ORIGINAL PAPER

Bounded Cutoff Window for the Non-backtracking Random Walk on Ramanujan Graphs

Evita Nestoridi[1] · Peter Sarnak[1]

Abstract
We prove that the non-backtracking random walk on Ramanujan graphs with large girth exhibits the fastest possible cutoff with a bounded window.

Keywords Non-backtracking random walk · Cutoff · Window · Almost diameter · Density hypothesis

Mathematics Subject Classification 60 · 11 · 60J10 · 20C30 · 05E99

1 Introduction

Fix $d \geq 3$, which we write as $p + 1$, and consider d-regular graphs X on n vertices with $n \to \infty$, that do not have self–loops and multiple edges. During the last decade, there has been a lot of interest in studying the simple random walk (SRW) and the non-backtracking random walk (NBRW) on such graphs. The focus has been to understand mixing times and related cutoff phenomena [1–6]. The non-backtracking random walk was introduced by Hashimoto [7], it mixes faster, has sharper transitions and has been very useful in multiple cases [8–15]. We focus exclusively on the NBRW on X, which is defined as follows:

$$K_t(x, y) = \#\left\{(x = x_0, x_1, \ldots, x_t = y)\,\middle|\,\begin{matrix} x_i \in X \\ x_i \sim x_{i+1} \\ x_{i-1} \neq x_{i+1} \end{matrix}\right\},$$

✉ Evita Nestoridi
    exn@princeton.edu

✉ Peter Sarnak
    sarnak@math.princeton.edu

[1] Princeton University, Princeton, USA

where $x_i \sim x_{i+1}$ indicates that (x_i, x_{i+1}) is an edge.

Let $N(t) := \sum_y K_t(x, y) = (p+1)p^{t-1}$ and let $P^t(x, y) = \frac{1}{N(t)} K_t(x, y)$ be the transition matrix of the non-backtracking random walk on X. The total variation distance of $P^t_x = P^t(x, \cdot)$ from the uniform measure is defined as $d_x(t) := \frac{1}{2} \sum_{y \in X} \left| P^t(x, y) - \frac{1}{n} \right|$. We will also consider the total variation distance when starting at the worst possible starting point

$$d(t) := \max_{x \in X} \{d_x(t)\}.$$

For $0 < \eta < 1$, the total variation mixing time is defined as

$$t_{\mathrm{mix}}(\eta) = \min\{t \geq 0 : d(t) \leq \eta\}.$$

The main focus of this paper is studying the cutoff phenomenon. We say that the NBRW on X exhibits cutoff at t_n with window $w_n = o(t_n)$ if

$$\lim_{c \to \infty} \lim_{n \to \infty} d(t_n - c w_n) = 1 \quad \text{and} \quad \lim_{c \to \infty} \lim_{n \to \infty} d(t_n + c w_n) = 0. \tag{1}$$

If $N(t) \leq n$ one checks that $d_x(t) \geq \frac{U_x(t)}{n}$ where $U_x(t)$ is the number of vertices that are not reached by the walk at time t, when starting at x. Hence,

$$d(t) \geq 1 - \frac{N(t)}{n},$$

if $N(t) \leq n$, which implies that

$$t_{\mathrm{mix}}(1 - \eta) \geq \log_p n - \log_p \eta^{-1} - \log_p \left(1 + \frac{1}{p}\right). \tag{2}$$

This gives an absolute lower limit in (1) for the cutoff time $t_n = \log_p n$ and bounded w_n and we are interested in graphs X for which this t_n is indeed the cutoff time for the NBRW.

We will search for such X among different types of expanders. For $\lambda < d$ an (n, d, λ) graph X is a d regular graph on n nodes for which the eigenvalues $\{\lambda_j\}_{j=0}^{n-1}$ of the adjacency matrix of X satisfy

$$\begin{cases} \lambda_0 = d := p + 1 & \text{if } j = 0 \\ |\lambda_j| \leq \lambda & \text{if } j \neq 0. \end{cases}$$

If $\lambda = 2\sqrt{p}$ then X is called a Ramanujan graph.

The key results in this direction are due to Lubetzky and Sly [16] and Lubetzky and Peres [3]. In the first, it is shown that w.h.p. the non-backtracking random walk on a random d-regular graph exhibits cutoff at

$$t_n = \log_p(dn) \text{with window of constant order.} \tag{3}$$

In the second one, X is assumed to be Ramanujan, and they show that the NBRW on any Ramanujan graph exhibits cutoff at $\log_p n$, but whether it occurs with a bounded window is not resolved.

As a corollary of their main results, they also prove a purely combinatorial fact about the almost diameter. For Ramanujan graphs, for any x we have that

$$\#\{y \in X : |\mathrm{dist}(x, y) - \log_p n| > 3 \log_p \log n\} = o(n). \tag{4}$$

Our main result shows that the NBRW on a Ramanujan graph with large girth g exhibits cutoff with a bounded window.

Theorem 1.1 *Fix $\delta > 0$. The NBRW on a Ramanujan graph with $g \geq \delta \log_p n$ satisfies*

$$t_{\mathrm{mix}}(\varepsilon) \leq \log_p n + 2 \log_p \varepsilon^{-1} + 2 \log_p(2 + 20\delta^{-1}),$$

for every $\varepsilon > 0$.

Remark 1.2 The girth condition of Theorem 1.1 is satisfied for the Ramanujan graphs of [17] with $\delta = 2/3$. This shows that the NBRW on these graphs exhibits cutoff with a bounded window, which was one of our goals in this note.

It is important to note that most of the examples (other than the result of Lubetzky and Sly [16]) that are known where the cutoff window is bounded are non-local Markov chains, such as riffle shuffles [18] and random transvections [19].

Next, we discuss what can be said about cutoff if we drop the Ramanujan condition. Writing the eigenvalues in the form

$$\lambda_j = 2\sqrt{p} \cos \theta_j,$$

where for $|\lambda_j| \leq 2\sqrt{p}$, we have a unique $\theta_j \in [0, \pi]$, and otherwise for the "exceptional" eigenvalues we choose θ_j uniquely in the form

$$\begin{cases} \theta_j = i\phi_j \log p & \text{if } \lambda_j > 2\sqrt{p} \\ \theta_j = \pi + i\psi_j \log p & \text{if } \lambda_j < -2\sqrt{p}, \end{cases}$$

with $\phi_j, \psi_j \in (0, \frac{1}{2})$.

We use the following notation. We say that two real functions f and g satisfy $f(x) \ll_\varepsilon g(x)$ if and only if there is a constant $C = C(\varepsilon)$ such that $f(x) \leq Cg(x)$.

Definition 1.3 A sequence of graphs X is said to satisfy the density hypothesis if for every $0 \leq \alpha < 1/2$ and $\varepsilon > 0$, the number of exceptional eigenvalues M satisfies

$$M(\alpha, X) := \#\{j : \phi_j \geq \alpha\} + \#\{j : \psi_j \geq \alpha\} \ll_\varepsilon n^{1-2\alpha+\varepsilon}.$$

For a discussion of this density hypothesis see [20] and [21]. The point is that this density can often be established in cases where the Ramanujan is not known or even fails.

In [22] and [21] it is shown that the density together with the assumption that X is an expander suffice to show that the SRW on X exhibits cutoff at $\frac{p+1}{p-1} \log_p n$. This indicates that the density hypothesis can be used as a replacement of the Ramanujan assumption as far as the shortest possible cutoff for the NBRW is concerned. Given its importance we give the precise statement and short proof.

Theorem 1.4 *Let X be a transitive sequence (that is the automorphisms act transitively on the vertices) of (n, d, λ) expander graphs which satisfy the density hypothesis. Then the NBRW on X exhibits cutoff at $\log_p n$. That is,*

$$d((1 + \eta) \log_p n) \to 0,$$

for every $\eta > 0$.

The next results focus on the diameter of (n, d, λ) graphs and strengthens (4). Let $\mathcal{N}_x(\ell)$ be the number of vertices $y \in X$ such that $d(x, y) > \ell$.

Theorem 1.5 *Let X be an (n, d, λ) graph; then for $\xi > 0$ we have that*

$$\max_{x \in X} \left\{ \frac{1}{n} \mathcal{N}_x \left(\frac{1}{2} \log_b n + \xi \right) \right\} \le \frac{4}{b^{2\xi}},$$

where $b = \frac{d}{\lambda} + \sqrt{\left(\frac{d}{\lambda}\right)^2 - 1}$.

We note that if we choose ξ (bounded) so that $4b^{-2\xi} < 1/2$, then given $x, y \in X$ we can find a common z with $d(x, z) < \frac{1}{2} \log_b n + \xi$ and $d(y, z) < \frac{1}{2} \log_b n + \xi$. Therefore, $d(x, y) < \log_b n + 2\xi$. This shows that the diameter is at most $\log_b n + 2\xi$. This matches the bounds for the diameter that were derived in [17] for Ramanujan graphs and in [23] for (n, d, λ) graphs. As in these papers, a crucial element in the analysis are the Chebyshev polynomials of the first kind.

Let $p = d - 1$. For the case where X is Ramanujan, we have that $\lambda = 2\sqrt{p}$ and $b = \sqrt{p}$. Theorem 1.5 gives the following.

Corollary 1.6 *Let X be a Ramanujan graph on n vertices, then for $\xi > 0$ we have that*

$$\max_{x \in X} \left\{ \frac{1}{n} \mathcal{N}_x (\log_p n + \xi) \right\} \le \frac{4}{p^{\xi}}.$$

Remark 1.7 Corollary 1.6 gives a bounded window strengthening (4) and if it is not optimal, it is very close to being so. In particular, it allows one to replace the $3 \log_p \log n$ term in (4) by any function $f(n)$ which goes to infinity with n.

In the context of d-regular graphs, the almost diameter bound of Corollary 1.6 is essentially the smallest it could be among all such graphs. On the other hand, the bound $2 \log_p n + 4$ for the diameter of a Ramanujan graph is probably not optimal. The random d-regular graph has diameter $(1 + o(1)) \log_p n$ (see [24]), however the Ramanujan graphs of [17] can have diameter at least $\frac{4}{3} \log_p n$, as was shown in [6]. We expect that this $\frac{4}{3} \log_p n$ is an upper bound for the diameter of a Ramanujan graph.

As is standard in cutoff analysis, our proofs involve the ℓ^2 distance $\|P_x^t - U\|_2^2 = \sum_{y \in X} \left| P^t(x, y) - \frac{1}{n} \right|^2$ and its average over x

$$d_2(t) := \frac{1}{n} \sum_x \|P_x^t - U\|_2^2.$$

Note that if X is transitive, then $d_2(t) = \|P_x^t - U\|_2^2$ for all x, as are all of the quantities defined in terms of the starting point x.

For the case of reversible Markov chains, such as the SRW on X, one can express $d_2(t)$ in terms of the eigenvalues and eigenfunctions of the transition matrix (see chapter 12 of [25]). Studying the spectrum of the transition has been a powerful tool for proving cutoff for many well-known Markov chains, such as [19, 26, 27]. We make judicious use of Chebyshev polynomials and the eigenvalues and eigenfunctions of the adjacency matrix of X to prove our results, and avoid using the NBRW on the edges of the graph.

Our analysis leads to the following basic conjecture.

Conjecture 1.8 *Fix δ and p. If X is a sequence of Ramanujan graphs and $t < (2 - \delta) \log_p n$, then*

$$d_2(t)N(t) \to 1, \tag{5}$$

as $n \to \infty$.

This is consistent with the model that in this window the $N(t)$ end points of walks of length t are placing themselves at random among the n vertices.

Our proofs involve approximations to (5). The source of the gain being that the Kesten measure on $[-2\sqrt{p}, 2\sqrt{p}]$ vanishes to second order at $-2\sqrt{p}$ and $2\sqrt{p}$ (see (33)). In [28] it is proven that the probability measure supported on $[-2\sqrt{p}, 2\sqrt{p}]$ corresponding to the eigenvalues of a Ramanujan graph, converges to the Kesten measure as $n \to \infty$. Conjecture 1.8 requires that this convergence holds with polynomials of degree as large as $4(1 - \delta) \log_p n$. In a forthcoming paper [29] this convergence and in particular Conjecture 1.8 is established for various arithmetic Ramanujan graphs. Our Conjecture 1.8 implies that the NBRW on these Ramanujan graphs exhibit cutoff with an explicit and tight bounded window, namely

$$t_x(\varepsilon) \le \log_p n + 2 \log_p \varepsilon^{-1},$$

for almost every starting point x.

2 Preliminaries

Let X be a connected, d regular graph on n vertices, where d is fixed. Let A denote the adjacency matrix of X. A is a symmetric matrix with real eigenvalues

$$-d \le \lambda_{n-1} \le \ldots \le \lambda_1 < \lambda_0 = d.$$

Denote the corresponding orthonormal basis of real eigenfunctions as $f_{n-1}, \ldots, f_0$, with $f_0(x) = \frac{1}{\sqrt{n}}$ for every $x \in X$. The fact that the $\{f_j\}$ are orthonormal means that

$$\sum_{x \in X} f_i(x) f_j(x) = \delta_{i,j}. \tag{6}$$

The fact that $\{f_j\}$ is an orthonormal basis gives that

$$\delta_x(y) = \sum_{j=0}^{n-1} \langle f_j, \delta_x \rangle f_j(y),$$

which translates to

$$\delta_x(y) = \sum_{j=0}^{n-1} f_j(x) f_j(y). \tag{7}$$

When considering the t-th power of A, we have that the (x, y) entry $A^t(x, y)$ is equal to the number of walks of length t starting at x and ending at y. Let P be a polynomial of the form

$$P(x) = a_0 + a_1 x + \ldots + a_\ell x^\ell.$$

We have that the matrix $P(A)$ can be expressed as

$$P(A)(x, y) = \sum_{j=0}^{n-1} P(\lambda_j) f_j(x) f_j(y).$$

The key quantity that we estimate is the variance W with respect to P, defined as

$$W(P, x) := \sum_y \left(P(A)(x, y) - \frac{P(\lambda_0)}{n} \right)^2, \tag{8}$$

which by (6) is equal to the spectral sum

$$\sum_{j \neq 0} |P(\lambda_j)|^2 f_j^2(x). \tag{9}$$

3 The Almost Diameter

To estimate the almost diameter of X, we use the following key lemma.

Lemma 3.1 *Let $\ell(P)$ be the degree of P, then for any x*

$$\left(\frac{P(\lambda_0)}{n}\right)^2 \mathcal{N}_x(\ell(P)) \leq \max_{\lambda \neq \lambda_0}\{|P(\lambda)|^2\},$$

where $\mathcal{N}_x(\ell(P))$ is the number of vertices $y \in X$ such that $d(x, y) > \ell(P)$.

Proof First of all, we note that since $A^t(x, y)$ is equal to the number of walks of length t starting at x and ending at y, we have that

$$\text{for every } x, y \in X, \text{ if } d(x, y) > \ell(P) \text{ then } P(A)(x, y) = 0. \tag{10}$$

Combining this with (8) and (9) we have that

$$\sum_{y:d(x,y)>\ell(P)} \left(\frac{P(\lambda_0)}{n}\right)^2 \leq W(P, x) \leq \max_{\lambda \neq \lambda_0}\{|P(\lambda)|^2\} \sum_{j \neq 0} |f_j(x)|^2. \tag{11}$$

Equation (7) gives that $\sum_{j \neq 0} |f_j(x)|^2 \leq 1$, which finishes the proof. $\qquad\square$

3.1 Chebyshev Polynomials of the First Kind

Let T_ℓ be the Chebyshev polynomials of the first kind of degree ℓ, that is $T_\ell(x) = \cos(\ell \arccos x)$ and therefore $T_\ell(x) \in [-1, 1]$ for every $x \in [-1, 1]$.

Lemma 3.2 *For $\lambda \leq \lambda_0$, the Chebyshev polynomials of the first kind satisfy*

$$T_\ell\left(\frac{\lambda_0}{\lambda}\right) \geq \frac{b^\ell}{2},$$

where $b = \left(\frac{\lambda_0}{\lambda} + \sqrt{\left(\frac{\lambda_0}{\lambda}\right)^2 - 1}\right)$.

Proof Using the fact that $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, we can write $\lambda_0 = \lambda\cos\theta_0$, where $\theta_0 = i\log\left(\frac{\lambda_0}{\lambda} + \sqrt{\left(\frac{\lambda_0}{\lambda}\right)^2 - 1}\right)$. This gives that

$$T_\ell\left(\frac{\lambda_0}{\lambda}\right) = T_\ell(\cos\theta_0) = \frac{1}{2}(b^\ell + b^{-\ell}) \geq \frac{b^\ell}{2}.$$

$\qquad\square$

3.2 The Almost Diameter for Expanders

In this section, we present the proof of Theorem 1.5 concerning the almost diameter of (n, d, λ) graphs.

Proof of Theorem 1.5 Let T_ℓ be the Chebyshev polynomial of the first kind of degree ℓ. We apply Lemma 3.1 to the polynomial

$$P(x) = T_\ell \left(\frac{x}{\lambda} \right),$$

where ℓ will be determined later. The right hand side of the equation in Lemma 3.1 satisfies that

$$\max_{\lambda_i \neq \lambda_0} \{|P(\lambda_i)|^2\} \leq 1, \tag{12}$$

since all $\lambda_i \neq \lambda_0$ satisfy that $|\lambda_i| \leq \lambda$ and $T_\ell(x) = \cos(\ell \arccos x)$ for $x \in [-1, 1]$. At the same time, Lemma 3.2 gives that

$$(P(\lambda_0))^2 = \left(T_\ell \left(\frac{\lambda_0}{\lambda} \right) \right)^2 \geq \frac{b^{2\ell}}{4}. \tag{13}$$

Lemma 3.1 and equations (12) and (13) give that

$$\frac{1}{n} \mathcal{N}_x(\ell) \leq \frac{4n}{b^{2\ell}}. \tag{14}$$

Let $\xi > 0$ and set $\ell = \frac{1}{2} \log_b n + \xi$. Then equation 14 gives the desired result. $\qquad\square$

4 The Mixing Time for the Non-backtracking Random Walk

In this section, we present our results concerning the mixing time of the NBRW on X.

4.1 Chebyshev Polynomials of the Second Kind

The NBRW can be expressed in terms of the Chebyshev polynomials of the second kind. In this section, we explain this connection and we prove some useful properties for the Chebyshev polynomials of the second kind.

Let U_ℓ be the Chebyshev polynomials of the second kind of degree ℓ, defined as

$$U_\ell(\cos \theta) = \frac{\sin((\ell + 1)\theta)}{\sin \theta}.$$

The Chebyshev polynomials of the second kind satisfy the following recurrence relation:

$$\begin{cases} U_0(x) & = 1 \\ U_1(x) & = 2x \\ U_{\ell+1}(x) & = 2xU_\ell(x) - U_{\ell-1}(x). \end{cases}$$

Set

$$P_\ell(x) = p^{\ell/2} U_\ell\left(\frac{x}{2\sqrt{p}}\right). \tag{15}$$

Lemma 4.1 *Let A be the adjacency matrix of a regular graph. We have that*

$$P_\ell(A)(x, y) = \sum_{0 \le j \le \ell/2} K_{\ell-2j}(x, y),$$

where $K_t(x, y)$ is the number of non-backtracking random walks of length t from x to y.

Proof The two sides have the following generating function

$$\sum_{\ell=0}^{\infty} P_\ell t^\ell = \frac{1}{1 - At + pt^2},$$

and therefore they are equal. For more details, we refer to Lemma 1.4.3 of [30]. ☐

We start with the following lemma. Set $\lambda_j = 2\sqrt{p}\cos\theta_j$. Notice that $\theta_0 = i\log\sqrt{p}$ and therefore

$$P_\ell(\lambda_0) = \frac{p^{\ell+1} - 1}{p - 1}. \tag{16}$$

Lemma 4.2 *Let g be the girth of X and let $\ell \le g/5$. For n large enough, we have that*

$$\sum_{j=1}^{n-1} \left(U_\ell(\cos\theta_j)\right)^2 f_j^2(x) \le 2,$$

for every $x \in X$.

Proof Since $\ell < g$, the ℓ first steps of the NBRW on X are the same as the ℓ first steps on a d regular tree. Therefore,

$$\sum_{1 \le j \le \ell/2} K_{\ell-2j}(x, y) = \begin{cases} 1 & d(x, y) \le \ell \text{ and } d(x, y) \equiv \ell \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Combined with (8), (16) and Lemma 4.1, this gives that

$$
\begin{aligned}
W(P_\ell, x) &= \sum_{\substack{d(x,y)>\ell \text{ or} \\ d(x,y)\equiv\ell+1 \mod 2}} \left(\frac{p^{\ell+1}-1}{n(p-1)}\right)^2 + \sum_{\substack{d(x,y)\leq\ell \\ d(x,y)\equiv\ell \mod 2}} \left(1-\frac{p^{\ell+1}-1}{n(p-1)}\right)^2 \\
&\leq \frac{1}{n}\left(\frac{p^{\ell+1}-1}{p-1}\right)^2 + \sum_{\substack{d(x,y)\leq\ell \\ d(x,y)\equiv\ell \mod 2}} 1 \\
&\leq \frac{1}{n}\left(\frac{p^{\ell+1}-1}{p-1}\right)^2 + \left(\frac{p^{\ell+1}-1}{p-1}\right).
\end{aligned}
$$

For n large, we use the fact that $2\ell + 2 \leq \frac{2}{5}g + 2 \leq \frac{4}{5}\log_p n + 2$ to get that

$$
W(P_\ell, x) \leq \frac{p^{\ell+1}}{p-1}. \tag{17}
$$

Equations (9) and (15) give that

$$
\sum_{j=1}^{n-1} \left(U_\ell(\cos\theta_j)\right)^2 f_j^2(x) \leq \frac{p}{p-1} \leq 2,
$$

as desired.   □

4.2 The Non-backtracking Random Walk

The first lemma gives K_t as an explicit polynomial in A (see also [31] and [8]).

Lemma 4.3 *Set* $Q_t(x) = p^{t/2}\left(\frac{p-1}{p}U_t\left(\frac{x}{2\sqrt{p}}\right) + \frac{2}{p}T_t\left(\frac{x}{2\sqrt{p}}\right)\right)$. *For* $t \geq 1$, *we have*

$$
Q_t(A)(x, y) = K_t(x, y),
$$

for every $x, y \in X$.

Proof Using Lemma 4.1, we can write that

$$
K_t(x, y) = P_t(A)(x, y) - P_{t-2}(A)(x, y). \tag{18}
$$

Using the following relationship between Chebyshev polynomials of the two types

$$
U_t = U_{t-2} + 2T_t
$$

and (15), we can rewrite (18) as

$$K_t(x, y) = p^{t/2} \left(\frac{p-1}{p} U_t \left(\frac{A}{2\sqrt{p}} \right) + \frac{2}{p} T_t \left(\frac{A}{2\sqrt{p}} \right) \right). \tag{19}$$

In other words,

$$Q_t(A)(x, y) = K_t(x, y). \tag{20}$$

$\square$

Recall that $\lambda_j = 2\sqrt{p} \cos\theta_j$. The fact that X is Ramanujan gives that θ_j is real. We now use Lemma 4.3, the facts that $|\sin\theta_j| \leq 1$ and $|\sin((t+1)\theta_j)| \leq |t+1||\sin(\theta_j)|$ to write the following expression for the variance.

$$W(Q_t(A), x) = p^t \sum_{j\neq0} \left(\frac{p-1}{p} \frac{\sin((t+1)\theta_j)}{\sin\theta_j} + \frac{2}{p}\cos(t\theta_j) \right)^2 f_j^2(x)$$
$$\leq p^t (t+1)^2, \tag{21}$$

which is the bound given in Lubetzky and Peres [3]. As they note in Remark 3.7 of [3], in order to get rid of the factor $(t+1)^2$ in (21), one needs some control on the distribution of the θ_j. To do so, we assume a lower bound on the girth g of X.

Lemma 4.4 *Fix $\delta > 0$ and assume that X has girth $g \geq \delta \log_p n$ and is Ramanujan, then*

$$W(Q_t(A), x) \leq 12 \left(\frac{10}{\delta} + 1 \right)^2 p^t,$$

for $\log_p n \leq t \leq 2 \log_p n$ and n large enough.

Proof Set $k = \lfloor \frac{10}{\delta} \rfloor + 1$. For $t \in [\log_p n, 2\log_p n]$, write $t + 1$ as $mk + r$ with $0 \leq r < k$. Notice that then $m \leq \frac{2}{k}\log_p n \leq g/5$ and so we can apply Lemma 4.2 with this m. According to Lemma 4.3 with $\lambda_j = 2\sqrt{p}\cos\theta_j$, we have that

$$W(Q_t(A), x) = p^t \sum_{j\neq0} \left(\frac{p-1}{p} \frac{\sin((t+1)\theta_j)}{\sin\theta_j} + \frac{2}{p}\cos(t\theta_j) \right)^2 f_j^2(x)$$
$$\leq 2p^t \sum_{j\neq0} \left(\left(\frac{\sin((mk+r)\theta_j)}{\sin\theta_j} \right)^2 + \frac{4}{p^2}\cos^2(t\theta_j) \right) f_j^2(x) \tag{22}$$

Using standard trigonometric identities we have that

$$
\begin{aligned}
(22) &\leq 2p^t \sum_{j\neq 0} \left(\frac{\sin(mk\theta_j)\cos r\theta_j + \cos(mk\theta_j)\sin(r\theta_j)}{\sin\theta_j} \right)^2 f_j^2(x) + 8p^{t-2} \\
&\leq 4p^t \sum_{j\neq 0} \left(\left(\frac{\sin(mk\theta_j)}{\sin\theta_j}\right)^2 + \left(\frac{\sin(r\theta_j)}{\sin\theta_j}\right)^2 \right) f_j^2(x) + 8p^{t-2} \\
&\leq 4p^t \sum_{j\neq 0} \left(\left(\frac{\sin(mk\theta_j)}{\sin\theta_j}\right)^2 + r^2 \right) f_j^2(x) + 8p^{t-2} \\
&\leq 4p^t \sum_{j\neq 0} \left(\left(\frac{\sin(mk\theta_j)}{\sin\theta_j}\right)^2 \right) f_j^2(x) + 4r^2 p^t + 8p^{t-2}
\end{aligned}
\tag{23}
$$

We use the fact that

$$
\left| \frac{\sin(mk\theta_j)}{\sin\theta_j} \right| = \left| \frac{\sin(mk\theta_j)}{\sin(m\theta_j)} \frac{\sin(m\theta_j)}{\sin\theta_j} \right| \leq k \left| \frac{\sin(m\theta_j)}{\sin\theta_j} \right|
$$

and Lemma 4.2 to get

$$
4p^t \sum_j \left(\frac{\sin(mk\theta_j)}{\sin\theta_j} \right)^2 f_j^2(x) \leq 8k^2 p^t.
$$

Equation (23) gives that

$$
W(Q_t(A), x) \leq 12k^2 p^t \leq 12 \left(\frac{10}{\delta} + 1 \right)^2 p^t,
$$

since $0 \leq r < k$. This completes the proof of Lemma 4.4.   □

4.3 The Bounded Window

In this section, we present the proof of Theorem 1.1. Let P be the transition matrix of the non-backtracking random walk on X. For $t \geq 0$, we have that $P^t(x, y) = \frac{1}{(p+1)p^{t-1}} K_t(x, y)$. Therefore, applying Cauchy-Schwartz we get that

$$
4d_x^2(t) \leq n \sum_y \left| \frac{1}{(p+1)p^{t-1}} K_t(x, y) - \frac{1}{n} \right|^2.
$$

Since $Q_t(d) = (p+1)p^{t-1}$, equation (8) gives that

$$W(Q_t(A), x) = \sum_y \left(K_t(x, y) - \frac{(p+1)p^{t-1}}{n} \right)^2.$$

Therefore,

$$4d_x^2(t) \leq \frac{n}{(p+1)^2 p^{2t-2}} W(Q_t(A), x). \tag{24}$$

Using Lemma 4.4, we get that for $\log_p n \leq t \leq 2\log_p n$,

$$d_x(t) \leq \frac{n^{1/2}}{2(p+1)p^{t-1}} (W(Q_t(A), x))^{1/2}$$

$$\leq 2\left(1 + \frac{10}{\delta}\right)\left(\frac{n}{p^t}\right)^{1/2}. \tag{25}$$

By taking $t = \log_p n + 2\log_p \varepsilon^{-1} + 2\log_p \left(2(1 + \frac{10}{\delta})\right)$, we get that

$$d_x(t) \leq \varepsilon,$$

and this holds uniformly for $x \in X$.

5 The Density Hypothesis

Let X be an (n, d, λ) graph satisfying the density property, as defined in 1.3. The goal of this section is to prove that the mixing time of the non-backtracking random walk on X is at most $(1 + \eta)\log_p n$ for every $\eta > 0$.

The following Lemma is key to proving Theorem 1.4.

Lemma 5.1 *Let X be an (n, d, λ) expander sequence, that satisfies the density hypothesis. We set $I_n = t^2 \sum_{j=1}^{n-1} \left(p^{-[\frac{1}{2}-\phi_j]2t} + p^{-[\frac{1}{2}-\psi_j]2t} \right)$. Then for $\eta > 0$ fixed,*

$$\lim_{n \to \infty} I_n = 0,$$

if $t \geq (1 + \eta)\log_p n$.

Proof Since X is an expander, we have that there is $\delta_1 > 0$ such that $0 \leq \phi_j \leq \frac{1}{2} - \delta_1$. We express the sums in I_n in terms of the function M in Definition 1.3;

$$\sum_{j=1}^{n-1} p^{-[\frac{1}{2}-\phi_j]2t} = -\int_0^{\frac{1}{2}-\delta_1} p^{-[\frac{1}{2}-\alpha]2t} dM(\alpha).$$

Integrating by parts, we get that

$$\sum_{j=1}^{n-1} p^{-[\frac{1}{2}-\phi_j]2t} = M(0)p^{-t} - M\left(\frac{1}{2}-\delta_1\right)p^{-2t\delta_1} + 2t\log p \int_0^{\frac{1}{2}-\delta_1} p^{-[\frac{1}{2}-\alpha]2t} M(\alpha)d\alpha.$$

The density hypothesis asserts that $M(\alpha) \ll_\varepsilon n^{1-2\alpha+\varepsilon}$ for $0 \le \alpha \le 1/2$. Therefore,

$$\sum_{j=1}^{n-1} p^{-[\frac{1}{2}-\phi_j]2t} \ll_\varepsilon n^{1+\varepsilon} p^{-t} + 2t \int_0^{\frac{1}{2}-\delta_1} p^{-[\frac{1}{2}-\alpha]2t} n^{1-2\alpha+\varepsilon} d\alpha$$

$$= n^{1+\varepsilon} p^{-t} \left(1 + 2t \int_0^{\frac{1}{2}-\delta_1} \left(\frac{p^{2t}}{n^2}\right)^\alpha d\alpha\right).$$

Since $t > \log_p n$, we have that

$$\sum_{j=1}^{n-1} p^{-[\frac{1}{2}-\phi_j]2t} \ll_\varepsilon n^{1+\varepsilon} p^{-t} \left(1 + 2t \left(\frac{p^t}{n}\right)^{1-2\delta_1}\right)$$

$$\ll_\varepsilon n^{1+\varepsilon} p^{-t} + 2t n^\varepsilon \left(\frac{n}{p^t}\right)^{2\delta_1}.$$

We can get a similar bound for $\sum_{j=1}^{n-1} p^{-[\frac{1}{2}-\psi_j]2t}$. Since $\varepsilon > 0$ is arbitrarily small and $\delta_1 > 0$ is fixed, it follows that $\lim_{n\to\infty} I_n = 0$ if $t \ge (1+\eta)\log_p n$. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4 We only consider the eigenvalues that satisfy $|\lambda| > 2\sqrt{p}$, since the rest of them can be studied just as in the Ramanujan case. We recall that when $\lambda_j > 2\sqrt{p}$, we have that $\theta_j = i\phi_j \log p$ for $\phi_j \in (0, 1/2]$. Then,

$$|\cos(t\theta_j)| = \left|\frac{1}{2}\left(p^{t\phi_j} + p^{-t\phi_j}\right)\right| \le p^{t\phi_j} \tag{26}$$

and

$$\left|U_t\left(\frac{\lambda_j}{2\sqrt{p}}\right)\right| = \left|\frac{p^{(t+1)\phi_j} - p^{-(t+1)\phi_j}}{p^{\phi_j} - p^{-\phi_j}}\right| \le t p^{(t+2)\phi_j}. \tag{27}$$

We can get similar bounds in terms of the ψ_j for the case $\lambda < -2\sqrt{p}$.

$$W(Q_t(A), x) = p^t \sum_{j\ne 0} \left(\frac{p-1}{p}\frac{\sin((t+1)\theta_j)}{\sin\theta_j} + \frac{2}{p}\cos(t\theta_j)\right)^2 f_j^2(x)$$

Summing over x and using the fact that X is transitive, we have that

$$W(Q_t(A), x) \leq \frac{p^t}{n} \sum_{j=1}^{n-1} \left(\frac{p-1}{p} \frac{\sin((t+1)\theta_j)}{\sin\theta_j} + \frac{2}{p} \cos(t\theta_j) \right)^2. \qquad (28)$$

We set

$$\phi_j' = \begin{cases} \phi_j & \text{if } \lambda_j > 2\sqrt{p}, \\ \psi_j & \text{if } \lambda_j < -2\sqrt{p}. \end{cases}$$

Considering the terms corresponding to all $|\lambda_j| \leq 2\sqrt{p}$ and using equations (26) and (27), we have

$$W(Q_t(A), x) \leq p^t (t+1)^2 + \frac{p^t}{n} \sum_{j=1}^{n-1} \left(\frac{p-1}{p} \left| \frac{\sin(t+1)\theta_j}{\sin\theta_j} \right| + \frac{2}{p} |\cos(\theta_j t)| \right)^2.$$

Using (26) and (27) yields

$$W(Q_t(A), x) \leq p^t (t+1)^2 + 3p^2 \frac{p^t}{n} t^2 \sum_{j=1}^{n-1} p^{2t\phi_j'}. \qquad (29)$$

Plugging this into the ℓ^2 bound, we have that

$$\begin{aligned} d_x(t) &\leq \frac{n^{1/2}}{2(p+1)p^{t-1}} (W(Q_t(A), x))^{1/2} \\ &\leq \frac{1}{2} \left(np^{-t} (t+1)^2 + 3p^2 t^2 \sum_{j=1}^{n-1} p^{-[\frac{1}{2} - \phi_j']2t} \right)^{1/2}, \qquad (30) \end{aligned}$$

for every $x \in X$. Lemma 5.1 finishes the proof of Theorem 1.4.   □

Remark 5.2 In Theorem 1.4, if X is not transitive then in as much as we summed over all $x \in X$ in the proof, the result remains true for almost all x in place of all x.

6 Conjecture 1.8

We end with some comments about our conjectured asymptotics of the variance for Ramanujan graphs. The spectral expansion (9) gives that

$$
\begin{aligned}
W_2(t) : &= \frac{1}{n} \sum_{x \in X} W(Q_t, x) \\
&= \sum_{j \neq 0} Q_t^2 (2\sqrt{p} \cos \theta_j) \\
&= \frac{p^t}{n} \sum_{j \neq 0} R_t^2(\theta_j),
\end{aligned}
\tag{31}
$$

where $R_t = \frac{p-1}{p} U_t + \frac{2}{p} T_t$. We write (31) as

$$
W_2(t) = p^t \mu_X(R_t^2),
\tag{32}
$$

where μ_X is the density of the eigenvalues on $[0, \pi]$:

$$
\mu_X = \frac{1}{n} \sum_{j \neq 0} \delta_{\theta_j}.
$$

For any sequence of Ramanujan graphs X, μ_X is known to converge to the Plancherel measure ν_p, as $n \to \infty$ [28]. That is for a fixed polynomial R

$$
\mu_X(R) \to \int_0^\pi R(\theta) d\nu_p(\theta)
$$

as $n \to \infty$. Here the Plancherel, or Kesten measure, ν_p is;

$$
d\nu_p = \frac{2(p+1) \sin^2 \theta}{\pi[(p^{1/2} + p^{-1/2})^2 - 4\cos^2 \theta]} d\theta.
\tag{33}
$$

For X's whose girth is at least $\delta \log_p n$, the calculation in Sect. 4.3, which was used to establish the bounded window for these, yields that for $t < g/5$,

$$
W_2(t) \sim (p+1) p^{t-1},
$$

as $n \to \infty$. Hence for these X's and in this range of t's

$$
\mu_X(R_t^2) \sim \frac{p+1}{p},
\tag{34}
$$

as $n \to \infty$. One can check that the R_t's are orthogonal polynomials for the measure v_p on $[0, \pi]$ (see [31] for example) and that for $t \geq 1$

$$\int_0^\pi R_t^2(\theta) dv_p(\theta) = \frac{p+1}{p} \tag{35}$$

Thus (34) reads that the large girth X's and in the range $t < g/5$

$$\mu_X(R_t^2) \sim v_p(R_t^2) = \frac{p+1}{p}, \tag{36}$$

as $n \to \infty$. Our conjecture is that (36) holds in general for any sequence of Ramanujan graphs and in the larger range $t < 2 \log_p n$. By (32), Conjecture 1.8 is equivalent to

$$W_2(t) \sim N(t),$$

for $t < 2 \log_p n$ as $n \to \infty$. In the forthcoming paper [29], Conjecture 1.8 is proven for various families of arithmetical Ramanujan graphs, such as the ones discussed in [32].

References

1. Chapman, M., Parzanchevski, O.: Cutoff on Ramanujan complexes and classical groups. arXiv:1901.09383 (2022). https://doi.org/10.4171/CMH/537
2. Hermon, J.: Cutoff for Ramanujan graphs via degree inflation. Electron. Commun. Probab. **22**, 1–10 (2017)
3. Lubetzky, E., Peres, Y.: Cutoff on all Ramanujan graphs. Geom. Funct. Anal. **26**(4), 1190–1216 (2016)
4. Lubetzky, E., Lubotzky, A., Parzanchevski, O.: Random walks on Ramanujan complexes and digraphs. J. Eur. Math. Soc. (JEMS) **22**(11), 3441–3466 (2020)
5. Ozawa, N.: An entropic proof of cutoff on Ramanujan graphs. Electron. Commun. Probab. (2020). https://doi.org/10.1214/20-ECP358
6. Sardari, N.T.: Diameter of Ramanujan graphs and random Cayley graphs. Combinatorica **39**(2), 427–446 (2019)
7. Hashimoto, Ki-ichiro.: Zeta functions of finite graphs and representations of p-adic groups. In *Automorphic forms and geometry of arithmetic varieties*, volume 15 of *Adv. Stud. Pure Math.*, pages 211–280. Academic Press, Boston, MA, (1989)
8. Alon, N., Benjamini, I., Lubetzky, E., Sodin, S.: Non-backtracking random walks mix faster. Commun. Contemp. Math. **9**(4), 585–603 (2007)
9. Bordenave, C., Lelarge, M., Massoulié, L.: Nonbacktracking spectrum of random graphs: community detection and nonregular Ramanujan graphs. Ann. Probab. **46**(1), 1–71 (2018)
10. Ben-Hamou, A., Salez, J.: Cutoff for nonbacktracking random walks on sparse random graphs. Ann. Probab. **45**(3), 1752–1770 (2017)
11. Fitzner, R., van der Hofstad, R.: Non-backtracking random walk. J. Stat. Phys. **150**(2), 264–284 (2013)
12. Glover, C., Kempton, M.: Spectral properties of the non-backtracking matrix of a graph. arXiv:2011.09385 (2020). https://doi.org/10.48550/arXiv.2011.09385
13. Hermon, J.: Reversibility of the non-backtracking random walk. Ann. Inst. Henri Poincaré Probab. Stat. **55**(4), 2295–2319 (2019)
14. Hermon, J., Sly, A., Sousi, P.: Universality of cutoff for graphs with an added random matching. Ann. Probab. **50**(1), 203–240 (2022)

15. Ortner, R., Woess, W.: Non-backtracking random walks and cogrowth of graphs. Canad. J. Math. **59**(4), 828–844 (2007)
16. Lubetzky, E., Sly, A.: Cutoff phenomena for random walks on random regular graphs. Duke Math. J. **153**(3), 475–510 (2010)
17. Lubotzky, A., Phillips, R., Sarnak, P.: Ramanujan graphs. Combinatorica **8**(3), 261–277 (1988)
18. Bayer, D., Diaconis, P.: Trailing the dovetail shuffle to its lair. Ann. Appl. Probab. **2**(2), 294–313 (1992)
19. Hildebrand, M.: Generating random elements in $SL_n(F_q)$ by random transvections. J. Algebraic Combin. **1**(2), 133–150 (1992)
20. Sarnak, Peter.: Letter to Miller and Tabelizadeh on optimal strong approximation by integral points on quadratics, appendix : Optimal lifting of integral points. https://publications.ias.edu/sarnak/paper/2637, (2015)
21. Golubev, K., Kamber, A.: Cutoff on graphs and the Sarnak–Xue density of eigenvalues. J. Combinat. **104**, 103530 (2022)
22. Bordenave, C., Lacoin, H.: Cutoff at the entropic time for random walks on covered expander graphs. J Inst. Math. Jussieu (2021). https://doi.org/10.1017/S1474748020000663
23. Chung, F.R.K., Faber, V., Manteuffel, T.A.: An upper bound on the diameter of a graph from eigenvalues associated with its Laplacian. SIAM J. Discrete Math. **7**(3), 443–457 (1994)
24. Bollobás, B., Fernandez de la Vega, W.: The diameter of random regular graphs. Combinatorica **2**(2), 125–134 (1982)
25. Levin, David A., Peres, Yuval.: *Markov chains and mixing times*. American Mathematical Society, Providence, RI, Second edition of [MR2466937], With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson (2017)
26. Bernstein, M., Nestoridi, E.: Cutoff for random to random card shuffle. Ann. Probab. **47**(5), 3303–3320 (2019)
27. Diaconis, P., Shahshahani, M.: Generating a random permutation with random transpositions. Z. Wahrsch. Verw. Gebiete **57**(2), 159–179 (1981)
28. Abért, M., Glasner, Y., Virág, B.: The measurable Kesten theorem. Ann. Probab. **44**(3), 1601–1646 (2016)
29. Sarnak, P., Zubrilina, N.: Convergence to Plancherel measure of Hecke eigenvalues. arXiv:2201.03523v1 (2022). https://doi.org/10.48550/arXiv.2201.03523
30. Davidoff, G., Sarnak, P., Valette, A.: Elementary number theory, group theory, and Ramanujan graphs, London Mathematical Society Student Texts. Cambridge University Press, Cambridge (2003)
31. Cartier, P.: Harmonic analysis on trees. In *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, pages 419–424, (1973)
32. Costache, A., Feigon, B., Lauter, K., Massierer, M., Puskás, A.: Ramanujan graphs in cryptography, Research Directions in Number Theory, p. 2019. Springer International Publishing, Cham (2019)