Regular languages for contracting geodesics

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Abstract
Let G be a finitely generated group. We show that for
any finite symmetric generating set A, the language con-
sisting of all geodesics in $\mathrm{Cay}(G, A)$ with the contract-
ing property is a regular language. An immediate con-
sequence is that the existence of an infinite contracting
geodesic in a Cayley graph of a finitely generated group
implies the existence of a contracting element. In par-
ticular, torsion groups cannot contain an infinite con-
tracting geodesic. As an application, this implies that
any finitely generated group containing an infinite con-
tracting geodesic must be either virtually $\mathbb{Z}$ or acylindri-
cally hyperbolic.

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1 | INTRODUCTION

The study of Gromov hyperbolic groups has been so fruitful that extending tools from this setting
to more general classes of groups is a central theme in geometric group theory.

Among the fundamental tools is the study of the geodesic language in hyperbolic groups. A
classic result by Cannon [4] shows that for any finitely generated hyperbolic group G, the lan-
guage consisting of geodesic words in $\mathrm{Cay}(G, S)$ is a regular language regardless of the chosen
generating set S. A regular language is simply a (typically infinite) set of words of low enough
complexity that it can be produced by a finite graph. The existence of such a language has beau-
tiful geometric, algebraic, analytical, and combinatorial consequences. For instance, a geometric
reflection of the existence of a regular language for geodesics in the group is the finiteness of
cone types in hyperbolic groups. This in turns implies the algebraic fact that the word problem is
solvable [7].

The existence of a regular language also has the analytical/combinatorial consequence that the growth function of the group is a linear recursive function. A beautiful, mind boggling example was given by Cannon of a finitely presented group where the language of all geodesic words is a regular language with respect to one generating set but not another. This is Example 4.4.2 in [7], and the group is $\mathbb{Z} \rtimes \mathbb{Z}_2 = \langle x, y, z \mid x^2, zyz^{-1}y^{-1}, yxz^{-1}x^{-1} \rangle$. Therefore, the regularity of the geodesic language in a finitely presented group G is not an intrinsic property of the group and is sensitive to the chosen generating set. As the example $\mathbb{Z} \rtimes \mathbb{Z}_2$ above shows, regularity of the geodesic language is not independent of the presentation even in virtually abelian groups.

We show that in a finitely generated group G, if one restricts their attention to the language consisting of all 'hyperbolic-like' geodesics, one gets a regular language for any generating set. The condition on a geodesic α to be 'hyperbolic-like' is that for any subsegment $\beta \subseteq \alpha$, the projection to β of any ball disjoint from β has diameter bounded above by D. A geodesic satisfying this condition is called *D-super-contracting*.

A related well-studied 'hyperbolic-like' property is that of a contracting geodesic. A geodesic α is said to be *D-contracting* if projections to α of balls disjoint from it have diameter at most D. In the literature, the exact value of D is seldom important. What is interesting is that such a parameter exists and is independent of the length of the geodesic. This means that projections onto such a geodesic are similar to projections in negatively curved geometry and different from projections in flat geometry. While our notion of D-super-contracting is technically stronger than D-contracting, the difference is only in the parameterization and completely independent of the length of the geodesic. In fact, we show the following.

Theorem 1.1. *Every D-contracting geodesic in a proper geodesic metric space X is D'-super-contracting where D' depends only on D. In fact, D' may be taken to be $D' = 54D + 10$. Since every D-super-contracting geodesic is D-contracting, an infinite geodesic is contracting if and only if it is super contracting.*

In light of the above theorem, the notions of super-contracting and contracting geodesics are equivalent, up to changing the contraction's parameter. Our main theorem is the following.

Theorem 1.2. *Let G be a finitely generated group, and let A be any finite symmetric generating set. Then the language L_D consisting of all D-super-contracting geodesic words in $\mathrm{Cay}(G, A)$ is a regular language for any D.*

Since a finitely generated group $G = \langle A \rangle$ is hyperbolic if and only if every geodesic in $\mathrm{Cay}(G, A)$ is D-super-contracting for a uniform D (see Remark 4.9), the above theorem recovers a classic result by Cannon where he shows that for a hyperbolic group G, and for a finitely generated set A, the language consisting of all geodesics in $\mathrm{Cay}(G, A)$ is a regular language. Since the generating function counting the number of words of length n in a regular language is always rational, this opens a host of combinatorial questions. For any finitely generated group, choice of generating set, and parameter D, we can ask how many geodesics of length n are D-super-contracting. All of these questions can be answered with a rational generating function. An immediate consequence to Theorem 1.1 and Theorem 1.2 is the following.

Corollary 1.3. *Let G be a finitely generated group and let A be a finite symmetric generating set for G. If $\mathrm{Cay}(G, A)$ contains an infinite contracting geodesic, then G contains a contracting isometry. In particular, torsion groups do not contain infinite contracting geodesics.*

As an interesting application to the above theorem, we answer the following question posed by Osin: Does the existence of an infinite contracting geodesic in a finitely generated group imply that the group is acylindrically hyperbolic?

The previous theorem yields a positive answer to this question:

Corollary 1.4. *Let G be a finitely generated group with a generating set A such that $Cay(G, A)$ contains an infinite contracting geodesic, then G must be acylindrically hyperbolic.*

The paper is organized as follows. In Section 2, we introduce an a priori stronger notion of a contracting geodesic that we call *super-contracting* and we set up the main tools needed for the proof of the main two theorems. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.1, Corollaries 1.3 and 1.4.

2 | SUPER-CONTRACTING GEODESICS

Throughout this section, X denotes a proper geodesic metric space. Also, if α is a path in X, we will generally use α to denote the image of α in the space X.

Definition 2.1 (projection). Let C be a closed subset of X. We define the *projection* of a point x onto C to be

$$\pi_C(x) = \{p \in C \mid d(x, p) = \min_{y \in C} d(x, y)\}.$$

In general, $\pi_C(x)$ may contain more than one point. For $B \subset X$, we define the *projection* to be $\pi_C(B) = \cup_{x \in B} \pi_C(x)$. We write $d(x, C)$ to denote the distance from x to its projection points.

Definition 2.2 (contracting). Let α be a continuous quasi-geodesic (possibly infinite). We say that α is *D-contracting* if for any closed metric ball B disjoint from α, $\operatorname{diam}(\pi_\alpha(B)) \leqslant D$. We say it is *contracting* if it is D-contracting for some D.

Definition 2.3 (super-contracting). Let α be a continuous quasi-geodesic in X. We say that α is *D-super-contracting* if every sub-segment of α is D-contracting in the above sense. That is, if for any subsegment $\gamma \subseteq \alpha$ and any closed metric ball B disjoint from γ, $\operatorname{diam}(\pi_\gamma(B)) \leqslant D$. We say α is *super-contracting* if it is D-super-contracting for some D. It is immediate from the definition of a D-super-contracting quasi-geodesic α that a subsegment of α is itself D-super-contracting.

It is clear from the above definitions that if a geodesic is D-super-contracting, then it is D-contracting. As mentioned in the table of contents, Section 4 is devoted to proving a converse of the above statement. In other words, we prove that for any proper geodesic metric space X and for any D-contracting geodesic $\alpha \subseteq X$, there exists D', depending only on D, such that α is D'-super-contracting. Also, note that projections on contracting/super-contracting quasi-geodesics are coarsely well defined. More precisely, if α is a D-contracting geodesic, then $\forall x \in X$ we must have $\operatorname{diam}(\pi_\alpha(x)) \leqslant D$.

Definition 2.4 (Quasi Isometric embedding). Let (X, d_X) and (Y, d_Y) be metric spaces. For constants $\lambda \geqslant 1$ and $\epsilon \geqslant 0$, we say a map $f : X \to Y$ is a (λ, ϵ)–*quasi-isometric embedding* if, for all points $x_1, x_2 \in X$

$$\frac{1}{\lambda} d_X(x_1, x_2) - \epsilon \leqslant d_Y(f(x_1), f(x_2)) \leqslant \lambda\, d_X(x_1, x_2) + \epsilon.$$

If, in addition, there exists a constant $C \geqslant 0$ such that every point in Y lies in the C-neighborhood of the image of f, then f is called a (λ, ϵ)-*quasi-isometry*. When such a map exists, X and Y are said to be *quasi-isometric*.

Definition 2.5 (Quasi-geodesics). A (λ, ϵ)-*quasi-geodesic* is a (λ, ϵ)-quasi-isometric embedding $\gamma : [a, b] \to X$. A quasi-geodesic is a (λ, ϵ)-*quasi-geodesic* for some $\lambda \geqslant 1$, and $\epsilon \geqslant 0$.

Definition 2.6 (Morse). A quasi-geodesic γ in a proper geodesic metric space is called *N-Morse*, where N is a function $N : [1, \infty) \times [0, \infty) \to [0, \infty)$, if for any (λ, ϵ)-quasi-geodesic σ with endpoints on γ, we have $\sigma \subseteq \mathcal{N}_{N(\lambda, \epsilon)}(\gamma)$. The function $N(\lambda, \epsilon)$ is called a *Morse gauge*.

The following is Lemma 3.3 in [8].

Lemma 2.7 (Contracting implies Morse). *For any proper geodesic metric space X, and for each $D \geqslant 0$, there exists an N, depending only on D such that every D-contracting geodesic is N-Morse. In particular, every D-super-contracting geodesic is N-Morse.*

Lemma 2.8. *Let M be a Morse gauge and let α be an M-Morse geodesic in $Cay(G, A)$ starting at the identity. If γ is any geodesic in the $Cay(G, A)$ starting at the identity and ending 1 apart from α, then γ is N-Morse where N depends only on M. Also, $N \geqslant M$.*

Proof. The proof of this lemma follows easily from Lemma 2.1 in [6]. $\qquad\square$

The following is Lemma 2.7 in [6]. It basically states that if you have two N-Morse geodesics with the same origin that end close to each other, then they have to be roughly uniformly close.

Lemma 2.9. *If $\alpha_1, \alpha_2 : [0, A] \to X$ are N-Morse geodesics with $\alpha_1(0) = \alpha_2(0)$ and $d(\alpha_1(s), \alpha_2) \leqslant K$ for some $s \in [0, A]$ and some $K > 0$, then $d(\alpha_1(t), \alpha_2(t)) \leqslant 8N(3, 0)$ for all $t < s - K - 4N(3, 0)$.*

Lemma 2.10. *If α is a D-super-contracting geodesic in $Cay(G, A)$ and β is another geodesic with the same starting point as α and ending at most 1 apart from where α ends, then there is a constant $C \geqslant 1$ only depending on D such that $d(\alpha(t), \beta(t)) \leqslant C$ for all $t < |\alpha| - 1 - \frac{C}{2}$.*

Proof. A D-super-contracting geodesic is D-contracting and therefore M-Morse by Lemma 2.7. Combining this with Lemmas 2.8 and 2.9, we get the following. If α is any D-super-contracting geodesic in $Cay(G, A)$ starting at the identity, then α has to be M-Morse where M depends only on D. By Lemma 2.8, if β is any other geodesic in $Cay(G, A)$, starting at the identity and ending 1 apart from where α ends, then β has to be N-Morse where N depends only on M which depends only on D. Now Lemma 2.9 gives us that $d(\alpha(t), \beta(t)) \leqslant 8N(3, 0)$ for all $t < |\alpha| - 1 - 4N(3, 0)$. Thus, the conclusion follows by taking $C = 8N(3, 0)$. $\qquad\square$

The following lemma relates projection onto a subsegment of a super-contracting geodesic to projection onto the full geodesic. It shows that if the projection onto the full geodesic is on one side of the subsegment, then projection onto the subsegment cannot jump too far toward the other side.

Lemma 2.11 (bounded jumps). *Let $\gamma : [a, b] \to X$ be a D-super-contracting geodesic. Let $c \in [a, b]$. Denote by α and β the subsegments of γ from $\gamma(a)$ to $\gamma(c)$ and from $\gamma(c)$ to $\gamma(b)$, respectively. For any $x \in X$, if $\pi_\gamma(x) \cap \alpha \neq \emptyset$, then every $\gamma(q) \in \pi_\beta(x)$ satisfies $d(\gamma(c), \gamma(q)) \leqslant D$.*

Proof. Let $x \in X$ be such that there exists a point $\gamma(p) \in \pi_\gamma(x) \cap \alpha \neq \emptyset$. Since γ is continuous, the map $f(t) = d(x, \gamma(t))$ is a continuous real-valued function. Let $\gamma(q)$ be in $\pi_\beta(x)$ and let $\gamma(q')$ be the point in α closest to $\gamma(c)$ satisfying $d(x, \gamma(q')) = d(x, \gamma(q))$. That is, let $q' = \sup\{s \leqslant c \mid f(s) = f(q)\}$. Such a point exists by the intermediate value theorem applied to the inequality $f(p) \leqslant f(q) \leqslant f(c)$. Taking the supremum guarantees that no point between $\gamma(q')$ and $\gamma(c)$ is closer to x than $\gamma(q')$ is. Thus, $\gamma(q)$ and $\gamma(q')$ are both in the projection of x onto $\gamma|_{[q',q]}$. Since γ is D-super-contracting, $d(\gamma(q'), \gamma(q)) \leqslant D$. Therefore, $d(\gamma(c), \gamma(q)) \leqslant D$. $\square$

3 | SUPER-CONTRACTING LANGUAGE IS A REGULAR LANGUAGE IN ANY FINITELY GENERATED GROUP

Definition 3.1 (A language over a finite alphabet). Let A be a finite set and $A^\star$ be the free monoid over A. We call an element $w \in A^\star$ a *word* in A. If $w \in A^\star$ and $w = a_1 \cdots a_n$ where each $a_i \in A$, then we call $a_1, \ldots, a_n$ the *letters* of the word w. A *language over A* is a set of words in $A^\star$. The *word length* of $w \in A^\star$ is the number of letters of A in the word w. We denote the word length of w by $|w|$.

The alphabets for the languages we will be working with will be finite generating sets for groups.

Definition 3.2. Let G be a finitely generated group and let A be a finite, symmetric generating set for G. The *Cayley graph of G with respect to A*, $\mathrm{Cay}(G, A)$, is the graph whose vertices are the elements of G and $g, h \in G$ are joined by an edge if $g^{-1}h \in A$. If $g^{-1}h \in A$, then we label the edge connecting g and h by $g^{-1}h$. The Cayley graph $\mathrm{Cay}(G, A)$ is a metric space by declaring each edge to have length 1.

Definition 3.3 (Left actions on Cayley graphs). Let G be a finitely generated group and let A be a finite symmetric generating set. The group G admits a natural left action on $\mathrm{Cay}(G, A)$ by isometries as follows. Let $g \in G$ and $x \in \mathrm{Cay}(G, A)$. If x is a vertex, then it is also a group element, so we can define $g \cdot x = gx$ where gx is simply the product of g and x in G. If x is a point on an edge connecting two vertices v_1 and v_2 at distance $t \in [0, 1]$ from v_1, let us denote it $x = (v_1, t, v_2)$. We define the group action on such a point to be $g \cdot x = (gv_1, t, gv_2)$. For a subset $B \subseteq \mathrm{Cay}(G, A)$, we define $g \cdot B := \{g \cdot x \mid x \in B\}$. For a continuous path $\sigma : [a, b] \to \mathrm{Cay}(G, A)$, we define a new path $g \cdot \sigma : [a, b] \to \mathrm{Cay}(G, A)$ by $(g \cdot \sigma)(t) = g \cdot \sigma(t)$. We will sometimes abuse notation and use $g \cdot \sigma$ to denote the image of the map $g \cdot \sigma$.

When the group G is generated by the finite set A, every path in $\mathrm{Cay}(G, A)$ produces a word in $A^\star$ by concatenating the labels of the edges in the order they appear along the path. Conversely,

every word in $A^\star$ produces a path in $\mathrm{Cay}(G, A)$ by starting at the identity e and traversing the edges in $\mathrm{Cay}(G, A)$ labeled by the letters of the word appearing from left to right. We will be particularly concerned with words in $A^\star$ that correspond with geodesic paths in $\mathrm{Cay}(G, A)$.

Definition 3.4. For a word $w \in A^\star$, let $\overline{w}$ denote the element of G obtained by viewing w as an element of G. A word $w \in A^\star$ is *geodesic* in $\mathrm{Cay}(G, A)$ if the path in $\mathrm{Cay}(G, A)$ from e to $\overline{w}$ labeled by the letters of w is a geodesic in $\mathrm{Cay}(G, A)$. This is equivalent to saying that $|w|$ is minimal among all words that represent $\overline{w}$. Further, for a word $w \in A^\star$, we define $|\overline{w}|_G$ to be the minimal word length among all possible words $u \in A^\star$ with $\overline{u} = \overline{w}$. In notation, we have

$$|\overline{w}|_G := \min\{|u| \text{ for all } u \in A^\star \text{such that } \overline{u} = \overline{w}\}.$$

Definition 3.5. A *finite state automaton (FSA)* over an alphabet A is a finite graph whose edges are directed and labeled by elements of A; the vertices of the graph are divided into two sets — 'accept' and 'reject' — and there is a distinguished vertex s_0 called the initial vertex. The *accepted language* of the automaton is the set of words which occur as labels on a directed edge path beginning at s_0 and ending at an accept vertex. Those vertices are often called the *states* of the automaton.

Definition 3.6 (Regular languages). Let A be a finite set and A^* be the set of all words with letters in A. Recall that a language over A is a subset $L \subset A^*$. A language over A is *regular* if it is the accepted language of some finite state automaton over A.

Definition 3.7. Let $u \in A^\star$ be a geodesic word in the $\mathrm{Cay}(G, A)$. We define the *D-super-contracting cone* of u, denoted by $\mathrm{Cone}^D(u)$, to be all $w \in A^\star$ so that the concatenation uw is a D-super-contracting geodesic in $\mathrm{Cay}(G, A)$. If u and v have the same D-super-contracting cone, we will say that u and v have the same *cone type*.

Example 3.8. In the free group on two letters $F_2 = \langle a, b \rangle$, if we take $D = 1$, the D-super-contracting cone of a is all geodesic words in the group that do not start with a^{-1} whereas in $\mathbb{Z} \oplus \mathbb{Z} = \langle a, b | [a, b] \rangle$, if $D = 1$, the D-super-contracting cone of a is empty. If we take $D = 2$, then in the free group example, the D-super-contracting cone of a will still be all geodesic words in the group that do not start with a^{-1} whereas in the $\mathbb{Z} \oplus \mathbb{Z}$ example the D-super-contracting cone of a is now the set $\{a, b, b^{-1}\}$.

Definition 3.9. Given an element u in $A^\star$ representing a geodesic in $\mathrm{Cay}(G, A)$, let $\overline{u}$ be the unique group element represented by the word u. Given $k \in \mathbb{N}$, we define the *k-tail* of u to be all elements $h \in G$ with $|h|_G \leqslant k$ such that $|\overline{u}h|_G < |\overline{u}|_G$. We denote the k-tail of u by $T_k(u)$.

The k-tail of a geodesic word u is all group elements in a ball of radius k of the identity that move $\overline{u}$ closer to the identity in the Cayley graph.

Definition 3.10. Given $u \in A^\star$ representing a geodesic in the $\mathrm{Cay}(G, A)$, $m \in \mathbb{N}$, and $D \geqslant 0$, we define the *m-local D-contracting type* of u to be all words $w \in A^\star$ with $|w| \leqslant m$ such that the concatenation uw is a D-super-contracting geodesic in $\mathrm{Cay}(G, A)$. We will denote the m-local D-contracting type of a geodesic word u by $\mathrm{Cone}_m^D(u)$.
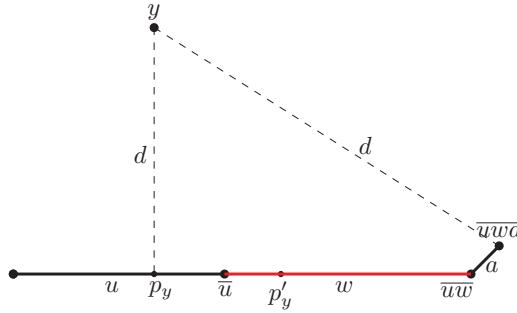
FIGURE 1 Possible projections of y to the geodesic uwa where uwa is D-super-contracting and $|w| \geqslant 3D + 1$

The following lemma is a slight modification of the Lemma 2.11. It will be used in the proof of Theorem 3.13.

Lemma 3.11. *Let G be a group with a finite symmetric generating set A. Let u and w be words in A and $a \in A$. Suppose that uwa is a geodesic word such that $|w| \geqslant 3D + 1$ and uw is D-super-contracting. For any vertex $y \in Cay(G, A)$, if $\pi_{uwa}(y) \cap u \neq \emptyset$, then $d(\overline{u}, p) \leqslant D$ for any $p \in \pi_{wa}(y)$.*

Proof. Let $X = Cay(G, A)$, note that since X is a graph, for any geodesic word v in X and any vertex $x \in X$, we have $d(x, \pi_v(x)) \in \mathbb{N} \cup \{0\}$. In other words, there exists at least one vertex $x' \in v$ such that $d(x, x') = d(x, \pi_v(x))$. That is to say, as X is a graph, no projection of x to v can lie on an interior of an edge in v.

Claim 3.12. Let uwa and y be as in the statement of the lemma, we have the following.

(1) All projections of y to uwa live on the subsegment uw. That is, $\pi_{uwa}(y) = \pi_{uw}(y)$.
(2) All projections of y to wa live on the subsegment w. That is, $\pi_{wa}(y) = \pi_w(y)$.

Before proving the above claims, we show how they imply the statement of the lemma. Note that by the assumption of the lemma, there exists a point $p_y \in \pi_{uwa}(y) \cap u$. As uw is a subsegment of uwa, we have $p_y \in \pi_{uw}(y) \cap u$. Since uw is D-super-contracting, using Lemma 2.11, all projections of y to w must be within D of $\overline{u}$. That is to say, for every $p \in \pi_w(y)$, we have $d(p, \overline{u}) \leqslant D$. However, using part (2) of Claim 3.12, we have $\pi_{wa}(y) = \pi_w(y)$ which finishes the proof.

Proof of (1): Let $p_y \in \pi_{uwa}(y) \cap u$ and let $d = d(y, p_y) = d(y, \pi_{uwa}(y))$. In order to prove the claim, it suffices to show that $d(y, \overline{uwa}) > d$. Suppose not, that is, suppose $d(y, uwa) \leqslant d$. As $d = d(y, \pi_{uwa}(y))$, we have $d(y, uwa) = d$. See Figure 1. Note that $d \leqslant d(y, \overline{uw}) \leqslant d + 1$. However, $d(y, \overline{uw}) \neq d$ as this would imply $\operatorname{diam}(\pi_{uw}(y)) \geqslant 3D + 1$ contradicting the assumption that uw is D-super-contracting. Hence, we have $d(y, \overline{uw}) = d + 1$. Now, note that since uw is a subsegment of uwa, the point p_y must also live in $\pi_{uw}(y)$. Let p'_y be the unique point in $\pi_{uw}(y)$ so that $d(e, p'_y) = \max_{p \in \pi_{uw}(y)} d(e, p)$. We claim that p'_y lives on the w subsegment of uw. Suppose not. That is, suppose $p'_y \in uw$ and $p'_y \notin w$. This implies that $d(y, p'_y) = d(y, \pi_{uw}(y)) \geqslant d$ and thus every point in w is at least $d + 1$ away from y. Consequently, we have $\overline{uw} \in \pi_w(y)$ contradicting Lemma 2.11 since $|w| \geqslant 3D + 1$. Hence, we have $p'_y \in w$. Let q be the unique vertex on uw so that $d(e, q) = d(e, p'_y) + 1$. Note that since uw is D-super-contracting, we have

$d(q, p_y) \leqslant d(q, p'_y) + d(p'_y, p_y) \leqslant 1 + D$. Furthermore, by our choice of q, we have $d(y, q) = d + 1$ and $d(y, q') \geqslant d + 1$ for any vertex q' on the geodesic uw with $d(e, q') \geqslant d(e, q)$. Now, let $\alpha = [q, \overline{uw}]_{uw}$ denote the subsegment of uw starting at the vertex q and ending at the vertex $\overline{uw}$. The points $q, \overline{uw}$ both live in $\pi_\alpha(y)$ as $d(y, q) = d(y, \overline{uw}) = d + 1$ where $d(y, z) \geqslant d + 1$ for any vertex $z \in \alpha$. On the other hand, $d(q, \overline{u}) = 1 + d(p'_y, \overline{u}) \leqslant 1 + d(p'_y, p_y) \leqslant 1 + D$. This yields that $d(q, \overline{uw}) = d(\overline{u}, \overline{uw}) - d(q, \overline{u}) \geqslant (3D + 1) - (D + 1) = 2D$, which contradicts the fact that uw is D-super-contracting. Therefore, the projection of y to uwa is contained entirely in the subsegment uw. Hence $\pi_{uw}(y) = \pi_{uwa}(y)$.

Proof of (2): Let $p_y \in \pi_{uwa}(y) \cap u$ and let $d = d(y, p_y) = d(y, \pi_{uwa}(y))$. Using part (1), we have $d(y, \overline{uwa}) > d$. Note that if $\pi_{uwa}(y) \cap w \neq \emptyset$, then we are done as if $z \in \pi_{uwa}(y) \cap w$, then $d(y, z) = d$ while $d(y, \overline{uwa}) > d$. Hence $\pi_w(y) = \pi_{wa}(y)$.

Otherwise, if $\pi_{uwa}(y) \cap w = \emptyset$, then all projections of y to uwa live in the u subsegment of uwa. In particular, since uw is a subword of uwa, all projections of y to uw live on the u subsegment of uw. Let p'' be the point with $d(\overline{u}, p'') = \max_{p \in \pi_w(y)} d(\overline{u}, p)$. Note that by Lemma 2.11, every projection of y to w is at most D away from $\overline{u}$, and since $|w| \geqslant 3D + 1$, we have $p'' \neq \overline{uw}$. Let q'' be the point on w with $d(\overline{u}, q'') = d(\overline{u}, p'') + 1$. By Lemma 2.11, since uw is D-super-contracting, we have $d(\overline{u}, p'') \leqslant D$ and hence $d(\overline{u}, q'') \leqslant D + 1$. Note that by our choice of p'' if we let $d' = d(y, p'') = d(y, \pi_w(y))$, then $d(y, q'') = d' + 1$. In order to finish the proof of this claim, we need to show that $d(y, \overline{uwa}) \neq d'$. Suppose for the sake of contradiction that $d(y, \overline{uwa}) = d'$, this implies that $d(y, \overline{uw})$ is either d' or $d' + 1$. However, it cannot be d' as this would contradict the assumption that uw is D-super-contracting. Thus $d(y, \overline{uw}) = d' + 1$. But that implies that both q'' and $\overline{uw}$ live on the projection of y to the subsegment of uw connecting q'' to $\overline{uw}$ given by $\beta = [q'', \overline{uw}]_{uw}$. This contradicts the assumption that uw is D-super-contracting as $\text{diam}(\pi_\beta(y)) = d(\overline{uw}, q'') = d(\overline{uw}, \overline{u}) - d(\overline{u}, q'') \geqslant (3D + 1) - (D + 1) = 2D$. $\square$

Let G be a group with a finite symmetric generating set A. The goal of this section is to show that the languages L_D consisting of all D-super-contracting geodesics are all regular languages regardless of the chosen generating set A. Before doing so, we will state an important key theorem that will provide us with the states needed for our FSA, we will refer to this theorem by 'the cone types theorem'. The theorem states that in order to determine D-super-contracting cone type of a geodesic word u, you need only to understand the local geometry around the vertex $\overline{u}$. To be more precise, it says that there exist a uniform m, depending only on D, such that the m-neighborhood around a vertex $\overline{u}$ encodes the information needed to determine what elements are in $\text{Cone}^D(u)$. This will imply that we have only finitely many cone types because there are only finitely many types of m-neighborhoods in $\text{Cay}(G, A)$.

Theorem 3.13. *For a given constant $D \geqslant 0$, there exists an integer m, depending only on D, such that if u, v are two geodesic words in $\text{Cay}(G, A)$ with $T_m(u) = T_m(v)$ and $\text{Cone}^D_m(u) = \text{Cone}^D_m(v)$, then $\text{Cone}^D(u) = \text{Cone}^D(v)$. In particular, there are only finitely many such cones.*

We will prove this theorem, but for now, let us show how it implies our first main theorem about the existence of a regular language for all D-super-contracting geodesics:

Theorem 3.14. *Let G be a group and A any finite symmetric generating set. Let L_D be the language of words in the alphabet A which, when interpreted as paths in the Cayley graph, are D-super-contracting geodesics. Then for any fixed D, the language L_D is a regular language.*

Proof. Consider the finite graph Γ whose vertices are the D-contracting cone types of G and which has a directed edge labeled $a \in A \cup A^{-1}$ connecting the D-super-contracting cone type of a geodesic word u to the D-super-contracting cone type of ua if and only if a belongs to the D-super-contracting cone of u. Otherwise the edge labeled a goes to the unique fail state. All non-empty D-super-contracting cone types are accept states. The initial state is the cone type of the empty word. The previous theorem shows that there are finitely many vertices, and the D-super-contracting cone type was defined precisely to pick out those continuations that are both geodesic and D-super-contracting. Also, observe that if $\text{Cone}^D(u) = \text{Cone}^D(v)$, then we have $\text{Cone}^D(ua) = \text{Cone}^D(va)$ for any $a \in A$. To see this, first, note that since $\text{Cone}^D(u) = \text{Cone}^D(v)$, the word ua is a D-super-contracting geodesic if and only if va is. Further, for any word w, if we let $w' = aw$, we have $w' \in \text{Cone}^D(u)$ if and only if $w' \in \text{Cone}^D(v)$. This implies that $uw' = uaw$ is a D-super-contracting geodesic if and only if $vw' = vaw$ is. Therefore, $\text{Cone}^D(ua) = \text{Cone}^D(va)$ whenever $\text{Cone}^D(u) = \text{Cone}^D(v)$. This shows that it is not possible for a single vertex of the directed graph Γ to have two distinct outgoing edges labeled with the same letter $a \in A$. Such a finite state automaton is called *deterministic*. $\qquad\square$

Now we prove Theorem 3.13:

Proof. Fix a super-contracting constant D. Using Lemma 2.10, there exists a constant C, which depends only on D, such that for any D-super-contracting geodesic α, if β is a geodesic starting at the same point as α and ending at most 1 away from α, then $d(\alpha(t), \beta(t)) \leqslant C$ for all $t < |\alpha| - 1 - \frac{C}{2}$.

We want to show that there exists some m large enough so that if two geodesic words $u, v \in \text{Cay}(G, A)$ satisfy $T_m(u) = T_m(v)$ and $\text{Cone}_m^D(u) = \text{Cone}_m^D(v)$, then $\text{Cone}^D(u) = \text{Cone}^D(v)$. Recall that for a word w in $A^\star$, the number of letters appearing in w is denoted by $|w|$ while the length of the group element $\overline{w}$ is denoted by $|\overline{w}|_G$.

Let $\overline{u}, \overline{v}$ be the unique group elements represented by u and v, respectively. Note that since u, v are assumed to be geodesic words, we must have $|\overline{u}|_G = |u|$ and $|\overline{v}|_G = |v|$. Choose $m > \max\{C + 1, 3D + 1\}$.

We proceed by induction on the length of the words in the cone. Since u and v satisfy $T_m(u) = T_m(v)$ and $\text{Cone}_m^D(u) = \text{Cone}_m^D(v)$, if $w \in A^\star$ with $|w| \leqslant m$, then $w \in \text{Cone}^D(u)$ if and only if $w \in \text{Cone}^D(v)$. This covers the base cases. For the induction step, let $w \in \text{Cone}^D(u) \cap \text{Cone}^D(v)$ and consider wa for $a \in A$. We want to show that $wa \in \text{Cone}^D(u)$ if and only if $wa \in \text{Cone}^D(v)$.

Suppose for contradiction that $wa \in \text{Cone}^D(v)$ but $wa \notin \text{Cone}^D(u)$. The cones agree for words of length at most m, so $|wa| \geqslant m + 1$. Since $wa \notin \text{Cone}^D(u)$, then by definition, either the word uwa is not a geodesic word or uwa is a geodesic word that is not D-super-contracting.

First we show that uwa must be a geodesic word. We remark that this part of the proof closely follows a proof in the cone types section of [3]. If uwa is not a geodesic word, then there must exist some geodesic word ℓ of length strictly less than $|u| + |w| + 1$ such that $\overline{\ell} = \overline{uwa}$. Write ℓ as a product $\ell_1 \ell_2$ such that $|\ell_1| = |u| - 1 = |\overline{u}|_G - 1$ and $|\ell_2| \leqslant |w| + 1$. Note that as $w \in \text{Cone}^D(u)$, the word uw is D-super-contracting and since uw and ℓ end 1 apart from each other, Lemma 2.10 gives us that $d(\overline{\ell}_1, \overline{u}) < C + 1$. Define $z := u^{-1}\ell_1$, hence the group element $\overline{z} = \overline{u^{-1}\ell_1}$ satisfies $|\overline{z}|_G \leqslant C + 1 \leqslant m$ and $|\overline{uz}|_G < |\overline{u}|_G$ which implies that $\overline{z} \in T_m(u)$. Recall that $T_m(u) = T_m(v)$ by assumption, so $\overline{z} \in T_m(v)$ and $|\overline{vz}|_G < |\overline{v}|_G$. Let α be any geodesic word connecting the identity to the group element $\overline{vz}$, so $|\alpha| < |\overline{v}|_G$. Now consider the concatenation of the geodesic α with the edge path labeled ℓ_2. On one hand, you get $\overline{\alpha \ell_2} = \overline{vz\ell_2} = \overline{vu^{-1}\ell_1\ell_2} = \overline{vu^{-1}uwa} = \overline{vwa}$. There-
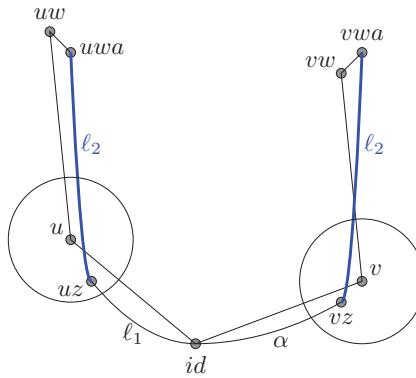
**FIGURE 2**   Geodesic cone is determined locally
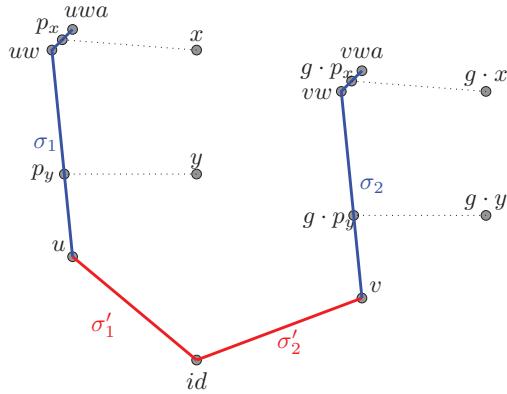
FIGURE 3   Contracting cone is determined locally

fore, the edge path $\alpha \ell_2$ ends at the same vertex as the geodesic word vwa. Consequently, since vwa is a geodesic, we have $|\alpha \ell_2| \geqslant |v| + |w| + 1$. But on the other hand, if we concatenate the geodesic word α with the edge path labeled ℓ_2 we get $|\alpha \ell_2| \leqslant |\alpha| + |\ell_2| < |v| + |w| + 1$ which is a contradiction. Therefore, uwa must be a geodesic word (Figure 2).

Now we consider the other possibility, that is, if uwa is a geodesic word that is not D-super-contracting. Denote this geodesic by β_1, its geodesic subsegments labeled wa, u by σ_1, σ_1', respectively. Similarly, denote the geodesic vwa by β_2, its geodesic subsegments labeled wa, v by σ_2, σ_2', respectively. Note that if we let $g := \overline{vu^{-1}}$, then we have $\sigma_2 = g \cdot \sigma_1$. So the assumption is that β_1 is not a D-super-contracting geodesic but β_2 is. This implies the existence of a subsegment γ of β_1 that is not D-contracting. In other words, there exists a ball B disjoint from γ, points $x, y \in B$, and projection points $p_x \in \pi_\gamma(x)$, $p_y \in \pi_\gamma(y)$ such that $d(p_x, p_y) > D$.

Note that since the geodesic uw is assumed to be D-super-contracting, then at least one of the points p_x and p_y, say p_x, is on the edge labeled a at the end of the geodesic β_1. Moreover, we have $p_x \in \pi_{\sigma_1}(x)$ (Figure 3).

The first thing to observe is that, since g is an isometry taking σ_1 to σ_2, $g \cdot p_x \in \pi_{\sigma_2}(g \cdot x)$ which is at the edge labeled a at the end of σ_2.

Now we consider the two different possibilities for projections of y to the subsegment γ of β_1. If all projections of y on the γ-subsegment of β_1 are on σ_1, then by our assumption, $\exists p_y \in \pi_{\sigma_1 \cap \gamma}(y)$ with $d(p_x, p_y) > D$. Let γ' denote the subsegment of γ which connects p_x to p_y. Note that since the ball B containing x, y is disjoint from γ, it must also be disjoint from γ'. By definition of γ', we have $\gamma' \subseteq \sigma_1$ and since g is an isometry taking σ_1 to σ_2, it must take $\gamma' \subseteq \sigma_1$ to $g \cdot \gamma' \subseteq \sigma_2$. Furthermore, since $p_x \in \pi_{\gamma'}(x), p_y \in \pi_{\gamma'}(y)$, we have

$$g \cdot p_x \in \pi_{g \cdot \gamma'}(g \cdot x), \ g \cdot p_y \in \pi_{g \cdot \gamma'}(g \cdot y),$$

$$g \cdot B \cap g \cdot \gamma' = \emptyset, \text{ and } d(g \cdot p_x, g \cdot p_y) = d(p_x, p_y) > D,$$

which contradicts the assumption that β_2 is D-super-contracting. The other possibility is that some projection of y on the γ-subsegment of β_1 meets σ_1'. In other words, assume there exists a point $p \in \pi_\gamma(y) \cap \sigma_1'$. If that is the case, then by removing σ_1' from β_1, using Lemma 3.11, we get a point $p_y \in \pi_{\sigma_1}(y)$ which is at most D away from $\bar{u}$. This implies $d(p_x, p_y) \geqslant (|\sigma_1| - 1) - D = |w| - D \geqslant m - D > D$. Since $\sigma_1 \subseteq \gamma$, and $B \cap \gamma = \emptyset$, we have $B \cap \sigma_1 = \emptyset$. Again, as g is an isometry with $g \cdot \sigma_1 = \sigma_2$, we have

$$g \cdot p_x \in \pi_{\sigma_2}(g \cdot x), \ g \cdot p_y \in \pi_{\sigma_2}(g \cdot y),$$

$$g \cdot B \cap g \cdot \sigma_1 = g \cdot B \cap \sigma_2 = \emptyset, \text{ and } d(g \cdot p_x, g \cdot p_y) = d(p_x, p_y) > D.$$

Hence, there exists a ball $B' = g \cdot B$ disjoint from $\sigma_2 = g \cdot \sigma_1$ containing two points $g \cdot x, g \cdot y$ whose projections to σ_2 include points $g \cdot p_x, g \cdot p_y$ with $d(g \cdot p_x, g \cdot p_y) > D$. This contradicts the assumption that β_2 is $D-$super-contracting. $\qquad\square$

Definition 3.15 (acylindrical action). The action of a group G on a metric space X is called *acylindrical* if for every $\epsilon > 0$, there exist $R, N > 0$ such that for every two points $x, y \in X$ with $d(x, y) > R$, there are at most N elements $g \in G$ satisfying:

$$d(x, gx) < \epsilon \ \text{ and } \ d(y, gy) < \epsilon.$$

Definition 3.16. A group G is said to be *acylindrically hyperbolic* if it admits a non-elementary acylindrical action on a hyperbolic space.

As a consequence of the above theorem, we will show that any finitely generated group which contains an infinite super-contracting geodesic must be acylindrically hyperbolic answering a question posed by Osin, but first, we state an easy corollary of Theorems H and I in [1]:

Corollary 3.17. *Let G be a finitely generated group which is not virtually cyclic, and let A be a generating set for G. If G contains an element g with a contracting axis in $Cay(G, A)$, then G must be acylindrically hyperbolic.*

Now we show how Theorem 3.14 along with the previous corollary imply the following:

Corollary 3.18. *Let G be a finitely generated group which is not virtually cyclic, and let A be a generating set for G. If $Cay(G, A)$ contains an infinite super-contracting geodesic, then it must contain*

a contracting isometry and hence G must be acylindrically hyperbolic. In particular, G cannot be a torsion group.

Proof. Let γ be an infinite D-super-contracting geodesic. Using Theorem 3.14, there exists a finite state automaton M that accepts every initial subsegment of γ. Denote its accepted language by L_D. Note that since γ is an infinite D-super-contracting geodesic, the language L_D must be infinite. Choose a large enough initial subsegment w of γ whose length is larger than the number of states in M. This implies the existence of a state that w passes twice. In other words, w has an initial subsegment of the form uv such that $uv^n \in L_D$ for all $n \in \mathbb{N}$ (this is a standard argument in formal languages which is often referred to as the *pumping lemma*). Since subsegments of D-super-contracting geodesics are themselves D-super-contracting, the subsegment v^n of uv^n is D-super-contracting. Hence, $\bar{v}$ is an element whose axis is super-contracting, and therefore, using the previous corollary, the group G must be acylindrically hyperbolic. □

3.1 | CONTRACTING IS ALSO SUPER-CONTRACTING

Throughout this section, let X denote a proper geodesic metric space. The goal of this section is to prove the following.

Theorem 4.1. *Every D-contracting geodesic in a proper geodesic metric space X is D'-super-contracting where D' depends only on D. In fact, D' may be taken to be $D' = 54D + 10$. Since every D-super-contracting geodesic is D-contracting, an infinite geodesic is contracting if and only if it is super contracting.*

To show this, we will use an additional hyperbolicity property, slimness, which we show in Lemma 4.4 holds for contracting geodesics.

Definition 4.2. A geodesic α is δ-*slim* if for any $x \in X$, $p \in \pi_\alpha(x)$, and geodesic β from x to a point on α, we have $d(p, \beta) \leqslant \delta$.

Contracting geodesics are known to also be slim in the setting of CAT(0) spaces. This property also holds in the generality of proper geodesic metric spaces, but the proof has to be adjusted slightly to accomodate the greater generality. In their proof that contracting implies slim, Bestvina and Fujiwara use the assumption that projection onto geodesics is coarsely distance decreasing [2]. This does not always hold for projections in a proper geodesic metric space, but it does hold for projections onto contracting geodesics, as we show in the following proposition.

Proposition 4.3. *Let α be a D-contracting geodesic. For any points $x_1, x_2 \in X$ and projection points $p_1 \in \pi_\alpha(x_1), p_2 \in \pi_\alpha(x_2)$,*

$$d(p_1, p_2) \leqslant d(x_1, x_2) + 4D.$$

Proof. First we consider the case of two points x and y connected by a geodesic $[x, y]$ such that $d(w, \alpha) > D$ for all $w \in [x, y]$. Subdivide $[x, y]$ into $n = \lceil \frac{d(x,y)}{D} \rceil$ subsegments $[z_i, z_{i+1}]$ of length $\leqslant D$ with $z_0 = x$ and $z_n = y$. So $nD \leqslant d(x, y) + D$. Let $z'_0 \in \pi_\alpha(x)$ and $z'_n \in \pi_\alpha(y)$ be any pair of

projection points. Pick $z_i' \in \pi_\alpha(z_i)$ for the rest. Since α is D-contracting, $d(z_i', z_{i+1}') \leqslant D$ for all i. Then

$$d(z_0', z_n') \leqslant \sum_{i=0}^{n-1} d(z_i', z_{i+1}') \leqslant nD \leqslant d(x, y) + D.$$

Now for the general case, fix any geodesic $[x_1, x_2]$. Let $y_1 \in [x_1, x_2]$ be the point closest to x_1 satisfying $d(y_1, \alpha) \leqslant D$ and y_2 the analogous point closest to x_2. If no such points exist, we are in the previous case and already done. Pick any two projection points $q_i \in \pi_\alpha(y_i)$. By the triangle inequality, $d(q_1, q_2) \leqslant d(y_1, y_2) + 2D$. The segments $[x_i, y_i]$ are at distance greater than D from α. Therefore we can apply the result of the previous paragraph to get

$$d(p_1, p_2) \leqslant d(p_1, q_1) + d(q_1, q_2) + d(q_2, p_2)$$
$$\leqslant d(x_1, y_1) + D + d(y_1, y_2) + 2D + d(y_2, x_2) + D$$
$$= d(x_1, x_2) + 4D. \qquad \square$$

Lemma 4.4. *Let α be a D-contracting geodesic in a proper geodesic metric space. Then α is also δ-slim where δ depends only on D.*

Proof. Proposition 4.3 shows that projections onto D-contracting geodesics satisfy the distance decreasing axiom of [2] with $C = 4D$. The proof of slimness is then the same as in Lemmas 3.5 and 3.6 of [2]. From this proof, $\delta = 7D + 1$ suffices. $\qquad \square$

Remark 4.5. Let α be a D-contracting (and thus δ-slim by lemma 4.4) geodesic. Let x be any point in X and $p \in \pi_\alpha(x)$. If p_t is a point on α at distance t away from p, and z is a point in some geodesic connecting x to p_t with $d(p, z) \leqslant \delta$, then $d(x, p_t) = d(x, z) + d(z, p_t) \geqslant (d(x, p) - \delta) + (t - \delta)$. Therefore, we have

$$d(x, p_t) \geqslant d(x, p) + t - 2\delta.$$

The following lemma looks very similar to Lemma 2.11. The main difference is that here we only assume γ is D-contracting, whereas Lemma 2.11 assumes that γ is D-super-contracting.

Lemma 4.6. *Let $\gamma : [a, b] \to X$ be a D-contracting geodesic and σ any subsegment of γ. Let $\delta = 7D + 1$. Suppose we have a point x and a projection point $q \in \pi_\gamma(x)$ so that $q \notin \sigma$. If u is the endpoint of σ closest to q, then for any $p \in \pi_\sigma(x)$, $d(p, u) \leqslant 2\delta$.*

Proof. By Lemma 4.4, γ is δ-slim. Let $p \in \pi_\sigma(x)$. The point u is between q and p on a geodesic, so $d(q, p) = d(p, u) + d(u, q)$, and $d(x, p) \leqslant d(x, u)$ because p is in the projection of x onto σ. By the triangle inequality, $d(x, u) \leqslant d(u, q) + d(x, q)$. Starting from the inequality in Remark 4.5 with $t = d(p, q)$, we have

$$d(x, q) + d(q, p) \leqslant d(x, p) + 2\delta$$
$$d(x, q) + d(p, u) + d(u, q) \leqslant d(x, u) + 2\delta$$
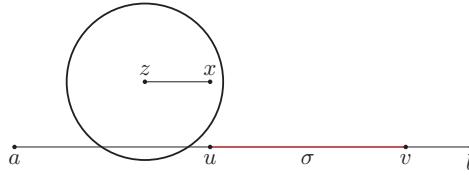
FIGURE 4   Case 2 of Lemma 4.7

$$d(x, q) + d(p, u) + d(u, q) \leqslant d(u, q) + d(x, q) + 2\delta$$

$$d(p, u) \leqslant 2\delta. \qquad \qquad \square$$

We are now ready to prove that subsegments of a contracting geodesic are also contracting. Our proof is essentially the same as that of Lemma 3.2 in [2]. We include it only to verify that is still valid assuming only that X is a proper geodesic metric space, using Proposition 4.3 in place of their distance decreasing axiom.

Lemma 4.7. *Let γ be a D-contracting geodesic. There is a constant D', depending only on D, so that any subsegment σ of γ is D'-contracting. Consequently, γ is D'-super-contracting.*

Proof. Let $\delta = \delta(D)$ be the slimness constant from Lemma 4.4. Let u and v be the endpoints of σ. In other words, $\sigma = [u, v]$ where $[u, v]$ denotes the the geodesic subsegment of γ connecting u to v. Let a, b be the first and last points of γ, respectively.

We claim that $D' = 26D + 4\delta + 6$ does the job. We may assume that length$(\sigma) > D'$ as otherwise, there would be nothing to prove. Let B be a ball centered at a point z and disjoint from our subsegment σ. From here we divide the proof into two cases. The constant D' is sufficient for both.

Case 1. $\pi_\gamma(z) \cap \sigma \neq \emptyset$. In this case, the ball B is also disjoint from γ, so we know that the diameter of $\pi_\gamma(B)$ is at most D. Let $I = [s_1, s_2]$ denote the smallest interval containing all of the points in $\pi_\gamma(B)$. The length of I is bounded above by D. We will argue that every point in $\pi_\sigma(B)$ is within a distance 2δ of $I \cap \sigma$.

If $x \in B$ has projection $\pi_\gamma(x) \cap \sigma \neq \emptyset$, then $\pi_\sigma(x) = \pi_\gamma(x) \cap \sigma$. In particular, $\pi_\sigma(x) \subset I$. Suppose instead $\pi_\gamma(x)$ contains a point $q \notin \sigma$. For concreteness, suppose u is the endpoint of σ nearest to q (otherwise apply the same argument with v instead of u). Then $u \in I$ and $d(u, p) \leqslant 2\delta$ for any $p \in \pi_\sigma(x)$ by Lemma 4.6. So everything in $\pi_\sigma(B)$ is within 2δ of I, an interval of length at most D. Therefore, the diameter of $\pi_\sigma(B)$ is at most $D + 4\delta$ for Case 1 (Figure 4).

Case 2. $\pi_\gamma(z) \cap \sigma = \emptyset$. Note that in this case it is possible that B intersects γ. Every point $p_z \in \pi_\gamma(z)$ must belong to precisely one of $[a, u), (v, b]$. In other words, since γ is D-contracting, it is not possible to have two points $p_z, p_z' \in \pi_\gamma(z)$ with $p_z \in [a, u)$ and $p_z' \in (v, b]$. Without loss of generality, suppose that every $p_z \in [a, u)$. We want to show, for all $x \in B$ and $p_x \in \pi_\sigma(x)$, that $d(p_x, u) \leqslant 13D + 2\delta + 3$. This would imply $d(p_x, p_y) \leqslant 26D + 4\delta + 6 = D'$ for all $x, y \in B$ and $p_x, p_y \in \pi_\sigma(x), \pi_\sigma(y)$, respectively.

To show the above, assume for the sake of contradiction that there exists $x \in B$ so that $\pi_\sigma(x)$ contains a point further than $13D + 2\delta + 3$ from u. Note that no projection point of x to γ can be on $[a, u)$ because Lemma 4.6 would imply $d(\pi_\sigma(x), u) \leqslant 2\delta$. So there is also at least one point $p_x \in \pi_\gamma(x)$ with $d(p_x, u) > 13D + 2\delta + 3$. Fix $[z, x]$ to be some geodesic connecting z to x. By subdividing $[z, x]$ into intervals of length at most one, and using Proposition 4.3, we can find a

point $y \in [z, x]$ such that the set $\pi_\gamma(y)$ is contained in the $1 + 4D$ neighborhood of u. Let B' be the closed ball centered at y with radius $r = d(y, \gamma) - 1$ and since γ is D-contracting, the set $\pi_\gamma(B')$ has diameter $\leqslant D$. Note that

$$d(y, u) \geqslant d(z, u) - d(z, y) = d(z, u) - d(z, x) + d(y, x) \geqslant d(y, x).$$

Now, since the set $\pi_\gamma(y)$ is contained in the $1 + 4D$ neighborhood of u, we have

$$d(y, x) \leqslant d(y, u) \leqslant d(y, \gamma) + 4D + 1 = r + 4D + 2$$

Therefore, by the previous inequality, the ball B' must contain a point w with $d(w, x) \leqslant 4D + 2$. Now note that for any $p_y \in \pi_\gamma(y)$ and $p_w \in \pi_\gamma(w)$, we have

$$d(p_y, p_x) \leqslant d(p_y, p_w) + d(p_w, p_x) \leqslant D + ((4D + 2) + 4D) = 9D + 2.$$

On the other hand, we have $d(u, p_y) \leqslant 1 + 4D$, which gives that $d(u, p_x) \leqslant d(u, p_y) + d(p_y, p_x) \leqslant (1 + 4D) + (9D + 2) = 13D + 3$, this is a contradiction since $d(u, p_x) > 13D + 2\delta + 3$. $\qquad\square$

In light of Theorem 4.1, we can apply Corollary 3.18 to show the following.

Corollary 4.8. *Let G be a finitely generated group which is not virtually cyclic, and let A be a generating set for G. If $\mathrm{Cay}(G, A)$ contains an infinite contracting geodesic, then G must be acylindrically hyperbolic.*

Remark 4.9. In the introduction, we stated that a geodesic metric space is hyperbolic if and only if there exists a constant D such that every geodesic is D-super-contracting. The forward direction is a standard fact about hyperbolic spaces. For example, the argument given in Theorem 2.14 of [5] proving that for a CAT(0) space X every δ-slim geodesic is D-contracting (and hence D'-super-contracting) still works when the CAT(0) space X is replaced by a hyperbolic space. Conversely, if every geodesic is D-super-contracting, then by Lemma 2.7, there exists an M such that every geodesic is M-Morse. Now, Lemma 2.2 of [6] states that if two edges of a triangle in a geodesic metric space are M-Morse, then the triangle is $4M(3, 0)$-thin. This implies that the space is $4M(3, 0)$-hyperbolic. $\qquad\square$

All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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