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On Promotion and Quasi-Tangled Labelings of Posets

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Abstract. In 2022, Defant and Kravitz introduced extended promotion (denoted ∂), a map that acts on the set of labelings of a poset. Extended promotion is a generalization of Schützenberger's promotion operator, a well-studied map that permutes the set of linear extensions of a poset. It is known that if L is a labeling of an n-element poset P, then $\partial^{n-1}(L)$ is a linear extension. This allows us to regard ∂ as a sorting operator on the set of all labelings of P, where we think of the linear extensions of P as the labelings which have been sorted. The labelings requiring $n-1$ applications of ∂ to be sorted are called *tangled*; the labelings requiring $n-2$ applications are called *quasi-tangled*. We count the quasi-tangled labelings of a relatively large class of posets called *inflated rooted trees with deflated leaves*. Given an n-element poset with a unique minimal element with the property that the minimal element has exactly one parent, it follows from the aforementioned enumeration that this poset has $2(n-1)!-(n-2)!$ quasi-tangled labelings. Using similar methods, we outline an algorithmic approach to enumerating the labelings requiring $n-k-1$ applications to be sorted for any fixed $k \in \{1,\dots,n-2\}$. We also make partial progress towards proving a conjecture of Defant and Kravitz on the maximum possible number of tangled labelings of an n-element poset.

1. Introduction

1.1. Background

Let P be an n-element poset, whose order relation we denote by $<_P$. A *labeling* of P is a bijection $L : P \to [n]$ (where $[n] = \{1,\dots,n\}$). A labeling L is called a *linear extension* if it preserves the order on P, i.e. if for all pairs $x, y \in P$ with $x <_P y$ we have $L(x) < L(y)$. Let $\Lambda(P)$ be the set of all labelings of P; let $\mathcal{L}(P) \subset \Lambda(P)$ be the subset consisting of all linear extensions.

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In [21–23], Schützenberger introduced an intriguing bijection on $\mathcal{L}(P)$ called *promotion*. Promotion has connections with various topics in algebraic combinatorics and representation theory, as seen in [12,15,19,20,24].

In [9], Defant and Kravitz extended the promotion map to an operator $\partial : \Lambda(P) \to \Lambda(P)$, not necessarily invertible, that is defined on all of $\Lambda(P)$. When the poset is a chain, extended promotion is dynamically equivalent to the bubble-sort map studied in [16]. Promotion can also be described in terms of Bender-Knuth involutions (first introduced by Haiman [13] as well as Malvenuto and Reutenauer [18]); in [9], Defant and Kravitz extended these Bender-Knuth involutions to arrive at an equivalent "toggle" definition of extended promotion. The following results of [9] are crucial properties of extended promotion:

1. When restricted to $\mathcal{L}(P)$, ∂ agrees with Schützenberger's promotion operator.
2. If L is a labeling of an n-element poset P, then $\partial^{n-1}(L) \in \mathcal{L}(P)$.

Thus, (extended) promotion[1] may be regarded as a sorting operator, where linear extensions are considered "sorted." Property (2) shows that promotion sorts every labeling after at most $n-1$ applications.

We define the *sorting time* of a labeling L to be the smallest $k \in \mathbb{N}$ (here we take $\mathbb{N}$ to include 0) such that $\partial^k(L) \in \mathcal{L}(P)$. Defant and Kravitz mainly studied *tangled* labelings—those labelings with sorting time $n-1$. In particular, they enumerated these tangled labelings for a large class of posets called *inflated rooted forests*.

Given any noninvertible combinatorial dynamical system, it is natural to try to study (especially, enumerate) those objects requiring a fixed number of iterations to reach a periodic point. In particular, given a sorting operator, it is natural to enumerate the objects requiring a certain number of applications to be sorted. For other sorting operators, substantial effort has been dedicated to performing such enumerations. For example, in [2], Chung-Claesson-Dukes-Graham gave explicit formulas for the permutations on n letters with d descents requiring a minimum of k applications of bubble sort to reach the identity permutation. In [5,7,8], Defant performed similar enumerations for the 2- and 3-stack-sortable permutations, and in [3], Claesson-Dukes-Steingrímsson counted the $(n-4)$-stack-sortable permutations. Similarly-flavored results are given in [4,6,14] for pop-stack sorting and in [11,17] for pop-tsack torsing.

In the context of promotion, even counting labelings that require no applications of promotion to be sorted is quite difficult, as this is just counting the linear extensions of a poset. Brightwell and Winkler showed in [1], that this problem is #P-complete. However, restricting to certain classes of posets, such as Young-diagram-shaped or rooted tree posets, can sometimes lead to the discovery of very nice formulas. For example, in [16], Knuth gave a hook-length formula counting the number of linear extensions of a rooted forest poset. In [9], Defant and Kravitz enumerated the tangled labelings of a large class of posets built from and generalizing rooted tree posets, called inflated rooted

[1] Henceforth, "promotion" always refers to ∂ rather than its restriction to $\mathcal{L}(P)$.

tree posets. The main result of this paper builds on the work of Defant and Kravitz. In particular, we give an explicit enumeration of the quasi-tangled labelings for a large subset of inflated rooted trees which, importantly, contains rooted tree posets.

1.2. Outline and Summary of Main Results

In Sect. 2, we present the main definitions and background results needed for the rest of the paper. The main result of the paper, an explicit enumeration of the labelings with sorting time $n-2$ for a large class of posets called inflated rooted trees with deflated leaves (Theorem 9), is given in Sect. 3. A corollary of this result is that an n-element poset with a unique minimal element with the property that the minimal element has exactly one parent has $2(n-1)!-(n-2)!$ quasi-tangled labelings. Another consequence of the main theorem is given in Sect. 3.4, where we present an algorithmic approach to enumerating the labelings of a rooted tree poset with sorting time $n-k-1$ for fixed $k \in \{1,\dots,n-2\}$. In Sect. 4, we make partial progress (Theorem 28) on the following conjecture:

Conjecture 1. ([9], Conjecture 5.1) *If P is an n-element poset, then P has at most $(n-1)!$ tangled labelings.*

We also prove that, for an inflated rooted tree poset with deflated leaves, there are more quasi-tangled labelings than tangled labelings, and we conjecture that this statement holds for an arbitrary poset. Finally, we conclude Sect. 4 with several open problems and further directions of inquiry.

2. Preliminaries and Some Special Classes of Posets

2.1. Extended Promotion

Let P be an n-element poset, and let L be a labeling of P. For $x \in P$ not maximal, the *L-successor* of x is the element greater than x with minimal label. Now, let $v_1 = L^{-1}(1)$. Let v_2 be the L-successor of v_1; let v_3 the L-successor of v_2, and so on until we get an element v_m that is maximal. The resulting chain $v_1 <_P v_2 <_P \cdots <_P v_m$ is called the *promotion chain* of L. Now, define $\partial(L)$ to be the labeling

$$\partial(L)(x) = \begin{cases} L(x)-1 & \text{if } x \notin \{v_1,\dots,v_m\}; \\ L(v_{i+1})-1 & \text{if } x = v_i \text{ for } i \in \{1,\dots,m-1\}; \\ n & \text{if } x = v_m. \end{cases}$$

In other words, promotion may be thought of as decreasing each label by 1 (working modulo n so that $0 = n$) and then cycling the promotion chain downwards one step. The following proposition captures a fundamental sorting property of promotion (Fig. 1).

Proposition 2. ([9], Proposition 2.7) *If P is an n-element poset, then $\partial^{n-1}(\Lambda(P)) = \mathcal{L}(P)$.*

2.2. Preliminary Definitions

A *lower order ideal* of a poset P is a subset $Q \subset P$ such that for every $x \in Q$ and $y \in P$ with $y <_P x$ we have $y \in Q$. Similarly, an *upper order ideal* of a poset P is a subset $Q \subset P$ such that for all $x \in Q$ and $y \in P$ with $y >_P x$ we have $y \in Q$. It is often useful to note that Q is a lower order ideal of P if and only if $P \backslash Q$ is an upper order ideal of P. For $x, y \in P$, we say that y *covers* x and write $x \lessdot y$ if $x <_P y$ and $\{z \in P \mid x <_P z <_P y\} = \emptyset$. In this case, we say that y is a *parent* of x and that x is a *child* of y.

The *Hasse diagram* of a poset P is a graphical illustration of its covering relations. Each element of P is represented by a vertex, and if $x <_P y$, then the vertex corresponding to x is drawn below that corresponding to y; there exists an edge between these vertices if and only if $x \lessdot_P y$. We say a poset is *connected* if its Hasse diagram is connected when regarded as a graph; the *connected components* of P are the subposets induced by the connected components of the Hasse diagram of P.

Suppose P is an n-element poset, and let $f : P \to \mathbb{Z}$ be an injective function. Then the *standardization* of f, denoted $\mathrm{st}(f)$, is the labeling $L : P \to [n]$ such that for all $x, y \in P$, $L(x) < L(y)$ if and only if $f(x) < f(y)$. Note that this labeling is unique. Equivalently, if $g : f(P) \to [n]$ is an order-preserving bijection, then $\mathrm{st}(f) = g \circ f$.

Proposition 2 motivates the following definitions:

Definition 1. Let L be a labeling of a poset P. The sorting time of L is the minimum number $k \in \mathbb{N}$ such that $\partial^k(L) \in \mathcal{L}(P)$. Note that the sorting time of a linear extension is 0.

Definition 2. For an n-element poset P, a labeling is called *tangled* if it has sorting time $n-1$. A labeling is called *quasi-tangled* if it has sorting time $n-2$.

If P is a poset, L a labeling of P, and $\gamma \in \mathbb{N}$, we will frequently use the shorthand L_γ to denote $\partial^\gamma(L)$. Note that $L_0 = L$.

Example 1. The following illustrates how extended promotion works:

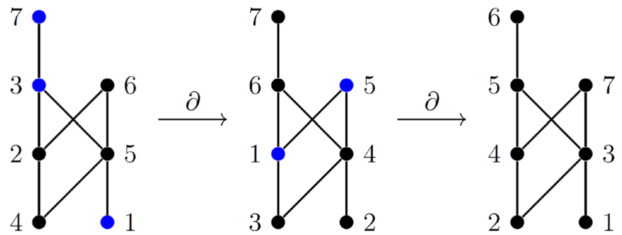

FIGURE 1. In the above, the promotion chain is colored in blue

2.3. Some Useful Results About Promotion

The following results will be helpful later:

Lemma 3. *Let P be an n-element poset. For $\gamma \in \mathbb{N}$ and any $x \in \{2, \ldots, n\}$, we have that $L_\gamma^{-1}(x) \geq_P L_{\gamma+1}^{-1}(x-1)$; equality holds if and only if $L_\gamma^{-1}(x)$ is not in the promotion chain of L_γ.*

Proof. Suppose first that $L_\gamma^{-1}(x)$ is not in the γth promotion chain. Then $L_{\gamma+1}^{-1}(x-1) = L_\gamma^{-1}(x)$, and we are done. Otherwise, since $x > 1$, there exists an element $a <_P L_\gamma^{-1}(x)$ such that a is in the promotion chain and $L_\gamma^{-1}(x)$ is the L_γ-successor of a. Hence, $a = L_{\gamma+1}^{-1}(x-1) <_P L_\gamma^{-1}(x)$, as desired. □

Lemma 4. *Let x and y be two elements of P with $y <_P x$. Fix some $\gamma \in \{1, \ldots, n-2\}$, and suppose that $L_\gamma(y) > L_\gamma(x)$. Then $L_\gamma(y) = L_{\gamma-1}(y) - 1$. Moreover, we have that $L_{\gamma-1}(y) > L_{\gamma-1}(x)$.*

Proof. Begin by letting $a = L_\gamma(y)$ and $b = L_\gamma(x)$, and note that $a > b$. Observe that the first part of the lemma holds if and only if y is not in the $(\gamma-1)$st promotion chain. Assume for a contradiction that y *is* in the promotion chain of $L_{\gamma-1}$. By Lemma 3, $L_{\gamma-1}^{-1}(b+1) \geq_P L_\gamma^{-1}(b) = x >_P y$, so $L_{\gamma-1}^{-1}(a+1)$ cannot be the $L_{\gamma-1}$-successor of y, since $a > b$. This is a contradiction. The second part of the lemma is simple: $L_{\gamma-1}(x) \leq b+1 < a+1 = L_{\gamma-1}(y)$. □

Theorem 5. ([9], Theorem 2.10) *For $0 \leq k \leq n-2$, an n-element poset P has a labeling with sorting time $n-k-1$ if and only if it has a lower order ideal of size $k+2$ that is not an antichain.*

2.4. Rooted Tree and Forest Posets

Definition 3. A *rooted forest poset* is a poset in which each element is covered by at most one other element. A *rooted tree poset* is a connected rooted forest poset. Given a rooted tree poset, we say a subset Q of P is a subchain (respectively, subtree) if the poset induced by Q is a chain (respectively, tree). The *leaves* of a rooted tree poset are its minimal elements (Fig. 2).

Note here that a rooted tree poset is a poset whose Hasse diagram is a rooted tree where the root is the unique maximal element. A rooted forest poset is a poset whose connected components are rooted tree posets.

2.5. Inflated Rooted Trees

Definition 4. Let Q be a finite poset. An *inflation* of Q is a poset P along with a surjective map $\varphi : P \to Q$ satisfying:

1. For all $v \in Q$, the set $\varphi^{-1}(v)$ has a unique minimal element.
2. If $x, y \in P$ are such that $\varphi(x) \neq \varphi(y)$, then $x <_P y$ if and only if $\varphi(x) <_Q \varphi(y)$.

An *inflated rooted tree poset* is an inflation of a rooted tree poset. An *inflated rooted tree poset with deflated leaves* is an inflation of a rooted tree poset Q such that for all leaves $\ell \in Q$ we have $|\varphi^{-1}(\ell)| = 1$.

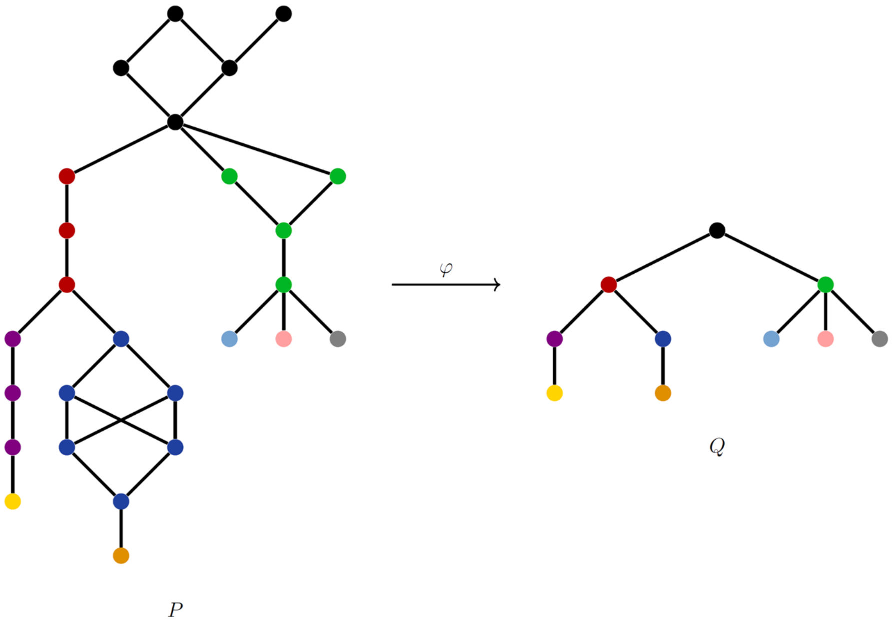

FIGURE 2. (P, φ) is an inflated rooted tree with deflated leaves, where P inflates Q. The colors illustrate the preimages of the elements of Q

Remark 1. We say that a rooted tree poset is *reduced* if no vertex has exactly one child. Since the composition of inflations is also an inflation, we see that every inflated rooted tree poset is the inflation of a reduced rooted tree poset. When we refer to an inflation $\varphi : P \to Q$ of a rooted tree poset with deflated leaves we will assume that the subposet of Q obtained by removing its leaves is reduced.

3. Quasi-Tangled Labelings of Inflated Rooted Trees with Deflated Leaves

3.1. Setup and Statement of the Main Theorem

In the following, we give a formula enumerating the quasi-tangled labelings of inflated rooted tree posets with deflated leaves. The following is shown in the proof of Theorem 5, but we state it as its own lemma here:

Lemma 6. ([9], Proof of Theorem 2.10) *Let P be an n-element poset and L a labeling of P such that $L_{n-k} \notin \mathcal{L}(P)$. Then $\{L_{n-k}^{-1}(1), \ldots, L_{n-k}^{-1}(k)\}$ forms a lower order ideal of size k that is not an antichain, and the restriction of L_{n-k} to this set is not a linear extension.*

Lemma 7. *If L is a quasi-tangled labeling of an n-element inflated rooted tree poset P with deflated leaves, then one of the following holds:*

1. *$L^{-1}(n-1)$ is minimal;*
2. *$L^{-1}(n)$ is minimal;*
3. *$L^{-1}(n)$ covers a minimal element.*

In each case, the element in question is involved in an inversion after $n-3$ promotions.

Proof. By Lemma 6, the set $Y = \{L_{n-3}^{-1}(1), L_{n-3}^{-1}(2), L_{n-3}^{-1}(3)\}$ forms a lower order ideal that is not an antichain. Moreover, L_{n-3} restricted to this set is not a linear extension.

Now, note that every non-antichain lower order ideal I with three elements occurring in an inflated rooted tree poset with deflated leaves is one of the following:

1. I is a chain with 3 elements;
2. I is the rooted tree poset on 3 elements with 2 leaves;
3. I is the disjoint union of a singleton and a chain of size 2.

Suppose Y is of type (1). If $L_{n-3}^{-1}(3)$ is maximal in Y, then $L_{n-3}^{-1}(2) \lessdot L_{n-3}^{-1}(1) \lessdot L_{n-3}^{-1}(3)$, and repeatedly applying Lemma 4 tells us that $L^{-1}(n-1)$ is minimal. If $L_{n-3}^{-1}(3)$ is not maximal, then it covers a minimal element or is minimal itself. We may again repeatedly apply Lemma 4 to see that $L^{-1}(n)$ either covers a minimal element or is minimal.

Suppose Y is of type (2). It follows that $L_{n-3}^{-1}(3)$ cannot be maximal, because then L_{n-3} would be a linear extension. So, $L_{n-3}^{-1}(3)$ is minimal. The element covering $L_{n-3}^{-1}(3)$ has a smaller label in L_{n-3}, so $L^{-1}(n)$ is minimal by Lemma 4.

Lastly, suppose Y is of type (3). If $L_{n-3}^{-1}(3)$ occupies its own component, then $L_{n-3}^{-1}(2) \lessdot L_{n-3}^{-1}(1)$ and $L^{-1}(n-1)$ is minimal. Otherwise, $L_{n-3}^{-1}(3)$ is covered by either $L_{n-3}^{-1}(1)$ or $L_{n-3}^{-1}(2)$, so $L^{-1}(n)$ must be minimal. □

Theorem 9 enumerates the quasi-tangled labelings of inflated rooted trees with deflated leaves. In particular, note that this relatively large class of posets includes rooted tree posets. While this particular class of posets seems artificial, it will be important that the posets we are working with have the property that if one removes a minimal element or an element covering a minimal element from the poset, then the resulting poset remains an inflated rooted tree.

We remark here that the quasi-tangled labelings of a poset are those labelings L such that L_{n-3} is not a linear extension but L_{n-2} is a linear extension, i.e., L is not tangled. To count these labelings, we condition on the positions of $L^{-1}(n-1)$ and $L^{-1}(n)$ and keep track of where $L_\gamma^{-1}(n-1-\gamma)$ and $L_\gamma^{-1}(n-2-\gamma)$ end up (after $n-3$ promotions these are the elements that are labeled 2 and 1, respectively). In particular, we consider a uniformly random labeling L of P with $L^{-1}(n)$ or $L^{-1}(n-1)$ fixed and calculate the probability that, at each "branch vertex," the elements in question "get pulled down" in the desired direction, i.e., in the direction such that L_{n-3} is not a linear extension. The proof of the theorem involves a lot of casework, which we prove beforehand in several technical lemmas.

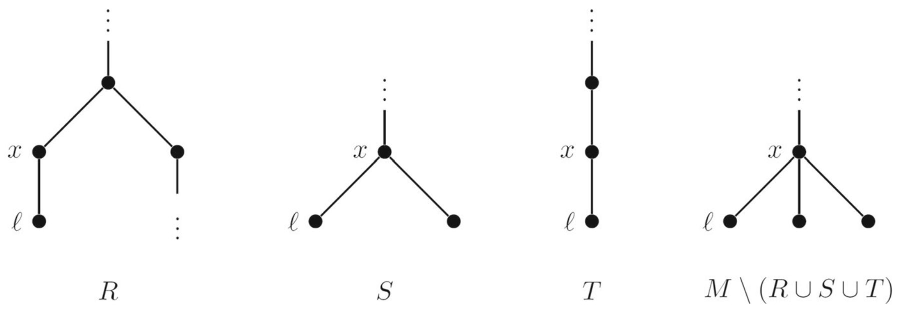

FIGURE 3. An illustration of what the elements of R, S, and T "look like"

Let (P, φ) be an inflation of a rooted tree poset Q. For each leaf ℓ of Q, there exists a unique path in the Hasse diagram of Q from ℓ to the root. Let the elements of this path be called $u_{\ell,0}, u_{\ell,1}, \ldots, u_{\ell,\omega(\ell)}$, where $u_{\ell,0} = \ell$ and for all $1 \leq j \leq \omega(\ell)$, $u_{\ell,j}$ covers $u_{\ell,j-1}$. Then define

$$b_{\ell,j} = \sum_{v \leq_Q u_{\ell,j-1}} |\varphi^{-1}(v)| \qquad \text{and} \qquad c_{\ell,j} = \sum_{v <_Q u_{\ell,j}} |\varphi^{-1}(v)|.$$

Note that $\frac{b_{\ell,j}}{c_{\ell,j}}$ is the fraction of elements below the minimal element of $\varphi^{-1}(u_{\ell,j})$ that "lie in the direction" of $\varphi^{-1}(\ell)$. The following enumerates the number of tangled labelings of P:

Theorem 8. ([9], Theorem 3.5) *Let P be an n-element inflation of a rooted tree poset Q, and assume $n \geq 2$. Then the number of tangled labelings of P is*

$$(n-1)! \sum_{i=1}^{s} \prod_{j=1}^{\omega(\ell_i)} \frac{b_{\ell_i,j} - 1}{c_{\ell_i,j} - 1},$$

where $\ell_1, \ldots, \ell_s$ are the leaves of Q.

Now, define M to be the set of minimal elements of P. We define subsets R, S, T of M: Let $\ell \in M$, and let x cover ℓ.

1. Put ℓ in R if ℓ is the only child of x and the parent of x exists and has multiple children.
2. Put ℓ in S if x has precisely two children;
3. Put ℓ in T if ℓ is the only child of x, the parent of x exists, and x is the only child of its parent.

Note here that R, S, and T are disjoint but do not partition P. However, it is not difficult to see that R, S, and T partition the set of minimal elements of P that, along with their parents, lie in non-antichain lower order ideals of size 3. See Fig. 3.

Note that a 2-element poset has no quasi-tangled labelings (this follows from Theorem 5). In the following, we will assume that our posets have at least 3 elements.

Theorem 9. *Let (P, φ) be an inflation of the rooted tree poset Q with deflated leaves. Suppose P has n elements, and let $n \geq 3$. Let M, R, S, T, and the $u_{\ell,j}$'s, $b_{\ell,j}$'s, and $c_{\ell,j}$'s be as defined above. Then the number of quasi-tangled labelings of P is*

$$(n-1)!\left(2\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}-\frac{1}{n-1}\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-2}{c_{\ell,j}-2}+2\sum_{\ell\in R}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}\right.$$
$$\left.+\sum_{\ell\in S}\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}\right).$$

Corollary 10. *Let P be a poset with n elements, and assume $n \geq 3$. Suppose that P has a unique minimal element and that this minimal element has exactly one parent. Then P has $2(n-1)! - (n-2)!$ quasi-tangled labelings.*

3.2. Computing Probabilities

In this section, we compute several probabilities that will help us in the proof of Theorem 9. Importantly, in Lemma 13, we generalize Lemma 3.11 of [9], which tells us that the probability of a certain label ending up in some subtree of our inflated rooted tree poset after a certain number of promotions is proportional to the size of the subtree.

Lemma 11. ([9], Lemma 3.9) *Let P be an N-element poset, and let $X = \{y \in P \mid y <_P x\}$ for some $x \in P$. Suppose every element of P that is comparable with some element of X is also comparable with x. If L and $\tilde{L}$ are labelings of P that agree on $P \backslash X$, then for every $\gamma \geq 1$, the labelings L_γ and $\tilde{L}_\gamma$ also agree on $P \backslash X$.*

Lemma 12. ([9], Lemma 3.10) *Let P, x, and X be defined as in Lemma 11. If L is a labeling of P and $\gamma \geq 0$, then the set $L_\gamma(X)$ depends only on the set $L(X)$ and the restriction $L|_{P\backslash X}$; it does not depend on the way in which the labels in $L(X)$ are distributed among the elements of X.*

For the rest of this subsection, let P, x, and X be defined as in Lemma 11. Suppose there is a partition of X into disjoint subsets A and B such that no element of A is comparable to an element of B. Note that both A and B are lower order ideals of P.

Definition 5. Let L be a labeling of P. Suppose that $k \in [N-1]$ and $m \in [N]$ are such that $m + k \leq N$, $L_k^{-1}(m) \in X$, and $L^{-1}(m+k) \notin X$. We say that γ *pulls down* $m+k$ if $\gamma < k$ is the largest index such that $L_\gamma^{-1}(m+k-\gamma) \notin X$. We see immediately from the definition that γ is the unique such index pulling down $m+k$ and that γ pulls down exactly one label.

With notation as above, note that in order to determine whether $L_k^{-1}(m) \in A$, it suffices to determine whether $L_\gamma^{-1}(1) \in A$: because $L_\gamma^{-1}(m+k-\gamma) \notin X$ and $L_{\gamma+1}^{-1}(m+k-\gamma-1) \in X$, we have that $L_\gamma^{-1}(m+k-\gamma)$ is in the γth promotion chain. Hence, $L_{\gamma+1}^{-1}(m+k-\gamma-1) \in A$ if and only if $L_\gamma^{-1}(1) \in A$ (recall A and B are disjoint lower order ideals).

To each such $m+k$, one can associate a decreasing sequence of indices $\gamma_0, \gamma_1, \dots, \gamma_r$ whose values depend only on $L|_{P\setminus X}$ as follows. Let γ_0 pull down $m+k$. By Lemma 12, the value γ_0 depends only on $L|_{P\setminus X}$. If $L^{-1}(\gamma_0+1) \in X$, we are done. Otherwise, let γ_1 pull down γ_0+1, where we let γ_0 take the role of k and 1 the role of m (note that $\gamma_1 < \gamma_0$). If $L^{-1}(\gamma_1+1) \in X$, we are done; otherwise, let γ_2 pull down γ_1+1. This process can be continued, where γ_{i+1} pulls down γ_i+1. Since $0 \leq \gamma_{i+1} < \gamma_i$, this process eventually terminates, yielding a decreasing sequence $\gamma_0, \dots, \gamma_r$. By Lemma 12, we see that the values $\gamma_0, \dots, \gamma_r$ depend only on the set $L(X)$ and $L|_{P\setminus X}$, not on the way in which the labels in $L(X)$ are distributed.

Let $a, b \in [N]$ and $k \in [N-1]$ be such that $a+k, b+k \leq N$, $L_k^{-1}(a), L_k^{-1}(b) \in X$, and $L^{-1}(a+k), L^{-1}(b+k) \notin X$. Suppose $a \neq b$. Let $\alpha_0, \dots, \alpha_r$ and $\beta_0, \dots, \beta_s$ be the sequences associated to $a+k$ and $b+k$, respectively. We claim that $\{\alpha_0, \dots, \alpha_r\} \cap \{\beta_0, \dots, \beta_s\} = \emptyset$. Assume the contrary. If $\alpha_i = \beta_j$ for some $0 \leq i \leq r$ and $0 \leq j \leq s$, then it follows from Definition 5 that $\alpha_r = \beta_s$.

Without loss of generality, suppose $r \leq s$. If $r < s$, then by definition, $\alpha_r = \beta_s$ pulls down $\alpha_{r-1}+1 = \beta_{s-1}+1$, $\alpha_{r-1} = \beta_{s-1}$ pulls down $\alpha_{r-2}+1 = \beta_{s-2}+1$, etc., until $\alpha_0 = \beta_{s-r}$ pulls down $a+k = \beta_{s-r-1}+1$. It follows that $a+k-1 = \beta_{s-r-1}$. Note that $a+k-1 \geq k$, but $\beta_{s-r-1} \leq \beta_0 < k$. This is a contradiction. The case where $s > r$ is identical. If $r = s$, then $\alpha_0 = \beta_0$, and it follows that $a+k = b+k$, contradicting our assumption that $a \neq b$.

The following is a generalization of Lemma 3.11 in [9]. For $d = 1$, the lemmas are exactly the same. This lemma will be applied repeatedly in the proof of Theorem 9. Informally, it states that given a list of d labels whose corresponding elements are in X after k promotions, the probability that all of these labels are in A is proportional to $(|A|)!/(|A|-d-1)!$.

Lemma 13. (Probability Lemma) *Let P, x, and X be defined in Lemma 11, and let A and B be defined as above. Let $k \in [N-1]$, and let $n_1, \dots, n_d \in [N]$ be such that $n_i + k \leq N$ for all $1 \leq i \leq d$. Fix an injective map $M : P\setminus X \to [N]$ such that every labeling L extending M has the property that $L_k^{-1}(n_1), \dots, L_k^{-1}(n_d) \in X$. If such an L is chosen uniformly at random among all such extensions of M, then the probability that $L_k^{-1}(n_1), \dots, L_k^{-1}(n_d) \in A$ is*

$$\frac{|A|(|A|-1)\cdots(|A|-d)}{|X|(|X|-1)\cdots(|X|-d)}.$$

Proof. Suppose that $n_1 < n_2 < \cdots < n_d$, and let $n_{i_1} < \cdots < n_{i_t}$ be the subset of labels such that $L^{-1}(n_i+k) \notin X$. For all $s \notin \{i_1, \dots, i_t\}$, because $L^{-1}(n_s+k) \in X$, Lemma 3 gives that $L_k^{-1}(n_s) \in A$ if and only if $L^{-1}(n_s+k) \in A$.

Now, by our discussion above, to each n_{i_j} we may associate a decreasing sequence of γ^j's given by $\gamma_0^j > \cdots > \gamma_{r(j)}^j$. Note that by Lemma 12, the set of γ^j's depends only on M, not on how the labels in $L(X)$ are distributed. Recall that the sets $\{\gamma_0^j, \dots, \gamma_{r(j)}^j\}$ are pairwise disjoint. Importantly, we have that the $\gamma_{r(j)}^j$'s are all distinct.

Fix some n_j for $j \in \{i_1, \ldots, i_t\}$. In this and the next paragraph, denote the associated sequence by $\gamma_0, \ldots, \gamma_r$. We claim that the probability that $L_k^{-1}(n_j) \in A$ is equal to the probability that $L_{\gamma_r}^{-1}(1) \in A$. To see why this is true, recall that γ_0 pulls down $n_j + k$. In the discussion above, we showed that $L_k^{-1}(n_j) \in A$ if and only if $L_{\gamma_0}^{-1}(1) \in A$. Now, γ_1 pulls down $\gamma_0 + 1$, so $L_{\gamma_0}^{-1}(1) \in A$ if and only if $L_{\gamma_1}^{-1}(1) \in A$. Clearly, we may continue in this manner until we see that $L_k^{-1}(n_j) \in A$ if and only if $L_{\gamma_r}^{-1}(1) \in A$.

By assumption, $L^{-1}(\gamma_r + 1) \in X$. Because A and B are disjoint lower order ideals, Lemma 3 implies that $L_{\gamma_r}^{-1}(1) \in A$ if and only if $L^{-1}(\gamma_r + 1) \in A$. Hence, the probability that $L_k^{-1}(n_j) \in A$ is equal to the probability that $L^{-1}(\gamma_r + 1) \in A$.

Thus, for all i, we have reduced calculating the probability that $L_k^{-1}(n_i) \in A$ to calculating the probability that $L^{-1}(a_i) \in A$ for some particular label $a_i \in [N]$. Note that our assumptions on M give $L^{-1}(a_i) \in X$ for all i. For each n_{i_j}, we have that $a_{i_j} = \gamma_{r(j)}^j + 1$. Recall that the $\gamma_{r(j)}^j$'s are all distinct; moreover note that for all $s \notin \{i_1, \ldots, i_t\}$, we have that $\gamma_{r(j)}^j + 1 \neq n_s + k$, since $\gamma_{r(j)}^j + 1 \leq \gamma_0^j + 1 \leq k < n_s + k$. For each $s \notin \{i_1, \ldots, i_t\}$, $a_s = n_s + k$. Thus, the a_i's are distinct. Since L is chosen uniformly at random from the labelings extending M, it follows that the probability $L_k^{-1}(n_i) \in A$ for all i is

$$\frac{|A|(|A|-1)\cdots(|A|-d)}{|X|(|X|-1)\cdots(|X|-d)},$$

as desired. □

3.3. Proof of the Main Theorem

Lemma 14. *Let P be an n-element poset, and let L be a labeling of P. Let $x_0 \in P$. Define $\tilde{P} = P \setminus \{x_0\}$ and $\tilde{L} = \mathrm{st}(L|_{\tilde{P}})$. Suppose that x_0 is not part of the promotion chain for any of the first γ promotions. Then $\mathrm{st}(L_\gamma|_{\tilde{P}}) = \tilde{L}_\gamma$.*

Proof. Recall that promotion depends only on the promotion chain, which in turn depends only on the relative order of the labels. Since x_0 is never in the promotion chain for the first γ promotions, the promotion chains of $L_0, \ldots, L_\gamma$ and $\tilde{L}_0, \ldots, \tilde{L}_\gamma$ are the same, as desired. □

Before proving Theorem 9, we define some notation. Let P be as in Theorem 9, and let $x_0 \in P$ either cover a unique minimal element or be minimal itself. If x_0 is minimal, let $\ell = x_0$; if it covers a minimal element, denote this minimal element by ℓ. Define $\tilde{P}$ and $\tilde{L}$ as in Lemma 14, and let $\tilde{\varphi} = \varphi|_{\tilde{P}}$. Note that $\omega(\ell) \geq 1$ unless Q is a one-element poset; if Q has only one element, then so does P. A one-element poset has no quasi-tangled labelings, so henceforth we assume Q has more than one element.

For $j \in \{2, \ldots, \omega(\ell)\}$, let x_j be the minimal element of $\tilde{\varphi}^{-1}(u_{\ell,j})$. Also define

$$X_j = \{y \in \tilde{P} \mid y <_{\tilde{P}} x_j\} \quad \text{and} \quad A_j = \bigcup_{v \leq_Q u_{\ell,j-1}} \tilde{\varphi}^{-1}(v).$$

Let A'_j be defined analogously but with φ instead of $\tilde{\varphi}$. Recall that for $j \in \{2, \ldots, \omega(\ell)\}$,

$$b_{\ell,j} = \sum_{v \leq_Q u_{\ell,j-1}} |\varphi^{-1}(v)| \qquad \text{and} \qquad c_{\ell,j} = \sum_{v <_Q u_{\ell,j}} |\varphi^{-1}(v)|,$$

where $u_{\ell,0}, u_{\ell,1}, \ldots, u_{\ell,\omega(\ell)}$ is the unique path in Q from $\varphi(\ell)$ to the root.

In order to count the quasi-tangled labelings of P, we condition on the label of x_0 and count the labelings L such that $L_{n-3} \notin \mathcal{L}(P)$ and there exists $y \in P$ such that $y >_P x_0$ and $L_{n-3}(y) < L_{n-3}(x_0)$. For example, when x_0 is minimal, we count the labelings L such that $L(x_0) = n-1$ and $L_{n-3}^{-1}(1) >_P x_0 = L_{n-3}^{-1}(2)$. Note here that $L_{n-3}^{-1}(1) >_P L_{n-3}^{-1}(2)$ if and only if $L_{n-3}^{-1}(1) \in A'_2$. Our strategy is to fix the label of x_0 and choose a labeling uniformly at random among the $(n-1)!$ such labelings of P; observe that this induces the uniform distribution on the labelings of $\tilde{P}$. Given such a random labeling, we want to calculate the probability that certain labels end up in A'_2.

We will show later that, in each case, calculating this probability can be reduced to calculating the probability that the labels in question end up in A_2. Thus, we make the following definitions: Let K be some nonempty subset of $\{1, 2\}$. For x_0 and ℓ as defined above and $j \in \{2, \ldots, \omega(\ell)\}$, let $E_{\ell,j}$ be the event that $K \subset \tilde{L}_{n-3}(A_j)$. In other words, $E_{\ell,j}$ is the event that every label in K ends up on the "correct side" of x_j after $n-3$ promotions. We would like to compute $\mathbb{P}(E_{\ell,2})$ for $\tilde{L}$. To do so, we note that

$$\mathbb{P}(E_{\ell,2}) = \mathbb{P}(E_{\ell,\omega(\ell)})\mathbb{P}(E_{\ell,\omega(\ell)-1} \mid E_{\ell,\omega(\ell)}) \cdots \mathbb{P}(E_{\ell,2} \mid E_{\ell,3}) \tag{1}$$

and compute the multiplicands on the right-hand side of the equation above.

Lemma 15. *Fix $r \in \{1, 2\}$. Let P, x_0, ℓ, and the A_j's, A'_j's, and X_j's be defined as above. If x_0 is minimal, fix $a \in \{n-1, n\}$. Otherwise fix $a = n$. Set $L(x_0) = a$. Then $L_{n-3}^{-1}(r) \in A'_2$ if and only if $\tilde{L}_{n-3}^{-1}(r) \in A_2$.*

Proof. Suppose x_0 is minimal. Then $L_{n-3}(x_0) = L(x_0) - n + 3 \in \{2, 3\}$, and x_0 is never in the promotion chain for the first $n-3$ promotions. The lemma follows immediately from applying Lemma 14 to P, L, and x_0.

Suppose x_0 covers a unique minimal element ℓ and $L(x_0) = n$. Also assume that $L_{n-3}^{-1}(r) \in A'_2$. We claim that this implies x_0 is not in the promotion chain for the first $n-3$ promotions. Suppose to the contrary that x_0 is in the γth promotion chain for some $0 \leq \gamma \leq n-4$. This forces $L_\gamma(\ell) = 1$ and implies that x_0 is the L_γ-successor of ℓ. Note that $L_{n-3}^{-1}(r) \in A'_2$ implies that $L_\alpha^{-1}(r+n-3-\alpha)$ is comparable to x_0 for all $0 \leq \alpha \leq n-3$. In particular, $L_\gamma^{-1}(r+n-3-\gamma)$ must be above x_0, since x_0 is above only ℓ and $L_\gamma(\ell) = 1$. It follows that x_0 cannot be the L_γ-successor of ℓ, since $r+n-3-\gamma < n-\gamma = L_\gamma(x_0)$. This is a contradiction, so x_0 is not in the promotion chains of $L, \ldots, L_{n-4}$. Hence, we may apply Lemma 14, and it follows that $\tilde{L}_{n-3}^{-1}(r) \in A_2$.

For the converse, assume that $L_{n-3}^{-1}(r) \notin A'_2$. We have two cases: (1) x_0 is not in the promotion chains of $L, \ldots, L_{n-4}$; (2) x_0 is in the promotion chain

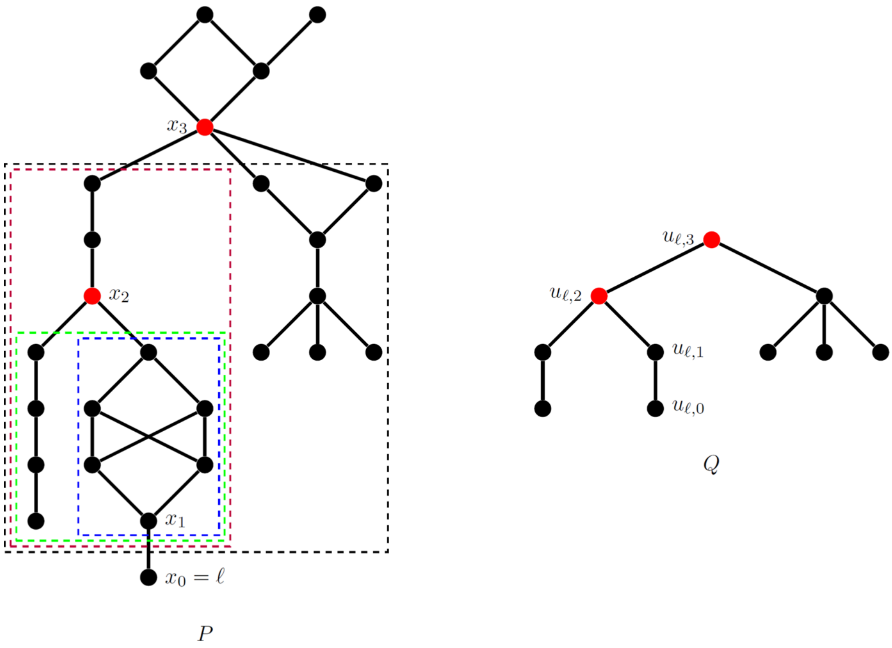

FIGURE 4. An illustration of the notation defined above, where P is the inflated rooted tree from Fig. 2. The black and green boxes denote X_3 and X_2, respectively, while the red and blue boxes denote A_3 and A_2, respectively

of L_γ for some $\gamma \in \{0, \dots, n-4\}$. For case (1), we simply apply Lemma 14 and are done.

For case (2), we note that for all $0 \le \alpha \le \gamma$, Lemma 14 implies that $\mathrm{st}(L_\alpha|_{\tilde{P}}) = \tilde{L}_\alpha$. In particular, we have $\mathrm{st}(L_\gamma|_{\tilde{P}}) = \tilde{L}_\gamma$. Since we are assuming that x_0 is in the promotion chain of L_γ, it follows that $L_\gamma(\ell) = 1$ and that x_0 is the L_γ-successor of ℓ. Hence, with respect to L_γ, there are no elements of P above x_0 with label smaller than $n-\gamma$. In particular, $L_\gamma^{-1}(r+n-3-\gamma)$ is not comparable to x_0 or ℓ. Since $\mathrm{st}(L_\gamma|_{\tilde{P}}) = \tilde{L}_\gamma$, it follows that $\tilde{L}_\gamma^{-1}(r+n-3-\gamma)$ is not comparable to ℓ in $\tilde{P}$. Therefore, $\tilde{L}_{n-3}^{-1}(r) \notin A_2$, as desired. $\square$

The next step is using this machinery to compute the conditional probabilities $\mathbb{P}(E_{\ell,j}|E_{\ell,j+1})$ as well as $\mathbb{P}(E_{\ell,\omega(\ell)})$.

Lemma 16. *Let P, x_0, ℓ, K, and the $E_{\ell,j}$'s, A_j's, X_j's, $b_{\ell,j}$'s, and $c_{\ell,j}$'s be defined as above. Let $j \in \{2, \dots, \omega(\ell)-1\}$, and fix any injective map*

$$M_j : \tilde{P} \setminus X_j \to [n-1]$$

such that every labeling $\tilde{L} : \tilde{P} \to [n-1]$ *extending* M_j *has the property that* $E_{\ell,j+1}$ *occurs. Consider the uniform distribution on such labelings* $\tilde{L}$*. Then*

$$\mathbb{P}(E_{\ell,j} \mid E_{\ell,j+1}) = \prod_{t=1}^{|K|} \frac{|A_j| - t + 1}{|X_j| - t + 1} = \prod_{t=1}^{|K|} \frac{b_{\ell,j} - t}{c_{\ell,j} - t}.$$

Proof. Recall that we may always assume Q has more than one element and thus that $\omega(\ell) \geq 1$. Also, by Remark 1, we have that $|A_2| \geq 2$.

By hypothesis, every labeling $\tilde{L} : \tilde{P} \to [n-1]$ extending M_j has the property that $E_{\ell,j+1}$ occurs. Recall that this implies $K \subset \tilde{L}_{n-3}(A_{j+1})$ and hence that $K \subset \tilde{L}_{n-3}(X_j)$, since $\{\tilde{L}_{n-3}^{-1}(1), \tilde{L}_{n-3}^{-1}(2)\}$ forms a lower order ideal of $\tilde{P}$. By Lemma 12, $\tilde{L}_\gamma$ depends only on $\tilde{L}|_{\tilde{P} \setminus X_j}$. Hence, $\mathbb{P}(E_{\ell,j} \mid E_{\ell,j+1})$ depends only on $\tilde{L}|_{\tilde{P} \setminus X_j}$. Apply the Probability Lemma (Lemma 13) with $N = n-1$, $x = x_j$, $X = X_j$, $A = A_j$, $M = M_j$, $k = n-3$, and $\{n_1, \ldots, n_d\} = K$. This tells us that

$$\mathbb{P}(E_{\ell,j} \mid E_{\ell,j+1}) = \prod_{t=1}^{|K|} \frac{|A_j| - t + 1}{|X_j| - t + 1}.$$

The lemma follows. □

Lemma 17. *With notation as in the previous lemma, fix any injective map*

$$M_{\omega(\ell)} : \tilde{P} \setminus X_{\omega(\ell)} \to [n-1],$$

and consider the uniform distribution on the labelings $\tilde{L} : \tilde{P} \to [n-1]$ *extending* $M_{\omega(\ell)}$*. Then*

$$\mathbb{P}(E_{\ell,\omega(\ell)}) = \prod_{t=1}^{|K|} \frac{|A_{\omega(\ell)}| - t + 1}{|X_{\omega(\ell)}| - t + 1} = \prod_{t=1}^{|K|} \frac{b_{\ell,\omega(\ell)} - t}{c_{\ell,\omega(\ell)} - t}.$$

Proof. We split into cases based on whether or not P has a unique minimal element. Suppose P has a unique minimal element. Then $A_j = X_j$, and, consequentially, $b_{\ell,j} = c_{\ell,j}$. Hence, it suffices to show that the probability in question is 1. Since $\{\tilde{L}_{n-3}^{-1}(1), \tilde{L}_{n-3}^{-1}(2)\}$ forms a lower order ideal of size 2, it is not difficult to see that when P has a unique minimal element, $\{\tilde{L}_{n-3}^{-1}(1), \tilde{L}_{n-3}^{-1}(2)\} \subset A_2$. Hence, $K \subset \tilde{L}_{n-3}(A_2)$. Because $A_2 \subset A_j$, the probability in question is 1, as desired.

Suppose P does not have a unique minimal element. The argument is identical to that in Lemma 16 as long as we show that for any such labeling $\tilde{L}$, $K \subset \tilde{L}_{n-3}(X_{\omega(\ell)})$. This simply follows from recalling that $\{\tilde{L}_{n-3}^{-1}(1), \tilde{L}_{n-3}^{-1}(2)\}$ forms a lower order ideal of size 2, because $|\tilde{P}| = n-1$. Since P does not have a unique minimal element, $x_{\omega(\ell)}$ is greater than at least two elements, implying $\{\tilde{L}_{n-3}^{-1}(1), \tilde{L}_{n-3}^{-1}(2)\} \subset X_{\omega(\ell)} = \{y \in \tilde{P} \mid y <_{\tilde{P}} x_{\omega(\ell)}\}$. □

The previous two lemmas only give us information about $\tilde{P}$. In the following, we use Lemma 15 to translate these results into information about P.

Lemma 18. *Let $\varphi : P \to Q$ be an inflation of a rooted tree poset with deflated leaves, and let n be the number of elements in P. Let x_0 cover a unique minimal element or be minimal itself. If x_0 is a minimal element, let $\ell = x_0$; otherwise let ℓ be the minimal element covered by x_0. If x_0 is minimal, fix $a \in \{n-1, n\}$. Otherwise fix $a = n$. Let the A_j's, X_j's, A'_j's, $E_{\ell,j}$'s, $b_{\ell,j}$'s, and $c_{\ell,j}$'s be defined as above. Let K be some nonempty subset of $\{1,2\}$. Suppose $L : P \to [n]$ is a labeling chosen uniformly at random among the $(n-1)!$ labelings with $L(x_0) = a$. Then the probability that every label in K is in $L_{n-3}(A'_2)$ is*

$$\prod_{j=2}^{\omega(\ell)} \prod_{t=1}^{|K|} \frac{b_{\ell,j} - t}{c_{\ell,j} - t}.$$

Proof. Note that L induces the uniform distribution on labelings $\tilde{L} : \tilde{P} \to [n-1]$. By Lemma 15, $K \subset L_{n-3}(A'_2)$ if and only if $K \subset \tilde{L}_{n-3}(A_2)$. Thus, we would like to calculate

$$\mathbb{P}(E_{\ell,2}) = \mathbb{P}(E_{\ell,\omega(\ell)})\mathbb{P}(E_{\ell,\omega(\ell)-1} \mid E_{\ell,\omega(\ell)}) \cdots \mathbb{P}(E_{\ell,2} \mid E_{\ell,3}).$$

The result follows from applying Lemma 17 and Lemma 16. □

The following three lemmas are applications of Lemma 18 to the configurations of interest for the proof of Theorem 9:

Lemma 19. *With notation as in Theorem 9, the number of labelings L of P with $L^{-1}(n-1) \notin S$, $L^{-1}(n-1)$ minimal, and $L^{-1}_{n-3}(2) <_P L^{-1}_{n-3}(1)$ is*

$$(n-1)! \left(\sum_{\ell \in T} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j} - 1}{c_{\ell,j} - 1} + \sum_{\ell \in R} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j} - 1}{c_{\ell,j} - 1} \right).$$

Proof. Begin by noting that $L^{-1}_{n-3}(2) = L^{-1}(n-1)$. Because $L^{-1}(n-1)$ is minimal, Lemma 6 gives us that $L^{-1}(n-1) \in T$, $L^{-1}(n-1) \in R$, or $L^{-1}(n-1) \in S$.

Case (1): Assume $L^{-1}(n-1) \in T$. We would like to compute the probability that $L^{-1}_{n-3}(1) >_P L^{-1}(n-1) = L^{-1}_{n-3}(2)$. Note that this event occurs if and only if $L^{-1}_{n-3}(1) \in A'_2$. Applying Lemma 18 with $x_0 = \ell = L^{-1}(n-1)$, we see that this probability is just

$$\prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j} - 1}{c_{\ell,j} - 1}.$$

Summing over the minimal elements in T, we get the summation corresponding to T in the formula.

Case (2): An analogous argument works for $L^{-1}(n-1) \in R$. The lemma follows. □

Lemma 20. *With notation as in Theorem 9, the number of labelings L of P with $L^{-1}(n)$ minimal and $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(1)$ or $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(2)$ is*

$$(n-1)!\left(2\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}+2\sum_{\ell\in R}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}\right.$$
$$\left.+\sum_{\ell\in S}\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}-\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}\right).$$

Proof. Recall that Lemma 6 implies that $L^{-1}(n)$ is in either R, S, or T. When $L^{-1}(n)$ is in R or T, the process of counting the number of such labelings (where $L^{-1}(n)$ is minimal and $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(1)$ or $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(2)$) is nearly identical to the one used in the proof of Lemma 19 (just apply Lemma 18). However, if $L^{-1}(n) \in T$, it is possible that $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(1)$ *and* $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(2)$; we are twice-counting such labelings. To count the labelings where $L^{-1}(n) \in T$, $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(1)$, and $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(2)$, we apply Lemma 18. Thus, for $L^{-1}(n) \in R \cup T$, there are

$$(n-1)!\left(2\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}+2\sum_{\ell\in R}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}-\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}\right)$$

labelings where $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(1)$ or $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(2)$. The term being subtracted in the above expression is the number of labelings with $L^{-1}(n) \in T$, $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(1)$, and $L^{-1}_{n-3}(3) <_P L^{-1}_{n-3}(2)$.

However, the process changes when $L^{-1}(n) \in S$. Let m be the other element covered by the parent of $L^{-1}(n)$; let Y be the lower order ideal of size 3 consisting of $L^{-1}(n)$, m, and their parent. Note that we must have $Y = \{L^{-1}_{n-3}(1), L^{-1}_{n-3}(2), L^{-1}_{n-3}(3)\}$. Setting $L^{-1}(n) = x_0 = \ell$, with notation as in Lemma 18, we have that $Y = \{L^{-1}_{n-3}(1), L^{-1}_{n-3}(2), L^{-1}_{n-3}(3)\}$ if and only if $L^{-1}_{n-3}(1)$ and $L^{-1}_{n-3}(2)$ are in A'_2. By Lemma 18, the probability that both $L^{-1}_{n-3}(1)$ and $L^{-1}_{n-3}(2)$ are in A'_2 is

$$\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}.$$

The formula follows from summing over all elements in S. □

Lemma 21. *With notation as in Theorem 9, the number of labelings L where $L^{-1}(n)$ covers a minimal element and $L^{-1}_{n-3}(1) <_P L^{-1}(n) <_P L^{-1}_{n-3}(2)$ or $L^{-1}_{n-3}(2) <_P L^{-1}(n) <_P L^{-1}_{n-3}(1)$ is*

$$(n-1)!\left(\sum_{m\in T}\prod_{j=2}^{\omega(m)}\frac{(b_{m,j}-1)(b_{m,j}-2)}{(c_{m,j}-1)(c_{m,j}-2)}\right).$$

Proof. Let $x_0 = L^{-1}(n)$, and let notation be as in Theorem 18 so that $L^{-1}(n)$ covers some minimal element ℓ. Note that $\ell \in T$ since $L^{-1}_{n-3}(1)$, $L^{-1}_{n-3}(2)$, and $L^{-1}_{n-3}(3) = L^{-1}(n)$ form a lower order ideal. Moreover, note that $L^{-1}_{n-3}(1)$ and $L^{-1}_{n-3}(2)$ are comparable to x_0 if and only if they are in A'_2. Hence, we may apply Lemma 18 to see that the probability both $L_{n-3}^{-1}(1)$ and $L_{n-3}^{-1}(2)$ are comparable to x_0 is

$$\prod_{j=2}^{\omega(m)} \frac{(b_{m,j}-1)(b_{m,j}-2)}{(c_{m,j}-1)(c_{m,j}-2)}.$$

Summing over all the elements in T will imply the lemma. □

Proof of Theorem 9. We begin by counting the number of tangled labelings of P. By Lemma 3.8 in [9], if L is a tangled labeling of an n-element poset, then $L^{-1}(n)$ is minimal. Moreover, a labeling is tangled if and only if $L_{n-2}^{-1}(1) >_P L_{n-2}^{-1}(2)$. If L is tangled, then it follows that $L^{-1}(n) \in R \cup T$, since $\{L_{n-2}^{-1}(1), L_{n-2}^{-1}(2)\}$ forms a lower order ideal and since $L_{n-2}^{-1}(1) >_P L_{n-2}^{-1}(2)$. Applying Lemma 18, we see that there are

$$(n-1)!\left(\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}+\sum_{\ell\in R}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}\right) \tag{2}$$

tangled labelings.

Now, we enumerate the labelings L such that $L_{n-3} \notin \mathcal{L}(P)$. Lemma 7 tells us that if $L_{n-3} \notin \mathcal{L}(P)$, then either $L^{-1}(n-1)$ is minimal, $L^{-1}(n)$ is minimal, or $L^{-1}(n)$ covers a minimal element. Moreover, we know that $Y = \{L_{n-3}^{-1}(1), L_{n-3}^{-1}(2), L_{n-3}^{-1}(3)\}$ forms a lower order ideal of size 3, and L_{n-3} restricted to Y is not a linear extension. We condition on the three cases given by Lemma 7.

Case (1): We count the labelings L such that $L^{-1}(n-1)$ is minimal, $L^{-1}(n-1) \notin S$, and $L_{n-3}^{-1}(1) >_P L_{n-3}^{-1}(2)$. By Lemma 19, there are

$$(n-1)!\left(\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}+\sum_{\ell\in R}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}\right) \tag{3}$$

such labelings. If $L^{-1}(n-1) \in S$, because $\{L_{n-3}^{-1}(1), L_{n-3}^{-1}(2), L_{n-3}^{-1}(3)\}$ is a lower order ideal, it follows from the definition of S that $L_{n-3}^{-1}(1)$ must be the unique parent of both $L_{n-3}^{-1}(2)$ and $L_{n-3}^{-1}(3)$. Now, repeatedly applying Lemma 4 to $L_{n-3}^{-1}(1)$ and $L_{n-3}^{-1}(3)$ tells us the position of $L^{-1}(n)$, namely that $L^{-1}(n) = L_{n-3}^{-1}(3) \in S$. Thus, this subcase can be excluded and will be addressed in Case (2) when we assume $L^{-1}(n)$ is minimal.

Case (2): We count of labelings L such that $L^{-1}(n)$ is minimal and $L_{n-3}^{-1}(3) <_P L_{n-3}^{-1}(1)$ or $L_{n-3}^{-1}(3) <_P L_{n-3}^{-1}(2)$. By Lemma 20, there are

$$\begin{aligned}(n-1)!\Bigg(&2\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}+2\sum_{\ell\in R}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}+\sum_{\ell\in S}\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}\\ &-\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}\Bigg)\end{aligned} \tag{4}$$

such labelings.

Case (3): We count the labelings L such that $L^{-1}(n)$ covers a minimal element ℓ, $L_{n-3}^{-1}(2) >_P L_{n-3}^{-1}(3)$ or $L_{n-3}^{-1}(1) >_P L_{n-3}^{-1}(3)$, and $L(\ell) \neq n-1$. (The case where $L(\ell) = n-1$ and $L_{n-3}^{-1}(1) >_P L_{n-3}^{-1}(3) >_P L_{n-3}^{-1}(2)$ was counted in Case (1).) We first count the labelings L such that $L_{n-3}^{-1}(2) >_P L_{n-3}^{-1}(3)$ or $L_{n-3}^{-1}(1) >_P L_{n-3}^{-1}(3)$. Since Y is a lower order ideal of size 3, it follows that $\ell \in Y$. Thus, we may assume that $\ell \in T$. Hence, it is sufficient to count the labelings L such that $L^{-1}(n)$ covers some $\ell \in T$ and $L_{n-3}^{-1}(1) <_P L^{-1}(n) <_P L_{n-3}^{-1}(2)$ or $L_{n-3}^{-1}(2) <_P L^{-1}(n) <_P L_{n-3}^{-1}(1)$. We have already done this—the number of such labelings is given in Lemma 21. Note that each such labeling is indeed quasi-tangled. To account for the condition $L(\ell) \neq n-1$, we enumerate the labelings with $L^{-1}(n-1) \in T$, $L^{-1}(n-1) \lessdot_P L^{-1}(n)$, and $L_{n-3}^{-1}(1) >_P L_{n-3}^{-1}(3)$. An adaptation of Lemma 18 allows us to enumerate these labelings, the number of which is given by the term being subtracted in the following expression:

$$(n-1)!\left(\sum_{m\in T}\prod_{j=2}^{\omega(m)}\frac{(b_{m,j}-1)(b_{m,j}-2)}{(c_{m,j}-1)(c_{m,j}-2)}-\frac{1}{n-1}\sum_{m\in T}\prod_{j=2}^{\omega(m)}\frac{(b_{m,j}-2)}{(c_{m,j}-2)}\right). \quad (5)$$

Note that the above enumerates the labelings L such that $L^{-1}(n)$ covers a minimal element ℓ, $L_{n-3}^{-1}(2) >_P L_{n-3}^{-1}(3)$ or $L_{n-3}^{-1}(1) >_P L_{n-3}^{-1}(3)$, and $L(\ell) \neq n-1$.

Summing (3), (4), and (5) and subtracting (2) gives that the number of quasi-tangled labelings of P is given by

$$(n-1)!\left(2\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}-\frac{1}{n-1}\sum_{\ell\in T}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-2}{c_{\ell,j}-2}+2\sum_{\ell\in R}\prod_{j=2}^{\omega(\ell)}\frac{b_{\ell,j}-1}{c_{\ell,j}-1}\right.$$
$$\left.+\sum_{\ell\in S}\prod_{j=2}^{\omega(\ell)}\frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)}\right)$$

as desired. □

3.4. Enumerating Labelings with Sorting Time $n-1-k$

In light of Lemma 7 and the Probability Lemma (Lemma 13), it is natural to ask if the methods used in Sect. 3 can be extended to enumerate the labelings of an n-element poset P with sorting time $n-1-k$, where we assume $n \geq k+2$ (Lemma 5). In the following, we give an algorithmic approach for doing so when P is an inflated rooted tree poset such that the leaves are deflated, the parents of leaves are deflated, and so on, up k levels. Importantly, we note that this class of posets contains rooted trees. While, theoretically, this approach could yield a general formula for the labelings with sorting time $n-1-k$, any such formula would be much too complicated to be practical. Instead, for a fixed k, we offer an algorithmic approach to enumerating the labelings with sorting time $n-1-k$. Using this method, it would be possible to write a computer program that computes the number of such labelings for a fixed poset.

In order to do so, we first note that Lemma 7 generalizes. In particular, for a fixed k, one can prove that if L has sorting time $n-1-k$, then one of the following holds:

- $L^{-1}(n-k)$ is minimal;
- $L^{-1}(n-k+1)$ is minimal or covers a minimal element;
- $L^{-1}(n-k+2)$ is minimal, covers a minimal element, or is greater than exactly 2 other elements;

$$\vdots$$

- $L^{-1}(n)$ is greater than at most k other elements.

Algorithm 1. Let P be an n-element rooted tree poset. Then we may enumerate the labelings L of P with sorting time $n-1-k$ in the following way:

1. List the possible lower order ideals of size k other than antichains appearing in P.
2. For each such lower order ideal occurring in P, use (1) and the Probability Lemma to count the labelings with sorting time $n-1-k$. This will involve lots of casework based on the positions of $L^{-1}(n-k), \ldots, L^{-1}(n)$ and lower order ideals of size k occurring in P. The proof of Theorem 9 illustrates this casework for the case $k=1$ in full generality.

4. Bounding Labelings with a Given Sorting Time

In this section, we focus on bounding the number of labelings with a given sorting time for an arbitrary poset. While proving our conjectural bounds seems to be quite difficult for general posets, we are able to make partial progress by restricting our attention to inflated rooted trees.

In [9], Defant and Kravitz conjectured the following (this was already stated as Conjecture 1, but we restate it here for convenience):

Conjecture 22. ([9], Conjecture 5.1) *If P is an n-element poset, then P has at most $(n-1)!$ tangled labelings.*

The above is not obvious even for classes of posets for which we can explicitly enumerate the tangled labelings (e.g., Theorem 8). In light of Theorem 9, we also conjecture the following:

Conjecture 23. *Let P be an n-element poset. Then the number of labelings $L : P \to [n]$ such that $L_{n-3} \notin \mathcal{L}(P)$ is at most $3(n-1)!$.*

Conjecture 24. *Let P be an n-element poset. Then the number of tangled labelings of P is strictly less than the number of quasi-tangled labelings of P.*

We can prove Conjecture 22 for inflated rooted forests. The following results from [9] will be useful:

Corollary 25. ([9], Corollary 3.7) *Let P be an n-element poset with r connected components, each having a unique minimal element. Then the number of tangled labelings of P is*

$$(n-r)(n-2)!.$$

Theorem 26. ([9], Theorem 3.4) *Let P be an n-element poset with connected components $P_1, \dots, P_r$. Let $n_i = |P_i|$, and let t_i denote the number of tangled labelings of P_i. The number of tangled labelings of P is*

$$(n-2)! \sum_{i=1}^{r} \frac{t_i}{(n_i - 2)!}.$$

Lemma 27. *Let P, P_i, n_i, and t_i be defined as in the above for $i = 1, \dots, r$. If there are at most $(n_i - 1)!$ tangled labelings of each P_i, then there are at most $(n-r)(n-2)!$ tangled labelings of P.*

Proof. Substituting $(n_i - 1)! \geq t_i$ into

$$(n-2)! \sum_{i=1}^{r} \frac{t_i}{(n_i - 2)!} \leq (n-2)! \sum_{i=1}^{r} (n_i - 1) = (n-2)!(n-r)$$

gives the bound. □

Theorem 28. *Let P be an n-element inflated rooted forest poset. Then P has at most $(n-1)!$ tangled labelings. Equality holds if and only if P has a unique minimal element.*

Proof. By Lemma 27, it suffices to prove this for P an inflated rooted tree, where Q is the rooted tree and $\varphi : P \to Q$ the inflation map. Assume without loss of generality that Q is reduced. We know that the number of tangled labelings of P is

$$(n-1)! \sum_{i=1}^{s} \prod_{j=1}^{\omega(i)} \frac{b_{i,j} - 1}{c_{i,j} - 1}. \tag{6}$$

Let $\ell_1, \dots, \ell_s$ denote the leaves of Q, and let $m_1, \dots, m_s$ be the unique minimal elements of $\varphi^{-1}(\ell_1), \dots, \varphi^{-1}(\ell_s)$, respectively. Suppose without loss of generality that ℓ_{s-1} and ℓ_s have the same parent in Q (such leaves exist because Q is assumed to be reduced). For all $j \neq 1$, $b_{s-1,j} = b_{s,j}$; for all j, $c_{s-1,j} = c_{s,j}$. Hence, we may rewrite (6) as

$$(n-1)! \left(\sum_{i=1}^{s-2} \prod_{j=1}^{\omega(i)} \frac{b_{i,j} - 1}{c_{i,j} - 1} + \prod_{j=2}^{\omega(s-1)} \frac{b_{s-1,j} - 1}{c_{s-1,j} - 1} \left(\frac{b_{s-1,1} + b_{s,1} - 2}{c_{s-1,1} - 1} \right) \right).$$

Now, let P' be the poset obtained from P by adding the additional relation $m_{s-1} \lessdot_{P'} m_s$. Note that the resulting poset is still an inflated rooted tree poset and that P' is an inflation of Q', where Q' is the (reduced) rooted tree poset formed by setting $\ell_s = \ell_{s-1}$ and reducing if necessary. Let $\psi : P' \to Q'$ be the corresponding inflation map. Moreover, note that Q' has $s-1$ leaves. For each leaf $\ell'_1, \dots, \ell'_{s-1}$ in Q', let $u'_{i,0}, \dots, u'_{i,\omega(i)}$ denote the unique path from ℓ'_i to the root of Q'. Let

$$b'_{i,j} = \sum_{v \leq_{Q'} u'_{i,j-1}} |\psi^{-1}(v)| \qquad \text{and} \qquad c'_{i,j} = \sum_{v <_{Q'} u'_{i,j}} |\psi^{-1}(v)|.$$

Note that for $i \in \{1, \ldots, s-2\}$ and $j \in \{1, \ldots, \omega(i)\}$, $b_{i,j} = b'_{i,j}$ and $c_{i,j} = c'_{i,j}$. Moreover, we also know that when $i = s-1$, $b_{i,j} = b'_{i,j}$ and $c_{i,j} = c'_{i,j}$ for $j = 2, \ldots, \omega(i)$. When $j = 1$, we have $b'_{s-1,1} = b_{s-1,1} + b_{s,1}$ and $c'_{i,j} = c_{i,j}$. It follows that the number of tangled labelings of P' is

$$(n-1)! \left(\sum_{i=1}^{s-2} \prod_{j=1}^{\omega(i)} \frac{b_{i,j}-1}{c_{i,j}-1} + \prod_{j=2}^{\omega(s-1)} \frac{b_{s-1,j}-1}{c_{s-1,j}-1} \left(\frac{b_{s-1,1}+b_{s,1}-1}{c_{s-1,1}-1} \right) \right).$$

Note that P' has more tangled labelings than P and that P' has $s-1$ minimal elements.

The result follows from induction and Corollary 25. □

We are also able to show Conjecture 24 for inflated rooted trees with deflated leaves. Note that Conjecture 24 is a special case of the following conjecture:

Conjecture 29. ([10], Conjecture 5.2) *Let P be an n-element poset, and let $a_k(P)$ denote the labelings of P with sorting time k. Then the sequence $a_0(P)$, $\ldots, a_{n-1}(P)$ is unimodal.*[2]

Theorem 30. *Let P be an n-element inflated rooted tree poset with deflated leaves, and assume $n \geq 3$. Then the number of tangled labelings of P is strictly less than the number of quasi-tangled labelings of P.*

Proof. We may refine Theorem 8 in the following way: the number of tangled labelings of P is given by

$$(n-1)! \sum_{\ell \in R \cup T} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1}. \tag{7}$$

The above follows from the simple observation that if $\ell \in M \backslash (R \cup T)$, then it cannot be an element of a lower order ideal of P of size 2 (recall Lemma 6 and the proof of Theorem 8). In other words, it suffices to sum over the minimal elements in lower order ideals of size 2.

By Theorem 9, P has

$$(n-1)! \left(2 \sum_{\ell \in T} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1} - \frac{1}{n-1} \sum_{\ell \in T} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-2}{c_{\ell,j}-2} + 2 \sum_{\ell \in R} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1} \right.$$
$$\left. + \sum_{\ell \in S} \prod_{j=2}^{\omega(\ell)} \frac{(b_{\ell,j}-1)(b_{\ell,j}-2)}{(c_{\ell,j}-1)(c_{\ell,j}-2)} \right) \tag{8}$$

[2] While this conjecture appears in [10] (a preprint), it does not appear in [9], which is the published version of the article.

quasi-tangled labelings. Because the summands in (7) and (8) are indexed by minimal elements in P, the result will follow if we can show that each of the terms in (7) is less than or equal to the corresponding term in (8).

Suppose $\ell \in R$. The term in (7) corresponding to ℓ is

$$(n-1)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1},$$

which is less than

$$2(n-1)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1},$$

the term in (8) corresponding to ℓ.

Suppose $\ell \in T$. Then the term in (7) corresponding to ℓ is

$$(n-1)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1}.$$

The term in (8) corresponding to ℓ is

$$2(n-1)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1} - (n-2)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-2}{c_{\ell,j}-2} \geq (n-1)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1}.$$

The inequality in the above follows immediately from the fact that $b_{\ell,j} \leq c_{\ell,j}$, which implies

$$\begin{aligned}(b_{\ell,j}-1)(c_{\ell,j}-2) = b_{\ell,j}c_{\ell,j} - 2b_{\ell,j} - c_{\ell,j} + 2 \geq b_{\ell,j}c_{\ell,j} - b_{\ell,j} - 2c_{\ell,j} \\ +2 = (b_{\ell,j}-2)(c_{\ell,j}-1) \end{aligned} \tag{9}$$

for all j.

Now, suppose the number of tangled labelings of P is equal to the number of quasi-tangled labelings. Then we know from the above that P cannot have any minimal elements in R or S. It follows that P has

$$2(n-1)! \sum_{\ell \in T} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1} - (n-2)! \sum_{\ell \in T} \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-2}{c_{\ell,j}-2}$$

quasi-tangled labelings. The fact that each of the terms in (8) is greater than or equal to the corresponding term in (7) implies that equality holds if and only if for each $\ell \in T$,

$$(n-1)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1} = (n-2)! \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-2}{c_{\ell,j}-2}.$$

If $b_{\ell,j} < c_{\ell,j}$ for any j, then (9) implies that

$$\prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-1}{c_{\ell,j}-1} > \prod_{j=2}^{\omega(\ell)} \frac{b_{\ell,j}-2}{c_{\ell,j}-2},$$

so we may assume that $b_{\ell,j} = c_{\ell,j}$ for all $2 \leq j \leq \omega(\ell)$.

This forces $\omega(\ell) = 2$ for all ℓ, which can be the case only if P has a unique minimal element and is an inflation of a 2-element chain. It follows from Theorem 8 and Corollary 10 that P has $(n-1)!$ tangled labelings and $2(n-1)! - (n-2)!$ quasi-tangled labelings. These two quantities are equal if and only if $n = 2$. However, this contradicts our assumption that $n \geq 3$. Therefore, equality can never hold and there are always more quasi-tangled labelings than tangled ones. □

Theorem 9, in conjunction with the enumeration of the tangled labelings of inflated rooted forests given in [9], seems to imply that the inflation operation on posets is very compatible with promotion. Thus, it would be a natural next step to study promotion on inflations of non-rooted trees. For example, it would be interesting to enumerate the tangled labelings of inflations of simple posets such as N-posets or M-posets. Doing so might generate new methods for attacking Conjecture 22, which may also be refined in the following way:

Conjecture 31. *Let P be an n-element poset with s minimal elements. Then P has at most $(n-s)(n-2)!$ tangled labelings.*

Note that Lemma 27 implies that it suffices to show the above for P connected. It is also possible to reframe Conjecture 22 in the following way: Let P be a connected, n-element poset, and let $m_1, \ldots, m_s$ be the minimal elements of P. Let $c : P \to \mathcal{P}([s])$ be a coloring of P given by $i \in c(x)$ if $x \geq_P m_i$ (here $\mathcal{P}([s])$ denotes the power set of $[s] = \{1, \ldots, s\}$). The following implies Conjecture 22.

Conjecture 32. *With notation as above,*

$$\mathbb{P}\left(c(L_{n-2}^{-1}(1)) = \{s\} \mid L(m_{s-1}) = n\right) \geq \mathbb{P}\left(c(L_{n-2}^{-1}(1)) = \{s\} \mid L(m_s) = n\right).$$

If the above holds, we may apply the same argument as in Theorem 28 to show that the number of tangled labelings increases when we make $m_{s-1} \lessdot m_s$. Applying this fact repeatedly would prove Conjecture 22, since posets with a unique minimal element have exactly $(n-1)!$ tangled labelings.

It would be interesting to see if the methods used in the proof of Theorem 28 can be adapted to show that Conjecture 23 holds for a smaller class of posets, such as inflated rooted trees with deflated leaves or rooted tree posets. Finally, we conclude with the following question, which is inspired by Conjecture 22:

Question 33. *What is the maximum number of quasi-tangled labelings a poset can have?*

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