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Inverse problem for the nonlinear long wave runup on a plane sloping beach ☆

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ABSTRACT

We put forward a simple but effective explicit method of recovering initial data for the nonlinear shallow water system from the reading at the shoreline. We then apply our method to the tsunami waves inverse problem.

1. Introduction

The process of the tsunami wave run-up on a coast, a very important problem of the tsunami science, is commonly studied in the framework of shallow water approximations [1]. A similar problem occurs when long swell waves come ashore forming rogue waves [2]. From the mathematical point of view it is the Cauchy problem for a system of nonlinear PDEs with initial conditions specified at the source of the tsunami wave (e.g. epicenter of an earthquake). That is, given initial water displacement and velocity field, find the motion of the shoreline. Unfortunately, even displacement data are never available and certain models for initial data are used instead (see e.g. [3]).

At the same time, mareographs installed at most of ports across the globe collect data in digital form. This readily suggests an inverse problem: given data read at specific locations, recover the characteristics of the source of a tsunami wave. Historically this problem was approached first using isochron [4] (see [5–8]

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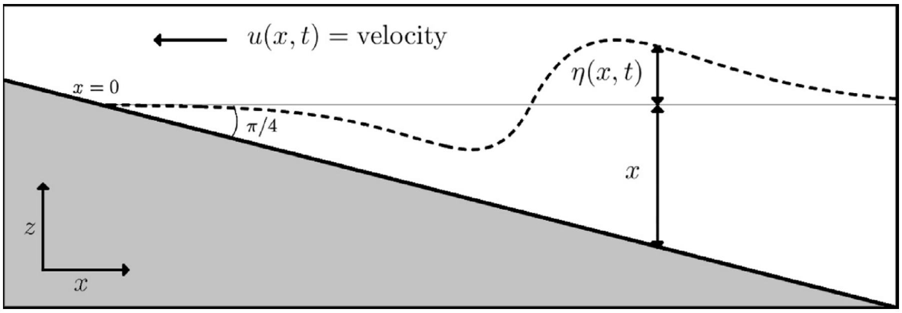

Fig. 1. A sketch of the beach in the $x - z$ plane where $\eta(x,t)$ is the water elevation over the unperturbed water level $z = 0$ and $H(x,t) = x + \eta(x,t)$ is the total perturbed water depth. The unperturbed shoreline occurs at $x = 0$, $z = 0$.

and particularly the review by K. Satake [9] for contemporary numerical methods and the literature cited therein).

This note is concerned with an inverse problem for the long wave run-up on a plane sloping beach. There is an extensive literature devoted to the direct problem, i.e. finding various explicit solutions of the nonlinear shallow-water equations and computing runup characteristics from the known initial wave in a source. See e.g. [2,10–20] ,[1,21–25] ,[9,26–32] and the literature cited therein. In our work we solve the inverse problem when the source characteristics can be found via the moving shoreline. More specifically, given the time of an earthquake and the equation of motion of the shoreline (shoreline equation), estimate the dimensions of the tsunami source. This would not be much of a problem in the linear theory but the run-up process is essentially nonlinear, which presents a real challenge as inverse problems in nonlinear settings are particularly poorly understood from both geophysical and mathematical points of view. What we do in this contribution is to find a model that can be effectively linearized by a suitable hodograph transformation while still retaining most important features. We take a popular model of the 1D nonlinear shallow water system for an (infinite) sloping beach and show that the well-known Carrier–Greenspan hodograph turns it into a linear model whose inverse problem can be solved explicitly by means of the Abel transform under mild additional assumption that are acceptable (in fact, common) from a physical point of view. This provides a convenient setting to identify and treat much more complicated models as well as a quick assessment of what needs to be known and done in more general situations.

2. Shallow water equations (SWE)

A tsunami wave is a motion of viscous fluid described by the Navier–Stokes equations, a highly nonlinear 3+1 (three spatial and one temporal derivatives) system, which is notoriously hard to analyze even numerically. Under certain assumptions (e.g. no vorticity, no friction, no dispersion, 1D velocity field, small depth to wavelength ratio, etc.) this system can be simplified to a 2+1 system of order one equations referred to as the shallow water equations (SWE) [33]. We also assume that our bathymetry represents an infinite sloping plane beach extending along the y axis infinitely far (see Fig. 1).

The SWE can be reduced down to a 1+1 system which in dimensionless variables reads [33]

$$\begin{array}{ll} \partial_t \eta + \partial_x[(x+\eta)u] = 0 & \text{(continuity equation)} \\ \partial_t u + u\partial_x u + \partial_x \eta = 0 & \text{(momentum equation)} \end{array}, \tag{2.1}$$

where: $\eta(x,t)$ is the water elevation over unperturbed water level $z = 0$ (unperturbed water depth). It need not be sign definite (can be positive or negative); $u(x,t)$ is the flow velocity averaged over the z axis. Since the positive x-axis is directed off-shore, $u < 0$ and $u > 0$ typically[1] corresponds respectively to an in-coming wave (i.e. moving towards the shore) and out-going wave (i.e. moving from the shore).

Note that in the physical literature (2.1) is typically given in dimensional units with the acceleration due to gravity g and the slope α are present explicitly. The substitution $\widetilde{x} = (H_0/\alpha)\ x$, $\widetilde{t} = \sqrt{H_0/g}\, t/\alpha$, $\widetilde{\eta} = H_0\eta$, $\widetilde{u} = \sqrt{H_0 g}\ u$, where H_0 is a typical height (or length), transforms our system (2.1) into a dimensional one.

Since $H(x,t) = x + \eta(x,t)$ is the total (perturbed) water depth along the main axis x. From Fig. 1, the equation

$$x + \eta(x,t) = 0 \text{ (shoreline equation in physical plane)} \tag{2.2}$$

describes the motion of the shoreline (wet/dry boundary) and its solution $x_0(t)$ describes the run-up and run-down (draw down) of tsunami waves.

We are concerned with the initial value problem (IVP) for (2.1) with

$$\eta(x,0) = \eta_0(x), \quad u(x,0) = u_0(x) \qquad \text{(initial conditions)} \tag{2.3}$$

and refer to such initial conditions (IC) as standard.

The IVP (2.1)–(2.3) has drawn an enormous attention (see, the literature cited in Introduction). The system (2.1) has a quadratic nonlinearity but its main feature is that it can be linearized by using the Carrier–Greenspan (CG) hodograph (transformation), originally introduced in [14]. We use its version given in [15]:

$$\begin{array}{ll} \varphi(\sigma,\tau) = u(x,t), & \psi(\sigma,\tau) = \eta(x,t) + u^2(x,t)/2 \\ \sigma^2 = x + \eta(x,t), & \tau = (t - u(x,t))/2 \end{array}, \tag{2.4}$$

which turns (2.1) into a linear hyperbolic system which, in turn, can be reduced to the wave equation

$$\partial_\tau^2 \psi = \partial_\sigma^2 \psi + \sigma^{-1}\partial_\sigma \psi. \tag{2.5}$$

Note that the simple scaling $\tau = \lambda/2$ turns (2.5) into the one from [15]. Our form is more convenient for our purposes.

The IC (2.3) transforms as follows. Let $x = \gamma(\sigma)$ solve the equation $x + \eta_0(x) = \sigma^2$. In the hodograph plane (σ,τ)(2.3) turns into

$$\varphi|_\Gamma = \varphi_0(\sigma),\ \psi|_\Gamma = \psi_0(\sigma), \tag{2.6}$$

where $\Gamma = \{(\sigma, -u_0(\gamma(\sigma))/2) \mid \ \sigma \ge 0\}$ is a curve in the (σ,τ) plane and $\varphi_0(\sigma) = u_0(\gamma(\sigma))$, $\psi_0(\sigma) = \eta_0(\gamma(\sigma)) + u_0(\gamma(\sigma))^2/2$.

Thus, the nonlinear system (2.1) with linear IC (2.3) is transformed into a linear wave equation (2.5) initialized on a (nonlinear) curve [28,29]; σ^2 being the wave height from the bottom, τ is a delayed time, φ being the flow velocity, and ψ being wave energy. As is well-known, the main advantage of the CG hodograph is that moving shoreline (wet/dry boundary) $x_0(t)$ given by (2.2) is fixed now at $\sigma = 0$; the main drawback is that the curve is nonlinear and common methods for solving IVP for linear systems fail. The latter is not an issue when $u_0(x) = 0$ (no initial velocity). Indeed, Γ becomes a vertical line $(\sigma, 0)$ and the IC in (2.6) read

$$\varphi(\sigma,0) = 0, \quad \psi(\sigma,0) = \eta_0(\gamma(\sigma)). \tag{2.7}$$

Note that in (2.1)$t \ge 0$ but x is not sign definite while in (2.5)$\sigma \ge 0$ but τ need not be positive. Also note that $\sigma = 0$ is a regular singular for (2.5) causing some computational difficulties at the shoreline. Finally we emphasize that the CG hodograph works as long as it is invertible, i.e. when $\det \dfrac{\partial(\sigma,\tau)}{\partial(x,t)} \ne 0$. Recall (see e.g. [29]) that the latter holds as long as the wave does not break. We shall always assume this condition.

[1] In some cases an incoming wave may generate regions where $u > 0$.

3. The shoreline equation

In this section we are concerned with what we call the shoreline equation, an explicit equation relating $\varphi(0,\tau)$, $\psi(0,\tau)$ with $\varphi(\sigma,0)=\varphi_0(\sigma)$, $\psi(\sigma,0)=\psi_0(\sigma)$. This is of course a classical Cauchy problem but τ need not be positive. Using standard techniques of the Hankel transform we can derive

$$\psi(0,\tau)=\frac{d}{d\tau}\operatorname{sgn}\tau\left[\int_0^{|\tau|}\frac{s\psi_0(s)}{\sqrt{\tau^2-s^2}}dx-\tau\int_0^{|\tau|}\frac{s\varphi_0(s)}{\sqrt{\tau^2-s^2}}ds\right], \tag{3.1}$$

which is the classical Poisson formula adjusted to the signed time τ. These formulas can be conveniently written in terms of the Abel transform pair [18]

$$(\mathcal{A}f)(x)=\int_0^x\frac{f(s)ds}{\sqrt{x^2-s^2}},\quad (\mathcal{A}^{-1}f)(x)=\frac{2}{\pi}\frac{d}{dx}\int_0^x\frac{sf(s)ds}{\sqrt{x^2-s^2}}.$$

Since negative τ in (3.1) may occur only for over critical flows [10,11] (e.g. when an extreme run-down at the initial moment occurs) we can safely assume that $\tau\geq 0$. Then

$$\begin{aligned}\psi(0,\tau)&=\frac{d}{d\tau}\int_0^{\tau}\frac{s\psi_0(s)}{\sqrt{\tau^2-s^2}}dx-\tau\frac{d}{d\tau}\int_0^{\tau}\frac{s\varphi_0(s)}{\sqrt{\tau^2-s^2}}ds-\int_0^{\tau}\frac{s\varphi_0(s)}{\sqrt{\tau^2-s^2}}ds\\&=\frac{\pi}{2}\left[\mathcal{A}^{-1}\psi_0-\tau\mathcal{A}^{-1}\varphi_0\right]-\mathcal{A}(s\varphi_0).\end{aligned}$$

Applying the Abel transform, we have

$$(\mathcal{A}\Psi)(\sigma)=\frac{\pi}{2}\left[\psi_0(\sigma)-\mathcal{A}\tau\left(\mathcal{A}^{-1}\varphi_0\right)(\sigma)\right]-\left(\mathcal{A}^2s\varphi_0\right)(\sigma),$$

where we recall that ψ_0 is a function of σ. We now compute the repeated Abel transforms seen above. Denote

$$K_n(x,s)=\int_s^x\frac{t^n dt}{\sqrt{(t^2-s^2)(x^2-t^2)}},\quad n=0,2,$$

which is an elliptic integral. By switching the order of integration, one obtains

$$\left(\mathcal{A}^2s\varphi_0\right)(\sigma)=\int_0^{\sigma}s\varphi_0(s)K_0(\sigma,s)ds.$$

The other repeated transformation is computed by integrating the innermost integral by parts and then switching the order of integration to obtain

$$\mathcal{A}\tau\left(\mathcal{A}^{-1}\varphi_0\right)(\sigma)=\sigma\varphi_0(0)+\int_0^{\sigma}K_2(\sigma,s)\varphi_0'(s)ds.$$

Thus

$$\begin{aligned}&\psi_0(\sigma)+\int_0^{\sigma}\left[\partial_sK_2(\sigma,s)-\frac{2s}{\pi}K_0(\sigma,s)\right]\varphi_0(s)\,ds\\&=\frac{2}{\pi}(\mathcal{A}\Psi)(\sigma),\ \text{(shoreline equation)}.\end{aligned} \tag{3.2}$$

This equation is fundamental to solving the inverse problem for tsunami waves. In different forms and for specific ψ_0,φ_0 it appears in e.g. [14,15,22,29,30].

4. Inverse problem

We are now concerned with the following inverse problem. Given a sloping plane bathymetry and assuming zero initial velocity[2] $u_0(x) = 0$, suppose we know the law of motion $x_0(t)$ of the shoreline. Find the initial displacement $\eta_0(x)$. In this section we give a complete solution to this problem. I.e. we show that (2.4) and (3.2) yield an explicit and totally elementary procedure to restore $\eta_0(x)$ from $x_0(t)$.

It follows from (2.4) that if $t = 0$ then $\tau = 0$ and $\varphi_0(\sigma) = \varphi(\sigma, 0) = 0$. Therefore (3.2) simplifies to

$$\psi_0(\sigma) = (2/\pi)(\mathcal{A}\Psi)(\sigma). \tag{4.1}$$

Note also that $dx_0(t)/dt = v_0(t)$ is the shoreline velocity. Apparently $v_0(t) = u(x_0(t), t)$ and at the shoreline ($\sigma = 0$) the CG hodograph (2.4) then reads

$$\begin{array}{cc} \varphi(0,\tau) = v_0(t), & \psi(0,\tau) = \eta(x_0(t), t) + v_0(t)^2/2 \\ x_0(t) + \eta(x_0(t), t) = 0, & \tau = (t - v_0(t))/2 \end{array}. \tag{4.2}$$

The inverse problem is now solved as follows:

(1) Compute $\Psi(\tau)$ for (4.1) from the shoreline data $x_0(t)$. It follows from (4.2) that $\Psi(\tau) = \psi(0, \tau)$ can be (implicitly) found from

$$\Psi(\tau) = -x_0(t) + v_0(t)^2/2, \quad \tau = (t - v_0(t))/2.$$

(2) By (4.1) we find $\psi(\sigma, 0) = \psi_0(\sigma)$.

(3) Setting in (2.4)$\tau = 0$ ($\sigma \geq 0$) we find the (implicit) equation for the initial displacement $\eta_0(x)$:

$$\eta_0(x) = \psi_0(\sigma), \quad \sigma = \sqrt{x + \eta_0(x)}.$$

The space limitation does not allow us to present numerical experiments here. However, since the direct problem is well-verified already [13] and the inverse problem does not contain additional approximations, the latter is as accurate as the former and a numerical test of our algorithm is unnecessary. Such numerics are nevertheless of interest for applications to geophysics (tsunami problems) and will be given elsewhere.

5. Conclusions

Steps 1–3 above provide a smoothly working algorithm for solving the inverse problem for tsunami waves under the assumption that the initial velocity is zero. A similar algorithm should be in order for zero initial displacement and nonzero initial velocity. The general case of when $u_0(x)$ and $\eta_0(x)$ are both nonzero[3] is of course of great interest. However, on a sloping bottom there is no rigorous description of the traveling wave (as opposed to a flat bottom), and only approximate results can be obtained. We believe that the important case $u_0(x) = -\eta_0(x)/\sqrt{x}$ and $u_0(x)^2 \ll 1$ may be successfully treated but the shoreline equation (3.2) becomes a Fredholm integral equation which no longer has an explicit solution. We hope to return to it elsewhere.

More complicated U and V shaped bathymetries may also be treated. In fact, we believe that the inverse problem can be solved in closed form in the case when the underlying direct problem can be linearized (see e.g. [16,19,27,29] for such cases). Considering general power shaped bays is work in progress.

The most practical problem is when the beach is plane for $x \leq L$ with some $L > 0$.[4] The inverse problem then requires to find the wave at L. The underlying problem is boundary and the techniques of [10,29] should be used to derive the shoreline equation in this case.

[2] Such an assumption is typical in the tsunami problem, when the bottom displacement is specified (in the framework of shallow water, it is equivalent to the displacement of the water surface) by the Okada seismic model [1].

[3] This situation corresponds to a wave approaching the shore and is often implemented in laboratory experiments.

[4] One of the reasons is that we may neglect dispersion only when the wave is close to the shoreline and catastrophic events are about to arise.

Data availability

No data was used for the research described in the article.

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