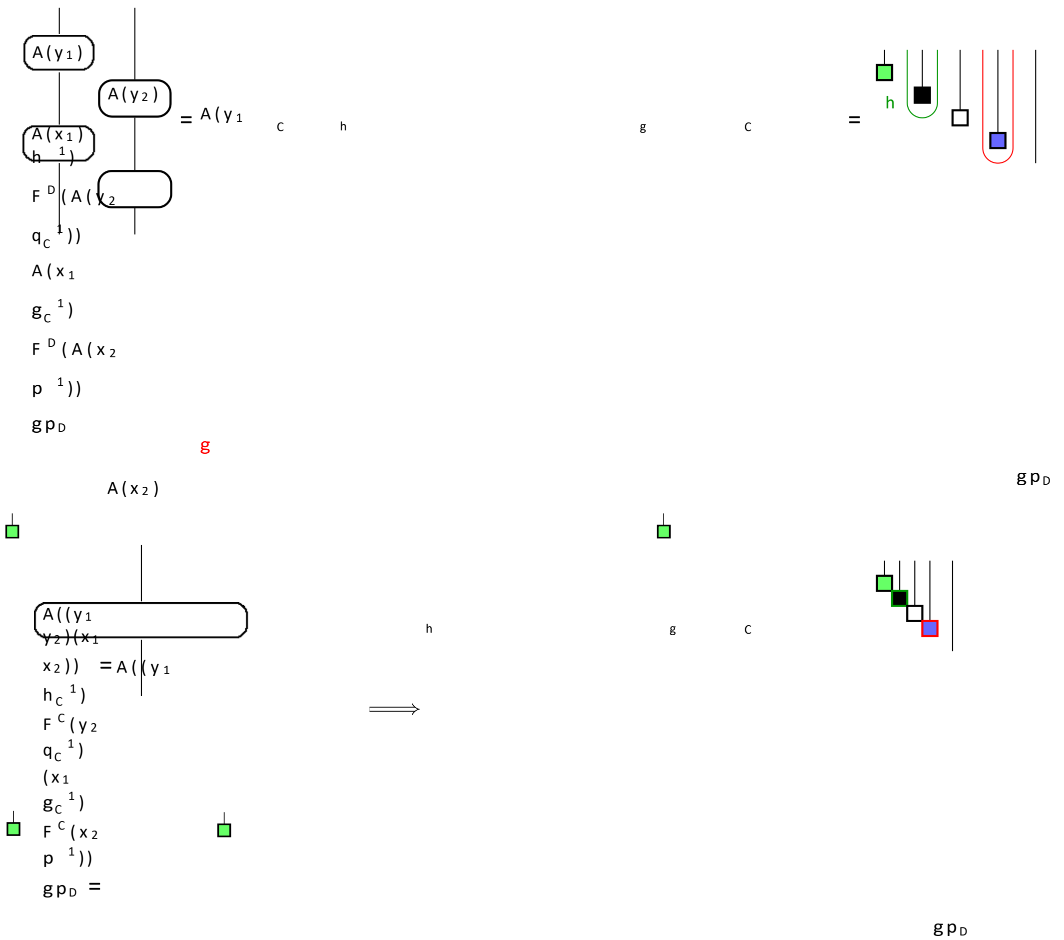
which commutes by unitality of $A^1; A^2$. The other triangle is similar. □

Proof of Thm. 4.21: $(A; {}^A; {}^A) \in 3\mathrm{Cat}^{st}_G(C \to D)$.

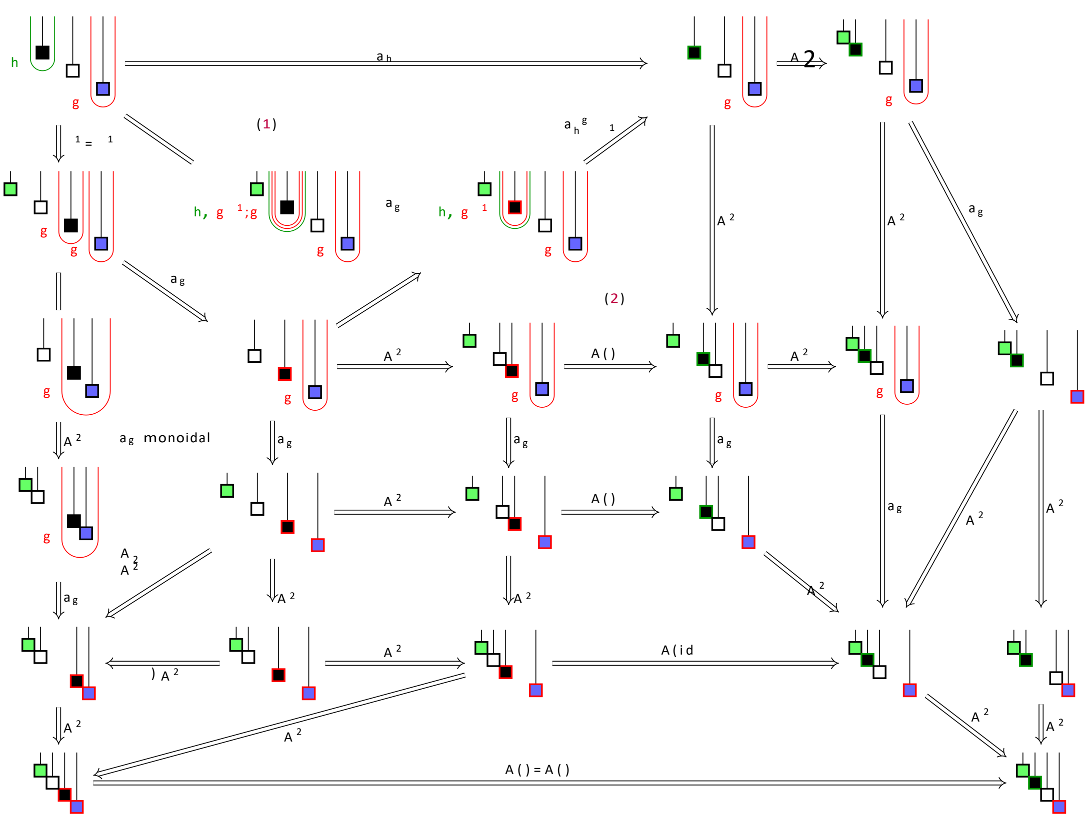
(F-II).i Each component in the definition of ${}_{y;x}^A$ is natural in x and y.

(F-II).ii For $g; h \in G$, $x_1 \in C(g_C \to h_C)$, $x_2 \in C(h_C \to k_C)$, $y_1 \in C(p_C \to q_C)$, and $y_2 \in C(q_C \to r_C)$, we use the following shorthand as in Notation B.1:

Observe that by the denition of A from A and the nudging convention (8), we have



Going around the outside of the diagram below corresponds to (F-II).ii, except we leave o the extra gp_D strand on the right hand side of each string diagram.



The faces without labels above commute either by naturality or by associativity of A^2.

(F-II).iii This follows since each F_g^D is strictly unital, and thus for all g 2 G,

$$A_e^1 = A^2_{id_e;id_e} (A_e{}^1 \quad 1 \quad 2 \quad {}_D \quad 1 \quad D \quad 1$$
$$A_e) = A_{id_e;F_g\ (id_e)} (A_e; F_g\ (A_e):$$

(F-III) This part is automatic as ${}_1{}^A = A_e^1$.

(F-IV) This follows by monoidality of a_g and associativity of A^2. We omit the full proof as it is much easier than (F-II).ii above.

(F-V) This reduces to unitality of A^1 and A^2, i.e., for all x 2 C(g_C ! h_C),

$$A^2_{id_e;x}$$
$$g_C{}^1\ (A1$$
$$id_{A(x}$$
$$g_C{}^1)) = id_{A(x}$$
$$g_C{}^1):$$

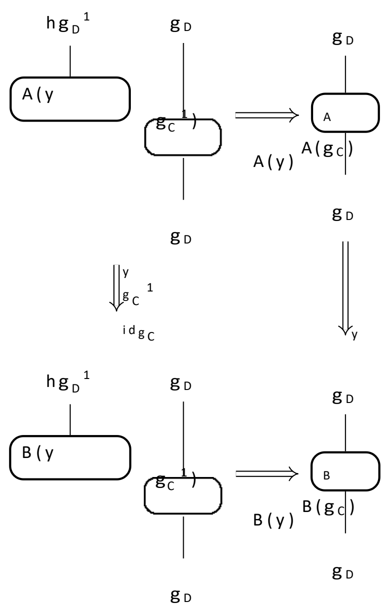
The other unitality axiom is similar.

(F-1) Every map is the identity map.

(F-2) Every map is the identity map. □

Proof of Thm. 4.22: the map $3\mathrm{Cat}^{st}_G(A \Rightarrow B) \to \mathrm{GCrsBrd}(A \Rightarrow B)$ is bijective.

Suppose that ; 2 $3\mathrm{Cat}^{st}_G(A \Rightarrow B)$ satisfy $_x$ = $_x$ for every x 2 C(1_C ! g_C) for all g 2 G. Since ; are 2-morphisms in $3\mathrm{Cat}^{st}_G$, we have = e_D = , $_g$ = id_{g_D} = $_g$ for all g 2 G. For an arbitrary y 2 C(g_C ! h_C), we have $^y_{g_C^{-1}}$ = $^y_{g_C^{-1}}$. By (T-IV) for : A ⇒ B, the following diagram commutes:



(22)

as does a similar diagram for replacing . Since $^y_{g_C^{-1}}$ = $^y_{g_C^{-1}}$ by assumption, $_{\mathrm{id}_g}$ = B^1 $(A^1)^{-1}$ = $_{\mathrm{id}_g}$ by (T-II).iii, and ^{A}; ^{B} are invertible 2-cells, we conclude that $_x$ = $_x$.

Now suppose h : A ⇒ B is a G-monoidal natural transformation. We define : A ⇒ B by = e_D, $_g$ = id_{g_D} for all g 2 G, and for y 2 C(g_C ! h_C), we use (22) above to define

$$_y := {}^B_{y\, g_C^{-1}; g_C} \, (h_{y\, g_C^{-1}} \, (B^1 \, (A_g)^{-1})) \, ({}^A_{y\, g_C^{-1}; g_C})^{-1}:$$

By construction, provided is a transformation ! h. It remains to verify that : A ⇒ B is indeed a transformation. We prove one of the coherences below, and we give a hint as how to proceed for the other coherences.

(T-II).i Every composite step in the definition of is natural.

(T-II).ii This follows from functoriality of 1-cell composition together with the fact that h is monoidal, and two instances each (one for each of A and B) of (F-I).ii, (21), and (F-IV).

(T-II).iii This follows using 2 instances of (F-II).iii (one for each of A and B) together with the fact that

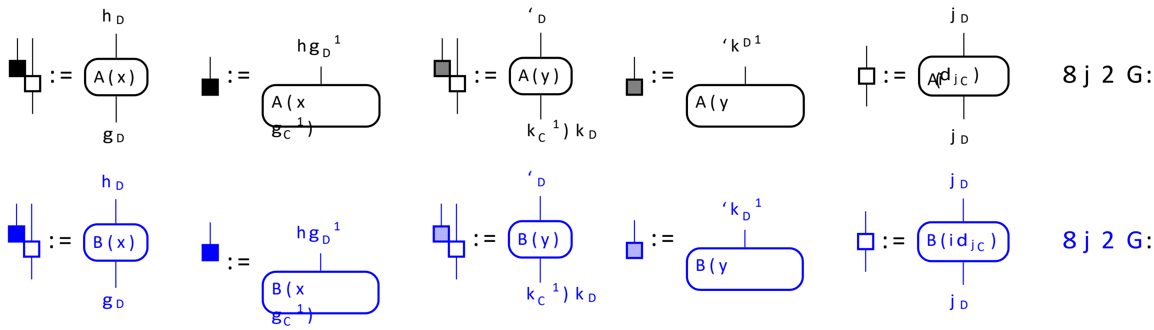
$$h_{\mathrm{id}_e} = B^1_e \, (A^1_e)^{-1} = B^1 \, (A^1)^{-1}$$

which is unitality for a monoidal natural transformation.

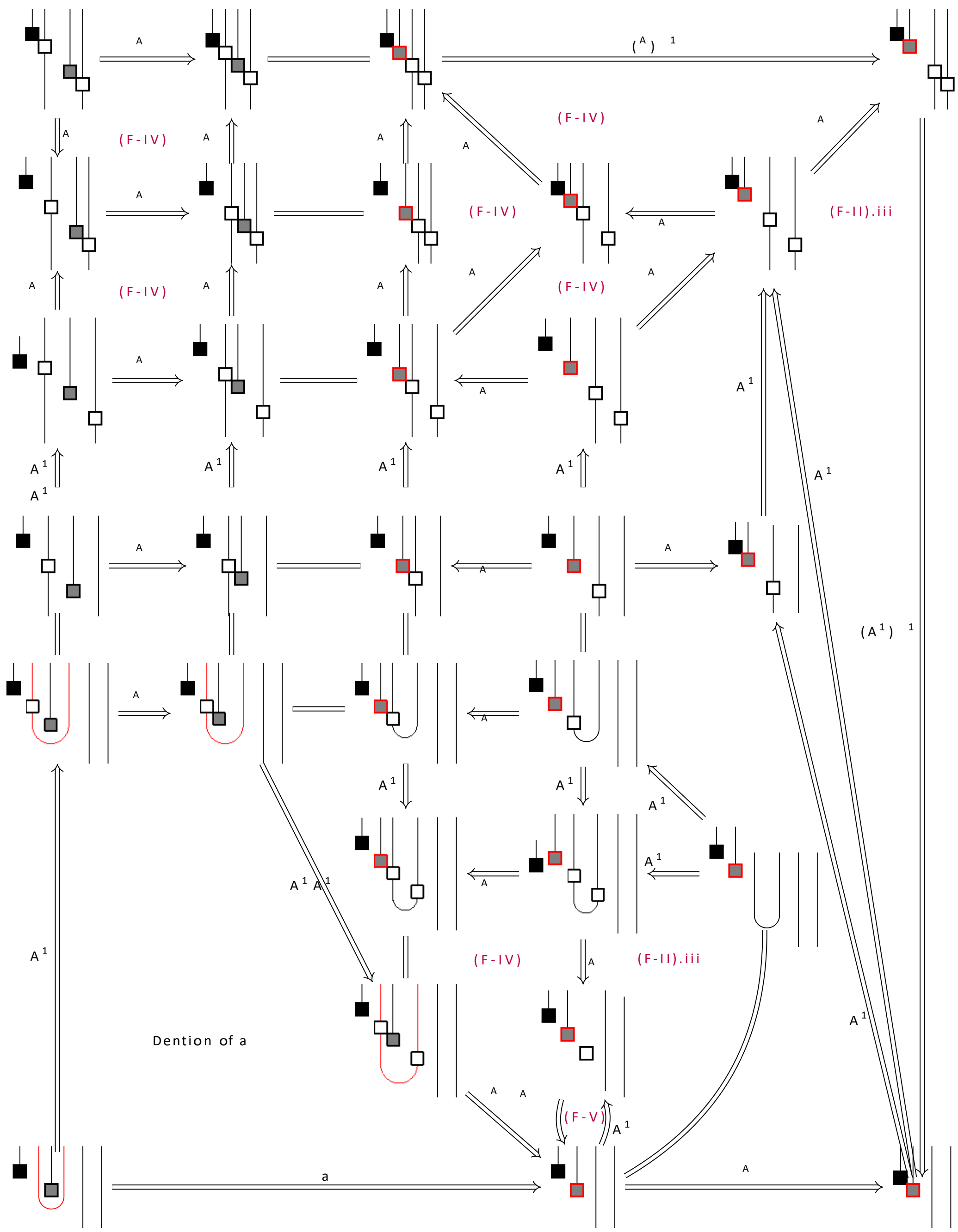
(T-III) This condition is automatically satisfied.

(T-IV) For $x \in \mathcal{C}(g_C \to h_C)$ and $y \in \mathcal{C}(k_C \to \ell_C)$, we use the following shorthand as in Notation B.1 and

Notation C.2:



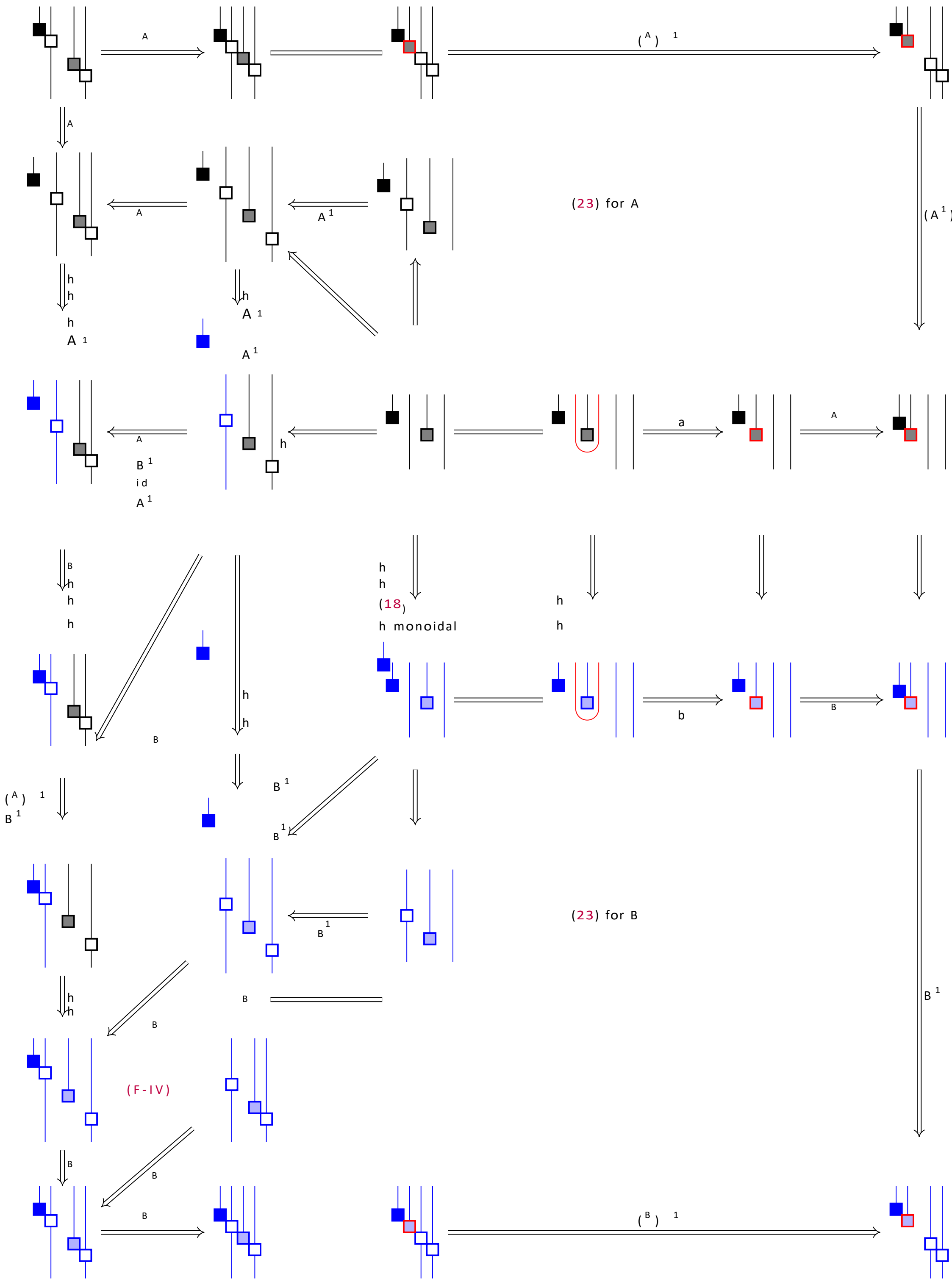
Suppose $x \in C(g_C \to h_C)$ and $y \in C(k_C \to \ell_C)$. We begin with the following observation that the following diagram commutes:



(23)

Observe that (23) above also holds with $(A; A^{1}; {}^{A}; A; a)$ replaced by $(B; B^{1}; {}^{B}; B; b)$.

Going around the outside of the diagram below corresponds to (T-IV), where we also use the abuse of notation of h for B^{1} (A^{1}) ^{1}.



The faces without labels above commute by functoriality of 1-cell composition or by the shorthand h = B^{1} (A^{1}) ^{1}.

(T-1) Every map is the identity map.

(T-2) Every map is the identity map.

□

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