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Haake–Lewenstein–Wilkens approach to spin-glasses revisited

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Abstract

We revisit the Haake–Lewenstein–Wilkens approach to Edwards–Anderson (EA) model of Ising spin glass (SG) (Haake *et al* 1985 *Phys. Rev. Lett.* **55** 2606). This approach consists in evaluation and analysis of the probability distribution of configurations of two replicas of the system, averaged over quenched disorder. This probability distribution generates squares of thermal copies of spin variables from the two copies of the systems, averaged over disorder, that is the terms that enter the standard definition of the original EA order parameter, q_{EA}. We use saddle point/steepest descent (SPSD) method to calculate the average of the Gaussian disorder in higher dimensions. This approximate result suggest that $q_{\mathrm{EA}} > 0$ at $0 < T < T_c$ in 3D and 4D. The case of 2D seems to be a little more subtle, since in the present approach energy increase for a domain wall competes with boundary/edge effects more strongly in 2D; still our approach predicts SG order at sufficiently low temperature. We speculate, how these predictions confirm/contradict widely spread opinions

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that: (i) There exist only one (up to the spin flip) ground state in EA model in 2D, 3D and 4D; (ii) there is (no) SG transition in 3D and 4D (2D). This paper is dedicated to the memories of Fritz Haake and Marek Cieplak.

Keywords: Edwards-Anderson order parameter,
Haake-Lewenstein-Wilkens approach, spin glass,
Saddle Point/Steppest Descend method

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Spin glass problem

Spin glasses (SGs) have entered solid state and statistical mechanics in the 1970s, and from the very beginning were considered to be one of the most outstanding and challenging problems of classical statistical physics and theory of disordered and complex systems [1, 2], not to mention their quantum version (see [3–5] and references therein). The most important and elaborated models of SGs are: the Edwards–Anderson (EA) model with short range interactions [6], and the Sherrington–Kirkpatrick (SK) model with infinite range interactions [7].

1.2. SK model

The SK model was solved approximately by its inventors using replica trick and replica symmetric solution of the equations that 'minimise' the free energy. This solution was clearly physically incorrect, leading to negative entropy at low temperature. This observation called for replica symmetry breaking, as suggested by de Almeida and Thouless [8]. G Parisi found an ingenious way to break the replica symmetry in a hierarchical way [9]. Parisi's solution of the SK model turned out to be exact, first as a local extremum of the free energy [10], and proven rigorously to be unique [11]. To deepen the understanding of this amazing results it is also recommended to consult the [12]. Parisi's solution and Parisi's order parameter, interpreted in terms of probability of overlaps between different frozen configurations of the SG is nowadays accepted commonly. For this achievement, and many others, G Parisi was awarded the Nobel Prize in physics in 2021 for 'the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales'.

1.3. EA model

In the case of EA model, we are very far from a rigorous solution. Most of our knowledge is based on numerical simulations on special purpose classical computers, going back to the 1980s [13, 14]. It is widely believed that for Ising EA model there is no SG transition at non-zero temperature in 2D, but there is in 3D and higher dimensions. It is not clear that Parisi's picture applies in these low dimensions; an alternative is provided by the 'droplet model' of [15], which predicts that there exist only one (up to the spin flip) ground state in EA model in 2D, 3D and 4D, but the domain walls (DWs), separating the flipped region, from not flipped one, are complex and might even have fractal dimension. Of course, there are some rigorous results concerning EA model (see [16]), but they are rather weak and very scarce. Thus, the question of the nature of the SG, as well as many other questions concerning the EA model, is open (see [2] and references therein). In recent years various aspects have been studied: ground

states in $J = \pm 1$ model [17], information theory approach to 3D EA models [18], absence of Almeida-Thouless line in 3D SG [19], or universality in such systems [20], and several other. The goal of this paper is to look at EA model more than 25 years after the publication of [21], revising the approach developed there.

1.4. HLW approach

In 1984 Universität Essen GHS initiated the extremely successful Sonderforschung Bereich 'Unordnung und groß e Fluktuazionen' with several neighboring centers. Fritz was a speaker of this initiative for the next 12 years. He convinced Maciej Lewenstein (his summer-time postdoc) and his new PhD student Martin Wilkens to study short-range SGs. They formulated a new approach to this problem, based on idea of studying disorder-averaged probability distribution for configurations of two replicas/copies of the system [21]. The idea of the Haake–Lewenstein–Wilkens (HLW) approach is as follows. We consider two replicas/copies of the system and evaluate the joint probability distribution of configurations averaged over the disorder:

$$P(\sigma,\sigma') = \left\langle\left\langle \exp\left\{-\beta\left[H(\sigma,\{K_{ij}\}_{\langle ij\rangle}) + H(\sigma',\{K_{ij}\}_{\langle ij\rangle})\right]\right\}/Z(\{K_{ij}\})^2\right\rangle\right\rangle, \tag{1}$$

where $H(\sigma,\{K_{ij}\}_{\langle ij\rangle}) = -\sum_{<ij>} K_{ij}\sigma_i\sigma_j$, $< ij >$ denotes nearest neighbors, $\langle\langle\cdot\rangle\rangle$ denotes average over disorder, and we denote for brevity $Z(\{K_{ij}\}) = Z(K)$. is the partition function calculated for a given configuration of the quenched disorder variables K_{ij}. We assume that K_{ij} are iidrv's (independent identically distributed random variables) with a Gaussian distribution, $P(K) = \exp(-K^2/2\Delta^2)/\sqrt{2\pi\Delta^2}$ or a binary distribution, $P(K = \pm\Delta) = 1/2$. Note that both distributions are even, that is, invariant under the change of sign of K_{ij}. The idea is to absorb the sign of $\sigma_i\sigma_j$ into $K_{ij} \to K_{ij}\sigma_i\sigma_j$, and introduce the spin overlap variables $\tau_i = \sigma_i\sigma_i'$. We obtain the effective probability distribution for τ's

$$P(\tau) = 2^N\left\langle\left\langle \exp\left[\beta\sum_{\langle ij\rangle} K_{ij}(1+\tau_i\tau_j)\right]/Z(K)^2\right\rangle\right\rangle. \tag{2}$$

Here the number of relevant variables is reduced as we summed over dummy variables. Note that magnetic order for τ's implies the non-zero EA order parameter q_{EA} and vice versa,

$$\left\langle\sum_i \tau_i\right\rangle_T N^{-1} = \left\langle\left\langle\sum_i \langle\sigma_i\sigma_i'\rangle_T/N\right\rangle\right\rangle = \sum_i\left\langle\left\langle \langle\sigma_i\rangle_T^2\right\rangle\right\rangle = q_{\mathrm{EA}}. \tag{3}$$

We term $\langle\cdot\rangle$ or $\langle\cdot\rangle_T$ the thermal average over possible configurations. Denoting $\alpha = \beta\Delta$, with Δ the parameter characterizing the probability distributions for the disorder, HLW used a convenient high temperature expansion to calculate (2) up to 12 order in the expansion parameters $\alpha^2/(1+\alpha^2)$ for the Gaussian, and $\tanh^2(\alpha)/(1+\tanh^2(\alpha))$ for binary case. In effect, they calculated

$$P(\tau) = \exp[-H_{\mathrm{eff}}(\alpha,\tau)]/Z_{\mathrm{eff}}, \tag{4}$$

where effective Hamiltonian contained nearest neighbors couplings $-K_1$, next nearest neighbors couplings K_2, and elementary plaquette terms, K_3. The coefficients of these terms were explicit functions of temperature (α in the notation of the present paper). The critical surface separating ferromagnetic from paramagnetic region was estimated then using (optimized) real

space renormalization group approach. It turned out that in 2D the H_{eff} never enters the ferromagnetic region, in 4D it enters the ferromagnetic region for sure, and in 3D the situation was not clear, suggesting that H_{eff} touches the critical region in a quadratic manner. That would imply that the critical exponents of the SG model are two times bigger than those of the standard Ising model, in agreement with the best numerical simulation available at that time.

1.5. HLW followers

The paper by HLW did not found too many followers, but some very prominent are worth mentioning. Indeed, R. Swendsen with collaborators published two papers on HLW method in *Phys. Rev.* B in the end of 1980s. In the first one by Wang and Swendsen [22], the authors studied Monte Carlo renormalization-group of Ising SGs. Application of this approach to the $\pm J$ Ising SG showed clear differences between 2D, 3D, and 4D models. The data were consistent with a zero-temperature transition in two dimensions, and non-zero temperature transitions in three and four dimensions. In another paper [23] Monte Carlo and high-temperature-expansion calculations of a spin-glass effective Hamiltonian were performed. The authors studied the quenched random-coupling spin-glass problem from the point of view of a nonrandom effective Hamiltonian, by Monte Carlo and high-temperature-expansion methods. It was found that the high-temperature series of the spin-glass effective Hamiltonian diverges below the ferromagnetic transition temperature. The Monte Carlo approach does give reliable results at low temperatures. The results were compared with the HLW picture of spin-glass phase transitions.

1.6. Present work

In this paper we revise HLW approach. The idea is to estimate $P(\tau)$, performing saddle point/steepest descent (SPSD) approximation in calculating the Gaussian average of the disorder, which should be correct in the limit $\alpha \to \infty$. We argue that the resulting spin model has couplings that are positive in the region where $\tau_i\tau_j = 1$'s, so it has tendency to order ferromagnetically on islands/domains, separated from other domains by negative couplings. In effect, boundary/edge effects start to play a role in estimates of various quantities that may characterize the order in our system.

We present here various arguments in favor or against the SG order (ferromagnetic order in overlap variables). First, we consider the original Peierls' argument [13, 25], and argue that in our situation, it can hardly be used. We turn then to an argument, studying sensitivity of the system to boundary conditions. This argument was originally proposed by Thouless [26–28] for models of electron propagation in the presence of disorder and subsequently adapted to study Ising models in random magnetic fields [29] (see also [30]), and also SGs [31]. This argument is relating the existence of the ferromagnetic phase transition to the sensitivity to boundary conditions. It can be trivially used for ferromagnetic spin models: it 'predicts' transition for $d \geqslant 2$ for Ising model, no transitions for $d = 2$ models with continuous symmetry (Mermin-Wagner-Hohenberg theorem), and transitions for $d \geqslant 3$ for systems with continuous symmetry, like XY or Heisenberg models (see [32]).

To apply this argument, we calculate $P_+ = P(\{\tau_i = 1\}_{\text{for all } i\text{'s}})$ on a cylinder in d dimensions of cross-section L^{d-1} and length L, and compare it to $P_- = P(\{\tau\} =$ corresponding to one domain wall). We analyze $\delta = \log(P_+/P_-)$ and argue that this quantity, within approximations used, is always positive and proportional to L^{d-1} in $d \geqslant 2$. We will argue that the situation in 2D seems to be a little more complex because of the stronger interplay between the boundary effects and the DW energy. This leads to significantly higher critical temperature in 2D than in higher dimensions.

2. SPSD calculations

We focus here on the case of Gaussian disorder, since we are going to use differential calculus. First, we rescale $K_{ij} = \alpha\Delta\kappa_{ij}$, so that both the logarithm of the distribution of κ_{ij}, and the logarithm of $P(\tau)$ become proportional to α^2 as $\alpha^2 \to \infty$. The HLW formula becomes

$$P(\tau) = 2^N \left\langle\left\langle \exp\left[\alpha^2 \sum_{\langle ij\rangle} \kappa_{ij}(1+\tau_i\tau_j)\right] / Z(\{\kappa_{ij}\})^2 \right\rangle\right\rangle, \tag{5}$$

where the average $\langle\langle\cdot\rangle\rangle$ is now with respect the distribution $P(\kappa) = \exp(-\alpha^2\kappa^2/2)/\sqrt{2\pi/\alpha^2}$.

2.1. Laplace's method

The idea is to calculate the asymptotic behavior of the disorder average using the Laplace method, also known as the SPSD method, which we expect to be asymptotically accurate for $\alpha \to \infty$. The SPSD equations equating to zero the first derivatives of the logarithm of the integrand with respect to κ_{ij}'s read:

$$0 = \alpha^2(-\kappa_{ij} + 1 + \tau_i\tau_j - 2\langle\sigma_i\sigma_j\rangle), \tag{6}$$

where $\langle\sigma_i\sigma_j\rangle$ is the thermal average of the neighboring spins correlator, calculated according to the canonical distribution $P(\sigma) = \exp[\alpha^2 \sum_{\langle ij\rangle} \kappa_{ij}\sigma_i\sigma_j]/Z(\kappa)$, where again for brevity we denote $Z(\{\kappa_{ij}\}) = Z(\kappa)$. There are two possibilities:

- $\tau_i\tau_j = 1$. In this case:

$$\kappa_{ij} = 2(1 - \langle\sigma_i\sigma_j\rangle) > 0, \tag{7}$$

  so that the corresponding coupling is clearly ferromagnetic.
- $\tau_i\tau_j = -1$. In this case

$$\kappa_{ij} = -2\langle\sigma_i\sigma_j\rangle, \tag{8}$$

  and the situation is more delicate. For α large, if $\langle\sigma_i\sigma_j\rangle > 0$, we expect the coupling κ_{ij} to be ferromagnetic, but the above equation implies the opposite. Likewise, if the correlation function is negative, the $\kappa_{ij} < 0$ should be ferromagnetic. The contradiction could be avoided if $\kappa_{ij} = 0$, but the true situation is more complex, as we will see below, by solving systematically mean field equations. This contradiction is really an expression of frustration in our system!

It follows that we can write the SPSD solutions as $\kappa_{ij} > 0$ on the domains, where neighboring $\tau_i\tau_j = 1$. This solution has a very clear meaning: the canonical ensemble that serves to calculate the correlation functions $\langle\sigma_i\sigma_j\rangle$ corresponds to ferromagnetic islands/domains (where $\tau_i\tau_j = 1$), separated by DWs, where the bonds $\kappa_{ij} \leqslant 0$, $\tau_i\tau_j = -1$, and the correlations between σ's from different DWs are still positive, but perhaps smaller at the border.

Note that the situation we consider is not as in the standard SG, where we look at $\langle\sigma_i\sigma_j\rangle$ for a fixed configuration of random κ_{ij}'s. There, it is quite common that the sign of κ_{ij} is not equal to the sign of $\langle\sigma_i\sigma_j\rangle$: this is actually how the frustration exhibits itself basically! Here, however, we consider a different situation: for a given configuration of τ's, we adjust the values of κ_{ij}'s to satisfy the SPSD equations. The natural expectation is a ferromagnetic order for τ's (i.e. SG order for σ's) in our system, with the energy (free energy/probability) cost of the energy wall to scale as L^{d-1}, as in, say, the standard Ising ferromagnet. At the same

time, we cannot exclude the existence of other solutions of SPSD equations that would inherit frustration more explicitly. We discuss this possibility, which goes beyond the scope of the present paper, in the outlook.

2.2. Hessian matrix

In the zeroth order one can calculate now $P(\tau)$, substituting for κ_{ij}'s their SPSD values. One can go one step further calculating the Gaussian correction to the SPSD. To this aim we calculate the Hessian matrix of the second derivatives of the logarithm of the integrand. Let us introduce the shortened notation $(ij) = \mu$, $(i'j') = \nu$, $\sigma_i\sigma_j = c_\mu$, $\sigma_{i'}\sigma_{j'} = c_\nu$, etc. The Hessian matrix reads

$$\mathscr{H}_{\mu\nu} = -\alpha^2[\delta_{\mu\nu} + \alpha^2[\langle c_\mu c_\nu\rangle - \langle c_\mu\rangle\langle c_\nu\rangle]]. \tag{9}$$

Note that the correlations matrix

$$\langle c_\mu c_\nu\rangle - \langle c_\mu\rangle\langle c_\nu\rangle = \langle (c_\mu - \langle c_\mu\rangle)(c_\nu - \langle c_\nu\rangle)\rangle, \tag{10}$$

i.e. it is explicitly positively semi-definite. In effect the Hessian matrix:

$$\hat{\mathscr{H}} < 0, \tag{11}$$

so that the logarithm of the integrated function, which we consider is a strictly convex function of many variables, is expected to have one maximum, corresponding to our SPSD solutions. Note also that eigenvalues of the Hessian matrix are all negative and will typically be of order α^4, and they are bounded in modulus from below by α^2. One should thus expect that SPSD method should become for $\alpha \to \infty$ asymptotically very precise, if not exact.

3. Peierls and Thouless approaches

In this section we examine if the τ variables of our effective model for two copies of the EA systems exhibite ferromagnetic order i.e. if the EA order parameter is nonzero, signifying SG order. We present two approaches: (i) Peierls approach; (ii) Thouless approach; in the latter case we first discuss several analytic estimates, and then present self-consistent calculations, using SPSD solutions for κ_{ij}'s as a point of departure for local mean field calculations of the averages of τ's and $\tau - \tau$ correlations.

3.1. Peierls approach

Peierls considers DWs in a square lattice in 2D, defining them in an unambiguous way. In a ferromagnetic Ising model with the uniform coupling K (with β absorbed into K), and with periodic boundary conditions on a square of side L, and number of sites $N = L^2$, with all spins $\tau_i = 1$ on the boundary, all DWs are closed. Let b denote the length of the domain's boundary; We classify them according to length b, and within a class of given length we give each a number i. A wall of the length b fits into a square of the side $b/4$ and area $b^2/16$. Let $m(b)$ be the number of DWs of length b; it is obviously bounded by $m(b) \leqslant 4N3^{b-1}$. The next step is to consider the quantity $X(b,i) = 1$, if the DW (b,i) occurs in that configuration, and $X(b,i) = 0$ otherwise. Clearly, the number of spins down fulfills:

$$N_-/N \leqslant \sum_b (b^2/16)3^b \sum_i^{m(b)} X(b,i). \tag{12}$$

Peierls estimates then the thermal average of $X(b,i)$ in the Gibbs–Boltzmann ensemble, bounding the partition function from below by the contribution of the configuration, in which all spins inside the considered domain were flipped, and obtaining the bound $\langle X(b,i)\rangle \leqslant \exp[-2\beta Kb]$. Stacking this in the above equation we get

$$\begin{aligned} N_-/N &\leqslant \sum_b (b^2/16)3^b \sum_i^{m(b)} X(b,i) \\ &\leqslant \sum_b (b^2/16)3^b \sum_i^{m(b)} \exp[-2\beta Kb]. \end{aligned} \tag{13}$$

The sum over b on RHS of equation (13) is obviously (i) convergent to a finite (independent of L for large L) value; (ii) and the value of the sum tend to zero as βK tends to infinity. This leads to the desired result that N_-/N becomes clearly smaller than one at sufficiently low temperatures. Notably, this argument can be generalized to higher dimensions, with a little extra effort to estimate the entropy of contours, see [33] for an elementary discussion and references therein for original work.

Unfortunately, we cannot use this reasoning, because in our case: (i) couplings are non-homogeneous; (ii) their values depend on DWs configurations, according to SPSD equations. We can estimate that the configuration C, in which the domain (b,i) occurs, has contribution to the 'energy' coming from two edges, $4\alpha^2\kappa_{\rm e}$, where $\kappa_{\rm e}$ is the coupling on the edge. The configuration $\tilde{C}$, in which the spins inside the wall are flipped, contributes to $Z(\kappa)$ with the energy larger by $3\alpha^2\kappa$, with κ being the coupling in the bulk, so that $\langle X(b,i)\rangle \leqslant \exp\{\alpha^2[4\kappa_{\rm e} - 6\ \kappa]b\}$. Since, according to mean field, $\kappa_{\rm e} > \kappa$, the question is to be able to estimate more precisely the interplay of the edge and bulk contributions. To this aim we turn, however, to a simpler Thouless argument, to decide about the existence of the magnetization, i.e. SG order.

3.2. Thouless argument

In order to investigate the sensitivity to boundary conditions, we calculate $P_+ = P(\{\tau_i = 1\}_{\text{for all } i\text{'s}})$ on a cylinder in d dimensions of cross-section L^{d-1} and length L, and compare it to $P_- = P(\{\tau_i = 1\}_{\text{on the left}}, \{\tau_i = -1\}_{\text{on the right}})$ with $\tau_i = \pm 1$ on the left (right) of a DW, correspondingly. We determine the parameter $\delta = \ln(P_+/P_-)$; Ferromagnetic order for τ's (SG order for σ's) is indicated by $\delta > 0$

We consider a lattice with coordination number f, with $f_{\rm out}$ bonds sticking out at any site of any L^{d-1}-dimensional hyper-plane (cross-section). As we will see, we will need to compare the effects of DW and boundary effects, since both scale as L^{d-1}. To this end we will also consider effective coordination number at the edge (boundary) hyper-planes, $f_{\rm e}$. Geometrically, $f_{\rm e} = f - f_{\rm out}/2$ for the left and right edge (boundary) hyper-planes—only half of $f_{\rm out}$ stick out to the right (left) from the left (right) edge. This is evidently a good estimate for $f_{\rm e}$ in higher dimensions, where we expect $f_{\rm out} \ll f$. On the other hand, boundary effects, in general, do extend to more than just the edge layer, but also the next, and next-to-next one, so the values of spins in all of these layers (entering the bulk) are affected; it is thus reasonable to approximate

$$f_{\rm e} \leqslant f - f_{\rm out}. \tag{14}$$

Obviously, it is a purely phenomenological, qualitative expectation without much quantitative rigor.

We stress that we are NOT considering here DWs in the disordered EA model, where they are believed to a have a very complex geometry, scaling and maybe even effective dimension, in accordance with the seminal droplet model [15]. We are studying here DWs in the effective, averaged over disorder, probability distribution of the τ variables. Just from the construction, there are no reasons for this probability distribution to break translation symmetry (everywhere, i.e. in d dimensions, if we apply global periodic boundary conditions, or at least in $(d-1)$ transverse dimensions, if we apply periodic boundary conditions there). It is thus natural to look in the first place for DWs that are just flat hyper-planes.

4. Self-consistent SPSD and local MF solutions

In the following we focus on hyper-cubic lattices in d-dimensions, with coordination number $f = 2d$, $f_{\text{out}} = 2$, and $f_{\text{e}} \leqslant 2d-1$. We leave the preliminary discussion of other lattices to appendix C and future publication. In this section, we estimate the bulk and the edge contributions applying SPSD and local mean field theory (MF) consistently from the beginning till the end. We consider a d-dimensional cylinder of spins with L layers with bonds distributed according to a Gaussian distribution $P(K) = \exp(-K^2/2\Delta^2)/\sqrt{2\pi\Delta^2}$, at an inverse temperature β. We denote as above $\alpha = \Delta\beta$.

4.1. Local mean field theory

We assume translation symmetry in $d-1$ transverse dimensions, so that magnetization depend only on one index, i, enumerating the layer, and the couplings depend on two indices, enumerating involved single layer (two neighboring layers). Using standard MF, we find the magnetization that is the thermal average of $\langle\sigma_i\rangle$ at the ith layer as

$$m_i = \tanh\left[\alpha^2\ \left(2\,(d-1)\,\kappa_{i,i}m_i + \kappa_{i,i+1}m_{i+1} + \kappa_{i-1,i}m_{i-1}\right)\right], \tag{15}$$

with

$$\kappa_{i,j} = 2 - 2m_i m_j, \tag{16}$$

and boundary conditions:

$$m_0 = m_{L+1} = 0, \tag{17}$$

$$\kappa_{0,1} = \kappa_{L,L+1} = 0. \tag{18}$$

Note that these local mean field equations are, in a certain sense, analogues of the Thouless–Anderson–Palmer equations for disordered models; at least they look indeed quite similar, see [38].

4.2. Quantities to be determined

Our aim is to calculate logarithm of the probability P_+, P_- and $\delta = \ln(P_+/P_-)$. We denote $\ln(P_\pm) = H_\pm$, and call it 'energy' in the following, so that

$$H_+ = L\ln(2) + \alpha^2 \sum_{i,j}[2\kappa_{i,j} - \kappa_{i,j}^2/2] - 2\ln Z(\kappa). \tag{19}$$

Being an extensive quantity, the energy of the system divided by the volume in all but one dimension is

$$\frac{H_+}{L^{d-1}} = L\ln(2) + \alpha^2 \left[(d-1)\sum_{i=1}^{L}[2\kappa_{i,i} - \kappa_{i,i}^2/2] + \sum_{i=1}^{L-1}[2\kappa_{i,i+1} - \kappa_{i,i+1}^2/2] \right] - \sum_{i=1}^{L} 2\ln\left[2\cosh(F_i(m))\right] - \frac{1}{2}\ln(\det(\hat{H}_+)), \tag{20}$$

with $F_i(m) = \alpha^2(2(d-1)\kappa_{i,i}m_i + \kappa_{i-1,i}m_{i-1} + \kappa_{i,i+1}m_{i+1})$. Note that we have included in this expression the term coming from the Gaussian fluctuations around the SPSD solution. The above quantity in the leading order should be a linear function of the cylinder's length,

$$H_+/L^{d-1} = A(\alpha)L + B_+(\alpha). \tag{21}$$

A similar expression holds for P_-, also including Gaussian fluctuations terms:

$$\frac{H_-}{L^{d-1}} = L\ln(2) + \alpha^2 \left[(d-1)\sum_{i=1}^{L}[2\kappa_{i,i} - \kappa_{i,i}^2/2] + \sum_{i=1}^{L/2-1}[2\kappa_{i,i+1} - \kappa_{i,i+1}^2/2] + \sum_{i=L/2+1}^{L-1}[2\kappa_{i,i+1} - \kappa_{i,i+1}^2/2] - \kappa_{L/2,L/2+1}^2/2 \right] - \sum_{i=1}^{L} 2\ln\left[2\cosh(F_i(m))\right] - \frac{1}{2}\ln(\det(\hat{H}_-)). \tag{22}$$

Since configuration contributing to P_- has connection between two layers in the middle of the cylinder given by a different expression, clearly

$$H_-/L^{d-1} = A(\alpha)L + B_-(\alpha), \tag{23}$$

with the same bulk contribution, but different boundary term; thus

$$\delta = B_+(\alpha) - B_-(\alpha). \tag{24}$$

Positive value of δ indicates ferromagnetic order for τ's and SG order for σ's.

To calculate H_-/L^{d-1} we repeat the above calculations using the same formulae as before, except that we use

$$\kappa_{L/2,L/2+1} = -2m_{L/2}m_{L/2+1}. \tag{25}$$

4.3. Gaussian fluctuation terms

Generally speaking, Gaussian fluctuation terms play a sub-leading role, as expected. We approximate $\ln(\det(\hat{H}_\pm)) = \sum_\mu \ln(\lambda_\mu) \approx \sum_\mu \ln(\hat{H}_{\mu\mu})$, that is the sum of logarithms of eigenvalues by the sum of logarithms of diagonal elements of the Hessian matrix. Noting that

$$\partial\langle\sigma_i\sigma_j\rangle/\partial\kappa_{ij} = \alpha^2\left(1 - m_i^2 m_j^2\right), \tag{26}$$

we obtain

$$-\frac{1}{2}\ln(\det(\hat{H}_\pm)) \approx -\frac{1}{2}\sum_{(ij)}\ln[\alpha^2(1+\alpha^2(1-m_i^2m_j^2)], \tag{27}$$

where the SPSD solutions for m's are calculated for the case $\pm$ accordingly. The above expression undergoes, obviously, further simplifications under the translation symmetry.

4.4. High α regime

Before going to numerical solutions, we first analyze the asymptotic regime $\alpha \to \infty$, where $A(\alpha)$ can be estimated analytically. We consider MF equations in the bulk of the d-dimensional hyper-cubic lattice. The corresponding self-consistent equations in the bulk are:

$$\kappa = 2(1-m^2), \tag{28}$$

$$m = \tanh(2\alpha^2 d\kappa m), \tag{29}$$

$$g = \ln(2\cosh(2\alpha^2 d\kappa m). \tag{30}$$

We transform the first two into an equation of $x = 2\alpha^2 d\kappa$.

$$x = 4\alpha^2 d/\cosh^2\left(x\sqrt{1-\kappa/(4\alpha^2 d2)}\right). \tag{31}$$

For large α we get $x = \ln(4\alpha^2 d)/2$, and $\kappa = \ln(4\alpha^2 d)/(4\alpha^2 d)$. As expected, $\kappa \to 0$ as $\alpha \to \infty$, and $m \to 1$, but $2\alpha^2 d\kappa$ diverges as $\ln(4\alpha^2 d)$. Elementary analysis leads to the result:

$$A(\alpha) \simeq \ln(2) - \frac{1}{2}\ln(4\alpha^2 d), \tag{32}$$

i.e. as expected $\ln(P_\pm) = H_\pm$ becomes negative at large L (when our analysis makes sense) and at large α (when SPSD should work well); $A(\alpha)$ diverges with α, but very slowly, only logarithmically.

In calculation of asymptotic behavior of $\alpha^2\kappa$. we typically set local magnetization to 1: they indeed tend to one, but in slightly different way in the bulk and on the ends, as the numeric illustrates below. If we set $m_i = 1$ in equation (20), and expand for large α, then we obtain a simple expression for

$$\frac{H_+}{L^{d-1}} = L\ln(2) - 2\alpha^2\left[(d-1)\sum_{i=1}^{L}\kappa_{i,i} + \sum_{i=1}^{L-1}\kappa_{i,i+1}\right], \tag{33}$$

neglecting sub-leading Gaussian corrections. Since our numerical analysis in the asymptotic regime is tough, we may and will use this expression there. The analysis is more complex in the case of $\frac{H_-}{L^{d-1}}$, where we need to take into account the dramatic change of the nature of SPSD solutions at the DW.

4.5. 'Phase transition' at moderate α

The solution of the MF equations change character as α grows from small values (when all $m_i = 0$) to larger values (when all $m_i \neq 0$). We infer the existence of this 'phase transition' at a

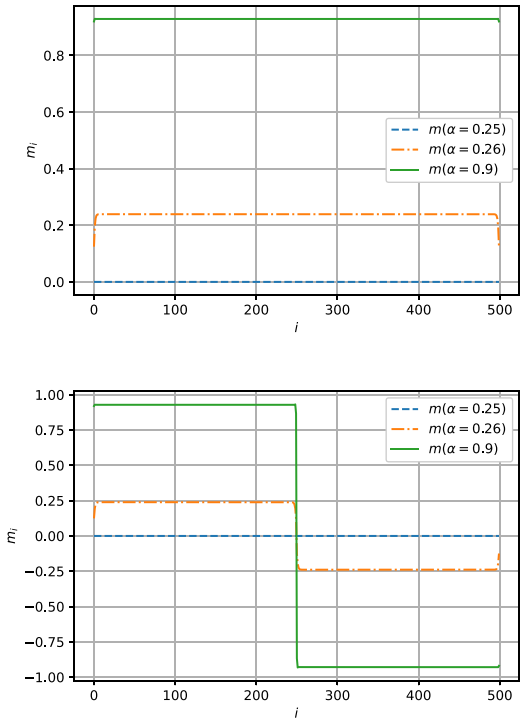

Figure 1. Numerical solutions of the system of equation (15) for $d = 4$, $L = 500$ at various temperatures, corresponding to the case of P_+ (upper panel) and P_- (lower panel). In the latter case, the solutions change the sign of m's in the middle (and the keep the sign of κ's positive). Solutions for d equal to 2 and 3 are qualitatively the same, and quantitatively very similar.

finite α by imposing that solutions get trivial at that point, α_T. This way we can approximate equation (15) for temperatures close to α_T as a series expansion for small m_i to get:

$$m_i \approx \alpha^2 \left(2\left(d-1\right)\kappa_i m_i + \kappa_{i-1,i} m_{i-1} \kappa_{i+1} m_{i+1}\right). \tag{34}$$

To first order, $\kappa_{i,j} = 2$ and $m_i = m\ \forall i$, so we find the critical temperature:

$$\alpha_T = \frac{1}{2\sqrt{d}}. \tag{35}$$

4.6. Numerical calculations

By numerically solving the system of equations $F(m) = m_i$ for $1 \leqslant i \leqslant L$ and taking into account that in positions $i = \{0, L+1\}$ there are no spins and therefore conditions equations (17) and (18) apply, we find non-trivial solutions above a certain temperature threshold, see figure 1.

We solve the system of equations for various lengths L and fit the obtained results in order to obtain A and B at different temperatures, figure 2. We do so for dimensions $d = 2, 3, 4$ and obtain similar behaviors. As expected, MF solutions for all three systems undergo a 'phase transition' from $m = 0$ to $m \neq 0$ at their respective critical temperatures, $\alpha_T^{d=(2,3,4)} = \left\{\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{3}}, \frac{1}{4}\right\}$.

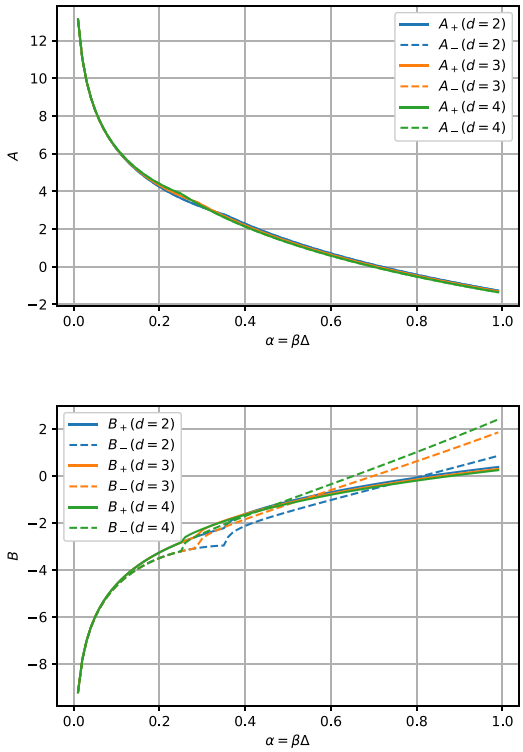

Figure 2. Numerical solutions for A (upper panel) and B_+ and B_- (lower panel) for d equal 2, 3, 4, and at various temperatures. Gaussian fluctuations contributions are included.

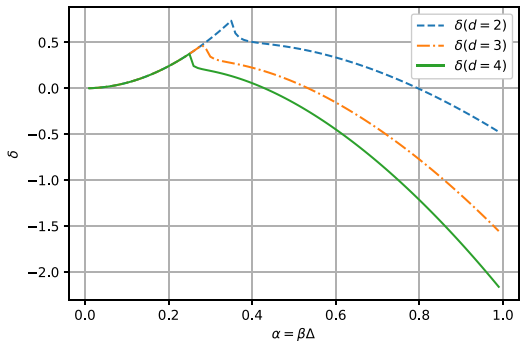

Figure 3. The parameter δ for $d=2,3,4$ as a function of α (temperature). Gaussian fluctuations contribution is included.

The results show in accordance with analytic calculations that $A(\alpha)$ tends to $-\infty$ logarithmically. On the other hand, $B_+(\alpha)$ tend to a positive constant for large α, while $B_-(\alpha)$ to infinity, indicating SG transition in $2\,D$ (unfortunately), $3\,D$ (fortunately), and $4\,D$ (fortunately). This is illustrated clearly in figure 3, Still, one observes quite a quantitative difference in behavior for $d=2$ and d larger.

5. Conclusions and outlook

In the short note we revised the HLW approach to EA model of Ising SG. The main results are the following:

(a) We have calculated the disorder averaged probability of spin configurations for two replicas, which reduces to a probability of overlaps between spins from the two replicas, $P(\tau)$. To this aim we used the SPSD method which seems to be asymptotically exact in the limit of $\alpha = \beta\Delta$ going to infinity. The integral we consider, has an integrand, whose logarithm has a well peaked single maximum, with the Hessian of order at least α^2, if not α. It would be challenging to study if one can control this result rigorously.
(b) We attempted to apply Peierls and Thouless approaches to decide whether there exist SG order in the low temperature (large α^2 limit). The results indicate that this indeed is the case in 2D and above, but we identified the reasons, why this does not have to be the case in 2D. Namely, the competing boundary effects might destroy the order. Our estimates, based on MF, clearly require improvement, for instance by studying precisely the solutions of boundary effects in SPSD equations etc. If we accept the proposed form of the solutions of the SPSD solutions, the simulating P_+ requires MC simulations of a finite size ferromagnetic model, while simulating P_-—also a finite size ferromagnetic model with a DW and a bump/dip in the couplings at the wall.
(c) In a nutshell: Our results predict SG transition in EA model in $4\,d$, $3\,d$, but unfortunately also in $2\,d$. There can be several reasons for that: (i) SPSD approximation is not precise enough; (ii) is completely incorrect; In the first case we can include Gaussian and maybe even beyond Gaussian corrections to SPSD solutions. In the second case, there might be many SPSD solutions contributing or something like that; Hessian result suggests this is not the case, but it is not rigorous; (iii) finally, local MF calculations of edge/boundary effects might be too rough.
(d) The paper contains four appendices: In appendix A we discuss shortly the exactly soluble 1D case, in appendix B—the normalization of $P(\tau)$ that implies nice properties of certain multidimensional integrals. Of course, the present results are compatible with the expectation that there exist only one (up to the spin flip) ground state in EA model in 2D and 3D [17]. Another interesting conclusion is that the existences of the SG transition in the present picture, might depend on the connectivity of the lattice. As discussed in appendix C, even within our SPSD and MF DWs have a certain width. This might depend crucially on the dimension and even on the coordination number (connectivity) of the lattice. Finally, alternative way of calculations combining SPSD method with the expected behavior of $Z(\kappa)$ for large α is discussed in appendix D. This method explicitly accounts for dependence of correlators $\langle\sigma_i\sigma_j\rangle$ on κ_{ij},

Our somewhat self-critical conclusions call for other approaches to the problem of precise and reliable calculations of the disorder averaged probability of spin configurations for two replicas, which reduces to a probability of overlaps between spins from the two replicas, $P(\tau)$. We thank the referees for these suggestions.

- *Ad* (C) We could improve our calculations of $P(\tau)$, coming back to the high temperature expansion, as used in [21], but using more sophisticated version of it. For instance, there exists a high-temperature (or high-dimensional) series expansion technique to construct the free energy in spin systems, disordered or not, and developed in [34]. This approach has been applied to the spin-glass model on hypercubic lattices in D dimension [35]. Although these

are still cumbersome mean-field calculations, one can hope to get the critical temperature value of the spin-glass transition at larger dimensions more accurately.

- *Ad* (B) We could improve our application of Peierls and Thouless approach to decide whether there exist SG order in the low temperature (large α^2 limit). For instance, we could adapt the method of defect wall renormalization group, as developed and applied by M. Gingras to anisotropic vector spin glass model in [36, 37].

We leave these two suggestions to the future investigations.
Clearly, this study requires further studies, but this goes beyond the present note.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Exact solution 1D

Calculation of $P(\tau)$ in 1D are elementary. We observe first that

$$Z(\kappa) = \prod_{i=1}^{i=L-1} 2\cosh(\alpha^2 \kappa_{i,i+1}), \tag{A1}$$

so that $P(\tau)$ can be written as

$$P(\tau) = \left\langle\left\langle \frac{2^L}{Z^2(\kappa)} \prod_{i=1}^{i=L-1} \left[\cosh(\alpha^2\kappa_{i,i+1}) + \sinh(\alpha^2\kappa_{i,i+1})\right] \right.\right. \\ \left.\left. \left[\cosh(\alpha^2\kappa_{i,i+1}) + \tau_i\tau_{i+1}\sinh(\alpha^2\kappa_{i,i+1})\right] \right\rangle\right\rangle. \tag{A2}$$

Since we average over the even distributions the terms $\cosh(.)\sinh(.)$ average zero, and we get

$$P(\tau) = 2\prod_{i=1}^{i=L-1} \left[1 + \tau_i\tau_{i+1}\langle\langle\tanh^2(\alpha^2\kappa)\rangle\rangle\right], \tag{A3}$$

where we skipped the subscript of κ. We can again estimate $\langle\langle\tanh^2(\alpha^2\kappa)\rangle\rangle$ using SPSD. Saddle point value for $2\alpha^2\kappa$ diverges again as $\ln(4\alpha^2)$, so the 1D system exhibits a 'phase transition' at zero temperature ($\alpha \to \infty$) with diverging correlation length $\xi \propto \alpha^2$.

Appendix B. Normalization issues—amazing formulae

Note that if we observe that $P(\tau)$, by definition is normalized

$$P(\tau) = 2^N \left\langle\left\langle \exp\left[\beta\sum_{\langle ij\rangle} K_{ij}(1+\tau_i\tau_j)\right] / Z(K)^2 \right\rangle\right\rangle, \tag{B1}$$

then by tracing over τ's we obtain

$$1 = 2^N \left\langle\left\langle \exp\left[\beta\sum_{\langle ij\rangle} K_{ij}\right] / Z(K) \right\rangle\right\rangle. \tag{B2}$$

The above expression is true for any even distribution of K's, Gaussian or not, discrete or continuous. It can be generalized to certain matrix models with couplings invariant with respect to local unitary transformations. The independent proof of this formula employs the fact that $2^N = \sum_\sigma 1$. Using the above formula and then incorporating each of the configurations of σ's into the averaging over disorder, gives the desired identity.

Appendix C. Domain wall width

It is worth noticing the DWs in the case of P_- have a finite width. This means that local magnetization m_i does not jump from nearly one to nearly minus one (see figure 4). In effect, κ's in the DW regions are not so close to zero, and the terms $\alpha^2\kappa$ simply behave in this region as α^2. This explain the rapid growth of B_- in figure 2.

Sharpening of the DW to the configuration that $m = \pm 1$ in the bulk, and $m = 0$ at the DW edges would lead presumably to instability of the ferromagnetic phases. In fact we have

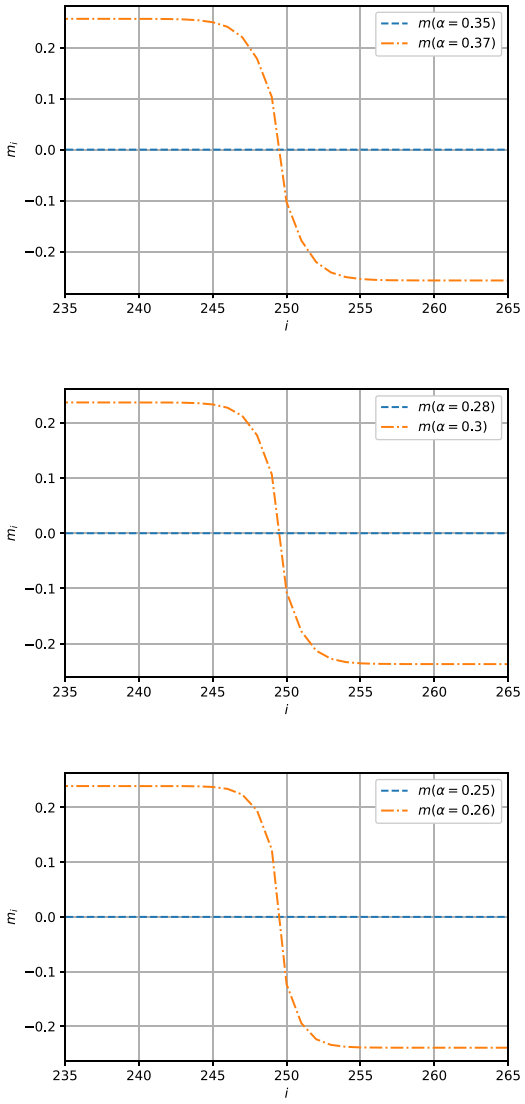

Figure 4. Numerical solutions for magnetization in the DW region for $d=2$ (upper panel), $d=3$ (middle panel), and for $d=4$ (c) at indicated temperatures.

originally postulated (incorrect) solutions of SPSD equations with $\kappa=0$ at the walls. Such solution leads to $B_-=2B_+$—it still predicts the ferromagnetic order, but with very different, much more milder behavior of δ. Conversely, widening the wall, more in the spirit of the 'droplet model' might also lead to unexpected behavior, since the assumption that $\ln(P_{\pm1})=AL+B_{\pm1}$ would then cease to hold.

Our numerical findings with the SPSD and MF approximations indicate that: (i) for fixed α^2, the DW reaches an L-independent limit for L large; (ii) for fixed L, the DW shinks from L (below phase transtion, where all m's are zero), to a very small values dictated by the very fast growth of $|m|$'s toward one, in accordance with the MF laws.

Appendix D. Alternative approach

Here we propose alternative way of calculating δ based on expected behavior of $\ln(Z(K))$ for low temperatures. Namely, we expect that

$$-2\ln(Z(K)) = 2\beta F \simeq 2\beta\langle U\rangle, \tag{D1}$$

where U is the internal energy. That means that in the SPSD method we need to analyze the logarithm of the integral kernel:

$$\alpha^2 \sum_{\langle ij\rangle} [\kappa_{ij}(1+\tau_i\tau_j - 2\langle\sigma_i\sigma_j\rangle) - \kappa_{ij}^2/2]. \tag{D2}$$

The equations for κ's are modified due to the explicit dependence of $\langle\sigma_k\sigma_l\rangle$ on κ_{ij}; in fact one easily gets

$$\partial\langle\sigma_k\sigma_l\rangle/\partial\kappa_{ij} = \alpha^2\left(\langle\sigma_k\sigma_l\sigma_i\sigma_j\rangle - \langle\sigma_k\sigma_l\rangle\langle\sigma_i\sigma_j\rangle\right). \tag{D3}$$

Fortunately, most of these correlators are negligible: in fact they vanish in the MF approximation for distinct, non-overlapping pairs (k,l) and (i,j). The non-vanishing and non-trivial are

$$\partial\langle\sigma_i\sigma_j\rangle/\partial\kappa_{ij} = \alpha^2\left(1 - m_i^2 m_j^2\right), \tag{D4}$$

and

$$\partial\langle\sigma_i\sigma_l\rangle/\partial\kappa_{ij} = \alpha^2\left(m_l m_j(1-m_i^2)\right), \tag{D5}$$

and its variations. We obtain then modified equations for κ_{ij} that have now to be solved in an iterative manner,

$$\kappa_{ij} = \frac{1+\tau_i\tau_j - 2m_i m_j - 2\Delta\kappa_{ij}}{1+2\alpha^2(1-m_i^2m_j^2)}, \tag{D6}$$

where

$$\begin{aligned}\Delta\kappa_{ij} = {}& \alpha^2 \sum_{l=\text{n.n.}} \kappa_{il} m_l m_j (1-m_i^2) \\ & + \alpha^2 \sum_{k=\text{n.n.}} \kappa_{kj} m_i m_k \left(1-m_j^2\right)\Big),\end{aligned} \tag{D7}$$

where l's (k's) and neighbors of i (j), different from j (i).

These expressions get simplified upon translation symmetry,

$$\begin{aligned}\Delta\kappa_{i,i} = {}& 2\alpha^2\left(\kappa_{i,i}(2d-3)m_i^2(1-m_i^2)\right. \\ & \left. + m_i(\kappa_{i,i+1}m_{i+1} + \kappa_{i-1,i}m_{i-1})(1-m_i^2)\right),\end{aligned} \tag{D8}$$

$$\begin{aligned}\Delta\kappa_{i,i+1} = {}& \alpha^2\left(2(d-1)\kappa_{i,i+1}m_{i+1}^2(1-m_i^2)\right. \\ & + 2(d-1)\kappa_{i,i+1}m_i^2(1-m_{i+1}^2) \\ & \left. + \kappa_{i-1,i}m_{i-1}m_{i+1}(1-m_i^2) + \kappa_{i,i+1}m_i m_{i+2}(1-m_{i+1}^2)\right).\end{aligned} \tag{D9}$$

For P_+ and P_- (away from the wall) we get

$$\kappa_{ij} = \frac{2-2m_i m_j - 2\Delta\kappa_{ij}}{1+2\alpha^2(1-m_i^2m_j^2)}, \tag{D10}$$

and at the wall

$$\kappa_{L/2,L/2+1} = \frac{-2m_{L/2}m_{L/2+1} - 2\Delta\kappa_{L/2,L/2+1}}{1 + 2\alpha^2\left(1 - m_{L/2}^2 m_{L/2+1}^2\right)}. \tag{D11}$$

Otherwise, all other expressions are valid. The calculations of P_+ and P_- reduces now to evaluation of

$$\frac{H_+}{L^{d-1}} = L\ln(2) + \alpha^2\left[(d-1)\sum_{i=1}^{L}[2\kappa_{i,i}(1-m_i^2) - \kappa_{i,i}^2/2]\right. \tag{D12}$$
$$\left. + \sum_{i=1}^{L-1}[2\kappa_{i,i+1}(1 - m_i m_{i+1}) - \kappa_{i,i+1}^2/2]\right],$$

and

$$H_-/L^{d-1} = \alpha^2 {\sum_{\langle ij\rangle}}'[2\kappa_{ij} - \kappa_{ij}^2/2 - 2\kappa_{ij}m_i m_j] \tag{D13}$$
$$+ \alpha^2[-\kappa_{L/2,L/2+1}^2/2 - 2\kappa_{L/2,L/2+1}m_{L/2}m_{L/2+1}].$$

Similarly,

$$\frac{H_-}{L^{d-1}} = L\ln(2) + \alpha^2\left[(d-1)\sum_{i=1}^{L}[2\kappa_{i,i}(1-m_i^2) - \kappa_{i,i}^2/2]\right.$$
$$+ \sum_{i=1}^{L/2-1}[2\kappa_{i,i+1}(1 - m_i m_{i+1}) - \kappa_{i,i+1}^2/2] \tag{D14}$$
$$+ \sum_{i=L/2+1}^{L-1}[2\kappa_{i,i+1}(1 - m_i m_{i+1}) - \kappa_{i,i+1}^2/2]$$
$$\left. - \kappa_{L/2,L/2+1}^2/2 - \kappa_{L/2,L/2+1}m_{L/2}m_{l/2+1}\right].$$

We have calculated δ, using the present approach, in which we neglected contributions from $\Delta\kappa$'s terms, leaving only the effect due to $\partial\langle\sigma_i\sigma_j\rangle/\partial\kappa_{ij} = \alpha^2\left(1 - m_i^2 m_j^2\right)$. The end results are quantitatively and qualitatively the very similar to those obtained with the 'pure' SPSD method.

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