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Highly Sorted Permutations with Respect to a 312-avoiding Stack

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Abstract: Let s_{312} be the generalized stack-sorting map, where instead of being required to increase, the stack avoids the pattern 312. In this note, we give a simple characterization of the set $s_{312}^{n-2}(S_n)$ for all n, the consequence of which is that $|s_{312}^{n-2}(S_n)| = 2^{n-1} + 1$. As a corollary, we settle a conjecture by Defant and Zheng.

Keywords: Pattern avoidance; Permutation; Stack sort
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1. Introduction

In 1990, West [11] introduced the stack-sorting map s, which sends the input permutation through a stack in a right-greedy manner, while insisting that the stack must be increasing when read from top to bottom. The stack-sorting map has now been studied extensively and has found connections with a variety of topics, including free probability theory [8]. In 2020, Cerbai, Claesson, and Ferrari [4] generalized the stack-sorting map s to s_σ for permutations σ. As in s, the map s_σ sends the input permutation through a stack in a right-greedy manner. But instead of insisting that the stack avoids descents when read from top to bottom, we insist that the stack avoids subsequences that are order-isomorphic to σ when read from top to bottom. For example, Figure 1 illustrates that $s_{312}(3241) = 2143$.

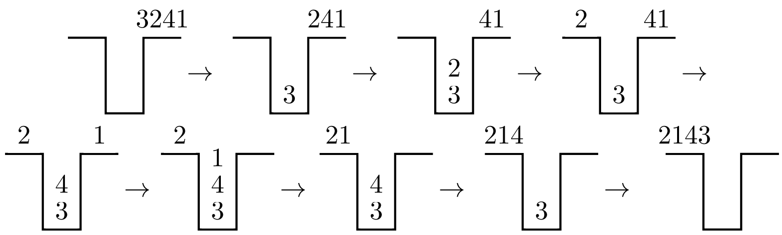

Figure 1: The stack-sorting map s_{312} on $\pi = 3241$.

Since then, Baril, Cerbai, Khalil, and Vajnovski [1] generalized s_σ to $s_{\sigma,\tau}$ for permutations σ and τ, in which the stack does not contain subsequences that are order-isomorphic to σ or τ. More recently, Berlow [2] extended s_σ to s_T for sets of permutations T. In s_T, the stack must not contain any subsequence that is order-isomorphic to any permutation in T at any time.

The namesake of the stack-sorting map s is an easily-verifiable statement that the identity permutation is the only permutation in $s^{n-1}(S_n)$. Naturally, the most classical questions on the stack-sorting map are the characterization and enumeration of *t-stack sortable* permutations, where a permutation π is t-stack-sortable if $s^t(\pi)$ is the identity permutation. Knuth [10] was the first to show that a permutation is $1-$stack-sortable if and only if it does not contain a subpermutation that is order-isomorphic to 231; he showed that the number of such permutations in S_n is the n^{th} Catalan number, $\frac{1}{n+1}\binom{2n}{n}$. West [11] then characterized the $2-$stack-sortable permutations and conjectured that the number of $2-$stack-sortable permutations in S_n is $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$, which Zeilberger [12] later affirmed. Only recently, a polynomial time algorithm to count the $3-$stack-sortable permutations [5] was discovered.

As Defant wrote in [7], the t-stack-sortable permutations, in some sense, are duals to the permutations in the image $s^t(S_n)$. In [3], Bousquet-Mélou defined a permutation to be *sorted* if it is in $s(S_n)$ and proved a recurrence relation that counts $|s(S_n)|$. In [6], Defant generalized the notion of sorted permutations to t-*sorted* permutations: a permutation is t-*sorted* if it is in the image of $s^t(S_n)$. Furthermore, he showed that the maximum number of descents in a t-sorted permutation of length n is $\lfloor \frac{n-t}{2} \rfloor$ and characterized the t-sorted permutations of length n that achieve the maximum number of descents when $n \equiv t \pmod 2$. Extending the notion of t-sorted permutations, for the rest of this article, we say that a permutation π is t-*sorted with respect to* s_σ if $\pi \in s_\sigma^t(S_n)$.

It is well–known that $t = n-1$ is the minimum t such that $s^t(S_n) = s^{T(S_n)}(S_n)$ for all $T(S_n) \geq t$. In another recent article [7], Defant defined a permutation in S_n to be *highly sorted* if it is t-sorted for some t close to n. In the same article, he characterized and enumerated $s^{n-m}(S_n)$ for all $n \geq 2m-3$. Likewise, it is clear that for any s_σ, there exists some minimum $T_\sigma(S_n)$ for which $s_\sigma^{T_\sigma(S_n)}(S_n) = s_\sigma^t(S_n)$ for all $t \geq T_\sigma(S_n)$, i.e., there exists a minimum positive integer $T_\sigma(S_n)$ such that $s_\sigma^{T_\sigma(S_n)}(S_n)$ is the set of the periodic points of s_σ. Therefore, we extend the notion of a highly sorted permutation and call a permutation in S_n *highly sorted with respect to* s_σ if it is t-sorted with respect to s_σ for some t that is close to $T_\sigma(S_n)$.

Recently, Defant and Zheng [9] showed that $T_{312}(S_n) = n-1$. Furthermore, they characterized and enumerated permutations that are $(n-1)$-sorted respect to s_{312}. In their theorem that follows, a permutation $\pi \in S_n$ is in $\mathrm{Av}_n(213, 312)$ if and only if it avoids subsequences that are order-isomorphic to 213 or 312.

Theorem 1.1 (Defant and Zheng, [9])**.** *For any $m \geq n-1$, a permutation π is in $s_{312}^m(S_n)$ if and only if $\pi \in \mathrm{Av}_n(213, 312)$. Consequently, when $m \geq n-1$,*

$$|s_{312}^m(S_n)| = 2^{n-1}.$$

Furthermore, if $\pi \in \mathrm{Av}_n(213, 312)$, then $s_{312}^2(\pi) = \pi$.

The main result of our paper is that we continue the study of highly sorted permutations with respect to s_{312} by characterizing and enumerating the permutations π that are $(n-2)$-sorted with respect to s_{312}. To state our theorem, we let $\zeta_n \in S_n$ be the permutation obtained by starting with 21, listing the remaining even numbers that are at most n in increasing order, and then listing the remaining odd numbers that are at most n in decreasing order. For example, $\zeta_5 = 21453$ and $\zeta_8 = 21468753$.

Theorem 1.2. *A permutation π is in $s_{312}^{n-2}(S_n)$ if and only if one of the following holds:*

- *$\pi \in \mathrm{Av}_n(213, 312)$;*
- *$\pi = \zeta_n$.*

Consequently,

$$|s_{312}^{n-2}(S_n)| = 2^{n-1} + 1.$$

Now, for any permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, we write $\mathrm{comp}(\sigma) = (n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$. As Defant and Zheng [9] noted, for any permutation σ, it follows directly from the definition of s_σ that $s_{\mathrm{comp}(\sigma)} = \mathrm{comp} \circ s_\sigma \circ \mathrm{comp}$. Therefore, the analog of Theorem 1.2 for s_{132} follows directly from Theorem 1.2.

Corollary 1.1. *A permutation π is in $s_{132}^{n-2}(S_n)$ if and only if one of the following holds:*

- *$\pi \in \mathrm{Av}_n(213, 312)$;*
- *$\pi = \mathrm{comp}(\zeta_n)$.*

Consequently,

$$|s_{132}^{n-2}(S_n)| = 2^{n-1} + 1.$$

As a direct corollary of Corollary 1.1, we settle a conjecture by Defant and Zheng (see Conjecture 8.4 of [9]). In what follows, let Ξ_n be the permutation given by listing the odd numbers that are at most n in increasing order and then listing the even numbers that are at most n in decreasing order. For example, $\Xi_6 = 135642$ and $\Xi_7 = 1357642$.

Corollary 1.2. *If $\pi \in S_n$ such that $s_{132}^{n-2}(\pi) \notin \mathrm{Av}_n(213, 312)$, then $s_{132}^{n-1}(\pi) = \mathrm{comp}(\Xi_n)$.*

We end with another direct corollary of Theorem 1.2.

Corollary 1.3. *For all $1 \leq m \leq n-1$, we have that $s_{312}^m(S_n) \subsetneq s_{312}^{m-1}(S_n)$.*

2. Preliminaries

A *permutation* $\pi = \pi_1\pi_2\ldots\pi_n$ is a finite ordered set of positive integers. As usual, let S_n be the set of permutations of $[n] = \{1, 2, \ldots n\}$, and let $\ell(\pi)$ be the length of π. A *descending run* of π is a maximal decreasing substring of π. For example, the descending runs of $\pi = 51763284$ are 51, 7632, and 84. In addition, for a permutation π, we say that the *reduction* of π is $\mathrm{red}(\pi) = \pi'_1 \cdots \pi'_n \in S_n$, where $\pi'_i = i$ if π_i is the i^{th} smallest number from the set $\{\pi_1, \pi_2, \ldots, \pi_n\}$. For example, if $\pi = 6243$, then $\mathrm{red}(\pi) = 4132$. Furthermore, let $\mathrm{big}_k(\pi)$ be the permutation given by deleting all but the k largest entries of π while keeping the relative ordering of the remaining entries invariant. For example, $\mathrm{big}_5(123) = 123$ and $\mathrm{big}_3(216845) = 685$.

Next, we say that the permutation π *contains* the pattern σ if there exists a sequence of indices $a(1) < \cdots < a(k)$ such that $\pi_{a(1)}\pi_{a(2)}\cdots\pi_{a(k)}$ is order-isomorphic to σ. Equivalently, we say that $\pi_{a(1)}\pi_{a(2)}\cdots\pi_{a(k)}$ is a σ *pattern of* π. For example, $\pi_1\pi_2\pi_4 = 524$ is a $\sigma = 312$ pattern of $\pi = 52143$. We say a permutation *avoids* a pattern σ if it does not contain it. For instance, the permutation $\pi = 23145$ avoids the pattern $\sigma = 312$. We say that $\pi \in \mathrm{Av}(213, 312)$ if π avoids the patterns 312 and 312. In addition, let $\mathrm{Av}_n(213, 312) = S_n \cap \mathrm{Av}_n(213, 312)$.

Next, we first define the *contraction map* c. For a permutation π, if $\ell(\pi) \geq 3$ and the two largest entries of π are adjacent in π, then let c take π to the permutation obtained by deleting the largest entry of π while keeping the relative ordering of every other entries invariant. Otherwise, let c take π to itself. For example, $c(31254) = 3124$ and $c(21534) = 21534$. Then for any permutation π, we let its *contraction length* $\mathrm{cl}(\pi)$ be defined as the smallest non-negative integer such that $c^{\mathrm{cl}(\pi)}(\pi) = c^{\mathrm{cl}(\pi)+1}(\pi)$. For example, $\mathrm{cl}(31254) = 2$, because $c^1(31254) \neq c^2(31254) = c^3(31254) = 312$, and $\mathrm{cl}(21534) = 0$, because $c^0(21534) = c^1(21534) = 21534$.

We end this section by citing a lemma by Defant and Zheng [9], which follows from their proof of their Theorem 7.1.

Lemma 2.1 (Defant and Zheng [9]). *For any permutation π, its largest entry appears in the rightmost descending run of $s_{312}(\pi)$.*

3. Proof of the Main Result

We begin by establishing auxiliary lemmas that lead up to the proof of Theorem 1.2. We first show for any k that the operators s_{312} and big_k commute.

Lemma 3.1. *For any permutation π and $k, t \geq 0$, we have that*

$$s_{312}^t(\mathrm{big}_k(\pi))) = \mathrm{big}_k(s_{312}^t(\pi)).$$

Proof. The statement is vacouously true for $t = 0$. It suffices to show that the statement holds true for $t = 1$; then the rest follows by induction.

Now, suppose that at some point of applying s_{312} to π, placing the next entry from the input permutation introduces a 312 pattern to the stack. Let σ be the permutation that we have when we read the stack from top to bottom after placing the next entry from the input permutation at the top of the stack. Then we claim there exists some $i \leq \ell(\sigma) - 2$ for which $\sigma_i\sigma_{\ell(\sigma)-1}\sigma_{\ell(\sigma)}$ form a 312 pattern of σ.

Suppose otherwise. Because placing $\sigma_{\ell(\sigma)}$ introduces a 312 pattern to the stack, there exist some i and $j (\neq \ell(\sigma) - 1)$ such that $\sigma_i\sigma_j\sigma_{\ell(\sigma)}$ form a 312 pattern of σ. First, because $\sigma_i\sigma_j\sigma_{\ell(\sigma)}$ form a 312 pattern, $\sigma_i < \sigma_{\ell(\sigma)}$. Next, the stack before placing $\sigma_{\ell(\sigma)}$ to the top does not contain a 312 pattern, so $\sigma_{\ell(\sigma)-1} < \sigma_i$ as otherwise, $\sigma_i\sigma_j\sigma_{\ell(\sigma)-1}$ form a 312 pattern of σ that does not involve $\sigma_{\ell(\sigma)}$. Therefore, $\sigma_{\ell(\sigma)-1} < \sigma_i < \sigma_{\ell(\sigma)}$, and so $\sigma_i\sigma_{\ell(\sigma)-1}\sigma_{\ell(\sigma)}$ form a 312 pattern of σ—a contradiction as sought.

Thus, when applying s_{312} to π, if placing the next entry from the input permutation introduces a 312 pattern to the stack, then after the next entry of the input permutation is placed at the top of the stack, there exists a 312 pattern in the stack in which the last two entries of the stack coincide with the last two entries of the pattern. Now, in a 312 pattern, if the middle entry appears in $\mathrm{big}_k(\pi)$, then all three entries of the pattern appear in $\mathrm{big}_k(\pi)$. Therefore, if the current top entry of the stack appears in $\mathrm{big}_k(\pi)$ and placing the next entry from the input permutation introduces a 312 pattern to the stack, then after the next entry of the input permutation is placed at the top of the stack, there exists a 312 pattern for which all three entries of the pattern appear in $\mathrm{big}_k(\pi)$. Because when the input permutation is not empty, we remove an entry from the stack if and only if it is at the top of the stack and placing the next entry from the input introduces a 312 pattern to the stack, the statement of the lemma follows. □

Next, we characterize the permutations π in $\mathrm{Av}(213, 312)$ by their contraction lengths.

Lemma 3.2. *A permutation π for which $\ell(\pi) \geq 2$ is in $\mathrm{Av}(213, 312)$ if and only if $\mathrm{cl}(\pi) = \ell(\pi) - 2$.*

Proof. We induct on $\ell(\pi)$. The statement of the lemma clearly holds for $\ell(\pi) \leq 3$. A permutation π for which $\ell(\pi) \geq 4$ is in $\text{Av}(213, 312)$ if and only if after deleting the largest element, the permutation is still in $\text{Av}(213, 312)$, and the largest two entries of π are adjacent. The statement of the lemma now follows from induction. □

Now, we show that the operators s_{312} and $c^{\text{cl}(\pi)}$ commute for any permutation $\pi \in S_n$.

Lemma 3.3. *For any permutation $\pi \in S_n$, we have that*

$$\text{c}^{\text{cl}(\pi)}(s_{312}(\pi)) = s_{312}(\text{c}^{\text{cl}(\pi)}(\pi)).$$

Proof. Because the entries in $\text{big}_{\text{cl}(\pi)+1}(\pi)$ are larger than every other entry in π, when the leftmost entry of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ enters the stack, the stack must be increasing from top to bottom. Furthermore, by the definition of cl, we must have $\text{big}_{\text{cl}(\pi)+1}(\pi) \in \text{Av}(213, 312)$. Therefore, after the leftmost entry of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ enters the stack, the rest of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ enters the stack without any entry from the stack removed. Now, because the stack is increasing from top to bottom until the first entry of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ and the entries of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ are greater than all the other entries of π, the entries of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ do not leave the stack until the input permutation is empty. Therefore, applying s_{312} to π keeps the entries in $\text{big}_{\text{cl}(\pi)+1}(\pi)$ adjacent to each other, except reverses their orders. Because every entry in $\text{big}_{\text{cl}(\pi)+1}(\pi)$ is greater than every other entry in π, the statement of the lemma now follows. □

Finally, we end our series of lemmas by showing that iterative applications of s_{312} increase the contraction length of π.

Lemma 3.4. *For $\pi \in S_n$ and $2 \leq t \leq n-1$ such that $\text{cl}(s_{312}^{t-1}(\pi)) < n-2$, we have that*

$$\text{cl}(s_{312}^{t}(\pi)) \geq \text{cl}(s_{312}^{t-1}(\pi)) + 1.$$

Proof. We first claim that the entry $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ appears to the left of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ in $s_{312}^{t-1}(\pi)$. Suppose otherwise. By Lemma 2.1, n appears in the rightmost descending run of $s_{312}^{t-1}(\pi)$. Therefore, if $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ appears to the right of $\text{big}_{\text{cl}(\pi)+1}(\pi)$, it must be adjacent to a number greater than it. But if so, then the contraction length of $s_{312}^{t-1}(\pi)$ must be at least $\text{cl}(s_{312}^{t-1}(\pi)) + 1$. Thus, we reach a contradiction, and $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ must appear to the left of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ in $s_{312}^{t-1}(\pi)$.

Now, when $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ enters the stack, it must be the largest number in the stack. Thus, when $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ enters the stack, the stack must be increasing from top to bottom. Therefore, $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ remains in the stack when the leftmost entry of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ enters the stack. In addition, because the entries of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ are larger than $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$, there cannot be any entry in between $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ and the leftmost entry of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ in the stack. Now, because $\text{big}_{\text{cl}(\pi)+1}(\pi) \in \text{Av}(213, 312)$ by the definition of cl, after the leftmost entry of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ enters the stack, the rest of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ enters the stack without any entry from the stack removed. Now, because the stack is increasing from top to bottom until $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ and every entry in $\text{big}_{\text{cl}(\pi)+1}(\pi)$ is greater than every other entry in π, the entry $n - \text{cl}(s_{312}^{t-1}(\pi)) - 1$ and the entries of $\text{big}_{\text{cl}(\pi)+1}(\pi)$ do not leave the stack until the input permutation is empty. Therefore, $\text{cl}(s_{312}^{t}(\pi)) \geq \text{cl}(s_{312}^{t-1}(\pi))+1$ as sought. □

We have the following as an immediate corollary of Lemma 3.4.

Corollary 3.1. *For $\pi \in S_n$ and $1 \leq t \leq n-1$, we have that $\text{cl}(s_{312}^{t}(\pi)) \geq t-1$.*

Now, we proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. From Theorem 1.1, if $\pi \in \text{Av}_n(213, 312)$, then $s_{312}^2(\pi) = \pi$. Therefore, if $\pi \in \text{Av}_n(213, 312)$, then $\pi \in s_{312}^{n-2}(S_n)$. Similarly, from the proof of Theorem 7.2 in Defant and Zheng [9], we have that $\zeta_n \in s_{312}^{n-2}(S_n)$. Therefore, it suffices to show that if $\pi \notin \text{Av}_n(213, 312)$ and $\pi \in s_{312}^{n-2}(S_n)$, then $\pi = \zeta_n$.

We induct on n. The statement clearly holds for $n \leq 4$. Now, suppose that $s_{312}^{n-2}(\pi) \notin \text{Av}_n(213, 312)$. If, for any $1 \leq t \leq n-2$, we have $\text{cl}(s_{312}^{t}(\pi)) \geq t$, then by Lemma 3.4, $\text{cl}(s_{312}^{n-2}(\pi)) \geq n-2$. Therefore, by Lemma 3.2, $s_{312}^{n-2}(\pi) \in \text{Av}_n(213, 312)$. Thus, $\text{cl}(s_{312}^{t}(\pi)) < t$ for all $1 \leq t \leq n-2$, and thus by Corollary 3.1,

$$\text{cl}(s_{312}^{t}(\pi)) = t - 1 \tag{1}$$

for all $1 \leq t \leq n-2$. In particular, setting $t = n-3$ in Equation (1), we have that

$$\text{cl}(s_{312}^{n-3}(\pi)) = n - 4. \tag{2}$$

Now, by Lemma 3.1,

$$\text{big}_{n-1}(s_{312}^{n-3}(\pi)) = s_{312}^{n-3}(\text{big}_{n-1}(\pi)).$$

Therefore,

$$\mathrm{red}(\mathrm{big}_{n-1}(s_{312}^{n-3}(\pi))) = \mathrm{red}(s_{312}^{n-3}(\mathrm{big}_{n-1}(\pi))) \in s_{312}^{n-3}(S_{n-1}).$$

Now, because of Equation (2) and Lemma 3.2, we have that $\mathrm{big}_{n-1}(s_{312}^{n-3}(\pi)) \notin \mathrm{Av}(213, 312)$. Thus, it follows from the induction hypothesis that

$$\mathrm{red}(\mathrm{big}_{n-1}(s_{312}^{n-3}(\pi))) = \zeta_{n-1}. \tag{3}$$

Now, from Equation (2), we have that

$$\ell(\mathrm{c}^{\mathrm{cl}(s_{312}^{n-3}(\pi))}(s_{312}^{n-3}(\pi))) = \ell(\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi))) = 4.$$

Furthermore, from Equation (3), we have that 3, 2, and 4 appear in that order in $\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi))$. Thus,

$$\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi)) \in \{1324, 3124, 3214, 3241\}. \tag{4}$$

Now, we claim that

$$\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi)) \in \{3124, 3241\}. \tag{5}$$

Suppose otherwise. Then by Equation (4), $\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi)) \in \{1324, 3214\}$. But then $s_{312}(\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi))) \in \{1243, 2431\}$, and so

$$\mathrm{cl}(s_{312}(\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi)))) = 2. \tag{6}$$

Now, by Lemma 3.3, we have that

$$\mathrm{c}^{n-4}(s_{312}^{n-2}(\pi)) = s_{312}(\mathrm{c}^{n-4}(s_{312}^{n-3}(\pi))). \tag{7}$$

Therefore, from Equation (6) and Equation (7), we have that

$$\mathrm{cl}(\mathrm{c}^{n-4}(s_{312}^{n-2}(\pi))) = 2.$$

Thus, $\mathrm{cl}(s_{312}^{n-2}(\pi)) = n-2$, which is a contradiction to Equation (1) when $t = n-2$. Therefore, we reach a contradiction as sought and Equation (5) must hold.

Now, from Equation (2), Equation (3), and Equation (5), we have that $s_{312}^{n-3}(\pi)$ is given either by

- starting with 32, listing all the odd numbers that are at least 5 and at most n in increasing order, then listing all the even numbers that are at least 4 and at most n in decreasing order, and finally, ending with 1 (for example, 32578641 when $n = 8$) or
- starting with 312, listing all the odd numbers that are at least 5 and at most n in increasing order, and then listing all the even numbers that are at least 4 and at most n in decreasing order (for example, 31257864 when $n = 8$).

It is easy to see that in either case, $s_{312}^{n-2}(\pi) = \zeta_n$, from which the statement of the theorem follows. □

The proof of Corollary 1.2 now follows immediately.

Proof of Corollary 1.2. If $s_{312}^{n-2}(\pi) \notin \mathrm{Av}_n(213, 312)$, then $s_{312}^{n-2}(\pi) = \zeta_n$ by Theorem 1.2. Therefore, $s_{312}^{n-1}(\pi) = s_{312}(s_{312}^{n-2}(\pi)) = s_{312}(\zeta_n) = \Xi_n$. As Defant and Zheng [9] noted, for any permutation σ, it follows directly from the definition of s_σ that $s_{\mathrm{comp}(\sigma)} = \mathrm{comp} \circ s_\sigma \circ \mathrm{comp}$. The statement of the corollary now follows. □

The proof of Corollary 1.3 also follows immediately.

Proof of Corollary 1.3. By Theorem 1.2, there exists $\pi \in S_n$ such that $s_{312}^{n-2}(\pi) \notin \mathrm{Av}_n(213, 312)$. Then for all $1 \le m \le n-1$, there does not exist $\sigma \in S_n$ such that $s_{312}^{m-1}(\pi) = s_{312}^{m}(\sigma)$, because if so, then $s_{312}^{n-2}(\pi) = s_{312}^{n-1}(\sigma) \in \mathrm{Av}_n(213, 312)$ by Theorem 1.1. Therefore, $s_{312}^{m-1}(\pi) \notin s_{312}^{m}(S_n)$, from which our claim follows. □

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