WARPED QUASI-ASYMPTOTICALLY CONICAL CALABI-YAU METRICS

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Abstract. We construct many new examples of complete Calabi-Yau metrics of maximal volume growth on certain smoothings of Cartesian products of Calabi-Yau cones with smooth cross-sections. A detailed description of the geometry at infinity of these metrics is given in terms of a compactification by a manifold with corners obtained through the notion of weighted blow-up for manifolds with corners. A key analytical step in the construction of these Calabi-Yau metrics is to derive good mapping properties of the Laplacian on some suitable weighted Hölder spaces.

Contents

1. Introduction

A complete Kähler manifold (X, g, J) is Calabi-Yau if it is Ricci-flat and has a nowhere vanishing parallel holomorphic volume form $\Omega_X \in H^0(X; K_X)$. This latter condition forces the holonomy of g to be contained in $\mathrm{SU}(m)$ with $m = \dim_{\mathbb{C}} X$. By the result of Yau [42], we know that a compact Kähler manifold admits a Calabi-Yau metric if and only if its canonical bundle is trivial, in which case a unique Calabi-Yau metric can be obtained in each Kähler class by solving a complex Monge-Ampère equation. On non-compact complete Kähler manifolds, the triviality of the canonical bundle is also a necessary condition for the existence of a Calabi-Yau metric, but one needs also to take into account the geometry at infinity. For instance, on $\mathbb{C}^2$, the flat metric and the Taub-NUT metric are two complete Calabi-Yau metrics in the same Kähler class, but with quite distinct geometry at infinity, the volume growth of the latter being only cubic instead of order 4.

In this paper, we will focus on Calabi-Yau metrics of maximal volume growth, that is, such that the volume of a ball of radius r is comparable to r^{2m} for r large with m the complex dimension of the manifold. A tangent cone at infinity of such a metric is then of the same dimension. When such a tangent cone at infinity has a smooth cross-section, or equivalently when the Calabi-Yau metric has quadratic curvature decay, then by [12], this is in fact the unique tangent cone at infinity of the metric. By [35], such Calabi-Yau metrics are asymptotically conical (AC-metrics for short), that is, Calabi-Yau metrics converging smoothly at infinity at a rate $\mathcal{O}(r^{-\epsilon})$ for some $\epsilon > 0$ to a Calabi-Yau cone with smooth cross-section. Various examples have been obtained over the years by solving a complex Monge-Ampère equation, notably in [27, 40, 41, 22, 15, 16]. For a fixed Calabi-Yau cone at infinity (C, g_C) with smooth cross-section, a complete classification of asymptotically conical Calabi-Yau manifolds with tangent cone at infinity (C, g_C) was obtained in [14], generalizing in particular Kronheimer's classification [30] of asymptotically locally Euclidean hyperKähler 4-manifolds. The upshot is that all such asymptotically conical Calabi-Yau metrics are obtained by considering a Kähler crepant resolution of a deformation of C seen as an affine variety.

Allowing the tangent cone at infinity to have a singular cross-section greatly opens up the possibilities of examples that can occur. When $C = \mathbb{C}^m/\Gamma$ with Γ a finite subgroup of $\mathrm{SU}(m)$ and g_C is the metric induced by the Euclidean metric on $\mathbb{C}^m$, Joyce [27] constructed examples of Calabi-Yau metrics on Kähler crepant resolutions of C, the so called quasi-asymptotically locally Euclidean metrics (QALE-metrics for short). This was extended in [13] to obtain Calabi-Yau quasi-asymptotically conical metrics (QAC-metrics for short) in the sense of [20]. If one considers instead smooth deformations of Calabi-Yau cones with singular cross-sections, then already on $\mathbb{C}^n$, many examples were obtained in [31, 18, 36], providing in particular counter-examples to a conjecture of Tian [38]. As pointed out in [18], these examples are not quite QAC-metrics, but they are very close to being so in that they are in some sense warped QAC-metrics. The Calabi-Yau metrics of [8] constructed on complex symmetric spaces seem to have a similar behavior at infinity. More recently, adapting the strategy of [36], new examples of Calabi-Yau metrics were obtained in [21] by smoothing $\mathbb{C}\times C$ when C is a complete intersection with smooth cross-section equipped with a Calabi-Yau cone metric g_C.

On $\mathbb{C}^n$, motivated by the conjecture of Tian [38], one could hope that a complete Calabi-Yau metric of maximal volume growth is completely determined up to scale and isometry by its tangent cone at infinity. When $C=\mathbb{C}\times A_1$ with A_1 the $(n-1)$-dimensional Stenzel cone with Calabi-Yau cone metric g_{A_1}, Székelyhidi [37] showed that this is indeed the case. However, already on $\mathbb{C}^3$, if one takes instead $C=\mathbb{C}\times A_2$ with A_2 the singular hypersurface

$$\{z_1^2+z_2^2+z_3^3=0\}\subset\mathbb{C}^3,$$

the Calabi-Yau metric with tangent cone at infinity $\mathbb{C}\times A_2$ constructed in [36] is not unique. Indeed, a 1-parameter family of distinct such metrics was recently constructed by Chiu [11].

In the present paper, we generalize the approach of [18] to construct new examples of complete Calabi-Yau metrics of maximal volume growth having a tangent cone at infinity with singular cross-section. To state our results, consider N Calabi-Yau cones $(W_1,g_1),\ldots,(W_N,g_N)$ with singular apex (i.e. not corresponding to the Euclidean space) but with smooth cross-sections. For each $q\in\{1,\ldots,N\}$, suppose that W_q is a complete intersection in $\mathbb{C}^{m_q+n_q}$,

$$W_q=\{z_q\in\mathbb{C}^{m_q+n_q}\mid P_{q,1}(z_q)=\cdots=P_{q,n_q}(z_q)=0\},$$

for n_q polynomials $P_{q,1},\ldots,P_{q,n_q}$, where $m_q=\dim_{\mathbb{C}}W_q$. Suppose that the natural $\mathbb{R}^+$-action on W_q is induced by a diagonal action

$$\begin{array}{rccl}\mathbb{R}^+\ni t: & \mathbb{C}^{m_q+n_q} & \to & \mathbb{C}^{m_q+n_q}\\ & z_q & \mapsto & t\cdot z_q=(t^{w_{q,1}}z_{q,1},\cdots,t^{w_{q,m_q+n_q}}z_{q,m_q+n_q})\end{array}$$

for some positive weights $w_{q,1},\ldots,w_{q,m_q+n_q}$. We will not assume that the cone is quasi-regular, so these weights are not necessarily rational. We will assume that each polynomial $P_{q,i}$ is homogeneous of some degree $d_{q,i}$ with respect to the $\mathbb{R}^+$-action,

$$P_{q,i}(t\cdot z_q)=t^{d_{q,i}}P_{q,i}(z)\quad\forall q\in\{1,\ldots,N\},\ \forall i\in\{1,\ldots,n_q\},$$

and that

$$d_{q,1}\le\cdots\le d_{q,n_q}.$$

Furthermore, we will assume that there is a $d>1$ such that for each $q\in\{1,\ldots,N\}$, there exists $k_q\in\{1,\ldots,N\}$ such that $d_{q,1}=\cdots=d_{q,k_q}=d$.

On W_q, the Kähler form of the metric g_q is given by $\omega_q=\frac{\sqrt{-1}}{2}\partial\overline{\partial}r_q^2$ with r_q the radial distance to the origin with respect to the Calabi-Yau metric g_q. On W_q, there is a holomorphic volume form Ω^{m_q} defined implicitly by

$$dz_{q,1}\wedge\cdots\wedge dz_{q,m_q+n_q}|_{W_q}=\Omega_q^{m_q}\wedge dP_{q,1}|_{W_q}\wedge\cdots\wedge dP_{q,n_q}|_{W_q}. \tag{1.1}$$

The fact that (W_q,g_q) is Calabi-Yau means that there is a constant $c_{m_q}\in\mathbb{C}\setminus\{0\}$ depending only on m_q such that

$$\omega_q^{m_q}=c_{m_q}\Omega_q^{m_q}\wedge\overline{\Omega}_q^{m_q}. \tag{1.2}$$

By (1.1), the holomorphic volume form $\Omega_q^{m_q}$ is homogeneous of degree $\sum_{j=1}^{m_q+n_q}w_{q,j}-n_qd$ with respect to the $\mathbb{R}^+$-action, while ω_q is of degree 2, so we deduce from (1.2) that

$$m_q=\left(\sum_{j=1}^{m_q+n_q}w_{q,j}\right)-n_qd. \tag{1.3}$$

Using the convention that $m_0=1$ and $n_0=0$, consider the Calabi-Yau cone $W_0:=\mathbb{C}^{m_0+n_0}=\mathbb{C}$ with canonical Euclidean metric g_0, radial function $r_0=|z_0|$ for $z_0\in\mathbb{C}$ and holomorphic volume form $\Omega_0^{m_0}=dz_0$.

The corresponding $\mathbb{R}^+$-action is the standard one, so the weights are just $w_0 = 1$. We will also be interested in the case $m_0 = 0$, in which case we will set $W_0 = \{0\}$.

We will consider the Cartesian product

$$C_0 := W_0 \times W_1 \times \cdots \times W_q \subset \mathbb{C}^{m_0+n_0} \times \cdots \times \mathbb{C}^{m_q+n_q} = \mathbb{C}^{m+n}, \quad m := \sum_{q=0}^{N} m_q,\ n := \sum_{q=0}^{N} n_q, \tag{1.4}$$

with product Calabi-Yau metric $g_{C_0} = g_0 \times \cdots g_N$. Thus, (C_0, g_{C_0}) is a Calabi-Yau cone with radial function

$$r := \sqrt{\sum_{q=0}^{N} r_q^2}, \tag{1.5}$$

Kähler form

$$\omega_{C_0} = \frac{\sqrt{-1}}{2}\partial\overline{\partial} r^2 \tag{1.6}$$

and holomorphic volume form

$$\Omega^m_{C_0} = (-1)^{\sum_{q=2}^{N} m_q(n_{q-1}+\cdots+n_1)}\Omega_0^{m_0} \wedge \Omega_1^{m_1} \wedge \ldots \wedge \Omega_N^{m_N}. \tag{1.7}$$

The possible sign in the right hand side of (1.7) is to ensure that this definition agrees with the implicit definition

$$\begin{aligned} dz_0|_{C_0} \wedge (dz_{1,1} \wedge \ldots \wedge dz_{1,m_1+n_1})|_{C_0} \wedge \ldots \wedge (dz_{N,1} \wedge \ldots \wedge dz_{N,m_N+n_N})|_{C_0} = \\ \Omega^m_{C_0} \wedge (dP_{1,1} \wedge \ldots dP_{1,n_1})|_{C_0} \wedge \ldots \wedge (dP_{N,1} \wedge \ldots \wedge dP_{N,n_N})|_{C_0}, \end{aligned} \tag{1.8}$$

where we use the convention that $dz_0|_{C_0} = 1$ if $W_0 = \{0\}$. Since W_q is singular at the origin for $q > 0$, notice that (C_0, g_0) has a singular cross-section. In fact, for each subset $\{0\} \subsetneq \mathfrak{q} \subset \{0, 1 \ldots, N\}$,

$$W_{\mathfrak{q}} := \{z = (z_0, z_1, \ldots, z_q) \in C_0 \mid z_q = 0 \text{ for } q \notin \mathfrak{q}\} \tag{1.9}$$

consists of singular points of C_0 with the singular locus of C_0 given by

$$C_{0,\text{sing}} = \bigcup_{\{0\}\subset\mathfrak{q}\subsetneq\{0,\ldots,N\}} W_{\mathfrak{q}}. \tag{1.10}$$

Moreover, for each such $\mathfrak{q}$,

$$C_0 = W_{\mathfrak{q}} \times W_{\mathfrak{q}}^{\perp} \tag{1.11}$$

with

$$W_{\mathfrak{q}}^{\perp} = W_{\mathfrak{q}^c} = \{z = (z_0, z_1, \ldots, z_q) \in C_0 \mid z_q = 0 \text{ for } q \in \mathfrak{q}\}, \tag{1.12}$$

where $\mathfrak{q}^c = \{0, 1, \ldots, N\} \setminus \mathfrak{q}$. The decomposition (1.11) is also Riemannian, namely the restriction of the Calabi-Yau metric g_{C_0} to $W_{\mathfrak{q}}$ and $W_{\mathfrak{q}^c}$ induces Calabi-Yau cone metrics $g_{\mathfrak{q}}$ and $g_{\mathfrak{q}^c}$ such that g_{C_0} is just the Cartesian product of $g_{\mathfrak{q}}$ and $g_{\mathfrak{q}^c}$. Let

$$V_{\mathfrak{q}} := \{z = (z_0, z_1, \ldots, z_q) \in \mathbb{C}^{m+n} \mid z_q = 0 \text{ for } q \notin \mathfrak{q}\}, \tag{1.13}$$

$$V_{\mathfrak{q}}^{\perp} = V_{\mathfrak{q}^c} = \{z = (z_0, z_1, \ldots, z_q) \in \mathbb{C}^{m+n} \mid z_q = 0 \text{ for } q \in \mathfrak{q}\}, \tag{1.14}$$

be the subspaces such that $W_{\mathfrak{q}} = C_0 \cap V_{\mathfrak{q}}$ and $W_{\mathfrak{q}^c} = C_0 \cap V_{\mathfrak{q}^c}$.

We will suppose that for $\epsilon \in \mathbb{C} \setminus \{0\}$ close to 0, the cone C_0 admits a smoothing C_ϵ of the form

$$C_\epsilon = \{z = (z_0, \ldots, z_N) \in \mathbb{C}^{m+n} \mid P_{q,i}(z_q) = \epsilon Q_{q,i}(z_0) \quad \text{for} \quad q \in \{1, \ldots, N\} \text{ and } i \in \{1, \ldots, n_q\}\}, \tag{1.15}$$

where each $Q_{q,i}$ is a polynomial in $z_0 \in \mathbb{C}^{m_0+n_0} = \mathbb{C}$ of (weighted) degree $\ell < d$ for some fixed $\ell \geq 0$ not depending on q and i. If $\ell = 0$, we will assume that $W_0 = \{0\}$. It comes with a natural holomorphic volume form $\Omega^m_{C_\epsilon}$ defined explicitly by

$$\begin{aligned} dz_0|_{C_\epsilon} \wedge (dz_{1,1} \wedge \ldots \wedge dz_{1,m_1+n_1})|_{C_\epsilon} \wedge \ldots \wedge (dz_{N,1} \wedge \ldots \wedge dz_{N,m_N+n_N})|_{C_\epsilon} = \\ \Omega^m_{C_\epsilon} \wedge (d(P_{1,1} - \epsilon Q_{1,1}) \wedge \ldots d(P_{1,n_1} - \epsilon Q_{1,n_1}))|_{C_\epsilon} \wedge \ldots \wedge (d(P_{N,1} - \epsilon Q_{N,1}) \wedge \ldots \wedge d(P_{N,n_N} - \epsilon Q_{N,n_N}))|_{C_\epsilon}, \end{aligned} \tag{1.16}$$

where again we use the convention that $dz_0|_{C_\epsilon} = 1$ if $W_0 = \{0\}$. We will make the following two assumptions on the polynomials $Q_{q,i}$.

Assumption 1.1. *The polynomials $Q_{q,i}$ will not be assumed to be homogeneous, but we will assume that the homogenous part of degree ℓ, denoted $[Q_{q,i}]_\ell$, is non-zero for each q and i. Moreover, if $\ell > 0$ and $N > 1$, then we will assume that $m_0 = 1$, so that the zero locus $[Q_{q,i}]_\ell(z_0) = 0$ is the origin in $\mathbb{C}$ and corresponds to the hyperplane*

$$V_{\{1,\dots,N\}} = \{0\} \times \mathbb{C}^{m_1+n_1} \times \cdots \times \mathbb{C}^{m_N+n_N} \subset \mathbb{C}^{m+n}$$

in $\mathbb{C}^{m+n}$.

Assumption 1.2. *For each subset $\{0\} \subset \mathfrak{q} \subset \{0, 1, \dots, N\}$, we suppose that for $\epsilon \in \mathbb{C} \setminus \{0\}$ sufficiently close to zero,*

$$W_{\mathfrak{q},\epsilon} := \{z_{\mathfrak{q}} \in V_{\mathfrak{q}} \mid P_{q,i}(z_q) = \epsilon Q_{q,i}(z_0) \quad \text{for } q \in \mathfrak{q} \setminus \{0\},\ i \in \{1, \dots, n_q\}\} \tag{1.17}$$

is a smoothing of $W_{\mathfrak{q}}$. If $m_0 = 1$ and $W_0 = \mathbb{C}$, then for $\epsilon \in \mathbb{C} \setminus \{0\}$ sufficiently close to zero and for all $\omega_{\mathfrak{q}} \in W_{\mathfrak{q}} \cap \mathbb{S}(V_{\mathfrak{q}})$ with $\mathbb{S}(V_{\mathfrak{q}})$ the unit sphere in $V_{\mathfrak{q}}$, we will also suppose that

$$\begin{aligned} W^{\perp}_{\mathfrak{q},\omega_{\mathfrak{q}},\epsilon} = W_{\mathfrak{q}^c,\omega_{\mathfrak{q}},\epsilon} = \{z_{\mathfrak{q}^c} \in V_{\mathfrak{q}^c} \mid \forall q \in \mathfrak{q}^c,\ i \in \{1, \dots, n_q\}, \\ P_{q,i}(z_q) = \epsilon[Q_{q,i}](\omega_0),\ i \le k_q, \quad P_{q,i}(z_q) = 0,\ i > k_q\} \end{aligned} \tag{1.18}$$

is a smoothing of $W^{\perp}_{\mathfrak{q}} = W_{\mathfrak{q}^c}$ for $\omega_0 \neq 0$, where ω_0 is the component of $\omega_{\mathfrak{q}}$ in V_0.

Remark 1.3. *The smoothing* (1.18) *is a Cartesian product of the smoothings $W_{q,\omega_{\mathfrak{q}},\epsilon}$ of W_q for $q \in \mathfrak{q}^c$.*

We can now state the main result of this paper; see also Corollaries 6.5 and 6.6 at the very end of the paper for more details.

Theorem 1.4. *Suppose that Assumptions 1.1 and 1.2 hold. If $\nu := \frac{\ell}{d}$ and $\beta := \min\{d, 2m_1, \dots, 2m_N\}$ are such that*

$$\beta > \frac{2}{1-\nu}, \tag{1.19}$$

then for $\epsilon \neq 0$ close to zero, C_ϵ admits a complete Calabi-Yau metric of maximal volume growth with tangent cone at infinity (C_0, g_{C_0}). Furthermore, if $N = 1$, this result still holds if instead of (1.19)*, ν and β are such that*

$$3 < \beta \le \frac{2}{1-\nu} < 2m_1 + 5. \tag{1.20}$$

When $N = 1$, $m_0 = 1$, $n_1 = 1$ and $\ell = 1$ with $Q_{1,1}(z_{0,1}) = z_{0,1}$, this theorem corresponds to most of the examples obtained in [31, 18, 36], while most of the examples of [21] corresponds to the case $N = 1$, $m_0 = 1$ and $\ell = 1$ with $Q_{1,i}$ homogeneous for each $i \in \{1, \dots, n_1\}$. Allowing other values of these parameters yields many new examples of Calabi-Yau metrics. Let us illustrate this with two classes of examples.

Example 1.5. *Letting $N = 1$, $m_0 = 1$ and $n_1 = 1$ as in* [31, 18, 36]*, we can obtain new examples of Calabi-Yau metrics by taking $\ell \neq 1$ or $Q_{1,1}$ not homogeneous. For instance, we can take $m_1 = 2$ with*

$$P_{1,1}(z_{1,1}, z_{1,2}, z_{1,3}) = z_{1,1}^2 + z_{1,2}^2 + z_{1,3}^2 \quad \text{and} \quad Q_{1,1}(z_{0,1}) = -z_{0,1}^2 + 1,$$

so that $C_\epsilon \subset \mathbb{C}^4$ is the affine hypersurface given by

$$z_{1,1}^2 + z_{1,2}^2 + z_{1,3}^2 + \epsilon z_{0,1}^2 = \epsilon.$$

In this case, we see from (1.3) *that $w_{i,i} = 2$ for all i, so $d = 4$, $\ell = 2$, $\nu = \frac{1}{2}$ and $\beta = 4$. In particular,* (1.20) *holds, so Theorem 1.4 yields a Calabi-Yau metric on C_ϵ with tangent cone at infinity $\mathbb{C} \times A_1$ with A_1 the Stenzel cone $z_{1,1}^2 + z_{1,2}^2 + z_{1,3}^2 = 0$. Notice that C_ϵ is also a smoothing of the Stenzel cone*

$$z_{1,1}^2 + z_{1,2}^2 + z_{1,3}^2 + \epsilon z_{0,1}^2 = 0, \tag{1.21}$$

so as such, it also admits an asymptotically conical Calabi-Yau metric with tangent cone at infinity (1.21)*.*

Example 1.6. *Taking $N > 1$, we can obtain Calabi-Yau metrics with tangent cone at infinity whose cross-section has singularities of depth N. For instance, we can take $m_0 = 1$ with $n_1 = \cdots = n_N = 1$ and $m_1 = \cdots = m_N = m$ with*

$$P_{q,1}(z_q) = z_{q,1}^k + \cdots + z_{q,m+1}^k \quad \forall q \in \{1, \ldots, N\}, \tag{1.22}$$

for some $2 \leq k \leq m$. It is well-known in this case, see for instance [18, Example 2.1]*, that W_q admits a Calabi-Yau cone metric. From* (1.3)*, we see that*

$$w_{q,i} = \frac{m}{m+1-k} \tag{1.23}$$

for $q > 0$, so $d = \frac{km}{m+1-k} > k$. Assuming $k > 3$, then $d > 3$ and to satisfy (1.19)*, it suffices to take $\ell < d-2$. To satisfy Assumption 1.2, we can take for instance $Q_{q,i}(z_0) = z_0$ for each q.*

Remark 1.7. *When $N = 1$, it would have been possible in principle to take $m_0 > 1$ in Theorem 1.4. One interesting special case would be to take $N = 1$, $\ell = 1$ and $n_1 > 1$ as in* [21]*, but with $k_1 = n_1 = m_0 > 1$ and $Q_{1,i}(z_0) = z_{0,i}$ for $i \in \{1, \ldots, n_1\}$, so that C_ϵ corresponds to the smoothing*

$$P_{1,i}(z_1) = \epsilon z_{0,i} \quad \text{for} \quad i \in \{1, \ldots, n_1\}. \tag{1.24}$$

Since it is the graph of a holomorphic map from $\mathbb{C}^{m_1+n_1}$ to $\mathbb{C}^{m_0} = \mathbb{C}^{n_1}$, the smoothing C_ϵ is in particular biholomorphic to $\mathbb{C}^{m_1+n_1}$. Unfortunately however, the problem is that (1.18) *of Assumption 1.2 is never satisfied in this case, so we cannot actually apply Theorem 1.4 to obtain new examples of Calabi-Yau metrics. Indeed, the singular locus $\mathcal{S}$ of the map $F := (P_{1,1}, \ldots, P_{1,n_1})$ is such that*

$$\dim_{\mathbb{C}} \mathcal{S} \geq m_1 + n_1 - (m_1 + 1) = n_1 - 1 > 0$$

as a subvariety of $\mathbb{C}^{m_1+n_1}$. Moreover, it contains the origin and is also clearly invariant under the $\mathbb{R}^+$-action. Hence, taking $s \in \mathcal{S} \setminus \{0\}$ close to the origin, then we can take $\omega_0 = (P_{1,1}(s), \ldots, P_{1,n_1}(s))$ in (1.18) *to obtain a singular variety. Since W_1 is assumed to be smooth outside the origin, notice that $\omega_0 \neq 0$ automatically.*

Remark 1.8. *When $\ell = 0$ and $m_0 = 0$, the Calabi-Yau metrics of Theorem 1.4 are QAC-metrics. In fact, they are more precisely a Cartesian product of asymptotically conical Calabi-Yau metrics (by* [28, Theorem 6.6]*, the class of* QAC*-metrics is closed under taking the Cartesian product), so this does not yield new examples of Calabi-Yau metrics.*

Since the Calabi-Yau metrics of Theorem 1.4 are obtained by solving a complex Monge-Ampère equation, they are not given by an explicit formula. However, as in [18], we provide a detailed description of their asymptotic behavior at infinity. In fact, the Calabi-Yau metrics of Theorem 1.4 are a higher depth version of the warped QAC-metrics of [18]. Indeed, the local models at infinity are again a warped version of the local model for QAC-metrics, namely a warped product of a cone metric $dr^2 + r^2 g_B$ and a warped QAC-metric of lower depth K_w,

$$dr^2 + r^2 g_B + r^{2\nu_K} K_w \tag{1.25}$$

with $\nu_K \in \{0, \nu\}$.

Compared to [18], a new feature of this higher depth version is that warped QAC-metrics are no longer necessarily conformal to a QAC-metric. They are in general only conformal to a slightly different class of metrics that we call weighted QAC-metrics. As for QAC-metrics, those come from a Lie structure at infinity in the sense of [4], so admit a nice global coordinate free description. They are in particular automatically complete of infinite volume with bounded geometry and the same holds true for warped QAC-metrics. There is also a conformally related class of metrics that we call weighted Qb-metrics. They play a similar role to Qb-metrics as in [13] by allowing to define the right weighted Hölder spaces in which to solve the complex Monge-Ampère equation. In terms of these Hölder spaces, we can in fact as in [18] adapt the arguments of [20] to derive nice mapping properties of the Laplacian of a warped QAC-metric; see in particular Corollary 3.23 below.

Remark 1.9. *We have not explored it here, but instead of* [20]*, one could try to develop a higher depth version of the approach of* [36] *to obtain mapping properties of the Laplacian. The approach of* [36] *allows in principle to consider weights for which the Laplacian is no longer an isomorphism, but is still surjective and Fredholm, which would possibly allow condition* (1.19) *in Theorem 1.4 to be weakened.*

All these classes of metrics are defined in terms of a compactification by a manifold with corners, so to define them on C_ϵ, we need to introduce a suitable compactification $\widehat{C}_\epsilon$ of C_ϵ. As in [18], our starting point is the radial compactification $\overline{\mathbb{C}^{m+n}_w}$ of $\mathbb{C}^{m+n}$ specified by the $\mathbb{R}^+$-action induced by the weights $w_{q,i}$. Let $\overline{C}_\epsilon$ be the closure of C_ϵ in $\overline{\mathbb{C}^{m+n}_w}$. This closure has singularities on the boundary $\partial\overline{\mathbb{C}^{m+n}_w}$ of $\overline{\mathbb{C}^{m+n}_w}$ that can be resolved by blowing them up. However, this cannot be achieved with the usual notion of blow-up [25, § 2.2] of a p-submanifold in a manifold with corners. We need instead to consider a weighted version of it, one special instance being the parabolic blow-up of Melrose [34, (7.4)]. Compared to the usual blow-up, this weighted version is sensitive to the choice of a tubular neighborhood of the p-submanifold, but in our setting, the p-submanifolds that need to be blown up in a weighted manner come with a natural choice of such tubular neighborhood, ensuring that each weighted blow-up is well-defined.

Once the natural compactification $\widehat{C}_\epsilon$ is given, each boundary hypersurface gives a model at infinity, essentially (1.17) and (1.18). Proceeding inductively on the depth of $\widehat{C}_\epsilon$ and using a convexity argument of [40, Lemma 4.3], we can first construct Kähler examples of warped QAC-metrics on C_ϵ. To obtain Calabi-Yau examples, we need to solve a complex Monge-Ampère equation. As in [13], proceeding by induction on the depth of $\widehat{C}_\epsilon$, we can first solve the complex Monge-Ampère equation on each model at infinity. Using the fixed point argument of [36, Proposition 25], we can then solve the complex Monge-Ampère equation outside a large compact set. From that point, we can then completely solve the complex Monge-Ampère equation using standard techniques.

The paper is organized as follows. in § 2, we introduce the notion of warped QAC-metrics, derive their main properties and introduce the weighted Hölder spaces that we will need. In § 3, following [20], we derive the mapping properties of the Laplacian of a warped QAC-metric that we will need. In § 4, we describe the notion of weighted blow-up for manifolds with corners and derive some of its features. In § 5, we introduce the compactification $\widehat{C}_\epsilon$ and construct examples of Kähler warped QAC-metrics on C_ϵ. Finally, in § 6, we solve a complex Monge-Ampère equation to obtain Theorem 1.4.

Acknowledgements. *The first author is supported by NSF grant DMS-1906466 and the second author is supported by NSERC and FRQNT.*

2. Warped quasi-asymptotically conical metrics

Since it will be a central tool in this paper, let us first recall the notion of manifold with fibered corners. Thus, let M be a manifold with corners. Unless otherwise stated, we will usually assume that M is compact. Denote by $\mathcal{M}_1(M)$ the set of boundary hypersurfaces of M, that is, the set of corners of codimension 1. As in [33], we will assume that its boundary hypersurfaces are all embedded. We denote by ∂M the union of all the boundary hypersurfaces of M. Suppose that each boundary hypersurface H of M is endowed with a fiber bundle $\phi_H : H \to S_H$ with base S_H and fibers also manifolds with corners. Denote by ϕ the collection of these fiber bundles.

Definition 2.1 ([3, 2, 19])**.** *We say that (M,ϕ) is a **manifold with fibered corners** or that ϕ is an **iterated fibration structure for** M if there is a partial order on $\mathcal{M}_1(M)$ such that*

- *Any subset $\mathcal{I} \subset \mathcal{M}_1(M)$ such that $\bigcap_{H\in\mathcal{I}} H \neq \emptyset$ is totally ordered;*
- *If $H < G$, then $H \cap G \neq \emptyset$, the map $\phi_H|_{H\cap G} : H \cap G \to S_H$ is a surjective submersion, $S_{GH} := \phi_G(H \cap G)$ is one of the boundary hypersurfaces of S_G and there is a surjective submersion $\phi_{GH} : S_{GH} \to S_H$ such that $\phi_{GH} \circ \phi_G = \phi_H$ on $H \cap G$;*
- *The boundary hypersurfaces of S_G are given by S_{GH} for $H < G$.*

One can check directly from the definition that the base S_H and the fibers of $\phi_H : H \to S_H$ are also naturally manifolds with fibered corners. If H is minimal with respect to the partial order, then S_H is in fact a closed manifold. Conversely, if H is maximal with respect to the partial order, then the fibers of $\phi_H : H \to S_H$ are closed manifolds. In various settings, this allows us to prove assertions by proceeding by induction on the **depth** of (M,ϕ), which is the largest codimension of a corner of M. Manifolds with fibered corners are intimately related with stratified spaces.

Definition 2.2. *A **stratified space** of dimension n is a locally separable metrizable space X together with a **stratification**, which is a locally finite partition $\mathcal{S} = \{s_i\}$ into locally closed subsets of X, called the **strata**,* *which are smooth manifolds of dimension* $\dim s_i \leq n$ *such that at least one is of dimension* n *and*

$$s_i \cap \overline{s}_j \neq \emptyset \quad \Longrightarrow \quad s_i \subset \overline{s}_j.$$

In this case, we write $s_i \leq s_j$ *and* $s_i < s_j$ *if* $s_i \neq s_j$. *A stratification induces a filtration*

$$\emptyset \subset X_0 \subset \cdots \subset X_n = X,$$

where X_j *is the union of all strata of dimension at most* j. *The strata included in* $X \setminus X_{n-1}$ *are said to be* ***regular****, while those included in* X_{n-1} *are said to be* ***singular****.*

Given a stratified space, notice that the closure of each of its strata is also naturally a stratified space. The **depth** of a stratified space is the largest integer k such that one can find $k+1$ different strata with

$$s_1 < \cdots < s_{k+1}.$$

As described in [3, 2, 19], a manifold with fibered corners (M, ϕ) arises as a resolution of the stratified space given by ${}^S M := M/\sim$, where $\sim$ is the equivalence relation

$$p \sim q \quad \Longleftrightarrow \quad p = q \quad \text{or} \quad p, q \in H \text{ with } \phi_H(p) = \phi_H(q) \text{ for some } H \in \mathcal{M}_1(M).$$

If $\beta : M \to {}^S M$ denotes the quotient map, which can be thought of as a blow-down map, then $\beta(M \setminus \partial M)$ yields the regular strata. In fact, the map β gives a one-to-one correspondence between the boundary hypersurfaces of M and the singular strata of ${}^S M$, namely $H \in \mathcal{M}_1(M)$ corresponds to the stratum s_H whose closure is given by $\overline{s}_H := \beta(H)$. In this correspondence, the base S_H of the fiber bundle ϕ_H is itself a resolution of $\overline{s}_H$ and $s_H = \beta(\phi_H^{-1}(S_H \setminus \partial S_H))$. Moreover, the depth of (M, ϕ) as a manifold with fibered corners is the same as the depth of ${}^S M$ and the partial order on $\mathcal{M}_1(M)$ matches the one on the strata of ${}^S M$. A stratified space admitting a resolution by a manifold with fibered corners is said to be **smoothly stratified**. Not all stratified spaces are smoothly stratified, but as discussed in [2, 19], the property of being smoothly stratified can be described intrinsically on a stratified space without referring to a manifold with fibered corners.

Recall from [33] that a **boundary defining function** for $H \in \mathcal{M}_1(M)$ is a function $x_H \in \mathcal{C}^\infty(M)$ such that $x_H \geq 0$, $H = x_H^{-1}(0)$ and dx_H is nowhere zero on H. Following [13, Definition 1.9], we will say that a boundary defining function x_H of H is **compatible** with the iterated fibration structure if for each $G > H$, the restriction of x_H to G is constant along the fibers of $\phi_G : G \to S_G$. By [19, Lemma 1.4], compatible boundary defining functions always exist. If $p \in \partial M$ is contained in the interior of a corner $H_1 \cap \cdots \cap H_k$ of codimension k, then without loss of generality, we can assume that $H_1 < \ldots < H_k$. If x_i is a choice of compatible boundary defining function for H_i, then by [13, Lemma 1.10], in a neighborhood of p where each fiber bundle $\phi_{H_i} : H_i \to H_i$ is trivial, we can consider tuples of functions $y_i = (y_i^1, \ldots, y_i^{k_i})$ and $z = (z_1, \ldots, z_q)$ such that

$$(x_1, y_1, \ldots, x_k, y_k, z) \tag{2.1}$$

provides coordinates near p with the property that on H_i, $(x_1, y_1, \ldots, x_{i-1}, y_{i-1}, y_i)$ induces coordinates on the base S_{H_i} with ϕ_{H_i} corresponding to the map

$$(x_1, y_1, \ldots, \widehat{x}_i, y_i, \ldots, x_k, y_k, z) \mapsto (x_1, y_1, \ldots, x_{i-1}, y_{i-1}, y_i),$$

where the notation " $\widehat{\ }$ " above a variable denotes its omission.

To describe the type of metrics we want to consider on a manifold with fibered corners (M, ϕ), let $\mathcal{A}_{\mathrm{phg}}(M)$ denote the space of bounded continuous functions on M that are smooth on $M \setminus \partial M$ and polyhomogeneous on M for some index family. We refer to [33] for more details on polyhomogeneous functions on a manifold with corners.

Definition 2.3. *A* ***weight function*** *for* (M, ϕ) *is a function*

$$\begin{array}{rccc} \mathfrak{n}: & \mathcal{M}_1(M) & \to & \{0, \nu\} \\ & H & \mapsto & \nu_H \end{array}$$

for some $\nu \in [0, 1)$ *such that for all* $G, H \in \mathcal{M}_1(M)$,

$$H < G \quad \Longrightarrow \quad \nu_H \leq \nu_G. \tag{2.2}$$

For choices of compatible boundary defining functions $x_H \in \mathcal{C}^\infty(M)$ for each $H \in \mathcal{M}_1(M)$, the corresponding $\mathfrak{n}$***-weighted distance*** *is the function*

$$\rho := \prod_{H \in \mathcal{M}_1(M)} x_H^{-\frac{1}{1-\nu_H}} \in \mathcal{A}_{\mathrm{phg}}(M).$$

Alternatively, we say that ρ^{-1} is a $\mathfrak{n}$***-weighted total boundary defining function****.*

A choice of $\mathfrak{n}$-weighted total boundary defining function specifies a Lie algebra of vector fields as follows.

Definition 2.4. *Let ρ^{-1} be an $\mathfrak{n}$-weighted total boundary defining function for (M,ϕ). For such a choice, a* $\mathfrak{n}$***-weighted quasi-fibered boundary vector field*** *($\mathfrak{n}\,$QFB-vector fields for short) is a b-vector field $\xi \in \mathcal{C}^\infty(M; {}^bTM)$ such that*

(1) $\xi|_H$ is tangent to the fibers of $\phi_H : H \to S_H$ for each $H \in \mathcal{M}_1(M)$;
(2) $\xi\rho^{-1} \in v\rho^{-1}\mathcal{A}_{\mathrm{phg}}(M)$, where $v = \prod_{H\in\mathcal{M}_1(M)} x_H$ is a total boundary defining function.

We denote by $\mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M)$ the space of all $\mathfrak{n}\,$QFB-vector fields.

Remark 2.5. *When $\mathfrak{n}$ is the trivial weight function given by $\mathfrak{n}(H) = 0$ for all $H \in \mathcal{M}_1(M)$, $\mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M) = \mathcal{V}_{\mathrm{QFB}}(M)$ corresponds to the Lie algebra of QFB-vector fields of* [13].

The first condition in Definition 2.4 is clearly closed under taking the Lie bracket, while for the second one, it is closed thanks to the fact that for any b-vector field ξ, $\xi v \in v\mathcal{C}^\infty(M)$. Thus, $\mathfrak{n}\,$QFB-vector fields indeed form a Lie subalgebra of the Lie algebra of b-vector fields.

The first condition means that ξ is an edge vector field in the sense of [32, 2, 1]. We denote by $\mathcal{V}_e(M)$ the Lie algebra of edge vector fields. In the local coordinates (2.1), it is locally generated over $\mathcal{C}^\infty(M)$ by

$$v_1\frac{\partial}{\partial x_1}, v_1\frac{\partial}{\partial y_1}, \ldots, v_k\frac{\partial}{\partial x_k}, v_k\frac{\partial}{\partial y_k}, \frac{\partial}{\partial z}, \tag{2.3}$$

where $v_i := \prod_{j=i}^k x_j$ and where $\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial z}$ stand for

$$\frac{\partial}{\partial y_i^1}, \ldots, \frac{\partial}{\partial y_i^{\ell_i}} \quad \text{and} \quad \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^q} \quad \text{respectively.}$$

Taking into account the second condition in Definition 2.4, $\mathfrak{n}\,$QFB-vector fields are locally generated over $\mathcal{C}^\infty(M)$ by

$$\begin{aligned} v_1x_1\frac{\partial}{\partial x_1}, v_1\frac{\partial}{\partial y_1}, v_2\left((1-\nu_2)x_2\frac{\partial}{\partial x_2} - (1-\nu_1)x_1\frac{\partial}{\partial x_1}\right), v_2\frac{\partial}{\partial y_2}, \ldots,\\ v_k\left((1-\nu_k)x_k\frac{\partial}{\partial x_k} - (1-\nu_{k-1})x_{k-1}\frac{\partial}{\partial x_{k-1}}\right), v_k\frac{\partial}{\partial y_k}, \frac{\partial}{\partial z}, \end{aligned} \tag{2.4}$$

where $\nu_i := \nu_{H_i}$. Indeed, the natural modification $v_ix_i\frac{\partial}{\partial x_i}$ of the edge vector fields 'normal' to the boundary hypersurfaces satisfy the second condition, but the difference $(1-\nu_i)x_i\frac{\partial}{\partial x_i} - (1-\nu_j)x_j\frac{\partial}{\partial x_j}$ even annihilates ρ^{-1}, while multiplied by $v_{\min(i,j)+1}$ makes it an edge vector field, so that (2.4) provides a generating set.

In a similar way that the Lie algebra of QFB-vector fields depends on a choice of total boundary defining function [29, Lemma 1.1], the Lie algebra of $\mathfrak{n}\,$QFB-vector fields depends on a choice of $\mathfrak{n}$-weight total boundary defining function ρ^{-1}. Two such functions will be said to be $\mathfrak{n}\,$QFB**-equivalent** if they yield the same Lie algebra of vector fields. The following generalization of [29, Lemma 1.1] gives a simple criterion for two $\mathfrak{n}$-weighted total boundary defining functions to be $\mathfrak{n}\,$QFB-equivalent.

Lemma 2.6. *Two $\mathfrak{n}$-weighted total boundary defining functions ρ^{-1} and $(\rho')^{-1}$ are $\mathfrak{n}\,$QFB-equivalent provided the function*

$$f := \log\left(\frac{\rho'}{\rho}\right) \in \mathcal{A}_{\mathrm{phg}}(M)$$

is such that near each boundary hypersurface $H \in \mathcal{M}_1(M)$, $f = \phi_H^* f_H + \mathcal{O}(x_H)$ for some $f_H \in \mathcal{A}_{\mathrm{phg}}(S_H)$.

Proof. Clearly, ρ^{-1} and $(\rho')^{-1}$ are $\mathfrak{n}$ QFB-equivalent if and only if for all $\xi \in \mathcal{V}_e(M)$,

$$\frac{d\rho}{\rho v}(\xi) \in \mathcal{A}_{\mathrm{phg}}(M) \iff \frac{d\rho'}{\rho' v} \in \mathcal{A}_{\mathrm{phg}}(M), \tag{2.5}$$

where $v = \prod_{H \in \mathcal{M}_1(M)} x_H$ is a total boundary defining function for M. Now, since $\rho' = e^f \rho$,

$$\frac{d\rho'}{\rho' v} = \frac{1}{v}\left(\frac{d\rho}{\rho} + df\right). \tag{2.6}$$

In particular, if near each $H \in \mathcal{M}_1(M)$, $f = \phi_H^* f_H + \mathcal{O}(x_H)$ for some $f_H \in \mathcal{A}_{\mathrm{phg}}(M)$, then (2.5) holds for each $\xi \in \mathcal{V}_e(M)$. $\square$

By the local description (2.4), we see that $\mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M)$ is a locally free sheaf of rank $m = \dim M$ over $\mathcal{C}^\infty(M)$. Thus, by the Serre-Swan theorem, there is a natural smooth vector bundle, the **$\mathfrak{n}$-weighted QFB-tangent bundle** ($\mathfrak{n}$ QFB-tangent bundle for short), denoted ${}^{\mathfrak{n}}TM$, and a natural bundle map $\iota_{\mathfrak{n}} : {}^{\mathfrak{n}}TM \to TM$, restricting to an isomorphism on $M \setminus \partial M$, such that

$$\mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M) = (\iota_{\mathfrak{n}})_* \mathcal{C}^\infty(M; {}^{\mathfrak{n}}TM). \tag{2.7}$$

In fact, at $p \in M$, the fiber of ${}^{\mathfrak{n}}TM$ above p is given by ${}^{\mathfrak{n}}T_pM = \mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M)/\mathcal{I}_p \cdot \mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M)$, where $\mathcal{I}_p$ is the ideal of smooth functions vanishing at p. The vector bundle dual to ${}^{\mathfrak{n}}TM$, denoted ${}^{\mathfrak{n}}T^*M$, will be called the $\mathfrak{n}$-weighted QFB-cotangent bundle ($\mathfrak{n}$ QFB-cotangent bundle for short). We see from (2.4) that in the coordinates (2.1), the $\mathfrak{n}$ QFB-cotangent bundle is locally spanned over $\mathcal{C}^\infty(M)$ by

$$\rho_1^{-\mathfrak{n}} d\rho_1, \frac{dy_1^1}{v_1}, \ldots, \frac{dy_1^{\ell_1}}{v_1}, \rho_2^{-\mathfrak{n}} d\rho_2, \frac{dy_2^1}{v_2}, \ldots, \frac{dy_2^{\ell_2}}{v_2}, \ldots, \rho_k^{-\mathfrak{n}} d\rho_k, \frac{dy_k^1}{v_k}, \ldots, \frac{dy_k^{\ell_k}}{v_k}, dz^1, \ldots, dz^q, \tag{2.8}$$

where $v_i = \prod_{j=i}^k x_j$, $\rho_i = \prod_{j=i}^k x_j^{-\frac{1}{1-\nu_j}}$ and $\rho_i^{-\mathfrak{n}} = \prod_{j=i}^k x_j^{\frac{\nu_j}{1-\nu_j}}$ with $\nu_j := \mathfrak{n}(H_j)$.

The natural map $\iota_{\mathfrak{n}} : {}^{\mathfrak{n}}TM \to TM$ gives ${}^{\mathfrak{n}}TM$ the structure of a Lie algebroid and indicates that $(M, \mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M))$ is a Lie structure at infinity for $M \setminus \partial M$ in the sense of [4, Definition 3.1]. As such, it comes with the following natural class of metrics.

Definition 2.7. *A* ***$\mathfrak{n}$-weighted quasi-fibered boundary metric*** *($\mathfrak{n}$ QFB-metric for short) is a choice of Euclidean metric $g_{\mathfrak{n}\,\mathrm{QFB}}$ for the vector bundle ${}^{\mathfrak{n}}TM$. A* ***smooth*** *$\mathfrak{n}$* ***QFB-metric*** *is a Riemannian metric on $M \setminus \partial M$ induced by some $\mathfrak{n}$ QFB-metric $g_{\mathfrak{n}\,\mathrm{QFB}}$ via the map $\iota_{\mathfrak{n}} : {}^{\mathfrak{n}}TM \to TM$. Trusting this will lead to no confusion, we will also denote by $g_{\mathfrak{n}\,\mathrm{QFB}}$ the smooth $\mathfrak{n}$ QFB-metric induced by $g_{\mathfrak{n}\,\mathrm{QFB}} \in \mathcal{C}^\infty(M; {}^{\mathfrak{n}}TM \otimes {}^{\mathfrak{n}}TM)$.*

In terms of a choice of $\mathfrak{n}$ QFB-metric, notice that the Lie algebra of $\mathfrak{n}$ QFB-vector fields can alternatively be defined by

$$\mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M) = \{\xi \in \mathcal{C}^\infty(M; TM) \mid \sup_{M \setminus \partial M} g_{\mathfrak{n}\,\mathrm{QFB}}(\xi, \xi) < \infty\}. \tag{2.9}$$

Since $\mathfrak{n}$ QFB-metrics are induced by a Lie structure at infinity, they come with the following geometric properties.

Lemma 2.8. *Any smooth $\mathfrak{n}$ QFB-metric is complete of infinite volume with bounded geometry.*

Proof. Since these metrics come from the Lie structure at infinity $(M, \mathcal{V}_{\mathfrak{n}\,\mathrm{QFB}}(M))$ in the sense of [4, Definition 3.3], the result follows from [4] and [9]. $\square$

As in [13], we will be mostly interested in the case where the manifold with fibered corners (M, ϕ) is such that for each maximal hypersurface H, $S_H = H$ and $\phi_H : H \to S_H$ is the identity map. We say in this case that (M, ϕ) is a QAC**-manifold with fibered corners** and that an $\mathfrak{n}$-weighted quasi-fibered boundary metric is a **$\mathfrak{n}$-weighted quasi-asymptotically conical metric** ($\mathfrak{n}$ QAC-metric for short). More generally, we will replace QFB by QAC and quasi-fibered boundary by quasi-asymptotically conical whenever (M, ϕ) is a QAC-manifold with fibered corners. In fact, from now on, unless otherwise specified, we will assume that (M, ϕ) is a QAC-manifold with fibered corners. Now, on such a manifold, the class of metrics we are interested in is not quite $\mathfrak{n}$ QAC-metrics, but a conformally related one.

Definition 2.9. *Let (M,ϕ) be a QAC-manifold with fibered corners together with some Lie algebra of $\mathfrak{n}$ QAC-vector fields associated to some weight function $\mathfrak{n}$ and some $\mathfrak{n}$-weighted total boundary defining function*

$$\rho^{-1} = \prod_{H\in\mathcal{M}_1(M)} x_H^{\frac{1}{1-\nu_H}}.$$

*In such a setting, a **smooth $\mathfrak{n}$-warped QAC-metric** is a Riemannian metric g_w on $(M\setminus\partial M)$ of the form*

$$g_w = \rho^{2\mathfrak{n}} g_{\mathfrak{n}\,\mathrm{QAC}} \tag{2.10}$$

for some smooth $\mathfrak{n}$ QAC-metric $g_{\mathfrak{n}\,\mathrm{QAC}}$, where

$$\rho^{\mathfrak{n}} := \prod_{H\in\mathcal{M}_1(M)} x^{-\frac{\nu_H}{1-\nu_H}}.$$

Example 2.10. *In the local basis (2.8), an example of $\mathfrak{n}$-warped QAC-metric is given by*

$$g_w = \sum_{i=1}^k \rho^{2\mathfrak{n}}\left(\rho_i^{-2\mathfrak{n}} d\rho_i^2 + \sum_{j=i}^{\ell_i} \frac{d(y_i^j)^2}{v_i^2}\right) + \rho^{2\mathfrak{n}}\sum_{j=1}^q d(z^j)^2,$$

where $\rho_i^{2\mathfrak{n}} = (\rho_i^{\mathfrak{n}})^2 = \prod_{j=i}^k x_j^{\frac{-2\nu_j}{1-\nu_j}}$.

In fact, $\mathfrak{n}$-warped QAC-metrics have a nice iterative structure as illustrated by the next example.

Example 2.11. *Near $H\in\mathcal{M}_1(M)$ and a local trivialization of $\phi_H : H\to S_H$ over $\mathcal{U}\subset S_H$, but away from G for $G<H$, a local model of an $\mathfrak{n}$-warped QAC-metric is given by*

$$d\rho_H^2 + \rho_H^2 g_{\mathcal{U}} + \rho_H^{2\nu_H}\kappa_H, \tag{2.11}$$

where $\rho_H := \prod_{G\geq H} x_G^{-\frac{1}{1-\nu_G}}$, $g_{\mathcal{U}}$ is a smooth metric in $\mathcal{U}$ and κ_H is an $\mathfrak{n}_{Z_H}$-warped QAC-metric on the fiber $Z_H = \phi_H^{-1}(s)$ for some $s\in\mathcal{U}$ with weight function $\mathfrak{n}_{Z_H}$ given by

$$\mathfrak{n}_{Z_H}(Z_H\cap G) := \frac{\nu_G-\nu_H}{1-\nu_H} \quad \text{for} \quad G>H.$$

With respect to a fixed point on Z_H, the $\mathfrak{n}_{Z_H}$-weighted distance function of κ_H in (2.11) is given by

$$\rho_{Z_H} := \prod_{G>H} x_G^{-\frac{1}{1-\mathfrak{n}_{Z_H}(G)}}.$$

In particular, in the model (2.11), to be close to H means that $x_H<c$ for some small constant $c>0$, which in terms of the functions ρ_H and ρ_{Z_H} corresponds to the inequality

$$\rho_{Z_H} < c\rho_H^{1-\nu_H}. \tag{2.12}$$

Remark 2.12. *The definition of $\mathfrak{n}$ QAC-metrics and $\mathfrak{n}$-warped QAC-metrics would still make sense for a weight function $\mathfrak{n}:\mathcal{M}_1(M)\to[0,1)$ satisfying (2.2), but we need to restrict the values of $\mathfrak{n}$ to $\{0,\nu\}$ to have a nice iterative structure as in Example 2.11.*

We can formally define the $\mathfrak{n}$-warped QAC-tangent bundle by

$$ {}^wTM := (\rho^{-\mathfrak{n}})({}^{\mathfrak{n}}TM) \quad \text{with space of sections} \quad \mathcal{C}^\infty(M;{}^wTM) := \rho^{-\mathfrak{n}}\mathcal{C}^\infty(M;{}^{\mathfrak{n}}TM) \tag{2.13}$$

and denote by ${}^wT^*M$ its dual, so that smooth $\mathfrak{n}$-warped QAC-metrics correspond to an element of $\mathcal{C}^\infty(M;S^2({}^wT^*M))$.

As in [13] and [18], it will be useful to introduce yet another class of metrics to describe the weighted Hölder spaces that we will use.

Definition 2.13. *Let (M,ϕ) be a* QAC*-manifold with fibered corners and let $x_{\max}$ be a product of boundary defining functions associated to all the maximal boundary hypersurfaces of M. Let $\mathfrak{n}$ be a weight function and let ρ be an $\mathfrak{n}$-weighted distance. In such a setting, a* ***smooth $\mathfrak{n}$-weighted quasi b-metric*** *($\mathfrak{n}\,$Qb-metric for short) on M is a Riemannian metric $g_{\mathfrak{n}\,\mathrm{Qb}}$ of the form*

$$g_{\mathfrak{n}\,\mathrm{Qb}} = x_{\max}^2 g_{\mathfrak{n}\,\mathrm{QAC}}$$

for some smooth $\mathfrak{n}$-weighted QAC*-metric $g_{\mathfrak{n}\,\mathrm{QAC}}$.*

Remark 2.14. *When $\mathfrak{n}$ is the trivial weight function given by $\mathfrak{n}(H)=0$ for all $H\in\mathcal{M}_1(M)$, a $\mathfrak{n}\,$Qb-metric corresponds to a* Qb*-metric in the sense of* [13].

As for $\mathfrak{n}\,$QAC-metrics, $\mathfrak{n}\,$Qb-metrics can be defined in terms of a Lie structure at infinity. To see this, consider the space

$$\mathcal{V}_{\mathfrak{n}\,\mathrm{Qb}}(M) := \{\xi\in\mathcal{C}^\infty(M;TM) \mid \sup_{M\setminus\partial M} g_{\mathfrak{n}\,\mathrm{Qb}}(\xi,\xi)<\infty\} \tag{2.14}$$

of smooth vector fields on M uniformly bounded with respect to some choice of $\mathfrak{n}\,$Qb-metric $g_{\mathfrak{n}\,\mathrm{Qb}}$. From Definition 2.4, Definition 2.13 and (2.9), the space (2.14) can alternatively be defined as the space of b-vector fields ξ on M such that

1. $\xi|_H$ is tangent to the fibers of $\phi_H : H\to S_H$ for all $H\in\mathcal{M}_1(M)$ not maximal with respect to the partial order;
2. $\xi\rho^{-1}\in \frac{v\rho^{-1}}{x_{\max}}\mathcal{C}^\infty(M)$.

As for $\mathfrak{n}\,$QAC-vector fields, one can check that these two conditions are closed under taking the Lie bracket, so that $\mathcal{V}_{\mathfrak{n}\,\mathrm{Qb}}(M)$ is a Lie subalgebra of the Lie algebra of b-vector fields. By (2.14) and (2.4), $\mathfrak{n}\,$Qb-vector fields are locally generated over $\mathcal{C}^\infty(M)$ by (2.4) when H_k is not maximal and by

$$\frac{v_1x_1}{x_k}\frac{\partial}{\partial x_1}, \frac{v_1}{x_k}\frac{\partial}{\partial y_1}, \frac{v_2}{x_k}\left((1-\nu_2)x_2\frac{\partial}{\partial x_2}-(1-\nu_1)x_1\frac{\partial}{\partial x_1}\right), \frac{v_2}{x_k}\frac{\partial}{\partial y_2},\ldots, \\ \left((1-\nu_k)x_k\frac{\partial}{\partial x_k}-(1-\nu_{k-1})x_{k-1}\frac{\partial}{\partial x_{k-1}}\right) \quad\text{and}\quad \frac{\partial}{\partial y_k} \tag{2.15}$$

otherwise. Thus, $\mathcal{V}_{\mathfrak{n}\,\mathrm{Qb}}(M)$ is a locally free sheaf of rank m over $\mathcal{C}^\infty(M)$. By the Serre-Swan theorem, there is a corresponding vector bundle ${}^{\mathfrak{n}\,\mathrm{Qb}}TM = x_{\max}^{-2}\,{}^{\mathfrak{n}}TM$ and a map $\iota_{\mathfrak{n}\,\mathrm{Qb}} : {}^{\mathfrak{n}\,\mathrm{Qb}}TM\to TM$ such that

$$\mathcal{V}_{\mathfrak{n}\,\mathrm{Qb}}(M) = (\iota_{\mathfrak{n}\,\mathrm{Qb}})_*\mathcal{C}^\infty(M;{}^{\mathfrak{n}\,\mathrm{Qb}}TM).$$

In other words, the map $\iota_{\mathfrak{n}\,\mathrm{Qb}}$ gives ${}^{\mathfrak{n}\,\mathrm{Qb}}TM$ the structure of a Lie algebroid over M and $(M,\mathcal{V}_{\mathfrak{n}\,\mathrm{Qb}}(M))$ is a Lie structure at infinity for $M\setminus\partial M$. In particular, by [4] and [9], smooth $\mathfrak{n}\,$Qb-metrics have the following geometric properties.

Lemma 2.15. *Any smooth $\mathfrak{n}\,$Qb-metric is automatically complete of infinite volume with bounded geometry.*

We will need various function spaces associated to these metrics. First, recall that if (X,g) is a Riemannian manifold and $E\to X$ is a Euclidean vector bundle over X together with a connection ∇ compatible with the Euclidean structure, we can for each $\ell\in\mathbb{N}_0$ associate the space $\mathcal{C}^\ell(X;E)$ of continuous sections $\sigma: X\to E$ such that

$$\nabla^j\sigma\in\mathcal{C}^0(X;T^0_jX\otimes E) \quad\text{and}\quad \sup_{p\in X}|\nabla^j\sigma|_g<\infty \quad \forall j\in\{0,\ldots,\ell\},$$

where ∇ denotes as well the connection induced by the Levi-Civita connection of g and the connection on E, while $|\cdot|_g$ is the norm induced by g and the Euclidean structure on E. This is a Banach space with norm

$$\|\sigma\|_{g,\ell} := \sum_{j=0}^{\ell}\sup_{p\in X}|\nabla^j\sigma(p)|_g.$$

The intersection of these spaces yields the Fréchet space

$$\mathcal{C}^\infty_g(X;E) := \bigcap_{\ell\in\mathbb{N}_0}\mathcal{C}^\ell_g(X;E).$$

For $\ell \in \mathbb{N}_0$ and $\alpha \in (0,1]$, there is also the Hölder space $\mathcal{C}^{\ell,\alpha}_g(X;E)$ consisting of sections $\sigma \in \mathcal{C}^\ell(X;E)$ such that

$$[\nabla^\ell \sigma]_{g,\alpha} := \sup \left\{ \frac{|P_\gamma(\nabla^\ell \sigma(\gamma(0))) - \nabla^\ell \sigma(\gamma(1))|}{\ell(\gamma)^\alpha} \mid \gamma \in \mathcal{C}^\infty([0,1];X),\ \gamma(0) \neq \gamma(1) \right\} < \infty,$$

where $\ell(\gamma)$ is the length of γ with respect to g and $P_\gamma : T^0_\ell X \otimes E|_{\gamma(0)} \to T^0_\ell X \otimes E|_{\gamma(1)}$ is parallel transport along γ. Again, this is a Banach space with norm given by

$$\|\sigma\|_{g,\ell,\alpha} := \|\sigma\|_{g,\ell} + [\nabla^\ell \sigma]_{g,\alpha}.$$

For $\mu \in \mathcal{C}^\infty(X)$ a positive function, there are corresponding weighted versions

$$\mu \mathcal{C}^{\ell,\alpha}_g(X;E) := \left\{ \sigma \mid \frac{\sigma}{\mu} \in \mathcal{C}^{\ell,\alpha}_g(X;E) \right\} \quad \text{with norm} \quad \|\sigma\|_{\mu \mathcal{C}^{\ell,\alpha}_g} := \left\| \frac{\sigma}{\mu} \right\|_{g,\ell,\alpha}.$$

When $X = M \setminus \partial M$ and $g = g_{\mathfrak{n}\,\mathrm{Qb}}$ is a $\mathfrak{n}\,$Qb-metric, we obtain the $\mathfrak{n}\,$Qb-Hölder space $C^{\ell,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M;E)$, as well as the space $\mathcal{C}^\ell_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M;E)$. Similarly, if $g = g_w$ is an $\mathfrak{n}$-warped QAC-metric, we can define the $\mathfrak{n}$-warped QAC-Hölder space $\mathcal{C}^{\ell,\alpha}_w(M \setminus \partial M;E)$. Since a $\mathfrak{n}\,$Qb-metric is conformally related to an $\mathfrak{n}$-warped QAC-metric via

$$g_w = \frac{g_{\mathfrak{n}\,\mathrm{Qb}}}{\chi^2} \quad \text{with} \quad \chi := \frac{x_{\max}}{\rho^{\mathfrak{n}}} \geq 0 \text{ bounded},$$

there is an obvious continuous inclusion

$$\mathcal{C}^{\ell,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus M;E) \subset \mathcal{C}^{\ell,\alpha}_w(M \setminus \partial M;E). \tag{2.16}$$

Conversely, there is the following partial counterpart.

Lemma 2.16. *For $0 < \delta < 1$, there is a continuous inclusion $\chi^\delta \mathcal{C}^{0,1}_w(M \setminus \partial M;E) \subset \mathcal{C}^{0,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M;E)$ for $\alpha \leq \delta$.*

Proof. Since χ is a product of powers of boundary defining functions, its logarithmic differential is automatically a b-differential in the sense of [34], so

$$\frac{d\chi}{\chi} \in \mathcal{A}_{\mathrm{phg}}(M;{}^bT^*M) \subset \mathcal{A}_{\mathrm{phg}}(M;{}^{\mathfrak{n}\,\mathrm{Qb}}T^*M) \subset \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M;{}^{\mathfrak{n}\,\mathrm{Qb}}T^*M).$$

Using this observation, we can run the same argument as in the proof of [18, Lemma 3.9] to obtain the result, the starting point of this proof being a similar observation, namely [18, (3.38)]. □

We can also consider the Sobolev space $H^\ell_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M)$ associated to a $\mathfrak{n}\,$Qb-metric. For an $\mathfrak{n}$-warped QAC-metric, instead of the natural Sobolev space associated to such a metric, we will consider the weighted version of the $\mathfrak{n}\,$Qb-Sobolev space

$$H^\ell_w(M \setminus \partial M) := \chi^{\frac{\dim M}{2}} H^\ell_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M)), \tag{2.17}$$

where the factor $\chi^{\frac{\dim M}{2}}$ ensures that we integrate with respect to the volume density of an $\mathfrak{n}$-warped QAC-metric, but with pointwise norms of the derivatives measured with respect to a $\mathfrak{n}\,$Qb-metric instead of an $\mathfrak{n}$-warped QAC-metric.

So far, we have only considered *smooth* $\mathfrak{n}\,$Qb-metrics and $\mathfrak{n}$-warped QAC-metrics, that is, metrics corresponding to elements of $\mathcal{C}^\infty(M;S^2({}^{\mathfrak{n}\,\mathrm{Qb}}T^*M))$ and $\mathcal{C}^\infty(M;S^2({}^wT^*M))$, but to look for Calabi-Yau examples, it will be important to be less restrictive on the regularity of these metrics at the boundary. More precisely, we will look at $\mathfrak{n}\,$Qb-metrics corresponding to sections of $\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M;S^2({}^{\mathfrak{n}\,\mathrm{Qb}}T^*M))$ (quasi-isometric to some fixed smooth $\mathfrak{n}\,$Qb-metric). For $\mathfrak{n}$-warped QAC-metrics, we could look at those corresponding to sections of $\mathcal{C}^\infty_w(M \setminus \partial M;S^2({}^wT^*M))$ (quasi-isometric to some smooth $\mathfrak{n}$-warped QAC metrics), but keeping in mind the continuous inclusion (2.16), we will in fact be stricter and consider $\mathfrak{n}$-warped QAC-metrics corresponding to sections of $\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M;S^2({}^wT^*M))$, that is, $g_w \in \mathcal{C}^\infty_w(M \setminus \partial M;S^2({}^wT^*M))$ of the form

$$g_w = \frac{g_{\mathfrak{n}\,\mathrm{Qb}}}{\chi^2}$$

for some $\mathfrak{n}\,$Qb-metric $g_{\mathfrak{n}\,\mathrm{Qb}} \in \mathcal{C}^\infty_{\mathrm{Qb}}(M \setminus \partial M;S^2({}^{\mathfrak{n}\,\mathrm{Qb}}T^*M))$. We will say that such an $\mathfrak{n}$-warped QAC-metric is $\mathfrak{n}\,$Qb-smooth. Clearly, Lemma 2.15 still holds for $\mathfrak{n}\,$Qb-metrics in $\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M;S^2({}^{\mathfrak{n}\,\mathrm{Qb}}T^*M))$, since by assumption we control the curvature and its derivatives. For $\mathfrak{n}$-warped QAC-metrics, we also have such a result.

Proposition 2.17. *Any $\mathfrak{n}$-warped* QAC-*metric* $g_w \in \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M; S^2({}^wT^*M))$ *is complete of infinite volume with bounded geometry.*

Proof. By assumption,

$$g_w = \frac{g_{\mathfrak{n}\,\mathrm{Qb}}}{\chi^2}$$

for some $\mathfrak{n}\,$Qb-metric $g_{\mathfrak{n}\,\mathrm{Qb}} \in \mathcal{C}^\infty_{\mathrm{Qb}}(M \setminus \partial M; S^2({}^{\mathfrak{n}\,\mathrm{Qb}}T^*M))$. By Lemma 2.15 and the comment above, we know that $g_{\mathfrak{n}\,\mathrm{Qb}}$ is complete of infinite volume with bounded geometry. On the other hand, χ is a bounded positive function, so g_w is automatically complete of infinite volume. By the continuous inclusion (2.16) and [7, Theorem 1.159], we see that the curvature of g_w is bounded, as well as its covariant derivatives. To see that it is of bounded geometry, it suffices then to show that the injectivity radius is positive, which follows from the facts that $g_{\mathfrak{n}\,\mathrm{Qb}}$ has positive injectivity radius and that χ is a bounded positive function on $M \setminus \partial M$. □

3. Mapping properties of the Laplacian

The mapping properties of the Laplacian of a QAC-metric obtained in [20] have a natural analogue for warped QAC-metrics. Indeed, such results were already obtained for some specific examples of warped QAC-metrics of depth one in [18]; see also [31, 36, 21] for different approaches. In this section, we will combine the arguments [20] and [18] to obtain mapping properties of the Laplacian for warped QAC-metrics of arbitrary depth. It would presumably be possible to obtain such mapping properties using barrier functions as in [27], though probably only ensuring invertibility of the Laplacian for a smaller range of weights. Thus, let

$$\mathfrak{n} : \mathcal{M}_1(M) \to \{0, \nu\}$$

be a weight function for some $\nu \in [0, 1)$. The set of boundary hypersurfaces of M therefore decomposes as

$$\mathcal{M}_1(M) = \mathcal{M}_{1,0}(M) \cup \mathcal{M}_{1,\nu}$$

with $\mathcal{M}_{1,a}(M) = \mathfrak{n}^{-1}(a)$. This is a disjoint union if $\nu > 0$, and otherwise $\mathcal{M}_1(M) = \mathcal{M}_{1,0}(M) = \mathcal{M}_{1,\nu}(M)$.

For the convenience of the reader, let us first recall the general strategy of [20]. If g is a complete Riemannian metric on a manifold Z and h is a positive smooth function on it, then we can introduce a measure $d\mu = h^2 dg$ using the volume density dg of g. The triple (Z, g, μ) is then a complete weighted Riemannian manifold in the sense of [24]. On such manifolds, the Riemannian metric g induces a distance function $d(p, q)$ between two points $p, q \in Z$. We will denote by

$$B(p, r) := \{q \in Z \mid d(p, q) < r\}$$

the geodesic ball of radius r centered at $p \in Z$. To measure the volume of such balls, we use however the measure μ, not the volume density of g. Similarly, the natural L^2-inner product of two functions is the one induced by the measure μ, namely

$$\langle u, v\rangle_\mu := \int_Z uv d\mu. \tag{3.1}$$

If ∇ is the Levi-Civita connection of g, we will be interested int studying the mapping properties of the corresponding Laplacian $\Delta = \operatorname{div} \circ \nabla$ of g. More generally, for $\mathcal{R}$ a function, we will consider the operator $\mathcal{L} := -\Delta + \mathcal{R}$, as well as its Doob transform with respect to h,

$$\widetilde{\mathcal{L}} := h^{-1} \circ \mathcal{L} \circ h = -\Delta_\mu + V + \mathcal{R}, \tag{3.2}$$

where $V := \frac{\Delta h}{h}$ and $-\Delta_\mu = \nabla^{*,\mu}\nabla$ with $\nabla^{*,\mu}$ the adjoint of ∇ with respect to the L^2-inner product (3.1) and the L^2-inner product on forms given by

$$\langle \eta_1, \eta_2\rangle_\mu := \int_Z (\eta_1, \eta_2)_g(z) d\mu(z). \tag{3.3}$$

Denote by $H_{\mathcal{L}}(t, z, z')$ and $H_{-\Delta+V}(t, z, z')$ the heat kernels of $\mathcal{L}$ and $-\Delta + V$ with respect to the volume density g and let

$$G_{\mathcal{L}}(z, z') = \int_0^\infty H_{\mathcal{L}}(t, z, z')dt \quad \text{and} \quad G_{-\Delta+V}(z, z') = \int_0^\infty H_{-\Delta+V}(t, z, z')dt \tag{3.4}$$

be the corresponding Green's functions. Similarly, let $H_{-\Delta_\mu}(t,z,z')$ be the heat kernel of $-\Delta_\mu$ with respect to the measure μ with corresponding Green's function

$$G_{-\Delta_\mu}(z,z') = \int_0^\infty H_{-\Delta_\mu}(t,z,z')dt.$$

By a result of [20], those heat kernels and Green functions are related as follows.

Lemma 3.1 (Theorem 3.12 in [20])**.** *If $\mathcal{R} \geq V$, then*

$$|H_{\mathcal{L}}(t,z,z')| \leq H_{-\Delta+V}(t,z,z') \leq h(z)h(z')H_{-\Delta_\mu}(t,z,z')$$

and

$$|G_{\mathcal{L}}(z,z')| \leq G_{-\Delta+V}(z,z') \leq h(z)h(z')G_{-\Delta_\mu}.$$

Thus, to obtain control on $H_{\mathcal{L}}(t,z,z')$ and $G_{\mathcal{L}}(z,z')$, it suffices to obtain control on $H_{-\Delta_\mu}$. This can be achieved by the method of Grigor'yan and Saloff-Coste invoking the following notions.

Definition 3.2. *The complete weighted Riemannian manifold (Z,g,μ) satisfies*

$(VD)_\mu$ ***the weighted volume doubling property*** *if there exists $C_D > 0$ such that*

$$\mu(B(p,2r)) \leq C_D\mu(B(p,r)) \quad \forall p \in Z,\ \forall r > 0;$$

$(PI)_{\mu,\delta}$ ***the uniform weighted Poincaré inequality with parameter*** $\delta \in (0,1]$ *if there exists a constant $C_P > 0$ such that*

$$\int_{B(p,r)} (f-\overline{f})^2 d\mu \leq C_P r^2 \int_{B(p,\delta^{-1}r)} |df|_g^2 d\mu \quad \forall f \in W^{1,2}_{loc}(Z),\ \forall p \in Z,\ \forall r > 0;$$

$(PI)_\mu$ ***the uniform weighted Poincaré inequality*** *if we can take $\delta = 1$ in the previous statement.*

In terms of these conditions, one of the main results of [24] is the following.

Theorem 3.3 (Theorem 2.7 in [24])**.** *Let (Z,g,μ) be a complete weighted Riemannian manifold satisfying $(VD)_\mu$ and $(PI)_\mu$. Then there are positive constants C and c such that*

$$c(\mu(B(z,\sqrt{t})),\mu(B(z',\sqrt{t})))^{-\frac{1}{2}} e^{-C\frac{d(z,z')^2}{t}} \leq H_{-\Delta_\mu}(t,z,z') \leq C(\mu(B(z,\sqrt{t})),\mu(B(z',\sqrt{t})))^{-\frac{1}{2}} e^{-c\frac{d(z,z')^2}{t}}$$

for all $(t,z,z') \in (0,\infty)\times Z\times Z$.

By Lemma 3.1, this yields the following.

Corollary 3.4. *If the complete weighted Riemannian manifold (Z,g,μ) satisfies $(VD)_\mu$ and $(PI)_\mu$, then there are positive constants c and C such that*

$$c(\mu(B(z,\sqrt{t})),\mu(B(z',\sqrt{t})))^{-\frac{1}{2}} e^{-C\frac{d(z,z')^2}{t}} \leq \frac{H_{-\Delta+V}(t,z,z')}{h(z)h(z')} \leq C(\mu(B(z,\sqrt{t})),\mu(B(z',\sqrt{t})))^{-\frac{1}{2}} e^{-c\frac{d(z,z')^2}{t}}$$

for all $(t,z,z') \in (0,\infty)\times Z\times Z$.

We want to apply this result when $Z = (M\setminus\partial M)$ is the interior of a manifold with fibered corners M of dimension m and g is a warped QAC-metric with weight $\mathfrak{n}$ for some choice of $\nu \in [0,1)$. The type of measure μ we will consider will be one of the form

$$d\mu_a = x^a dg \quad \text{with} \quad x^a = \prod_{H\in\mathcal{M}_1(M)} x_H^{a_H} \tag{3.5}$$

for some

$$\begin{array}{rccc} a: & \mathcal{M}_1(M) & \to & \mathbb{R} \\ & H & \mapsto & a_H. \end{array}$$

It will be convenient to distinguish between maximal and non-maximal boundary hypersurfaces. To this end, we will use the notation

$$\mathcal{M}_{\max}(M) := \{H\in\mathcal{M}_1(M) \mid H \text{ is maximal}\} \quad \text{and} \quad \mathcal{M}_{\mathrm{nm}}(M) := \mathcal{M}_1(M)\setminus\mathcal{M}_{\max}(M).$$

We will denote by

$$v := \prod_{H\in\mathcal{M}_1(M)} x_H$$

a total boundary defining function for M. The distance function of g with respect to a fixed point $o \in M\setminus\partial M$ is comparable to

$$\rho = \prod_{H\in\mathcal{M}_1(M)} x^{-\frac{1}{1-\nu_H}}. \tag{3.6}$$

Near H, but away from G for $G < H$, the distance function with respect to a fixed point is also comparable to $\rho_H = \prod_{G\geq H} x_G^{-\frac{1}{1-\nu_G}}$. It will be convenient also to have the following weighted version of the function v_H, namely

$$\widetilde{v}_H := \rho_H^{-(1-\nu_H)} = \begin{cases} \prod_{G\geq H} x_G, & \nu_H = \nu; \\ \prod_{G\geq H} x_G^{\frac{1}{1-\nu_G}}, & \nu_H = 0, \end{cases} \tag{3.7}$$

as well as the function

$$\sigma = \begin{cases} \prod_{H\in\mathcal{M}_{1,\nu}(M)} x_H^{-\frac{1}{1-\nu}}, & \nu > 0, \\ 1, & \nu = 0, \end{cases} \tag{3.8}$$

such that $\sigma^\nu = \rho^{\mathfrak{n}}$.

We will also denote by

$$m_H = \dim M - \dim S_H - 1$$

the dimension of the fibers of $\phi_H : H \to S_H$ and by $b_H = \dim S_H$ the dimension of the base.

To apply the previous result to this setting, we will need to check that $(VD)_\mu$ and $(PI)_\mu$ hold for suitable choices of weight $a : \mathcal{M}_1(M) \to \mathbb{R}$. To that end, recall the following notation used in [20].

Definition 3.5. *Fix once and for all a basepoint $o \in M \setminus \partial M$. A ball of radius R at o is called **anchored** and we denote its volume by*

$$\mathcal{A}(R; a) = \mu_a(B(o, R)).$$

*Fix $c \in (0, 1)$. With respect to this choice, a ball $B(p, r)$ is said to be **remote** if $r < cd(o, p)$, in which case we use the notation*

$$\mathcal{R}(p, r; a) := \mu_a(B(p, r)).$$

If $B(p, r)$ is any ball, possibly neither anchored nor remote, we use the notation

$$\mathcal{V}(p, r; a) = \mu_a(B(p, r)).$$

Following the strategy of [20], we will derive an estimate of the volume of anchored balls in terms of the volume of remote balls. We start with the following estimate.

Proposition 3.6. *Provided $(1 - \nu_H)(a_H + m_H) \neq m$ for all $H \in \mathcal{M}_1(M)$ and $(1 - \nu_H)(a_H + m_H) \neq (1 - \nu_G)(a_G + m_G)$ for all $G, H \in \mathcal{M}_1(M)$ with $G < H$, we have that for $R \geq 1$,*

$$\mathcal{A}(R; a) \asymp 1 + R^m \sum_{H\in\mathcal{M}_1(M)} R^{(\nu_H-1)(a_H+m_H)}, \tag{3.9}$$

where the notation $f_1 \asymp f_2$ means that there exists positive constants c and C such that $cf_2 \leq f_1 \leq Cf_2$. In particular, in the QAC *setting, that is, when $\nu = 0$, this gives*

$$\mathcal{A}(R; a) \asymp 1 + R^m \sum_{H\in\mathcal{M}_1(M)} R^{-a_H-m_H}. \tag{3.10}$$

Proof. When M is of depth 1, we are in the AC-setting with $v \asymp \rho^{\nu-1}$, so we have

$$\mathcal{A}(R; a) \asymp 1 + \sum_{H\in\mathcal{M}_1(M)} R^{m+(\nu-1)(a_H+m_H)}$$

by taking into account the contribution of each asymptotically conical end and taking into account that $m_H = 0$ in this case. We can therefore proceed by induction on the depth of M to obtain the result. More precisely, in an open set $\mathcal{V}$ where the local model (2.11) is valid, we need to show that

$$\mu_a(B(o, R) \cap \mathcal{V}) \asymp 1 + R^m \sum_{G\geq H} R^{(\nu_G-1)(a_G+m_G)}.$$

Now, using that $\rho \asymp \rho_H$ in this region, we see that

$$\mu_a(B(o,R)\cap\mathcal{V}) \asymp 1 + \int_1^R \rho^{(\nu_H-1)a_H}\rho^{b_H}\rho^{\nu_H m_H}\mathcal{A}(\rho^{1-\nu_H};\widetilde{a})d\rho,$$

where $\widetilde{a} : \{G \in \mathcal{M}_1(M) \mid G > H\} \to \mathbb{R}$ given by

$$\widetilde{a}(G) = \widetilde{a}_G := \begin{cases} a_G - a_H, & \nu_H = \nu, \\ a_G - \frac{a_H}{1-\nu_G}, & \nu_H = 0. \end{cases} \tag{3.11}$$

On the fibers of $\phi_H : H \to S_H$, recall that the model metrics are actually QAC-metrics (even if g is a warped QAC-metric with factor $\nu > 0$) when $\nu_H = \nu$. Thus assuming by induction on the depth that (3.10) holds for these metrics, we compute that when $\nu_H = \nu$,

$$\begin{aligned}
\mu_a(B(o,R)\cap\mathcal{V}) &\asymp 1 + \int_1^R \rho^{(\nu-1)a_H}\rho^{b_H}\rho^{\nu m_H}\mathcal{A}(\rho^{1-\nu};\widetilde{a})d\rho \\
&\asymp 1 + \int_1^R \rho^{(\nu-1)a_H+b_H+\nu m_H}\left(1+\rho^{(1-\nu)m_H}\sum_{G>H}\rho^{(1-\nu)(-(a_G-a_H)-m_G)}\right)d\rho \\
&\asymp 1 + R^{(\nu-1)a_H+b_H+1+\nu m_H}\left(1+R^{(1-\nu)m_H}\sum_{G>H}R^{(1-\nu)(-(a_G-a_H)-m_G)}\right) \\
&\asymp 1 + R^m\sum_{G\geq H}R^{(\nu-1)(a_G+m_G)} \\
&\asymp 1 + R^m\sum_{G\geq H}R^{(\nu_G-1)(a_G+m_G)}.
\end{aligned}$$

If instead $\nu_H = 0$, then

$$\begin{aligned}
\mu_a(B(o,R)\cap\mathcal{V}) &\asymp 1 + \int_1^R \rho_H^{-a_H+b_H}\mathcal{A}(\rho_H;\widetilde{a})d\rho \\
&\asymp 1 + \int_1^R \rho_H^{-a_H+b_H}\left(1+\rho_H^{m_H}\sum_{G>H}\rho_H^{(\nu_G-1)(a_G-\frac{a_H}{1-\nu_G}+m_G)}\right)d\rho_H \\
&\asymp 1 + R^{-a_H+b_H+1}\left(1+R^{m_H}\sum_{G>H}R^{(\nu_G-1)(a_G+m_G)+a_H}\right) \\
&\asymp 1 + R^m\left(\sum_{G\geq H}R^{\nu_G-1(a_G+m_G)}\right).
\end{aligned}$$

In both computations, the conditions on the weights a_H and a_G have been used to ensure there are no logarithmic terms when we integrate in ρ.

□

Remark 3.7. *When we drop the condition that $(1-\nu_H)(a_H+m_H) \neq m$ for all $H \in \mathcal{M}_1(M)$ and $(1-\nu_H)(a_H+m_H) \neq (1-\nu_G)(a_G+m_G)$ for all $G, H \in \mathcal{M}_1(M)$ with $G < H$, a similar result holds, but with some powers of R multiplied by some positive integer power of* $(\log R)$.

Corollary 3.8. *Suppose that a is such that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H+m_H)$ for all $H \in \mathcal{M}_{\mathrm{nm}}(M)$. In this case,*

$$\mathcal{A}(R;a) \asymp R^{m+(\nu-1)a_{\max}} \quad \textit{for } R > 1.$$

For remote balls, we have the following preliminary estimate.

Proposition 3.9. *Suppose that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$. Fix $p \in M \setminus \partial M$ and suppose that we have chosen a remote parameter $c \in (0,1)$ so that in fact $c \in (0, \frac{1}{5})$. If $x_G(p) \geq 1-3c$ for all $G \in \mathcal{M}_{\rm nm}(M)$, then for $r \in (0, c\rho(p))$,*

$$\mathcal{R}(p,r;a) \asymp \rho(p)^{(\nu-1)a_{\max}} r^m. \tag{3.12}$$

If instead $x_H(p) < 1-3c$ for some $H \in \mathcal{M}_{\rm nm}(M)$ and $x_G(p) \geq 1-3c$ for $G < H$, then

$$\mathcal{R}(p,r;a) \asymp \rho(p)^{(\nu_H-1)a_H+\nu_H m_H} r^{b_H+1} \mathcal{V}(p_{Z_H}, \frac{r}{\rho(p)^{\nu_H}}; \widetilde{a}), \tag{3.13}$$

where p_{Z_H} is the projection of p onto the factor Z_H in the decomposition (2.11) *and $\widetilde{a}$ is given by* (3.11).

Proof. First, notice that $\rho(z) \asymp \rho(p)$ for $z \in B(p, c\rho(p))$. If $x_G(p) \geq 1-3c$ for all $G \in \mathcal{M}_{\rm nm}(M)$, we are in a region where the metric behaves like an AC-metric with $x_H \asymp \rho^{\nu-1}$ for $H \in \mathcal{M}_{\max}(M)$, so the result follows by a simple rescaling argument as in [20, Proposition 4.3]. If instead $x_H(\rho) < 1-3c$ for some $H \in \mathcal{M}_{\rm nm}(M)$, but $x_G(p) \geq 1-3c$ for $G < H$, then $B(p,r) \subset V_H$ with

$$V_H = \{p \in M \setminus \partial M \mid x_H(p) < 1-c\}.$$

Indeed, if we set $r_2 := \rho^{\nu_H}\rho_{Z_H} = \rho x_H$, then using the triangle inequality in terms of the model metric (2.11), we see that if $q \in B(p,r)$, then

$$\rho(q) \geq \rho(p) - d(p,q) \geq \rho(p) - r \geq \rho(p) - c\rho(p) = (1-c)\rho(p) \tag{3.14}$$

and

$$r_2(q) \leq r_2(p) + d(p,q) \leq r_2(p) + c\rho(p), \tag{3.15}$$

so that

$$x_H(q) = \frac{r_2(q)}{\rho(q)} \leq \frac{r_2(p)+c\rho(p)}{(1-c)\rho(p)} \leq \frac{(1-3c)\rho(p)+c\rho(p)}{(1-c)\rho(p)} = \frac{1-2c}{1-c} = 1 - \frac{c}{1-c} < 1-c.$$

Similarly, for $G < H$, setting instead $r_2 = \rho^{\nu_G}\rho_{Z_G} = \rho x_G$, we have the inequalities

$$\rho(q) \leq \rho(p) + d(p,q) \leq (1+c)\rho(p)$$

and

$$r_2(q) \geq r_2(p) - d(p,q) \geq r_2(p) - c\rho(p)$$

that we can use to show that $x_G(q) \geq 1-5c > 0$ for $q \in B(p,r)$. So we can use the model metric (2.11) with κ_H a warped QAC-metric on the fiber Z_H of $\phi_H : H \to S_H$ to estimate $\mu_a(B(p,r))$ with $r \in (0, c\rho(p))$. If p corresponds to the point $(p_1, p_2) \in (\mathbb{R}^+ \times S_H) \times Z_H$, let $B_1(p_1, r)$ be the geodesic ball in $C_H = \mathbb{R}^+ \times S_H$ and $B_2(p_2, r)$ denote the geodesic ball on (Z_H, g_{Z_H}). Since $\rho \asymp \rho(p)$ on $B(p,r)$, notice that the weighted volume of $B(p,r)$ is comparable to that of the product of balls

$$B_1(p_1, r) \times B_2(p_2, \frac{r}{\rho(p)^{\nu_H}}).$$

Now, we compute that

$$\begin{aligned} \mu_a(B_1(p_1,r) \times B_2(p_2, \frac{r}{\rho(p)^{\nu_H}})) &= \int_{B_1(p,r)} \left(\int_{B_2(p_2, \frac{r}{\rho(p)^{\nu_H}})} x^{\widetilde{a}} d\kappa_H \right) \rho^{(\nu_H-1)a_H+\nu_H m_H} dg_{C_H} \\ &\asymp \rho(p)^{(\nu_H-1)a_H+\nu_H m_H} r^{b_H+1} \mathcal{V}(p_2, \frac{r}{\rho(p)^{\nu_H}}; \widetilde{a}) \end{aligned}$$

as claimed, where $g_{C_H} = d\rho^2 + \rho^2 g_{S_H}$ is the natural cone metric on the cone $C_H = \mathbb{R}^+ \times S_H$.

□

On the other hand, for the volume of non-remote balls, we have the following estimate.

Proposition 3.10. *Suppose that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H + m_H)$ for all $H \in \mathcal{M}_{nm}(M)$. If $c \in (0, \frac{1}{5})$ is a remote parameter, then for $p \in M \setminus \partial M$,*

$$\mathcal{V}(p,r,a) \asymp r^{m+(\nu-1)a_{\max}} \tag{3.16}$$

for $r \geq c\rho(p)$.

Proof. We can proceed by induction on the depth of M using the previous proposition. If $C > 2$ is a fixed constant, then for $r > C\rho(p)$,

$$\{q \mid \rho(q) \le (1 - C^{-1})r\} \subset B(p, r) \subset \{q \mid \rho(q) \le (C^{-1} + 1)r\},$$

so for such r,

$$\mathcal{V}(p, r; a) \asymp \mathcal{A}(r; a) \asymp r^{m+(\nu-1)a_{\max}}$$

by Corollary 3.8. If instead $c\rho(p) \le r \le C\rho(p)$, then

$$\mathcal{R}(p, c\rho(p); a) \le \mathcal{V}(p, r; a) \le \mathcal{A}((1 + C)\rho(p); a).$$

By Corollary 3.8, the right hand side behaves like $r^{m+(\nu-1)a_{\max}}$. For the left hand side, notice that since $r \asymp \rho(p)$, it behaves like $r^{m+(\nu-1)a_{\max}}$ by Proposition 3.9. Indeed, if $x_G(p) \ge 1 - 3c$ for all $G \in \mathcal{M}_{\mathrm{nm}}(M)$, this follows from (3.12). If instead $x_H(p) < 1-3c$ for some $H \in \mathcal{M}_{\mathrm{nm}}(M)$ and $x_G(p) \ge 1-3c$ for all $G < H$, then in the notation of the proof of Proposition 3.9,

$$\rho(p) \ge \rho_{Z_H}(p) \quad \Longrightarrow \quad \frac{r}{\rho(p)^{\nu_H}} \ge c\rho(p)^{1-\nu_H} = \frac{c\rho_{Z_H}(p)}{x_H} \ge c\rho_{Z_H}(p_{Z_H}),$$

so we can apply (3.16) to Z_H. By (3.13), we know that

$$\mathcal{R}(p, c\rho(p); a) \asymp \rho(p)^{(\nu_H-1)a_H+\nu_H m_H} r^{b_H+1} \mathcal{V}(p_{Z_H}, \frac{r}{\rho(p)^{\nu_H}}; \widetilde{a}),$$

so if $\nu_H = \nu$, this gives

$$\begin{aligned} \mathcal{R}(p, c\rho(p); a) &\asymp \rho(p)^{(\nu-1)a_H+\nu m_H} r^{b_H+1} \left(\frac{r}{\rho(p)^{\nu}}\right)^{m_H-(a_{\max}-a_H)} \quad && \text{by (3.16) on } Z_H \text{ with } \nu = 0, \\ &\asymp r^{m+(\nu-1)a_{\max}}, \quad \text{since } r \asymp \rho(p), \end{aligned}$$

while if $\nu_H = 0$, we obtain that

$$\begin{aligned} \mathcal{R}(p, c\rho(p); a) &\asymp \rho(p)^{-a_H} r^{b_H+1} (r)^{m_H-(1-\nu)(a_{\max}-\frac{a_H}{1-\nu})} \quad && \text{by (3.16) on } Z_H, \\ &\asymp r^{m+(\nu-1)a_{\max}}, \quad \text{since } r \asymp \rho(p). \end{aligned}$$

□

The last two propositions can be used to obtain a sharper estimate of the volume of remote balls, a technical result needed later. We need however to take into account how close the point p is to a corner of M. In [20], this is achieved with the notion of remote chains, but we will proceed differently, taking advantage of the fact that we have a compactification by a manifold with fibered corners.

Definition 3.11. *Fix a remote parameter $c \in (0, \frac{1}{3})$ and boundary hypersurfaces $H_1, \ldots, H_k \in \mathcal{M}_{\mathrm{nm}}(M)$ with $H_1 < \cdots < H_k$. For such a choice, we say that a point $p \in M \setminus \partial M$ is* ***close to the non-maximal corner*** *$H_1 \cap \cdots \cap H_k$ if $x_{H_i}(p) \le 1 - 3c$ for all $i \in \{1, \ldots, k\}$, but $x_H(p) > 1 - 3c$ for $H \in \mathcal{M}_{\mathrm{nm}}(M) \setminus \{H_1, \ldots, H_k\}$. On the other hand, if $x_H(p) > 1 - 3c$ for $H \in \mathcal{M}_{\mathrm{nm}}(M)$, we say that p is* ***far from all non-maximal corners****. Close to a non-maximal corner $H_1 \cap \cdots H_k$, we will consider the function*

$$\sigma := x_{\max}^{-\frac{1}{1-\nu}} \prod_{\nu_{H_i}=\nu} x_i^{-\frac{1}{1-\nu}} \tag{3.17}$$

corresponding to (3.8).

In terms of this notion, we have the following finer estimate on the volume of remote balls.

Proposition 3.12. *Suppose that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H+m_H)$ for all $H \in \mathcal{M}_{nm}(M)$. If $c \in (0, \frac{1}{5})$ is a remote parameter and $p \in M \setminus \partial M$ is close to the non-maximal corner $H_1 \cap \cdots H_k$ and*

$$\frac{c\sigma^{\nu_{i+1}}(p)}{\widetilde{v}_{i+1}(p)} \le r \le \frac{c\sigma^{\nu_i}(p)}{\widetilde{v}_i(p)}$$

for some $i \in \{1, \ldots, k+1\}$*, where* $\widetilde{v}_i = \widetilde{v}_{H_i}$ *in terms of* (3.7) *and* $\nu_i = \nu_{H_i}$ *with the conventions that* $\widetilde{v}_{k+1} = x_{\max}$*,* $\nu_{k+1} = \nu$ *and* $\frac{c\sigma^{\nu_{k+1}}(p)}{v_{k+1}(p)} := 0$*. Then*

$$\mathcal{R}(p,r;a) \asymp \left(\prod_{j=1}^{i-1} x_j^{a_j}(p)\right) \widetilde{v}_i^{a_i}(p)\sigma^{\nu_i(a_{\max}-a_i)}(p) r^{m-(a_{\max}-a_i)+(\nu-\nu_i)a_{\max}} \tag{3.18}$$

with the conventions that $a_{k+1} = a_{\max}$ *and* $a_0 = 0$*. With these conventions,* (3.18) *also holds with* $k = 0$ *and* $i = 1$ *when* p *is far from all non-maximal corners and* $r \leq \frac{c\sigma(p)^{\nu_1}}{\widetilde{v}_1(p)} = cx_{\max}^{-\frac{1}{1-\nu}}$*.*

Proof. We can proceed by induction on the depth of M. If M is of depth 1, we are in the AC setting, so $k = 0$ and $i = 1$. For a remote ball, $r \leq c\rho(p) = c\rho_1(p) = c\rho_{\max}(p)$, so that (3.18) follows from (3.12). If M is of higher depth and $k \geq 1$, we can then assume that (3.18) holds for QAC-manifolds of lower depth. We need to distinguish two cases. If

$$\frac{c\sigma^{\nu_2}(p)}{\widetilde{v}_2(p)} \leq r \leq \frac{c\sigma^{\nu_1}(p)}{\widetilde{v}_1(p)},$$

then we can apply Proposition 3.9 together with Proposition 3.10 to obtain the claimed estimate, namely, with the notation that $m_i = m_{H_i}$, if $\nu_1 = \nu$, then $\sigma \asymp \rho$ and

$$\begin{aligned}
\mathcal{R}(p,r;a) &\asymp \rho(p)^{(\nu-1)a_1+\nu m_1} r^{m-m_1} \mathcal{V}(p_{Z_{H_1}}, \frac{r}{\rho(p)^{\nu}}; \widetilde{a}) \quad \text{by Proposition 3.9,} \\
&\asymp \rho(p)^{a_1(\nu-1)+\nu m_1} r^{m-m_1} \left(\frac{r}{\rho^{\nu}}\right)^{m_1-(a_{\max}-a_1)} \quad \text{by Proposition 3.10 with } \nu = 0, \\
&\asymp \rho(p)^{\nu a_{\max}-a_1} r^{m-(a_{\max}-a_1)} = \rho(p)^{(\nu-1)a_1} \rho(p)^{\nu(a_{\max}-a_1)} r^{m-(a_{\max}-a_1)} \\
&\asymp \widetilde{v}_1^{a_1} \rho(p)^{\nu(a_{\max}-a_1)} r^{m-(a_{\max}-a_1)} \asymp \widetilde{v}_1^{a_1} \rho(p)^{\nu_1(a_{\max}-a_1)} r^{m-(a_{\max}-a_1)+(\nu-\nu_1)a_{\max}},
\end{aligned}$$

while if $\nu_1 = 0$, then

$$\begin{aligned}
\mathcal{R}(p,r;a) &\asymp \rho(p)^{-a_1} r^{m-m_1} \mathcal{V}(p_{Z_{H_1}}, r; \widetilde{a}) \quad \text{by Proposition 3.9,} \\
&\asymp \rho(p)^{-a_1} r^{m-m_1} r^{m_1+(\nu-1)(a_{\max}-\frac{a_1}{1-\nu})} \quad \text{by Proposition 3.10,} \\
&\asymp \rho(p)^{-a_1} r^{m+(\nu-1)a_{\max}+a_1} \\
&\asymp \widetilde{v}(p)^{a_1} r^{m-(a_{\max}-a_1)+(\nu-\nu_1)a_{\max}},
\end{aligned}$$

yielding the claimed result.

If instead $\frac{c\sigma^{\nu_{i+1}}(p)}{\widetilde{v}_{i+1}(p)} \leq r \leq \frac{c\sigma^{\nu_i}(p)}{\widetilde{v}_i(p)}$ with $i \geq 2$, we can combine Proposition 3.9 with (3.18) on Z_{H_1} to obtain the estimate. First, for $\nu_1 = \nu$, $\sigma \asymp \rho$ and

$$\begin{aligned}
\mathcal{R}(p,r;a) &\asymp \rho(p)^{(\nu-1)a_1+\nu m_1} r^{m-m_1} \mathcal{V}(p_{Z_{H_1}}, \frac{r}{\rho(p)^{\nu}}; \widetilde{a}) \quad \text{by Proposition 3.9,} \\
&\asymp \rho(p)^{(\nu-1)a_1+\nu m_1} r^{m-m_1} \left(\prod_{j=2}^{i-1} x_j^{a_j-a_1}\right) \widetilde{v}_i^{a_i-a_1} \left(\frac{r}{\rho(p)^{\nu}}\right)^{m_1-(a_{\max}-a_i)} \quad \text{by (3.18) with } \nu = 0, \\
&\asymp \rho(p)^{(\nu-1)a_1+\nu(a_{\max}-a_i)} r^{m-(a_{\max}-a_i)} \left(\prod_{j=2}^{i-1} x_j^{a_j-a_1}\right) \widetilde{v}_i^{a_i-a_1} \\
&\asymp \widetilde{v}_1^{a_1} \rho(p)^{\nu(a_{\max}-a_i)} r^{m-(a_{\max}-a_i)} \left(\prod_{j=2}^{i-1} x_j^{a_j-a_1}\right) \widetilde{v}_i^{a_i-a_1} \\
&\asymp \left(\left(\prod_{j=1}^{i-1} x_j\right) \widetilde{v}_i\right)^{a_1} \rho(p)^{\nu(a_{\max}-a_i)} r^{m-(a_{\max}-a_i)} \left(\prod_{j=2}^{i-1} x_j^{a_j-a_1}\right) \widetilde{v}_i^{a_i-a_1} \\
&\asymp \left(\prod_{j=1}^{i-1} x_j^{a_j}\right) \widetilde{v}_i^{a_i} \rho(p)^{\nu_i(a_{\max}-a_i)} r^{m-(a_{\max}-a_i)+(\nu-\nu_1)a_{\max}}.
\end{aligned}$$

If instead $\nu_1 = 0$, then

$$\begin{aligned}
\mathcal{R}(p,r;a) &\asymp \rho(p)^{-a_1} r^{m-m_1} \mathcal{V}(p,r;\widetilde{a}), \quad \text{by Proposition 3.9,} \\
&\asymp \rho(p)^{-a_1} r^{m-m_1} \left(\prod_{j=2}^{i-1} x_j^{\widetilde{a}_j} \right) \widetilde{v}_i^{\widetilde{a}_i} \sigma^{\nu_i(\widetilde{a}_{\max}-\widetilde{a}_i)} r^{m_1-(\widetilde{a}_{\max}-\widetilde{a}_i)+(\nu-\nu_i)\widetilde{a}_{\max}}, \quad \text{by (3.18),}
\end{aligned}$$

and we need to distinguish two cases. If $\nu_i = \nu$, then in fact

$$\begin{aligned}
\mathcal{R}(p,r;a) &\asymp \rho(p)^{-a_1} \left(\prod_{j=2}^{i-1} x_j^{a_j - \frac{a_1}{1-\nu_j}} \right) \widetilde{v}_i^{a_i - \frac{a_1}{1-\nu}} \sigma^{\nu(a_{\max}-a_i)} r^{m-(a_{\max}-a_i)} \\
&\asymp \left(\prod_{j=2}^{i-1} x_j^{a_j - \frac{a_1}{1-\nu_j}} \right) \left(\prod_{j=1}^{i-1} x_j^{\frac{a_1}{1-\nu_j}} \right) \widetilde{v}_i^{a_i} \sigma(p)^{\nu(a_{\max}-a_i)} r^{m-(a_{\max}-a_i)} \\
&\asymp \left(\prod_{j=1}^{i-1} x_j^{a_j} \right) \widetilde{v}_i^{a_i} \sigma^{\nu_i(a_{\max}-a_i)} r^{m-(a_{\max}-a_i)},
\end{aligned}$$

yielding the claimed result. If instead $\nu_i = 0$, then

$$\begin{aligned}
\mathcal{R}(p,r;a) &\asymp \rho(p)^{-a_1} \left(\prod_{j=2}^{i-1} x_j^{a_j - a_1} \right) \widetilde{v}_i^{a_i - a_1} r^{m-(a_{\max}-\frac{a_1}{1-\nu}-a_i+a_1)+\nu(a_{\max}-\frac{a_1}{1-\nu})} \\
&\asymp \left(\prod_{j=1}^{i-1} x_j^{a_1} \right) \left(\prod_{j=2}^{i-1} x_j^{a_j - a_1} \right) \widetilde{v}_i^{a_i} r^{m-(a_{\max}-a_i)+\nu a_{\max}}, \\
&\asymp \left(\prod_{j=1}^{i-1} x_j^{a_j} \right) \widetilde{v}_i^{a_i} r^{m-(a_{\max}-a_i)+\nu a_{\max}},
\end{aligned}$$

again yielding the claimed result. □

For the moment however, just using Propositions 3.9 and 3.10, we can deduce the volume doubling property.

Corollary 3.13. *Suppose that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H + m_H)$ for all $H \in \mathcal{M}_{nm}(M)$. Then $(VD)_\mu$ holds.*

Proof. For non remote balls, this follows from Proposition 3.10. For remote balls, we can assume by induction that the result holds for QAC-manifolds of lower depth. By Proposition 3.9, $(VD)_\mu$ holds when we have (3.12), while when (3.13) holds,

$$\mathcal{R}(p,r;a) \asymp \rho(p)^{(\nu_H - 1)a_H + \nu_H m_H} r^{b_H + 1} \mathcal{V}(p_{Z_H}, \frac{r}{\rho(p)^{\nu_H}}; \widetilde{a})$$

and the result follows by $(VD)_\mu$ on $(Z_H, \kappa_H, \mu_{\widetilde{a}})$. □

We can also estimate the volume of anchored balls in terms of the volume of remote balls as follows.

Corollary 3.14. *Suppose that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H + m_H)$ for all $H \in \mathcal{M}_{nm}(M)$. For $c \in (0, \frac{1}{5})$ a choice of remote parameter, there exists a constant C_V such that*

$$\mathcal{A}(\rho(p); a) \leq CV\mathcal{R}(p, c\rho(p); a) \quad \forall\, p \in M \setminus \partial M.$$

Proof. This follows from Corollary 3.8 with $R = \rho(p)$ and Proposition 3.10 with $r = c\rho(p)$. □

These estimates will help obtain the weighted Poincaré inequality via the following result.

Corollary 3.15. *Suppose that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H+m_H)$ for all $H \in \mathcal{M}_{nm}(M)$. Suppose moreover that ∂M is connected. If the complete weighted Riemannian manifold $(M\setminus \partial M, g, \mu_a)$ satisfies $(VD)_\mu$ and $(PI)_{\mu,\delta}$ with parameter $\delta \in (0,1]$ for all remote balls, then $(VD)_\mu$ and $(PI)_{\mu,\delta}$ hold for all balls.*

Proof. By the previous corollaries and [24, Theorem 5.2], it suffices to check that $(M\setminus\partial M, g)$ satisfies the property of relatively connected annuli (RCA) with respect to the base point o, that is, there exists $C_A > 1$ such that for all $r > C_A^2$ and for all $p, q \in M \setminus \partial M$ with $d(o,p) = d(o,q) = r$, there exists a continuous path $\gamma : [0,1] \to M\setminus\partial M$ starting at p and ending at q with image contained in the shell $B(o, C_A r)\setminus B(o, C_A^{-1} r)$. But since ∂M is assumed to be connected, the RCA property clearly holds. □

Thus, assuming ∂M is connected, it remains to check that $(PI)_{\mu,\delta}$ holds on remote balls to conclude it holds on every ball. To be able to run an argument by induction on the depth of M, we need to assume as well that the boundary ∂Z_H of the fibers of $\phi_H : H \to S_H$ is connected for all $H \in \mathcal{M}_{\mathrm{nm}}(M)$.

Theorem 3.16. *Suppose that ∂M is connected as well as ∂Z_H for each $H \in \mathcal{M}_{\mathrm{nm}}(M)$. Suppose also that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H + m_H)$ for all $H \in \mathcal{M}_{nm}(M)$. Then the properties $(VD)_\mu$ and $(PI)_\mu$ hold on $(M\setminus\partial M, g, \mu_a)$.*

Proof. We will closely follow the proof of [18, Theorem 5.15]. By the argument of Jerison [26], if $(VD)_\mu$ and $(PI)_{\mu,\delta}$ hold for all balls for some $\delta \in (0,1]$, then $(PI)_\mu$ also holds for all balls. Hence, by the discussion above, it suffices to check that $(PI)_{\mu,\delta}$ holds for all remote balls. Let $c \in (0, \frac{1}{6})$ be our remote parameter and let $B(p,r)$ be a remote ball. If $x_G(p) \geq 1-4c$ for all $G \in \mathcal{M}_{\mathrm{nm}}(M)$, we are in a region where the metric behaves like an AC-metric, so we can apply the rescaling argument of [20, Proposition 4.13] to conclude that $(PI)_\mu$ holds on $B(p,r)$. In particular, proceeding by induction on the depth of M, we can now assume that the statement of the theorem holds for warped QAC-metrics on manifolds with corners of lower depth.

If instead $x_H(p) < 1-4c$ for some $H \in \mathcal{M}_{\mathrm{nm}}(M)$, but $x_G(p) \geq 1-4c$ for $G < H$, then as discussed in the proof of Proposition 3.9, we can assume that $B(p,r)$ is included in a region where g is of the form (2.11). Regarding $B(p,r)$ as a subset of $C_H \times Z_H$ with p corresponding to the point $(p_1,p_2) \in C_H \times Z_H$ and $C_H = \mathbb{R}^+ \times S_H$, notice that it is contained in the product of balls

$$Q(r) = B_1 \times B_2 := B_1(p_1, r) \times B_2\Big(p_2, \frac{r}{((1-c)\rho(p))^{\nu_H}}\Big), \tag{3.19}$$

where $B_2(p_2, r)$ is a geodesic ball in (Z_H, κ_H). Let us first prove the uniform weighted Poincaré inequality on $Q(r)$ by writing $d\mu_a = d\mu_1 d\mu_2$ with

$$d\mu_1 = \rho^{(\nu_H - 1)a_H + \nu m_H} dg_{C_H} \quad \text{and} \quad d\mu_2 = x^{\widetilde{a}} d\kappa_H.$$

Given a function f on $Q(r)$, we define the partial averages by

$$\overline{f}_i := \frac{1}{\mu_i(B_i)} \int_{B_i} f d\mu_i, \quad i \in \{1,2\}, \quad \overline{f}_Q := \frac{1}{\mu_a(Q(r))} \int_{Q(r)} f d\mu_a,$$

so that in particular $\overline{f}_Q = \overline{(\overline{f}_1)}_2 = \overline{(\overline{f}_2)}_1$. Since Z_H is of lower depth, we see by induction that $(PI)_{\mu_2}$ already holds on $(Z_H \setminus \partial Z_H, \kappa_H, \mu_2)$, so there is a constant $C_2 > 0$ such that

$$\begin{aligned}
\int_{Q(r)} |f - \overline{f}_Q|^2 d\mu_1 d\mu_2 &= 2\int_{Q(r)} (|f - \overline{f}_2|^2 + |\overline{f}_2 - \overline{f}_Q|^2) d\mu_1 d\mu_2 \\
&\leq 2\int_{B_1} \left(C_2 \left(\frac{r}{((1-c)\rho(p))^{\nu_H}} \right)^2 \int_{B_2} |d_2 f|^2_{\kappa_H} d\mu_2 + \int_{B_2} |\overline{f}_2 - \overline{f}_Q|^2 d\mu_2 \right) d\mu_1 \\
&= 2\int_{B_1} \left(C_2 r^2 \int_{B_2} |d_2 f|^2_{((1-c)\rho(p))^{\nu_H}\kappa_H} d\mu_2 + \int_{B_2} |\overline{f}_2 - \overline{f}_Q|^2 d\mu_2 \right) d\mu_1,
\end{aligned} \tag{3.20}$$

where d_i is the exterior differential taken on the factor B_i. Since $(1-c)\rho(p) \leq \rho(q) \leq (1+c)\rho(p)$ for $q \in Q(r)$, the first term on the right of (3.20) is bounded by

$$2C_2 \left(\frac{1+c}{1-c} \right)^{2\nu_H} r^2 \int_{Q(r)} |df|^2_g d\mu_a.$$

For the second term, since (C_H, g_{C_H}) is a cone, we can apply $(PI)_{\mu_1}$ on B_1, so that there is a constant $C_1 > 0$ such that

$$\int_{Q(r)} |\overline{f}_2 - \overline{f}_Q|^2 d\mu_1 d\mu_2 \le \int_{B_2} \left(C_1 r^2 \int_{B_1} |d_1 \overline{f}_2|^2_{g_1} d\mu_1 \right) d\mu_2, \tag{3.21}$$

where $g_1 = g_{C_H}$. On the other hand, using the fact that

$$d_1 \overline{f}_2 = \frac{1}{\mu_2(B_2)} \int_{B_2} (d_1 f) d\mu_2,$$

we deduce from the Cauchy-Schwarz inequality that

$$|d_1 \overline{f}_2|_{g_1} \le \frac{1}{\mu_2(B_2)} \int_{B_2} |d_1 f|^2_{g_1} d\mu_2. \tag{3.22}$$

Inserting (3.22) in (3.21), we thus obtain that

$$\begin{aligned} \int_{Q(r)} |\overline{f}_2 - \overline{f}_Q|^2 d\mu_1 d\mu_2 &\le \int_{B_2} \left(C_1 r^2 \int_{B_1} \left(\frac{1}{\mu_2(B_2)} \int_{B_2} |d_1 f|^2_{g_1} d\mu_2 \right) d\mu_1 \right) d\mu_2 \\ &\le C_1 r^2 \int_{Q(r)} |d_1 f|^2_{g_1} d\mu_a \le C_1 r^2 \int_{Q(r)} |df|^2_g d\mu_a, \end{aligned} \tag{3.23}$$

showing that $(PI)_{\mu_a}$ holds on $Q(r)$. For $B(p,r)$ now, notice that we have the sequence of inclusions

$$B(p,r) \subset Q(r) \subset B\left(p, r + \left(\frac{1+c}{1-c}\right)^{\nu_H} r\right) \subset B(p, 3r) \tag{3.24}$$

for $c \in (0, \frac{1}{6})$ sufficiently small, in which case

$$\begin{aligned} \int_{B(p,r)} |f - \overline{f}_{B(p,r)}|^2 d\mu_a = \inf_c \int_{B(p,r)} |f - c|^2 d\mu_a &\le \int_{B(p,r)} |f - \overline{f}_Q|^2 d\mu_a \le \int_{Q(r)} |f - \overline{f}_Q|^2 d\mu_a \\ &\le C r^2 \int_{Q(r)} |df|^2_g d\mu, \quad \text{by } (PI)_{\mu_a} \text{ on } Q(r), \\ &\le C r^2 \int_{B(p,\delta^{-1} r)} |df|^2_g d\mu_a, \quad \text{with } \delta = \frac{1}{3}. \end{aligned} \tag{3.25}$$

□

By the results of [24], this gives the following bound on the heat kernel.

Corollary 3.17. *Let g be a warped* QAC*-metric on the interior of a manifold with fibered corners M. Suppose that ∂M is connected as well as ∂Z_H for each $H \in \mathcal{M}_{\mathrm{nm}}(M)$. Suppose also that $a_H = a_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H + m_H)$ for all $H \in \mathcal{M}_{nm}(M)$. Then the heat kernel $H_{\mathcal{L}}$ of $\mathcal{L} = -\Delta + \mathcal{R}$ with $\mathcal{R} \ge V := \frac{\Delta(x^{\frac{a}{2}})}{x^{\frac{a}{2}}}$ satisfies the estimate*

$$|H_{\mathcal{L}}(t,z,z')| \le H_{-\Delta+V}(t,z,z') \preceq \frac{x^{\frac{a}{2}}(z) x^{\frac{a}{2}}(z') e^{-\frac{c d(z,z')^2}{t}}}{\left(\mu_a(B(z,\sqrt{t}))\mu_a(B(z',\sqrt{t}))\right)^{\frac{1}{2}}}$$

for a positive constant c, where $f \preceq g$ if there exists a positive constant C such that $f \le Cg$.

This implies in particular the Sobolev inequality for warped QAC-metrics.

Corollary 3.18 (Sobolev inequality)**.** *If g is a warped* QAC*-metric as in Corollary 3.17, then there is a constant $C_S > 0$ such that*

$$\left(\int_{M \setminus \partial M} |u|^{\frac{m}{\frac{m}{2}-1}} dg \right)^{1-\frac{2}{m}} \le C_S \int |df|^2_g dg, \quad \forall u \in \mathcal{C}^\infty_c(M \setminus \partial M). \tag{3.26}$$

Proof. When $a = 0$ and $V = 0$, we can take $\mathcal{R} = 0$ in Corollary 3.17, yielding a Gaussian bound for $H_{-\Delta}$, which is well-known to be equivalent to the Sobolev inequality (3.26); see for instance [23]. □

Through Lemma 3.1, (3.4) and [20, Theorem 3.22], Corollary 3.17 yields a corresponding estimate for the Green's function, namely

$$\begin{aligned}|G_{\mathcal{L}}(z,z')| &\le G_{-\Delta+V}(z,z') \le G_{-\Delta+V}(z,z') \\ &\asymp x(z)^{\frac{a}{2}}x(z')^{\frac{a}{2}}\int_{d(z,z')}^{\infty}\frac{sds}{\sqrt{\mathcal{V}(z,s;a)\mathcal{V}(z',s;a)}}.\end{aligned} \tag{3.27}$$

This is the estimate, together with our estimates on the volume of balls, that will allow us to determine spaces on which the operator $\mathcal{L} = -\Delta + \mathcal{R}$ can be inverted. In fact, what we really need is to estimate the integral

$$\int_{d(z,z')}^{\infty}\frac{sds}{\sqrt{\mathcal{V}(z,s;a)\mathcal{V}(z',s;a)}}.$$

Since $s \ge d(z,z')$, we can use the volume doubling property as in [20, § 5.2] to conclude that this integral is equivalent to the slightly simpler integral

$$\mathbb{I}(z,z') = \int_{d(z,z')}^{\infty}\frac{sds}{\mathcal{V}(z,s;a)}.$$

Using Propositions 3.10 and 3.12, this simpler integral can be estimated as follows.

Lemma 3.19. *Let M, g and a be as in Corollary 3.17 and let $c \in (0,\frac{1}{5})$ be a choice of remote parameter. Suppose also that $(1-\nu)a_{\max} < m-2$ and $m > 2$. Let $z, z' \in M \setminus \partial M$ be given. If $d(z,z') > c\rho(z)$,then*

$$\mathbb{I}(z,z') \asymp d(z,z')^{2-m-(\nu-1)a_{\max}}.$$

If instead $d(z,z') < c\rho(z)$ and z is far from all non-maximal corners, then

$$\mathbb{I}(z,z') \asymp \rho(z)^{-(\nu-1)a_{\max}}d(z,z')^{2-m}.$$

Finally, if $d(z,z') < c\rho(z)$ and z is close to the non-maximal corner $H_1\cap\ldots\cap H_k$ with $\frac{c\sigma^{\nu_{i+1}(z)}}{v_{i+1}(z)} \le d(z,z') \le \frac{\sigma^{\nu_i(z)}}{v_i(z)}$ for some $i \in \{1,\ldots,k+1\}$ (using the conventions of Proposition 3.12), then

$$\mathbb{I}(z,z') \asymp \left(\prod_{j=1}^{i-1}x_j(z)^{-a_j}\right)\widetilde{v}_i(z)^{-a_i}\sigma(z)^{\nu_i(a_i-a_{\max})}d(z,z')^{2-m+a_{\max}-a_i-(\nu-\nu_i)a_{\max}}.$$

In terms of (3.27), this gives the following estimate.

Proposition 3.20. *Let M, g and a be as in Lemma 3.19 and let $c \in (0,\frac{1}{5})$ be a choice of remote parameter. Let $z, z' \in M \setminus \partial M$ be given. If $d(z,z') > c\rho(z)$,then*

$$G_{-\Delta+V}(z,z') \asymp x(z)^{\frac{a}{2}}x(z')^{\frac{a}{2}}d(z,z')^{2-m-(\nu-1)a_{\max}}. \tag{3.28}$$

If instead $d(z,z') < c\rho(z)$ and z is far from all non-maximal corners, then

$$G_{-\Delta+V}(z,z') \asymp x(z)^{\frac{a}{2}}x(z')^{\frac{a}{2}}\rho(z)^{-(\nu-1)a_{\max}}d(z,z')^{2-m}. \tag{3.29}$$

Finally, if $d(z,z') < c\rho(z)$ and z is close to the non-maximal corner $H_1\cap\ldots\cap H_k$ with $\frac{c\sigma^{\nu_{i+1}(z)}}{v_{i+1}(z)} \le d(z,z') \le \frac{c\sigma^{\nu_i(z)}}{v_i(z)}$ for some $i \in \{1,\ldots,k+1\}$ (using the conventions of Proposition 3.12), then

$$G_{-\Delta+V}(z,z') \asymp x(z)^{\frac{a}{2}}x(z')^{\frac{a}{2}}\left(\prod_{j=1}^{i-1}x_j(z)^{-a_j}\right)\widetilde{v}_i(z)^{-a_i}\sigma(z)^{\nu_i(a_i-a_{\max})}d(z,z')^{2-m+a_{\max}-a_i-(\nu-\nu_i)a_{\max}}. \tag{3.30}$$

As in [20], we can extract mapping properties from these estimates using the Schur test. This amounts to obtain the following estimates.

Proposition 3.21. *Let M, g and a be as in Lemma 3.19. Let b be a multiweight with $b_H = b_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$. Suppose that*

$$\frac{2}{1-\nu}+\frac{a_{\max}}{2} < b_{\max} < \frac{m}{1-\nu}-\frac{a_{\max}}{2}$$

and that

$$\left(\frac{1-\nu}{1-\nu_H}\right)\left(\frac{a_{\max}}{2}+b_{\max}\right)-\frac{a_H}{2}-m_H<b_H<\frac{a_H}{2}+\left(\frac{1-\nu}{1-\nu_H}\right)\left(b_{\max}-\frac{a_{\max}}{2}\right)-2$$

for all $H\in\mathcal{M}_{\rm nm}(M)$. *Then*

$$\int_{M\setminus\partial M}G_{-\Delta+V}(z,z')x(z')^b dg(z')\preceq x(z)^b(\rho(z)w(z))^2, \tag{3.31}$$

where $w=\prod_{H\in\mathcal{M}_{\rm nm}(M)}x_H$.

Proof. We follow the strategy of the proof of [20, Lemma 6.5]. Thus, fix a remote parameter $c\in(0,\frac{1}{5})$ and decompose the region of integration into three subdomains: $\mathcal{D}_1:=(M\setminus\partial M)\setminus B(o,2\rho(z))$, $\mathcal{D}_2:=B(o,2\rho(z))\setminus B(z,c\rho(z))$ and $\mathcal{D}_3:=B(z,c\rho(z))$, the latter being a maximal remote ball. On the first region, $d(z,z')\asymp\rho(z')$, so by (3.28),

$$\begin{aligned}\int_{\mathcal{D}_1}G_{-\Delta+V}(z,z')x(z')^b dg(z')&\preceq x(z)^{\frac{a}{2}}\int_{\mathcal{D}_1}x(z')^{\frac{a}{2}+b}d(z,z')^{2-m-(\nu-1)a_{\max}}dg(z')\\&\preceq x(z)^{\frac{a}{2}}\int_{\mathcal{D}_1}x(z')^{\frac{a}{2}+b}\rho(z')^{2-m-(\nu-1)a_{\max}}dg(z')\\&\preceq x(z)^{\frac{a}{2}}\int_{2\rho(z)}^{\infty}d\mathcal{A}\left(s;\frac{a}{2}+b+\frac{m-2-(1-\nu)a_{\max}}{1-\mathfrak{n}}\right)\\&\preceq x(z)^{\frac{a}{2}}\int_{2\rho(z)}^{\infty}s^{m-1}\left(\sum_{H\in\mathcal{M}_1(M)}s^{(\nu_H-1)(\frac{a_H}{2}+b_H+\frac{m-2-(1-\nu)a_{\max}}{1-\nu_H}+m_H)}\right)ds,\end{aligned} \tag{3.32}$$

where in the last line we have used Proposition 3.6. By our assumption on b, we see that

$$m-1+(\nu-1)(b_{\max}+\frac{m-2}{1-\nu}-\frac{a_{\max}}{2})<-1$$

and

$$m-1+(\nu_H-1)\left(\frac{a_H}{2}+b_H+\frac{m-2-(1-\nu)a_{\max}}{1-\nu_H}+m_H\right)<m-1+(\nu-1)(b_{\max}+\frac{m-2}{1-\nu}-\frac{a_{\max}}{2})<-1,$$

so the integral at the very end of (3.32) does not diverge and

$$\begin{aligned}\int_{\mathcal{D}_1}G_{-\Delta+V}(z,z')x(z')^b dg(z')&\preceq x(z)^{\frac{a}{2}}\rho(z)^{m+(\nu-1)(b_{\max}-\frac{a_{\max}}{2}+\frac{m-2}{1-\nu})}\\&\preceq x(z)^{\frac{a}{2}}\rho(z)^{2+(\nu-1)(b_{\max}-\frac{a_{\max}}{2})}\\&\preceq x(z)^{\frac{\widehat{a}}{2}}\rho(z)^{2+(\nu-1)b_{\max}},\quad\text{where }\widehat{a}_H=a_H-\left(\frac{1-\nu}{1-\nu_H}\right)a_{\max}.\end{aligned} \tag{3.33}$$

But if $\widehat{b}$ is the multiweight such that $\widehat{b}_H=b_H-\left(\frac{1-\nu}{1-\nu_H}\right)b_{\max}$, we see from our assumption on b that

$$\widehat{b}_H+2<\frac{\widehat{a}_H}{2}\quad\text{for}\quad H\in\mathcal{M}_{\rm nm}(M), \tag{3.34}$$

so that

$$\int_{\mathcal{D}_1}G_{-\Delta+V}(z,z')x(z')^b dg(z')\preceq x(z)^{\widehat{b}}w^2\rho(z)^{2+(\nu-1)b_{\max}}=x(z)^b(\rho(z)w(z))^2$$

as claimed. In the second region, $d(z,z')\asymp\rho(z)$, so again, by (3.28),

$$\begin{aligned}\int_{\mathcal{D}_2}G_{-\Delta+V}(z,z')x(z')^b dg(z')&\preceq x(z)^{\frac{a}{2}}\int_{\mathcal{D}_2}d(z,z')^{2-m-(\nu-1)a_{\max}}x(z')^{\frac{a}{2}+b}dg(z')\\&\preceq x^{\frac{a}{2}}(z)\rho(z)^{2-m-(\nu-1)a_{\max}}\int_{B(o,2\rho(z))}x(z')^{\frac{a}{2}+b}dg(z')\\&\preceq x(z)^{\frac{a}{2}}\rho(z)^{2-m-(\nu-1)a_{\max}}\mathcal{A}\left(\rho(z),\frac{a}{2}+b\right).\end{aligned}$$

Now, by our assumption on b, the multiweight $\frac{a}{2}+b$ satisfies the assumption of Corollary 3.8, so

$$\begin{aligned}\int_{\mathcal{D}_2} G_{-\Delta+V}(z,z')x(z')^b dg(z') &\preceq x(z)^{\frac{a}{2}}\rho(z)^{2-m-(\nu-1)a_{\max}}\rho(z)^{m+(\nu-1)(\frac{a_{\max}}{2}+b_{\max})} \\ &\preceq x(z)^{\frac{a}{2}}\rho(z)^{2+(\nu-1)(b_{\max}-\frac{a_{\max}}{2})} \\ &\preceq x(z)^{\frac{\widehat{a}}{2}}\rho(z)^{2+(\nu-1)b_{\max}} \\ &\preceq x(z)^{\widehat{b}}w(z)^2\rho(z)^{2+(\nu-1)b_{\max}} \quad \text{by (3.34)}, \\ &\preceq x(z)^b(\rho(z)w(z))^2.\end{aligned}$$

Finally, in the last region, suppose that z is close to the non-maximal corner $H_1\cap\ldots\cap H_k$. Then by (3.30), setting $B_i := B(z, \frac{c\sigma(z)^{\nu_i}}{v_i(z)})$, we see that

$$\int_{\mathcal{D}_3} G_{-\Delta+V}(z,z')dg(z') \asymp \sum_{i=1}^{k+1} I_i \tag{3.35}$$

with

$$I_{k+1} = \int_{B_{k+1}} x(z)^{\frac{a}{2}}x(z')^{\frac{a}{2}}\left(\prod_{j=1}^k x_j(z)^{-a_j}\right)x_{\max}(z)^{-a_{\max}}d(z,z')^{2-m}x(z')^b dg(z')$$

and

$$I_i = \int_{B_i\setminus B_{i+1}} x(z)^{\frac{a}{2}}x(z')^{\frac{a}{2}}\left(\prod_{j=1}^{i-1} x_j(z)^{-a_j}\right)\widetilde{v}_i(z)^{-a_i}\sigma(z)^{\nu_i(a_i-a_{\max})}d(z,z')^{2-m+a_{\max}-a_i-(\nu-\nu_i)a_{\max}}x(z')^b dg(z')$$

for $i\le k$. For I_{k+1}, we can use our assumption on b and (3.18) with $i=k+1$ to obtain

$$\begin{aligned}I_{k+1} &\preceq x(z)^{\frac{a}{2}}\left(\prod_{j=1}^k x_j(z)^{-a_j}\right)x_{\max}(z)^{-a_{\max}}\int_0^{\frac{c\sigma(z)^\nu}{x_{\max}(z)}} r^{2-m}d\mathcal{R}(z,r;\frac{a}{2}+b) \\ &\preceq x(z)^{\frac{a}{2}}\left(\prod_{j=1}^k x_j(z)^{-a_j}\right)x_{\max}(z)^{-a_{\max}}\int_0^{\frac{c\sigma(z)^\nu}{x_{\max}(z)}} r^{1-m}\left(\prod_{j=1}^k x_j(z)^{\frac{a_j}{2}+b_j}\right)x_{\max}(z)^{\frac{a_{\max}}{2}+b_{\max}}r^m dr \\ &\preceq x(z)^{\frac{a}{2}}\left(\prod_{j=1}^k x_j(z)^{-\frac{a_j}{2}+b_j}\right)x_{\max}(z)^{-\frac{a_{\max}}{2}+b_{\max}}\int_0^{\frac{c\sigma(z)^\nu}{x_{\max}(z)}} rdr \\ &\preceq x(z)^{\frac{a}{2}}\left(\prod_{j=1}^k x_j(z)^{-\frac{a_j}{2}+b_j}\right)x_{\max}(z)^{-\frac{a_{\max}}{2}+b_{\max}}\left(\frac{\sigma(z)^\nu}{x_{\max}(z)}\right)^2 \asymp x(z)^b\sigma(z)^2\left(\frac{\sigma(z)^{\nu-1}}{x_{\max}(z)}\right)^2 \\ &\preceq x(z)^b(\rho(z)w(z))^2.\end{aligned}$$

For $i \leq k$, again using our hypothesis on b, we see from Proposition 3.12 that

$$
\begin{aligned}
I_i &\preceq x^{\frac{a}{2}}(z)\left(\prod_{j=1}^{i-1} x_j(z)^{-a_j}\right)\widetilde{v}_i(z)^{-a_i}\sigma(z)^{\nu_i(a_i-a_{\max})}\int_{\frac{c\sigma(z)^{\nu_{i+1}}}{v_{i+1}}}^{\frac{c\sigma(z)^{\nu_i}}{v_i}} r^{2-m+a_{\max}-a_i-(\nu-\nu_i)a_{\max}}d\mathcal{R}(z,r,b+\frac{a}{2})\\
&\preceq x^{\frac{a}{2}}(z)\left(\prod_{j=1}^{i-1} x_j(z)^{-a_j}\right)\widetilde{v}_i(z)^{-a_i}\sigma(z)^{\nu_i(a_i-a_{\max})}\int_{\frac{c\sigma(z)^{\nu_{i+1}}}{v_{i+1}}}^{\frac{c\sigma(z)^{\nu_i}}{v_i}} r^{1-m+a_{\max}-a_i-(\nu-\nu_i)a_{\max}}\cdot\\
&\qquad\left(\left(\prod_{j=1}^{i-1} x_j(z)^{\frac{a_j}{2}+b_j}\right)\widetilde{v}_i(z)^{\frac{a_i}{2}+b_i}\sigma(z)^{\nu_i(\frac{a_{\max}}{2}+b_{\max}-\frac{a_i}{2}-b_i)}r^{m-(\frac{a_{\max}}{2}+b_{\max}-\frac{a_i}{2}-b_i)+(\nu-\nu_i)(\frac{a_{\max}}{2}+b_{\max})}\right)dr\\
&\preceq x^{\frac{a}{2}}(z)\left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j}\right)\widetilde{v}_i(z)^{-\frac{a_i}{2}+b_i}\sigma(z)^{\nu_i(\frac{a_i}{2}-\frac{a_{\max}}{2}+b_{\max}-b_i)}\cdot\\
&\qquad\int_{\frac{c\sigma(z)^{\nu_{i+1}}}{v_{i+1}}}^{\frac{c\sigma(z)^{\nu_i}}{v_i}} r^{1+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+(\nu-\nu_i)(b_{\max}-\frac{a_{\max}}{2})}dr\\
&\preceq x^{\frac{a}{2}}(z)\left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j}\right)\widetilde{v}_i(z)^{-\frac{a_i}{2}+b_i}\sigma(z)^{\nu_i(\frac{a_i}{2}-\frac{a_{\max}}{2}+b_{\max}-b_i)}\cdot\\
&\qquad\left|\left(\frac{\sigma(z)^{\nu_i}}{\widetilde{v}_i(z)}\right)^{2+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+(\nu-\nu_i)(b_{\max}-\frac{a_{\max}}{2})}-\left(\frac{\sigma(z)^{\nu_{i+1}}}{\widetilde{v}_{i+1}(z)}\right)^{2+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+(\nu-\nu_i)(b_{\max}-\frac{a_{\max}}{2})}\right|.
\end{aligned}
$$

Using the facts that $\frac{c\sigma^{\nu_{i+1}}(z)}{\widetilde{v}_{i+1}(z)} \leq \frac{c\sigma^{\nu_i}(z)}{\widetilde{v}_i(z)}$ and that $2+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+(\nu-\nu_i)(b_{\max}-\frac{a_{\max}}{2})<0$, this implies that

$$
\begin{aligned}
I_i \preceq x^{\frac{a}{2}}(z)\left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j}\right)\widetilde{v}_i(z)^{-\frac{a_i}{2}+b_i}\sigma(z)^{\nu_i(\frac{a_i}{2}-\frac{a_{\max}}{2}+b_{\max}-b_i)}&\\
\left(\frac{\sigma(z)^{\nu_{i+1}}}{\widetilde{v}_{i+1}(z)}\right)^{2+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+(\nu-\nu_i)(b_{\max}-\frac{a_{\max}}{2})}&.
\end{aligned}
\tag{3.36}
$$

If $\nu_i = \nu$, this gives

$$\begin{aligned}
I_i &\preceq x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j} \right) \widetilde{v}_i(z)^{-\frac{a_i}{2}+b_i} \sigma(z)^{2\nu} \widetilde{v}_{i+1}(z)^{-2-\frac{a_{\max}}{2}+\frac{a_i}{2}+b_{\max}-b_i} \\
&\preceq x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j} \right) x_i(z)^{-\frac{a_i}{2}+b_i} \sigma(z)^{2\nu} \widetilde{v}_{i+1}(z)^{-2-\frac{a_{\max}}{2}+b_{\max}} \\
&\preceq x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i} x_j(z)^{-\frac{a_j}{2}+b_j} \right) \sigma(z)^{2\nu} \widetilde{v}_{i+1}(z)^{-2-\frac{a_{\max}}{2}+b_{\max}} \\
&\preceq \left(\prod_{j=i+1}^{k+1} x_j(z)^{\frac{a_j}{2}-b_j} \right) x(z)^b \sigma(z)^2 \left(\frac{\sigma(z)^{\nu-1}}{x_{\max}} \right)^2 x_{\max}(z)^2 \widetilde{v}_{i+1}(z)^{-2-\frac{a_{\max}}{2}+b_{\max}} \\
&\asymp x(z)^b (\rho(z) w(z))^2 \left(\prod_{j=i+1}^{k+1} x_j(z)^{\frac{a_j}{2}-b_j} \right) x_{\max}(z)^2 \widetilde{v}_{i+1}(z)^{-2-\frac{a_{\max}}{2}+b_{\max}} \\
&\preceq x(z)^b (\rho(z) w(z))^2 \left(\prod_{j=i+1}^{k+1} x_j(z)^{\frac{a_j}{2}-b_j-2-\frac{a_{\max}}{2}+b_{\max}} \right) x(z)^2_{\max} \\
&\preceq x(z)^b (\rho(z) w(z))^2, \quad \text{by (3.34)},
\end{aligned}$$

completing the proof in this case. If instead $\nu_i = 0$ and $\nu_{i+1} = \nu$, then

$$\begin{aligned}
I_i &\preceq x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j} \right) \widetilde{v}_i^{-\frac{a_i}{2}+b_i} \left(\frac{\sigma(z)^{\nu}}{\widetilde{v}_{i+1}(z)} \right)^{2+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+\nu\left(b_{\max}-\frac{a_{\max}}{2}\right)} \\
&\asymp x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j} \right) \left(x_i^{-\frac{a_i}{2}+b_i} (\widetilde{v}_{i+1}^{\frac{1}{1-\nu}})^{-\frac{a_i}{2}+b_i} \right) \left(\widetilde{v}_{i+1}^{-\frac{1}{1-\nu}} \right)^{2+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+\nu\left(b_{\max}-\frac{a_{\max}}{2}\right)} \\
&\asymp x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i} x_j(z)^{-\frac{a_j}{2}+b_j} \right) (\widetilde{v}_{i+1}^{-\frac{1}{1-\nu}})^{2+(\nu-1)(b_{\max}-\frac{a_{\max}}{2})} \\
&\asymp x(z)^b \left(\prod_{j=i+1}^{k+1} x_j(z)^{\frac{a_j}{2}-b_j} \right) \widetilde{v}_{i+1}(z)^{b_{\max}-\frac{a_{\max}}{2}-\frac{2}{1-\nu}} \asymp x(z)^b \left(\prod_{j=i+1}^{k+1} x_j(z)^{\frac{\widehat{a}_j}{2}-\widehat{b}_j} \right) \widetilde{v}_{i+1}(z)^{-\frac{2}{1-\nu}} \\
&\preceq x(z)^b \widetilde{v}_{i+1}(z)^2 \widetilde{v}_{i+1}(z)^{-\frac{2}{1-\nu}} \quad \text{by (3.34)}, \\
&\asymp x(z)^b (\widetilde{v}_{i+1}(z)^{-\frac{\nu}{1-\nu}})^2 \preceq x(z)^b (x_{\max}(z)^{-1} \widetilde{v}_{i+1}(z)^{-\frac{\nu}{1-\nu}})^2 \asymp x(z)^b (\rho(z) w(z))^2,
\end{aligned}$$

while if $\nu_i = \nu_{i+1} = 0$, then

$$\begin{aligned} I_i &\preceq x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j} \right) \widetilde{v}_i(z)^{-\frac{a_i}{2}+b_i} \left(\frac{1}{\widetilde{v}_{i+1}(z)} \right)^{2+\frac{a_{\max}}{2}-\frac{a_i}{2}-b_{\max}+b_i+\nu\left(b_{\max}-\frac{a_{\max}}{2}\right)} \\ &\asymp x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i-1} x_j(z)^{-\frac{a_j}{2}+b_j} \right) x_i(z)^{-\frac{a_i}{2}+b_i} \widetilde{v}_{i+1}(z)^{-\frac{a_i}{2}+b_i} \widetilde{v}_{i+1}(z)^{-2-\frac{a_{\max}}{2}+\frac{a_i}{2}+b_{\max}-b_i-\nu\left(b_{\max}-\frac{a_{\max}}{2}\right)} \\ &\asymp x(z)^{\frac{a}{2}} \left(\prod_{j=1}^{i} x_j(z)^{-\frac{a_j}{2}+b_j} \right) \widetilde{v}_{i+1}(z)^{-2+(1-\nu)\left(b_{\max}-\frac{a_{\max}}{2}\right)} \asymp x(z)^b \left(\prod_{j=i+1}^{k+1} x_j(z)^{\frac{\widehat{a}_j}{2}-\widehat{b}_j} \right) \widetilde{v}_{i+1}^{-2} \\ &\preceq x(z)^b \left(\prod_{j=i+1}^{k+1} x_j(z)^2 \right) \widetilde{v}_{i+1}^{-2} \quad \text{by (3.34)}, \\ &\asymp x(z)^b \left(\prod_{j=i+1}^{k+1} x_j(z)^{2-\frac{2}{1-\nu_j}} \right) \asymp x(z)^b \left(\prod_{j=i+1}^{k+1} x_j(z)^{-\frac{\nu_j}{1-\nu_j}} \right)^2 \preceq x(z)^b \left(x_{\max}^{-1}(z) \prod_{j=i+1}^{k+1} x_j(z)^{-\frac{\nu_j}{1-\nu_j}} \right)^2 \\ &\asymp x(z)^b (\rho(z) w(z))^2. \end{aligned}$$

Finally, if z is away from all non-maximal corners, we can apply the estimate for I_{k+1} with $k = 0$. □

Relying on this estimate, we can now apply the Schur test to obtain the main result of this section.

Theorem 3.22. *Let g be an $\mathfrak{n}$-warped* QAC*-metric of weight function $\mathfrak{n} : \mathcal{M}_1(M) \to \{0, \nu\}$ for $\nu \in [0, 1)$ on the interior of a manifold with fibered corners M of dimension $m > 2$. Suppose that ∂M is connected, as well as ∂Z_H for each $H \in \mathcal{M}_{\mathrm{nm}}(M)$. Suppose that a is a multiweight such that $a_H = a_{\max}$ for $H \in \mathcal{M}_{\max}(M)$, that $a_{\max} < \frac{m-2}{1-\nu}$ and that $(1-\nu)a_{\max} < (1-\nu_H)(a_H + m_H)$ for all $H \in \mathcal{M}_{\mathrm{nm}}(M)$. Let also δ be a multiweight such that $\delta_H = \delta_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$ with*

$$\frac{a_{\max}}{2} < \delta_{\max} < \frac{m-2}{1-\nu} - \frac{a_{\max}}{2} \tag{3.37}$$

and

$$\left(\frac{1-\nu}{1-\nu_H}\right)\left(\delta_{\max} + \frac{a_{\max}}{2}\right) - \frac{a_H}{2} + 2 - m_H < \delta_H < \frac{a_H}{2} + \left(\frac{1-\nu}{1-\nu_H}\right)\left(-\frac{a_{\max}}{2} + \delta_{\max}\right). \tag{3.38}$$

If $\mathcal{R} \in \rho^{-2} w^{-2} \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M)$ is such that $\mathcal{R} \ge V := -\frac{-\Delta x^a}{x^a}$, then for all $\ell \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, the mappings

$$\begin{aligned} -\Delta + \mathcal{R} &: x^{\delta+\mathfrak{w}} \sigma^{\frac{\nu m}{2}} H^{\ell+2}_w(M) \to (\rho w)^{-2} x^{\delta+\mathfrak{w}} \sigma^{\frac{\nu m}{2}} H^\ell_w(M), \\ -\Delta + \mathcal{R} &: x^\delta \mathcal{C}^{\ell+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M) \to (\rho w)^{-2} x^\delta \mathcal{C}^{\ell,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M \setminus \partial M), \end{aligned} \tag{3.39}$$

are isomorphism, where $H^\ell_w(M)$ was introduced in (2.17) *and $\mathfrak{w}(H) := \frac{m_H - m}{2}$ is the multiweight such that $x^{\mathfrak{w}} \sigma^{\frac{\nu m}{2}} L^2_w(M) = L^2_b(M)$.*

Proof. For the mapping on Sobolev spaces, it suffices by local elliptic estimates to show that the Green's function $G_{\mathcal{L}}$ defines a bounded map

$$G_{\mathcal{L}} : (\rho w)^{-2} x^{\delta+\mathfrak{w}} \sigma^{\frac{\nu m}{2}} L^2_w(M) \to x^{\delta+\mathfrak{w}} \sigma^{\frac{\nu m}{2}} L^2_w(M). \tag{3.40}$$

This in turn corresponds to the boundedness of

$$\mathcal{K} : L^2_w(M) \to L^2_w(M) \tag{3.41}$$

with

$$\mathcal{K}(z, z') = x(z)^{-\delta-\mathfrak{w}} \sigma(z)^{-\frac{\nu m}{2}} G_{\mathcal{L}}(z, z') (\rho(z') w(z'))^{-2} x(z')^{\delta+\mathfrak{w}} \sigma(z')^{\frac{\nu m}{2}}.$$

By the Schur test, this will be the case provided we can find positive measurable functions f_1 and f_2 such that

$$\left|\int_{M\setminus\partial M}\mathcal{K}(z,z')f_1(z)dg(z)\right|\preceq f_2(z') \quad \text{and} \quad \left|\int_{M\setminus\partial M}\mathcal{K}(z,z')f_2(z')dg(z)\right|\preceq f_1(z)$$

for all $z,z'\in M\setminus\partial M$. We will take $f_1=f_2=x^{-\mathfrak{w}}\sigma^{-\frac{m\nu}{2}}$. Indeed, by Lemma 3.1, we have that

$$\begin{aligned}\left|\int_{M\setminus\partial M}\mathcal{K}(z,z')x^{-\mathfrak{w}}(z')\sigma(z')^{-\frac{m\nu}{2}}dg(z')\right| &\leq x(z)^{-\delta-\mathfrak{w}}\sigma(z)^{-\frac{\nu m}{2}}\int_{M\setminus\partial M}G_{-\Delta+V}x(z')^{\delta}(\rho(z')w(z'))^{-2}dg(z')\\ &\preceq x(z)^{-\delta-\mathfrak{w}}\sigma(z)^{-\frac{\nu m}{2}}\int_{M\setminus\partial M}G_{-\Delta+V}x(z')^{b}dg(z')\end{aligned}$$

with

$$x^b:=x^{\delta}(\rho w)^{-2}=\left(\prod_{H\in\mathcal{M}_1(M)}(x_H^{\delta_H+\frac{2}{1-\nu_H}})\right)w^{-2},$$

so in particular with multiweight b such that $b_{\max}=\delta_{\max}+\frac{2}{1-\nu}$ and $b_H=\delta_H+\frac{2}{1-\nu_H}-2$. In particular, by our assumption on δ, we can apply Proposition 3.21 so that

$$\begin{aligned}\left|\int_{M\setminus\partial M}\mathcal{K}(z,z')x^{-\mathfrak{w}}(z')\rho(z')^{-\frac{m\nu}{2}}dg(z')\right| &\preceq x(z)^{-\delta-\mathfrak{w}}\sigma(z)^{-\frac{\nu m}{2}}x(z)^{b}(\rho(z)w(z))^{2}\\ &\preceq x(z)^{-\mathfrak{w}}\sigma(z)^{-\frac{\nu m}{2}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\int_{M\setminus\partial M}\mathcal{K}(z,z')x^{-\mathfrak{w}}(z)\sigma(z)^{-\frac{m\nu}{2}}dg(z)\right| &\leq (\rho(z')w(z'))^{-2}x(z')^{\delta+\mathfrak{w}}\sigma(z')^{\frac{\nu m}{2}}\cdot\\ &\qquad\int_{M\setminus\partial M}G_{-\Delta+V}(z,z')x(z)^{-\delta-2\mathfrak{w}}\sigma(z)^{-m\nu}dg(z)\\ &=(\rho(z')w(z'))^{-2}x(z')^{\delta+\mathfrak{w}}\sigma(z')^{\frac{\nu m}{2}}\int_{M\setminus\partial M}G_{-\Delta+V}(z,z')x(z)^{b}dg(z)\end{aligned}$$

with this time multiweight b such that

$$b_H:=m-m_H+\frac{m\nu_H}{1-\nu_H}-\delta_H=\frac{m}{1-\nu_H}-m_H-\delta_H,$$

that is, such that

$$x^b=x^{-\delta-2\mathfrak{w}}\sigma(z)^{-m\nu}=\prod_H x_H^{-\delta_H-m_H+m+\frac{m\nu_H}{1-\nu_H}}.$$

By our assumptions on δ and since $G_{-\Delta+V}(z,z')=G_{-\Delta+V}(z',z)$, we can apply Proposition 3.21 to conclude that

$$\begin{aligned}\left|\int_{M\setminus\partial M}\mathcal{K}(z,z')x^{-\mathfrak{w}}(z)\sigma(z)^{-\frac{m\nu}{2}}dg(z)\right| &\preceq (\rho(z')w(z'))^{-2}x(z')^{\delta+\mathfrak{w}}\sigma(z')^{\frac{\nu m}{2}}x(z')^{b}(\rho(z')w(z'))^{2}\\ &\preceq x(z')^{-\mathfrak{w}}\sigma(z')^{-\frac{\nu m}{2}}.\end{aligned}$$

By the Schur test, the mapping (3.40) is indeed bounded and the result follows. For the map on Hölder spaces, we can use our estimates on Green's functions essentially as above to show that the Green's function $G_{\mathcal{L}}$ induces a bounded mapping

$$G_{\mathcal{L}}:x^{\delta}(\rho w)^{-2}\mathcal{C}^{\ell,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M\setminus\partial M)\to x^{\delta}L^{\infty}(M\setminus\partial M).$$

Using Schauder estimates and the fact that a $\mathfrak{n}\,$Qb-metric has bounded geometry, we then see that in fact it defines an inverse

$$G_{\mathcal{L}}:x^{\delta}(\rho w)^{-2}\mathcal{C}^{\ell,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M\setminus\partial M)\to x^{\delta}\mathcal{C}^{\ell+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M\setminus\partial M)$$

for the map

$$-\Delta+\mathcal{R}:x^{\delta}\mathcal{C}^{\ell+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M\setminus\partial M)\to(\rho w)^{-2}x^{\delta}\mathcal{C}^{\ell,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(M\setminus\partial M).$$

□

As in [18], we will be mostly interested in the case where $a = 0$ and $\mathcal{R} = 0$, in which case we have the following.

Corollary 3.23. *Let g be an $\mathfrak{n}$-warped QAC-metric of weight function $\mathfrak{n} : \mathcal{M}_1(M) \to \{0, \nu\}$ for some $\nu \in [0, 1)$ on the interior of a manifold with fibered corners M. Suppose that ∂M is connected, as well as ∂Z_H for each $H \in \mathcal{M}_{\mathrm{nm}}(M)$. Let δ be a multiweight such that $\delta_H = \delta_{\max}$ for all $H \in \mathcal{M}_{\max}(M)$. Then for all $\ell \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, the mappings*

$$\begin{aligned} &\Delta : x^{\delta+\mathfrak{w}}\rho^{\frac{\nu m}{2}} H_w^{\ell+2}(M) \to (\rho w)^{-2} x^{\delta+\mathfrak{w}}\rho^{\frac{\nu m}{2}} H_w^{\ell}(M), \\ &\Delta : x^{\delta}\mathcal{C}_{\mathrm{Qb}}^{\ell+2,\alpha}(M \setminus \partial M) \to (\rho w)^{-2} x^{\delta}\mathcal{C}_{\mathrm{Qb}}^{\ell,\alpha}(M \setminus \partial M), \end{aligned} \tag{3.42}$$

are isomorphisms provided

$$0 < \delta_{\max} < \frac{m-2}{1-\nu} \quad \text{and} \quad \left(\frac{1-\nu}{1-\nu_H}\right)\delta_{\max} + 2 - m_H < \delta_H < \left(\frac{1-\nu}{1-\nu_H}\right)\delta_{\max} \quad \forall\, H \in \mathcal{M}_{\mathrm{nm}}(M).$$

Remark 3.24. *In the case $x^\delta = \rho^s w^\tau$, this means that $\delta_{\max} = -\frac{s}{1-\nu}$ and $\delta_H = \tau - \frac{s}{1-\nu_H}$ for $H \in \mathcal{M}_{\mathrm{nm}}(M)$, so the maps in Corollary 3.23 will be isomorphisms provided*

$$2 - m < s < 0 \quad \text{and} \quad 2 - m_H < \tau < 0, \quad \forall\, H \in \mathcal{M}_{\mathrm{nm}}(M),$$

which is consistent with [20, Theorem 8.4] *and* [18, Corollary 5.19].

4. Weighted blow-ups

In our construction of examples of Calabi-Yau $\mathfrak{n}$-warped QAC-metrics on smoothing of cones, a key role will be played by weighted blow-ups. For this reason, let us take some time to collect basic results. First, recall that a n-tuple of positive integers $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ induces a $\mathbb{C}^*$-action on $\mathbb{C}^n$, namely the action defined by

$$t \cdot z := (t^{w_1} z_1, \ldots, t^{w_n} z_n) \tag{4.1}$$

for $t \in \mathbb{C}^*$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. By definition, the **weighted projective space** $\mathbb{CP}_w^{n-1}$ associated to w is the quotient of $\mathbb{C}^n \setminus \{0\}$ by this action,

$$\mathbb{CP}_w^{n-1} := (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*. \tag{4.2}$$

Unless $w = (1, \ldots, 1)$, in which case $\mathbb{CP}_w^{n-1}$ is just the usual complex projective space, the weighted projective space $\mathbb{CP}_w^{n-1}$ is not smooth, but it is a complex orbifold if the greatest common divisor of $w_1, \ldots, w_n$ is 1. As for the usual projective space, one can blow up the origin in $\mathbb{C}^n$ by replacing $\{0\}$ by $\mathbb{CP}_w^{n-1}$. More precisely, with respect to the weight w, the weighted blow-up of $\mathbb{C}^n$ at the origin is the quotient

$$\mathrm{Bl}_{\{0\}}(\mathbb{C}^n, w) := \left((\mathbb{C} \times \mathbb{C}^n) \setminus (\mathbb{C} \times \{0\})\right)/\mathbb{C}^* \tag{4.3}$$

with $\mathbb{C}^*$ acting on $\mathbb{C}^{n+1}$ with weight $(-1, w)$, that is,

$$t \cdot (z_0, z) = (t^{-1} z_0, t^{w_1} z_1, \ldots, t^{w_n} z_n)$$

for $t \in \mathbb{C}^*$ and $(z_0, z) \in \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}$. When $w = (1, \ldots, 1)$, one can readily check that (4.3) corresponds to the usual blow-up of a point. If we pick $w_0 \in \mathbb{N}$ and consider the more general $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ given by

$$t \cdot (z_0, z) = (t^{-w_0} z_0, t^{w_1} z_1, \ldots, t^{w_n} z_n),$$

notice that the corresponding quotient $\mathbb{CP}^n_{(-w_0, w)} := \left((\mathbb{C} \times \mathbb{C}^n) \setminus (\mathbb{C} \times \{0\})\right)/\mathbb{C}^*$, called the non-compact weighted projective space in [5], can also be thought of as the blow-up of the origin in $\mathbb{C}^n/\Gamma$, where Γ is the cyclic group of w_0-roots of unity with generator $e^{\frac{2\pi i}{w_0}}$ acting on $\mathbb{C}^n$ by

$$e^{\frac{2\pi i}{w_0}} \cdot z := (e^{\frac{2\pi i w_1}{w_0}} z_1, \ldots, e^{\frac{2\pi i w_n}{w_0}} z_n).$$

Over the real numbers, the weighted blow-up admits two interesting versions. The first one, the immediate algebraic analog, is to replace $\mathbb{C}$ by $\mathbb{R}$ and a weighted action of $\mathbb{C}^*$ by the weighted action of $\mathbb{R}^*$. The second one, which is the one on which we will focus, consist in replacing $\mathbb{C}$ by $\mathbb{R}$ again, but $\mathbb{C}^*$ by $\mathbb{R}^+ = (0, \infty)$. Only considering the action of $\mathbb{R}^+$ adds some flexibility, since we can then consider actions with weight $w \in (\mathbb{R}^+)^n$

given by a n-tuple of positive real numbers that are not necessarily integers. Moreover, we can replace $\mathbb{R}^n$ by the manifold with corners

$$\mathbb{R}^n_k := [0,\infty)^k \times \mathbb{R}^{n-k} \tag{4.4}$$

for some $k \in \{0,1,\ldots,n\}$. On this space, a choice of weight $w \in (\mathbb{R}^+)^n$ specifies a $\mathbb{R}^+$-action given by

$$t \cdot x := (t^w x_1, \ldots, t^{w_n} x_n) \tag{4.5}$$

for $t \in \mathbb{R}^+$ and $x = (x_1,\ldots,x_n) \in \mathbb{R}^n_k$. Clearly, the corresponding quotient $(\mathbb{R}^n_k \setminus \{0\})/\mathbb{R}^+$ can be identified with the unit sphere

$$\mathbb{S}^{n-1}_k := \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n_k \mid \sum_{i=1}^n x_i^2 = 1\}. \tag{4.6}$$

Definition 4.1. *With respect to a choice of weight $w \in (\mathbb{R}^+)^n$, the **weighted blow-up** of $\{0\}$ in $\mathbb{R}^n_k$ is the quotient*

$$[\mathbb{R}^n_k; \{0\}]_w := (([0,\infty) \times \mathbb{R}^n_k) \setminus ([0,\infty) \times \{0\}))/\mathbb{R}^+$$

with respect to the $\mathbb{R}^+$-action given by

$$t \cdot (x_0, x) = (t^{-1}x_0, t^{w_1}x_1, \ldots, t^{w_n}x_n)$$

for $t \in \mathbb{R}^+$ and $(x_0,x) \in [0,\infty) \times \mathbb{R}^n_k$. The corresponding blow-down map $\beta_{\{0\}} : [\mathbb{R}^n_k; \{0\}]_w \to \mathbb{R}^n_k$ is given by

$$\beta_{\{0\}}([x_0 : x]) = (x_0^{w_1}x_1, \ldots, x_0^{w_n}x_n),$$

where $[x_0 : x]_w$ denotes the class in $[\mathbb{R}^n_k; \{0\}]_w$ corresponding to $(x_0,x) \in ([0,\infty) \times \mathbb{R}^n_k) \setminus ([0,\infty) \times \{0\})$.

Clearly, the weighted blow-up $[\mathbb{R}^n_k; \{0\}]_w$ is naturally a manifold with corners diffeomorphic to $\mathbb{S}^{n-1}_k \times [0,\infty)$ via the map

$$\begin{array}{rccc} F: & \mathbb{S}^{n-1}_k \times [0,\infty) & \to & [\mathbb{R}^n_k; \{0\}]_w \\ & (\omega, t) & \mapsto & [t : \omega]_w. \end{array}$$

Closely related to the notion of blow-up is the notion of radial compactification.

Definition 4.2. *For $w = (w_1,\ldots,w_n) \in (\mathbb{R}^+)^n$, the **weighted radial compactification** of $\mathbb{R}^n_k$ is the quotient*

$$\overline{\mathbb{R}^n_{k,w}} := (([0,\infty) \times \mathbb{R}^n_k) \setminus \{0\})/\mathbb{R}^+ \tag{4.7}$$

with respect to the $\mathbb{R}^+$-action on $[0,\infty) \times \mathbb{R}^n_k$ given by

$$t \cdot (x_0, x) = (tx_0, t^{w_1}x_1, \ldots, t^{w_n}x_n) \quad \text{for} \quad t \in \mathbb{R}^+,\ (x_0,x) \in [0,\infty) \times \mathbb{R}^n_k.$$

Comparing Definitions 4.1 and 4.2, we see that the weighted radial compactification can alternatively be obtained by gluing $[\mathbb{R}^n_k; \{0\}]$ and $\mathbb{R}^n_k$ via the map

$$\begin{array}{rccc} \varphi: & [\mathbb{R}^n_k; \{0\}] \setminus H_{\{0\}} & \to & \mathbb{R}^n_k \setminus \{0\} \\ & [x_0 : x] & \mapsto & (x_0^{-w_1}x_1, \ldots, x_0^{-w_n}x_n), \end{array}$$

where $H_{\{0\}} \cong \mathbb{S}^{n-1}_k$ is the boundary hypersurface in $[\mathbb{R}^n_k; \{0\}]_w$ created by the weighted blow-up of the origin. The weighted radial compactification is also naturally diffeomorphic to the new boundary hypersurface created by the weighted blow-up

$$[[0,\infty) \times \mathbb{R}^n_k; \{0\}]_{(1,w)}$$

of $\{0\}$ in $[0,\infty) \times \mathbb{R}^n_k$ with respect to the weight $(1,w)$.

More generally, if M is a manifold with corners and $Y \subset M$ is a p-submanifold, we can do a weighted blow-up of M along the p-submanifold Y provided some extra data is given. More precisely, suppose that the inner pointing normal bundle N^+Y of Y admits a decomposition

$$N^+Y = \bigoplus_{i=1}^{\ell} V_i^+ \tag{4.8}$$

with V_i^+ a subbundle of N^+Y with fiber $\mathbb{R}^{n_i}_{k_i}$. For each V_i^+, let $w_i \in \mathbb{R}^+$ be a choice of weight. Let $\mathcal{U}$ be an open neighborhood of Y inside N^+Y and let

$$c : \mathcal{U} \to \mathcal{V} \subset M \tag{4.9}$$

be a collar neighborhood of Y in M, that is, c is a diffeomorphism between $\mathcal{U}$ and an open neighborhood $\mathcal{V}$ of Y in M. On each fiber of N^+Y, we can consider the weighted blow-up of the origin with respect to the decomposition (4.8) and the weight $w = (w_1, \ldots, w_\ell) \in (\mathbb{R}^+)^\ell$. These naturally combine to give the weighted blow-up

$$[N^+Y; Y]_w$$

of the zero section of N^+Y with blow-down map $\beta_{N^+Y} : [N^+Y; Y] \to N^+Y$. It comes with a natural fiber bundle $\pi : [NY^+; Y]_w \to Y$ with fiber $[\mathbb{R}^n_k; \{0\}]_w$, where

$$n = \sum_{i=1}^{\ell} n_i, \quad k = \sum_{i=1}^{\ell} k_i.$$

Using the collar neighborhood (4.9), one can then define the weighted blow-up of Y in M by

$$[M; Y]_{w,c} := M \setminus Y \cup_c \widetilde{\mathcal{U}}, \tag{4.10}$$

where $\widetilde{\mathcal{U}} = \beta^{-1}_{N^+Y}(\mathcal{U})$, that is, $[M; Y]_{w,c}$ is obtained by replacing $\mathcal{U}$ by $\widetilde{\mathcal{U}}$ in M via the gluing map given by (4.9). Unfortunately, compared to the usual blow-up, the definition (4.10) seems to be sensitive to the choice of collar neighborhood (4.9) and we will need to be careful about the class of collar neighborhoods we will allow.

Definition 4.3. *Two collar neighborhoods $c_i : \mathcal{U}_i \to \mathcal{V}_i$ of Y in M for $i \in \{1, 2\}$ are **equivalent** with respect to the decomposition* (4.8) *provided there are open neighborhoods $\mathcal{U}_0$ and $\mathcal{V}_0$ of Y in N^+Y contained in $\mathcal{U}_1$ and $\mathcal{U}_2$ such that*

$$c_2^{-1} \circ c_1 : \mathcal{U}_0 \to \mathcal{V}_0 \subset N^+Y \tag{4.11}$$

preserves (not necessarily linearly) the fibers of N^+Y and the decomposition (4.8)*. If moreover the map* (4.11) *is linear in each fiber, we say that the collar neighborhoods c_1 and c_2 are **linearly equivalent**.*

Lemma 4.4. *If c_1 and c_2 are equivalent collar neighborhoods with respect to the decomposition* (4.8)*, then there is a natural diffeomorphism*

$$[M; Y]_{w,c_1} \cong [M; Y]_{w,c_2}.$$

Proof. In this case, one can check in local coordinates that $c_2^{-1} \circ c_1$ extends to a diffeomorphism

$$\widetilde{c_2^{-1} \circ c_1} : \widetilde{\mathcal{U}}_0 \to \widetilde{\mathcal{V}_0}$$

with $\widetilde{\mathcal{U}}_0 = \beta^{-1}_{N^+Y}(\mathcal{U}_0)$ and $\widetilde{\mathcal{V}}_0 = \beta^{-1}_{N^+Y}(\mathcal{V}_0)$ the lifts of $\mathcal{U}_0$ and $\mathcal{V}_0$ to $[N^+Y; Y]_w$ respectively. Using this diffeomorphism, we see that the identity map $\mathrm{Id} : M \setminus Y \to M \setminus Y$ extends uniquely to a diffeomorphism

$$[M; Y]_{w,c_1} \cong [M; Y]_{w,c_2}.$$

□

Thus, we will be able to define the weighted blow-up of Y in M provided we can specify in some natural way a class of equivalent collar neighborhoods for Y in M. This will be possible for specific choices of M and Y. In fact, we will even be able to pick a natural linear equivalence class. Let us consider the simplest possible situation where $M = \mathbb{R}^n_k$ with weighted $\mathbb{R}^+$-action specified by a weight $w = (w_1, \ldots, w_n)$. In terms of the canonical coordinates on $\mathbb{R}^n_k$, suppose that Y is a p**-linear subspace** of $\mathbb{R}^n_k$, that is, a p-submanifold of the form

$$Y = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_k \mid x_i = 0 \quad \text{for } i \in \mathcal{I}\}$$

for some subset $\mathcal{I} \subset \{1, \ldots, n\}$. Using the linear structure on $\mathbb{R}^n_k$, we have a canonical decomposition

$$\mathbb{R}^n_k = N^+Y = Y \times \mathbb{R}^m_\ell \tag{4.12}$$

for some m and ℓ. Moreover, the weight w induces a weight on N^+Y, since the coordinates on the fiber $\mathbb{R}^m_\ell$ of N^+Y are given by x_i for $i \in \mathcal{I}$ to which we can assign the weight w_i. In this case, (4.12) naturally defines a linear equivalence class of collar neighborhoods with respect to the decomposition $\mathbb{R}^m_\ell = [0, \infty)^\ell \times \mathbb{R}^{m-\ell}$. With respect to this choice of linear equivalence class of collar neighborhoods, we therefore have a well-defined weighted blow-up

$$[\mathbb{R}^n_k; Y]_w = [\mathbb{R}^m_\ell; \{0\}]_w \times Y \quad \text{with blow-down map} \quad \beta_Y : [\mathbb{R}^n_k; Y]_w \to \mathbb{R}^n_k. \tag{4.13}$$

More importantly, let $Z \subset \mathbb{R}^n_k$ be another p-linear subspace of the form

$$Z = \{(x_1, \dots, x_n) \in \mathbb{R}^n_k \mid x_i = 0 \text{ for } i \in \mathcal{J}\}$$

for a subset $\mathcal{J}$ strictly contained in $\mathcal{I}$, so that Y is strictly included in Z. Then the **lift** of Z in $[\mathbb{R}^n_k; Y]_w$ is the p-submanifold

$$\widetilde{Z} = \overline{\beta_Y^{-1}(Z \setminus Y)}.$$

We claim that the inner pointing normal bundle $N^+\widetilde{Z}$ of $\widetilde{Z}$ has a natural decomposition induced by the $\mathbb{R}^+$-action on $\mathbb{R}^n_k$ and that there is a corresponding linear equivalence class of collar neighborhoods for $\widetilde{Z}$ in $[\mathbb{R}^n_k; Y]_w$. To see this, let us first relabel the coordinates $(x_1, \dots, x_n) \in \mathbb{R}^n_k$ so that $\mathcal{J} = \{1, \dots, m_J\}$ and $\mathcal{I} = \{1, \dots, m\}$ with $m > m_J$. Then for $i \in \mathcal{I} \setminus \mathcal{J}$, $x_i^{\frac{1}{w_i}}$ (and also $(-x_i)^{\frac{1}{w_i}}$ when x_i can take negative values) defines a local boundary defining function for the boundary hypersurface $H_Y := \beta_Y^{-1}(Y)$ created by the weighted blow-up of Y. It does so in a region overlapping with $\widetilde{Z} \cap H_Y$, where we can use the coordinates

$$\zeta_{ij,+} = \begin{cases} (x_i)^{\frac{1}{w_i}}, & j = i, \\ x_i^{-\frac{w_j}{w_i}} x_j, & j \in \mathcal{I} \setminus \{i\}, \\ x_j, & \text{otherwise}, \end{cases} \tag{4.14}$$

as well as

$$\zeta_{ij,-} = \begin{cases} (-x_i)^{\frac{1}{w_i}}, & j = i, \\ (-x_i)^{-\frac{w_j}{w_i}} x_j, & j \in \mathcal{I} \setminus \{i\}, \\ x_j, & \text{otherwise}, \end{cases} \tag{4.15}$$

when x_i can take negative values.

Under the identification (4.13), $(\zeta_{i1,\pm}, \dots, \zeta_{in,\pm})$ corresponds to the point

$$([\zeta_{ii,\pm} : \zeta_{i1,\pm} : \dots : \zeta_{i(i-1),\pm} : \pm 1 : \zeta_{i(i+1),\pm} : \dots : \zeta_{im,\pm}], (\zeta_{i(m+1),\pm}, \dots, \zeta_{in,\pm})) \in [\mathbb{R}^m_\ell; \{0\}]_w \times Y, \tag{4.16}$$

where $[x_0 : \dots : x_m] \in [\mathbb{R}^m_\ell; \{0\}]$ is the class corresponding to $(x_0, x_1, \dots, x_m) \in [0, \infty) \times \mathbb{R}^m_\ell$. In these coordinates, $\widetilde{Z}$ is locally given by

$$\zeta_{i1,\pm} = \dots = \zeta_{im_J,\pm} = 0.$$

For $i, j \in \mathcal{I} \setminus \mathcal{J}$ with $i \neq j$ and $p, q \in \{+, -\}$, the change for the coordinates $\zeta_{i,p}$ to the coordinates $\zeta_{j,q}$ is given by

$$\zeta_{jk,q} = \begin{cases} p(q\zeta_{ij,p})^{-\frac{w_i}{w_j}}, & k = i, \\ (q\zeta_{ij,p})^{\frac{1}{w_j}} \zeta_{ii,p}, & k = j, \\ (q\zeta_{ij,p})^{-\frac{w_k}{w_j}} \zeta_{ik,p}, & \text{otherwise}. \end{cases} \tag{4.17}$$

Lemma 4.5. *The coordinates $(x_1, \dots, x_n)$ and $(\zeta_{i1,\pm}, \dots, \zeta_{in,\pm})$ for $\mathcal{I} \setminus \mathcal{J}$ induce a decomposition of $N^+\widetilde{Z}$ and a linear equivalence class of collar neighborhoods for $\widetilde{Z}$ in $[\mathbb{R}^n_k; Y]_w$.*

Proof. In the coordinates $(x_1, \dots, x_n)$, the inner pointing normal bundle is trivialized by the coordinates $x_1, \dots, x_{m_J}$, while in the coordinates $(\zeta_{i1,\pm}, \dots, \zeta_{in,\pm})$ for $i \in \mathcal{I} \setminus \mathcal{J}$, it is trivialized by $\zeta_{i1,\pm}, \dots, \zeta_{im_J,\pm}$. In both cases, we have a corresponding local collar neighborhood for $\widetilde{Z}$ in $[\mathbb{R}^n_k; Y]$ and a decomposition. If we denote them by c_0 and $c_{i,\pm}$ respectively, then (4.14), (4.15) and (4.17) show that on their overlaps, they are linearly equivalent in the sense of Definition 4.3. Since these coordinates cover $\widetilde{Z}$ in $[\mathbb{R}^n_k; Y]_w$, this means that they induce a decomposition

$$N^+\widetilde{Z} = \bigoplus_{j=1}^{m_J} V_j$$

with V_j spanned locally by x_j and $\zeta_{ij,\pm}$ for $i \in \mathcal{I} \setminus \mathcal{J}$, as well as a corresponding natural linear equivalence class of collar neighborhoods for $\widetilde{Z}$ in $[\mathbb{R}^n_k; Y]_w$. $\square$

Of course, the weight w naturally induces a weight on V_j, namely w_j. This means that we can consider the weighted blow-up of $\widetilde{Z}$ in $[\mathbb{R}^n; Y]_w$, namely

$$[[\mathbb{R}^n_k; Y]_w; \widetilde{Z}]_w$$

with the linear equivalence class of collar neighborhood that of Lemma 4.5. As for the usual blow-up, it is convenient to use the notation

$$[\mathbb{R}^n_k; Y, Z]_w := [[\mathbb{R}^n_k; Y]_w; \widetilde{Z}]_w.$$

Clearly, if $\mathcal{I}_p \subsetneq \cdots \subsetneq \mathcal{I}_1$ is a sequence of strictly embedded subsets of $\{1, \ldots, n\}$ and $Y_1 \subsetneq \cdots \subsetneq Y_p$ the corresponding sequence of strictly embedded p-linear subspaces, then the above argument can be used to define the sequence of weighted blow-ups of $Y_1, \ldots, Y_p$

$$[\mathbb{R}^n_k; Y_1, \ldots, Y_p]_w := [\cdots[[\mathbb{R}^n_k; Y_1]_w; \widetilde{Y}_2^{(1)}]_w \cdots; \widetilde{Y}_{p-1}^{(p-2)}]_w; \widetilde{Y}_p^{(p-1)}]_w \tag{4.18}$$

with $\widetilde{Y}_i^{(i-1)} = \overline{\beta_{i-1}^{-1}(Y_i \setminus Y_{i-1})}$ the **lift** of Y_i to $[\mathbb{R}^n_k; Y_1, \ldots, Y_{i-1}]_w$ with blow-down map

$$\beta_{i-1} : [\mathbb{R}^n_k; Y_1, \ldots, Y_{i-1}] \to \mathbb{R}^n_k.$$

More generally, let $Y_1, \ldots, Y_p$ be a sequence of distinct p-linear subspaces of $\mathbb{R}^n_k$ with $\mathcal{I}_1, \ldots, \mathcal{I}_p$ the corresponding subsets of $\{1, \ldots, n\}$.

Lemma 4.6. *Suppose that $\{Y_1, \ldots, Y_p\}$ is closed under taking intersections, or equivalently, $\{\mathcal{I}_1, \ldots, \mathcal{I}_p\}$ is closed under taking unions. Then the weighted iterated blow-up*

$$[\mathbb{R}^n_k; Y_1, \ldots, Y_p]_w$$

is well-defined provided

$$Y_i \subset Y_j \quad \Longrightarrow \quad i \leq j. \tag{4.19}$$

Moreover, as long as (4.19) *holds, the definition will not depend on the ordering of the p-submanifolds $Y_1, \ldots, Y_p$.*

Proof. Let Y_i and Y_j be two of the p -submanifolds that are blown up. If one is included in the other, then the order in which they are blown up is completely determined by (4.19). On the other hand, if $Y_i \cap Y_j = Y_q$ for some q with $Y_q \subsetneq Y_i$ and $Y_q \subsetneq Y_j$, then by (4.19), $q < i$ and $q < j$, so the blow-up of Y_q is performed before those of Y_i and Y_j. But after the blow-up of Y_q, the lifts of Y_i and Y_j will be disjoint, so their weighted blow-ups commute, that is, the two orders in which we can blow-up Y_i and Y_j lead to the same space. The fact that they are disjoint also indicates more generally that we can essentially reduce to the case of sequences of strictly included p-submanifolds of (4.18). □

The manifolds on which we want to perform weighted iterated blow-ups is not quite $\mathbb{R}^n_k$, but almost. We want instead to apply this construction to the weighted radial compactification of $\overline{\mathbb{R}^n_{k,w}}$ with respect to some weight $w \in (\mathbb{R}^+)^n$. Let H_∞ be the boundary hypersurface of $\overline{\mathbb{R}^n_{k,w}}$ such that $\mathbb{R}^n_k = \overline{\mathbb{R}^n_{k,w}} \setminus H_\infty$.

Lemma 4.7. *Let $\{Y_1, \ldots, Y_p\}$ be a finite set of p-linear subspaces of $\mathbb{R}^n_k$ and $\mathcal{I}_1, \ldots, \mathcal{I}_p$ the corresponding subsets of $\{1, \ldots, n\}$. Let $\overline{Y}_i$ be the closure of Y_i in $\overline{\mathbb{R}^n_{k,w}}$ and consider the p-submanifolds $\overline{Y}_1 \cap H_\infty, \ldots, \overline{Y}_p \cap H_\infty$. Suppose that $\{\overline{Y}_1 \cap H_\infty, \ldots, \overline{Y}_p \cap H_\infty\}$ is closed under taking non-empty intersections and that*

$$\overline{Y}_i \cap H_\infty \subset \overline{Y}_j \cap H_\infty \implies i \leq j.$$

Then the weighted iterated blow-up

$$[\overline{\mathbb{R}^n_{k,w}}; \overline{Y}_1 \cap H_\infty, \ldots, \overline{Y}_p \cap H_\infty]_w \tag{4.20}$$

of $\overline{Y}_1 \cap H_\infty, \ldots, \overline{Y}_p \cap H_\infty$ in $\overline{\mathbb{R}^n_{k,w}}$ is well-defined, namely, at each step, the lifts of the p-submanifolds that must be blown-up come with a natural decomposition of their inner-pointing normal bundle and a natural linear equivalence class of collar neighborhood in the sense of Definition 4.3.

Proof. Regarding $\overline{\mathbb{R}^n_{k,w}}$ as the new boundary hypersurface in $[[0,\infty)\times\mathbb{R}^n_k;\{0\}]_{(1,w)}$ created by the blow-up of the origin, we see that this iterated blow-up essentially corresponds to the consideration the iterated blow-ups of the lifts of $\{0\}\times Y_1,\ldots,\{0\}\times Y_p$ in $[[0,\infty)\times\mathbb{R}^n_k;\{0\}]_{(1,w)}$. More precisely, if we consider

$$[[0,\infty)\times\mathbb{R}^n_k;\{0\},\{0\}\times Y_1,\ldots,\{0\}\times Y_p]_{(1,w)}, \tag{4.21}$$

which is well-defined by Lemma 4.6, then the boundary hypersurface created by the first weighted blow-up corresponds to the weighted iterated blow-up (4.21) that we want to define. ☐

5. Compactification of smoothings of Calabi-Yau cones

In this section, we will construct a suitable compactification of the smoothing (1.15) by a manifold with fibered corners. To do this, we will suppose that Assumptions 1.1,1.2 hold. When $\ell>0$ and $N>1$, the zero locus $V_{\{1,\ldots,N\}}$ of $[Q_{q,i}]_\ell(z_0)$ described in Assumption 1.1 will play an important role. On the other hand, if $\ell=0$, there is no zero locus, while if $N=1$, there is no need to consider this zero locus, so let us set

$$V_{\ell,\mathrm{zero}}=\begin{cases}\emptyset, & \ell=0 \text{ or } N=1,\\ V_{\{1,\ldots,N\}}, & \ell\ge 1 \text{ and } N>1.\end{cases} \tag{5.1}$$

Let $\overline{\mathbb{C}^{m+n}_w}$ be the radial compactification of $\mathbb{C}^{m+n}$ with respect to the $\mathbb{R}^+$-action specified by the weight $w=(w_{0,1},\ldots,w_{0,m_0+n_0},w_{1,1},\ldots,w_{1,m_1+n_1},\ldots,w_{N,1},w_{N,m_N+n_N})$,

$$t\cdot(z_0,z_1,\ldots,z_N)=(tz_0,t^{w_{1,1}}z_{1,1},\ldots,t^{w_{1,m_1+n_1}}z_{1,m_1+n_1},\ldots,t^{w_{N,1}}z_{N,1},t^{w_{N,m_N+n_N}}z_{N,m_N+n_N})\quad\text{for } t>0.$$

The boundary of $\overline{\mathbb{C}^{m+n}_w}$ is the sphere

$$\mathbb{S}^{2m+2n-1}:=\{z\in\mathbb{C}^{m+n}\mid\sum_{q=0}^N\sum_{j=1}^{m_q+n_q}|z_{q,j}|^2=1\}.$$

In terms of this sphere, $\overline{\mathbb{C}^{m+n}_w}$ is obtained by gluing $\mathbb{S}^{2m+2n-1}\times[0,\infty)$ and $\mathbb{C}^{m+n}$ via the map

$$\begin{array}{rccc}\nu: & \mathbb{S}^{2m+2n-1}\times(0,\infty) & \to & \mathbb{C}^{m+n}\\ & (\omega,\xi) & \mapsto & \left(\frac{\omega_0}{\xi^{w_0}},\frac{\omega_{1,1}}{\xi^{w_{1,1}}},\ldots,\frac{\omega_{N,m_N+n_N}}{\xi^{w_{N,m_N+n_N}}}\right).\end{array} \tag{5.2}$$

In fact, this map extends to give a tubular neighborhood

$$\nu:\mathbb{S}^{2m+2n-1}\times[0,\infty)\to\overline{\mathbb{C}^{m+n}_w}$$

of $\mathbb{S}^{2m+2n-1}$ in $\mathbb{C}^{m+n}$. In terms of the coordinates induced by this tubular neighborhood, we see that the defining equations of C_ϵ are

$$\xi^{-d}P_{q,i}(\omega_q)=\epsilon Q_{q,i}(\xi^{-1}\cdot\omega_0)\quad\text{for } q\in\{1,\ldots,N\},\ i\in\{1,\ldots,n_q\}, \tag{5.3}$$

that is, as $\xi\searrow 0$,

$$P_{q,i}(\omega_q)=\xi^{d-\ell}[Q_{q,i}]_\ell(\omega_0)+o(\xi^{d-\ell})\quad\text{for } i\in\{1,\ldots,m_q+n_q\}. \tag{5.4}$$

The closure $\overline{C}_\epsilon$ of C_ϵ in $\overline{\mathbb{C}^{m+n}_w}$ is obtained by taking also $\xi=0$ in (5.3). In particular, in light of (5.4), on $\partial\overline{\mathbb{C}^{m+n}_w}$, we see that $\partial\overline{C}_\epsilon=\overline{C}_\epsilon\cap\partial\overline{\mathbb{C}^{m+n}_w}$ is given by the equations

$$P_{q,i}(\omega_q)=0,\quad \omega\in\mathbb{S}^{2m+2n-1}=\partial\overline{\mathbb{C}^{m+n}_w},\quad q\in\{1,\ldots,N\},i\in\{1,\ldots,n\}.$$

This does not depend on ϵ and coincides with $\partial\overline{C}_0=\overline{C}_0\cap\partial\overline{\mathbb{C}^{m+n}_w}$, which corresponds to the fact that the polynomial $Q_{q,i}$ has degree strictly smaller than the one of $P_{q,i}$ for each i. In particular, if C_0 has a singular cross-section, that is, if $\partial\overline{C}_0$ is singular, then $\partial\overline{C}_\epsilon$ is also singular, even if C_ϵ itself is smooth. To resolve these singularities, consider in $\mathbb{C}^{m+n}$ the subsets

$$\mathbb{C}^{m+n}_{w,\mathrm{sing}}:=\bigcup_{\{0\}\subsetneq\mathfrak{q}\subset\{1,\ldots,N\}}V_{\mathfrak{q}} \tag{5.5}$$

and $V_{\ell,\mathrm{zero}}$ defined in (5.1). Let $\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}$ and $\overline{V}_{\ell,\mathrm{zero}}$ be their closure in $\overline{\mathbb{C}^{m+n}_w}$. If we set $\partial\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}=\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}\cap\partial\overline{\mathbb{C}^{m+n}_w}$ and $\partial\overline{V}_{\ell,\mathrm{zero}}=\overline{V}_{\ell,\mathrm{zero}}\cap\partial\overline{\mathbb{C}^{m+n}_w}$, then $\partial\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}$ is naturally a stratified space in $\partial\overline{\mathbb{C}^{m+n}_w}$. Our strategy to resolve the singularities of $\overline{C}_\epsilon$ will essentially be to blow up the strata of $\partial\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}$ in an order compatible with the partial ordering of the strata. However, when $\partial\overline{V}_{\ell,\mathrm{zero}}$ is not empty, the part of $\partial\overline{V}_{\ell,\mathrm{zero}}$ inside $\partial\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}$ needs to be blown up differently.

For this reason, the stratification we will consider on $\partial\mathbb{C}^{m+n}_{w,\mathrm{sing}}$ is as follows. If $\partial\overline{V}_{\ell,\mathrm{zero}}=\emptyset$, the closed strata will be given by $\partial\overline{V}_{\mathfrak{q}}:=\overline{V}_{\mathfrak{q}}\cap\partial\overline{\mathbb{C}^{m+n}_{w}}$ for $\{0\}\subsetneq\mathfrak{q}\subset\{0,\ldots,N\}$, where $\overline{V}_{\mathfrak{q}}$ is the closure of $V_{\mathfrak{q}}$ in $\overline{\mathbb{C}^{m+n}_{w}}$. If instead $\partial\overline{V}_{\ell,\mathrm{zero}}\neq\emptyset$, we will also consider for $\{0\}\subsetneq\mathfrak{q}\subset\{0,\ldots,N\}$ the closed strata

$$\partial\overline{V}_{\mathfrak{q}}\cap\partial\overline{V}_{\ell,\mathrm{zero}}=\partial\overline{V}_{\mathfrak{q}\setminus\{0\}}.$$

In other words, when $\partial\overline{V}_{\ell,\mathrm{zero}}\neq\emptyset$, the closed strata of $\partial\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}$ are given by

$$\partial\overline{V}_{\mathfrak{q}}\quad\text{for non-empty subsets}\quad\mathfrak{q}\subset\{0,\ldots,N\}\quad\text{with}\quad\mathfrak{q}\neq\{0\}.$$

In both cases, the partial order on strata is given by inclusion,

$$\partial\overline{V}_{\mathfrak{q}}\le\partial\overline{V}_{\mathfrak{p}}\quad\Longleftrightarrow\quad\partial\overline{V}_{\mathfrak{q}}\subseteq\partial\overline{V}_{\mathfrak{p}}\quad\Longleftrightarrow\quad\mathfrak{q}\subseteq\mathfrak{p}.\tag{5.6}$$

If $\partial\overline{V}_{\ell,\mathrm{zero}}\neq\emptyset$, that is, if $\ell>0$ and $N>1$, then by Lemma 4.7, we can unambiguously consider the weighted iterated blow-up

$$X:=[\overline{\mathbb{C}^{m+n}_{w}};\{\partial V_{\mathfrak{q}}\mid\mathfrak{q}\subset\{1,\ldots,N\},\ \mathfrak{q}\neq\emptyset\}]_{w}\tag{5.7}$$

provided we blow-up the subspaces $\partial\overline{V}_{\mathfrak{q}}$ in an order compatible with the partial order (5.6). If instead $\partial V_{\ell,\mathrm{zero}}=\emptyset$, that is, if instead $\ell=0$, we simply set

$$X:=\overline{\mathbb{C}^{m+n}_{w}}.\tag{5.8}$$

In either case, let $H_{\max}$ be the boundary hypersurface of X corresponding to (the lift of) $\partial\overline{\mathbb{C}^{m+n}_{w}}$. By Lemmas 4.5 and 4.7, $H_{\max}$ comes with a natural linear equivalence class of collar neighborhoods in the sense of Definition 4.3. Let $x_{\max}$ be a choice of boundary defining function for $H_{\max}$ compatible with this linear equivalence class of collar neighborhoods. Let $\widetilde{X}$ be the manifold with corners, which, as a topological space, is identified with X, but with smooth functions on $\widetilde{X}$ corresponding to smooth functions on $X\setminus H_{\max}$ having a smooth expansion at $H_{\max}$ in integer powers of $x_{\max}^{\frac{d-\ell}{d}}$ (instead of integer powers of $x_{\max}$). Since we require $x_{\max}$ to be compatible with the natural linear equivalence class of collar neighborhoods of $H_{\max}$, notice that $\widetilde{X}$ is well-defined in that it does not depend on the choice of $x_{\max}$. Let us denote by $\widetilde{H}_{\max}$ the boundary hypersurface $H_{\max}$ seen as a boundary hypersurface of $\widetilde{X}$. Let $\widetilde{x}_{\max}:=x_{\max}^{\frac{d-\ell}{d}}$ be the corresponding boundary defining function. Let $\widetilde{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}$ be the closure of $\mathbb{C}^{m+n}_{w,\mathrm{sing}}$ in $\widetilde{X}$. For $\{0\}\subsetneq\mathfrak{q}\subset\{0,\ldots,N\}$, let $\widetilde{V}_{\mathfrak{q}}$ be the closure of $V_{\mathfrak{q}}$ in $\widetilde{X}$. Then the lift of the strata $\partial\overline{V}_{\mathfrak{q}}$ of $\partial\overline{\mathbb{C}^{m+n}_{w,\mathrm{sing}}}$ to $\widetilde{H}_{\max}$ for $\{0\}\subsetneq\mathfrak{q}\subset\{0,\ldots,N\}$ are given by

$$\widetilde{V}_{\mathfrak{q}}\cap\widetilde{H}_{\max}.$$

Relying again on Lemma 4.7, we can then consider the weighted iterated blow-up

$$\widehat{X}:=[\widetilde{X};\{\widetilde{V}_{\mathfrak{q}}\cap\widetilde{H}_{\max}\mid\{0\}\subsetneq\mathfrak{q}\subset\{0,\ldots,N\}\}]_{\widetilde{w}}$$

provided the blow-ups are performed in an order compatible with the partial ordering of the strata, where the lift of the coordinate $\omega_{q,j}$ has weight $\widetilde{w}_{q,j}=w_{q,j}$, while (the lift of) the boundary defining function $\widetilde{x}_{\max}$ has weight 1. When $\nu:=\frac{\ell}{d}$ is positive, the set $\mathcal{M}_1(\widehat{X})$ of boundary hypersurfaces of $\widehat{X}$ decomposes as

$$\mathcal{M}_1(\widehat{X})=M_{1,0}(\widehat{X})\cup M_{1,\nu}(\widehat{X}),\tag{5.9}$$

where $\mathcal{M}_{1,0}(\widehat{X})$ is the set of boundary hypersurfaces corresponding to the lift of boundary hypersurfaces of $\widetilde{X}$ coming from blow-ups of the strata $\partial\overline{V}_{\mathfrak{q}}\cap\partial\overline{V}_{\ell,\mathrm{sing}}$ for subsets $\{0\}\subsetneq\mathfrak{q}\subset\{0,\ldots,N\}$, while $\mathcal{M}_{1,\nu}(\widehat{X})$ corresponds to the remaining boundary hypersurfaces. Notice that $\mathcal{M}_{1,0}(\widehat{X})$ will be empty when $\partial\overline{V}_{\ell,\mathrm{zero}}$ is. When $\nu=\frac{\ell}{d}=0$, $\partial\overline{V}_{\ell,\mathrm{zero}}$ is empty, but it will be convenient in this case to use the notation

$$\mathcal{M}_{1,\nu}(\widehat{X})=\mathcal{M}_{1,0}(\widehat{X}):=\mathcal{M}_1(\widehat{X})\quad\text{if }\nu=\frac{\ell}{d}=0.$$

For each blow-up performed to construct $\widehat{X}$, notice that the corresponding blow-down map induces a fiber bundle on the corresponding boundary hypersurface. On the other hand, on the lift $\widehat{H}_{\max}$ of the boundary hypersurface $\partial\overline{\mathbb{C}^{m+n}_{w}}$ to $\widehat{X}$, the natural fiber bundle we consider is that given by the identity map. All these fiber bundles combine to confer $\widehat{X}$ with an iterated fibration structure. In other words, $\widehat{X}$ is naturally a QAC-manifold with fibered corners. Given the order in which we blowed up strata, one important feature of the induced partial order on $\mathcal{M}_1(\widehat{X})$ when $\nu>0$ is that if $H\in\mathcal{M}_{1,0}(\widehat{X})$ and $G\in\mathcal{M}_{1,\nu}(\widehat{X})$ are such that $H\cap G\neq\emptyset$, then $H<G$.

Let $\widehat{C}_\epsilon$ be the closure of $C_\epsilon\subset\mathbb{C}^{m+n}_w$ in $\widehat{X}$. For each $\widehat{H}\in\mathcal{M}_1(\widehat{X})$, there is a corresponding boundary hypersurface $\widehat{H}_\epsilon:=\widehat{H}\cap\widehat{C}_\epsilon$ of $\widehat{C}_\epsilon$. For $\epsilon=0$, C_0 is a singular affine variety and $\widehat{H}_0$ will be singular as well. Since the base $S_{\widehat{H}}$ of the fiber bundle $\phi_{\widehat{H}}:\widehat{H}\to S_{\widehat{H}}$ corresponds to a resolution of a singular stratum of $\partial\overline{C}_0$, notice that ϕ_H restricts to $\widehat{C}_0$ and to $\widehat{C}_\epsilon$ for $\epsilon\neq 0$ to induce a fiber bundle

$$\phi_{\widehat{H}_\epsilon}:\widehat{H}_\epsilon\to S_{\widehat{H}}. \tag{5.10}$$

We need to distinguish two situations. First, if $\widehat{H}\in\mathcal{M}_{1,0}(\widehat{X})$ corresponds to the subspace $V_{\mathfrak{q}}$ for $\mathfrak{q}\subset\{1,\ldots,N\}$ with $\mathfrak{q}\neq\emptyset$, then on the interior of $\widehat{H}$, the interior of the fibers of $\phi_{\widehat{H}}:\widehat{H}\to S_{\widehat{H}}$ are naturally identified with the complementary subspace

$$V_{\mathfrak{q}}^\perp=V_{\mathfrak{q}^c}$$

with natural coordinates given by z_q for $q\in\mathfrak{q}^c$. Intuitively, this corresponds to the fact that the weighted blow-up of the stratum $\mathfrak{s}_{\widehat{H}}$ corresponding to $\widehat{H}$ 'undoes' the radial compactification along $V_{\mathfrak{q}}$ in the directions transverse to $V_{\mathfrak{q}}$. For $\omega_{\mathfrak{q}}\in\partial\overline{V}_{\mathfrak{q}}$ corresponding to an interior point of $S_{\widehat{H}}$, the interior of the fiber $\phi_{\widehat{H}}(\omega_{\mathfrak{q}})$ corresponds to $V_{\mathfrak{q}^c}$, with coordinate $z_{\mathfrak{q}^c}$. Such a fiber will have a non-empty intersection with $\widehat{C}_\epsilon$ provided

$$P_{q,i}(\omega_q)=\xi^d\epsilon Q_{q,i}(z_0)\quad\text{for}\quad q\in\mathfrak{q} \tag{5.11}$$

at $\xi=0$, that is, provided

$$P_{q,i}(\omega_q)=0\quad\text{for}\quad q\in\mathfrak{q}, \tag{5.12}$$

where ω_q is the component of $\omega_{\mathfrak{q}}$ in $V_q\subset V_{\mathfrak{q}}$. The interior of the fiber of $\phi^{-1}_{\widehat{H}_\epsilon}(\omega_{\mathfrak{q}})$ is then the affine variety $W_{\mathfrak{q}^c,\epsilon}\subset V_{\mathfrak{q}^c}$ of (1.17) with $V_{\mathfrak{q}^c}$ seen as the interior of $\phi^{-1}_{\widehat{H}}(\omega_{\mathfrak{q}})$. In particular, by Assumption 1.2, the interior of $\phi^{-1}_{\widehat{H}_\epsilon}(\omega_{\mathfrak{q}})$ is smooth for $\epsilon\in\mathbb{C}\setminus\{0\}$ close to zero.

If instead $\widehat{H}\in\mathcal{M}_{1,\nu}(\widehat{X})$ (with $\nu=\frac{\ell}{d}>0$) corresponds to the subspace $V_{\mathfrak{q}}$ with subset $\{0\}\subsetneq\mathfrak{q}\subset\{0,\ldots,N\}$, then, in the interior of $\widehat{H}$, the interiors of the fibers of $\phi_{\widehat{H}}:\widehat{H}\to S_{\widehat{H}}$ are again identified with $V_{\mathfrak{q}^c}$, this time however using the rescaled coordinates

$$\zeta_{q,j}:=\frac{\omega_{q,j}}{\xi^{\frac{d-\ell}{d}w_{q,j}}}=\frac{\xi^{w_{q,j}}z_{q,j}}{\xi^{\frac{d-\ell}{d}w_{q,j}}}=\xi^{\frac{\ell}{d}w_{q,j}}z_{q,j}\quad\text{for } q\in\mathfrak{q}^c,\ j\in\{1,\ldots,m_q+n_q\}. \tag{5.13}$$

Again, such a fiber $\phi^{-1}_{\widehat{H}}(\omega_{\mathfrak{q}})$ will have a non-empty intersection with $\widehat{C}_\epsilon$ provided

$$P_{q,i}(\omega_q)=0\quad\text{for}\quad q\in\mathfrak{q}\setminus\{0\}. \tag{5.14}$$

Correspondingly, in terms of the coordinates (5.13), the interior of $\phi_{\widehat{H}_\epsilon^{-1}}(\omega_{\mathfrak{q}})$ is the affine variety

$$W^\perp_{\mathfrak{q},\omega_{\mathfrak{q}},\epsilon}=W_{\mathfrak{q}^c,\omega_{\mathfrak{q}},\epsilon}=\{\zeta_{\mathfrak{q}^c}\in V_{\mathfrak{q}^c}\mid P_{q,i}(\zeta_q)=\epsilon[Q_{q,i}](\omega_0),\quad q\in\mathfrak{q}^c,\ i\in\{1,\ldots,n_q\}\} \tag{5.15}$$

corresponding to (1.18). Again, by Assumption 1.2, the interior of $\phi^{-1}_{\widehat{H}_\epsilon}(\omega_{\mathfrak{q}})$ will be smooth when $\epsilon\neq 0$ is sufficiently close to zero. In fact, in both cases, the interior of $\phi^{-1}_{\widehat{H}_\epsilon}(\omega_{\mathfrak{q}})$ is a smoothing of the cone $W_{\mathfrak{q}^c}$ identified with the interior of $\phi^{-1}_{\widehat{H}_0}(\omega_{\mathfrak{q}})$.

Now, in both cases, $\phi^{-1}_{\widehat{H}}(\omega_{\mathfrak{q}})$ provides a compactification for $V_{\mathfrak{q}}$ in the same way that $\widehat{X}$ provides a compactification for $\mathbb{C}^{m+n}_w$. In the case where $\widehat{H}\in\mathcal{M}_{1,\nu}(\widehat{X})$, this corresponds to such a compactification for $\mathbb{C}^{m+n}_w$ when $\nu=0$. Moreover, the closure of the interior of $\phi^{-1}_{\widehat{H}_\epsilon}(\omega_{\mathfrak{q}})$ in $\phi^{-1}_{\widehat{H}}(\omega_{\mathfrak{q}})$ is precisely $\phi^{-1}_{\widehat{H}_\epsilon}(\omega_{\mathfrak{q}})$. For $\epsilon\in\mathbb{C}\setminus\{0\}$, this discussion can be extended to show that $\widehat{C}_\epsilon$ is a manifold with corners inheriting from $\widehat{X}$ a natural iterated fibration structure. However, as a manifold with corners, it will typically not be of class $\mathcal{C}^\infty$, but of class $\mathcal{C}^{\lfloor d\rfloor}$ or lower. For instance, if d is not an integer and the polynomials $Q_{q,i}$ are all homogeneous, then it will be of class $\mathcal{C}^{\lfloor d\rfloor}$. Nevertheless, by restriction from $\widehat{X}$, there is on $C_\epsilon=\widehat{C}_\epsilon\setminus\partial\widehat{C}_\epsilon$ a natural ring of 'smooth' functions $\mathcal{C}^\infty(\widehat{C}_\epsilon)$, a ring $\mathcal{A}_{\mathrm{phg}}(\widehat{C}_\epsilon)$ of bounded polyhomogeneous functions and a ring of $\mathfrak{n}\,$Qb-smooth functions $\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(\widehat{C}_\epsilon)$.

Now, the set of boundary hypersurfaces of $\widehat{C}_\epsilon$ is in bijection with that of $\widehat{X}$. In particular, there is a decomposition

$$\mathcal{M}_1(\widehat{C}_\epsilon) = \mathcal{M}_{1,0}(\widehat{C}_\epsilon) \cup \mathcal{M}_{1,\nu}(\widehat{C}_\epsilon). \tag{5.16}$$

Again, when $\nu = \frac{\ell}{d} > 0$, this is a partition, while when $\nu = 0$, $\mathcal{M}_{1,0}(\widehat{C}_\epsilon) = \mathcal{M}_{1,\nu}(\widehat{C}_\epsilon) = \mathcal{M}_1(\widehat{C}_\epsilon)$. This suggests to consider the function

$$\begin{array}{rccc} \mathfrak{n}: & \mathcal{M}_1(\widehat{C}_\epsilon) & \to & \{0,\nu\} \\ & \widehat{H}_\epsilon & \mapsto & \nu_{\widehat{H}_\epsilon} \end{array} \tag{5.17}$$

given by

$$\nu_{\widehat{H}_\epsilon} := \begin{cases} \nu, & \widehat{H}_\epsilon \in \mathcal{M}_{1,\nu}(\widehat{C}_\epsilon), \\ 0, & \text{otherwise.} \end{cases}$$

By definition of (5.16),

$$\widehat{H}_\epsilon < \widehat{G}_\epsilon \quad \Longrightarrow \quad \nu_{\widehat{H}_\epsilon} \le \nu_{\widehat{G}_\epsilon},$$

so $\mathfrak{n}$ is a weight function in the sense of Definition 2.3. This induces a corresponding weight function $\mathfrak{n} : \mathcal{M}_1(\widehat{X}) \to \{0,\nu\}$. Let $\beta : \widehat{X} \to \overline{\mathbb{C}^{m+n}_w}$ denote the blow-down map and let $u \in \mathcal{C}^\infty(\overline{\mathbb{C}^{m+n}_w})$ be a choice of boundary defining function compatible with the $\mathbb{R}^+$-action near the boundary in the sense that $u(t\cdot z) = \frac{u(z)}{t}$ for $t \in \mathbb{R}^+$ and $z \in \mathbb{C}^{m+n}$ sufficiently large. In other words, choose u to be compatible with the natural linear equivalence class of collar neighborhood of $\overline{\mathbb{C}^{m+n}_w}$. Then

$$\rho^{-1} := \beta^* u \tag{5.18}$$

is an $\mathfrak{n}$-weighted total boundary defining function. Using Lemma 2.6, we readily see that the $\mathfrak{n}$ QAC-equivalence class of the $\mathfrak{n}$-weighted total boundary defining function does not depend on the choice of u, so there is a well-defined notion of $\mathfrak{n}$-warped QAC-metrics on $\overline{\mathbb{C}^{m+n}_w}$. By restriction, this induces a class of metrics on $\widehat{C}_\epsilon$. We will declare those to be **the class of $\mathfrak{n}$-warped QAC-metrics on $\widehat{C}_\epsilon$**. It will be smooth, polyhomogeneous or $\mathfrak{n}$ Qb-smooth if the associated metric on $\widehat{X}$ is. Proceeding in this way, we avoid having to deal with the fact that $\widehat{C}_\epsilon$ is possibly not of class $\mathcal{C}^\infty$. Still, since the differential of the polynomial $P_{q,i} - Q_{q,i}$, seen as a b-differential on $\widehat{X}$, is always at least of class $\mathcal{C}^0$, we can also define ${}^{\mathfrak{n}}T\widehat{C}_\epsilon$ (respectively ${}^{w}T\widehat{C}_\epsilon$ and ${}^{\mathfrak{n}\,\mathrm{Qb}}T\widehat{C}_\epsilon$) by considering the elements of ${}^{\mathfrak{n}}T\widehat{X}|_{\widehat{C}_\epsilon}$ (respectively ${}^{w}T\widehat{X}|_{\widehat{C}_\epsilon}$ and ${}^{\mathfrak{n}\,\mathrm{Qb}}T\widehat{X}|_{\widehat{C}_\epsilon}$) that are in the kernels of the differentials of the polynomials $(P_{q,i} - Q_{q,i})$ for all q and i. We say that $\widehat{X}$ and $\widehat{C}_\epsilon$ are **$\mathfrak{n}$ QAC-compactifications** for respectively $\mathbb{C}^{m+n}$ and C_ϵ. Similarly, we have natural $\mathfrak{n}$ QAC-compactifications for V_q, $W_{\mathfrak{q},\epsilon}$ and $W^\perp_{\mathfrak{q},\omega_\mathfrak{q},\epsilon} = W_{\mathfrak{q}^c,\omega_\mathfrak{q},\epsilon}$ in (1.17) and (1.18) that we denote by $\widehat{V}_q$, $\widehat{W}_{\mathfrak{q},\epsilon}$ and $\widehat{W}^\perp_{\mathfrak{q},\omega_\mathfrak{q},\epsilon} = \widehat{W}_{\mathfrak{q}^c,\omega_\mathfrak{q},\epsilon}$ respectively.

We are interested in examples of warped QAC-metrics that are Kähler. To study those and see in particular that they exist, it will be helpful to have complex coordinates adapted to the geometry. Near $\widehat{H} \in \mathcal{M}_{1,0}(\widehat{X})$ corresponding to $V_\mathfrak{q}$, we see from the discussion above that we can take the coordinates

$$z_q, \quad q \in \mathfrak{q}^c$$

in the fibers of $\phi_{\widehat{H}} : \widehat{H} \to S_{\widehat{H}}$, while on the base, instead of using the real coordinates ω_q, we can use the holomorphic coordinates

$$\varpi_{q,j} = \xi^{w_{q,j}} z_{q,j}, \quad q \in \mathfrak{q}, \quad j \in \{1,\ldots,m_q+n_q\}, \tag{5.19}$$

with $(\xi, \varpi_\mathfrak{q})$ corresponding to $z_\mathfrak{q}$ with $z_{q,j} = \xi^{-w_{q,j}}\varpi_{q,j}$, where ξ is now taking values in a sector of the complex plane, so that its logarithm and its complex powers can be well-defined. To obtain a holomorphic coordinate chart near $\widehat{H}$, we can then take z_q for $q \in \mathfrak{q}^c$, the coordinate ξ and all the coordinates $\varpi_{q,j}$ for $q \in \mathfrak{q}$ and $j \in \{1,\ldots,m_q,n_q\}$ (except one such $\varpi_{q,j}$ that we declare to be equal to one). In this case, away from the boundary of $\widehat{H}$, the 1-forms

$$\frac{d\xi}{\xi^2}, \frac{d\varpi_{q,i}}{\xi}, dz_{p,j}, \quad q \in \mathfrak{q},\ i \in \{1,\ldots,m_q+n_q\}, \quad p \in \mathfrak{q}^c,\ j \in \{1,\ldots,m_p+n_p\}, \tag{5.20}$$

except one of the $d\varpi_{q,i}$ omitted, form, together with their complex conjugates, a local basis of sections of ${}^{\mathfrak{n}}T^*\widehat{X} \otimes_\mathbb{R} \mathbb{C}$, giving a complex analog of (2.8) with $k=1$. Since $\mathfrak{n}(\widehat{H}) = 0$, we are also in the setting of (2.11) with $\nu_H = 0$, so it is also a local basis of sections of ${}^{w}T^*\widehat{X} \otimes \mathbb{C}$.

If instead $\widehat{H} \in \mathcal{M}_{1,\nu}(\widehat{X})$ corresponds to $V_{\mathfrak{q}}$ with $\{0\} \subsetneq \mathfrak{q} \subset \{0, 1, \ldots, N\}$ and $\nu > 0$, then again regarding ξ as taking values in a sector of the complex plane where its logarithm can be well-defined, we can regard the coordinate (5.13) as holomorphic. Using again the coordinates (5.19) on the base with one $\varpi_{0,j}$ omitted, as well as (5.13), we get holomorphic coordinates near $\widehat{H}$, and in this case the 1-forms

$$\xi^{\nu}\frac{d\xi}{\xi^2}, \xi^{\nu}\frac{d\varpi_{q,i}}{\xi}, d\zeta_{p,j}, \quad q \in \mathfrak{q},\ i \in \{1, \ldots, m_q + n_q\}, \quad p \in \mathfrak{q}^c,\ j \in \{1, \ldots, m_p + n_p\}, \tag{5.21}$$

with one $\xi^{\nu}\frac{d\varpi_{0,i}}{\xi}$ omitted, combined with their complex complex conjugates, form a local basis of sections of ${}^{\mathfrak{n}}T^*\widehat{X} \otimes_{\mathbb{R}} \mathbb{C}$ and are a complex analog of (2.8) with $k = 1$. Correspondingly,

$$\frac{d\xi}{\xi^2}, \frac{d\varpi_{q,i}}{\xi}, \xi^{-\nu}d\zeta_{p,j}, \quad q \in \mathfrak{q},\ i \in \{1, \ldots, m_q + n_q\}, \quad p \in \mathfrak{q}^c,\ j \in \{1, \ldots, m_p + n_p\}, \tag{5.22}$$

with one $\frac{d\varpi_{0,i}}{\xi}$ omitted, combined with their complex conjugates, form a local basis of sections of ${}^{w}T^*\widehat{X} \otimes_{\mathbb{R}} \mathbb{C}$.

More generally, we can introduce holomorphic coordinates near an intersection of boundary hypersurfaces as follows. Let $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_k \subset \{0, \ldots, N\}$ be a sequence of embedded subsets and let $\widehat{H}_i \in \mathcal{M}_1(\widehat{X})$ be the boundary hypersurface corresponding to $V_{\mathfrak{q}_i}$ so that

$$\widehat{H}_i < \widehat{H}_j \quad \Longrightarrow \quad i < j.$$

In particular, the intersection $\cap_{i=1}^k \widehat{H}_i$ is not empty. Pick as before $\xi_1 = \xi$ to be the coordinate of the $\mathbb{R}^+$-action of $V_{\mathfrak{q}_1}$ seen as taking values in some sector of the complex plane so that its logarithm can be well-defined. For $q \in \mathfrak{q}_1$, consider again the homogeneous coordinates

$$\varpi_{q,j} = \xi^{w_{q,j}} z_{q,j}, \quad q \in \mathfrak{q}_1,\ j \in \{1, \ldots, m_q + n_q\}. \tag{5.23}$$

In the fibers of $\widehat{H}_1$, we can initially consider the coordinates $z_{\mathfrak{q}_1^c}$ if $0 \notin \mathfrak{q}_1$ or else the rescaled coordinates $\zeta_{\mathfrak{q}_1^c}$ of (5.13) if $0 \in \mathfrak{q}_1$. In terms of those, we can take ξ_2 to be the coordinates of the $\mathbb{R}^+$-action of $V_{\mathfrak{q}_2 \setminus \mathfrak{q}_1}$ (seen as taking values in some sector of the complex plane) and define corresponding homogenous coordinates $\varpi_{q,j}$. Iterating this construction, we obtain holomorphic coordinates

$$\xi_1, \varpi_{\mathfrak{q}_1}, \ldots, \xi_k, \varpi_{\mathfrak{q}_k}, \zeta_k \tag{5.24}$$

with ζ_k denoting possibly rescaled coordinates in the fibers of $\widehat{H}_k$, where for each $\varpi_{\mathfrak{q}_i}$, one $\varpi_{q,j}$ with $q \in \mathfrak{q}_i \setminus \mathfrak{q}_{i-1}$ is omitted, where we use the convention that $\mathfrak{q}_0 = \emptyset$. More precisely, if $0 \in \mathfrak{q}_k$, let k' be the smallest integer such that $0 \in \mathfrak{q}_{k'}$, and otherwise set $k' = k + 1$. Then ξ_i is the holomorphic coordinate corresponding to the $\mathbb{R}^+$-action in $V_{\mathfrak{q}_i \setminus \mathfrak{q}_{i-1}}$, while

$$\varpi_{q,j} = \begin{cases} \xi_i^{w_{q,j}} z_{q,j}, & q \in (\mathfrak{q}_i \setminus \mathfrak{q}_{i-1}) \subset \mathfrak{q}_{k'}, \\ \xi_i^{w_{q,j}} (\xi_{k'}^{\nu w_{q,j}} z_{q,j}), & q \in \mathfrak{q}_i \setminus \mathfrak{q}_{i-1}, \quad k' < i, \end{cases} \tag{5.25}$$

and $\zeta_k = \{z_{q,j} \mid q \in \mathfrak{q}_k^c,\ j \in \{1, \ldots, m_q + n_q\}\}$ if $k' = k + 1$ and otherwise corresponds to the rescaled coordinates

$$\xi_{k'}^{\nu w_{q,j}} z_{q,j} \quad \text{for } q \in \mathfrak{q}_k^c, \quad j \in \{1, \ldots, m_q + n_q\}.$$

When $k' \le k$, we will further assume that in (5.24), the coordinate in $\varpi_{\mathfrak{q}_{k'}}$ that is omitted is $\varpi_0 = \xi_{k'} z_0$. This will ensure that $\xi_{k'}$ can be seen as a holomorphic coordinate coming from the $\mathbb{R}^+$-action of $V_0 \subset V_{\mathfrak{q}_{k'} \setminus \mathfrak{q}_{k'-1}}$.

In terms of (5.24), when $k' \le k$, a local basis of sections of ${}^{\mathfrak{n}}T^*\widehat{X} \otimes_{\mathbb{R}} \mathbb{C}$ is given by

$$\xi_{k'}^{\nu}\frac{d\xi_1}{\xi_1^2}, \xi_{k'}^{\nu}\frac{d\varpi_{\mathfrak{q}_1}}{\xi_1}, \ldots, \xi_{k'}^{\nu}\frac{d\xi_{k'}}{\xi_{k'}^2}, \xi_{k'}^{\nu}\frac{d\varpi_{\mathfrak{q}_{k'}}}{\xi_{k'}}, \frac{d\xi_{k'+1}}{\xi_{k'+1}^2}, \frac{d\varpi_{\mathfrak{q}_{k'+1}}}{\xi_{k'+1}}, \ldots, \frac{d\xi_k}{\xi_k^2}, \frac{d\varpi_{\mathfrak{q}_k}}{\xi_k}, d\zeta_k \tag{5.26}$$

and their complex conjugates, where to lighten notation, $d\varpi_{\mathfrak{q}_i}$ stands for

$$\{d\varpi_{q,j} \mid q \in \mathfrak{q}_i \setminus \mathfrak{q}_{i-1},\ j \in \{1, \ldots, m_q + n_q\}\}$$

with one $d\varpi_{q,j}$ omitted for each $d\varpi_{\mathfrak{q}_i}$. From (5.26), we see that the 1-forms

$$\frac{d\xi_1}{\xi_1^2}, \frac{d\varpi_{\mathfrak{q}_1}}{\xi_1}, \ldots, \frac{d\xi_{k'}}{\xi_{k'}^2}, \frac{d\varpi_{\mathfrak{q}_{k'}}}{\xi_{k'}}, \xi_{k'}^{-\nu}\frac{d\xi_{k'+1}}{\xi_{k'+1}^2}, \xi_{k'}^{-\nu}\frac{d\varpi_{\mathfrak{q}_{k'+1}}}{\xi_{k'+1}}, \ldots, \xi_{k'}^{-\nu}\frac{d\xi_k}{\xi_k^2}, \xi_{k'}^{-\nu}\frac{d\varpi_{\mathfrak{q}_k}}{\xi_k}, \xi_{k'}^{-\nu}d\zeta_k, \tag{5.27}$$

together with their complex conjugates, form a local basis of ${}^{w}T\widehat{X} \otimes_{\mathbb{R}} \mathbb{C}$.

If instead $k' = k+1$, then instead of (5.26) we see that

$$\frac{d\xi_1}{\xi_1^2}, \frac{d\varpi_{\mathfrak{q}_1}}{\xi_1}, \ldots, \frac{d\xi_k}{\xi_k^2}, \frac{d\varpi_{\mathfrak{q}_k}}{\xi_k}, d\zeta_k, \tag{5.28}$$

together with their complex conjugates, form a local basis of sections of ${}^{\mathfrak{n}}T^*\widehat{X} \otimes_{\mathbb{R}} \mathbb{C}$ and ${}^{w}T\widehat{X} \otimes_{\mathbb{R}} \mathbb{C}$. Notice that the coordinates (5.24) are only valid in the interior of $\widehat{X}$ and are not coordinates on $\widehat{X}$ as a manifold with corners. However, as in (2.8), the sections (5.26) and (5.28) naturally extend to the boundary of $\widehat{X}$ as sections of ${}^{\mathfrak{n}}T^*\widehat{X} \otimes \mathbb{C}$.

Lemma 5.1. *The complex structure J of $\mathbb{C}^{m+n}$ naturally extends to a section*

$$J \in \mathcal{C}^\infty(\widehat{X}; \operatorname{End}({}^{w}T\widehat{X})) \quad \textit{with} \quad J^2 = -\operatorname{Id}.$$

Proof. This is clear from the local descriptions (5.27) and (5.28). □

Remark 5.2. *Since* $\mathfrak{n}\,\mathrm{Qb}$*-metrics and* $\mathfrak{n}\,\mathrm{QAC}$*-metrics are conformal to* $\mathfrak{n}$*-warped* QAC*-metrics, notice that in fact*

$$J \in \mathcal{C}^\infty(\widehat{X}; \operatorname{End}({}^{w}T\widehat{X})) = \mathcal{C}^\infty(\widehat{X}; \operatorname{End}({}^{\mathfrak{n}}T\widehat{X})) = \mathcal{C}^\infty(\widehat{X}; \operatorname{End}({}^{\mathfrak{n}\,\mathrm{Qb}}T\widehat{X})).$$

By restriction to $\widehat{C}_\epsilon$, Lemma 5.1 shows that the complex structure J_ϵ of C_ϵ extends to a section

$$J_\epsilon \in \mathcal{C}^\infty(\widehat{C}_\epsilon, \operatorname{End}({}^{w}T\widehat{C}_\epsilon)) \tag{5.29}$$

which is 'smooth' up to the boundary in the sense that it comes from the restriction of J to $\widehat{C}_\epsilon$.

To show that a Kähler metric on C_ϵ is a warped QAC-metric, Lemma 5.1 indicates that it suffices to check that its Kähler form is an element of $\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(\widehat{C}_\epsilon; \Lambda^{1,1}({}^{w}T^*\widehat{C}_\epsilon \otimes_{\mathbb{R}} \mathbb{C}))$. In the remainder of this section, we will construct examples of Kähler warped QAC-metrics with $\partial\overline{\partial}$-exact Kähler forms, that is, with Kähler form $\omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial}U$ for some potential U. To do so, as in [18, Corollary 4.4], we will use a convexity argument of van Coevering [40, Lemma 4.3] to extend a model at infinity to a Kähler form on the entire space.

Lemma 5.3. *Let $\widehat{Z}_\epsilon$ denote $\widehat{C}_\epsilon$, the* $\mathfrak{n}\,\mathrm{QAC}$*-compactification $\widehat{W}_{\mathfrak{q},\epsilon}$ of* (1.17) *or the* $\mathfrak{n}\,\mathrm{QAC}$*-compactification $\widehat{W}^{\perp}_{\mathfrak{q},\omega_{\mathfrak{q}},\epsilon}$ of* (1.18) *for $\epsilon \neq 0$. Suppose that u_ϵ is a smooth positive proper function on $Z_\epsilon := \widehat{Z}_\epsilon \setminus \partial\widehat{Z}_\epsilon$ such that*

$$\omega_\epsilon = \frac{\sqrt{-1}}{2}\partial\overline{\partial}u_\epsilon$$

is the Kähler form of a Kähler $\mathfrak{n}$*-warped* QAC*-metric $g_\epsilon \in \mathcal{C}^\infty(\overline{\widehat{Z}_\epsilon \setminus K_\epsilon}; S^2({}^{w}T^*\widehat{Z}_\epsilon))$ on the complement of a compact set $K_\epsilon \subset Z_\epsilon$. Then there exists a potential $\widetilde{u}_\epsilon$ defined on Z_ϵ and agreeing with u_ϵ outside a compact set such that*

$$\widetilde{\omega}_\epsilon = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{u}_\epsilon$$

is the Kähler form of a global Kähler $\mathfrak{n}$*-warped* QAC*-metric $g_\epsilon \in \mathcal{C}^\infty(\widehat{Z}_\epsilon; S^2({}^{w}T^*\widehat{Z}_\epsilon))$ on Z_ϵ.*

Proof. By assumption $\frac{\sqrt{-1}}{2}\partial\overline{\partial}u_\epsilon > 0$ outside the compact set K_ϵ. Since u_ϵ is proper, this can be formulated as saying that there exists a constant $C > 0$ such that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}u_\epsilon > 0$ for all $p \in Z_\epsilon$ such that $u_\epsilon(p) > C$. Let $\eta \in \mathcal{C}^\infty(\mathbb{R})$ be a non-decreasing convex function such that

$$\eta(t) = \begin{cases} t, & \text{if } t \geq C+2, \\ C + \frac{3}{2}, & \text{if } t \leq C+1. \end{cases} \tag{5.30}$$

For such a choice, $\eta \circ u_\epsilon$ agrees with u_ϵ outside a compact set. Since $\eta', \eta'' \geq 0$, we see also that

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\eta \circ u_\epsilon = \frac{\sqrt{-1}}{2}\eta''(u_\epsilon)\partial u_\epsilon \wedge \overline{\partial}u_\epsilon + \frac{\sqrt{-1}}{2}\eta'(u_\epsilon)\partial\overline{\partial}u_\epsilon \geq \frac{\sqrt{-1}}{2}\eta'(u_\epsilon)\partial\overline{\partial}u_\epsilon \geq 0. \tag{5.31}$$

Now, Z_ϵ is an affine variety in $\mathbb{C}^k$ for some $k \in \mathbb{N}$. Let w_ϵ be the restriction to Z_ϵ of the Euclidean potential $\sum_{i=1}^k |z_i|^2$ on $\mathbb{C}^k$ and let $\phi_\epsilon \in \mathcal{C}^\infty_c(Z_\epsilon)$ be a nonnegative function such that

$$\phi_\epsilon(p) = \begin{cases} 1, & \text{if } u_\epsilon(p) \leq C+2, \\ 0, & \text{if } u_\epsilon \geq C+3. \end{cases}$$

We claim that it suffices to take

$$\widetilde{u}_\epsilon := \eta \circ u_\epsilon + \delta \phi w_\epsilon$$

with $\delta > 0$ sufficiently small. Indeed, by construction, $\widetilde{u}_\epsilon$ is equal to u_ϵ outside a compact set. By (5.31), at a point p where $u_\epsilon(p) < C + 2$,

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{u}_\epsilon \geq \delta \frac{\sqrt{-1}}{2}\partial\overline{\partial} w_\epsilon > 0,$$

while at a point p where $u_\epsilon > C + 3$,

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{u}_\epsilon = \frac{\sqrt{-1}}{2}\partial\overline{\partial} u_\epsilon > 0.$$

On the other hand, in the compact region where

$$C + 2 \leq u_\epsilon(p) \leq C + 3,$$

notice that

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}(\eta \circ u_\epsilon) = \frac{\sqrt{-1}}{2}\partial\overline{\partial} u_\epsilon > 0,$$

so taking $\delta > 0$ sufficiently small, we can ensure that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{u}_\epsilon > 0$ in this region as well. □

If we set

$$r_{\mathfrak{q}} := \sqrt{\sum_{q \in \mathfrak{q}} r_q^2},$$

then

$$\omega_{W_{\mathfrak{q}}} := \frac{\sqrt{-1}}{2}\partial\overline{\partial} r_{\mathfrak{q}}^2$$

is the Kähler form of the natural Calabi-Yau cone metric on $W_{\mathfrak{q}}$. For each q, consider a smooth extension r'_q of r_q to $V_q \setminus \{0\}$ as a homogeneous positive function of degree 1 with respect to the $\mathbb{R}^+$-action. Cut it off near $\{0\}$ using a partition of unity and denote by $\widetilde{r}_q$ the resulting function. This function is smooth, but no longer homogeneous. However, it agrees with r'_q outside a compact set. More generally, let

$$\widetilde{r}_{\mathfrak{q}} = \sqrt{\sum_{q \in \mathfrak{q}} \widetilde{r}_q^2}$$

be the corresponding function on $V_{\mathfrak{q}}$. For $\mathfrak{q} = \{0, \ldots, N\}$, we will also use the notation

$$\widetilde{r} = \sqrt{\sum_{q=0}^{N} \widetilde{r}_q^2}.$$

Lemma 5.4. *The function $\frac{1}{\widetilde{r}}$ is an $\mathfrak{n}$-weighted total boundary defining function $\mathfrak{n}\,$QAC-equivalent to that of* (5.18).

Proof. Let $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_k \subset \{0, \ldots, N\}$ be a sequence of embedded subsets and let $\widehat{H}_i \in \mathcal{M}_1(\widehat{X})$ be the boundary hypersurface corresponding to $V_{\mathfrak{q}_i}$. Then near a compact region of the interior of $\widehat{H}_1 \cap \ldots \cap \widehat{H}_k$, the function $\widetilde{r}_{\mathfrak{q}_k}$ is homogeneous of degree 1, so $\frac{1}{\widetilde{r}_{\mathfrak{q}_k}}$ is clearly $\mathfrak{n}\,$QAC-equivalent to (5.18). On the other hand,

$$\frac{1}{\widetilde{r}} = \frac{1}{\sqrt{\widetilde{r}_{\mathfrak{q}_k}^2 + \widetilde{r}_{\mathfrak{q}_k^c}^2}} = \frac{1}{\widetilde{r}_{\mathfrak{q}_k}} \frac{1}{\sqrt{1 + \frac{\widetilde{r}_{\mathfrak{q}_k^c}^2}{\widetilde{r}_{\mathfrak{q}_k}^2}}}$$

and clearly $\frac{\widetilde{r}_{\mathfrak{q}_k^c}^2}{r_{\mathfrak{q}}^2} = 0$ on $\widehat{H}_1, \ldots, \widehat{H}_{k-1}$ and $\widehat{H}_k$. By Lemma 2.6, $\frac{1}{\widetilde{r}}$ is $\mathfrak{n}\,$QAC-equivalent to $\frac{1}{\widetilde{r}_{\mathfrak{q}_k}}$ and (5.18) near a compact region of the interior of $\widehat{H}_1 \cap \ldots \cap \widehat{H}_k$. Since $\widehat{H}_1 \cap \ldots \cap \widehat{H}_k$ was an arbitrary corner of $\widehat{X}$, the result follows.

□

On $\widehat{C}_0$, notice that $\widetilde{r}^2$ agrees with r^2 near the maximal boundary hypersurface $\widehat{H}_{\max}$ of $\widehat{X}$, so

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2 = \omega_{C_0}$$

on $\widehat{C}_0$ near $\widehat{H}_{\max}$. In fact, in the coordinates (5.24) near $H_{\max}$, so with $H_k = H_{\max}$ and with no coordinate ζ_k, the potential $\widetilde{r}^2$ takes the form

$$\widetilde{r}^2 = \sum_{i=1}^{k'} |\xi_i|^{-2} f_i(\varpi_{\mathfrak{q}_i}) + |\xi_{k'}|^{-2\nu} \sum_{i=k'+1}^{k} f_i(\varpi_{\mathfrak{q}_i}) \tag{5.32}$$

for some smooth functions f_i. Clearly, in terms of (5.27), using the fact that $1-\nu > 0$, we see that

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2 \in \mathcal{C}^\infty(\widehat{X}; \Lambda^{1,1}({}^wT^*\widehat{X} \otimes_{\mathbb{R}} \mathbb{C})).$$

Since $\widehat{C}_\epsilon$ and $\widehat{C}_0$ are tangent to order $\widehat{x}_{\max}^d$ at $\widehat{H}_{\max}$, we see also that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2$ remains positive definite on $\widehat{C}_\epsilon$ near $\widehat{H}_{\max}$, hence defines there a Kähler $\mathfrak{n}$-warped QAC-metric. However, there is no reason a priori for $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2$ to be positive definite everywhere on $\widehat{C}_\epsilon$. Nevertheless, using Lemma 5.3, we will modify $\widetilde{r}^2$ to obtain the potential of a Kähler $\mathfrak{n}$-warped QAC-metric on $\widehat{C}_\epsilon$.

Theorem 5.5. *There exists a smooth positive function ϕ_ϵ on $\widehat{C}_\epsilon$ agreeing with $\widetilde{r}^2$ near $\widehat{H}_{\max}$ such that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_\epsilon$ is the Kähler form of an $\mathfrak{n}$-warped* QAC*-metric on $\widehat{C}_\epsilon$. Moreover, if $\widehat{H}$ corresponds to the subset $\mathfrak{q}$, then in terms of the decomposition $\mathbb{C}^{m+n} = V_{\mathfrak{q}} \times V_{\mathfrak{q}^c}$, near $\widehat{H}$,*

$$\phi_\epsilon = \widetilde{r}_{\mathfrak{q}}^2 + \phi_{\mathfrak{q}^c}(z_{\mathfrak{q}^c}, \overline{z}_{\mathfrak{q}^c}) \tag{5.33}$$

if $H \in \mathcal{M}_{1,0}(\widehat{X})$, where $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\mathfrak{q}^c}(z_{\mathfrak{q}^c}, \overline{z}_{\mathfrak{q}^c})$ is the Kähler form of an $\mathfrak{n}$-warped QAC*-metric on $W_{\mathfrak{q}^c,\epsilon}$. If instead $H \in \mathcal{M}_{1,\nu}(\widehat{X})$ with $\nu > 0$, then*

$$\phi_\epsilon = \widetilde{r}_{\mathfrak{q}}^2 + |z_0|^{2\nu}\phi_{\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c}, \overline{\zeta}_{\mathfrak{q}^c}), \tag{5.34}$$

where $\zeta_{\mathfrak{q}^c}$ are the rescaled coordinates of (5.13) *in $V_{\mathfrak{q}^c}$ and $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c}, \overline{\zeta}_{\mathfrak{q}^c})$ is the Kähler form of a* QAC*-metric on $W_{\mathfrak{q}^c,\omega_{\mathfrak{q}},\epsilon}$.*

Proof. Let $\widehat{H} \in \mathcal{M}_1(\widehat{X})$ be a boundary hypersurface such that $\widehat{H} < \widehat{H}_{\max}$ and

$$\widehat{H} < \widehat{G} \implies \widehat{G} = \widehat{H}_{\max}.$$

The fibers of $\phi_{\widehat{H}} : \widehat{H} \to S_{\widehat{H}}$ are then manifolds with boundary and $\widehat{H} \in \mathcal{M}_{1,\nu}(\widehat{X})$. If $\widehat{H}$ corresponds to the subset $\mathfrak{q} \subset \{0, \ldots, N\}$, then in terms of the decomposition

$$\mathbb{C}^{m+n} = V_{\mathfrak{q}} \times V_{\mathfrak{q}^c},$$

we have that

$$\widetilde{r}^2 = \widetilde{r}_{\mathfrak{q}}^2 + \widetilde{r}_{\mathfrak{q}^c}^2. \tag{5.35}$$

Now, the function $\widetilde{r}_{\mathfrak{q}_c}$ agrees with $r_{\mathfrak{q}^c}$ outside a compact set and so is homogenous there. In terms of the coordinates (5.24) with $k=1$, $\widehat{H}_1 = \widehat{H}$ and such that ξ_1 is the function of the $\mathbb{R}^+$-action on $V_{\{0\}}$, this means that in this region,

$$\widetilde{r}_{\mathfrak{q}_c}(z_{\mathfrak{q}^c}) = |\xi_1|^{-2\nu}\widetilde{r}_{\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c}).$$

By Lemma 5.3, we can find $\phi_{\mathfrak{q}}(\zeta_{\mathfrak{q}^c}, \overline{\zeta}_{\mathfrak{q}^c})$ with $\phi_{\mathfrak{q}^c} = \widetilde{r}_{\mathfrak{q}^c}^2$ outside a compact set such that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c}, \overline{\zeta}_{\mathfrak{q}^c})$ is the Kähler form of a QAC-metric, in fact of an AC-metric, on $W_{\mathfrak{q}^c,\omega_{\mathfrak{q}},\epsilon}$ for each $\omega_{\mathfrak{q}}$ and $\epsilon \neq 0$. Now, replacing $\widetilde{r}_{\mathfrak{q}}^2$ in (5.35) by $|\xi_1|^{-2\nu}\phi_{\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c}, \overline{\zeta}_{\mathfrak{q}^c})$, we obtain a potential

$$\psi_\epsilon := \widetilde{r}_{\mathfrak{q}}^2 + |\xi_1|^{-2\nu}\phi_{\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c}, \overline{\zeta}_{\mathfrak{q}^c})$$

on $\widehat{C}_\epsilon$ which agrees with $\widetilde{r}^2$ near $\widehat{H}_{\max}$ and such that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi_\epsilon$ is the Kähler form of an $\mathfrak{n}$-warped QAC-metric not only near $\widehat{H}_{\max}$, but also near $\widehat{H}$. Indeed, in any coordinates (5.24) near $\widehat{H}$, $\xi_{k'}$ corresponds to the coordinate of the $\mathbb{R}^+$-action on V_0 and we compute that

$$\begin{aligned}\frac{\sqrt{-1}}{2}\partial\overline{\partial}|\xi_{k'}|^{-2\nu}\phi_{\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c},\overline{\zeta}_{\mathfrak{q}^c}) &= |\xi_{k'}|^{-2\nu}\frac{\sqrt{-1}}{2}\Bigg(\partial\overline{\partial}\phi_{\mathfrak{q}^c} + -\nu\partial\phi_{\mathfrak{q}^c}\wedge\overline{\xi}_{k'}\frac{d\overline{\xi}_{k'}}{\overline{\xi}^2_{k'}} - \nu\xi_{k'}\frac{d\xi_{k'}}{\xi^2_{k'}}\wedge\overline{\partial}\phi_{\mathfrak{q}^c} \\ &\qquad + \nu^2\phi_{\mathfrak{q}^c}\xi_{k'}\frac{d\xi_{k'}}{\xi^2_{k'}}\wedge\overline{\xi}_{k'}\frac{d\overline{\xi}_{k'}}{\overline{\xi}^2_{k'}}\Bigg) \\ &= |\xi_{k'}|^{-2\nu}\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\mathfrak{q}^c} + |\xi_{k'}|^{1-\nu}x^{-1}_{\max}\mathcal{A}_{\mathrm{phg}}(\widehat{X};\Lambda^{1,1}({}^wT^*\widehat{X}\otimes_{\mathbb{R}}\mathbb{C})) \\ &= |\xi_{k'}|^{-2\nu}\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\mathfrak{q}^c} + x_{\widehat{H}}\mathcal{A}_{\mathrm{phg}}(\widehat{X};\Lambda^{1,1}({}^wT^*\widehat{X}\otimes_{\mathbb{R}}\mathbb{C})).\end{aligned} \tag{5.36}$$

In particular, since $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\mathfrak{q}^c}$ is a Kähler form on $W_{\mathfrak{q}^c,\omega_{\mathfrak{q}},\epsilon}$, this shows that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi_\epsilon$ is an $\mathfrak{n}$-warped QAC-Kähler form when $x_{\widehat{H}}$ is small enough, that is, for points sufficiently closed to $\widehat{H}$. We can iterate this construction near each boundary hypersurface of $\widehat{C}_\epsilon$ proceeding in a non-increasing order with respect to the partial order of $\mathcal{M}_1(\widehat{X})$. More precisely, fix $\widehat{H}\in\mathcal{M}_1(\widehat{X})$ and assume that we have a potential ψ_ϵ agreeing with $\widetilde{r}^2$ near $\widehat{H}_{\max}$ and such that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi_\epsilon$ is a warped QAC-Kähler form near $\widehat{G}$ for each $\widehat{G}>\widehat{H}$. Suppose that $\widehat{H}$ corresponds to the subset $\mathfrak{q}\subset\{0,\ldots,N\}$. In terms of the decomposition $\mathbb{C}^{m+n}=V_{\mathfrak{q}}\times V_{\mathfrak{q}^c}$, we can assume by induction that near $\widehat{H}\cap\left(\bigcup_{\widehat{G}>\widehat{H}}\widehat{G}\right)$,

$$\psi_\epsilon = \widetilde{r}^2_{\mathfrak{q}} + \psi_{\epsilon,\mathfrak{q}^c} \tag{5.37}$$

with $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2_{\mathfrak{q}}$ a warped QAC-Kähler form on $\widehat{W}_{\mathfrak{q}}$ near $S_{\widehat{H}}$ seen as a boundary hypersurface of $\widehat{V}_{\mathfrak{q}}$. If $0\in\mathfrak{q}$, then using the coordinates (5.24) with $\widehat{H}_k=\widehat{H}$, $\xi_{k'}$ a coordinate corresponding to the $\mathbb{R}^+$-action in $V_{\{0\}}$ and $\zeta_k=\zeta_{\mathfrak{q}^c}$ corresponding to a rescaled coordinate on $V_{\mathfrak{q}^c}$, we can suppose more precisely that outside a compact set of $V_{\mathfrak{q}^c}$,

$$\psi_{\epsilon,\mathfrak{q}^c}(,\xi_{k'},\overline{\xi}_{k'},\zeta_{\mathfrak{q}^c},\overline{\zeta}_{\mathfrak{q}^c}) := |\xi_{k'}|^{-2\nu}\psi_{\epsilon,\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c},\overline{\zeta}_{\mathfrak{q}^c})$$

with $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi_{\epsilon,\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c},\overline{\zeta}_{\mathfrak{q}^c})$ a Kähler form of a QAC-metric outside a compact set of $W_{\mathfrak{q}^c,\omega_{\mathfrak{q}},\epsilon}$. Using Lemma 5.3, we can change $\psi_{\epsilon,\mathfrak{q}^c}$ to a function $\phi_{\epsilon,\mathfrak{q}^c}$ within a compact set so that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\epsilon,\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c},\overline{\zeta}_{\mathfrak{q}^c})$ is positive definite everywhere on $W_{\mathfrak{q}^c,\omega_{\mathfrak{q}},\epsilon}$ for each $\omega_{\mathfrak{q}}$ and $\epsilon\neq 0$ small. We can then take the new potential

$$\widetilde{\psi}_\epsilon = \widetilde{r}^2_{\mathfrak{q}} + |\xi_{k'}|^{-2\nu}\phi_{\epsilon,\mathfrak{q}^c}(\zeta_{\mathfrak{q}^c},\overline{\zeta}_{\mathfrak{q}^c}).$$

By construction, $\widetilde{\psi}_\epsilon=\psi_\epsilon$ near $\bigcup_{\widehat{G}>\widehat{H}}\widehat{G}$ and as in (5.36), one can check that

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{\psi}_\epsilon = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2_{\mathfrak{q}} + |\xi_{k'}|^{-2\nu}\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\mathfrak{q}^c} + x_{\widehat{H}}\mathcal{A}_{\mathrm{phg}}(\widehat{X};\Lambda^{1,1}({}^wT^*\widehat{X}\otimes_{\mathbb{R}}\mathbb{C})), \tag{5.38}$$

so is an $\mathfrak{n}$-warped QAC-Kähler form on $\widehat{C}_\epsilon$ near $\widehat{H}$. If instead $0\notin\mathfrak{q}$, then we still have a decomposition (5.37), but with the difference that on $V_{\mathfrak{q}^c}$, we can directly use the coordinates $z_{\mathfrak{q}^c}$ instead of the rescaled coordinates $\zeta_{\mathfrak{q}^c}$ and assume that

$$\psi_{\epsilon,\mathfrak{q}^c} = \psi_{\epsilon,\mathfrak{q}^c}(z_{\mathfrak{q}^c},\overline{z}_{\mathfrak{q}^c}).$$

Moreover, this time, $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi_{\epsilon,\mathfrak{q}^c}(z_{\mathfrak{q}^c},\overline{z}_{\mathfrak{q}^c})$ is an $\mathfrak{n}$-warped QAC-metric outside a compact set of $W_{\mathfrak{q}^c,\epsilon}$. We can therefore use again Lemma 5.3 to change $\psi_{\epsilon,\mathfrak{q}^c}$ on a compact set and obtain a new potential $\phi_{\epsilon,\mathfrak{q}^c}$ with $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_{\epsilon,\mathfrak{q}^c}$ positive definite everywhere on $W_{\mathfrak{q}^c,\epsilon}$. It suffices then to take

$$\widetilde{\psi}_\epsilon = \psi_{\epsilon,\mathfrak{q}} + \phi_{\epsilon,\mathfrak{q}^c}.$$

Indeed, by construction, $\widetilde{\psi}_\epsilon=\psi_\epsilon$ near $\bigcup_{\widehat{G}>\widehat{H}}\widehat{G}$ and we can easily check that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{\psi}_\epsilon$ is also the Kähler form of an $\mathfrak{n}$-warped QAC-metric on $\widehat{C}_\epsilon$ near $\widehat{H}$.

This completes the inductive step and shows that we can find a potential ψ_ϵ agreeing with $\widetilde{r}^2$ near $\widehat{H}_{\max}$ and such that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi_\epsilon$ is the Kähler form of an $\mathfrak{n}$-warped QAC-metric on $\widehat{C}_\epsilon$ outside a compact set. Using Lemma 5.3 one last time, we can thus find a potential ϕ_ϵ agreeing with ψ_ϵ outside a compact set such that $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_\epsilon$ is the Kähler form of an $\mathfrak{n}$-warped QAC-metric on $\widehat{C}_\epsilon$.

□

Let us formalize the type of metrics that we have obtained in the following analog of [13, Definition 3.6].

Definition 5.6. *A Kähler $\mathfrak{n}$-warped* QAC*-metric $g \in \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon; {}^wT^*\widehat{C}_\epsilon \otimes {}^wT^*\widehat{C}_\epsilon)$ is **asymptotic with rate** δ to the Calabi-Yau cone metric ω_{C_0} if:*

(1) Near $\widehat{H}_{\max}$, $\omega - \frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2 \in \widehat{x}^\delta_{\max}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon; \Lambda^{1,1}({}^wT^*\widehat{C}_\epsilon))$;

(2) Near $\widehat{H} \in \mathcal{M}_1(\widehat{X})$ corresponding to the subset $\mathfrak{q}$,

$$\omega - \frac{\sqrt{-1}}{2}\partial\overline{\partial}\widetilde{r}^2_{\mathfrak{q}} - |z_0|^{2\nu_{\widehat{H}}}\omega_{\widehat{H}} \in x_{\widehat{H}}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(\widehat{C}_\epsilon; \Lambda^{1,1}({}^wT^*\widehat{C}_\epsilon))$$

with $\omega_{\widehat{H}}$ a closed $(1,1)$-form on $\widehat{H}$ which restricts on each fiber $\widehat{W}_{\mathfrak{q}^c,\varpi_{\mathfrak{q}},\epsilon}$ of $\phi_{\widehat{H}_\epsilon} : \widehat{H}_\epsilon \to S_{\widehat{H}}$ to the Kähler form of a Kähler $\mathfrak{n}_{Z_{\widehat{H}_\epsilon}}$-warped QAC*-metric asymptotic with rate δ to $g_{W_{\mathfrak{q}^c}}$, the natural Calabi-Yau cone metric on $W_{\mathfrak{q}^c}$ obtained by restricting g_{C_0}. Moreover, as a family of $(1,1)$-forms parametrized by $S_{\widehat{H}_\epsilon}$, $\omega_{\widehat{H}}$ is smooth up to $\partial S_{\widehat{H}_\epsilon}$.*

Notice that the definition is not circular, since by induction on the depth of $\widehat{C}_\epsilon$, we can assume that the notion of a Kähler $\mathfrak{n}_{Z_{\widehat{H}_\epsilon}}$-warped QAC*-metric asymptotic to $g_{W_{\mathfrak{q}^c}}$ with rate δ has already been defined.*

This yields the following characterization of the Kähler metrics of Theorem 5.5.

Corollary 5.7. *If $d > 1$, then the Kähler $\mathfrak{n}$-warped* QAC*-metrics of Theorem 5.5 are asymptotic with rate d to the Calabi-Yau cone metric g_{C_0}.*

Proof. By construction, ϕ_ϵ agrees with $\widetilde{r}^2$ near $\widehat{H}_{\max}$, which implies (1) in Definition 5.6. Using computations as in (5.38), (5.33) and (5.34) imply (2) of Definition 5.6 so the result follows. □

To construct Calabi-Yau examples, the key property of the Kähler metrics of Definition 5.6 that we will use can be formulated in terms of the following analog of [13, Definition 5.1].

Definition 5.8. *For $\widehat{H} < \widehat{H}_{\max}$, let $\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(\widehat{H}_\epsilon/S_{\widehat{H}_\epsilon})$ be the space of smooth functions on*

$$\widehat{H}_\epsilon \setminus \left(\bigcup_{\widehat{G}_\epsilon > \widehat{H}_\epsilon} \widehat{G}_\epsilon \cap \widehat{H}_\epsilon \right)$$

*which restricts on each fiber $\phi^{-1}_{\widehat{H}_\epsilon}(s)$ to a function in $\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(\phi^{-1}_{\widehat{H}_\epsilon}(s))$. A function $f \in \widehat{x}^\delta_{\max}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ is said to **restrict** to $\partial\widehat{C}_\epsilon$ to order $r > 0$ if for each $\widehat{H} < \widehat{H}_{\max}$, there is $f_{\widehat{H}_\epsilon} \in \widehat{x}^\delta_{\max}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(\widehat{H}_\epsilon/S_{\widehat{H}_\epsilon})$ such that*

$$f - f_{\widehat{H}_\epsilon} \in \widehat{x}^\delta_{\max}x^r_{\widehat{H}}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon).$$

We denote by $\widehat{x}^\delta_{\max}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},r}(C_\epsilon)$ the subspace of functions in $\widehat{x}^\delta_{\max}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ that restrict to $\partial\widehat{C}_\epsilon$ to order r.

Indeed, as the next lemma shows, the Ricci potential of the Kähler metrics of Definition 5.6 is an example of such a function.

Lemma 5.9. *If $d > 1$ and ω_ϵ is the Kähler form of a Kähler $\mathfrak{n}$-warped* QAC*-metric asymptotic to g_{C_0} with rate d, then its Ricci potential*

$$\mathfrak{r}_\epsilon = \log\left(\frac{\omega^m_\epsilon}{c_m\Omega^m_{C_\epsilon} \wedge \overline{\Omega}^m_{C_\epsilon}}\right) \tag{5.39}$$

is an element of $\widehat{x}^d_{\max}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon)$.

Proof. Let us first check that $\mathfrak{r}_\epsilon$ restricts to $\partial\widehat{C}_\epsilon$ and consider the forms

$$dz := dz_0 \wedge (dz_{1,1} \wedge \ldots \wedge dz_{1,m_1+n_1}) \wedge \ldots \wedge (dz_{N,1} \wedge \ldots \wedge dz_{N,m_N+n_N}) \tag{5.40}$$

and

$$\mathcal{P}_\epsilon := (d(P_{1,1} - \epsilon Q_{1,1}) \wedge \ldots d(P_{1,n_1} - \epsilon Q_{1,n_1})) \wedge \ldots \wedge (d(P_{N,1} - \epsilon Q_{N,1}) \wedge \ldots \wedge d(P_{N,n_N} - \epsilon Q_{N,n_N})). \tag{5.41}$$

For $q \in \{1, \ldots, N\}$, consider also the form

$$\mathcal{P}_{\epsilon,q} := d(P_{q,1} - \epsilon Q_{q,1}) \wedge \ldots d(P_{q,n_q} - \epsilon Q_{q,n_q}).$$

Then from (1.16) and the definition of $\mathfrak{r}_\epsilon$, we see that there exists a constant $c_{m,n} \in \mathbb{C}^*$ depending only on m and n such that

$$\mathfrak{r}_\epsilon = \log\left(\frac{\omega_\epsilon^m \wedge \mathcal{P}_\epsilon \wedge \overline{\mathcal{P}}_\epsilon}{c_{m,n} dz \wedge d\overline{z}}\right)|_{\widehat{C}_\epsilon}. \tag{5.42}$$

To study the behavior of $\mathfrak{r}_\epsilon$ near $\widehat{H} \in \mathcal{M}_1(\widehat{X})$, let $\mathfrak{q} \subset \{0, \ldots, N\}$ be the subset corresponding to $\widehat{H}$. If $0 \notin \mathfrak{q}$, that is, if $\widehat{H} \in \mathcal{M}_{1,0}(\widehat{X})$, then in the coordinates (5.19), we see that for $q \in \mathfrak{q}^c \setminus \{0\}$, the 1-form

$$d(P_{q,j}(z_q) - \epsilon Q_{q,j}(z_0))$$

naturally restricts to $\widehat{H}$ while for $q \in \mathfrak{q}$,

$$\mathcal{P}_{\epsilon,q} = \mathcal{P}_{0,q} + \mathcal{O}(\xi^d),$$

where $\mathcal{O}(\xi^d)$ corresponds to a sum of n_q-forms, each homogeneous in ξ of degree at least $-(n_q - 1)d$ (instead of $-n_q d$ for $\mathcal{P}_{q,0}$). Keeping in mind (1.3), the latter is to be compared with

$$dz_{q,j} = d(\xi^{-w_{q,j}} \varpi_{q,j}).$$

If instead $0 \in \mathfrak{q}$, then in terms of the coordinates used in (5.22), we see that for $q \in \mathfrak{q}^c$,

$$\begin{aligned} d(P_{q,j}(z_q) - \epsilon Q_{q,j}(z_0)) &= d\left(\xi^{-\nu d} P_{q,j}(\zeta_q) - \epsilon Q_{q,j}(\xi^{-1}\varpi_0)\right) = d\left(\xi^{-\ell} P_{q,j}(\zeta_q) - \epsilon Q_{q,j}(\xi^{-1}\varpi_0)\right) \\ &= d\left(\xi^{-\ell}(P_{q,j}(\zeta_q) - \epsilon[Q_{q,j}](\xi^{-1}\varpi_0))\right) + \mathcal{O}(\xi^{\alpha+1-\ell}), \quad \text{for some } \alpha > 0, \\ &= d\left(\xi^{-\ell}(P_{q,j}(\zeta_q) - \epsilon[Q_{q,j}](\xi^{-1}\varpi_0))\right) + \mathcal{O}(x_{\widehat{H}}^{\frac{(\alpha+1)d}{d-\ell}} \xi^{-\ell}), \end{aligned} \tag{5.43}$$

where $\mathcal{O}(\xi^{\alpha+1-\ell}) = \mathcal{O}(x_{\widehat{H}}^{\frac{(\alpha+1)d}{d-\ell}} \xi^{-\ell})$ is with respect to the local sections of 1-forms $\frac{d\xi}{\xi^2}$ and $\frac{d\varpi_{0,j}}{\xi}$ in (5.22). For $q \in \mathfrak{q}$, we have instead that

$$d(P_{q,j}(z_q) - \epsilon Q_{q,j}(z_0)) = d(\xi^{-d} P_{q,j}(\varpi)) - \epsilon dQ_{q,j}(\xi^{-1}\varpi_0), \tag{5.44}$$

so that

$$\mathcal{P}_{\epsilon,q} = \mathcal{P}_{0,q} + \mathcal{O}(\xi^{d-\ell}) = \mathcal{P}_{0,q} + \mathcal{O}(x_{\widehat{H}}^d) \tag{5.45}$$

with $\mathcal{O}(\xi^{d-\ell}) = \mathcal{O}(x_{\widehat{H}}^d)$ corresponding to a sum of n_q-forms, each homogeneous in ξ of degree at most $-n_q d + d - \ell$, instead of $-n_q d$ for $\mathcal{P}_{0,q}$. Thanks to property (2) of Definition 5.6 and keeping in mind (1.3), we see from (5.42) that $\mathfrak{r}_\epsilon$ restricts to ∂C_ϵ to order 1 in both coordinate charts. Similar computations can be done in the coordinates (5.24), from which the result follows.

At $\widehat{H}_{\max}$, notice that $\mathfrak{q} = \{0, \ldots, N\}$, so $\mathfrak{q}^c = \emptyset$, that is, (5.43) does not arise and we only need to consider (5.45). On the other hand, since ω_{C_0} is Calabi-Yau, we know that

$$\mathfrak{r}_0 = \log\left(\frac{\omega_{C_0}^m}{c_m \Omega_{C_0}^m \wedge \overline{\Omega}_{C_0}^m}\right) = 0. \tag{5.46}$$

Combined with property (1) of Definition 5.6, this gives the desired decay at $\widehat{H}_{\max}$. □

6. Solving the complex Monge-Ampère equation

Suppose that $d > 1$ and let g_ϵ be a Kähler $\mathfrak{n}$-warped QAC-metric asymptotic to the Calabi-Yau cone metric g_{C_0} with rate d. By Theorem 5.5 and Corollary 5.7, such a metric exists provided $d > 1$. If ω_ϵ is the Kähler form of g_ϵ, then $\widetilde{\omega}_\epsilon := \omega_\epsilon + \sqrt{-1}\partial\overline{\partial}u$ will be the Kähler form of a Calabi-Yau metric provided u is a solution of the complex Monge-Ampère equation

$$\log\left(\frac{(\omega_\epsilon + \sqrt{-1}\partial\overline{\partial}u)^m}{\omega_\epsilon^m}\right) = -\mathfrak{r}_\epsilon, \tag{6.1}$$

where $\mathfrak{r}_\epsilon$ is the Ricci potential of g_ϵ as defined in (5.39). By Lemma 5.9, this is a particular case of the more general complex Monge-Ampère equation

$$\log\left(\frac{(\omega_\epsilon + \sqrt{-1}\partial\overline{\partial}u)^m}{\omega_\epsilon^m}\right) = f \quad \text{for } f \in \widehat{x}_{\max}^d \mathcal{C}^\infty_{\mathfrak{n}\mathrm{Qb},1}(\widehat{C}_\epsilon). \tag{6.2}$$

As in [13, § 5], we will solve this equation by first solving the corresponding equation on each non-maximal boundary hypersurface of $\widehat{C}_\epsilon$. This will allow us to proceed by induction on the depth of $\widehat{C}_\epsilon$. In the induction step, we can in fact assume that $f \in \widehat{x}^d_{\max} w \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$, where

$$w = \prod_{\widehat{H}\in\mathcal{M}_{\mathrm{nm}}(\widehat{X})} x_{\widehat{H}}.$$

Lemma 6.1. *Let β be a real number such that $\frac{2}{(1-\nu)} < \beta < \frac{2m}{(1-\nu)}$ and $\beta \le 2m_{\widehat{H}_\epsilon}$ for each $\widehat{H}_\epsilon \in \mathcal{M}_{\mathrm{nm}}(\widehat{C}_\epsilon)$, where $2m_{\widehat{H}_\epsilon}$ is the real dimension of the fibers of $\phi_{\widehat{H}_\epsilon} : \widehat{H}_\epsilon \to S_{\widehat{H}_\epsilon}$. Let δ be a multiweight such that $\delta_{\max} = \beta - \frac{2}{1-\nu}$, $-\frac{2\nu_{\widehat{H}}}{1-\nu_{\widehat{H}}} < \delta_{\widehat{H}} < 1 - \frac{2\nu_{\widehat{H}}}{1-\nu_{\widehat{H}}}$ and*

$$\left(\frac{1-\nu}{1-\nu_{\widehat{H}}}\right)\beta - \frac{2}{1-\nu_{\widehat{H}}} + 2 - 2m_{\widehat{H}_\epsilon} < \delta_{\widehat{H}} < \left(\frac{1-\nu}{1-\nu_{\widehat{H}}}\right)\beta - \frac{2}{1-\nu_{\widehat{H}}}$$

for $\widehat{H} \in \mathcal{M}_{\mathrm{nm}}(\widehat{X})$. If $f \in \widehat{x}^{\beta'}_{\max} w \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ for some $\beta' > \beta$, then there exists $v \in x^\delta \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ such that $\widetilde{\omega}_\epsilon = \omega_\epsilon + \sqrt{-1}\partial\overline{\partial} v$ is the Kähler form of an $\mathfrak{n}$-warped QAC*-metric $\widetilde{g}_\epsilon$ asymptotic to g_{C_0} with rate* $\min\{d,\beta\}$ *and with*

$$\widetilde{f} := f - \log\left(\frac{\widetilde{\omega}^m_\epsilon}{\omega^m_\epsilon}\right) \in \mathcal{C}^\infty_c(C_\epsilon). \tag{6.3}$$

Proof. We follow the approach of [36, Proposition 25] and apply a fixed point argument outside a sufficiently large compact set of C_ϵ. For $A > 0$, consider the closed subset

$$\rho^{-1}([A,\infty)) \subset C_\epsilon.$$

Denote by $x^\delta \mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty)))$ the subspace of functions on $\rho^{-1}([A,\infty))$ obtained by restriction of $x^\delta \mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ to $\rho^{-1}([A,\infty))$. The norm of a function in $x^\delta \mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty)))$ can be defined as the infimum over the corresponding norms of the possible extensions in $x^\delta \mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$. For δ as in the statement of the lemma, consider the subset

$$\mathcal{B} := \{u \in x^\delta \mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty))) \mid \|u\|_{x^\delta \mathcal{C}^{k+2,\delta}_{\mathfrak{n}\,\mathrm{Qb}}} \le \epsilon_0\}$$

for some $\epsilon_0 > 0$ chosen sufficiently small so that $\omega_\epsilon + \sqrt{-1}\partial\overline{\partial} u$ is positive definite in $\rho^{-1}([A,\infty))$. To solve the complex Monge-Ampère-equation near infinity, consider the nonlinear operator

$$\begin{array}{cccc} F: & \mathcal{B} & \to & (\rho w)^{-2} x^\delta \mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty))) \\ & u & \mapsto & \log\left(\frac{(\omega_\epsilon+\sqrt{-1}\partial\overline{\partial}u)^m}{\omega^m_\epsilon}\right) - f. \end{array}$$

This can be rewritten as

$$F(u) = F(0) + \frac{1}{2}\Delta_{g_\epsilon} u + Q(u),$$

where $\Delta_{g_\epsilon} = g^{ij}_\epsilon \nabla_i \nabla_j$ is the Laplacian associated to g_ϵ and $Q(u)$ is the nonlinear part of $F(u)$. Let $\mu \in \mathcal{C}^\infty(\mathbb{R})$ be a function such that $\mu(t) \equiv 0$ for $t < 1$ and $\mu(t) \equiv 1$ for $t > 2$. To solve (6.2), it suffices then to solve the equation

$$u = 2\Delta^{-1}_{g_\epsilon}\left[\mu(\rho - A)(-F(0) - Q(u))\right], \tag{6.4}$$

where $\Delta^{-1}_{g_\epsilon}$ is the inverse provided by Corollary 3.23. Indeed, if $u \in x^\delta \mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty)))$ is a solution of (6.4), then it is a solution of (6.2) on $\rho^{-1}([A+2,\infty))$. Now, to solve (6.4), we can look for a fixed point of the operator

$$N(u) := 2\Delta^{-1}_{g_\epsilon}\left[\mu(\rho - A)(-F(0) - Q(u))\right]. \tag{6.5}$$

This can be achieved by showing that N is a contraction. First, as in the proof of [36, Proposition 25], we see from the explicit formula for Q that for $u, v \in \mathcal{B}$,

$$\begin{aligned}\|Q(u)-Q(v)\|_{(\rho w)^{-2}x^{\delta}\mathcal{C}^{k,\alpha}} &\le C\left(\|\partial\overline{\partial}u\|_{\mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty);\Lambda^{1,1}({}^wT^*\widehat{C}_\epsilon))}+\|\partial\overline{\partial}v\|_{\mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty);\Lambda^{1,1}({}^wT^*\widehat{C}_\epsilon))}\right)\cdot \\ &\qquad\qquad\qquad\qquad \|u-v\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} \\ &\le \widetilde{C}\left(\|u\|_{(\rho w)^2\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}}+\|u\|_{(\rho w)^2\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}}\right)\|u-v\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}}\end{aligned}$$

for some positive constants C and $\widetilde{C}$ depending on g_ϵ, but independent of the choice of ϵ_0 in the definition of $\mathcal{B}$ if we assume without loss of generality that $\epsilon_0 < 1$. Since $\delta_{\max} > \frac{-2}{1-\nu}$ and $\delta_{\widehat{H}} > -\frac{2\nu_{\widehat{H}}}{1-\nu_{\widehat{H}}}$, there is a continuous inclusion

$$x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty))\subset(\rho w)^2\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}(\rho^{-1}([A,\infty)),$$

so changing $\widetilde{C}$ if necessary,

$$\|Q(u)-Q(v)\|_{(\rho w)^{-2}x^{\delta}\mathcal{C}^{k,\alpha}} \le \widetilde{C}\left(\|u\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}}+\|u\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}}\right)\|u-v\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} \quad \forall u,v\in\mathcal{B}. \tag{6.6}$$

Since the $\mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}$-norms of $\mu(\rho-A)$ can be controlled independently of the choice of A, we see from (6.5), (6.6) and Corollary 3.23 that by taking $\epsilon_0 > 0$ sufficiently small in the definition of $\mathcal{B}$ but independent of the choice of A, we will have that

$$\|N(u)-N(v)\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} < \frac{1}{2}\|u-v\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} \quad \forall u,v\in\mathcal{B}.$$

To apply the Banach fixed point theorem, we also need to check that N maps $\mathcal{B}$ to itself. First note that

$$F(0) = -f \in \widehat{x}_{\max}^{\beta'}w\mathcal{C}^{\infty}_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon).$$

Since $\beta' > \beta$ and $\delta_{\widehat{H}} < 1-\frac{2\nu_{\widehat{H}}}{1-\nu_{\widehat{H}}}$, this means that there are constants $C > 0$ and $\tau > 0$ such that

$$\|F(0)\|_{(\rho w)^{-2}x^{\delta}\mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} < CA^{-\tau}.$$

Hence, by Corollary 3.23, for $u \in \mathcal{B}$, there is a constant $K > 0$ independent of A such that

$$\begin{aligned}\|N(u)\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} &\le \|N(0)\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}}+\|N(u)-N(0)\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} \\ &\le K\|F(0)\|_{(\rho w)^{-2}x^{\delta}\mathcal{C}^{k,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}}+\frac{1}{2}\|u\|_{x^{\delta}\mathcal{C}^{k+2,\alpha}_{\mathfrak{n}\,\mathrm{Qb}}} \\ &\le KCA^{-\tau}+\frac{\epsilon_0}{2}.\end{aligned}$$

Thus, for $A > 0$ sufficiently large, $N(\mathcal{B}) \subset \mathcal{B}$ and N has a fixed point u as required. Initially, u is only defined on $\rho^{-1}([A,\infty))$, but using Lemma 5.3, we can find a new function v agreeing with u outside a compact set such that

$$\widetilde{\omega}_\epsilon = \omega_\epsilon + \sqrt{-1}\partial\overline{\partial}v$$

is positive definite everywhere. Using Corollary 3.23 and (6.2), we can bootstrap to see that in fact $v \in x^{\delta}\mathcal{C}^{\infty}_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$.

□

The decay in (6.3) is exactly what we need to solve the complex Monge-Ampère equation via the approach of [39] or its parabolic version [10].

Theorem 6.2. *For the Kähler form $\widetilde{\omega}_\epsilon$ and the function $\widetilde{f}$ of Lemma 6.1, the complex Monge-Ampère equation*

$$\log\left(\frac{(\widetilde{\omega}_\epsilon+\sqrt{-1}\partial\overline{\partial}u)^m}{\widetilde{\omega}_\epsilon^m}\right) = \widetilde{f} \tag{6.7}$$

has a unique solution u in $x^{\widetilde{\delta}}\mathcal{C}^{\infty}_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$, where $\widetilde{\delta}$ is any multiweight satisfying the conditions of Corollary 3.23 such that $\widetilde{\delta}_{\max} > 0$ and $\widetilde{\delta}_{\widehat{H}} > 0$ for $\widehat{H} \in \mathcal{M}_{\mathrm{nm}}(\widehat{X})$.

Proof. The proof is quite similar to those of [13, Theorem 5.4] and [18, Theorem 6.2]. For the convenience of the reader, we will go over the argument putting emphasis on the new features. The overall strategy is to apply the continuity method to the equation

$$\log\left(\frac{(\widetilde{\omega}_\epsilon+\sqrt{-1}\partial\overline{\partial}u)^m}{\widetilde{\omega}_\epsilon^m}\right)=t\widetilde{f} \tag{6.8}$$

for $t\in[0,1]$. This will be achieved by showing that the set

$$S:=\{s\in[0,1]\mid \text{there is a solution } u_\epsilon\in x^{\widetilde{\delta}}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon) \text{ of (6.8) for } t=s\}$$

is all of $[0,1]$, that is, by showing that S is nonempty, open and closed. Clearly, $u_0=0$ is a solution of (6.8) with $t=0$, so S is not empty. On the other hand, the openness of S follows from Corollary 3.23.

To see that S is also closed, suppose that $[0,\tau)\subset S$ for some $0<\tau\le 1$. We need to show that (3.23) has a solution $u_\tau\in x^{\widetilde{\delta}}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ for $t=\tau$. By Corollary 3.18, the Sobolev inequality holds for $\mathfrak{n}$-warped QAC-metrics on C_ϵ, so we can apply Moser iteration to derive an a priori C^0-bound on solutions of (6.8). The argument of Yau then provides uniform bounds on $\sqrt{-1}\partial\overline{\partial}u_t$, which by the result of Evans-Krylov, yields a priori $C^{2,\gamma}_w(C_\epsilon)$-bounds on solutions. Taking an increasing sequence $t_i\nearrow\tau$, we can apply the Arzelà-Ascoli theorem to extract a subsequence of $\{u_{t_i}\}$ converging in $\mathcal{C}^2_w(C_\epsilon)$ to some solution u_τ of (6.8) for $t=\tau$. Bootstrapping shows that $u_\tau\in\mathcal{C}^\infty_w(C_\epsilon)$.

To show that $u_\tau\in x^\delta\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$, we can first apply a Moser iteration with weight as in [27, § 8.6.2 and § 9.6.2] to obtain an a priori bound in $\rho^{-\mu_1}\mathcal{C}^0_w(C_\epsilon)$ for $\mu_1>0$ such that $\frac{\mu_1}{1-\nu_{\widehat{H}}}<\delta_{\widehat{H}}$ for each $\widehat{H}\in\mathcal{M}_1(\widehat{X})$. As in [13, (5.11)], equation (6.8) for $t=\tau$ can be rewritten as

$$\Delta_u u=\tau\widetilde{f}, \tag{6.9}$$

where

$$\Delta_u v=\frac{1}{2}\int_0^1(\Delta_{u,t}v)dt$$

with $\Delta_{u,t}$ the Laplacian associated to the Kähler form $\widetilde{\omega}_\epsilon+t\sqrt{-1}\partial\overline{\partial}u$. By Proposition 2.17, we can apply the Schauder estimate to (6.9) in terms of $\mathfrak{n}$-warped QAC-metrics to bootstrap and obtain that in fact $u_\tau\in\rho^{-\mu_1}\mathcal{C}^\infty_w(C_\epsilon)$. Now by Lemma 2.16, we know that

$$\rho^{-\mu_1}\mathcal{C}^1_w(C_\epsilon)\subset C^{0,\mu_1}_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon).$$

In particular, this implies that $\|\partial\overline{\partial}u_\tau\|_{\widetilde{g}_\epsilon}\in\mathcal{C}^{0,\mu_1}_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$. Rewriting (6.9) in terms of an elliptic $\mathfrak{n}\,$Qb-operator

$$(\rho w)^2\Delta_u u=(\rho w)^2\tau\widetilde{f} \tag{6.10}$$

and using Lemma 2.15, we can apply the Schauder estimate and bootstrap to see that $u_\tau\in\rho^{-\mu_1}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$. Finally, using the inclusion $x^{\widetilde{\delta}}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)\subset\rho^{-\mu_1}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ for $\mu_1>0$ small enough, we see from Corollary 3.23 applied with $\widetilde{\delta}$ above and with a positive multiweight δ' small enough such that $\rho^{-\mu_1}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)\subset x^{\delta'}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ that $u\in x^\delta\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$. This shows that S is closed and completes the proof of existence. For uniqueness, we can proceed as in [6, Proposition 7.13] but using the isomorphism (3.42) instead of the maximum principle. □

This allows us to solve the complex Monge-Ampère (6.2) when $f\in\widehat{x}^\beta_{\max}w\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$.

Corollary 6.3. *Suppose that β is a real number such that*

$$\frac{2}{(1-\nu)}<\beta<\frac{2m}{(1-\nu)}$$

and $\beta\le 2m_{\widehat{H}_\epsilon}$ for each $\widehat{H}_\epsilon\in\mathcal{M}_{\mathrm{nm}}(\widehat{C}_\epsilon)$. Let δ be the multiweight as in Lemma 6.1. If $f\in\widehat{x}^{\beta'}_{\max}w\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$ for some $\beta'>\beta$, then the complex Monge-Ampère equation (6.2) has a unique solution $u\in x^\delta\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$.

Proof. The existence is given by Lemma 6.1 and Theorem 6.2. Uniqueness follows again from [6, Proposition 7.13] using the isomorphism (3.42) instead of the maximum principle. □

Using these results, we can finally solve the complex Monge-Ampère equation (6.2) in full generality.

Theorem 6.4. *Suppose that* $\beta := \min\{d, 2m_1, \ldots, 2m_N\}$ *is such that*

$$\beta > \frac{2}{1-\nu}.$$

Then for $f \in x_{\max}^d \mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon)$, *the complex Monge-Ampère equation* (6.2) *has a unique solution*

$$u \in \widehat{x}_{\max}^{\beta-2}\sigma^{2\nu}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon),$$

where $\sigma := \prod_{\widehat{H}\in\mathcal{M}_{1,\nu}(\widehat{X})} x_{\widehat{H}}^{-\frac{1}{1-\nu}}$ *if* $\nu > 0$ *and* $\sigma := 1$ *otherwise, so that* $\sigma^\nu = \rho^{\mathfrak{n}}$.

Proof. To construct u, we will first construct what should be the restriction of $\sigma^{-2\nu}u$ to $\widehat{H}_\epsilon$ for $\widehat{H}_\epsilon \in \mathcal{M}_{\mathrm{nm}}(\widehat{C}_\epsilon)$ such that

$$\widehat{H} < \widehat{G} \implies \widehat{G} = \widehat{H}_{\max}.$$

By property (2) of Definition 5.6, using the rescaled complex coordinates (5.13) in the fibers of $\phi_{\widehat{H}_\epsilon} : \widehat{H}_\epsilon \to S_{\widehat{H}_\epsilon}$, this restriction that we will denote by $u_{\widehat{H}_\epsilon}$, should satisfy fiberwise the complex Monge-Ampère equation

$$\log\left(\frac{(\omega_{\widehat{H}} + \sqrt{-1}\partial\overline{\partial}u_{\widehat{H}})^{m_{\widehat{H}_\epsilon}}}{\omega_{\widehat{H}_\epsilon}^{m_{\widehat{H}_\epsilon}}}\right) = f|_{\widehat{H}_\epsilon}. \tag{6.11}$$

This is a family of complex Monge-Ampère equations in the asymptotically conical setting. Applying the asymptotically conical analog of Corollary 6.3, see for instance [15, Theorem 2.1], we can solve it in each fiber to find a unique solution $u_{\widehat{H}_\epsilon} \in \widehat{x}_{\max}^{\beta-2}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(\widehat{H}_\epsilon/S_{\widehat{H}_\epsilon})$. Extend $u_{\widehat{H}_\epsilon}$ smoothly off $\widehat{H}_\epsilon$ to a function $\widehat{u}_{\widehat{H}_\epsilon}$ and consider the new closed $(1,1)$-form

$$\omega_{\epsilon,\widehat{H}} = \omega_\epsilon + \sqrt{-1}\partial\overline{\partial}(\sigma^{2\nu}\widehat{u}_{\widehat{H}_\epsilon}). \tag{6.12}$$

By property (2) of Definition 5.21 and (6.11), it is positive definite outside a compact set. By Lemma 5.3, changing $\widehat{u}_{\widehat{H}_\epsilon}$ on a compact set of C_ϵ, we can assume that $\omega_{\epsilon,\widehat{H}}$ is positive definite everywhere. By (6.11) and computing as in (5.36), using the fact that

$$\sigma^{2\nu}\widehat{x}_{\max}^{\beta-2+\frac{2}{1-\nu}} = \mathcal{O}(x_{\max}^\beta.(\sigma^{2\nu}\widehat{x}_{\max}^{\frac{2\nu}{1-\nu}})),$$

we see that

$$f_1 := f - \log\left(\frac{\omega_{\epsilon,\widehat{H}}^m}{\omega_\epsilon^m}\right) \in x_{\max}^\beta x_{\widehat{H}}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon). \tag{6.13}$$

Thus, replacing ω_ϵ by $\omega_{\epsilon,\widehat{H}}$ and f by f_1, the complex Monge-Ampère equation (6.2) corresponds to solving

$$\log\left(\frac{(\omega_{\epsilon,\widehat{H}} + \sqrt{-1}\partial\overline{\partial}u)^m}{\omega_{\epsilon,\widehat{H}}^m}\right) = f_1 \in \widehat{x}_{\max}^\beta x_{\widehat{H}}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon). \tag{6.14}$$

Let $\mathcal{K}$ be the subset of $\mathcal{M}_{\mathrm{nm}}(\widehat{X})$ consisting of boundary hypersurfaces $\widehat{H}$ such that

$$\widehat{H} < \widehat{G} \implies \widehat{G} = \widehat{H}_{\max}.$$

Performing this argument at each $\widehat{H} \in \mathcal{K}$, we can reduce to the case where

$$f \in \widehat{x}_{\max}^\beta\left(\prod_{\widehat{H}\in\mathcal{K}} x_{\widehat{H}}\right)\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon). \tag{6.15}$$

Knowing that (6.15) holds, this argument can be iterated. Namely, if $\widehat{H} \in \mathcal{M}_{\mathrm{nm}}(\widehat{X}) \setminus \mathcal{K}$ is such that

$$\widehat{H} < \widehat{G} \implies \widehat{G} \in \mathcal{K} \cup \{\widehat{H}_{\max}\},$$

then we can find the restriction of $\sigma^{-2\nu}u$ by solving (6.11) again, this time however with Corollary 6.3 (with $\nu = 0$ if $\widehat{H} \in \mathcal{M}_{1,\nu(\widehat{X})}$) instead of [15, Theorem 2.1]. Proceeding in an order which is non-increasing with respect to the partial order on $\mathcal{M}_{\mathrm{nm}}(\widehat{X})$, this argument can be iterated to reduce to the case where $f \in \widehat{x}_{\max}^\beta w\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$. One can then apply Corollary 6.3 once more to get the desired solution on C_ϵ.

To show the solution is unique, we again apply [6, Proposition 7.13] with the maximum principle replaced by the isomorphism (3.42) with multiweight δ such that $\delta_{\widehat{H}} = -\frac{2\nu_H}{1-\nu_H}$ for $\widehat{H} \in \mathcal{M}_{\mathrm{nm}}(\widehat{X})$ and

$$\delta_{\max} = \beta - \frac{2}{1-\nu} - \lambda$$

with $\lambda \geq 0$ possibly positive but small.

□

Applying this result to the complex Monge-Ampère equation (6.1) yields the main result of this paper.

Corollary 6.5. *Suppose that $d > 1$ and that $\beta := \min\{d, 2m_1, \ldots, 2m_N\}$ is such that*

$$\beta > \frac{2}{1-\nu}.$$

Then for $\epsilon \neq 0$, C_ϵ admits a Calabi-Yau $\mathfrak{n}$-warped QAC-metric asymptotic to g_{C_0} with rate β.

Proof. Since $d > 1$, we know by Corollary 5.7 that there exists on C_ϵ a Kähler $\mathfrak{n}$-warped QAC-metric asymptotic to g_{C_0} with rate d. By Lemma 5.9, its Ricci potential $\mathfrak{r}_\epsilon$ is in $\widehat{x}^d_{\max}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon)$, so by Theorem 6.4, the complex Monge-Ampère equation (6.1) has a unique solution $u \in \widehat{x}^{\beta-2}_{\max}\sigma^{2\nu}\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb},1}(C_\epsilon)$ and

$$\widetilde{\omega}_\epsilon = \omega_\epsilon + \sqrt{-1}\partial\overline{\partial}u$$

is the desired Calabi-Yau metric.

□

When $N = 1$, it is possible to improve slightly the result as follows.

Corollary 6.6. *Suppose that $N = 1$, that $d > 1$ and that $\beta := \min\{d, 2m_1\}$ is such that either $\beta > \frac{2}{1-\nu}$, or else*

$$3 < \beta \leq \frac{2}{1-\nu} < 2m_1 + 5. \tag{6.16}$$

Then for $\epsilon \neq 0$, C_ϵ admits a Calabi-Yau $\mathfrak{n}$-warped QAC-metric asymptotic to g_{C_0} with rate β.

Proof. If $\beta > \frac{2}{1-\nu}$, this is a particular case of Corollary 6.5. If instead (6.16) holds, we can follow the strategy of [18, § 7]. More precisely, as in the first part of the proof of Theorem 6.4, we can first solve the Monge-Ampère equation on the fibers of $\widehat{H}_\epsilon \in \mathcal{M}_{\mathrm{nm}}(C_\epsilon)$ to reduce to the case where $\mathfrak{r}_\epsilon \in \widehat{x}^\beta_{\max}w\mathcal{C}^\infty_{\mathfrak{n}\,\mathrm{Qb}}(C_\epsilon)$. By [17, Corollary 5.4], we know in fact that $\mathfrak{r}_\epsilon \in \widehat{x}^\beta_{\max}w\mathcal{A}_{\mathrm{phg}}(\widehat{C}_\epsilon)$. Proceeding as in [18, Lemma 7.1], we can then eliminate the part of the polyhomogeneous expansion of $\mathfrak{r}_\epsilon$ at $\widehat{H}_{\max}$ of order $\frac{2}{1-\nu}$ or less. Indeed, if $(\widehat{x}^\mu_{\max}w)e$ is a term of order $\mu \leq \frac{2}{1-\nu}$ in this expansion at $\widehat{H}_{\max}$, then since

$$\widehat{x}^\mu_{\max}w = v^\mu w^{1-\mu} = \rho^{-\mu(1-\nu)}w^{1-\mu},$$

we need to solve $I(B,\lambda)f_\lambda = w^{1-\mu}e$ as in [18, (7.6)] with $\lambda = \mu(1-\nu) - 2$, which can be achieved through [18, Corollary A.5] with

$$a - 2 = 1 - \mu.$$

But in our case, $f = 2m_1 - 1$ in the statement of [18, Corollary A.5], so for this corollary to apply, we need $-f + 1 < a < 0$, which in terms of μ and m_1 translates into

$$3 < \mu < 2m_1 + 5. \tag{6.17}$$

By assumption, $\beta \leq \mu \leq \frac{2}{1-\nu}$, so (6.17) holds thanks to (6.16). We can therefore proceed as in [18, Lemma 7.1] to reduce to the case

$$\mathfrak{r}_\epsilon \in \widehat{x}^\mu_{\max}w\mathcal{A}_{\mathrm{phg}}(C_\epsilon)$$

with $\mu > \frac{2}{1-\nu}$. We can then rely on Corollary 6.3 to conclude the proof. □

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