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Speed of Convergence of Time Euler Schemes for a Stochastic 2D Boussinesq Model

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Abstract: We prove that an implicit time Euler scheme for the 2D Boussinesq model on the torus D converges. The various moments of the $W^{1,2}$-norms of the velocity and temperature, as well as their discretizations, were computed. We obtained the optimal speed of convergence in probability, and a logarithmic speed of convergence in $L^2(\Omega)$. These results were deduced from a time regularity of the solution both in $L^2(D)$ and $W^{1,2}(D)$, and from an $L^2(\Omega)$ convergence restricted to a subset where the $W^{1,2}$-norms of the solutions are bounded.

Keywords: Boussinesq model; implicit time Euler schemes; convergence in probability; strong convergence

MSC: Primary 60H15; 60H35; 65M12; Secondary 76D03; 76M35

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1. Introduction

The Boussinesq equations have been used as a model in many geophysical applications. They have been widely studied in both deterministic and stochastic settings. We take random forces into account and formulate the Bénard convection problem as a system of stochastic partial differential equations (SPDEs). The need to take stochastic effects into account for modeling complex systems has now become widely recognized. Stochastic partial differential equations (SPDEs) arise naturally as mathematical models for nonlinear macroscopic dynamics under random influences. The Navier–Stokes equations are coupled with a transport equation for the temperature and with diffusion. The system is subjected to a multiplicative random perturbation, which will be defined later. Here, u describes the fluid velocity field, whereas θ describes the temperature of the buoyancy-driven fluid, and p is the fluid's pressure.

We study the multiplicative stochastic Boussinesq equations

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \theta + G(u)\,dW \quad \text{in} \quad (0,T)\times D, \tag{1}$$

$$\partial_t \theta - \kappa\Delta\theta + (u\cdot\nabla\theta) = \tilde{G}(\theta)\,d\tilde{W} \quad \text{in} \quad (0,T)\times D, \tag{2}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad (0,T)\times D,$$

where $T > 0$. The processes $u : \Omega\times(0,T)\times D \to \mathbb{R}^2$ and $\theta : \Omega\times(0,T)\times D \to \mathbb{R}$ have initial conditions u_0 and θ_0 in D, respectively. The parameter $\nu > 0$ denotes the kinematic viscosity of the fluid, and $\kappa > 0$ denotes its thermal diffusivity. These fields satisfy periodic boundary conditions $u(t, x + Lv_i) = u(t, x)$, $\theta(t, x + Lv_i) = \theta(t, x)$ on $(0,T)\times\partial D$, where v_i, $i = 1, 2$ denotes the canonical basis of $\mathbb{R}^2$, and $p : \Omega\times(0,T)\times D \to \mathbb{R}$ is the pressure.

In dimension 2 without any stochastic perturbation, this system has been extensively studied with a complete picture about its well-posedness and long-time behavior. In the deterministic setting, more investigations have been extended to the cases where $n = 0$ and/or $k = 0$, with some partial results.

If the $(L^2)^2$ (resp., L^2) norms of u_0 and q_0 are square integrable, it is known that the random system (1)–(2) is well-posed, and that there exists a unique solution $(u \ q)$ in $C([0, T]; (L^2)^2 \ L^2) \setminus L^2(W; (H^1)^2 \ H^1)$; see, e.g., [1,2].

Numerical schemes and algorithms have been introduced to best approximate the solution to non-linear PDEs. The time approximation is either an implicit Euler or a time-splitting scheme coupled with a Galerkin approximation or finite elements to approximate the space variable. The literature on numerical analysis for SPDEs is now very extensive. In many papers, the models are either linear, have global Lipschitz properties, or, more generally, have some monotonicity property. In this case, the convergence was proven to be in mean square. When nonlinearities are involved that are not of Lipschitz or monotone type, then a rate of convergence in mean square is more difficult to obtain. Indeed, because of the stochastic perturbation, one may not use the Gronwall lemma after taking the expectation of the error bound, since it involves a nonlinear term that is often quadratic; such a nonlinearity requires some localization.

In a random setting, the discretization of the Navier–Stokes equations on the torus has been intensively investigated. Various space–time numerical schemes have been studied for the stochastic Navier–Stokes equations with a multiplicative or an additive noise, where, in the right hand side of (1) (with no q), we have either $G(u)\,dW$ or dW. We refer to [3–7], where the convergence in probability is stated with various rates of convergence in a multiplicative setting for a time implicit Euler scheme, and [8] for a time splitting scheme. As stated previously, the main tool used to obtain the convergence in probability is the localization of the nonlinear term over a space of large probability. We studied the strong (that is, $L^2(W)$) rate of convergence of the time-implicit Euler scheme (resp., space–time-implicit Euler scheme coupled with finite element space discretization) in our previous papers [9] (resp., [10]) for an H^1-valued initial condition. The method is based on the fact that the solution (and the scheme) have finite moments (bounded uniformly on the mesh). For a general multiplicative noise, the rate is logarithmic. When the diffusion coefficient is bounded (which is a slight extension of an additive noise), the supremum of the H^1-norm of the solution has exponential moments; we used this property in [9,10] to obtain an explicit polynomial strong rate of convergence. However, this rate depends on the viscosity and the strength of the noise, and is strictly less than $1/2$ for the time parameter (resp., less than 1 for the spatial one). For a given viscosity, the time rates on convergence increase to $1/2$ when the strength of the noise converges to 0. For an additive noise, if the strength of the noise is not too large, the strong ($L^2(W)$) rate of convergence in time is the optimal one, and is almost $1/2$ (see [11]). Once more, this is based on exponential moments of the supremum of the H^1-norm of the solution (and of its scheme for the space discretization); this enabled us to have strong polynomial time rates.

In the current paper, we study the time approximation of the Boussinesq Equations (1) and (2) in a multiplicative setting. To the best of our knowledge, it is the first result where a time-numerical scheme is implemented for a more general hydrodynamical model with a multiplicative noise. We use a fully implicit time Euler scheme and once more assume that the initial conditions u_0 and q_0 belong to $H^1(D)$ in order to prove a rate of convergence in $L^2(D)$ uniformly in time. We prove the existence of finite moments of the H^1-norms of the velocity and the temperature uniformly in time. Since we are on the torus, this is quite easy for the velocity. However, for the temperature, due to the presence of the velocity in the bilinear term, the argument is more involved and has to be carried out in two steps. It requires higher moments on the H^1-norm of the initial condition. The time regularity of the solutions u, q is the same as that of u in the Navier–Stokes equations. We then study rates of convergence in probability and in $L^2(W)$. The rate of convergence in probability is optimal (almost $1/2$); we have to impose higher moments on the initial conditions than what is needed for the velocity described by stochastic Navier–Stokes equations. Once more, we first obtain an $L^2(\Omega)$ convergence on a set where we bound the L^2 norm of the gradients of both the velocity and the temperature. We deduce an optimal rate of convergence in probability that is strictly less than 1/2. When the H^1-norm of the initial condition has all moments (for example, it is a Gaussian H^1-valued random variable), the rate of convergence in $L^2(\Omega)$ is any negative exponent of the logarithm of the number of time steps. These results extend those established for the Navier–Stokes equations subject to a multiplicative stochastic perturbation.

The paper is organized as follows. In Section 2, we describe the model and the assumptions on the noise and the diffusion coefficients, and describe the fully implicit time Euler scheme. In Section 3, we state the global well-posedness of the solution to (1)–(2), moment estimates of the gradient of u and θ uniformly in time and the existence of the scheme. We then formulate the main results of the paper about the rates of the convergence in probability and in $L^2(\Omega)$ of the scheme to the solution. In Section 4, we prove moment estimates in H^1 of u and θ uniformly on the time interval $[0, T]$ if we start with more regular (H^1) initial conditions. This is essential in order to be able to deduce a rate of convergence from the localized result. Section 5 states the time regularity results of the solution (u, θ) both in $L^2(D)$ and $H^1(D)$; this a crucial ingredient of the final results. In Section 6, we prove that the time Euler scheme is well-defined and prove its moment estimates in L^2 and H^1. Section 7 deals with the localized convergence of the scheme in $L^2(\Omega)$. This preliminary step is necessary due to the bilinear term, which requires some control of the H^1 norm of u and θ. In Section 8, we prove the rate of convergence in probability and in $L^2(\Omega)$. Finally, Section 9 summarizes the interest of the model and describes some further necessary/possible extensions of this work.

As usual, except if specified otherwise, C denotes a positive constant that may change throughout the paper, and $C(a)$ denotes a positive constant depending on some parameter a.

2. Preliminaries and Assumptions

In this section, we describe the functional framework, the driving noise, the evolution equations, and the fully implicit time Euler scheme.

2.1. The Functional Framework

Let $D = [0, L]^2$ with periodic boundary conditions $\mathbb{L}^p := L^p(D)^2$ (resp., $\mathbb{W}^{k,p} := W^{k,p}(D)^2$) be the usual Lebesgue and Sobolev spaces of vector-valued functions endowed with the norms $\|\cdot\|_{\mathbb{L}^p}$ (resp., $\|\cdot\|_{\mathbb{W}^{k,p}}$).

Let $V^0 := \{u \in \mathbb{L}^2 : \mathrm{div}(u) = 0 \text{ on } D\}$. Let $P : \mathbb{L}^2 \to V^0$ denote the Leray projection, and let $A = -P\Delta$ denote the Stokes operator, with domain $\mathrm{Dom}(A) = \mathbb{W}^{2,2} \cap V^0$.

Let $\tilde{A} = -\Delta$ acting on $L^2(D)$. For any non-negative real number k, let

$H^k = \mathrm{Dom}\big(\tilde{A}^{\frac{k}{2}}\big)$, $V^k = \mathrm{Dom}\big(A^{\frac{k}{2}}\big)$ endowed with the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{V^k}$.

Thus, $H^0 = L^2(D)$ and $H^k = W^{k,2}$. Moreover, let V^{-1} be the dual space of V^1 with respect to the pivot space V^0, and $\langle \cdot, \cdot \rangle$ denote the duality between V^1 and V^{-1}.

Let $b : (V^1)^3 \to \mathbb{R}$ denote the trilinear map defined by

$$b(u_1, u_2, u_3) := \int_D \big(u_1(x) \cdot \nabla u_2(x)\big) \cdot u_3(x)\, dx.$$

The incompressibility condition implies that $b(u_1, u_2, u_3) = -b(u_1, u_3, u_2)$ for $u_i \in V^1$, $i = 1, 2, 3$. There exists a continuous bilinear map $B : V^1 \times V^1 \to V^{-1}$ such that

$$\langle B(u_1, u_2), u_3 \rangle = b(u_1, u_2, u_3), \quad \text{for all } u_i \in V^1,\ i = 1, 2, 3.$$

Therefore, the map B satisfies the following antisymmetry relations:

$$\langle B(u_1,u_2),u_3\rangle = -\langle B(u_1,u_3),u_2\rangle, \quad \langle B(u_1,u_2),u_2\rangle = 0 \quad \text{for all} \quad u_i \in V^1. \tag{3}$$

For $u, v \in V^1$, we have $B(u,v) := P\,(u\cdot\nabla v)$.
Furthermore, since $D = [0,L]^2$ with periodic boundary conditions, we have (see e.g., [12])

$$\langle B(u,u), Au\rangle = 0, \quad \forall u \in V^2. \tag{4}$$

Note that, for $u \in V^1$ and $\theta_1, \theta_2 \in H^1$, if $(u.\nabla)\theta = \sum_{i=1,2} u_i \partial_i \theta$, we have

$$\langle [u.\nabla]\theta_1, \theta_2\rangle = -\langle [u.\nabla]\theta_2, \theta_1\rangle, \tag{5}$$

so that $\langle [u.\nabla]\theta, \theta\rangle = 0$ for $u \in V^1$ and $\theta \in H^1$.

In dimension 2, the inclusions $H^1 \subset L^p$ and $V^1 \subset L^p$ for $p \in [2,\infty)$ follow from the Sobolev embedding theorem. More precisely, the following Gagliardo–Nirenberg inequality is true for some constant $\bar{C}_p$:

$$\|u\|_{L^p} \le \bar{C}_p \|A^{\frac{1}{2}} u\|_{L^2}^{\alpha} \|u\|_{L^2}^{1-\alpha} \quad \text{for} \quad \alpha = 1 - \frac{2}{p}, \quad \forall u \in V^1. \tag{6}$$

Finally, let us recall the following estimate of the bilinear terms $(u.\nabla)v$ and $(u.\nabla)\theta$.

Lemma 1. Let α, ρ be positive numbers and $\delta \in [0,1)$ be such that $\delta + \rho > \frac{1}{2}$ and $\alpha + \delta + \rho \ge 1$. Let $u \in V^{\alpha}$, $v \in V^{\rho}$ and $\theta \in H^{\rho}$; then,

$$\|A^{-\delta} P[(u.\nabla)v]\|_{V^0} \le C\|A^{\alpha} u\|_{V^0} \|A^{\rho} v\|_{V^0}, \tag{7}$$

$$\|\tilde{A}^{-\delta}[(u.\nabla)\theta]\|_{H^0} \le C\|A^{\alpha} u\|_{V^0} \|\tilde{A}^{\rho} \theta\|_{H^0}, \tag{8}$$

for some positive constant $C := C(\alpha, \delta, \rho)$.

Proof. The upper estimate (7) is Lemma 2.2 in [13]. The argument, which is based on the Sobolev embedding theorem and Hölder's inequality, clearly proves (8). □

2.2. The Stochastic Perturbation

Let K (resp., $\tilde{K}$) be a Hilbert space and let $(W(t), t \ge 0)$ (resp., $(\tilde{W}(t), t \ge 0)$) be a K-valued (resp., $\tilde{K}$-valued) Brownian motion with covariance Q (resp., $\tilde{Q}$), which is a trace-class operator of K (resp., $\tilde{K}$) such that $Qz_j = q_j z_j$ (resp., $\tilde{Q}\tilde{z}_j = \tilde{q}_j \tilde{z}_j$), where $\{z_j\}_{j\ge0}$ (resp., $\{\tilde{z}_j\}_{j\ge0}$) is a complete orthonormal system of K (resp., $\tilde{K}$), $q_j, \tilde{q}_j > 0$, and $\mathrm{Tr}(Q) = \sum_{j\ge0} q_j < \infty$ (resp., $\mathrm{Tr}(\tilde{Q}) = \sum_{j\ge0} \tilde{q}_j < \infty$). Let $\{\beta_j\}_{j\ge0}$ (resp., $\{\tilde{\beta}_j\}_{j\ge0}$) be a sequence of independent one-dimensional Brownian motions on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \ge 0), \mathbb{P})$. Then,

$$W(t) = \sum_{j\ge0} \sqrt{q_j}\, \beta_j(t)\, z_j, \quad \tilde{W}(t) = \sum_{j\ge0} \sqrt{\tilde{q}_j}\, \tilde{\beta}_j\, \tilde{z}_j.$$

For details concerning these Wiener processes, we refer to [14].

Projecting the velocity on divergence-free fields, we consider the following SPDEs for processes modeling the velocity $u(t)$ and the temperature $\theta(t)$. The initial conditions u_0 and θ_0 are $\mathcal{F}_0$-measurable, taking values in V^0 and H^0, respectively, and

$$\partial_t u(t) + \big[\nu A u(t) + B(u(t), u(t))\big]dt = P(\theta(t) v_2) + G(u(t))\, dW(t), \tag{9}$$

$$\partial_t \theta(t) + \big[\kappa \tilde{A}\theta(t) + (u(t).\nabla)\theta(t)\big]\, dt = \tilde{G}(\theta(t))\, d\tilde{W}(t), \tag{10}$$

where ν, κ are strictly positive constants, and $v_2 = (0,1) \in \mathbb{R}^2$.

We make the following classical linear growth and Lipschitz assumptions on the diffusion coefficients G and $\tilde{G}$. For technical reasons, we will have to require $u_0 \in V^1$ and $q \in H^1$ and prove estimates similar to (19) and (20), raising the space regularity of the processes by one step in the scale of Sobolev spaces. Therefore, we have to strengthen the regularity of the diffusion coefficients.

Condition (C-u) (i) Let $G : V^0 \to L(K; V^0)$ be such that

$$\|G(u)\|^2_{L(K,V^0)} \le K_0 + K_1\|u\|^2_{V^0}, \qquad \forall u \in V^0, \tag{11}$$

$$\|G(u_1) - G(u_2)\|^2_{L(K,V^0)} \le L_1\|u_1 - u_2\|^2_{V^0}, \qquad \forall u_1, u_2 \in V^0. \tag{12}$$

(ii) Let also $G : V^1 \to L(K; V^1)$ satisfy the growth condition

$$\|G(u)\|^2_{L(K;V^1)} \le K_2 + K_3\|u\|^2_{V^1}, \qquad \forall u \in V^1. \tag{13}$$

and

Condition (C-q) (i) Let $\tilde{G} : H^0 \to L(\tilde{K}; H^0)$ be such that

$$\|\tilde{G}(q)\|^2_{L(\tilde{K},H^0)} \le \tilde{K}_0 + \tilde{K}_1\|q\|^2_{H^0}, \qquad \forall q \in H^0, \tag{14}$$

$$\|\tilde{G}(q_1) - \tilde{G}(q_2)\|^2_{L(\tilde{K},H^0)} \le \tilde{L}_1\|q_1 - q_2\|^2_{H^0}, \qquad \forall q_1, q_2 \in H^0. \tag{15}$$

(ii) Let also $\tilde{G} : H^1 \to L(K; H^1)$ satisfy the growth condition

$$\|\tilde{G}(q)\|^2_{L(\tilde{K};H^1)} \le \tilde{K}_2 + \tilde{K}_3\|q\|^2_{H^1}, \qquad \forall q \in H^1. \tag{16}$$

2.3. The Fully Implicit Time Euler Scheme

Fix $N \in \{1, 2, ...\}$, let $h := \frac{T}{N}$ denote the time mesh, and, for $j = 0, 1, \dots, N$, set $t_j := j\frac{T}{N}$. The fully implicit time Euler scheme $\{u^k; k = 0, 1, ..., N\}$ and $\{q^k; k = 0, 1, ..., N\}$ is defined by $u^0 = u_0$, $q^0 = q_0$, and, for $\varphi \in V^1$, $\psi \in H^1$ and $l = 1, ..., N$,

$$\begin{aligned}\left(u^l - u^{l-1} + h\nu A u^l + hB\left(u^l, u^l\right), \varphi\right) &= \left(Pq^{l-1}v_2, \varphi\right)h \\ &\quad + \left(G(u^{l-1})[W(t_l) - W(t_{l-1})], \varphi\right),\end{aligned} \tag{17}$$

$$\left(q^l - q^{l-1} + h\kappa\tilde{A}q^l + h[u^{l-1}.\nabla]q^l, \psi\right) = \left(\tilde{G}(q^{l-1})[W(t_l) - W(t_{l-1})], \psi\right). \tag{18}$$

3. Main Results

In this section, we state the main results about the well-posedness of the solutions (u, q), the scheme $\{u^k; k = 0, 1, ..., N\}$ and the rate of the convergence of the scheme $\{(u^k, q^k); k = 0, 1, ..., N\}$ to (u, q).

3.1. Global Well-Posedness and Moment Estimates of (u, q)

The first results state the existence and uniqueness of a weak pathwise solution (that is a strong probabilistic solution in the weak deterministic sense) of (9) and (10). It is proven in [1] (see also [2]).

Theorem 1. Let $u_0 \in L^{2p}(\Omega; V^0)$ and $q_0 \in L^{2p}(\Omega; H^0)$ for $p = 1$ or $p \in [2, \infty)$. Let the coefficients G and $\tilde{G}$ satisfy the conditions **(C-u)(i)** and **(C-q)(i)**, respectively. Then, Equations (9) and (10) have a unique pathwise solution, i.e.,

- u (resp., q) is an adapted V^0-valued (resp., H^0-valued) process that belongs a.s. to $L^2(0, T; V^1)$ (resp., to $L^2(0, T; H^1)$);

- P a.s. we have $u \in C([0,T];V^0)$, $\theta \in C(0,T];H^0)$ and

$$\big(u(t),\varphi\big) + \nu\int_0^t \big(A^{\frac12}u(s), A^{\frac12}\varphi\big)ds + \int_0^t \big\langle [u(s)\cdot\nabla]u(s),\varphi\big\rangle ds$$
$$= \big(u_0,\varphi\big) + \int_0^t \big(\Pi\theta(t)v_2,\varphi\big)ds + \int_0^t \big(\varphi, G(u(s))dW(s)\big),$$
$$\big(\theta(t),\psi\big) + \kappa\int_0^t \big(\tilde A^{\frac12}\theta(s),\tilde A^{\frac12}\psi\big)ds + \int_0^t \big\langle [u(s)\cdot\nabla]\theta(s),\psi\big\rangle ds$$
$$= \big(\theta_0,\psi\big) + \int_0^t \big(\psi,\tilde G(\theta(s))d\tilde W(s)\big),$$

for every $t \in [0,T]$ and every $\varphi \in V^1$ and $\psi \in H^1$.

Furthermore,

$$E\Big(\sup_{t\in[0,T]}\|u(t)\|^{2p}_{V^0} + \int_0^T \|A^{\frac12}u(t)\|^2_{V^0}\big[1+\|u(t)\|^{2(p-1)}_{V^0}\big]dt\Big) \le C\big[1+E(\|u_0\|^{2p}_{V^0})\big], \tag{19}$$
$$E\Big(\sup_{t\in[0,T]}\|\theta(t)\|^{2p}_{H^0} + \int_0^T \|\tilde A^{\frac12}\theta(t)\|^2_{H^0}\big[1+\|\theta(t)\|^{2(p-1)}_{H^0}\big]dt\Big) \le C\big[1+E(\|\theta_0\|^{2p}_{H^0})\big]. \tag{20}$$

The following result proves that, if $u_0 \in V^1$, the solution u to (9) and (10) is more regular.

Proposition 1. Let $u_0 \in L^{2p}(\Omega;V^1)$ and $\theta_0 \in L^{2p}(\Omega;H^0)$ for $p = 1$ or some $p \in [2,\infty)$, and let G satisfy condition **(C-u)** and $\tilde G$ satisfy condition **(C-θ)**. Then, the solution u to (9) and (10) belongs a.s. to $C([0,T];V^1) \cap L^2([0,T];V^2)$. Moreover, for some constant C,

$$E\Big(\sup_{t\in[0,T]}\|u(t)\|^{2p}_{V^1} + \int_0^T \|Au(t)\|^2_{V^0}\big[1+\|A^{\frac12}u(t)\|^{2(p-1)}_{V^0}\big]dt\Big) \le C\big[1+E\big(\|u_0\|^{2p}_{V^1}+\|\theta_0\|^{2p}_{H^0}\big)\big]. \tag{21}$$

The next result proves similar bounds for moments of the gradient of the temperature uniformly in time.

Proposition 2. Let $u_0 \in L^{8p+\epsilon}(\Omega;V^1)$ and $\theta_0 \in L^{8p+\epsilon}(\Omega;H^1)$ for some $\epsilon > 0$ and $p = 1$ or $p \in [2,+\infty)$. Suppose that the coefficients G and $\tilde G$ satisfy the conditions **(C-u)** and **(C-θ)**. There exists a constant C such that

$$E\Big[\sup_{t\le T}\|\tilde A^{\frac12}\theta(t)\|^{2p}_{H^0} + \int_0^T \|\tilde A\theta(s)\|^2_{H^0}\,\|\tilde A^{\frac12}\theta(s)\|^{2(p-1)}_{H^0}ds\Big] \le C. \tag{22}$$

3.2. Global Well-Posedness of the Time Euler Scheme

The following proposition states the existence and uniqueness of the sequences $\{u^k\}_{k=0,\dots,N}$ and $\{\theta^k\}_{k=0,\dots,N}$.

Proposition 3. Let condition **(G-u)(i)** and **(C-θ)(i)** be satisfied, $u_0 \in V^0$ and $\theta^0 \in H^0$ a.s. The time fully implicit scheme (17) and (18) has a unique solution $\{u^l\}_{l=1,\dots,N} \in V^1$, $\{\theta^l\}_{l=1,\dots,N} \in H^1$.

3.3. Rates of Convergence in Probability and in $L^2(\Omega)$

The following theorem states that the implicit time Euler scheme converges to the pair (u,θ) in probability with the "optimal" rate "almost 1/2". It is the main result of the paper. For $j = 0,\dots,N$, set $e_j := u(t_j) - u^j$ and $\tilde e_j := \theta(t_j) - \theta^j$; then, $e_0 = \tilde e_0 = 0$.

Theorem 2. Suppose that the conditions **(C-u)** and **(C-θ)** hold. Let $u_0 \in L^{32+\varepsilon}(\Omega; V^1)$ and $\theta_0 \in L^{32+\varepsilon}(\Omega; H^1)$ for some $\varepsilon > 0$, u, θ be the solution to (9) and (10) and $\{u^j, \theta^j\}_{j=0,\dots,N}$ be the solution to (17) and (18). Then, for every $\eta \in (0,1)$, we have

$$\lim_{N\to\infty} P\Big(\max_{1\le J\le N} \big(\|e_J\|^2_{V^0} + \|\tilde{e}_J\|^2_{H^0} \big) + \frac{T}{N}\sum_{j=1}^{N} \big(\|A^{\frac12} e_j\|^2_{V^0} + \|A^{\frac12}\tilde{e}_j\|^2_{H^0} \big) \ge N^{-\eta} \Big) = 0. \tag{23}$$

We finally state that the strong (i.e., in $L^2(\Omega)$) rate of convergence of the implicit time Euler scheme is some negative exponent of $\ln N$. Note that, if the initial conditions u_0 and θ_0 are deterministic, or if their V^1 and H^1-norms have moments of all orders (for example, if u_0 and θ_0 are Gaussian random variables), the strong rate of convergence is any negative exponent of $\ln N$. More precisely, we have the following result.

Theorem 3. Suppose that the conditions **(C-u)** and **(C-θ)(i)** hold. Let $u_0 \in L^{2^q+\varepsilon}(\Omega; V^1)$ and $\theta_0 \in L^{2^q+\varepsilon}(\Omega; H^1)$ for $q \in [5, \infty)$ and some $\varepsilon > 0$. Then, for some constant C such that

$$E\Big(\max_{1\le J\le N} \big(\|e_J\|^2_{V^0} + \|\tilde{e}_J\|^2_{H^0} \big) + \frac{T}{N}\sum_{j=1}^{N} \big(\|A^{\frac12} e_j\|^2_{V^0} + \|A^{\frac12}\tilde{e}_j\|^2_{H^0} \big) \Big) \le C \big(\ln(N)\big)^{-(2^{q-1}+1)} \tag{24}$$

for large enough N.

4. More Regularity of the Solution

4.1. Moments of u in $L^\infty(0,T;V^1)$

In this section, we prove that, if $u_0 \in V^1$ and $\theta_0 \in H^0$, the H^1-norm of the velocity has bounded moments uniformly in time.

Proof of Proposition 1. Apply the operator $A^{\frac12}$ to (9) and use (formally) Itô's formula for the square of the $\|.\|_{V^0}$-norm of $A^{\frac12}u(t)$. Then, using (4), we obtain

$$\begin{aligned} \|A^{\frac12}u(t)\|^2_{V^0} + 2\nu\int_0^t \|Au(s)\|^2_{V^0}\,ds &= \|A^{\frac12}u_0\|^2_{V^0} + 2\int_0^t \big(A^{\frac12}P\theta(s)v_2, A^{\frac12}u(s)\big)ds \\ &\quad + 2\int_0^t \big(A^{\frac12}G(u(s))dW(s), A^{\frac12}u(s)\big) + \int_0^t \|A^{\frac12}G(u(s))\|^2_{L(K;V^0)}\,\mathrm{Tr}(Q)ds. \end{aligned} \tag{25}$$

Let $\tau_M := \inf\{t : \|u(t)\|_{V^1} \ge M\}$; using (13), integration by parts and the Cauchy–Schwarz and Young inequalities, we deduce, for $M > 0$ and $t \in [0,T]$,

$$\begin{aligned} E\Big(\|A^{\frac12}u(t\wedge\tau_M)\|^2_{V^0} + 2\nu\int_0^{t\wedge\tau_M} \|Au(s)\|^2_{V^0}ds \Big) &\le E\big(\|u_0\|^2_{V^0}\big) \\ &\quad + 2E\int_0^{t\wedge\tau_M} \|\theta(s)\|_{H^0}\|Au(s)\|_{V^0}ds + \mathrm{Tr}(Q)E\int_0^{t\wedge\tau_M} \big(K_2 + K_3\|u(s)\|^2_{V^1}\big)ds \\ &\le E\big(\|u_0\|^2_{V^0}\big) + \nu\int_0^{t\wedge\tau_M} \|Au(s)\|^2_{V^0}ds + \frac{1}{\nu}E\int_0^{t\wedge\tau_M} \|\theta(s)\|^2_{H^0}ds + K_2T \\ &\quad + K_3TE\Big(\sup_{t\in[0,T]} \|u(t)\|^2_{V^0}\Big) + K_3\int_0^t E\Big(\|A^{\frac12}u(s\wedge\tau_M)\|^2_{V^0}\Big)ds. \end{aligned}$$

Indeed the stochastic integral in the right hand side of (25) is a square integrable, and hence a centered martingale. Neglecting the time integral in the left hand side, using (19) and the Gronwall lemma, we deduce

$$\sup_M \sup_{t\in[0,T]} E\|A^{\frac12}u(t\wedge\tau_M)\|^2_{V^0} \le C < \infty. \tag{26}$$

As $M \to \infty$, this implies that $E\int_0^T \|Au(s)\|^2_{V^0}ds < \infty$.

Furthermore, the Davis inequality and Young's inequality imply

$$
\begin{aligned}
E\Big(\sup_{s\le t}\Big|\int_0^{s\wedge t_M}\big(A^{\frac12}G(u(r))dW(r),A^{\frac12}u(r)\big)\Big|\Big)
&\le 3E\Big\{\int_0^t \|A^{\frac12}u(r\wedge t_M)\|^2_{V^0}\,\mathrm{Tr}(Q)\,\|A^{\frac12}G(u(r\wedge t_M))\|^2_{L(K;V^0)}dr\Big\}^{\frac12}\\
&\le 3E\Big(\sup_{s\le t}\|A^{\frac12}u(s\wedge t_M)\|_{V^0}\Big\{\mathrm{Tr}(Q)\int_0^t[K_2+K_3\|u(s\wedge t_M)\|^2_{V^1}]ds\Big\}^{\frac12}\Big)\\
&\le \frac12 E\Big(\sup_{s\le t}\|A^{\frac12}u(s\wedge t_M)\|^2_{V^0}\Big)+9\mathrm{Tr}(Q)E\Big(\int_0^t[K_2+K_3\|u(r\wedge t_M)\|^2_{V^1}]ds\Big).
\end{aligned}
$$

The upper estimates (19), (20), (25) and (26) imply that, for some constant C depending on $E\big(\int_0^T \big(\|u(t)\|^2_{V^0}+\|A^{\frac12}u(t)\|^2_{V^0}+\|q(t)\|^2_{H^0}\big)ds\big)<\infty$,

$$
\begin{aligned}
\sup_M E\Big(\frac12\sup_{t\le T}\|A^{\frac12}u(t\wedge t_M)\|^2_{V^0}+\int_0^{T\wedge t_M}\|Au(s)\|^2_{V^0}ds\Big)
&\le C+CE\int_0^T\big(\|A^{\frac12}u(t)\|^2_{V^1}+\|q(t)\|^2_{H^0}\big)ds<\infty.
\end{aligned}
$$

As $M\to\infty$, we deduce

$$
E\Big(\sup_{t\in[0,T]}\|A^{\frac12}u(t)\|^2_{V^0}\Big)+E\int_0^T\|Au(s)\|^2_{V^0}ds\le C<\infty.
$$

This proves (21) for $p=1$.

Given $p\in[2,\infty)$ and using Itô's formula for the map $x\to x^p$ in (25), we obtain

$$
\begin{aligned}
\|A^{\frac12}u(t\wedge t_M)\|^{2p}_{V^0}&+2p\nu\int_0^{t\wedge t_M}\|Au(s)\|^2_{V^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds=\|A^{\frac12}u_0\|^{2p}_{V^0}\\
&+2p\int_0^{t\wedge t_M}\big(A^{\frac12}Pq(s)v_2,A^{\frac12}u(s)\big)\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds\\
&+2p\int_0^{t\wedge t_M}\big(A^{\frac12}G(u(s))dW(s),A^{\frac12}u(s)\big)\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}\\
&+p\mathrm{Tr}(Q)\int_0^{t\wedge t_M}\|G(u(s)\|^2_{L(K;V^1)}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds\\
&+2p(p-1)\mathrm{Tr}(Q)\int_0^{t\wedge t_M}\|\big(A^{\frac12}G(u(s))\big)^*A^{\frac12}u(s)\|^2_K\|A^{\frac12}u(s)\|^{2(p-2)}_{V^0}ds. \qquad (27)
\end{aligned}
$$

Integration by parts and the Cauchy–Schwarz, Hölder and Young inequalities imply that

$$
\begin{aligned}
\int_0^t\big(A^{\frac12}Pq(s)v_2,A^{\frac12}u(s)\big)\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds
&\le\int_0^t\|Au(s)\|_{V^0}\|q(s)\|_{H^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds\\
&\le\Big\{\int_0^t\|Au(s)\|^2_{V^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds\Big\}^{\frac12}\Big\{\int_0^t\|q(s)\|^2_{H^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds\Big\}^{\frac12}\\
&\le\frac{p\nu}{2}\int_0^t\|Au(s)\|^2_{V^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds+\frac{1}{2p\nu}\int_0^t\|q(s)\|^2_{H^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds\\
&\le\epsilon\int_0^t\|Au(s)\|^2_{V^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0}ds+C\int_0^t\|q(s)\|^{2p}_{H^0}ds+C\int_0^t\|A^{\frac12}u(s)\|^{2p}_{V^0}ds. \qquad (28)
\end{aligned}
$$

Since $a^{p-1} \le 1 + a^p$ for any $a \ge 0$, the growth condition (13) implies that

$$\begin{aligned}\int_0^t \|A^{\frac12} G(u(s))\|^2_{L(K,V^0)} \|A^{\frac12} u(s)\|^{2(p-1)}_{V^0} ds &\le \int_0^t \big[K_2 + K_3\|u(s)\|^2_{V^0} + K_3\|A^{\frac12}u(s)\|^2_{V^0}\big]\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0} ds \\ &\le C\Big(T + \int_0^T \|u(s)\|^{2p}_{V^0} + \int_0^t \|A^{\frac12}u(s)\|^{2p}_{V^0} ds\Big). \end{aligned} \tag{29}$$

Furthermore, since $\|A^{\frac12} G(u(s)) A^{\frac12} u(s)\|^2_{V^0} \le [K_2 + K_3\|u(s)\|^2_{V^1}]\|A^{\frac12}u(s)\|^2_{V^0}$, the upper estimate of the corresponding integral is similar to that of (29). Since the stochastic integral $\int_0^{t\wedge\tau_M}\big(A^{\frac12}G(u(s))dW(s), A^{\frac12}u(s)\big)\|u(s)\|^{2(p-1)}_{V^0}$ is square integrable, it is centered. Therefore, (27) and the above upper estimates (28) and (29) imply that

$$\begin{aligned}\sup_M \mathbb{E}\|A^{\frac12}u(t\wedge\tau_M)\|^{2p}_{V^0} &+ p\eta \int_0^{t\wedge\tau_M} \|Au(s)\|^2_{V^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0} \\ &\le CT + \mathbb{E}\int_0^t \big(\|u(s)\|^{2p}_{V^0} + \|q(s)\|^{2p}_{H^0}\big) ds + \int_0^t \mathbb{E}\|A^{\frac12}u(s\wedge\tau_M)\|^{2p}_{V^0} ds.\end{aligned}$$

Using Gronwall's lemma we obtain

$$\sup_M \sup_{t\in[0,T]} \mathbb{E}\big(\|A^{\frac12}u(s\wedge\tau_M)\|^{2p}_{V^0}\big) = C < \infty, \tag{30}$$

$$\sup_M \mathbb{E}\int_0^{T\wedge\tau_M}\|Au(s)\|^2_{V^0}\|A^{\frac12}u(s)\|^{2(p-1)}_{V^0} ds = C < \infty. \tag{31}$$

Finally, using the Davis inequality, the Hölder and Young inequalities, we deduce

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]} 2p\Big|\int_0^{s\wedge\tau_M}&\big(A^{\frac12}G(u(r))dW(r), A^{\frac12}u(r)\big)\|A^{\frac12}u(r)\|^{2(p-1)}_{V^0}\Big| \\ &\le 6p\,\mathbb{E}\Big\{\int_0^{t\wedge\tau_M} \mathrm{Tr}(Q)\|A^{\frac12}G(u(s))\|^2_{L(K;V^0)}\|A^{\frac12}u(s)\|^{4p-2}_{V^0}ds\Big\}^{\frac12} \\ &\le 6p\,\mathrm{Tr}(Q)^{\frac12}\,\mathbb{E}\Big[\sup_{s\le t\wedge\tau_M}\|A^{\frac12}u(s)\|^p_{V^0} \\ &\qquad\Big\{\int_0^t \|A^{\frac12}G(u(s\wedge\tau_N))\|^2_{L(K;V^0)}\|A^{\frac12}u(s\wedge\tau_M)\|^{2p-2}_{V^0}ds\Big\}^{\frac12}\Big] \\ &\le \frac12\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_M]}\|A^{\frac12}u(s)\|^{2p}_{V^0}\Big) + C\mathbb{E}\Big(1+\int_0^t\|u(s)\|^{2p}_{V^0}ds + \int_0^t\|A^{\frac12}u(s)\|^{2p}_{V^0}ds\Big).\end{aligned} \tag{32}$$

The upper estimates (27), (19) and (32) imply that

$$\begin{aligned}\sup_M \mathbb{E}\Big(\sup_{s\in[0,T\wedge\tau_M]}\|A^{\frac12}u(s)\|^{2p}_{V^0}\Big) \le C\Big[1+\sup_M \mathbb{E}\int_0^T\big(\|q(s\wedge\tau_M)\|^{2p}_{H^0} + \|u(s\wedge\tau_M)\|^{2p}_{V^1}\big)ds\Big] < \infty.\end{aligned}$$

As $M\to\infty$ in this inequality and in (31), the monotone convergence theorem concludes the proof of (21). □

4.2. Moment Estimates of q in $L^\infty(0,T;H^1)$

We next give upper estimates for moments of $\sup_{t\in[0,T]}\|\hat{A}^{\frac12}q(t)\|_{H^0}$, i.e., prove Proposition 2.

However, since $\langle [u(s).\nabla]\theta(s), \tilde{A}\theta(s)\rangle = 0$, unlike what we have in the proof of the previous result, we keep the bilinear term. This creates technical problems and we proceed in two steps. First, using the mild formulation of the weak solution θ of (10), we prove that the gradient of the temperature has finite moments. Then, going back to the weak form, we prove the desired result.

Let $\{S(t)\}_{t\geq 0}$ be the semi-group generated by $-\nu A$, $\{\tilde{S}(t)\}_{t\geq 0}$ be the semi-group generated by $-\kappa\tilde{A}$, which is $S(t) = \exp(-\nu t A)$, and $\tilde{S}(t) = \exp(-\kappa t\tilde{A})$ for every $t \geq 0$. Note that, for every $\alpha > 0$,

$$\|A^{\alpha}S(t)\|_{L(V^0;V^0)} \leq Ct^{-\alpha}, \qquad \forall t > 0 \tag{33}$$

$$\|A^{-\alpha}\big(Id - S(t)\big)\|_{L(V^0;V^0)} \leq Ct^{\alpha}, \qquad \forall t > 0. \tag{34}$$

Similar upper estimates are valid when we replace A with $\tilde{A}$, $S(t)$ with $\tilde{S}(t)$ and V^0 with H^0.

Note that if $u_0 \in L^2(\Omega; V^1)$ and $\theta_0 \in L^2(\Omega; H^0)$, we deduce $u \in L^2(\Omega; C([0,T]; V^0) \cap L^{\infty}([0,T]; V^1))$ and $\theta \in L^2(\Omega; C([0,T]; H^0)) \cap L^2(\Omega \times [0,T]; H^1)$. We can write the solutions of (9) and (10) in the following mild form:

$$\begin{aligned} u(t) = {} & S(t)u_0 - \int_0^t S(t-s)B(u(s),u(s))\,ds + \int_0^t S(t-s)\big(P\theta(t)v_2\big)\,ds \\ & + \int_0^t S(t-s)G(u(s))dW(s), \end{aligned} \tag{35}$$

$$\theta(t) = \tilde{S}(t)\theta_0 - \int_0^t \tilde{S}(t-s)\big([u(s).\nabla]\theta(s)\big)\,ds + \int_0^t \tilde{S}(t-s)\tilde{G}(\theta(s))d\tilde{W}(s), \tag{36}$$

where the first equality holds a.s. in V^0 and the second one in H^0.

Indeed, since $\|A^{\alpha}u\|_{V^0} \leq C\|A^{\frac{1}{2}}u\|_{V^0}^{2\alpha}\|u(s)\|_{V^0}^{1-2\alpha}$, the upper estimate (7) for $\delta + \rho > \frac{1}{2}$, $\delta + \alpha + \rho = 1$ and the Minkowski inequality imply that

$$\begin{aligned} \left\| \int_0^t S(t-s)B(u(s),u(s))ds \right\|_{V^0} & \leq \int_0^t \|A^{\delta}A^{-\delta}B(u(s),u(s))\|_{V^0}ds \\ & \leq C\int_0^t (t-s)^{-\delta}\|A^{\alpha}u(s)\|_{V^0}\|A^{\rho}u(s)\|_{V^0}ds \\ & \leq C\sup_{s\in[0,t]}\|u(s)\|_{V^1}^2 \int_0^t (t-s)^{-\delta}ds \end{aligned}$$

Since $\|S(t)\|_{L(V^0;V^0)} \leq 1$, it is easy to see that

$$\left\| \int_0^t S(t-s)P\theta(t)v_2 ds \right\|_{V^0} \leq C\int_0^t \|\theta(t)\|_{H^0}ds.$$

Furthermore,

$$E\left| \int_0^t S(t-s)G(u(s))dW(s) \right|_{V^0}^2 \leq Tr(Q)E\int_0^t \big[K_0 + K_1\|u(t)\|_{V^0}^2\big]\,ds < \infty.$$

Therefore, the stochastic integral $\int_0^t S(t-s)G(u(s))dW(s) \in V^0$ a.s., and the identity (35) is true a.s. in V^0.

A similar argument shows that (36) holds a.s. in H^0. We only show that the convolution involving the bilinear term belongs to H^0. Using the Minkowski inequality and the upper estimate (8) with positive constants δ, α, ρ such that $\alpha, \rho \in (0, \frac{1}{2})$, $\delta + \rho > \frac{1}{2}$ and $\delta + \alpha + \rho = 1$, we obtain

$$\Big\| \int_0^t S(t-s)[(u(s).\nabla)\theta(s)]ds \Big\|_{H^0} \le \int_0^t \|A^{\delta} S(t-s) A^{-\delta}[(u(s).\nabla)\theta(s)]\|_{H^0} ds$$
$$\le C \int_0^t (t-s)^{-\delta} \|A^{\alpha} u(s)\|_{V^0} \|\tilde{A}^{\rho}\theta(s)\|_{H^0} ds$$
$$\le C \sup_{s\in[0,t]} \|u(s)\|_{V^1} \sup_{s\in[0,t]} \|\theta(s)\|_{H^0}^{1-2\rho} \Big(\int_0^t (t-s)^{-\frac{\delta}{1-\rho}} ds\Big)^{1-\rho} \Big(\int_0^t \|\tilde{A}^{\frac{1}{2}}\theta(s)\|^2_{H^0} ds\Big)^{\rho} < \infty,$$

where the last upper estimate is deduced from Hölder's inequality and $\frac{\delta}{1-\rho} < 1$.

The following result shows that, for fixed t, the L^2-norm of the gradient of $\theta(t)$ has finite moments.

Lemma 2. Let $p \in [0, +\infty)$, $u_0 \in L^{4p+\epsilon}(\Omega; V^1)$ and $\theta_0 \in L^{4p+\epsilon}(\Omega; H^1)$ for some $\epsilon \in (0, \frac{1}{2})$. Let the diffusion coefficient G and $\tilde{G}$ satisfy the condition **(C)** and **(C̃)**, respectively. For every N, let $\tau_N := \inf\{t \ge 0 : \|A^{\frac{1}{2}}\theta(t)\|_{H^0} \ge N\} \wedge T$; then,

$$\sup_{N>0} \sup_{t\in[0,T]} \mathbb{E}\, \|\tilde{A}^{\frac{1}{2}}\theta(t\wedge\tau_N)\|^{2p}_{H^0} < \infty. \tag{37}$$

Proof. Write $\theta(t)$ using (36); then, $\|\tilde{A}^{\frac{1}{2}}\theta(t)\|_{H^0} \le \sum_{i=1}^{3} T_i(t)$, where

$$T_1(t) = \|\tilde{A}^{\frac{1}{2}}\tilde{S}(t)\theta_0\|_{H^0}, \quad T_2(t) = \Big\| \int_0^t \tilde{A}^{\frac{1}{2}}\tilde{S}(t-s)[(u(s).\nabla)\theta(s)]ds \Big\|_{H^0},$$
$$T_3(t) = \Big\| \int_0^t \tilde{A}^{\frac{1}{2}}\tilde{S}(t-s)\tilde{G}(\theta(s))d\tilde{W}(s) \Big\|_{H^0}.$$

The Minkowski inequality implies that, for $\beta \in (0, \frac{1}{2})$,

$$T_2(t) \le \int_0^t \|\tilde{A}^{\frac{1}{2}}\tilde{S}(t-s)[(u(s).\nabla)\theta(s)]\|_{H^0} ds$$
$$\le \int_0^t \|\tilde{A}^{1-\beta}\tilde{S}(t-s)\|_{L(H^0;H^0)} \|\tilde{A}^{-(\frac{1}{2}-\beta)}[(u(s).\nabla)\theta(s)\|_{H^0} ds.$$

Apply (8) with $\delta = \frac{1}{2} - \beta$, $\alpha = \frac{1}{2}$ and $\rho \in (\beta, \frac{1}{2})$. A simple computation proves that $\|\tilde{A}^{\rho} f\|_{H^0} \le \|\tilde{A}^{\frac{1}{2}} f\|_{H^0}^{2\rho} \|f\|_{H^0}^{1-2\rho}$ for any $f \in H^1$. Therefore,

$$\|\tilde{A}^{-(\frac{1}{2}-\beta)}[(u(s).\nabla)\theta(s)]\|_{H^0} \le C\|A^{\frac{1}{2}}u(s)\|_{V^0}\|\tilde{A}^{\rho}\theta(s)\|_{H^0}$$
$$\le C\|A^{\frac{1}{2}}u(s)\|_{V^0}\|\tilde{A}^{\frac{1}{2}}\theta(s)\|^{2\rho}_{H^0}\|\theta(s)\|^{1-2\rho}_{H^0}.$$

This upper estimate and (33) imply that

$$T_2(t) \le C \sup_{s\in[0,T]} \|A^{\frac{1}{2}}u(s)\|_{V^0} \sup_{s\in[0,t]} \|\theta(s)\|^{1-2\rho}_{H^0} \int_0^t (t-s)^{-1+\beta} \|\tilde{A}^{\frac{1}{2}}\theta(s)\|^{2\rho}_{H^0} ds.$$

For $p \in [1, \infty)$, Hölder's inequality with respect to the measure $(t-s)^{-(1-\beta)} 1_{[0,t)}(s)ds$ implies that

$$T_2(t)^{2p} \le C \sup_{s\in[0,t]} \|A^{\frac{1}{2}}u(s)\|^{2p}_{V^0} \sup_{s\in[0,t]} \|\theta(s)\|^{2p(1-2\rho)}_{H^0} \Big(\int_0^t (t-s)^{-(1-\beta)} ds\Big)^{2p-1}$$
$$\int_0^t (t-s)^{-(1-\beta)} \|\tilde{A}^{\frac{1}{2}}\theta(s)\|^{4p\rho}_{H^0} ds.$$

Let $p_1 = \frac{2(1-\rho)}{1-2\rho}$, $p_2 = \frac{2(1-\rho)}{(1-2\rho)^2}$ and $p_3 = \frac{1}{2\rho}$. Then, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $4\rho p p_3 = 2p$ and $pp_1 = p(1-2\rho)p_2 := \tilde{p}$. Young's and Hölder's inequalities imply that

$$T_2(t)^{2p} \le C\Big[\frac{1}{p_1}\sup_{s\in[0,t]}\|A^{\frac12}u(s)\|_{V^0}^{2\tilde p} + \frac{1}{p_2}\sup_{s\in[0,t]}\|\theta(s)\|_{H^0}^{2\tilde p} + \frac{1}{p_3}\int_0^t (t-s)^{-1+\beta}\|A^{\frac12}\theta(s)\|_{H^0}^{2p}\,ds\Big(\int_0^t (t-s)^{-1+\beta}ds\Big)^{p_3-1}\Big].$$

Note that the continuous function $\rho \in (0,\frac12) \to \frac{2(1-\rho)}{1-2\rho}$ increases with $\lim_{\rho\to 0}\frac{2(1-\rho)}{1-2\rho} = 2$. Given $\epsilon > 0$, choose $\rho \in (0,\frac12)$ close enough to 0 to have $2\tilde p = 2p\frac{2(1-\rho)}{1-2\rho} = 4p+\epsilon$, and then choose $\beta \in (0,\rho)$. The above computations yield

$$T_2(t)^{2p} \le C\Big[\sup_{s\in[0,t]}\|A^{\frac12}u(s)\|_{V^0}^{4p+\epsilon} + \sup_{s\in[0,t]}\|\theta(s)\|_{H^0}^{4p+\epsilon}\Big] + C\int_0^t (t-s)^{-1+\beta}\|A^{\frac12}\theta(s)\|_{H^0}^{2p}\,ds. \tag{38}$$

Finally, Burhholder's inequality, the growth condition (16) and Hölder's inequality imply that, for $t \in [0,T]$,

$$\begin{aligned}
E\Big\|\int_0^{t\wedge\tau_N}\tilde A^{\frac12}\tilde S(t-s)\tilde G(\theta(s))\,d\tilde W(s)\Big\|_{H^0}^{2p} &\le C_p\,\mathrm{Tr}(Q)^p\,E\Big(\int_0^{t\wedge\tau_N}\|\tilde A^{\frac12}\tilde G(\theta(s))\|^2_{L(K;\tilde H^0)}ds\Big)^p\\
&\le C_p\,\mathrm{Tr}(Q)^p\,E\Big(\int_0^{t\wedge\tau_N}\big[\tilde K_2 + \tilde K_3\|\theta(s)\|^2_{H^0} + \tilde K_3\|\tilde A^{\frac12}\theta(s)\|^2_{H^0}\big]ds\Big)^p\\
&\le C(p,\tilde K_2,\tilde K_3,\mathrm{Tr}(Q))T^p\Big[1 + E\sup_{s\in[0,T]}\|\theta(s)\|_{H^0}^{2p}\Big]\\
&\quad + C_p\,\mathrm{Tr}(Q)^p\tilde K_3^pT^{p-1}\int_0^t E\|A^{\frac12}\theta(s\wedge\tau_N)\|_{H^0}^{2p}ds.
\end{aligned} \tag{39}$$

The upper estimates (38), (39) and $T_1(t) \le \|A^{\frac12}\theta_0\|_{H^0} \le \|\theta_0\|_{H^1}$ used with $t\wedge\tau_N$ instead of t imply that, for every $t\in[0,T]$,

$$\begin{aligned}
E\|\tilde A^{\frac12}\theta(t\wedge\tau_N)\|_{H^0}^{2p} &\le C_p\Big[1 + E\|A^{\frac12}\theta_0\|_{H^0}^{2p} + \sup_{s\in[0,T]}\|A^{\frac12}u(s)\|_{V^0}^{4p+\epsilon} + \sup_{s\in[0,T]}\|\theta(s)\|_{H^0}^{4p+\epsilon}\Big]\\
&\quad + C_p\int_0^t\big[(t-s)^{-1+\beta} + \tilde K_3T^{p-1}\big]E\|A^{\frac12}\theta(s\wedge\tau_N)\|_{H^0}^{2p}ds,
\end{aligned}$$

where the constant C_p does not depend on t and N. Theorem 1, Proposition 1 and the version of Gronwall's lemma proved in the following lemma 3 imply that (37) for some constant C depending on $E(\|u_0\|_{V^1}^{4p+\epsilon})$ and $E(\|\theta_0\|_{H^0}^{4p+\epsilon})$. The proof of the Lemma is complete. □

The following lemma is an extension of Lemma 3.3, p. 316 in [15]. For the sake of completeness, its proof is given at the end of this section.

Lemma 3. Let $\epsilon \in (0,1)$, a, b, c be positive constants and φ be a bounded non-negative function such that

$$\varphi(t) \le a + \int_0^t\big[b + c(t-s)^{-1+\epsilon}\big]\varphi(s)\,ds, \quad \forall t\in[0,T]. \tag{40}$$

Then, $\sup_{t\in[0,T]}\varphi(t) \le C$ for some constant C depending on a, b, c, T and ϵ.

Proof of Proposition 2. We next prove that the gradient of the temperature has bounded moments uniformly in time.

We only prove (22) for $p \in [2, +\infty)$; the other argument is similar and easier.

Applying the operator $\tilde{A}^{\frac{1}{2}}$ to Equation (10), and writing Itô's formula for the square of the corresponding H^0-norm, we obtain

$$\|\tilde{A}^{\frac{1}{2}} q(t)\|^2_{H^0} + 2\kappa \int_0^t \|\tilde{A} q(s)\|^2_{H^0} ds = \|\tilde{A}^{\frac{1}{2}} q_0\|^2_{H^0} - 2\int_0^t \langle (u(s).\nabla) q(s), \tilde{A} q(s)\rangle ds$$
$$+ 2\int_0^t \big(\tilde{A}^{\frac{1}{2}} \tilde{G}(q(s)) d\tilde{W}(s), \tilde{A}^{\frac{1}{2}} q(s)\big) + \mathrm{Tr}(\tilde{Q}) \int_0^t \|\tilde{A}^{\frac{1}{2}} \tilde{G}(q(s))\|^2_{H^0} ds.$$

Then, apply Itô's formula for the map $x \mapsto x^p$. This yields, using integration by parts,

$$\|\tilde{A}^{\frac{1}{2}} q(t)\|^{2p}_{H^0} + 2p\kappa \int_0^t \|\tilde{A} q(s)\|^2_{H^0} \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds = \|\tilde{A}^{\frac{1}{2}} q_0\|^{2p}_{H^0}$$
$$- 2p \int_0^t \langle (u(s).\nabla) q(s), \tilde{A} q(s)\rangle \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds$$
$$+ 2p \int_0^t \big(\tilde{A}^{\frac{1}{2}} \tilde{G}(q(s)) d\tilde{W}(s), \tilde{A}^{\frac{1}{2}} q(s)\big) \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0}$$
$$+ p\mathrm{Tr}(\tilde{Q}) \int_0^t \|\tilde{A}^{\frac{1}{2}} \tilde{G}(q(s))\|^2_{H^0} \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds$$
$$+ 2p(p-1)\mathrm{Tr}(\tilde{Q}) \int_0^t \big\| \big(\tilde{A}^{\frac{1}{2}} \tilde{G}(q(s))\big)^* \tilde{A}^{\frac{1}{2}} q(s)\big\|^2_K \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-2)}_{H^0} ds. \quad (41)$$

The Gagliardo–Nirenberg inequality (6) and the inclusion $V^1 \subset L^4$ implies that

$$\int_0^t |\langle (u(s).\nabla) q(s), \tilde{A} q(s)\rangle| \, \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds$$
$$\le C \int_0^t \|\tilde{A} q(s)\|_{H^0} \|u(s)\|_{L^4} \|\tilde{A}^{\frac{1}{2}} q(s)\|_{L^4} \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds$$
$$\le C \int_0^t \|\tilde{A} q(s)\|^{\frac{3}{2}}_{H^0} \|u(s)\|_{V^1} \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2p-\frac{3}{2}}_{H^0} ds.$$

Then, using the Hölder and Young's inequalities, we deduce

$$2p \int_0^t |\langle (u(s).\nabla) q(s), \tilde{A} q(s)\rangle| \, \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds$$
$$\le (2p-1)\kappa \int_0^t \|\tilde{A} q(s)\|^2_{H^0} \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds$$
$$+ C(\kappa, p) \sup_{s \in [0,T]} \|u(s)\|^4_{V^1} \int_0^t \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2p}_{H^0} ds. \quad (42)$$

The growth condition (16) and Hölder's and Young inequalities imply that

$$\int_0^t \|\tilde{A}^{\frac{1}{2}} \tilde{G}(q(s))\|^2_{H^0} \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-1)}_{H^0} ds \le C \int_0^t \big[1 + \|q(s)\|^{2p}_{H^0} + \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2p}_{H^0}\big] ds, \quad (43)$$

and a similar computation yields

$$\int_0^t \big\| \big(\tilde{A}^{\frac{1}{2}} \tilde{G}(q(s))\big)^* \tilde{A}^{\frac{1}{2}} q(s)\big\|^2_K \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2(p-2)}_{H^0} ds$$
$$\le C \int_0^t \big[1 + \|q(s)\|^{2p}_{H^0} + \|\tilde{A}^{\frac{1}{2}} q(s)\|^{2p}_{H^0}\big] ds. \quad (44)$$

Let $\tau_N := \inf\{t \ge 0 : \|\tilde{A}^{\frac{1}{2}} q(t)\|_{H^0} \ge N\}$. The upper estimates (41)–(44) written for $t \wedge \tau_N$ instead of t imply

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|\tilde{A}^{\frac12} q(t\wedge\tau_N)\|^{2p}_{H^0} + \kappa \int_0^{T\wedge\tau_N} \|\tilde{A} q(s)\|^2_{H^0} \|\tilde{A}^{\frac12} q(s)\|^{2(p-1)}_{H^0}\, ds \le \|\tilde{A}^{\frac12} q_0\|^{2p}_{H^0} \\
&\quad + C \sup_{s\in[0,T]} \|u(s)\|^4_{V^1} \int_0^{T\wedge\tau_N} \|\tilde{A}^{\frac12} q(s)\|^{2p}_{H^0}\, ds + C \int_0^{T\wedge\tau_N} \Big(1 + \|q(s)\|^{2p}_{H^0} + \|\tilde{A}^{\frac12} q(s)\|^{2p}_{H^0}\Big) ds \\
&\quad + 2p \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\tau_N} \Big(\tilde{A}^{\frac12} \tilde{G}(q(s))\, d\tilde{W}(s), \tilde{A}^{\frac12} q(s) \Big) \|\tilde{A}^{\frac12} q(s)\|^{2(p-1)}_{H^0} \Big|.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, Fubini's theorem, (21) and (37), we deduce

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,T]} \|u(s)\|^4_{V^1} \int_0^{T\wedge\tau_N} \|\tilde{A}^{\frac12} q(s)\|^{2p}_{H^0}\, ds \Big) \\
&\qquad \le \Big\{ \mathbb{E}\Big(\sup_{s\in[0,T]} \|u(s)\|^8_{V^1} \Big) \Big\}^{\frac12} \Big\{ \int_0^T \mathbb{E}\, \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^{4p}_{H^0}\, ds \Big\}^{\frac12} \le C. \qquad (45)
\end{aligned}
$$

The Davis inequality, the growth condition (16) and the Cauchy–Schwarz, Young and Hölder inequalities imply that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]} \Big| \int_0^{t\wedge\tau_N} \Big(\tilde{A}^{\frac12} \tilde{G}(q(s))\, d\tilde{W}(s), \tilde{A}^{\frac12} q(s) \Big) \|\tilde{A}^{\frac12} q(s)\|^{2(p-1)}_{H^0} \Big| \Big) \\
&\quad \le C\, \mathbb{E}\Big\{ \int_0^T \mathrm{Tr}(\tilde{Q}) \Big(\tilde{K}_2 + \tilde{K}_3 \|q(s\wedge\tau_N)\|^2_{H^1} \Big) \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^{4p-2}_{H^0}\, ds \Big\}^{\frac12} \\
&\quad \le C\, \mathbb{E}\Big[\sup_{s\le T} \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^{p}_{H^0} \big(\mathrm{Tr}(\tilde{Q})\big)^{\frac12} \\
&\qquad \Big\{ \int_0^T \Big(\tilde{K}_2 + \tilde{K}_3 \|q(s\wedge\tau_N)\|^2_{H^0} + \tilde{K}_3 \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^2_{H^0} \Big) \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^{2(p-1)}_{H^0}\, ds \Big\}^{\frac12} \Big] \\
&\quad \le \frac{1}{4p}\, \mathbb{E}\Big(\sup_{s\le T} \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^{2p}_{H^0} \Big) + C\, \mathbb{E}\Big(\int_0^T \Big(1 + \|q(s\wedge\tau_N)\|^{2p}_{H^0} + \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^{2p}_{H^0} \Big) ds \Big).
\end{aligned}
$$

Therefore, the upper estimates (20), (37) and (45) imply that

$$
\frac12\, \mathbb{E} \sup_{s\le T} \|\tilde{A}^{\frac12} q(s\wedge\tau_N)\|^{2p}_{H^0} + \kappa\, \mathbb{E} \int_0^{T\wedge\tau_N} \|\tilde{A} q(s\wedge\tau_N)\|^2_{H^0} \|\tilde{A}^{\frac12} q(s)\|^{2(p-1)}_{H^0}\, ds \le C
$$

for some constant C independent of N. As $N \to +\infty$, we deduce (22); this completes the proof of Proposition 3. □

We conclude this section with the proof of an extension of the Gronwall lemma.

Proof of Lemma 3. For $t \in [0,T]$, iterating (40) and using the Fubini theorem, we obtain

$$
\begin{aligned}
\varphi(t) &\leq a + \int_0^t \big(b + c(t-s)^{-1+\epsilon}\big)\Big[a + \int_0^s \big(b + c(s-r)^{-1+\epsilon}\big)\varphi(r)\,dr\Big]ds \\
&\leq a\Big[1 + \int_0^t [b + c(t-s)^{-1+\epsilon}]ds\Big] \\
&\quad + \int_0^t \int_r^t [b + c(t-s)^{-1+\epsilon}][b + c(s-r)^{-1+\epsilon}]ds\,\varphi(r)\,dr \\
&\leq A_1 + \int_0^t \Big[b^2(t-r) + \frac{2bc}{\epsilon}(t-r)^{\epsilon} + c^2 \int_r^t (t-s)^{-1+\epsilon}(s-r)^{-1+\epsilon}ds\Big]\varphi(r)\,dr \\
&\leq A_1 + \int_0^t \Big[B_1 + C_1(t-r)^{-1+2\epsilon}\int_0^1 \lambda^{-1+\epsilon}(1-\lambda)^{-1+\epsilon}d\lambda\Big]\varphi(r)\,dr,
\end{aligned}
$$

for positive constants A_1 (depending on a, b, c, T, ϵ), B_1 (depending on b, c, T, ϵ) and C_1 (depending on c and ϵ). One easily proves by induction on k that, for every integer $k \geq 1$,

$$
\begin{aligned}
\varphi(t) &\leq A_k + \int_0^t \Big[B_k + cC_{k-1}\int_r^t (t-s)^{-1+k\epsilon}(s-r)^{-1+\epsilon}ds\Big]\varphi(r)\,dr \\
&\leq A_k + \int_0^t \big[B_k + C_k(t-r)^{-1+(k+1)\epsilon}\big]\varphi(r)\,dr,
\end{aligned}
$$

for some positive constants A_k, B_k and C_k depending on a, b, c, T and ϵ. Indeed, a change in variables implies that

$$
\begin{aligned}
\int_r^t (t-s)^{-1+k\epsilon}(s-r)^{-1+\epsilon}ds &= (t-r)^{-1+(k+1)\epsilon}\int_0^1 \lambda^{-1+k\epsilon}(1-\lambda)^{-1+\epsilon}d\lambda \\
&= \tilde{C}_k(t-r)^{-1+(k+1)\epsilon},
\end{aligned}
$$

for some constant $\tilde{C}_k$ depending on k and ϵ.

Let k be the largest integer such that $k\epsilon < 1$; that is, $k\epsilon < 1 \leq (k+1)\epsilon$. Then, since $(t-r)^{-1+(k+1)\epsilon} \leq T^{-1+(k+1)\epsilon}$, we deduce that

$$\varphi(t) \leq A + \int_0^T B\,\varphi(r)\,dr,$$

for some positive constants A and B depending on the parameters a, b, c, T and ϵ. The classical Gronwall lemma concludes the proof of the lemma. □

5. Moment Estimates of Time Increments of the Solution

In this section ,we prove moment estimates for various norms of time increments of the solution to (9) and (10). This will be crucial for deducing the speed of the convergence of numerical schemes. We first prove the time regularity of the velocity and temperature in L^2.

Proposition 4. Let u_0, θ_0 be $\mathcal{F}_0$-measurable; suppose that G and $\tilde{G}$ satisfy **(C-u)** and **(C-θ)**, respectively.

(i) Let $u_0 \in L^{4p}(\Omega; V^1)$ and $\theta_0 \in L^{2p}(\Omega; H^0)$. Then for $0 \leq t_1 < t_2 \leq T$,

$$\mathbb{E}\big(\|u(t_2) - u(t_1)\|_{V^0}^{2p}\big) \leq C|t_2 - t_1|^p. \tag{46}$$

(ii) Let $u_0 \in L^{8p+\epsilon}(\Omega; V^1)$, $\theta_0 \in L^{8p+\epsilon}(\Omega; H^1)$ for some $\epsilon > 0$. Then, for $0 \leq t_1 < t_2 \leq T$,

$$\mathbb{E}\big(\|\theta(t_2) - \theta(t_1)\|_{H^0}^{2p}\big) \leq C|t_2 - t_1|^p. \tag{47}$$

Proof. Recall that $S(t) = e^{-\nu t A}$ is the analytic semi group generated by the Stokes operator A multiplied by the viscosity ν and that $\tilde{S}(t) = e^{-\kappa t \tilde{A}}$ is the semi group generated by $\tilde{A} = -\Delta$. We use the mild formulation of the solutions stated in (35) and (36).

(i) Let $0 \le t_1 < t_2 \le T$; then, $u(t_2) - u(t_1) = \sum_{i=1}^{4} T_i(t_1, t_2)$, where

$$
\begin{aligned}
T_1(t_1,t_2) &= S(t_2)u_0 - S(t_1)u_0 = [S(t_2-t_1) - \mathrm{Id}]S(t_1)u_0,\\
T_2(t_1,t_2) &= \int_0^{t_2} S(t_2-s)B(u(s),u(s))\,ds - \int_0^{t_1} S(t_1-s)B(u(s),u(s))\,ds,\\
T_3(t_1,t_2) &= \int_0^{t_2} S(t_2-s)Pq(s)v_2\,ds - \int_0^{t_1} S(t_1-s)Pq(s)v_2\,ds\\
T_4(t_1,t_2) &= \int_0^{t_2} S(t_2-s)G(u(s))\,dW(s) - \int_0^{t_1} S(t_1-s)G(u(s))\,dW(s).
\end{aligned}
\tag{48}
$$

The arguments used in the proof of Lemma 2.1 [11], using (7), (33), (34) and (21) yield

$$
\mathbb{E}\Big(\|T_1(t_1,t_2)\|_{V^0}^{2p} + \|T_2(t_1,t_2)\|_{V^0}^{2p}\Big) \le C\big[1 + \mathbb{E}(\|u_0\|_{V^1}^{4p})\big]|t_2-t_1|^p. \tag{49}
$$

Let $T_3(t_1,t_2) = T_{3,1}(t_1,t_2) + T_{3,2}(t_1,t_2)$, where

$$
\begin{aligned}
T_{3,1}(t_1,t_2) &= \int_0^{t_1} [S(t_2-t_1) - \mathrm{Id}]S(t_1-s)Pq(s)v_2\,ds,\\
T_{3,2}(t_1,t_2) &= \int_{t_1}^{t_2} S(t_2-s)Pq(s)v_2\,ds.
\end{aligned}
$$

Since the family of sets $\{A(t,M)\}_t$ is decreasing, using the Minkowski inequality, (33) and (34), we obtain

$$
\begin{aligned}
\|T_{3,1}(t_1,t_2)\|_{V^0} &\le \int_0^{t_1} \|A^{\frac12}S(t_1-s)\|_{L(V^0;V^0)}\,\|A^{-\frac12}[S(t_2-t_1)-\mathrm{Id}]\|_{L(V^0;V^0)}\,\|Pq(s)v_2\|_{V^0}\,ds\\
&\le C|t_2-t_1|^{\frac12}\sup_{s\in[0,T]}\|q(s)\|_{H^0},
\end{aligned}
$$

and

$$
\|T_{3,2}(t_1,t_2)\|_{V^0} \le \int_{t_1}^{t_2} \|S(t-s)Pq(s)v_2\|_{V^0}\,ds \le |t_2-t_1|\sup_{s\in[0,T]}\|q(s)\|_{H^0}.
$$

The inequality (20) implies that

$$
\mathbb{E}\|T_3(t_1,t_2)\|_{V^0}^{2p} \le C|t_2-t_1|^p\,\mathbb{E}(\|q_0\|_{H^0}^{2p}). \tag{50}
$$

Finally, decompose the stochastic integral as follows:

$$
T_{4,1}(t_1,t_2) = \int_0^{t_1} [S(t_2-t_1)-\mathrm{Id}]S(t_1-s)G(s)\,dW(s), \quad T_{4,2}(t_1,t_2) = \int_{t_1}^{t_2} S(t_2-s)G(s)\,dW(s).
$$

The Burkholder inequality, (34), Hölder's inequality and the growth condition (13) yield

$$
\begin{aligned}
\mathbb{E}\|T_{4,1}\|_{V^0}^{2p} &\le C_p\,\mathbb{E}\Big(\int_0^{t_1}\|[S(t_2-t_1)-\mathrm{Id}]S(t_1-s)G(u(s))\|_{V^0}^2\,\mathrm{Tr}(Q)\,ds\Big)^p\\
&\le C(\mathrm{Tr}(Q))^p\,\mathbb{E}\Big(\int_0^{t_1}\|A^{-\frac12}[S(t_2-t_1)-\mathrm{Id}]\|_{L(V^0;V^0)}^2\,\|A^{\frac12}G(u(s))\|_{V^0}^2\,ds\Big)^p\\
&\le C\,\mathbb{E}\Big(\int_0^{t_1}|t_2-t_1|\big(K_2+K_3\|u(s)\|_{V^1}^2\big)ds\Big)^p\\
&\le C\big[1+\mathbb{E}(\|u_0\|_{V^1}^{2p})\big]|t_2-t_1|^p,
\end{aligned}
\tag{51}
$$

where the last upper estimate is a consequence of (19) and (21). A similar easier argument implies that

$$E\|T_{4,2}\|_{V^0}^{2p} \le C_p E\Big(\int_{t_1}^{t_2} \|\tilde S(t_2-s)G(u(s))\|^2_{V^0}\,\mathrm{Tr}(Q)\,ds\Big)^p \le C\big(1+E(\|u_0\|_{V^0}^{2p})\big)|t_2-t_1|^p. \tag{52}$$

The inequalities (49)–(52) complete the proof of (46).

(ii) As in the proof of (i), for $0\le t_1<t_2\le T$, let $q(t_2)-q(t_1)=\sum_{i=1}^3 \tilde T_i(t_1,t_2)$, where

$$\tilde T_1(t_1,t_2) = [\tilde S(t_2-t_1)-Id]\tilde S(t_1)q_0,$$
$$\tilde T_2(t_1,t_2) = -\int_0^{t_2}\tilde S(t_2-s)\,[u(s).\nabla]q(s)\,ds + \int_0^{t_1}\tilde S(t_1-s)\,[u(s).\nabla]q(s)\,ds$$
$$\tilde T_3(t_1,t_2) = \int_0^{t_2}\tilde S(t_2-s)\tilde G(q(s))\,d\tilde W(s) - \int_0^{t_1}\tilde S(t_1-s)\tilde G(q(s))\,d\tilde W(s). \tag{53}$$

The inequality (34) implies that

$$\|\tilde T_1(t_1,t_2)\|_{H^0} = \|\tilde A^{-\frac12}[\tilde S(t_2-t_1)-Id]\tilde S(t_1)\tilde A^{\frac12}q_0\|_{H^0} \le C|t_2-t_1|^{\frac12}\|q_0\|_{H^1}. \tag{54}$$

Decompose $\tilde T_2(t_1,t_2)=\tilde T_{2,1}(t_1,t_2)+\tilde T_{2,2}(t_1,t_2)$, where

$$\tilde T_{2,1}(t_1,t_2) = -\int_0^{t_1}[\tilde S(t_2-t_1)-Id]\tilde S(t_1-s)\,[u(s).\nabla]q(s)\,ds,$$
$$\tilde T_{2,2}(t_1,t_2) = -\int_{t_1}^{t_2}\tilde S(t_2-s)\,[u(s).\nabla]q(s)\,ds.$$

Let $\delta\in(0,\frac12)$; the Minkowski inequality, (33), (34) and (8) applied with $a=r=\frac12$ imply that

$$\begin{aligned}\|\tilde T_{2,1}(t_1,t_2)\|_{H^0} &\le \int_0^{t_1}\|\tilde S(t_1-s)[\tilde S(t_2-t_1)-Id]\,[u(s).\nabla]q(s)\|_{H^0}\,ds\\ &\le \int_0^{t_1}\|\tilde A^{\frac12+\delta}\tilde S(t_1-s)\|_{L(H^0;H^0)}\|\tilde A^{-\frac12}[\tilde S(t_2-t_1)-Id]\|_{L(H^0;H^0)}\\ &\qquad \|\tilde A^{-\delta}[u(s).\nabla]q(s)\|_{H^0}\,ds\\ &\le C\int_0^{t_1}(t_1-s)^{-(\frac12+\delta)}|t_2-t_1|^{\frac12}\|A^{\frac12}u(s)\|_{V^0}\|\tilde A^{\frac12}q(s)\|_{H^0}\,ds\\ &\le C|t_2-t_1|^{\frac12}\sup_{s\in[0,T]}\|A^{\frac12}u(s)\|_{V^0}\int_0^{t_1}(t_1-s)^{-(\frac12+\delta)}\|\tilde A^{\frac12}q(s)\|_{H^0}\,ds.\end{aligned}$$

Let $p_1\in\big(2,2+\frac{4}{\varepsilon}\big)$ and let $\delta\in\big(0,\frac12-\frac1{p_1}\big)$. Let p_2 be the conjugate exponent of p_1; we have $(\frac12+\delta)p_2<1$. Thus, Hölder's inequality for the finite measure $(t_1-s)^{-(\frac12+\delta)}1_{[0,t_1)}(s)ds$ with exponents $2p$ and $\frac{2p}{2p-1}$, and then, ds with conjugate exponents p_1 and p_2 imply

$$\begin{aligned}\|\tilde T_{2,1}(t_1,t_2)\|^{2p}_{H^0} &\le C|t_2-t_1|^p\sup_{s\in[0,T]}\|A^{\frac12}u(s)\|^{2p}_{V^0}\int_0^{t_1}(t_1-s)^{-(\frac12+\delta)}\|\tilde A^{\frac12}q(s)\|^{2p}_{H^0}\,ds\\ &\qquad\Big\{\int_0^{t_1}(t_1-s)^{-(\frac12+\delta)}ds\Big\}^{2p-1}\\ &\le C|t_2-t_1|^p\sup_{s\in[0,T]}\|A^{\frac12}u(s)\|^{2p}_{V^0}\Big\{\int_0^{t_1}\|\tilde A^{\frac12}q(s)\|^{2pp_1}_{H^0}\,ds\Big\}^{\frac1{p_1}}\\ &\qquad\Big\{\int_0^{t_1}(t_1-s)^{-(\frac12+\delta)p_2}\,ds\Big\}^{\frac1{p_2}}.\end{aligned}$$

Since $2pp_1 < 4p + \frac{\epsilon}{2}$ and $2pp_2 < 4p$, Hölder's inequality and Fubini's theorem, together with the upper estimates (21) and (37), imply that

$$\mathbb{E}\big(\|\tilde{T}_{2,1}(t_1,t_2)\|_{H^0}^{2p}\big) \leq C|t_2-t_1|^p \Big\{\mathbb{E}\Big(\sup_{s\in[0,T]}\|A^{\frac{1}{2}}u(s)\|_{V^0}^{2pp_2}\Big)\Big\}^{\frac{1}{p_2}} \Big\{\int_0^{t_1}\mathbb{E}\big(\|A^{\frac{1}{2}}\theta(s)\|_{H^0}^{2pp_1}\big)ds\Big\}^{\frac{1}{p_1}} \leq C|t_2-t_1|^p. \tag{55}$$

A similar argument proves that for $\eta \in (0,1)$,

$$\begin{aligned}\|\tilde{T}_{2,2}(t_1,t_2)\|_{H^0} &\leq \int_{t_1}^{t_2}\|A^{1-\eta}S(\tilde{t}_2-s)\|_{\mathcal{L}(H^0;H^0)}\|\tilde{A}^{-(1-\eta)}[u(s).\nabla]\theta(s)\|_{H^0}ds\\ &\leq C\int_{t_1}^{t_2}(t_2-s)^{-1+\eta}\|A^{\frac{1}{2}}u(s)\|_{V^0}\|\tilde{A}^{\frac{1}{2}}\theta(s)\|_{H^0}ds\\ &\leq C|t_2-t_1|^{\eta}\sup_{s\in[0,T]}\|A^{\frac{1}{2}}u(s)\|_{V^0}\int_{t_1}^{t_2}(t_2-s)^{-1+\eta}\|\tilde{A}^{\frac{1}{2}}\theta(s)\|_{H^0}ds.\end{aligned}$$

Let $\eta \in \big(\frac{p}{2p-1}, 1\big)$; for $\epsilon > 0$, let $p_1, p_2 \in (1,+\infty)$ be conjugate exponents such that $\frac{1}{\eta} \vee \frac{4p+\epsilon}{2p+\epsilon} < p_1 < 2$; then $(1-\eta)p_2 < 1$. Hölder's inequality implies that

$$\begin{aligned}\|\tilde{T}_{2,2}(t_1,t_2)\|_{H^0}^{2p} &\leq C|t_2-t_1|^{(2p-1)\eta}\sup_{s\in[0,T]}\|A^{\frac{1}{2}}u(s)\|_{V^0}^{2p}\int_{t_1}^{t_2}(t_2-s)^{-1+\eta}\|\tilde{A}^{\frac{1}{2}}\theta(s)\|_{H^0}^{2p}ds\\ &\leq C|t_2-t_1|^{(2p-1)\eta}\sup_{s\in[0,T]}\|A^{\frac{1}{2}}u(s)\|_{V^0}^{2p}\Big\{\int_{t_1}^{t_2}(t_2-s)^{-(1-\eta)p_2}ds\Big\}^{\frac{1}{p_2}}\\ &\quad\Big\{\int_{t_1}^{t_2}\|A^{\frac{1}{2}}\theta(s)\|_{H^0}^{2pp_1}ds\Big\}^{\frac{1}{p_1}}.\end{aligned}$$

Since $(2p-1)\eta > p$, $\frac{1}{\eta} < 2$; furthermore, $2pp_2 < 4p + \frac{\epsilon}{2}$ and $2pp_1 \leq 4p$. Hölder's inequality together with the upper estimates (21) and (22) imply that

$$\mathbb{E}\big(\|\tilde{T}_{2,2}(t_1,t_2)\|_{H^0}^{2p}\big) \leq C|t_2-t_1|^p \Big\{\mathbb{E}\Big(\sup_{s\in[0,T]}\|A^{\frac{1}{2}}u(s)\|_{V^0}^{2pp_2}\Big)\Big\}^{\frac{1}{p_2}} \Big\{\int_{t_1}^{t_2}\mathbb{E}\big(\|A^{\frac{1}{2}}\theta(s)\|_{H^0}^{2pp_1}\big)ds\Big\}^{\frac{1}{p_1}} \leq C|t_2-t_1|^p. \tag{56}$$

This inequality and (55) yield

$$\mathbb{E}\big(\|\bar{T}_2(t_1,t_2)\|_{H^0}^{2p}\big) \leq C|t_2-t_1|^p. \tag{57}$$

Finally, an argument similar to that used to prove (51) and (52), using the growth condition (16) and (20), implies that

$$\mathbb{E}\big(\|\bar{T}_3(t_1,t_2)\|_{H^0}^{2p}\big) \leq C|t_2-t_1|^p. \tag{58}$$

The upper estimates (54), (57) and (58) complete the proof of (47). □

We next prove some time regularity for the gradient of the velocity and the temperature.

Proposition 5. Let $N \geq 1$ be an integer and, for $k = 0, \dots, N$, set $t_k = \frac{kT}{N}$, where G and $\tilde{G}$ satisfy conditions **(C-u)** and **(C-θ)**, respectively, and let $\eta \in (0,1)$.

(i) Let $p \in [2,\infty)$, $u_0 \in L^{4p}(\Omega; V^1)$ and $\theta_0 \in L^{2p}(\Omega; H^0)$. Then, there exists a positive constant C (independent of N) such that

$$\mathbb{E}\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\big[\|u(s)-u(t_j)\|_{V^1}^2+\|u(s)-u(t_{j-1})\|_{V^1}^2\big]ds\Big)^p \le C\Big(\frac{T}{N}\Big)^{\eta p} \tag{59}$$

(ii) Let $p \in [2,\infty)$, $u_0 \in L^{16p+\epsilon}(\Omega; V^1)$ and $\theta_0 \in L^{16p+\epsilon}(\Omega; H^0)$ for some $\epsilon > 0$. Then,

$$\mathbb{E}\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\big[\|\theta(s)-\theta(t_j)\|_{H^1}^2+\|\theta(s)-\theta(t_{j-1})\|_{H^1}^2\big]ds\Big)^p \le C\Big(\frac{T}{N}\Big)^{\eta p} \tag{60}$$

Proof. (i) For $j = 1,\dots,N$, write the decomposition (48) of $u(t_j) - u(s)$ used in the proof of Lemma 4 (that is, $t_1 = s$, $t_2 = t_j$), and apply $A^{\frac{1}{2}}$. The upper estimates of the sum of terms $A^{\frac{1}{2}}T_1(s,t_j)$ and $A^{\frac{1}{2}}T_2(s,t_j)$ obtained in the proof of Lemma 2.2 in [11] imply that, for $\eta \in (0,1)$,

$$\mathbb{E}\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\big[\|A^{\frac{1}{2}}T_1(s,t_j)\|_{V^0}^2+\|A^{\frac{1}{2}}T_2(s,t_j)\|_{V^0}^2\big]ds\Big)^p \le C\big(\mathbb{E}(\|u_0\|_{V^1}^{4p})\big)\Big(\frac{T}{N}\Big)^{\eta p} \tag{61}$$

The Minkowski inequality and the upper estimates (33) and (34) imply, for $\delta \in (0,\frac{1}{2})$

$$\begin{aligned}\|A^{\frac{1}{2}}T_{3,1}(s,t_j)\|_{V^0} &\le \int_0^{t_j}\|A^{\frac{1}{2}+\delta}S(t_j-s)\|_{L(V^0;V^0)}\,\|A^{-\delta}[S(t_j-s)-Id]\|_{L(V^0;V^0)}\\ &\quad \|P\theta(s)v_2\|_{V^0}\,ds\\ &\le C|t_j-s|^{\delta}\sup_{s\in[0,t_j]}\|\theta(s)\|_{H^0}\int_0^{t_j}(t_1-s)^{-(\frac{1}{2}+\delta)}ds,\end{aligned}$$

Hence, we deduce

$$\begin{aligned}\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\|A^{\frac{1}{2}}T_{3,1}(s,t_j)\|_{V^0}^2\,ds\Big)^p &\le C\Big(\frac{T}{N}\Big)^{2p\delta}\sup_{s\in[0,T]}\|\theta(s)\|_{H^0}^{2p}\\ &\quad\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Big(\int_0^s(s-r)^{-(\frac{1}{2}+\delta)}dr\Big)^2ds\Big)^p\\ &\le C\Big(\frac{T}{N}\Big)^{2p\delta}\sup_{s\in[0,T]}\|\theta(s)\|_{H^0}^{2p}\Big(\int_0^T s^{1-2\delta}ds\Big)^p \le C\Big(\frac{T}{N}\Big)^{2p\delta}\sup_{s\in[0,T]}\|\theta(s)\|_{H^0}^{2p}.\end{aligned}$$

Using the Minkowski inequality and (33) once more, we obtain

$$\begin{aligned}\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\|A^{\frac{1}{2}}T_{3,2}(s,t_j)\|_{V^0}^2\,ds\Big)^p &\le \Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Big(\int_s^{t_j}\|A^{\frac{1}{2}}S(t_j-r)P\theta(r)v_2\|_{V^0}\,dr\Big)^2ds\Big)^p\\ &\le C\sup_{r\in[0,T]}\|\theta(r)\|_{H^0}^{2p}\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Big(\int_s^{t_j}(t_j-s)^{-\frac{1}{2}}dr\Big)^2ds\Big)^p\\ &\le C\sup_{r\in[0,T]}\|\theta(r)\|_{H^0}^{2p}\Big(\frac{T}{N}\Big)^p.\end{aligned}$$

The above estimates of $T_{3,1}$ and $T_{3,2}$, together with (20), imply, for $\eta \in (0,1)$, that

$$\mathbb{E}\Big(\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\|A^{\frac{1}{2}}T_3(s,t_j)\|_{V^0}^2\,ds\Big)^p \le C\Big(\frac{T}{N}\Big)^{\eta p}. \tag{62}$$

We next study the stochastic integrals. Using Hölder's inequality, the Burkholder inequality, (33), (34) and the growth condition (13) twice, we obtain for $\delta \in (0, \frac{1}{2})$

$$
\begin{aligned}
E\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|A^{\frac12} T_{4,1}(s,t_j)\|^2_{V^0} ds\Big)^p &\leq N^{p-1} \sum_{j=1}^N E\Big(\int_{t_{j-1}}^{t_j} \|A^{\frac12} T_{4,1}(s,t_j)\|^2_{V^0} ds\Big)^p \\
&\leq N^{p-1}\Big(\frac{T}{N}\Big)^{p-1} \sum_{j=1}^N \int_{t_{j-1}}^{t_j} E\Big\|\int_0^s A^{\delta} S(s-r) A^{-\delta}[S(t_j-s) - Id] A^{\frac12} G(u(r))dW(r)\Big\|^{2p}_{V^0} ds \\
&\leq C_p T^{p-1} \sum_{j=1}^N \int_{t_{j-1}}^{t_j} E\Big(\int_0^s (s-r)^{-2\delta} (t_j-s)^{2\delta} \|A^{\frac12} G(u(r))\|^2_{\mathcal{L}(K,V^0)} \mathrm{Tr}(Q) dr\Big)^p ds \\
&\leq C\Big(\frac{T}{N}\Big)^{2\delta p} \int_0^T E\Big(\int_0^s (s-r)^{-2\delta}\big[K_2 + K_3\|u(r)\|^2_{V^1}\big] dr\Big)^p ds \\
&\leq C\Big(\frac{T}{N}\Big)^{2\delta p}\Big[1 + E\int_0^T \|u(r)\|^{2p}_{V^1} \int_r^T (s-r)^{-2\delta} ds\, dr\Big] \leq C\Big(\frac{T}{N}\Big)^{2\delta p},
\end{aligned} \tag{63}
$$

where the last upper estimates are deduced from the Fubini theorem, and from the upper estimates (19) and (21).

A similar argument proves that

$$
\begin{aligned}
E\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|A^{\frac12} T_{4,2}(s,t_j)\|^2_{V^0} ds\Big)^p &\leq T^{p-1} \sum_{j=1}^N \int_{t_{j-1}}^{t_j} E\Big\|\int_s^{t_j} S(t_j-s) A^{\frac12} G(u(r))dW(r)\Big\|^{2p}_{V^0} ds \\
&\leq C_p T^{p-1} \mathrm{Tr}(Q)^p \sum_{j=1}^N \int_{t_{j-1}}^{t_j} E\Big(\int_s^{t_j} \big[K_2 + K_3\|u(r)\|^2_{V^1}\big] dr\Big)^p ds \\
&\leq C \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \Big(\frac{T}{N}\Big)^{p-1} \int_s^{t_j} \big[K_2^p + K_3^p E(\|A^{\frac12} u(r)\|^{2p}_{V^0})\big] dr\, ds \\
&\leq C\Big(\frac{T}{N}\Big)^{p-1} \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \big[K_2^p + K_3^p E(\|A^{\frac12} u(r)\|^{2p}_{V^0})\big] \Big(\int_r^{t_j} ds\Big) dr \\
&\leq C\Big(\frac{T}{N}\Big)^p \Big[1 + \int_0^T E(\|A^{\frac12} u(s)\|^{2p}_{V^0}) ds\Big] \leq C\Big(\frac{T}{N}\Big)^p.
\end{aligned} \tag{64}
$$

The inequalities (63) and (64) imply that, for $\eta \in (0,1)$,

$$
E\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|A^{\frac12} T_4(s,t_j)\|^2_{V^0} ds\Big)^p \leq C\Big(\frac{T}{N}\Big)^{\eta p}. \tag{65}
$$

The above arguments (61), (62) and (65) prove similar inequalities when replacing $T_i(s,t_j)$ with $T_i(t_{j-1},s)$ for $i = 1, \dots, 4$ and $j = 1, \dots, N$. Using (46), this concludes the proof of (59).

(ii) As above, we apply $\tilde A^{\frac12}$ to the terms $\tilde T_i(s,t_j)$, $i = 1,2,3$ of the decomposition (53) of $q(t_j) - q(s)$ introduced in the proof of Proposition 4 (ii). For $\delta \in (0,\frac12)$, the inequalities (33) and (34) imply that

$$\begin{aligned}
\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\tilde A^{\frac12} \tilde S(s)[\tilde S(\tilde t_j - s) - \mathrm{Id}] q_0\|_{H^0}^2\Big)^p
&\le \Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\tilde A^{\delta}\tilde S(s) \tilde A^{-\delta}[\tilde S(t_j - s) - \mathrm{Id}] \tilde A^{\frac12} q_0\|_{H^0}^2\Big)^p \\
&\le C\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} s^{-2\delta}\Big(\frac{T}{N}\Big)^{2\delta} \|\tilde A^{\frac12} q_0\|_{H^0}^2\, ds\Big)^p \\
&\le C \Big(\frac{T}{N}\Big)^{2\delta p} \|\tilde A^{\frac12} q_0\|_{H^0}^{2p} \Big(\int_0^T s^{-2\delta} ds\Big)^p.
\end{aligned}$$

Hence, for $\eta \in (0,1)$,

$$\mathbb{E}\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\tilde A^{\frac12} \tilde S(s)[\tilde S(\tilde t_j - s) - \mathrm{Id}] q_0\|_{H^0}^2\Big)^p \le C\, \mathbb{E}\|\tilde A^{\frac12} q_0\|_{H^0}^{2p} \Big(\frac{T}{N}\Big)^{\eta p}. \tag{66}$$

Let $\beta \in (0,\frac12)$ and $\delta \in (0,\frac12 - \beta)$. The Minkowski inequality, (33), (34) and (8) applied with $\alpha = \rho = \frac12$ imply that, for $s \in [t_{j-1}, t_j]$,

$$\begin{aligned}
&\Big\| \int_0^s \tilde A^{\frac12} \tilde S(s-r)[\tilde S(t_j - s) - \mathrm{Id}]\, [u(r).\nabla] q(r)\, dr \Big\|_{H^0} \\
&\le \int_0^s \|\tilde A^{\frac12+\beta+\delta} \tilde S(s-r)\, \tilde A^{-\beta}[\tilde S(t_j - s) - \mathrm{Id}]\, \tilde A^{-\delta} [u(r).\nabla] q(r)\|_{H^0}\, dr \\
&\le C \int_0^s (s-r)^{-(\frac12+\beta+\delta)} \Big(\frac{T}{N}\Big)^{\beta} \|A^{\frac12} u(r)\|_{V^0}\, \|\tilde A^{\frac12} q(r)\|_{H^0}\, dr.
\end{aligned}$$

Therefore, using the Cauchy–Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\tilde A^{\frac12} \tilde T_{2,1}(s,\tilde t_j)\|_{H^0}^2\, ds\Big)^p \\
&\le C \Big(\frac{T}{N}\Big)^{2\beta p} \mathbb{E}\Big[\sup_{s\in[0,T]} \|A^{\frac12} u(s)\|_{V^0}^{2p} \Big(\sum_{j=1}^N \int_{t_j}^{t_{j+1}} \int_0^s (s-r)^{-(\frac12+\beta+\delta)} \|\tilde A^{\frac12} q(r)\|_{H^0}^2 \\
&\qquad\qquad \int_0^s (s-r)^{-(\frac12+\beta+\delta)} dr\, ds\Big)^p \Big] \\
&\le C \Big(\frac{T}{N}\Big)^{2\beta p} \mathbb{E}\Big[\sup_{s\in[0,T]} \|A^{\frac12} u(s)\|_{V^0}^{2p} \Big(\int_0^T \|\tilde A^{\frac12} q(r)\|_{H^0}^2 \int_r^T (s-r)^{-(\frac12+\beta+\delta)} ds\, dr\Big)^p \Big] \\
&\le C \Big(\frac{T}{N}\Big)^{2\beta p} \Big\{ \mathbb{E}\Big(\sup_{s\in[0,T]} \|A^{\frac12} u(s)\|_{V^0}^{4p}\Big) \Big\}^{\frac12} \Big\{ \int_0^T \mathbb{E}\big(\|\tilde A^{\frac12} q(r)\|_{H^0}^{4p}\big) dr \Big\}^{\frac12}.
\end{aligned}$$

The upper estimates (21) and (37) imply, for $\eta \in (0,1)$, that

$$\mathbb{E}\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\tilde A^{\frac12} \tilde T_{2,1}(s,\tilde t_j)\|_{H^0}^2\, ds\Big)^p \le C \Big(\frac{T}{N}\Big)^{\eta p}. \tag{67}$$

Using the Minkowski inequality, (33) and (8) with $\alpha = \rho = \frac12$, and Fubini's theorem, we obtain, for $\delta \in (0,\frac12)$,

$$
\begin{aligned}
&\Big(\Big\|\sum_{j=1}^N\int_{t_{j-1}}^{t_j} A^{\frac12}\tilde T_{2,2}(s,\tilde t_j)\,ds\Big\|_{H^0}^2\Big)^p \\
&\quad\le \Big(\Big[\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\int_s^{t_j}(t_j-r)^{-(\frac12+\delta)}\|A^{\frac12}u(r)\|_{V^0}\|\tilde A^{-\frac12}q(r)\|_{H^0}\,dr\,ds\Big]^2\Big)^p \\
&\quad\le C\sup_{r\in[0,T]}\|A^{\frac12}u(r)\|_{V^0}^{2p}\Big(\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\int_s^{t_j}(t_j-r)^{-(\frac12+\delta)}\|\tilde A^{-\frac12}q(s)\|_{H^0}^2\,dr\int_s^{t_j}(t_j-r)^{-(\frac12+\delta)}dr\,ds\Big)^p \\
&\quad\le C\sup_{r\in[0,T]}\|A^{\frac12}u(r)\|_{V^0}^{2p}\Big(\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|\tilde A^{-\frac12}q(r)\|_{H^0}^2\int_{t_{j-1}}^{t_j}ds\,dr\Big)^p \\
&\quad\le C\sup_{r\in[0,T]}\|A^{\frac12}u(r)\|_{V^0}^{2p}\Big(\frac{T}{N}\Big)^p\int_0^T\|\tilde A^{-\frac12}q(s)\|_{H^0}^{2p}\,dr
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, (21) and (37), we obtain

$$
E\Big(\Big\|\sum_{j=1}^N\int_{t_{j-1}}^{t_j}A^{\frac12}\tilde T_{2,2}(s,\tilde t_j)\,ds\Big\|_{H^0}^2\Big)^p\le C\Big(\frac{T}{N}\Big)^p. \tag{68}
$$

Finally, arguments similar to those used to prove (65) imply, for $\eta\in(0,1)$, that

$$
E\Big(\Big\|\sum_{j=1}^N\int_{t_{j-1}}^{t_j}A^{\frac12}\tilde T_3(s,\tilde t_j)\,ds\Big\|_{V^0}^2\Big)^p\le C\Big(\frac{T}{N}\Big)^{\eta p}. \tag{69}
$$

The upper estimates (66)–(69) conclude the proof of

$$
E\Big(\Big\|\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\tilde A^{-\frac12}\big(q(t_j)-q(s)\big)\Big\|_{H^0}^2ds\Big)^p\le C\Big(\frac{T}{N}\Big)^{\eta p},\quad \eta\in(0,1).
$$

Using (47), a similar argument completes the proof of (60). □

6. The Implicit Time Euler Scheme

We first prove the existence of the fully time-implicit time Euler scheme $\{u^k;\, k=0,1,\dots,N\}$ and $\{q^k;\, k=0,1,\dots,N\}$ defined by (17) and (18). Set $\Delta_l W := W(t_l)-W(t_{l-1})$ and $\Delta_l\tilde W=\tilde W(t_l)-\tilde W(t_{l-1})$, $l=1,\dots,N$.

6.1. Existence of the Scheme

Proof of Proposition 3. The proof is divided into two steps.

Step 1 For technical reasons, we consider a Galerkin approximation. Let $\{e_l\}_l$ denote an orthonormal basis of V^0 made of elements of V^2 that are orthogonal in V^1 (resp., let $\{\tilde e_l\}_l$ denote an orthonormal basis of H^0 made of elements of H^2 that are orthogonal in H^1).

For $m=1,2,\dots$, let $V_m=\text{span}(e_1,\dots,e_m)\subset V^2$ and let $P_m: V^0\to V_m$ denote the projection from V^0 to V_m. Similarly, let $\tilde H_m=\text{span}(\tilde e_1,\dots,\tilde e_m)\subset H^2$ and let $\tilde P_m: H^0\to\tilde H_m$ denote the projection from H^0 to $\tilde H_m$.

In order to find a solution to (17) and (18), we project these equations onto V_m and $\tilde{H}_m$, respectively, which we define by induction as $\{u^k(m)\}_{k=0,\dots,N} \in V_m$ and $\{\theta^k(m)\}_{k=0,\dots,N} \in \tilde{H}_m$ such that $u^0(m) = P_m(u_0)$, $\theta^0(m) = \tilde{P}_m(\theta_0)$, and, for $k = 1,\dots,N$, $\varphi \in V_m$ and $\psi \in \tilde{H}_m$,

$$\big(u^k(m) - u^{k-1}(m), \varphi\big) + h\Big[\nu\big(A^{\frac{1}{2}}u^k(m), A^{\frac{1}{2}}\varphi\big) + \big\langle B\big(u^k(m), u^k(m)\big), \varphi\big\rangle\Big] = h\big(\Pi\theta^{k-1}v_2, \varphi\big) + \big(G(u^{k-1}(m))D_k W, \varphi\big) \tag{70}$$

$$\big(\theta^k(m) - \theta^{k-1}(m), \psi\big) + h\Big[\kappa\big(\tilde{A}^{\frac{1}{2}}\theta^k(m), \tilde{A}^{\frac{1}{2}}\psi\big) + \big\langle [u^{k-1}(m).\nabla]\theta^k(m), \psi\big\rangle\Big] = \big(\tilde{G}(\theta^{k-1}(m))D_k\tilde{W}, \psi\big) \tag{71}$$

For almost every ω set, $R(0,\omega) = \|u_0(\omega)\|_{V^0}$ and $\tilde{R}(0,\omega) = \|\theta_0(\omega)\|_{H^0}$. Fix $k = 1,\dots,N$ and suppose that, for $j = 0,\dots,k-1$, the $\mathcal{F}_{t_j}$- measurable random variables $u^j(m)$ and $\theta^j(m)$ have been defined, and that

$$R(j,\omega) := \sup_{m \ge 1} \|u^j(m,\omega)\|_{L^2} < \infty \quad \text{and} \quad \tilde{R}(j,\omega) := \sup_{m \ge 1} \|\theta^j(m,\omega)\|_{L^2} < \infty$$

for almost every ω. We prove that $u^k(m)$ and $\theta^k(m)$ exist and satisfy $\sup_{m\ge 1} \|u^k(m,\omega)\|_{V^0} < \infty$ and $\sup_{m \ge 1} \|\theta^k(m,\omega)\|_{H^0} < \infty$ a.s.

For $\omega \in \Omega$, let $F^k_{m,\omega} : V_m \to V_m$ (resp., $\tilde{F}^k_{m,\omega}$) be defined for $f \in V_m$ (resp., for $\tilde{f} \in \tilde{H}_m$) as the solution of

$$\big(F^k_{m,\omega}(f), \varphi\big) = \big(f - u^{k-1}(m,\omega), \varphi\big) + h\Big[\nu\big(A^{\frac{1}{2}}f, A^{\frac{1}{2}}\varphi\big) + \big\langle P_m B(f,f), \varphi\big\rangle - \big(\Pi\theta^{k-1}(m)v_2, \varphi\big)\Big] - \big(P_m G(u^{k-1}(m,\omega))D_k W(\omega), \varphi\big), \quad \forall \varphi \in V_m,$$

$$\big(\tilde{F}^k_{m,\omega}(\tilde{f}), \psi\big) = \big(\tilde{f} - \theta^{k-1}(m,\omega), \psi\big) + h\Big[\kappa\big(\tilde{A}^{\frac{1}{2}}\tilde{f}, \tilde{A}^{\frac{1}{2}}\psi\big) + \big\langle [u^{k-1}(m).\tilde{A}^{\frac{1}{2}}]\tilde{f}, \psi\big\rangle\Big] - \big(\tilde{P}_m\tilde{G}(\theta^{k-1}(m,\omega))D_k\tilde{W}(\omega), \psi\big), \quad \forall \psi \in \tilde{H}_m.$$

Then, the Cauchy–Schwarz and Young inequalities imply

$$\big|\big(u^{k-1}(m,\omega), f\big)\big| \le \frac{1}{4}\|f\|^2_{V^0} + \|u^{k-1}(m,\omega)\|^2_{V^0},$$

$$\big|\big(\theta^{k-1}(m,\omega), \tilde{f}\big)\big| \le \frac{1}{4}\|\tilde{f}\|^2_{H^0} + \|\theta^{k-1}(m,\omega)\|^2_{H^0},$$

$$\big|\big(\Pi\theta^{k-1}(m,\omega), f\big)\big| \le \frac{1}{4}\|f\|^2_{V^0} + \|\theta^{k-1}(m,\omega)\|^2_{H^0},$$

$$\begin{aligned}\big|\big(G(u^{k-1}(m,\omega))D_k W, f\big)\big| &\le \frac{1}{4}\|f\|^2_{V^0} + \|G(u^{k-1}(m,\omega))\|^2_{L(K,V^0)}\|D_k W\|^2_K \\ &\le \frac{1}{4}\|f\|^2_{V^0} + \big(K_0 + K_1\|u^{k-1}(m,\omega)\|^2_{V^0}\big)\|D_k W\|^2_K,\end{aligned}$$

$$\begin{aligned}\big|\big(\tilde{G}(\theta^{k-1}(m,\omega))D_k\tilde{W}, \tilde{f}\big)\big| &\le \frac{1}{4}\|\tilde{f}\|^2_{H^0} + \|\tilde{G}(\theta^{k-1}(m,\omega))\|^2_{L(\tilde{K},H^0)}\|D_k\tilde{W}\|^2_{\tilde{K}} \\ &\le \frac{1}{4}\|\tilde{f}\|^2_{H^0} + \big(\tilde{K}_0 + \tilde{K}_1\|\theta^{k-1}(m,\omega)\|^2_{H^0}\big)\|D_k\tilde{W}\|^2_{\tilde{K}}.\end{aligned}$$

If

$$\|f\|^2_{V^0} = R^2(k,\omega) := 4\Big[R^2(k-1,\omega) + h\tilde{R}(k-1,\omega)^2 + \big(K_0 + K_1 R^2(k-1,\omega)\big)\|D_k W(\omega)\|^2_K\Big],$$

$$\|\tilde{f}\|^2_{H^0} = \tilde{R}^2(k,\omega) := 2\Big[\tilde{R}^2(k-1,\omega) + \big(\tilde{K}_0 + \tilde{K}_1\tilde{R}^2(k-1,\omega)\big)\|D_k\tilde{W}(\omega)\|^2_{\tilde{K}}\Big],$$

we deduce

$$\begin{aligned}
\big(\mathcal{F}^k_{m,\omega}(f), f\big) &\ge \frac{1}{4}\|f\|^2_{L^2} - \|u^{k-1}(m,\omega)\|^2_{L^2} + h\nu\|A^{\frac12} f\|^2_{V^0} - h^2\|q^{k-1}(m,\omega)\|^2_{H^0} \\
&\quad - \big(K_0 + K_1\|u^{k-1}(m,\omega)\|^2_{V^0}\big)\|D_k W(\omega)\|^2_K \ge 0, \\
\big(\tilde{\mathcal{F}}^k_{m,\omega}(f), f\big) &\ge \frac{1}{2}\|f\|^2_{H^0} - \|q^{k-1}(m,\omega)\|^2_{H^0} + h\kappa\|\tilde{A}^{\frac12} f\|^2_{H^0} \\
&\quad - \big(\tilde{K}_0 + \tilde{K}_1\|q^{k-1}(m,\omega)\|^2_{H^0}\big)\|D_k\tilde{W}(\omega)\|^2_{\tilde{K}} \ge 0.
\end{aligned}$$

Using ([16], Cor 1.1) page 279, which can be deduced from Brouwer's theorem, we deduce the existence of an element $u^k(m,\omega) \in V(m)$ (resp., $q^k(m,\omega) \in \tilde{H}(m)$), such that $\mathcal{F}^k(m,\omega)(u^k(m,\omega)) = 0$ (resp., $\tilde{\mathcal{F}}^k(m,\omega)(q^k(m,\omega)) = 0$) and $\|u^k(m,\omega)\|^2_{V^0} \le R^2(k,\omega)$ (resp., $\|q^k(m,\omega)\|_{H^0} \le \tilde{R}^2(k,\omega)$) a.s. Note that these elements $u^k(m,\omega)$ and $q^k(m,\omega)$ need not be unique. Furthermore, the random variables $u^k(m)$ and $q^k(\omega)$ are $\mathcal{F}_{t_k}$-measurable.

The definition of $u^k(m)$ (resp., $q^k(m)$) implies that it is a solution to (70) (resp., (71)). Taking $\varphi = u^k(m)$ in (70), using the antisymmetry property (3) and the Young inequality, we obtain

$$\begin{aligned}
\|u^k(m)\|^2_{V^0} + h\nu\|A^{\frac12}u^k(m)\|^2_{V^0} &= \big(u^{k-1}(m), u^k(m)\big) + h\big(Pq^{k-1}(m)v_2, u^k(m)\big) \\
&\quad + \big(G(u^{k-1}(m))D_kW, u^k(m)\big) \\
&\le \frac{3}{4}\|u^k(m)\|^2_{V^0} + \|u^{k-1}(m)\|^2_{V^0} + \|q^{k-1}(m)\|^2_{H^0} + \big(K_0 + K_1\|u^{k-1}(m)\|^2_{V^0}\big)\|D_kW\|^2_K.
\end{aligned}$$

Hence, a.s.,

$$\begin{aligned}
\sup_{m\ge1}\Big[\frac{1}{4}\|u^k(m,\omega)\|^2_{V^0} + h\nu\|A^{\frac12}u^k(m,\omega)\|^2_{V^0}\Big] &\le R^2(k-1,\omega) + \tilde{R}^2(k-1,\omega) \\
&\quad + \big(K_0 + K_1R^2(k-1,\omega)\big)\|D_kW(\omega)\|^2_K.
\end{aligned}$$

A similar computation using $\psi = q^k(m)$ in (71) implies that

$$\sup_{m\ge1}\Big[\frac{1}{2}\|q^k(m,\omega)\|^2_{H^0} + h\kappa\|\tilde{A}^{\frac12}q^k(m,\omega)\|^2_{H^0}\Big] \le \tilde{R}^2(k-1) + \big(\tilde{K}_0 + \tilde{K}_1\tilde{R}^2(k-1)\big)\|D_k\tilde{W}\|^2_{\tilde{K}}.$$

Therefore, for fixed k and almost every ω, the sequence $\{u^k(m,\omega)\}_m$ is bounded in V^1; it has a sub-sequence (still denoted as $\{u^k(m,\omega)\}_m$) that converges weakly in V^1 to $\phi_k(\omega)$. The random variable ϕ_k is $\mathcal{F}_{t_k}$-measurable. Similarly, for fixed k and almost every ω, the sequence $\{q^k(m,\omega)\}_m$ is bounded in H^1; it has a sub-sequence (still denoted as $\{q^k(m,\omega)\}_m$) that converges weakly in H^1 to $\tilde{\phi}_k(\omega)$, which is $\mathcal{F}_{t_k}$-measurable.

Since D is bounded, the embedding of V^1 in V^0 (resp., of H^1 in H^0) is compact; hence, the sub-sequence $\{u^k(m,\omega)\}_m$ converges strongly to $\phi_k(\omega)$ in V^0 (resp., $\{q^k(m,\omega)\}_m$ converges strongly to $\tilde{\phi}_k(\omega)$ in H^0).

Step 2 We next prove that the pair $(\phi_k, \tilde{\phi}_k)$ is a solution to (17) and (18). By definition, $u^0(m)$ converges strongly to u_0 in V^0, and $q^0(m)$ converges strongly to q_0 in H^0. We next prove by induction on k that the pair $(\phi^k, \tilde{\phi}^k)$ solves (17) and (18). Fix a positive integer m_0 and consider the Equation (70) for $k = 1, \dots, N$, $\varphi \in V_{m_0}$ and $m \ge m_0$. As $m \to \infty$, we have, a.s.,

$$\begin{aligned}
&\big(u^k(m) - u^{k-1}(m), \varphi\big) \to \big(\phi^k - \phi^{k-1}, \varphi\big), \qquad \big(A^{\frac12}u^k(m), A^{\frac12}\varphi\big) \to \big(A^{\frac12}\phi^k, A^{\frac12}\varphi\big), \\
&\big(Pq^{k-1}(m)v_2, \varphi\big) = \big(q^{k-1}(m)v_2, \varphi\big) \to \big(\tilde{\phi}^{k-1}v_2, \varphi\big).
\end{aligned}$$

Furthermore, the antisymmetry of B (3) and the Gagliardo–Nirenberg inequality (6) yield, a.s.,

$$\big(B(u^k(m),u^k(m)) - B(\phi^k,\phi^k),\varphi\big)$$
$$= -\big(B(u^k(m)-\phi^k,\varphi), u^k(m)\big) - \big(B(\phi^k,\varphi), u^k(m)-\phi^k\big)$$
$$\le \|A^{\frac12}\varphi\|_{V^0}\|u^k(m)-\phi^k\|_{L^4}\big(\|u^k(m)\|_{L^4}+\|\phi^k\|_{L^4}\big)$$
$$\le C\|\varphi\|_{V^0}\max_m\big[\|u^k(m)\|_{V^1}+\|\phi^k\|_{V^1}\big]\|A^{\frac12}(u^k(m)-\phi^k)\|^{\frac12}_{V^0}\|u^k(m)-\phi^k\|^{\frac12}_{V^0}\to 0$$

as $m\to\infty$.

Finally, the Cauchy–Schwarz inequality and the Lipschitz condition (12) imply that

$$\big|\big(G(u^{k-1}(m)) - G(\phi^{k-1})\Delta_kW,\varphi\big)\big| \le \|\varphi\|_{V^0}\|G(u^{k-1}(m)) - G(\phi^{k-1})\|_{L(K;V^0)}\|\Delta_kW\|_K$$
$$\le \sqrt{L_1}\,\|\varphi\|_{L^2}\|u^{k-1}(m)-\phi^{k-1}\|_{L^2}\|\Delta_kW\|_K\to 0$$

as $m\to\infty$. Therefore, letting $m\to\infty$ in (70), we deduce that

$$\big(\phi^k-\phi^{k-1}+h\nu A\phi^k + hB(\phi^k,\phi^k),\varphi\big) = \big(\Pi\theta^{k-1}v_2,\varphi\big) + \big(G(\phi^{k-1})\Delta_kW,\varphi\big), \qquad \forall\varphi\in V_{m_0}.$$

Since $\cup_{m_0}V_{m_0}$ is dense in V, we deduce that $\{\phi^k\}_{k=0,\dots,N}$ is a solution to (17). A similar argument proves that $\tilde\phi^k$ is a solution to (18). This concludes the proof. □

6.2. Moments of the Euler Scheme

We next prove the upper bounds of moments of u^k and θ^k uniformly in $k=1,\dots,N$.

Proposition 6. Let G and $\tilde G$ satisfy the condition **(C-u)(i)** and **(C-θ)(i)**, respectively. Let $K\ge 1$ be an integer, and let $u_0\in L^{2^K}(\Omega;V^0)$ and $\theta_0\in L^{2^K}(\Omega;H^0)$, respectively. Let $\{u^k\}_{k=0,\dots,N}$ and $\{\theta_k\}_{k=0,\dots,N}$ be the solution of (17) and (18), respectively. Then,

$$\sup_N \mathbb{E}\Big(\max_{0\le L\le N}\|u^L\|^{2^K}_{V^0} + \max_{0\le L\le N}\|\theta^L\|^{2^K}_{H^0}\Big) < \infty \tag{72}$$

$$\sup_{N\ge 1}\mathbb{E}\Big[h\sum_{l=1}^N\|A^{\frac12}u^l\|^2_{V^0}\|u^l\|^{2^K-2}_{V^0} + h\sum_{l=1}^N\|A^{\frac12}\theta^l\|^2_{H^0}\|\theta^l\|^{2^K-2}_{H^0}\Big] < \infty, \tag{73}$$

Proof. Write (17) with $\varphi = u^l$, (18) with $\psi=\theta^l$ and use the identity $(f,f-g)=\frac12\big[\|f\|^2_{L^2}-\|g\|^2_{L^2}+\|f-g\|^2_{L^2}\big]$. Using the Cauchy–Schwarz and Young inequalities, the antisymmetry (3) and the growth condition (11) yields, for $l=1,\dots,N$,

$$\frac12\Big[\|u^l\|^2_{V^0}-\|u^{l-1}\|^2_{V^0}+\|u^l-u^{l-1}\|^2_{V^0}\Big] + h\nu\|A^{\frac12}u^l\|^2_{V^0} = h(\Pi\theta^{l-1}e_2,u^l) + \big(G(u^{l-1})\Delta_lW,u^l\big), \tag{74}$$

$$\frac12\Big[\|\theta^l\|^2_{H^0}-\|\theta^{l-1}\|^2_{H^0}+\|\theta^l-\theta^{l-1}\|^2_{H^0}\Big] + h\kappa\|A^{\frac12}\theta^l\|^2_{H^0} = \big(\tilde G(\theta^{l-1})\Delta_l\tilde W,\theta^l\big). \tag{75}$$

Fix $L = 1, \dots, N$ and add both equalities for $l = 1, \dots, L$; this yields

$$\frac{1}{2}\|u^L\|^2_{V^0} - \|u_0\|^2_{V^0} + \|\theta^L\|^2_{H^0} - \|\theta_0\|^2_{H^0} + \frac{1}{2}\Big[\sum_{l=1}^{L}\|u^l - u^{l-1}\|^2_{V^0} + \sum_{l=1}^{L}\|\theta^l - \theta^{l-1}\|^2_{H^0}\Big]$$
$$+ h\sum_{l=1}^{L}\Big[\nu_k\|A^{\frac{1}{2}}u^l\|^2_{V^0} + \kappa_k\|A^{\frac{1}{2}}\tilde{\theta}^l\|^2_{H^0}\Big] \le h\,\frac{1}{2}\sum_{l=0}^{L-1}\|\theta^l\|^2_{H^0} + h\sum_{l=1}^{L-1}\|\theta^l\|^2_{V^0} + h\|u^L\|^2_{V^0}$$
$$+ \sum_{l=1}^{L}\|G(u^{l-1})\|^2_{L(K;V^0)}\|D_l W\|^2_K + \frac{1}{4}\|u^l - u^{l-1}\|^2_{V^0} + \sum_{l=1}^{L}\big(G(u^{l-1})D_l W, u^{l-1}\big)$$
$$+ \sum_{l=1}^{L}\|\tilde{G}(\theta^{l-1})\|^2_{L(\tilde{K};H^0)}\|D_l\tilde{W}\|^2_{\tilde{K}} + \frac{1}{4}\|\theta^l - \theta^{l-1}\|^2_{H^0} + \sum_{l=1}^{L}\big(\tilde{G}(\tilde{\theta}^{l-1})D_l\tilde{W}, \tilde{\theta}^{l-1}\big). \tag{76}$$

Let N be large enough to have $h = \frac{T}{N} \le \frac{1}{8}$. Taking the expected values, we obtain

$$E\Big[\|u^L\|^2_{V^0} + \|\theta^L\|^2_{H^0} + \frac{1}{2}\sum_{l=1}^{L}\big(\|u^l - u^{l-1}\|^2_{V^0} + \|\theta^l - \theta^{l-1}\|^2_{H^0}\big)$$
$$+ 2h\sum_{l=1}^{N}\big(\nu_k\|A^{\frac{1}{2}}u^l\|^2_{V^0} + \kappa_k\|A^{\frac{1}{2}}\theta^l\|^2_{H^0}\big)\Big] \le E\big(\|u^0\|^2_{V^0} + \|\theta^0\|^2_{H^0}\big) + 2TK_0\mathrm{Tr}(Q) + \tilde{K}_0\mathrm{Tr}(\tilde{Q})$$
$$+ h\big(4 + 2\max(K_1\mathrm{Tr}(Q), \tilde{K}_1\mathrm{Tr}(\tilde{Q}))\big)\sum_{l=0}^{L-1}E\big(\|u^l\|^2_{V^0} + \|\theta^l\|^2_{H^0}\big).$$

Neglecting both sums in the left hand side and using the discrete Gronwall lemma, we deduce that

$$\sup_{1\le L\le N} E\big(\|u^L\|^2_{V^0} + \|\theta^L\|^2_{H^0}\big) \le C, \tag{77}$$

where

$$C = 2\Big(E\big(\|u^0\|^2_{V^0} + \|\theta^0\|^2_{H^0}\big) + 2TK_0\mathrm{Tr}(Q) + \tilde{K}_0\mathrm{Tr}(\tilde{Q})\Big)e^{T(4+2\max(K_1\mathrm{Tr}(Q), \tilde{K}_1\mathrm{Tr}(\tilde{Q})))}$$

is independent of N. This implies

$$\sup_{N\ge 1} E\sum_{l=1}^{N}\Big[\|Au^l\|^2_{V^0} + \|A\theta^l\|^2_{H^0} + \|u^l - u^{l-1}\|^2_{V^0} + \|\theta^l - \theta^{l-1}\|^2_{H^0}\Big] < \infty,$$

which proves (73) for $K = 1$. For $s \in [t_j, t_{j+1})$, $j = 0, \dots, N-1$, and set $\underline{s} = t_j$. The Davis inequality, and then the Cauchy-Schwarz and Young inequalities, imply that for any $\epsilon > 0$,

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{1\le L\le N}\sum_{l=1}^{L}\big[\big(G(u^{l-1})D_lW, u^{l-1}\big) + \big(\tilde G(\theta^{l-1})D_l\tilde W, \theta^{l-1}\big)\big]\Big)\\
&\le \mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t \big(G(u^s)dW(s), u^s\big)\Big|\Big) + \mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t \big(\tilde G(\theta^s)d\tilde W(s), \theta^s\big)\Big|\Big)\\
&\le 3\mathbb{E}\Big(\int_0^T \|G(u^s)\|^2_{\mathcal{L}(K;V^0)}\|u^s\|^2_{V^0}\mathrm{Tr}(Q)ds\Big)^{\frac12}\\
&\quad + 3\mathbb{E}\Big(\int_0^T \|\tilde G(\theta^s)\|^2_{\mathcal{L}(K;H^0)}\|\theta^s\|^2_{H^0}\mathrm{Tr}(\tilde Q)ds\Big)^{\frac12}\\
&\le 3\mathrm{Tr}(Q)^{\frac12}\mathbb{E}\Big[\max_{1\le l\le N}\|u^l\|_{V^0}\Big\{h\sum_{l=0}^{N}\big[K_0 + K_1\|u^{l-1}\|^2_{V^0}\big]\Big\}^{\frac12}\Big]\\
&\quad + 3\mathrm{Tr}(\tilde Q)^{\frac12}\mathbb{E}\Big[\max_{1\le l\le N}\|\theta^l\|_{H^0}\Big\{h\sum_{l=0}^{N}\big[\tilde K_0 + \tilde K_1\|\theta^{l-1}\|^2_{H^0}\big]\Big\}^{\frac12}\Big]\\
&\le \epsilon\,\mathbb{E}\Big(\max_{1\le l\le N}\|u^l\|^2_{V^0} + \mathbb{E}\|u^0\|^2_{V^0}\Big) + \frac{9\,\mathrm{Tr}(Q)}{4\epsilon}\,h\sum_{l=1}^{N}\big[K_0 + K_1\mathbb{E}(\|u^{l-1}\|^2_{V^0})\big]\\
&\quad + \epsilon\,\mathbb{E}\Big(\max_{1\le l\le N}\|\theta^l\|^2_{H^0} + \mathbb{E}\|\theta^0\|^2_{H^0}\Big) + \frac{9\,\mathrm{Tr}(\tilde Q)}{4\epsilon}\,h\sum_{l=1}^{N}\big[\tilde K_0 + \tilde K_1\mathbb{E}(\|\theta^{l-1}\|^2_{H^0})\big]. \qquad (78)
\end{aligned}
$$

Taking the maximum over L in (76) and using (78), we deduce

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{1\le L\le N}\|u^L\|^2_{V^0} + \|\theta^L\|^2_{H^0}\Big) \le 2\mathbb{E}\Big(\|u^0\|^2_{V^0} + \|\theta^0\|^2_{H^0} + h\sum_{l=1}^{N}\big[\|\theta^{l-1}\|^2_{H^0} + \|u^{l-1}\|^2_{V^0}\big]\Big) + 2\epsilon\,\mathbb{E}\Big(\max_{1\le L\le N}\|u^L\|^2_{V^0} + \|\theta^L\|^2_{H^0}\Big)\\
&\quad + \frac{9}{4\epsilon}\mathrm{Tr}(Q)\,h\sum_{l=1}^{N}\big(K_0 + K_1\mathbb{E}(\|u^{l-1}\|^2_{V^0})\big) + \frac{9}{4\epsilon}\mathrm{Tr}(\tilde Q)\,h\sum_{l=1}^{N}\big(\tilde K_0 + \tilde K_1\mathbb{E}(\|\theta^{l-1}\|^2_{H^0})\big).
\end{aligned}
$$

For $\epsilon = \frac14$, (77) proves that

$$
\sup_{N}\Big[\mathbb{E}\Big(\sup_{1\le L\le N}\|u^L\|^2_{V^0}\Big) + \mathbb{E}\Big(\sup_{1\le L\le N}\|\theta^L\|^2_{H^0}\Big)\Big] < \infty, \quad N \ge 1
$$

which proves (72) for $K = 1$.

We next prove (72) and (73) by induction on K. Multiply (74) by $\|u^l\|^2_{V^0}$ and (75) by $\|\theta^l\|^2_{H^0}$. Using the identity $a(a-b) = \frac12\big[a^2 - b^2 + |a-b|^2\big]$ for $a = \|u^l\|^2_{V^0}$ (resp., $a = \|\theta^l\|^2_{H^0}$) and $b = \|u^{l-1}\|^2_{V^0}$ (resp., $b = \|\theta^{k-1}\|^2_{H^0}$), we deduce, for $l = 1,\dots,N$, that

$$
\begin{aligned}
&\frac14\Big[\|u^l\|^4_{V^0} - \|u^{l-1}\|^4_{V^0} + \big(\|u^l\|^2_{V^0} - \|u^{l-1}\|^2_{V^0}\big)^2 + \|\theta^l\|^4_{H^0} - \|\theta^{l-1}\|^4_{H^0}\\
&\qquad + \big(\|\theta^l\|^2_{H^0} - \|\theta^{l-1}\|^2_{H^0}\big)^2\Big]\\
&\quad + \frac12\Big[\|u^l - u^{l-1}\|^2_{V^0}\|u^l\|^2_{V^0} + \|\theta^l - \theta^{l-1}\|^2_{H^0}\|\theta^l\|^2_{H^0}\Big] + h\nu\|A^{\frac12}u^l\|^2_{V^0}\|u^l\|^2_{V^0}\\
&\quad + h\kappa\|A^{\frac12}\theta^l\|^2_{H^0}\|\theta^l\|^2_{H^0} = h\big(P\theta^{l-1}v_2, u^l\big)\|u^{l-1}\|^2_{V^0} + \sum_{i=1}^{4}T_i(l), \qquad (79)
\end{aligned}
$$

where

$$
\begin{aligned}
T_1(l) &= \big(G(u^{l-1})D_lW, u^{l-1}\big)\|u^l\|^2_{V^0}, & T_2(l) &= \big(G(u^{l-1})D_lW, u^l - u^{l-1}\big)\|u^l\|^2_{V^0},\\
T_3(l) &= \big(\tilde G(\theta^{l-1})D_l\tilde W, \theta^{l-1}\big)\|\theta^{l-1}\|^2_{H^0}, & T_4(l) &= \big(\tilde G(\theta^{l-1})D_l\tilde W, \theta^l - \theta^{l-1}\big)\|\theta^l\|^2_{H^0}.
\end{aligned}
$$

The Cauchy–Schwarz and Young inequalities imply that

$$P(\theta^{l-1}v_2, u^l)\|u^l\|^2_{V'} \le \|\theta^{l-1}\|_{H'}\|u^l\|^3_{V'} \le \frac{1}{4}\|\theta^{l-1}\|^4_{H'} + \frac{3}{4}\|u^l\|^4_{V'}. \tag{80}$$

Using once more the Cauchy–Schwarz and Young inequalities, we deduce that for $\epsilon, \bar\epsilon > 0$,

$$\begin{aligned}
|T_2(l)| &\le \|G(u^{l-1})\|_{L(K;V')}\|u^l - u^{l-1}\|^2_{V'}\|u^l\|^2_{V'}\\
&\le \epsilon\|u^l - u^{l-1}\|^2_{V'}\|u^l\|^2_{V'}\\
&\quad + \frac{1}{4\epsilon}\|G(u^{l-1})\|^2_{L_2(K;V')}\|D_lW\|^2_K\|u^{l-1}\|^2_{V'} + \|u^l\|^2_{V'} - \|u^{l-1}\|^2_{V'}\\
&\le \epsilon\|u^l - u^{l-1}\|^2_{V'}\|u^l\|^2_{V'} + \frac{1}{4\epsilon}\|G(u^{l-1})\|^2_{L_2(K;V')}\|D_lW\|^2_K\|u^{l-1}\|^2_{V'}\\
&\quad + \bar\epsilon\big(\|u^l\|^2_{V'} - \|u^{l-1}\|^2_{V'}\big)^2 + \frac{1}{16\epsilon^2}\frac{1}{4\bar\epsilon}\|G(u^{l-1})\|^4_{L_2(K;V')}\|D_lW\|^4_K.
\end{aligned} \tag{81}$$

A similar argument proves, for $\epsilon, \bar\epsilon > 0$, that

$$\begin{aligned}
|T_4(l)| &\le \epsilon\|\theta^l - \theta^{l-1}\|^2_{H'}\|\theta^l\|^2_{H'} + \frac{1}{4\epsilon}\|\tilde G(\theta^{l-1})\|^2_{L_2(\tilde K;H')}\|D_l\tilde W\|^2_{\tilde K}\|\theta^{l-1}\|^2_{H'}\\
&\quad + \bar\epsilon\big(\|\theta^l\|^2_{H'} - \|\theta^{l-1}\|^2_{H'}\big)^2 + \frac{1}{16\epsilon^2}\frac{1}{4\bar\epsilon}\|\tilde G(\theta^{l-1})\|^4_{L_2(\tilde K;H')}\|D_l\tilde W\|^4_{\tilde K}.
\end{aligned} \tag{82}$$

A similar argument shows, for $\bar\epsilon > 0$, that

$$\begin{aligned}
|T_1(l)| &\le \|G(u^{l-1})D_lW\|_{V'}\|u^{l-1}\|^3_{V'}\\
&\quad + \|G(u^{l-1})D_lW\|_{V'}\|u^{l-1}\|_{V'}\big(\|u^l\|^2_{V'} - \|u^{l-1}\|^2_{V'}\big)\\
&\le \frac{1}{4}\|G(u^{l-1})\|^4_{L(K;V')}\|D_lW\|^4_K + \frac{3}{4}\|u^{l-1}\|^4_{V'} + \bar\epsilon\big(\|u^l\|^2_{V'} - \|u^{l-1}\|^2_{V'}\big)^2\\
&\quad + \frac{1}{4\bar\epsilon}\|G(u^{l-1})\|^2_{L(K;V')}\|D_lW\|^2_K\|u^{l-1}\|^2_{V'},
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
|T_3(l)| &\le \frac{1}{4}\|\tilde G(\theta^{l-1})\|^4_{L(\tilde K;H')}\|D_l\tilde W\|^4_{\tilde K} + \frac{3}{4}\|\theta^{l-1}\|^4_{H'} + \bar\epsilon\big(\|\theta^l\|^2_{H'} - \|\theta^{l-1}\|^2_{H'}\big)^2\\
&\quad + \frac{1}{4\bar\epsilon}\|\tilde G(\theta^{l-1})\|^2_{L_2(\tilde K;H')}\|D_l\tilde W\|^2_{\tilde K}\|\theta^{l-1}\|^2_{H'}.
\end{aligned} \tag{84}$$

Add the inequalities (79)–(84) for $l = 1$ to $L \le N$, choose $\epsilon = \frac{1}{4}$ and $\bar\epsilon = \frac{1}{16}$ and use the growth conditions (11) and (14). This yields

$$\begin{aligned}
&\|u^L\|^4_{V'} + \|\theta^L\|^4_{H'} + \frac{1}{2}\sum_{l=1}^{L}\Big[\big(\|u^l\|^2_{V'} - \|u^{l-1}\|^2_{V'}\big)^2 + \|u^l - u^{l-1}\|^2_{V'}\|u^l\|^2_{V'}\\
&\quad + \big(\|\theta^l\|^2_{H'} - \|\theta^{l-1}\|^2_{H'}\big)^2 + \|\theta^l - \theta^{l-1}\|^2_{H'}\|\theta^l\|^2_{H'}\\
&\quad + 4h\sum_{l=1}\big(\nu\|A^{1/2}u^l\|^2_{V'}\|u^l\|^2_{V'} + \kappa\|A^{1/2}\theta^l\|^2_{H'}\|\theta^l\|^2_{H'}\big)\Big]\\
&\le \|u_0\|^4_{V'} + \|\theta_0\|^4_{H'} + \frac{1}{4}h\sum_{l=0}^{L-1}\|\theta^l\|^4_{H'} + \frac{3}{4}h\sum_{l=1}^{L}\|u^l\|^4_{V'}\\
&\quad + C\sum_{l=0}\Big[\big(K_0 + K_1\|u^{l-1}\|^2_{V'}\big)\|u^{l-1}\|^2_{V'}\|D_lW\|^2_K + \big(K_0 + K_1\|u^{l-1}\|^2_{V'}\big)^2\|D_lW\|^4_K\Big]\\
&\quad + C\sum_{l=0}\Big[\big(K_0 + K_1\|\theta^{l-1}\|^2_{H'}\big)\|\theta^{l-1}\|^2_{H'}\|D_l\tilde W\|^2_{\tilde K} + \big(K_0 + K_1\|\theta^{l-1}\|^2_{H'}\big)^2\|D_l\tilde W\|^4_{\tilde K}\Big].
\end{aligned} \tag{85}$$

Taking expected values, we deduce, for every $L = 1,\dots,N$ and $h = \frac{T}{N} \le 1$, that

$$\begin{aligned}
&\mathbb{E}\Big[\|u^L\|_{V^0}^4 + \|\theta^L\|_{H^0}^4 + \frac{1}{2}\sum_{l=1}^{L}\Big\{\big(\|u^l\|_{V^0}^2 - \|u^{l-1}\|_{V^0}^2\big)^2 + \|u^l - u^{l-1}\|_{V^0}^2\|u^l\|_{V^0}^2 \\
&\quad + \big(\|\theta^l\|_{H^0}^2 - \|\theta^{l-1}\|_{H^0}^2\big)^2 + \|\theta^l - \theta^{l-1}\|_{H^0}^2\|\theta^l\|_{H^0}^2\Big\} \\
&\quad + 4h\sum_{l=1}^{L}\Big(\nu\|A^{\frac12}u^l\|_{V^0}^2\|u^l\|_{V^0}^2 + \kappa\|A^{\frac12}\theta^l\|_{H^0}^2\|\theta^l\|_{H^0}^2\Big)\Big] \\
&\le \mathbb{E}\big(\|u_0\|_{V^0}^4 + \|\theta_0\|_{H^0}^4\big) + 3h\,\mathbb{E}\big(\|u^L\|_{V^0}^4\big) + C + Ch\sum_{l=0}^{L-1}\mathbb{E}\big(\|u^l\|_{V^0}^4 + \|\theta^l\|_{H^0}^4\big)
\end{aligned}$$

for some constant C depending on K_i, $\tilde{K}_i$, $\mathrm{Tr}(Q)$, $\mathrm{Tr}(\tilde{Q})$ and T that does not depend on N. Let N be large enough to have $3h < \frac{1}{2}$. Neglecting the sums in the left hand side and using the discrete Gronwall lemma, we deduce, for $\mathbb{E}\big[\|u_0\|_{V^0}^4 + \|\theta_0\|_{H^0}^4\big] < \infty$, that

$$\sup_{N\ge 1}\max_{0\le L\le N}\mathbb{E}\Big[\|u^L\|_{V^0}^4 + \|\theta^L\|_{H^0}^4\Big] < \infty. \tag{86}$$

This yields

$$\sup_{N\ge 1}\mathbb{E}\,h\sum_{l=1}^{N}\Big[\|A^{\frac12}u^l\|_{V^0}^2\|u^l\|_{V^0}^2 + \|A^{\frac12}\theta^l\|_{H^0}^2\|\theta^l\|_{H^0}^2\Big] < \infty, \tag{87}$$

which proves (73) for $K = 2$. The argument used to prove (78) implies

$$\begin{aligned}
&\mathbb{E}\Big[\max_{1\le L\le N}\sum_{l=1}^{L}\big(G(u^{l-1})\Delta_l W, u^{l-1}\big)\|u^{l-1}\|_{V^0}^2\Big] \\
&\quad \le \epsilon\,\mathbb{E}\Big[\max_{1\le L\le N}\|u^L\|_{V^0}^4\Big] + C(\epsilon)\Big[1 + \max_{1\le L\le N}\mathbb{E}\big(\|u^L\|_{V^0}^4\big)\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\max_{1\le L\le N}\sum_{l=1}^{L}\big(\tilde{G}(\theta^{l-1})\Delta_l \tilde{W}, \theta^{l-1}\big)\|\theta^{l-1}\|_{H^0}^2\Big] \\
&\quad \le \epsilon\,\mathbb{E}\Big[\max_{1\le L\le N}\|\theta^L\|_{H^0}^4\Big] + C(\epsilon)\Big[1 + \max_{1\le L\le N}\mathbb{E}\big(\|\theta^L\|_{H^0}^4\big)\Big]
\end{aligned}$$

Taking the maximum for $L = 1, \dots, N$ and using (86), we deduce (72) for $K = 2$. The details of the induction step, similar to the proof in the case $K = 2$, are left to the reader. □

7. Strong Convergence of the Localized Implicit Time Euler Scheme

Due to the bilinear terms $[u.\nabla]u$ and $[u.\nabla]\theta$, we first prove an $L^2(\Omega)$ convergence of the $L^2(D)$-norm of the error, uniformly on the time grid, restricted to the set $\Omega_M(N)$ defined below for some $M > 0$:

$$\Omega_M(j) := \Big\{\sup_{s\in[0,t_j]}\|A^{\frac12}u(s)\|_{V^0}^2 \le M\Big\} \cap \Big\{\sup_{s\in[0,t_j]}\|\tilde{A}^{\frac12}\theta(s)\|_{H^0}^2 \le M\Big\}, \quad \forall j = 0,\dots,N, \tag{88}$$

and let $\Omega_M := \Omega_M(N)$. Recall that, for $j = 0,\dots,N$, set $e_j := u(t_j) - u^j$ and $\tilde{e}_j := \theta(t_j) - \theta^j$; then, $e_0 = \tilde{e}_0 = 0$. Using (9), (10), (17) and (18), we deduce, for $j = 1,\dots,N$, $\varphi \in V^1$ and $\psi \in H^1$, that

$$\begin{aligned}\big(e_j - e_{j-1}, \varphi\big) &+ \nu \int_{t_{j-1}}^{t_j} \big(A^{\frac{1}{2}}[u(s) - u^j], A^{\frac{1}{2}}\varphi\big) ds + \int_{t_{j-1}}^{t_j} \big\langle B(u(s),u(s)) - B(u^j,u^j), \varphi\big\rangle ds \\ &= \int_{t_j}^{t_{j+1}} \big(P[\theta(s) - \theta^{j-1}]v_2, \varphi\big) ds + \int_{t_{j-1}}^{t_j} \big([G(u(s)) - G(u^{j-1})]dW(s), \varphi\big), \end{aligned} \tag{89}$$

and

$$\begin{aligned}\big(\tilde{e}_j - \tilde{e}_{j-1}, \psi\big) &+ \kappa \int_{t_{j-1}}^{t_j} \big(\tilde{A}^{\frac{1}{2}}[\theta(s) - \theta^j], \tilde{A}^{\frac{1}{2}}\psi\big) ds + \int_{t_{j-1}}^{t_j} \big\langle [u(s).\nabla]\theta(s) - [u^j\cdot\nabla]\theta^j], \psi\big\rangle ds \\ &= \int_{t_{j-1}}^{t_j} \big([\tilde{G}(\theta(s)) - \tilde{G}(\theta^{j-1})]d\tilde{W}(s), \psi\big). \end{aligned} \tag{90}$$

In this section, we will suppose that N is large enough to have $h := \frac{T}{N} \in (0,1)$. The following result is a crucial step towards the rate of convergence of the implicit time Euler scheme.

Proposition 7. Suppose that the conditions **(C-u)** and **(C-θ)** hold. Let $u_0 \in L^{32+\epsilon}(\Omega; V^1)$ and $\theta_0 \in L^{32+\epsilon}(\Omega; H^1)$ for some $\epsilon > 0$, u, θ be the solution to (9) and (10) and $\{u^j, \theta^j\}_{j=0,\dots,N}$ be the solution to (17) and (18). Fix $M > 0$ and let $\Omega_M = \Omega_M(N)$ be defined by (88). Then, for $h \in (0,1)$, there exists a positive constant C, independent of N, such that, for large enough N,

$$\begin{aligned}\mathbb{E}\Big(1_{\Omega_M}\Big[\max_{1\le j\le N} \|u(t_j) - u^j\|^2_{V^0} + \|\theta(t_j) - \theta^j\|^2_{H^0} + \frac{T}{N}\sum_{j=1}^N \|A^{\frac{1}{2}}[u(t_j) - u^j]\|^2_{V^0} \\ + \|\tilde{A}^{\frac{1}{2}}[\theta(t_j) - \theta^j]\|^2_{H^0}\Big]\Big) \le C(1+M)e^{C(M)T}\,\frac{h}{N}, \end{aligned} \tag{91}$$

where

$$C(M) = \frac{9(1+\gamma)\bar{C}_4^2}{8} \max\Big(\frac{5}{\nu}, \frac{1}{\kappa}\Big) M$$

for some $\gamma > 0$, and $\bar{C}_4$ is the constant in the right hand side of the Gagliardo–Nirenberg inequality (6).

Proof. Write (89) with $\varphi = e_j$ and (90) with $\psi = \theta^j$; using the equality $(f, f - g) = \frac{1}{2}\big[\|f\|^2_{L^2} - \|g\|^2_{L^2} + \|f - g\|^2_{L^2}\big]$, we obtain for $j = 1,\dots,N$

$$\frac{1}{2}\Big(\|e_j\|^2_{V^0} - \|e_{j-1}\|^2_{V^0} + \|e_j - e_{j-1}\|^2_{V^0}\Big) + \nu h \|A^{\frac{1}{2}}e_j\|^2_{V^0} \le \sum_{l=1}^{7} T_{j,l}, \tag{92}$$

$$\frac{1}{2}\Big(\|\tilde{e}_j\|^2_{H^0} - \|\tilde{e}_{j-1}\|^2_{H^0} + \|\tilde{e}_j - \tilde{e}_{j-1}\|^2_{H^0}\Big) + \kappa h \|\tilde{A}^{\frac{1}{2}}\tilde{e}_j\|^2_{H^0} \le \sum_{l=1}^{6} \tilde{T}_{j,l}, \tag{93}$$

where, by the antisymmetry property (3), we have that

$$T_{j,1} = \int_{t_{j-1}}^{t_j} \langle B(e_j, u^j), e_j \rangle ds = \int_{t_{j-1}}^{t_j} \langle B(e_j, u(t_j)), e_j \rangle ds,$$
$$T_{j,2} = \int_{t_{j-1}}^{t_j} \langle B(u(s) - u(t_j), u(t_j)), e_j \rangle ds,$$
$$T_{j,3} = \int_{t_{j-1}}^{t_j} \langle B(u(s), u(s) - u(t_j)), e_j \rangle ds = -\int_{t_{j-1}}^{t_j} \langle B(u(s), e_j), u(s) - u(t_j) \rangle ds,$$
$$T_{j,4} = -\nu \int_{t_{j-1}}^{t_j} \langle A^{\frac12}(u(s) - u(t_j)), A^{\frac12} e_j \rangle ds, \quad T_{j,5} = \int_{t_{j-1}}^{t_j} \langle \Pi[(\theta(s) - \theta^{j-1}) v_2], e_j \rangle ds,$$
$$T_{j,6} = \Big\langle \int_{t_{j-1}}^{t_j} [G(u(s)) - G(u^{j-1})] dW(s), e_j - e_{j-1} \Big\rangle,$$
$$T_{j,7} = \Big\langle \int_{t_{j-1}}^{t_j} [G(u(s)) - G(u^{j-1})] dW(s), e_{j-1} \Big\rangle$$

and

$$\tilde{T}_{j,1} = -\int_{t_{j-1}}^{t_j} \langle [e_{j-1}.\nabla]\theta^j, \tilde{e}_j \rangle ds = -\int_{t_{j-1}}^{t_j} \langle [e_{j-1}.\nabla]\theta(t_j), \tilde{e}_j \rangle ds,$$
$$\tilde{T}_{j,2} = -\int_{t_{j-1}}^{t_j} \langle [(u(s) - u(t_{j-1})).\nabla]\theta(t_j), \tilde{e}_j \rangle ds,$$
$$\tilde{T}_{j,3} = -\int_{t_{j-1}}^{t_j} \langle [u(s).\nabla](\theta(s) - \theta(t_j)), \tilde{e}_j \rangle ds = \int_{t_{j-1}}^{t_j} \langle [u(s).\nabla]\tilde{e}_j, (\theta(s) - \theta(t_j)) \rangle ds,$$
$$\tilde{T}_{j,4} = -\kappa \int_{t_{j-1}}^{t_j} \langle \tilde{A}^{\frac12}(\theta(s) - \theta(t_j)), \tilde{A}^{\frac12} \tilde{e}_j \rangle ds,$$
$$\tilde{T}_{j,5} = \Big\langle \int_{t_{j-1}}^{t_j} [\tilde{G}(\theta(s)) - \tilde{G}(\theta^{j-1})] d\tilde{W}(s), \tilde{e}_j - \tilde{e}_{j-1} \Big\rangle,$$
$$\tilde{T}_{j,6} = \Big\langle \int_{t_{j-1}}^{t_j} [G(u(s)) - G(u^{j-1})] dW(s), e_{j-1} \Big\rangle.$$

We next prove upper estimates of the terms $T_{j,l}$ for $l = 1, \dots, 5$ and $\tilde{T}_{j,l}$ for $l = 1, \dots, 4$, and of the expected value of $T_{j,6}$, $T_{j,7}$ $\tilde{T}_{j,5}$ and $\tilde{T}_{j,6}$.

The Hölder and Young inequalities and the Gagliardo–Nirenberg inequality (6) imply, for $d_1 > 0$, that

$$\begin{aligned} |T_{j,1}| &\le \bar{C}_4 \tau \|e_j\|_{V^0} \|A^{\frac12} e_j\|_{V^0} \|A^{\frac12} u(t_j)\|_{V^0} \\ &\le d_1 \nu \tau \|A^{\frac12} e_j\|_{V^0}^2 + \frac{\bar{C}_4^2}{4 d_1 \nu} \tau \|A^{\frac12} u(t_j)\|_{V^0}^2 \|e_j\|_{V^0}^2, \end{aligned} \tag{94}$$

and, for $\tilde{d}_1, d_2 > 0$, that

$$\begin{aligned} |\tilde{T}_{j,1}| &\le \bar{C}_4 \tau \|A^{\frac12} e_{j-1}\|_{V^0}^{\frac12} \|e_{j-1}\|_{V^0}^{\frac12} \|\tilde{A}^{\frac12} \tilde{e}_j\|_{H^0}^{\frac12} \|\tilde{e}_j\|_{H^0}^{\frac12} \|\tilde{A}^{\frac12} \theta(t_j)\|_{H^0} \\ &\le d_2 \nu \tau \|A^{\frac12} e_{j-1}\|_{V^0}^2 + d_1 \tau \kappa \|\tilde{A}^{\frac12} \tilde{e}_j\|_{H^0}^2 \\ &\quad + \frac{\bar{C}_4^2}{16 d_2 \nu} \tau \|\tilde{A}^{\frac12} \theta(t_j)\|_{H^0}^2 \|e_{j-1}\|_{V^0}^2 + \frac{\bar{C}_4^2}{16 \tilde{d}_1 \kappa} \tau \|\tilde{A}^{\frac12} \theta(t_j)\|_{H^0}^2 \|\tilde{e}_j\|_{H^0}^2. \end{aligned} \tag{95}$$

Hölder's inequality and the Sobolev embedding $V^1 \subset L^4$ imply, for $d_3 > 0$, that

$$|T_{j,2}| \le C \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{V^1} \|A^{\frac12} u(t_j)\|_{V^0} \|A^{\frac12} e_j\|_{V^0}^{\frac12} \|e_j\|_{V^0}^{\frac12}\, ds$$

$$\le d_3 n \left(h \|A^{\frac12} e_j\|_{V^0}^2 + h \|e_j\|_{V^0}^2 \right) + \frac{C}{\sqrt{n}\, d_3} \|A^{\frac12} u(t_j)\|_{V^0}^2 \int_{t_{j-1}}^{t_j} \|u(t_j) - u(s)\|_{V^1}^2\, ds, \tag{96}$$

whereas, for $\tilde d_2 > 0$,

$$|\tilde T_{j,2}| \le \tilde d_2 \kappa \left(h \|A^{\frac12} \tilde e_j\|_{H^0}^2 + h \|e_j\|_{H^0}^2 \right) + \frac{C}{\kappa \tilde d_2} \left(\int_{t_{j-1}}^{t_j} \|A^{\frac12} u(s) - u(t_{j-1})\|_{V^0}^2\, ds \right)^2$$

$$+ \frac{C}{\kappa \tilde d_2} \|A^{\frac12} q(t_j)\|_{H^0}^4 \int_{t_{j-1}}^{t_j} \|u(s) - u(t_{j-1})\|_{V^0}^2\, ds. \tag{97}$$

Similar arguments prove, for $d_4, \tilde d_3 > 0$, that

$$|T_{j,3}| \le d_4 n h \|A^{\frac12} e_j\|_{V^0}^2 + \frac{C}{n d_4} \sup_{s \in [0,T]} \|u(s)\|_{V^1}^2 \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{V^1}^2\, ds, \tag{98}$$

$$|\tilde T_{j,3}| \le \tilde d_3 \kappa h \|A^{\frac12} \tilde e_j\|_{H^0}^2 + \frac{C}{\kappa \tilde d_3} \sup_{s \in [0,T]} \|u(s)\|_{V^1}^2 \int_{t_{j-1}}^{t_j} \|q(s) - q(t_j)\|_{H^1}^2\, ds. \tag{99}$$

The Cauchy–Schwarz and Young inequalities imply, for $d_5, \tilde d_4 > 0$, that

$$|T_{j,4}| \le d_5 n h \|A^{\frac12} e_j\|_{V^0}^2 + \frac{n}{4 d_5} \int_{t_{j-1}}^{t_j} \|A^{\frac12} [u(s) - u(t_j)]\|_{V^0}^2\, ds, \tag{100}$$

$$|\tilde T_{j,4}| \le \tilde d_4 \kappa h \|A^{\frac12} \tilde e_j\|_{H^0}^2 + \frac{\kappa}{4 \tilde d_4} \int_{t_{j-1}}^{t_j} \|A^{\frac12} [q(s) - q(t_j)]\|_{H^0}^2\, ds. \tag{101}$$

Using once more the Cauchy–Schwarz and Young inequalities, we deduce

$$|T_{j,5}| \le \int_{t_{j-1}}^{t_j} \|q(s) - q(t_{j-1})\|_{H^0} \, \|\tilde e_{j-1}\|_{H^0} \, \|e_j\|_{V^0}\, ds$$

$$\le \frac{h}{2} \|e_j\|_{V^0}^2 + \frac{h}{2} \|\tilde e_{j-1}\|_{H^0}^2 + \frac{1}{2} \int_{t_{j-1}}^{t_j} \|q(s) - q(t_{j-1})\|_{H^0}^2\, ds. \tag{102}$$

Note that the sequence of subsets $\{W_M(j)\}_{0 \le j \le N}$ is decreasing. Therefore, since $e_0 = \tilde e_0 = 0$, given $L = 1, \dots, N$, we obtain

$$\max_{1 \le J \le L} \sum_{j=1}^{J} 1_{W_M(j-1)} \left(\|e_j\|_{V^0}^2 - \|e_{j-1}\|_{V^0}^2 + \|\tilde e_j\|_{H^0}^2 - \|\tilde e_{j-1}\|_{H^0}^2 \right)$$

$$= \max_{1 \le J \le L} \Big[1_{W_M(J-1)} \left(\|e_J\|_{V^0}^2 + \|\tilde e_J\|_{H^0}^2 \right) + \sum_{j=2}^{J} \left(1_{W_M(j-2)} - 1_{W_M(j-1)} \right) \left(\|e_{j-1}\|_{V^0}^2 + \|\tilde e_{j-1}\|_{H^0}^2 \right) \Big]$$

$$\ge \max_{1 \le J \le L} 1_{W_M(J-1)} \left(\|e_J\|_{V^0}^2 + \|\tilde e_J\|_{H^0}^2 \right).$$

Hence, for $\sum_{j=1}^{5} d_j \le \frac13$ and $\sum_{j=1}^{4} \tilde d_j < \frac13$, using Young's inequality, we deduce, for every $a > 0$, that

$$
\begin{aligned}
&\le \frac{1}{6}\max_{1\le J\le L} 1_{\Omega_M(J-1)}\big(\|e_J\|_{V^0}^2+\|\tilde e_J\|_{H^0}^2\big) \\
&\quad+\frac{1}{6}\sum_{j=1}^{L} 1_{\Omega_M(j-1)}\big(\|e_j-e_{j-1}\|_{V^0}^2+\|\tilde e_j-\tilde e_{j-1}\|_{H^0}^2\big) \\
&\quad+\sum_{j=1}^{L}1_{\Omega_M(j-1)}\, h\Big[\nu\,\frac13\sum_{i=1}^{5}\delta_i\,\|A^{\frac12}e_j\|_{V^0}^2+\kappa\,\frac13\sum_{i=1}^{4}\tilde\delta_i\,\|\tilde A^{\frac12}\tilde e_j\|_{H^0}^2\Big] \\
&\quad+h\sum_{j=1}^{L}1_{\Omega_M(j-1)}\|e_j\|_{V^0}^2\Big(\frac{(1+\alpha)C^2}{4\delta_1\nu}\|A^{\frac12}u(t_{j-1})\|_{V^0}^2+\frac{(1+\alpha)\bar C^2}{16\delta_2\nu}\|\tilde A^{\frac12}\theta(t_{j-1})\|_{H^0}^2+\frac{3}{2}\Big) \\
&\quad+h\sum_{j=1}^{L}1_{\Omega_M(j-1)}\|\tilde e_j\|_{H^0}^2\,\frac{(1+\alpha)\bar C^2}{16\tilde\delta_1\kappa}\|\tilde A^{\frac12}\theta(t_{j-1})\|_{H^0}^2+\frac{3}{2}Z_L \\
&\quad+\max_{1\le J\le L}\sum_{j=1}^{J}1_{\Omega_M(j-1)}\big(T_{j,6}+\tilde T_{j,5}\big)+\max_{1\le J\le L}\sum_{j=1}^{J}1_{\Omega_M(j-1)}\big(T_{j,7}+\tilde T_{j,6}\big),
\end{aligned}
\tag{103}
$$

where

$$
\begin{aligned}
Z_L &= C\,h\sum_{j=1}^{L}\|e_j\|_{V^0}^2\Big(\|A^{\frac12}[u(t_j)-u(t_{j-1})]\|_{V^0}^2+\|\tilde A^{\frac12}[\theta(t_j)-\theta(t_{j-1})]\|_{H^0}^2\Big) \\
&\quad+C\,h\sum_{j=1}^{L}\|\tilde e_j\|_{V^0}^2\,\|\tilde A^{\frac12}[\theta(t_j)-\theta(t_{j-1})]\|_{H^0}^2 \\
&\quad+C\sum_{j=1}^{L}\sup_{s\in[0,T]}\big(1+\|u(s)\|_{V^1}^2\big)\int_{t_{j-1}}^{t_j}\big(\|u(t_j)-u(s)\|_{V^1}^2+\|u(s)-u(t_{j-1})\|_{V^1}^2\big)\,ds \\
&\quad+C\sum_{j=1}^{L}\|\tilde A^{\frac12}\theta(t_{j-1})\|_{H^0}^4\int_{t_{j-1}}^{t_j}\|u(s)-u(t_{j-1})\|_{V^0}^2\,ds \\
&\quad+C\sum_{j=1}^{L}\sup_{s\in[0,T]}\big(1+\|u(s)\|_{V^1}^2\big)\int_{t_{j-1}}^{t_j}\|\theta(s)-\theta(t_j)\|_{H^1}^2\,ds.
\end{aligned}
\tag{104}
$$

The Cauchy–Schwarz and Young inequalities imply that

$$
\begin{aligned}
\sum_{j=1}^{L}1_{\Omega_M(j-1)}|T_{j,6}| &\le \frac16\sum_{j=1}^{L}1_{\Omega_M(j-1)}\|e_j-e_{j-1}\|_{V^0}^2 \\
&\quad+\frac32\sum_{j=1}^{L}1_{\Omega_M(j-1)}\Big\|\int_{t_{j-1}}^{t_j}\big(G(u(s))-G(u^{j-1})\big)dW(s)\Big\|_{V^0}^2,
\end{aligned}
\tag{105}
$$

$$
\begin{aligned}
\sum_{j=1}^{L}1_{\Omega_M(j-1)}|\tilde T_{j,5}| &\le \frac16\sum_{j=1}^{L}1_{\Omega_M(j-1)}\|\tilde e_j-\tilde e_{j-1}\|_{H^0}^2 \\
&\quad+\frac32\sum_{j=1}^{L}1_{\Omega_M(j-1)}\Big\|\int_{t_{j-1}}^{t_j}\big(\tilde G(\theta(s))-\tilde G(\theta^{j-1})\big)d\tilde W(s)\Big\|_{H^0}^2.
\end{aligned}
\tag{106}
$$

Using the upper estimates (103)–(106), taking expected values and using the Cauchy–Schwarz and Young inequalities, as well as the inequalities (19), (20), (37), (46), (59), (60) and (72), we deduce that, for $h\in(0,1)$ and every $L=1,\dots,N$,

$$
\begin{aligned}
E(Z_L) \le\; & C\Big\{E\Big[\sup_{s\in[0,T]}\|u(s)\|^4_{V^0} + \max_{0\le j\le N}\|u^j\|^4_{V^0}\Big]\Big\}^{\frac12} \Big\{E\Big[h\sum_{j=1}^N \Big(\|A^{\frac12}(u(t_j)-u(t_{j-1}))\|^2_{V^0} + \|\tilde{A}^{\frac12}(q(t_j)-q(t_{j-1}))\|^2_{V^0}\Big)\Big]^2\Big\}^{\frac12} \\
&+ C\Big\{E\Big[\sup_{s\in[0,T]}\|q(s)\|^4_{H^0} + \max_{0\le j\le N}\|q^j\|^4_{H^0}\Big]\Big\}^{\frac12} \Big\{E\Big[h\sum_{j=1}^N \|A^{\frac12}(q(t_j)-q(t_{j-1}))\|^2_{V^0}\Big]^2\Big\}^{\frac12} \\
&+ C\Big\{E\Big[1+\sup_{0\le s\le T}\|u(s)\|^4_{V^1}\Big]\Big\}^{\frac12}\Big\{E\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\Big[\|u(s)-u(t_j)\|^2_{V^0} \\
&\qquad + \|u(s)-u(t_{j-1})\|^2_{V^0} + \|q(s)-q(t_j)\|^2_{H^0}\Big]^2 ds\Big\}^{\frac12} \\
&+ C\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\Big\{E\|\tilde{A}^{\frac12}q(t_j)\|^8_{H^0}\Big\}^{\frac12}\Big\{E\|u(s)-u(t_{j-1})\|^4_{V^0}\Big\}^{\frac12} ds \le C h^{\eta},
\end{aligned}
\tag{107}
$$

for some constant C independent of L and N. Furthermore, the Lipschitz conditions (12) and (15), the inclusion $W_M(j-1) \subset W_M(j-2)$ for $j = 2, \ldots, N$ and the upper estimates (46) and (47) imply that

$$
\begin{aligned}
E\Big\|\sum_{j=1}^L 1_{W_M(j-1)}\int_{t_{j-1}}^{t_j}\big[G(u(s)) - G(u^{j-1})\big]dW(s)\Big\|^2_{V^0}
&\le \sum_{j=1}^L E\int_{t_{j-1}}^{t_j} 1_{W_M(j-1)} L_1 \|u(s)-u^{j-1}\|^2_{V^0}\,\mathrm{Tr}(Q)\,ds \\
&\le 2L_1\mathrm{Tr}(Q)\,h\sum_{j=2}^L E\big(1_{W_M(j-2)}\|e_{j-1}\|^2_{V^0}\big) + C\sum_{j=1}^L E\int_{t_{j-1}}^{t_j}\|u(s)-u(t_{j-1})\|^2_{V^0}ds \\
&\le 2L_1\mathrm{Tr}(Q)\,h\sum_{j=2}^L E\big(1_{W_M(j-2)}\|e_{j-1}\|^2_{V^0}\big) + Ch,
\end{aligned}
\tag{108}
$$

$$
E\Big\|\sum_{j=1}^L 1_{W_M(j-1)}\int_{t_{j-1}}^{t_j}\big[\tilde{G}(q(s)) - \tilde{G}(q^{j-1})\big]d\tilde{W}(s)\Big\|^2_{H^0}
\le 2\tilde{L}_1\mathrm{Tr}(\tilde{Q})\,h\sum_{j=2}^L E\big(1_{W_M(j-2)}\|\tilde{e}_{j-1}\|^2_{H^0}\big) + Ch.
\tag{109}
$$

Finally, the Davis inequality, the inclusion $W_M(J-1)\subset W_M(j-1)$ for $j\le J$, the local property of stochastic integrals, the Lipschitz condition (12), the Cauchy–Schwarz and Young inequalities and the upper estimate (46) imply, for $\lambda>0$, that

$$\begin{aligned}
E\Big(\max_{1\le J\le L} 1_{W_M(J-1)}\sum_{j=1}^J T_{j,7}\Big)
&\le 3\sum_{j=1}^L E\Big(1_{W_M(j-1)}\Big\{\int_{t_{j-1}}^{t_j}\|G(u(s))-G(u^{j-1})\|^2_{\mathcal{L}(K;V^0)}\mathrm{Tr}(Q)\|e_{j-1}\|^2_{V^0}\,ds\Big\}^{\frac12}\Big)\\
&\le 3\sum_{j=1}^L E\Big[\max_{1\le j\le L}1_{W_M(j-1)}\|e_{j-1}\|_{V^0}\Big\{\int_{t_{j-1}}^{t_j}L_1\mathrm{Tr}(Q)\|u(s)-u^{j-1}\|^2_{V^0}\,ds\Big\}^{\frac12}\Big]\\
&\le \lambda E\Big(\max_{1\le j\le L}1_{W_M(j-1)}\|e_{j-1}\|^2_{V^0}\Big)+CE\sum_{j=1}^L\int_{t_{j-1}}^{t_j}L_1\mathrm{Tr}(Q)\|u(s)-u^{j-1}\|^2_{V^0}\,ds\\
&\le \lambda E\Big(\max_{1\le j\le L}1_{W_M(j-2)}\|e_{j-1}\|^2_{V^0}\Big)+Ch\sum_{j=1}^L E\big(\|e_{j-1}\|^2_{V^0}\big)+Ch.
\end{aligned}\tag{110}$$

A similar argument, using the Lipschitz condition (15) and (47), yields, for $\lambda>0$,

$$\begin{aligned}
E\Big(\max_{1\le J\le L}1_{W_M(J-1)}\sum_{j=1}^J\tilde T_{j,7}\Big)
\le \lambda E\Big(\max_{1\le j\le L}1_{W_M(j-2)}\|\tilde e_{j-1}\|^2_{H^0}\Big)+Ch\sum_{j=1}^L E\big(\|\tilde e_{j-1}\|^2_{V^0}\big)+Ch.
\end{aligned}\tag{111}$$

Collecting the upper estimates (94)–(111), we obtain, for $\sum_{i=1}^5\delta_i<\frac13$, $\sum_{i=1}^4\tilde\delta_i<\frac13$, $h\in(0,1)$ and $\alpha,\lambda>0$,

$$\begin{aligned}
&E\Big(\max_{1\le J\le N}1_{W_M(J-1)}\big[\|e_J\|^2_{V^0}+\|\tilde e_J\|^2_{H^0}\big]\Big)\\
&\quad+E\sum_{j=1}^N 1_{W_M(j-1)}h\Big[\nu\Big(2-6\sum_{i=1}^5\delta_i\Big)\|A^{\frac12}e_j\|^2_{V^0}+k\Big(2-6\sum_{i=1}^4\tilde\delta_i\Big)\|\tilde A^{\frac12}\tilde e_j\|^2_{V^0}\Big]\\
&\le h\sum_{j=1}^{N-1}E1_{W_M(j-1)}\|e_j\|^2_{V^0}\Big[\frac{3(1+\alpha)\bar C_4^2}{2\nu\delta_1}+\frac{1}{4\delta_2}M+C\Big]\\
&\quad+h\sum_{j=1}^{N-1}E\,1_{W_M(j-1)}\|\tilde e_j\|^2_{H^0}\Big[\frac{3(1+\alpha)\bar C_4^2}{8\tilde\delta_1 k}M+C\Big]\\
&\quad+C(1+M)hE\Big[\sup_{t\in[0,T]}\big(\|u(t)\|^2_{V^0}+\|q(t)\|^2_{H^0}\big)+\max_{1\le j\le N}\big(\|u^j\|^2_{V^0}+\|q^j\|^2_{H^0}\big)\Big]\\
&\quad+12\lambda E\Big(\max_{1\le j\le N}1_{W_M(j-1)}\big[\|e_{j-1}\|^2_{V^0}+\|\tilde e_j\|^2_{H^0}\big]\Big)+Ch^{h}.
\end{aligned}\tag{112}$$

Therefore, given $\gamma\in(0,1)$, choosing $\lambda\in(0,\frac{1}{12})$ and $\alpha>0$ such that $\frac{1+\alpha}{1-12\lambda}<1+\gamma$, neglecting the sum in the left hand side and using the discrete Gronwall lemma, we deduce, for $h\in(0,1)$, that

$$E\Big(\max_{1\le j\le N}1_{W_M(j-1)}\big[\|e_j\|^2_{V^0}+\|\tilde e_j\|^2_{H^0}\big]\Big)\le C(1+M)e^{TC(M)}h^{h},\tag{113}$$

where

$$C(M):=\frac{3(1+\gamma)\bar C_4^2}{2}\max\Big(\frac{1}{\delta_1\nu}+\frac{1}{4\delta_2\nu},\ \frac{1}{4\tilde\delta_1 k}\Big)M,$$

for $\sum_{i=1}^{2} \delta_i < \frac{1}{3}$ and $\tilde\delta_1 < \frac{1}{3}$ (and choosing δ_i, $i = 3,4,5$ and $\tilde\delta_i$, $i = 2,3,4$ such that $\sum_{i=1}^{5} \delta_i < \frac{1}{3}$ and $\sum_{i=1}^{4} \tilde\delta_i < \frac{1}{3}$). Let $\delta_2 < \frac{1}{15}$ and $\delta_1 = 4\delta_2$. Then, for some $\gamma > 0$, we have that

$$C(M) = \frac{9(1+\gamma)C_4^2}{8} \max\Big\{\frac{5}{\kappa}, 1\Big\} M.$$

Plugging the upper estimate (113) in (112), we conclude the proof of (91). □

8. Rate of Convergence in Probability and in $L^2(\Omega)$

In this section, we deduce from Proposition 7 the convergence in probability of the implicit time Euler scheme with the "optimal" rate of convergence of "almost 1/2" and a logarithmic speed of convergence in $L^2(\Omega)$. The presence of the bilinear term in the Itô formula for $\|\tilde A^{\frac{1}{2}} q(t)\|^2_{H^0}$ does not enable us to prove exponential moments for this norm, which prevents us from using the general framework presented in [10] to prove a polynomial rate for the strong convergence.

8.1. Rate of Convergence in Probability

In this section, we deduce the rate of the convergence in probability (defined in [17]) from Propositions 1, 2, 6 and 7.

Proof of Theorem 2. For $N \geq 1$ and $\eta \in (0,1)$, let

$$A(N,\eta) := \Big\{ \max_{1\leq J\leq N} \big(\|e_J\|^2_{V^0} + \|\tilde e_J\|^2_{H^0} \big) + \frac{T}{N}\sum_{j=1}^{N} \big(\|A^{\frac{1}{2}} e_j\|^2_{V^0} + \|\tilde A^{\frac{1}{2}} \tilde e_j\|^2_{H^0} \big) \geq N^{-\eta} \Big\}.$$

Let $\tilde\eta \in (\eta, 1)$, $M(N) = \ln(\ln N)$ for $N \geq 3$. Then,

$$P\big(A(N,\eta)\big) \leq P\big(A(N,\eta) \cap \Omega_{M(N)}\big) + P\big((\Omega_{M(N)})^c\big),$$

where $\Omega_{M(N)} = \Omega_{M(N)}(N)$ is defined in Proposition 7. The inequality (91) implies that

$$\begin{aligned}
P\big(A(N,\eta) \cap \Omega_{M(N)}\big)
&\leq N^{\eta}\, \mathbb{E}\Big[1_{\Omega_{M(N)}} \Big(\max_{1\leq J\leq N} \|e_J\|^2_{V^0} + \|\tilde e_J\|^2_{H^0} + \frac{T}{N}\sum_{j=1}^{N} \|A^{\frac{1}{2}} e_j\|^2_{V^0} + \|\tilde A^{\frac{1}{2}} \tilde e_j\|^2_{H^0} \Big) \Big] \\
&\leq N^{\eta}\, C\big(1+\ln(\ln N)\big) e^{TC\ln(\ln N)} \Big(\frac{T}{N}\Big)^{\tilde\eta} \\
&\leq C\big(1+\ln(\ln N)\big) (\ln N)^{CT} N^{-\tilde\eta+\eta} \to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

The inequalities (20)–(22) imply that

$$P\big((\Omega_{M(N)})^c\big) \leq \frac{1}{M(N)} \mathbb{E}\Big(\sup_{t\in[0,T]} \|u(t)\|^2_{V^1} + \sup_{t\in[0,T]} \|q(t)\|^2_{H^1} \Big) \to 0 \quad \text{as } N \to \infty.$$

The two above convergence results complete the proof of (23). □

8.2. Rate of Convergence in $L^2(\Omega)$

We finally prove the strong rate of convergence, which is also a consequence of Propositions 1, 2, 6 and 7.

Proof of Theorem 3. For any integer $N \ge 1$ and $M \in [1,+\infty)$, let $W_M = W_M(N)$ be defined by (88). Let p be the conjugate exponent of 2^q. Hölder's inequality implies that

$$E\Big(1_{(W_M)^c}\max_{1\le j\le N}\big(\|e_j\|_V^2+\|e_j\|_H^2\big)\Big)\le \Big\{P\big((W_M)^c\big)\Big\}^{\frac{1}{p}}$$
$$\Big\{E\Big(\sup_{s\in[0,T]}\|u(s)\|_V^{2^q}+\sup_{s\in[0,T]}\|q(s)\|_H^{2^q}+\max_{1\le j\le N}\|u^j\|_V^{2^q}+\max_{1\le j\le N}\|q^j\|_H^{2^q}\Big)\Big\}^{2^{-q}}$$
$$\le C\Big\{P\big((W_M)^c\big)\Big\}^{\frac{1}{p}}, \tag{114}$$

where the last inequality is a consequence of (19), (20) and (72).

Using (21) and (22), we deduce that

$$P\big((W_M)^c\big)\le M^{-2^{q-1}}E\Big(\sup_{s\in[0,T]}\|u(s)\|_V^{2^q}+\sup_{s\in[0,T]}\|q(s)\|_H^{2^q}\Big)\le C M^{-2^{q-1}}. \tag{115}$$

Using (91), we choose $M(N)\to\infty$ as $N\to\infty$ such that, for $h\in(0,1)$ and $g>0$,

$$N^{-h}\exp\Big[\frac{9(1+g)\bar{C}_4^2T}{8}\Big(\frac{5}{n}-\frac{1}{k}\Big)M(N)\Big]M(N)\le M(N)^{-2^{q-1}}$$

which, taking logarithms, yields

$$-h\ln(N)+\frac{9(1+g)\bar{C}_4^2}{8}T\Big(\frac{5}{n}-\frac{1}{k}\Big)M(N)\le -2^{q-1}\ln(M(N)).$$

Set

$$M(N)=\frac{8}{9(1+g)\bar{C}_4^2\big(\frac{5}{n}-\frac{1}{k}\big)T}\Big[h\ln(N)-\big(2^{q-1}+1\big)\ln\big(\ln(N)\big)\Big]$$
$$\sim\frac{8}{9(1+g)\bar{C}_4^2\big(\frac{5}{n}-\frac{1}{k}\big)T}\,h\ln(N).$$

Then,

$$-h\ln(N)+\frac{9(1+g)\bar{C}_4^2}{8}T\Big(\frac{5}{n}-\frac{1}{k}\Big)M(N)+\ln(M(N))\le-\big(2^{q-1}+1\big)\ln\big(\ln(N)\big)+O(1),$$
$$\le-\big(2^{q-1}+1\big)\ln\big(M(N)\big)\le-\big(2^{q-1}+1\big)\ln\ln(N)+O(1).$$

This implies that

$$E\Big(1_{(W_M)^c}\max_{1\le j\le N}\big(\|e_j\|_V^2+\|e_j\|_H^2\big)\Big)\le C\big(\ln(N)\big)^{-(2^{q-1}+1)}.$$

The inequalities (21) and (22) for $p=1$ and (73) for $K=1$ imply

$$\sup_{N\ge 1}E\Big(\frac{T}{N}\sum_{j=1}^{N}\|A^{\frac12}u(t_j)\|_V^2+\|A^{\frac12}u^j\|_V^2+\|A^{\frac12}q(t_j)\|_H^2+\|A^{\frac12}q^j\|_H^2\Big)<\infty.$$

Using a similar argument, we obtain

$$E\Big(\frac{T}{N}\sum_{j=1}^{N}\|A^{\frac12}e_j\|_V^2+\|A^{\frac12}e_j\|_H^2\Big)\le C\big(\ln(N)\big)^{-(2^{q-1}+1)}.$$

This yields (24) and completes the proof. □

9. Conclusions

This paper provides the first result on the rate of the convergence of a time discretization of the Navier–Stokes equations coupled with a transport equation for the temperature, driven by a random perturbation; this is the so-called Boussinesq/Bénard model. The perturbation may depend on both the velocity and temperature of the fluid. The rates of the convergence in probability and in $L^2(\Omega)$ are similar to those obtained for the stochastic Navier–Stokes equations. The Boussinesq equations model a variety of phenomena in environmental, geophysical and climate systems (see, e.g., [18,19]). Even if the outline of the proof is similar to that used for the Navier–Stokes equations, the interplay between the velocity and the temperature is more delicate to deal with in many places. This interplay, which appears in Bénard systems, is crucial for describing more general hydrodynamical models. The presence of the velocity in the bilinear term describing the dynamics of the temperature makes it more difficult to prove bounds of moments for the H^1-norm of the temperature uniformly in time and requires higher moments of the initial condition. Such bounds are crucial to deduce rates of convergence (in probability and in $L^2(\Omega)$) from the localized one.

This localized version of the convergence is the usual first step in a non-linear (non-Lipschitz and non-monotonous) setting. Numerical simulations, which are the ultimate aim of this study since there is no other way to "produce" trajectories of the solution, would require a space discretization, such as finite elements. This is not dealt with in this paper and will be carried out in a forthcoming work. This new study is likely to provide results similar to those obtained for the 2D Navier–Stokes equations.

In addition, note that another natural continuation of this work would be to consider a more general stochastic 2D magnetic Bénard model (as discussed in [1]) that describes the time evolution of the velocity, temperature and magnetic field of an incompressible fluid.

It would also be interesting to study the variance of the $L^2(D)$-norm of the error term, in both additive and multiplicative settings, for the Navier-0Stokes equations and more general Bénard systems. This would give some information about the accuracy of the approximation. Proving a.s. the convergence of the scheme for Bénard models is also a challenging question.

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