Probability Surveys
Vol. 20 (2023) 113–169
ISSN: 1549-5787
https://doi.org/10.1214/22-PS11

The method of stochastic characteristics for linear second-order hypoelliptic equations*

Juraj Földes

University of Virginia,
Department of Mathematics,
322 Kerchof Hall,
Charlottesville, VA USA 22904
e-mail: jf8dc@virginia.edu

David P. Herzog

Iowa State University,
Department of Mathematics,
411 Morrill Rd.,
Ames, IA USA 50011
e-mail: dherzog@iastate.edu

Abstract: We study hypoelliptic stochastic differential equations (SDEs) and their connection to degenerate-elliptic boundary value problems on bounded or unbounded domains. In particular, we provide probabilistic conditions that guarantee that the formal stochastic representation of a solution is smooth on the interior of the domain and continuously approaches the prescribed boundary data at a given boundary point. The main general results are proved using fine properties of the process stopped at the boundary of the domain combined with hypoellipticity of the operators associated to the SDE. The main general results are then applied to deduce properties of the associated Green's functions and to obtain a generalization of Bony's Harnack inequality. We moreover revisit the transience and recurrence dichotomy for hypoelliptic diffusions and its relationship to invariant measures.

MSC2020 subject classifications: Primary 60H30, 35H10, 35G15, 60H10; secondary 34B27, 35A01, 35A02.
Keywords and phrases: Hypoelliptic operator, boundary-valued PDE, stochastic characteristics.

Received December 2021.

Contents

*J.F. and D.P.H. graciously acknowledge support from National Science Foundation grants DMS-1816408 (J.F.) and DMS-1855504 (D.P.H.).

1. Introduction

1.1. Overview

Let $\mathfrak{m} \in \mathbf{N}$ and $U \subset \mathbf{R}^{\mathfrak{m}}$ be a nonempty, open set with nonempty boundary ∂U. Let $\mathcal{X} \subset \mathbf{R}^{\mathfrak{m}}$ be an open set containing $\overline{U} = U \cup \partial U$ and let $M_{\mathfrak{m}\times r}$ denote the set of $\mathfrak{m} \times r$ matrices with entries in $\mathbf{R}$. Consider a linear second-order differential operator of the form

$$L = \sum_{i=1}^{\mathfrak{m}} (b(x))_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{\mathfrak{m}} (\sigma(x)\sigma(x)^T)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.1}$$

where $b \in C^\infty(\mathcal{X}; \mathbf{R}^{\mathfrak{m}})$ and $\sigma \in C^\infty(\mathcal{X}; M_{\mathfrak{m}\times r})$. In this paper, we study the formal stochastic representation u_{stoc} (see (1.4) below) corresponding to the combined Dirichlet and Poisson problems for L on U:

$$\begin{cases} Lu = -f & \text{on } U, \\ u(x) \to g(x_*) & \text{as } x \to x_* \in \partial U,\ x \in U, \end{cases} \tag{1.2}$$

where $f \in C^\infty(U)$ and g is continuous in a neighborhood of ∂U. Importantly, our assumptions allow for U to be unbounded and for L to be *hypoelliptic* on U (see Definition 2.6 below).

Formally, the method of stochastic characteristics (see Section 1.2 below) produces the formula u_{stoc} as a candidate expression for a classical solution of (1.2). However, in this setting, u_{stoc} may not be defined let alone be a classical solution of (1.2). The goal of this paper is to provide practical, probabilistic conditions under which u_{stoc} is well defined and satisfies the properties required by the problem (1.2):

- $u_{\text{stoc}} \in C^2(U)$ and $Lu_{\text{stoc}} = f$ on U in the classical sense;
- $u(x) \to g(x_*)$ as $x \to x_* \in \partial U$, $x \in U$.

Results for classical well-posedness of (1.2) in the setting of a bounded domain U are also obtained.

Although we employ tools from analysis, the manuscript primarily focuses on probabilistic methods. One may compare our work with the classical work of Oksendal [52, Chapter 9] adapted to our setting, but we aim to be more self-contained on this particular topic. Specifically, we start from a probabilistic construction of fundamental solutions and build up to a solution theory for the boundary-value problem (1.2). As a consequence, one may also compare this work to a portion of the classical elliptic theory text of Gilbarg and Trudinger [27], but we use probabilistic techniques in the possibly unbounded U, hypoelliptic L setting.

1.2. The method of stochastic characteristics

In order to employ probabilistic techniques, we associate to L a diffusion process x_t on $\mathcal{X}$ with infinitesimal generator L. That is, we suppose that x_t satisfies an Itô stochastic differential equation (SDE) of the form

$$dx_t = b(x_t)\,dt + \sigma(x_t)\,d\mathbf{W}_t\,, \tag{1.3}$$

where $\mathbf{W}_t = (W_t^1, W_t^2, \ldots, W_t^r)^T$ is a standard, r-dimensional Brownian motion defined on, and adapted to, a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P}, \mathbf{E})$. The *formal stochastic representation* u_{stoc} corresponding to the problem (1.2) then has the form

$$u_{\text{stoc}}(x) = \mathbf{E}_x \int_0^\tau f(x_s)\,ds + \mathbf{E}_x g(x_\tau), \qquad x \in U, \tag{1.4}$$

where

$$\tau = \inf\{t > 0 \,:\, x_t \notin U\} \tag{1.5}$$

is the first positive exit time from U. Note that in (1.4), the symbol $\mathbf{E}_x$ means the expected value for the law $\mathbf{P}_x$ of the process x_t with $x_0 = x$. The formula (1.4) is formally derived using Dynkin's/Itô's formula (2.8) applied to a sufficiently

nice classical solution of (1.2), which may or may not exist in this setting. Specifically, in Dynkin's formula (2.8) below, formally set $\phi(t,x) = u(x)$ and $\sigma = \tau$ to arrive at (1.4). Compared with the usual method of characteristics for first-order partial differential equations, this is the analogous method for linear second-order equations, but this time the characteristic trajectories x_t are random.

Aside from it being unclear its precise relationship to (1.2), observe that the expression u_{stoc} in (1.4) is itself formal in two ways. First, depending on the set U and the coefficients b, σ, both the process x_t and the expected values in (1.4) may not be defined, so u_{stoc} in turn is not defined. Second, even if u_{stoc} is defined on U, it may not satisfy the equation (1.2) in the classical sense. However, employing this formal argument via Dynkin's formula (2.8), u_{stoc} is a *best guess* at a solution of (1.2), and so it is natural to study u_{stoc} in relation to (1.2).

If the operator L is hypoelliptic on U, then it has a smoothing effect reminiscent of second-order uniformly elliptic operators on U with smooth coefficients. However, even if U is a bounded domain with boundary ∂U satisfying the *exterior cone condition* (see Definition 3.10 below) there are many examples where the problem (1.2) is ill-posed in the classical sense (see Example 3.8 below). This is different compared to equation (1.2) when L is a uniformly elliptic operator of the form (1.1). From a probabilistic perspective, this difference can be explained intuitively, especially as it relates to satisfaction of the boundary condition in equation (1.2). Indeed, if L is uniformly elliptic, then the noise in (1.3) is present in each direction of the equation, and therefore for short times the particle visits all points in a small ball. See [16] for a rigorous formulation of this statement. This is, however, not necessarily the case for a general hypoelliptic diffusion. In particular, even if the process x_t initiated at any point $x \in U$ is defined, it may not hit certain portions of the boundary with positive probability, and consequently the values of g on these portions do not influence u_{stoc} in (1.4). Moreover, when started on the boundary ∂U, the process may have a positive probability of re-entering U prior to exiting, so that u_{stoc} may not satisfy the boundary condition. Thus, we can arbitrarily define the boundary conditions on a part of the boundary that is not hit by the process, and from the abstract perspective it means that the operator that maps boundary conditions to the solution has infinite dimensional kernel and infinite dimensional co-kernel. Thus the classical spectral theory is not applicable and tools from the functional analysis are limited.

1.3. Previous results and layout of the paper

Despite these issues in the hypoelliptic setting, understanding when the equation or boundary conditions in (1.2) are satisfied in the classical sense by u_{stoc} is of notable importance, as hypoelliptic operators, and their corresponding boundary-value problems like (1.2), play a key role in a number of problems in science and engineering. See, for example, hypoelliptic diffusions arising in

finite-dimensional models of turbulence [4, 5, 6, 7, 18, 19, 23, 32, 58] or in statistical mechanics and machine learning [10, 15, 26, 31, 46, 61]. Direct applications also occur in the ergodic theory of SDEs, where functionals of return times to compact sets can be seen as the formal stochastic representations corresponding to equations of the form (1.2). Such functionals are essential to understand precise rates of convergence to equilibrium [17, 29, 56].

Historically, various aspects of the problem (1.2) in the hypoelliptic setting have drawn interest from researchers dating back to Kolmogorov [41], who gave the first known example of a hypoelliptic diffusion that is not uniformly elliptic. Later, the seminal work of Hörmander [33] provided an efficient tool to determine hypoellipticity of L by calculating the Lie algebra of vector fields that define the operator L in the form (1.1). See Section 2.4 for further information. For classical well-posedness of the problem (1.2), the pioneering work of Bony [9] is fundamental. There, Bony gives conditions on a bounded open set U and a hypoelliptic operator M of the form $M = L-a$, where $a \in C^\infty(U)$ is positive and bounded away from zero on U, so that the problem (1.2) with M replacing L is well-posed in the classical sense. Apart from the presence of the positive function a, which in particular aides in the existence part of the problem (1.2), a critical assumption guaranteeing continuity on $\overline{U}$ is that at every point on ∂U, the noise in the equation points in the normal direction. Such an assumption is often not satisfied for many hypoelliptic operators of interest, for example if the noise is additive (that is, spatially constant) [6, 7, 10, 15, 18, 19, 23, 31, 32, 46, 58]. Thus, it is natural to investigate if classical well-posedness holds under weaker assumptions. More recently, the hypoelliptic Dirichlet problem ($f \equiv 0$ in (1.2)) in a bounded domain U was revisited in [55], where Bony's result was extended to operators L of the form (1.1); that is, with $a \equiv 0$, assuming L satisfies a maximum principle and again that at every point on ∂U there is noise pointing in the normal direction. Similar to Bony's work, the results in [55] fail to apply in many problems of interest.

Perhaps more closely related to the present paper is the work of Kogoj [39], which uses methods from potential theory to establish a cone-type criterion for the existence of a generalized solution, in the sense of Perron-Wiener, of the Dirichlet problem for a second-order hypoelliptic operator in the form $L - \partial_t$ where L is as in (1.1). Importantly, [39] assumes the existence of a *well-behaved* fundamental solution and that the diagonal of $\sigma(x)\sigma(x)^T$ never vanishes. Note that, probabilistically, the cone-type condition is natural as it gives particle a space to exit the domain, and indeed for uniformly elliptic operators, such condition is sufficient to guarantee that every point on the boundary is regular (see Section 3 for a probabilistic definition of a regular point). As described above, additional conditions are required in the hypoelliptic setting, and the supplementary criteria provided in [39] appear difficult to check in concrete scenarios. We refer the reader to [22, 45, 51] for earlier, related results and to the interesting, recent work of Carfagnini and Gordina [11] in the context of sub-Laplacians on homogeneous Carnot groups. There, an analytical criteria is given for a boundary point to be regular. We also refer to the recent work of the authors [16] which provides probabilistic methods for determining regular

points on the boundary using the theory of large deviations and laws of the iterated logarithm. Note that [16] was partly inspired by the work of Lachal [44], where a functional law of the iterated logarithm was obtained for the iterated Kolmogorov stochastic differential equation. However, applications to solvability of (1.2) were not discussed in detail in [16] or [44].

In order to solve (1.2) for a hypoelliptic, second-order operator L, one can alternatively employ the methods of Dirichlet forms and/or more classical analytical methods. Although the work on this topic is vast, we refer the reader to [8, 12, 20, 25, 28, 34, 47, 57, 59]. In these contexts, the operator L is assumed to have a simpler form, i.e. be nonnegative with respect to a natural inner product [47], e.g. $L^2(dx)$, or be a sum of squares of vector fields [59], often with simpler boundary conditions, e.g. zero boundary conditions [47]. These simplifications allow one to employ tools from geometry as well as arguments similar to the ones employed when L is the standard Laplacian. However, even though these results work in hypoelliptic settings, the results do not apply in the case when L has a drift vector field and the drift is needed to generate directions in the Lie algebra. These, more difficult situations where the drift vector field is indispensably needed in the Lie algebraic calculations are referred to as the *weakly hypoelliptic* settings (see Remark 2.9 for a precise definition). One of the goals of this work is to produce results applicable in such cases.

Moving in the weakly hypoelliptic direction, let us also mention the works [2, 13, 53, 54] where upper and lower Gaussian-type density bounds are obtained for certain, weakly hypoelliptic diffusions. Such results are generalizations of the classical works of Kusuoka and Stroock [42, 43]. In these works, upper Gaussian bounds are obtained under general conditions [42] while lower Gaussian bounds [43] are obtained under the *strong Hörmander* condition, i.e. the drift is not needed in the bracket calculation. While we do not seek to obtain Gaussian estimates in this setting, the calculations used in this setting are reminiscent of the derivation of the laws of the iterated logarithm for the corresponding diffusion [16].

In addition to generalizing the results mentioned above, another goal of the present work is to provide a relatively self-contained presentation on the topic in this paper, referring primarily to graduate-level textbooks to obtain needed results. Our hope is that the level of the paper is similar to the level in Oksendal's book [52], and that the interested graduate student who has had an introductory course in stochastic analysis will find the paper readable. We did this for two reasons. First, prior to starting this project, at times we found it difficult to locate results for (1.2) in the hypoelliptic setting and this sentiment was also confirmed in conversations with colleagues. Second, there were claims in the literature about (1.2) that seemed both correct and intuitive, but we could not locate a proof in the existing literature. Thus, we hope this paper will serve as a resource to which the (stochastic) analysis community can refer as needed, since either there is a physical proof in the paper or the statement is easy to locate in the literature using the provided references.

The organization of this paper is as follows. In Section 2, we introduce further notation, assumptions, terminology and make preliminary remarks about

hypoellipticity. Hörmander's theorem, in particular, will be discussed in detail in Section 2. In Section 3, we elaborate on the importance of boundary behavior for x_t as its relates to solvability of (1.2) in the classical sense. In particular, in Section 3 one can find our definition of *regular* and *irregular* points on $\overline{U}$. Section 4 focuses on special equations of the form (1.2), in particular those involving functionals of τ, and these special equations are used later to formulate our main general results in Section 6. Section 6 provides conditions on L, τ, and u_{stoc} that guarantee $Lu_{\text{stoc}} = -f$ on U in the sense of distributions, hence in the classical sense if L is hypoelliptic and f smooth. Green's functions and a generalization of Bony's form of the Harnack inequality [9, Theorem 7.1] are discussed in Section 5, while in Section 7, as an application of these results, we re-derive the transience and recurrence dichotomy for degenerate diffusions.

2. Notation and preliminary remarks

In this section, we introduce notation, assumptions and terminology used throughout the paper. We also make a few preliminary remarks. First, we fix some standard notation.

2.1. Basic notation

For Borel sets $V, V_1 \subset \mathbf{R}^k$, $V_2 \subset \mathbf{R}^n$, and $W \subset \mathbf{R}^\ell$, we use the following notation.

- $C(V;W)$ denotes the set of continuous functions $\phi : V \to W$;
- $C^j(V;W)$, $j \geq 1$, denotes the set of j-times continuously differentiable functions $\phi : V \to W$;
- $C^\infty(V;W) = \bigcap_{j=1}^\infty C^j(V;W)$;
- $C^{j_1,j_2}(V_1 \times V_2; W)$ denotes the set of functions $\phi = \phi(x,y) : V_1 \times V_2 \to W$ which are j_1-times continuously differentiable in x and j_2-times continuously differentiable in y;
- $B(V;W)$ denotes the set of bounded, Borel measurable functions $\phi : V \to W$;
- In any of the above functions spaces, a subscript of 0 indicate that the function is moreover compactly supported in its domain of definition, e.g. C_0^j versus C^j;
- For $1 \leq p \leq \infty$, $L^p(V)$ denotes the set of measurable functions $\phi : V \to \mathbf{R}$ such that $\|\phi\|_{L^p(V)}^p := \int_V |\phi|^p \, dx < \infty$ if $p \in [1,\infty)$ and $\|\phi\|_{L^\infty(V)} := \text{ess sup}_{x \in V} |\varphi(x)| < \infty$ if $p = \infty$;
- $\text{Lip}(V;W)$ denotes the set of functions $\phi : V \to W$ which are Lipschitz continuous on V;
- For any $s \in \mathbf{R}$, $H^s(\mathbf{R}^k)$ denotes the usual Sobolev space $W^{s,2}(\mathbf{R}^k)$;
- If the target W is clear from context or not important, we write $C^j(V) = C^j(V;W)$, $C_0^j(V) = C_0^j(V;W)$, $B(V) = B(V;W)$, etc. When the context is clear, we may also drop V and write C^j, C_0^j, etc.
- $\mathcal{B}_V$ denotes the set of Borel measurable subsets of V;

- $\mathcal{B}$ denotes the set of Borel measurable subsets of the open set $\mathcal{X}$, where $\mathcal{X}$ is as in the introduction.
- For any $B \in \mathcal{B}$ bounded with $\overline{B} \subset \mathcal{X}$, we define

$$\tau_B = \inf\{t > 0 \,:\, x_t \in B^c\}, \tag{2.1}$$

  where $\inf \emptyset := \infty$.
- We define a sequence $\mathcal{X}_n \subset \mathcal{X}$, $n \geq 1$ of bounded open sets with $\overline{\mathcal{X}_n} \subset \mathcal{X}_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} \mathcal{X}_n = \mathcal{X}$.
- At times, we will also need a sequence $U_n \subset U$, $n \geq 1$, of bounded open sets with $\overline{U_n} \subset U_{n+1}$ and $\bigcup_{n=1}^{\infty} U_n = U$.
- We define

$$\tau = \tau_U := \lim_{k\to\infty} \tau_{U\cap\mathcal{X}_k}. \tag{2.2}$$

  and

$$\tau_{\mathcal{X}} = \lim_{k\to\infty} \tau_{\mathcal{X}_k}. \tag{2.3}$$

2.2. Nonexplosivity of x_t

Depending on the behavior of the coefficients b and σ, the solution x_t of equation (1.3) evolving on the neighborhood $\mathcal{X}$ of $\overline{U}$ is only a priori defined locally in time until the process exits $\mathcal{X}$. That is, by the standard existence and uniqueness theorem for SDEs [52], for any initial condition $x \in \mathcal{X}$ at time $t = 0$, equation (1.3) has a unique (pathwise) solution x_t for all times $0 \leq t < \tau_{\mathcal{X}}$. However, throughout the paper, we will assume that $\tau_{\mathcal{X}}$ is almost surely infinite as in the following definition.

Definition 2.1. We say that the process x_t is *nonexplosive* if

$$\mathbf{P}_x\{\tau_{\mathcal{X}} < \infty\} = 0 \quad \text{for all } x \in \mathcal{X}, \tag{2.4}$$

where $\mathbf{P}_x$ denotes the probability $\mathbf{P}$ but indicates that $x_0 = x$.

The process being nonexplosive implies that for all initial conditions $x \in \mathcal{X}$, equation (1.3) has a unique pathwise solution x_t defined on $\mathcal{X}$ for all finite times $t \geq 0$ almost surely. Furthermore, $t \mapsto x_t : [0, \infty) \to \mathcal{X}$ is continuous almost surely.

Remark 2.2. Suppose that U is bounded. Since b and σ are smooth and defined on an open neighborhood of $\overline{U}$, one can extend b and σ to functions $\hat{b} \in C^\infty(\mathbf{R}^{\mathfrak{m}}; \mathbf{R}^{\mathfrak{m}})$ and $\hat{\sigma} \in C^\infty(\mathbf{R}^{\mathfrak{m}}; M_{\mathfrak{m}\times r})$, respectively, such that $\hat{b} = b$ on $\overline{U}$ and $\hat{\sigma} = \sigma$ on $\overline{U}$ and such that $\hat{b}, \hat{\sigma}$ both have bounded derivatives of all orders, e.g. extend b, σ to be zero outside of a larger neighborhood and mollify. Consequently, replacing $\mathcal{X}$ with $\mathbf{R}^{\mathfrak{m}}$ and b, σ in (1.3) with $\hat{b}, \hat{\sigma}$ we find that the resulting solution $\hat{x}_t$ is nonexplosive. Note that the problem (1.2) and its

formal stochastic representation (1.4) remain unchanged regardless of the chosen extension, so nonexplosivity is implicitly present. On the other hand, if U is unbounded, then one must verify nonexplosivity separately, typically using Lyapunov methods. See Lemma 2.4 below for further details.

If x_t is nonexplosive, then x_t is a well-defined Markov process on $\mathcal{X}$. It thus induces a corresponding Markov semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ which acts on functions $\phi \in B(\mathcal{X};\mathbf{R})$ via

$$\mathcal{P}_t\phi(x) := \mathbf{E}_x\phi(x_t) \tag{2.5}$$

and dually on $\mathcal{B}$-measures ν via

$$\nu\mathcal{P}_t(B) = \int_{\mathcal{X}} \mathcal{P}_t\mathbf{1}_B(x)\nu(dx),\ \ B \in \mathcal{B}. \tag{2.6}$$

We let

$$\mathcal{P}_t(x,B) := \mathcal{P}_t\mathbf{1}_B(x), \qquad B \in \mathcal{B},\ x \in \mathcal{X}, \tag{2.7}$$

denote the corresponding Markov transitions.

Note that if L is as in (1.1) and $\phi \in C_0^{1,2}([0,\infty)\times\mathcal{X};\mathbf{R})$, then *Dynkin's formula*

$$\mathbf{E}_x\phi(\sigma,x_\sigma) = \phi(0,x) + \mathbf{E}_x\int_0^\sigma (\partial_t + L)\phi(s,x_s)\,ds \tag{2.8}$$

holds for any bounded stopping time σ with respect to the filtration $\mathcal{F}_t$. Dynkin's formula allows one to study various properties of L by analyzing path properties of the stochastic process x_t.

Remark 2.3. By a standard localization procedure applied to the equation (1.3) on $\mathcal{X}_{k+1}$, the same formula (2.8) holds for any bounded stopping time $\sigma \leq \tau_{\mathcal{X}_k}$ and any $\phi \in C^{1,2}([0,\infty)\times\mathcal{X}_{k+1})$ regardless if x_t is nonexplosive.

As a simple application of Dynkin's formula, we briefly recall the following basic method for checking that x_t is nonexplosive. See also [36, 49, 56].

Lemma 2.4. *Suppose there exists $w \in C^2(\mathcal{X};[0,\infty))$ and constants $C, D > 0$ such that*

$$w_k := \inf_{x\in\partial\mathcal{X}_k} w(x) \to \infty \ \ \textit{as}\ \ k\to\infty \qquad \textit{and} \qquad Lw \leq Cw + D \ \ \textit{on}\ \ \mathcal{X}.$$

Then x_t is nonexplosive as in Definition 2.1.

Proof. Remark 2.3, nonnegativity of w and Dynkin's formula together imply that if $\tau_k = \tau_{\mathcal{X}_k}$, then

$$\begin{aligned} w_k e^{-Ct}\mathbf{P}_x\{\tau_k \leq t\} &\leq \mathbf{E}_x w(x_{t\wedge\tau_k})e^{-C(t\wedge\tau_k)} \\ &= w(x) + \mathbf{E}_x\int_0^{t\wedge\tau_k} -Cw(x_s)e^{-Cs} + e^{-Cs}Lw(x_s)\,ds \end{aligned}$$

$$\leq w(x) + \int_0^t e^{-Cs} D \, ds \leq w(x) + D.$$

Rearranging the above we obtain $\mathbf{P}_x\{\tau_k \leq t\} \leq e^{Ct}(w(x) + D)/w_k$. Passing $k \to \infty$, we find that $\mathbf{P}_x\{\tau_{\mathcal{X}} \leq t\} = 0$ for any given $t \geq 0$ and $x \in \mathcal{X}$ and (2.4) follows. □

2.3. The process x_t stopped on ∂U

We will often use the process x_t *stopped* on the boundary ∂U. That is, define the stopping time

$$\tau_0 = \inf\{t \geq 0 : x_t \notin U\}. \tag{2.9}$$

Then the *stopped process* $\tilde{x}_t$ is defined by

$$\tilde{x}_t := x_{t \wedge \tau_0}, \ t \geq 0. \tag{2.10}$$

Because x_t is assumed to be nonexplosive, the stopped process $\tilde{x}_t$ is a well-defined Markov process distributed on $\overline{U}$ for all times $t \geq 0$. We let $\tilde{\mathcal{P}}_t$ and $\tilde{\mathcal{P}}_t(x, \cdot)$, $x \in \overline{U}$, respectively denote the Markov semigroup and transitions associated to $\tilde{x}_t$.

Remark 2.5. For the process x_t with $x_0 \in \overline{U}$, τ_0 is the first time x_t hits the boundary. Note that τ_0 is in general different from the first positive exit time τ from U when x_t initiates on the boundary ∂U, as the process started there may first enter the region U before exiting.

2.4. Hypoellipticity

Let us first define precisely the term *hypoelliptic*.

Definition 2.6. Let $V \subset \mathbf{R}^k$ be nonempty, open set and M be a differential operator with coefficients belonging to $C^\infty(V)$. We say that M is *hypoelliptic* on V if for any distribution v on V with $Mv \in C^\infty(W)$ for some $W \subset V$ nonempty open, we have $v \in C^\infty(W)$.

Fundamental to our analysis are the smoothing properties of the differential operator L afforded by hypoellipticity. In the classical paper [33], Hörmander (see Theorem 2.10 below) provides a sufficient condition for hypoellipticity on an open set $V \subset \mathbf{R}^k$ for operators M of the form

$$M = a + X_0 + \tfrac{1}{2} \sum_{\ell=1}^{j} X_\ell^2, \tag{2.11}$$

where $a \in C^\infty(V; \mathbf{R})$ and $X_0, X_1, \ldots, X_j$ are C^∞ vector fields on V.

After standard algebraic manipulations, note that

$$L, L^*, \partial_t \pm L, \partial_t \pm L^*,$$

where L is as in (1.1) and L^* denotes the formal $L^2(dx)$-adjoint of L, can all be written in the form (2.11) on the respective open sets $\mathcal{X}, \mathcal{X}, (0,\infty)\times\mathcal{X}, (0,\infty)\times\mathcal{X}$. For example, observe that if we let $X_i = \sum_{j=1}^{\mathsf{m}} \sigma_{ji}(x)\partial_{x_j}$, $i = 1,\dots,r$, and $Y_0 = \sum_{j=1}^{\mathsf{m}} b_j(x)\partial_{x_j}$, then

$$L = Y_0 - \sum_{\ell=1}^{m}\left[\sum_{i=1}^{r}\sum_{j=1}^{m}\sigma_{ji}(x)\partial_{x_j}(\sigma_{\ell i}(x))\right]\frac{\partial}{\partial x_\ell} + \tfrac{1}{2}\sum_{j=1}^{r} X_j^2 \tag{2.12}$$

$$=: X_0 + \tfrac{1}{2}\sum_{j=1}^{r} X_j^2. \tag{2.13}$$

To introduce Hörmander's condition which implies hypoellipticity, we first define the *Lie bracket* $[X,Y]$ of differentiable vector fields $X = \sum_i X^i\partial_{x_i}$ and $Y = \sum_j Y^j\partial_{x_j}$ on an open set in $\mathbf{R}^k$ by

$$[X,Y] = \sum_{j=1}^{k}\sum_{i=1}^{k}(X^i\partial_{x_i}(Y^j) - Y^i\partial_{x_i}(X^j))\partial_{x_j}.$$

Definition 2.7. Suppose $V \subset \mathbf{R}^k$ is an open set and M is an operator of the form (2.11), where $X_0, X_1, \dots, X_m$ are C^∞ vector fields on V and $a \in C^\infty(V)$. Define the following C^∞ vector fields on V:

$$\begin{array}{ll} X_{i_0}, & i_0 \in \{0,1,\dots,m\} \\ [X_{i_0}, X_{i_1}], & i_0, i_1 \in \{0,1,\dots,m\} \\ [[X_{i_0}, X_{i_1}], X_{i_2}], & i_0, i_1, i_2 \in \{0,1,2,\dots,m\} \\ \vdots & \vdots \end{array} \tag{2.14}$$

If the vector fields in (2.14) span the tangent space at all points in V, we say M satisfies the *Hörmander condition* on V.

Remark 2.8. If M of the form (2.11) is elliptic, then $X_1, X_2, \dots, X_m$ span the tangent space at all points in V. Thus the brackets in (2.14) can be viewed as a generalization of ellipticity. Relating this back to L in (1.1) and the equation (1.3), one can interpret Lie brackets in (2.13) as a propagation of the randomness implicitly through the equation.

Remark 2.9. Consider an operator M of the form (2.11) and the subclass of vector fields

$$\begin{array}{ll} X_{i_0} & i_0 \in \{1,2,\dots,m\} \\ [X_{i_0}, X_{i_1}] & i_0, i_1 \in \{1,2,\dots,m\} \\ [[X_{i_0}, X_{i_1}], X_{i_2}] & i_0, i_1, i_2 \in \{1,2,\dots,m\} \\ \vdots & \vdots \end{array} \tag{2.15}$$

Note here that, different from (2.14), the list (2.15) does not contain X_0 or any brackets with X_0. If, however, on an open set $V \subset \mathbf{R}^k$, the sublist (2.15) spans the tangent space at all points in V, then we call the operator M *strongly hypoelliptic* on V, or that it satisfies the *strong Hörmander condition* on V. Otherwise, if the full list (2.14) must be used and it spans the tangent space at all points in V, then M is called *weakly hypoelliptic* on V and we say that it satisfies the *weak Hörmander condition* on V. In the present manuscript, we consider more general operators M that are weakly hypoelliptic.

Hörmander's theorem provides refined estimates on the smoothing effect along every bracket in the list (2.14). Essentially, the more brackets one takes to reach a certain direction, the smaller the smoothing effect occurs along that direction. While we do not state this general result rigorously, we need below the following simpler version to establish a generalization of Bony's Harnack inequality in Section 5.

Theorem 2.10 (Hörmander 1967 [33])**.** *If M defined in (2.11) satisfies the Hörmander condition on a nonempty open set $V \subset \mathbf{R}^k$, then M is hypoelliptic on V. More specifically, there exists $\delta > 0$ such that for any $\psi_1, \psi_2 \in C_0^\infty(V)$ with $\psi_2 = 1$ on an open neighborhood of* $\operatorname{supp}(\psi_1)$ *we have*

$$\|\psi_1 u\|_{H^{s+\delta}(\mathbf{R}^k)} \leq C_s(\|\psi_2 M u\|_{H^s(\mathbf{R}^k)} + \|\psi_2 u\|_{H^s(\mathbf{R}^k)}) \tag{2.16}$$

for any $s \in \mathbf{R}$ and any distribution u on V with $\psi_2 u, \psi_2 M u \in H^s(\mathbf{R}^k)$. In (2.16), C_s is a constant depending only on s, ψ_1, ψ_2, and the domain V and all functions are assumed to be zero outside of their compact supports.

Remark 2.11. Note that hypoellipticity is a consequence of the inequality (2.16) by a bootstrapping argument. As remarked above, although we mostly use hypoellipticity somewhat independently of Hörmander's result, we need the precise estimate (2.16) for the proof of a generalization of Bony's Harnack inequality [9].

Often, instead of listing all of the operators we need to be hypoelliptic, we make a simper hypothesis ensuring that all of them are hypoelliptic. See the *parabolic Hörmander condition* below, which ensures that an entire list of operators is hypoelliptic.

Definition 2.12. Suppose that $V \subset \mathbf{R}^k$ is open and M is an operator of the form (2.11) where the $X_0, X_1, \ldots, X_m$ are C^∞ vector fields on V and $a \in C^\infty(V)$. Let $\mathcal{V}_0 = \{X_1, X_2, \ldots, X_m\}$ and for $k \geq 1$ inductively define

$$\mathcal{V}_k = \{[X_i, X] \,:\, X \in \mathcal{V}_{k-1}, i = 0, 1, 2, \ldots, m\} \qquad \text{and} \quad \mathcal{V} = \bigcup_{k=0}^{\infty} \mathcal{V}_k \,.$$

If $\mathcal{V}$ spans $\mathbf{R}^{\mathbf{m}}$ at all points in V, then we say that M satisfies the *parabolic Hörmander condition* on V.

Remark 2.13. Observe that for M to satisfy the parabolic Hörmander condition, one cannot include X_0 itself in the list of a spanning set. Rather, X_0 must be first commuted with another vector field, for example $[X_0, X_1]$.

Example 2.14. Let

$$M = \partial_{x_2}^2 + \partial_{x_1} =: X_1^2 + X_0.$$

Then, M satisfies the Hörmander condition on $\mathbf{R}^2$ but M does not satisfy the parabolic Hörmander condition on any open set in $\mathbf{R}^2$.

Because the parabolic Hörmander condition limits the fields that can be taken in a spanning set, a routine calculation shows that if $V \subset \mathcal{X}$ is nonempty, open and L as in (1.1) is written in the form (2.13) and satisfies the parabolic Hörmander condition on V, then all operators

$$L,\ L^*,\ L+\beta,\ L^*+\beta,\ \partial_t \pm L,\ \partial_t \pm L^* \tag{2.17}$$

where $\beta \in \mathbf{R}$ are hypoelliptic on the respective sets

$$V,\ V,\ V,\ V,\ (0,\infty)\times V,\ (0,\infty)\times V. \tag{2.18}$$

Below, we will see that hypoellipticity of the operators in (2.17) decides the existence and regularity of densities related to the law of x_t.

2.5. Assumption list

Here we provide an almost complete list of assumptions used in the paper. It is meant as a reference except for Remark 2.15. Thus, aside from Remark 2.15, the reader should skip the rest of this section and come back to consult particular assumptions used later in the paper.

Remark 2.15. Throughout, we assume that $b \in C^\infty(\mathcal{X}; \mathbf{R}^{\mathfrak{m}})$ and $\sigma \in C^\infty(\mathcal{X}; M_{\mathfrak{m}\times r})$ without explicitly mentioning it. Also, if U is bounded, one does *not* need to explicitly assume that x_t is nonexplosive using the extension argument from Remark 2.2.

Depending on the context, we apply the following assumptions as needed.
(U00) $U \subset \mathbf{R}^{\mathfrak{m}}$ is nonempty, open set and $V := U^c$ is nonempty, closed set with $\mathcal{X} \supset \overline{U}$ open.
(U0) $U \subset \mathbf{R}^{\mathfrak{m}}$ is nonempty, open set and $\mathcal{X} \supset \overline{U}$ is an open set.
(L1) L is hypoelliptic on U.
(L2) $\partial_t - L^*$ is hypoelliptic on $(0,\infty)\times U$.
(L3) $\partial_t - L$ is hypoelliptic on $(0,\infty)\times U$.
(L4) L^* is hypoelliptic on U.
(PH) L satisfies the parabolic Hörmander condition on U as in Definition 2.12.
(UID(g, x_*)) Fix $g : \overline{U} \to \mathbf{R}$ measurable and $x_* \in \partial U$. Then for some $\delta > 0$ the family

$$\mathcal{G}_{g,\delta}(x_*) := \{g(x_\tau(x)) \,:\, |x - x_*| < \delta,\ x \in U\} \tag{2.19}$$

is uniformly integrable. Here, τ is as in (2.2) and x in $x_\tau(x)$ means $x_0 = x$.
(UID(g)) Fix $g : \overline{U} \to \mathbf{R}$ measurable. Then, for every $x_* \in \partial U$, condition **(UID(g, x_*))** is satisfied.
(UIP(f, x_*)) Fix $x_* \in \partial U$ and $f : \overline{U} \to \mathbf{R}$ measurable. Then for some $\delta > 0$,

$$\mathcal{G}_\delta^f(x_*) := \{\textstyle\int_0^\tau f(x_s(x))\, ds \,:\, |x - x_*| < \delta,\, x \in U\} \tag{2.20}$$

is uniformly integrable.
(UIP(f)) Fix $f : \overline{U} \to \mathbf{R}$ measurable. Then for every $x_* \in \partial U$, condition **(UIP(f, x_*))** is satisfied.
(CE(x_*)) Fix $x_* \in \partial U$ and recall $\mathcal{X}_n$ introduced in (2.3). For every $\delta_1 > 0$ there exists $n \in \mathbf{N}$ and $\delta_2 > 0$ such that $|x - x_*| < \delta_2$, $x \in U$, implies

$$\mathbf{P}_x\{\tau_{\mathcal{X}_n} < \tau\} < \delta_1. \tag{2.21}$$

(CE) For every $x_* \in \partial U$, condition **(CE(x_*))** is satisfied.

In Remark 6.7, we provide sufficient conditions for **(CE(x_*))** to hold.

3. Remarks on boundary behavior

In order to solve the equation (1.2) in the classical sense, understanding the behavior of the process x_t satisfying the equation (1.3) near the boundary ∂U is critical. In this section, we explore various conditions related to boundary behavior used in the literature to ensure well-posedness of equation (1.2) (in the classical sense) when L fails to be uniformly elliptic.

3.1. Nice points and regular points

Definition 3.1. We call $x_* \in \partial U$ *nice* if there exists an open neighborhood $U_{x_*} \subset \mathcal{X}$ of x_* and a function $w \in C^2(U_{x_*})$ satisfying the following conditions:

(i) $w > 0$ on $U_{x_*} \setminus \{x_*\}$ and $w(x_*) = 0$;
(ii) $Lw < 0$ on U_{x_*}.

Remark 3.2. We show in Proposition 3.9 below that if $x_* \in \partial U$ is nice and the relevant hypoellipticity is satisfied, then the process x_t exits U instantaneously when started from x_*. However, proving that x_* is nice; that is, finding a Lyapunov-type function w as in Definition 3.1, can be highly nontrivial or even impossible. One can ensure $x_* \in \partial U$ is nice provided ∂U has an exterior normal vector to U at x_* and randomness pointing in the direction of the normal vector (see Remark 3.11 below). Intuitively, the process x_t projected onto this normal direction for small times behaves like a scaled one-dimensional Brownian motion. Then, the process must exit the domain instantaneously as the one-dimensional Brownian motion has no preferred direction and dominates the motion in small times. See [16, Section 7] for further details.

Remark 3.3. If U is furthermore assumed to be bounded and L is assumed to be hypoelliptic, then one can show that the Dirichlet problem (1.2) with $f \equiv 0$, has a unique classical solution if all points on the boundary are nice. Although the main result in [55] establishes this fact, it is not explicitly stated in this way.

In this paper, we find it more convenient to phrase our hypotheses in terms of stopping times for the process x_t solving (1.3). Hence, we define

$$\overline{\tau} = \inf\{t \geq 0 \,:\, x_t \notin \overline{U}\}\,, \tag{3.1}$$

and recall the first positive exit time τ from U defined in equation (2.2).

Definition 3.4. We call an interior point $x \in U$ *regular* if $\mathbf{P}_x\{\tau = \overline{\tau}\} = 1$. A boundary point $x_* \in \partial U$ is called *regular* if $\mathbf{P}_{x_*}\{\overline{\tau} = 0\} = 1$. Points in $\overline{U}$ are called *irregular* otherwise. We call the set U *interior regular* if all $x \in U$ are regular. We call U *boundary regular* if all $x_* \in \partial U$ are regular.

Remark 3.5. Observe that if $x \in U$ is regular, then the process x_t started from x exits U and $\overline{U}$ at the same time. In particular, x_t cannot reach ∂U and return to the interior of U with positive probability. On the other hand, $x_* \in \partial U$ being regular means x_t initiated at x_* must exit $\overline{U}$ instantaneously.

Remark 3.6. Note that the event $\{\overline{\tau} = 0\}$ belongs to the *germ sigma field* $\bigcap_{t>0} \mathcal{F}_t$, hence has probability 0 or 1 by Blumenthal's 0-1 law. Thus, $x_* \in \partial U$ being regular is equivalent to $\mathbf{P}_{x_*}\{\overline{\tau} = 0\} > 0$.

Using the strong Markov property for x_t, the next result states that boundary regular implies interior regular.

Proposition 3.7. *If U is boundary regular, then U is interior regular.*

Proof. Let $x \in U$ and suppose $\tau = \infty$. Then $\overline{\tau} \geq \tau = \infty$, so $\tau = \overline{\tau}$. If the event $\{\tau < \infty\}$ has positive probability, then the strong Markov property and boundary regularity gives

$$\begin{aligned}\mathbf{P}_x\{\tau = \overline{\tau},\, \tau < \infty\} = \mathbf{E}_x[\mathbf{E}_x \mathbf{1}_{\{\tau=\overline{\tau},\, \tau<\infty\}} | \mathcal{F}_\tau] &= \mathbf{E}_x \mathbf{1}_{\{\tau<\infty\}} \mathbf{P}_{x_\tau}\{\overline{\tau} = 0\} \\ &= \mathbf{P}_x\{\tau < \infty\}.\end{aligned}$$

This finishes the proof since

$$\begin{aligned}\mathbf{P}_x\{\tau = \overline{\tau}\} &= \mathbf{P}_x\{\tau = \overline{\tau},\, \tau = \infty\} + \mathbf{P}_x\{\tau = \overline{\tau},\, \tau < \infty\} \\ &= \mathbf{P}_x\{\tau = \infty\} + \mathbf{P}_x\{\tau < \infty\} = 1.\end{aligned}$$ □

As the next example shows, the converse of Proposition 3.7 is false even if the parabolic Hörmander condition is satisfied.

Example 3.8. Let $U \subset \mathbf{R}^2$ be the interior of the open square with vertices $(-1, 1)$, $(1, 1)$, $(1, -1)$, $(-1, -1)$, and consider the following SDE on U

$$\begin{aligned} dx_t^1 &= -(x_t^2)^2\, dt\,, \\ dx_t^2 &= \sqrt{2}\, dB_t\,, \end{aligned}$$

where B_t is a standard, one-dimensional Brownian motion. Let I be the open segment connecting $(1,-1)$ and $(1,1)$. Note that all points on $I \subset \partial U$ are not regular since x_t^1 is decreasing for all times. On the other hand, $x_t = (x_t^1, x_t^2)$ has generator

$$L = -(x^2)^2 \partial_{x^1} + \partial_{x^2}^2 =: X_0 + X_1^2,$$

which satisfies the parabolic Hörmander condition on $\mathbf{R}^2$ since

$$X_1 = \partial_{x^2} \quad \text{and} \quad [X_1, [X_1, X_0]] = -2\partial_{x^1}$$

span the tangent space at all points in $\mathbf{R}^2$. Also, observe that the process initiated at $x \in U$ can only exit U on $\partial U \setminus I$. Hence, U is interior regular since every point on $\partial U \setminus I$ is regular.

Next, we clarify a the relationship between boundary regular and nice.

Proposition 3.9. *Suppose $x_* \in \partial U$ is nice and that $\partial_t \pm L$ and $\partial_t \pm L^*$ are hypoelliptic on $(0,\infty) \times B_\delta(x_*)$ for some $\delta > 0$ such that $B_\delta(x_*) \subset \mathcal{X}$. Then, x_* is regular.*

Proof. Let $x_* \in \partial U$ be nice and fix $\delta > 0$ such that $\partial_t \pm L^*$, $\partial_t \pm L^*$ are hypoelliptic on $(0,\infty) \times B_\delta(x_*)$. Choose an open neighborhood U_{x_*} of x_* such that there exists a function $w \in C^2(U_{x_*})$ satisfying properties (i) and (ii) in Definition 3.1. Without loss of generality, we may assume that $V := U_{x_*} = B_\delta(x_*)$. It follows that the distribution μ_t of the stopped process $x_t^V := x_{t\wedge\tau_V}$ (see Theorem 4.3) satisfies the equation $(\partial_t + L^*)(\mu_t) = 0$ in the sense of distributions on $(0,\infty) \times V$. Thus, when restricted to subsets of $(0,\infty) \times V$, μ_t is absolutely continuous with respect to Lebesgue measure with density p_t which is smooth on $(0,\infty) \times V$. In other words, the law of x_t^V restricted to subsets of V, for fixed t, is absolutely continuous with respect to Lebesgue measure on V.

Now if x_* is not regular, then $\mathbf{P}_{x_*}\{\overline{\tau} > 0\} = 1$. Thus, there exists $\epsilon > 0$ such that $B_\epsilon(x_*) \subset V$ and $\mathbf{P}_{x_*}\{\tau_{B_\epsilon(x_*)} < \overline{\tau}\} > 0$, for otherwise the distribution of x_t^V started at x_* would have nonzero mass concentrated at x_*, violating the absolute continuity of μ_t above. For $\sigma(t) := t \wedge \tau_{B_\epsilon(x_*)} \wedge \overline{\tau}$, Dynkin's formula (see (2.8)) and positivity of w on $\partial B_\epsilon(x_*)$ gives

$$c\mathbf{P}_{x_*}\{\tau_{B_\epsilon(x_*)} < \overline{\tau} \wedge t\} \le \mathbf{E}_{x_*} w(x_{\sigma(t)}) \le w(x_*) = 0$$

for some constant $c > 0$ independent of t. Passing $t \to \infty$ we obtain $\mathbf{P}_{x_*}\{\tau_{B_\epsilon(x_*)} < \overline{\tau}\} = 0$, a contradiction. □

Recall the following properties of ∂U.

Definition 3.10. We say that the boundary ∂U of an open set U satisfies the *exterior sphere condition* at $x_* \in \partial U$ if there exists $\lambda > 0$ such that $x_* \in \overline{B_\lambda(x')}$ and $B_\lambda(x') \subseteq \overline{U}^c$, where $B_\lambda(x')$ is the open ball centered at $x' = x_* + \lambda\nu(x_*)$ with $\nu(x_*)$ being a unit exterior normal vector to ∂U at x_*.

We say ∂U satisfies the *exterior cone condition* at $x_* \in \partial U$, if there exists a basis $x_1, x_2, \ldots, x_{\mathfrak{m}}$ of $\mathbf{R}^{\mathfrak{m}}$ and a parameter $\lambda_* > 0$ such that

$$\mathrm{Cone}(x_*, \lambda_*) := \{x_* + \textstyle\sum_{i=1}^{\mathfrak{m}} \lambda_i x_i \,:\, \lambda_i \in (0, \lambda_*)\} \subset \overline{U}^c.$$

Remark 3.11. Fix $x_* \in \partial U$ and suppose that ∂U satisfies the exterior sphere condition at x_*. In addition, assume $\partial_t \pm L$ and $\partial_t \pm L^*$ are hypoelliptic on $(0,\infty) \times B_\delta(x_*)$ for some $\delta > 0$ such that $B_\delta(x_*) \subset \mathcal{X}$. It is not hard to see that we can decrease λ in the definition of exterior sphere condition if necessary. Then, if the noise has a nonzero component in $\nu(x_*)$ at x_*, then x_* is regular (see for example [9]). Such an assumption in fact implies that x_* is nice. Indeed, suppose

$$\sum_{i,j=1}^{\mathfrak{m}} (\sigma(x_*)\sigma^T(x_*))_{ij} v_i(x_*) v_j(x_*) > 0. \tag{3.2}$$

Then, by choosing $\beta > 0$ large enough, the function

$$w(x) = e^{-\beta|x'-x_*|^2} - e^{-\beta|x'-x|^2} \tag{3.3}$$

satisfies Definition 3.1 at x_* on $U_{x_*} = B(x_*, |x'-x_*|/2) \cap U$ if $x' = x_* + \lambda\nu(x_*)$ and $\lambda > 0$ is sufficiently small. Note that (3.2) and large $\beta > 0$ allows one to to disregard terms in Lw of order β.

3.2. Modification of the domain

We conclude this section by restating [16, Corollary 7.10] in our context. Intuitively, it asserts that starting from a *reasonable* domain U, provided there is noise in a fixed direction for all points on the boundary ∂U (see Remark 3.11), one can slightly modify U to produce an approximate domain V which is boundary regular. Thus, by Proposition 3.7, V is both interior regular and boundary regular.

In order to state the result, we say that U has *nonflat* boundary ∂U if for each $x_* \in \partial U$ and every $r > 0$, the set $\partial U \cap B_r(x_*)$ is not a subset of a hyperplane. Then the following result was proved in [16, Corollary 7.10].

Theorem 3.12. *Suppose that $U \subset \mathbf{R}^{\mathfrak{m}}$, $\mathfrak{m} \geq 2$, is a bounded, convex, nonflat domain with C^1 boundary ∂U. Suppose, furthermore, that there exists a fixed unit vector $v \in \mathbf{R}^{\mathfrak{m}}$ such that v belongs to the column space of $\sigma(x)$ for all $x \in \partial U$. Then, for every $\epsilon > 0$, there exists a nonempty, open convex domain $V_\epsilon \subset U$ with piecewise linear boundary ∂V_ϵ such that V_ϵ and V_ϵ^c are boundary regular for x_t and $|U| - |V_\epsilon| < \epsilon$. Here, $|\cdot|$ denotes Lebesgue measure on $\mathbf{R}^{\mathfrak{m}}$.*

To prove Theorem 3.12, one defines V_ϵ to be the interior of the convex hull of a sufficient number of points $x_1, x_2, \ldots, x_{n(\epsilon)} \in \partial U$ selected independently and randomly according to Hausdorff measure on ∂U. One can then show that with probability one, the faces generated by these points on ∂V_ϵ are not parallel to the fixed vector v in Theorem 3.12, so that the process exits $\overline{V_\epsilon}$ instantaneously when started there. The fact that the volumes of V_ϵ and U are close is intuitive provided $n(\epsilon)$ is large enough. This fact was rigorously proved in [60].

4. Interior regularity of functionals of τ

Our goal in this section is to obtain interior smoothness of various functionals of the first positive exit time τ from U. Given the relevant hypoellipticity, the key first step is to show that the law of the stopped process $\tilde{x}_t$, introduced in Section 2.3, when restricted to Borel subsets of U, denoted by $\mathcal{B}_U$, has a density $\tilde{p}_t(x,y)$ with respect to Lebesgue measure on U. Furthermore, the mapping $(t,x,y) \mapsto \tilde{p}_t(x,y) \in C^\infty((0,\infty)\times U\times U)$ and satisfies, respectively, the forward and backward Kolmogorov equations in the classical sense:

$$\partial_t \tilde{p}_t(x,y) = L_y^* \tilde{p}_t(x,y) \quad \text{on} \quad (0,\infty)\times U,\ x\in U \text{ fixed}, \tag{4.1}$$

$$\partial_t \tilde{p}_t(x,y) = L_x \tilde{p}_t(x,y) \quad \text{on} \quad (0,\infty)\times U,\ y\in U \text{ fixed}. \tag{4.2}$$

This is done in Section 4.2. In Section 4.3, we deduce interior smoothness and the equations satisfied by

$$x \mapsto \mathbf{E}_x \varphi(\tau) : U \to \mathbf{R} \tag{4.3}$$

for some choices of smooth $\varphi : [0,\infty) \to \mathbf{R}$, for example $\varphi(x) = x^k$, $k \in \mathbf{N}$ or $\varphi(x) = \mathbf{1}_{(t,\infty)}(x)$ for fixed t.

First, however, we need to establish a few auxiliary results.

4.1. Auxiliary results

Lemma 4.1. *Suppose that the condition **(U0)** is satisfied and that the process x_t is nonexplosive as in Definition 2.1. Then, the following assertions hold true.*

(i) For any $t>0$, $x\in\mathcal{X}$, and $M>0$ we have

$$\lim_{y\to x} \mathbf{E}\Big\{ \sup_{s\in[0,t]} |x_s(x) - x_s(y)|^2 \wedge M \Big\} = 0\,,$$

where $s\mapsto x_s(z)$ satisfies (1.3) with $x_0(z)=z$.

(ii) If $x_*\in\partial U$ is regular, then for any $\epsilon,\delta>0$, there is $\rho>0$ such that $\mathbf{P}_x\{\tau\ge\delta\}\le\epsilon$ for any $x\in U$ with $|x-x_*|<\rho$. That is, as $x\to x_*\in\partial U, x\in U$, $\tau(x)\to\tau(x_*)$ in probability.

Proof. The proof of part (i) follows a standard Grönwall-type comparison argument. Let $\tau_k = \tau_{\mathcal{X}_k}$ and we denote the dependence of τ_k on the initial condition x of x_t by writing $\tau_k(x)$. Observe that for $x,y\in\mathcal{X}$ and $s\le\tau_k(x)\wedge\tau_k(y)$, the processes $x_s(x)$ and $x_s(y)$ respectively agree pathwise with processes $x_{s,k}(x)$ and $x_{s,k}(y)$ satisfying Itô SDEs with globally Lipschitz coefficients (cf. Remark 2.2). Thus if $\Delta_t = x_t(x) - x_t(y)$ and $\Delta_{t,k} = x_{t,k}(x) - x_{t,k}(y)$, then for any $\sigma \le t\wedge\tau_k(x)\wedge\tau_k(y)$ with $t\ge 0$ deterministic, we have the estimate

$$\mathbf{E}\sup_{s\in[0,\sigma]} |\Delta_s|^2 \le \mathbf{E}\sup_{s\in[0,t]} |\Delta_{s,k}|^2.$$

Hence, to estimate $\mathbf{E}\sup_{s\in[0,\sigma]}|\Delta_s|^2$ it suffices, for a given deterministic time $t\geq 0$ to estimate $\mathbf{E}\sup_{s\in[0,t]}|\Delta_s|^2$, where $\Delta_t = x_t(x) - x_t(y)$ satisfies

$$\Delta_t = x - y + \int_0^t [b(x_s(x)) - b(x_s(y))]\,ds + \int_0^t [\sigma(x_s(x)) - \sigma(x_s(y))]\,d\mathbf{W}_s \quad (4.4)$$

with globally Lipschitz b, σ on $\mathbf{R}^{\mathrm{m}}$ and Lipschitz constants depending on k. Hence, there exists a constant $Q_k > 0$ such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq Q_k|x - y| \quad \text{for all} \quad x, y \in \mathbf{R}^{\mathrm{m}}.$$

Using Doob's maximal inequality and the Itô isometry, we have

$$\begin{aligned}\mathbf{E}\sup_{s\in[0,t]}|\Delta_s|^2 &\leq 9|x-y|^2 + 9Q_k^2\mathbf{E}\sup_{s\in[0,t]}\left(\int_0^s |x_u(x) - x_u(y)|\,du\right)^2 \\ &\quad + 9\mathbf{E}\sup_{s\in[0,t]}\left|\int_0^s (\sigma(x_u(x)) - \sigma(x_u(y)))d\mathbf{W}_u\right|^2 \\ &\leq 9|x-y|^2 + C_k(t)\int_0^t \mathbf{E}\sup_{s\in[0,u]}|\Delta_s|^2 du\end{aligned}$$

for some constant $C_k(t)$ depending only on k, t. Consequently by Grönwall's inequality, we have

$$\mathbf{E}\sup_{s\in[0,t]}|\Delta_s|^2 \leq 9|x-y|^2 e^{C_k(t)t} =: |x-y|^2 D_k(t). \quad (4.5)$$

Now if b, σ are no longer globally Lipschitz and $\delta_k = \operatorname{dist}(\mathcal{X}_k, \partial\mathcal{X}_{k+1}) > 0$, we have by Chebyshev's inequality and (4.5)

$$\begin{aligned}&\mathbf{E}\sup_{s\in[0,t]}|\Delta_s|^2 \wedge M \\ &= \mathbf{E}\sup_{s\in[0,t]}|\Delta_s|^2 \wedge M(\mathbf{1}\{\tau_k(x) < t\} + \mathbf{1}\{\tau_k(x) \geq t\}) \\ &\leq M\mathbf{P}_x\{\tau_k < t\} + \mathbf{E}_x\sup_{s\in[0,t]}|\Delta_{s,k}|^2\mathbf{1}\{\tau_k(x) \geq t, \tau_{k+1}(y) \geq t\} \\ &\quad + M\mathbf{P}\{\sup_{s\in[0,t]}|\Delta_{s,k+1}|^2 > \delta_k^2\} \\ &\leq M\mathbf{P}_x\{\tau_k < t\} + D_k(t)|x-y|^2 + \frac{|x-y|^2 D_{k+1}(t)}{\delta_k^2}M.\end{aligned}$$

Passing $y \to x$ and then $k \to \infty$ using that $x_t(x)$ is nonexplosive, we obtain (i).

To obtain (ii), suppose there exist $\epsilon_0, \delta_0 > 0$ and a sequence $\{x_n\} \subset U$ with $x_n \to x_*$ such that

$$\mathbf{P}_{x_n}\{\tau \geq \delta_0\} \geq \epsilon_0$$

for all n. For any $\epsilon > 0$, let U_ϵ^+ be given by

$$U_\epsilon^+ = \{x \in \mathcal{X} : \operatorname{dist}(x, \overline{U}) < \epsilon\}. \quad (4.6)$$

By path continuity, $\overline{\tau} = \lim_{\epsilon\to 0} \tau_{U_\epsilon^+}$ almost surely. Since x_* is regular, there is $\epsilon \in (0,1)$ so that

$$\mathbf{P}_{x_*}\{\tau_{U_\epsilon^+} < \delta_0/2\} \geq 1 - \epsilon_0/2\,,$$

and consequently for any n

$$\mathbf{P}\{\tau(x_n) \geq \delta_0, \tau_{U_\epsilon^+}(x_*) < \delta_0/2\} \geq \epsilon_0/2\,,$$

where the initial condition is indicated in the stopping times above. By conclusion (i) with $M = 1$, and $\epsilon < 1$ for any large enough n one has $\mathbf{P}\{\sup_{s\in[0,\delta_0/2]} |x_s(x_n) - x_s(x_*)| > \epsilon/2\} \leq \epsilon_0/4$. But then

$$\frac{\epsilon_0}{2} \leq \mathbf{P}\left\{\tau(x_n) \geq \delta_0, \tau_{U_\epsilon^+}(x_*) < \frac{\delta_0}{2}, \sup_{s\in[0,\delta_0/2]} |x_s(x_n) - x_s(x_*)| \leq \frac{\epsilon}{2}\right\}$$
$$+ \mathbf{P}\left\{\sup_{s\in[0,\delta_0/2]} |x_s(x_n) - x_s(x_*)| > \frac{\epsilon}{2}\right\} \leq 0 + \frac{\epsilon_0}{4},$$

a contradiction. □

In addition to Lemma 4.1, we need one more auxiliary result. Although this result is basic, it is used repeatedly throughout the paper.

Lemma 4.2. *Let $V \subset \mathbf{R}^k$ be open and let $\mathcal{L}$ and $\mathcal{L}_n$, $n \in \mathbf{N}$, be linear second-order differential operators with $C^\infty(V)$ coefficients. Suppose that $\{v_n\}$ be a uniformly bounded sequence of measurable functions $v_n : V \to \mathbf{R}$ and that the following conditions are satisfied:*

- *For every $\phi \in C_0^\infty(V)$, $(\mathcal{L}_n - \mathcal{L})^*\phi \to 0$ as $n \to \infty$ in $L^1(V, dx)$ where the $*$ denotes the formal adjoint with respect to $L^2(V, dx)$-inner product;*
- *For some measurable function f on V which is bounded on compact subsets of V, $\mathcal{L}_n v_n \to -f$ as $n \to \infty$ on V in the sense of distributions.*
- *v_n converges to v_∞ in the sense of distributions on V as $n \to \infty$.*

Then, $\mathcal{L}v_\infty = -f$ on V in the sense of distributions. Furthermore, if $\mathcal{L}$ is hypoelliptic on V and $f \in C^\infty(V)$, then $v_\infty \in C^\infty(V)$ and $\mathcal{L}v_\infty = -f$ on V in the classical sense.

Proof. For fixed $\varphi \in C_0^\infty(V)$, the conditions of the statement and the Dominated Convergence Theorem imply

$$\int_V v_\infty \mathcal{L}^*\varphi\, dx = \lim_{n\to\infty} \int_V v_n \mathcal{L}^*\varphi\, dx = \lim_{n\to\infty} \int_V v_n \mathcal{L}_n^*\varphi - \lim_{n\to\infty} \int_U v_n(\mathcal{L}_n - \mathcal{L})^*\varphi\, dx$$
$$= \int_V -f\varphi\, dx.$$

Thus $\mathcal{L}v_\infty = -f$ on V in the sense of distributions. The remaining assertion is an immediate consequence of hypoellipticity. □

4.2. The forward and backward Kolmogorov equations for the stopped process

Recall that if x_t is nonexplosive, then the stopped process $\tilde{x}_t$, see (2.10), is a well-defined, continuous-time Markov process distributed on $\overline{U}$. Moreover, the associated semigroup and transitions are respectively denoted by $\tilde{\mathcal{P}}_t$ and $\tilde{\mathcal{P}}_t(x, \cdot\,)$.

Our main result in this section is the following.

Theorem 4.3. *Suppose that the process x_t is nonexplosive as in Definition 2.1 and that conditions **(U0)** and **(L2)** are satisfied.*

(i) For all $x \in U$ and $t > 0$, the restriction of the measure $\tilde{\mathcal{P}}_t(x, \cdot\,)$ to Borel subsets of U is absolutely continuous with respect to Lebesgue measure on U with density $\tilde{p}_t(x, y)$, and for fixed $x \in U$ the mapping $(t, y) \mapsto \tilde{p}_t(x, y) \in C^\infty((0, \infty) \times U)$. Furthermore, the forward Kolmogorov equation (4.1) is satisfied in the classical sense.

*(ii) If **(L3)** is furthermore satisfied, then $(t, x, y) \mapsto \tilde{p}_t(x, y)$ belongs to $C^\infty((0, \infty) \times U \times U)$ and the backward Kolmogorov equation (4.2) is satisfied in the classical sense.*

Remark 4.4. The above result is natural and understood by experts in the field. However, we found it difficult to locate a complete proof of part (ii) as it is much more subtle than part (i). While part (i) follows almost immediately from Dynkin's formula, part (ii) requires several nontrivial approximations and steps.

Remark 4.5. Intuitively the result above holds because, while the measure $\tilde{\mathcal{P}}_t(x, \cdot\,)$ has a singular component on the boundary if the process exits U by time t with positive probability, when restricted to subsets of U this singularity is not seen.

Remark 4.6. Note that in the statement above we may take $U = \mathcal{X}$, in which case $\tilde{x}_t$ coincides with the original process x_t on $\mathcal{X}$, and thus the statement is about the law of x_t in $\mathcal{X}$.

Proof of Theorem 4.3. To prove part (i), fix $t > 0$, $x \in U$, and take any $\psi \in C^\infty([0, t] \times U) \cap C_0^\infty((0, t) \times U)$. Thus, in particular, ψ is compactly supported in $(0, t) \times U$. Then, Dynkin's formula yields

$$\begin{aligned} 0 = \mathbf{E}_x \psi(t, \tilde{x}_t) = \mathbf{E}_x \psi(t, x_{t \wedge \tau_0}) &= \psi(0, x) + \mathbf{E}_x \int_0^{t \wedge \tau_0} (L + \partial_s)\psi(s, x_s)\, ds \\ &= 0 + \mathbf{E}_x \int_0^t (L + \partial_s)\psi(s, \tilde{x}_s)\, ds\,, \end{aligned}$$

where τ_0 from (2.9) is the first exit time from U. Hence,

$$\mathbf{E}_x \int_0^t (L + \partial_s)\psi(s, \tilde{x}_s) = \int_0^t \int_U (L + \partial_s)\psi(s, y)\tilde{p}_s(x, dy)ds = 0$$

for any $\psi \in C_0^\infty((0,t)\times U)$. That is, $(L_y^* - \partial_t)(\tilde{p}_t(x,dy)) = 0$ on $(0,\infty)\times U$ in the sense of distributions. By hypoellipticity of $\partial_t - L^*$ on $(0,\infty)\times U$ as in **(L2)**, part (i) follows.

Part (ii) is more involved as there are several layers of approximations. We divide the proof into two main steps.

Step 1. Let $\psi \in C_0^\infty(U)$ and consider the mapping $(t,x) \mapsto u(t,x) := \tilde{\mathcal{P}}_t\psi(x)$. We claim that $u \in C^\infty((0,\infty)\times U)$ and $\partial_t u = Lu$ in the classical sense on $(0,\infty)\times U$.

Step 2 .We prove part (ii) by taking a sequence of approximations of $\tilde{p}_t(x,y)$ of the form $\tilde{\mathcal{P}}_t\psi_n(x)$, $\psi_n \in C_0^\infty(U)$.

Proof of Step 1. First observe that since x_t is nonexplosive, the stopped process $\tilde{x}_t = x_{t\wedge\tau_0}$ with initial condition $\tilde{x}_0 = x \in U$ satisfies the Itô SDE

$$d\tilde{x}_t = b(\tilde{x}_t)\mathbf{1}_U(\tilde{x}_t)\,dt + \sigma(\tilde{x}_t)\mathbf{1}_U(\tilde{x}_t)\,d\mathbf{W}_t.$$

We first show $u \in C^\infty((0,\infty)\times U)$ and $\partial_t u = Lu$ in the classical sense on $(0,\infty)\times U$ in the special case when $b,\sigma \in C^\infty(\mathcal{X})$ are bounded with bounded derivatives of all orders.

Recall that U_n is a sequence of bounded open sets with $\overline{U}_n \subset U$ and $U_n \uparrow U$ as $n\to\infty$. Suppose that $\varphi_n \in C^\infty(\overline{U};[0,1])$ satisfies $\varphi_n(x) = 1$ on U_n and $\varphi_n(x) = 0$ for $x \in U_{n+1}^c$. Consider a sequence of approximating processes x_t^n with $x_0^n = x$ and

$$dx_t^n = b(x_t^n)\varphi_n(x_t^n)\,dt + \sigma(x_t^n)\varphi_n(x_t^n)\,d\mathbf{W}_t. \tag{4.7}$$

Claim. *If $b,\sigma \in C^\infty(\mathcal{X})$ are bounded with bounded derivatives of all orders, then for any $t>0$ fixed,*

$$\mathbf{E}_x \sup_{r\in[0,t]} |\tilde{x}_r - x_r^n|^2 \to 0 \quad \text{as} \quad n\to\infty. \tag{4.8}$$

Proof of (4.8). Observe that

$$\begin{aligned}\tilde{x}_t - x_t^n = &\int_0^t \varphi_n(x_s^n)[b(\tilde{x}_s) - b(x_s^n)]\,ds + \int_0^t \varphi_n(x_s^n)[\sigma(\tilde{x}_s) - \sigma(x_s^n)]\,d\mathbf{W}_s \\ &+\int_0^t (1-\varphi_n(x_s^n))\mathbf{1}_U(\tilde{x}_s)b(\tilde{x}_s)\,ds + \int_0^t (1-\varphi_n(x_s^n))\mathbf{1}_U(\tilde{x}_s)\sigma(\tilde{x}_s)\,d\mathbf{W}_s.\end{aligned}$$

Hence, squaring both sides, taking the supremum and using Doob's maximal inequality we find that for all $t \le T$

$$\mathbf{E}\sup_{r\in[0,t]} |\tilde{x}_r - x_r^n|^2 \le C_1\int_0^t \mathbf{E}\sup_{r\in[0,s]} |\tilde{x}_r - x_r^n|^2\,ds + C_2\int_0^t \mathbf{E}[(1-\varphi_n(x_s^n))\mathbf{1}_U(\tilde{x}_s)]^2\,ds\,,$$

where the constants $C_1 = C_1(T, \|b\|_{\mathrm{Lip}}, \|\sigma\|_{\mathrm{Lip}}) > 0$ and $C_2 = C_2(T, \|b\|_{L^\infty}, \|\sigma\|_{L^\infty}) > 0$ do *not* depend on n. Grönwall's inequality then implies that for all $t \le T$

$$\mathbf{E}\sup_{r\in[0,t]} |\tilde{x}_r - x_r^n|^2 \le C_2 e^{tC_1}\int_0^t \mathbf{E}[(1-\varphi_n(x_s^n))\mathbf{1}_U(\tilde{x}_s)]^2\,ds\,.$$

Next, observe that if $\tilde{x}_s(\omega) \in U$ for some $s \geq 0$, then by path continuity and the definition of the stopped process, $\tilde{x}_u(\omega) \in U_{n_0}$ for all $u \in [0,s]$ for some $n_0 = n_0(\omega, s) \in \mathbf{N}$. Hence, by definition, $x_u^n(\omega) \equiv \tilde{x}_u(\omega)$ for all $u \in [0,s]$ and $n \geq n_0$, and consequently by the Bounded Convergence Theorem

$$\lim_{n\to\infty} \int_0^t \mathbf{E}[(1-\varphi_n(x_s^n))\mathbf{1}_U(\tilde{x}_s)]^2\, ds = 0\,.$$

Hence, $\mathbf{E}\sup_{r\in[0,t]} |\tilde{x}_r - x_r^n|^2 \to 0$ as $n \to \infty$, establishing (4.8). □

Let $u_n(t,x) := \mathbf{E}_x\psi(x_t^n)$. It follows that for fixed $t > 0$, the mapping $x \mapsto u_n(t,x)$ belongs to the space $C^2(\mathbf{R}^{\mathfrak{m}})$ [21, Theorem 5.5]. Define an extended version A_n of the generator of x_t^n by

$$A_n g(x) := \lim_{t\to 0} \frac{\mathbf{E}_x g(x_t^n) - g(x)}{t}\,, \tag{4.9}$$

where the domain of A_n consists of all measurable functions $g : \mathbf{R}^{\mathfrak{m}} \to \mathbf{R}$ such that the limit in (4.9) exists for each x. Next, by [52, Theorem 8.1.1], for fixed $x \in \mathbf{R}^{\mathfrak{m}}$, $t \mapsto u_n(t,x) \in C^1(\mathbf{R}^{\mathfrak{m}})$ and by the Markov property u_n satisfies on $(0,\infty)\times\mathbf{R}^{\mathfrak{m}}$ the generalized backward equation

$$\partial_t u_n = A_n u_n.$$

Since $x \mapsto u_n(t,x)$ is globally C^2, Dynkin's formula implies that $A_n u = L_n u$ where L_n is the *classical* generator of x_t^n, which is a second-order differential operator with $C^\infty(\mathbf{R}^{\mathfrak{m}})$ coefficients. Moreover, these coefficients agree with the coefficients of L on U_n. In particular, by **(L3)**, $L - \partial_t$ is hypoelliptic on $(0,\infty)\times U_n$, and therefore $u_n \in C^\infty((0,\infty)\times U_n)$ and $\partial_t u_n = L u_n$ on $(0,\infty)\times U_n$ in the classical sense. To obtain a similar result for $u(t,x) = \tilde{\mathcal{P}}_t\psi(x) = \mathbf{E}_x\psi(\tilde{x}_t)$, we note that u_n is uniformly bounded. Furthermore, $u_n \to u$ pointwise as $n \to \infty$ since by the claim above

$$|u_n(t,x) - u(t,x)|^2 \leq \|\psi\|_{\mathrm{Lip}}^2 \mathbf{E}_x|\tilde{x}_t - x_t^n|^2 \to 0 \tag{4.10}$$

as $n \to \infty$. By Lemma 4.2, $u \in C^\infty((0,\infty)\times U)$ and $\partial_t u = Lu$ in the classical sense on $(0,\infty)\times U$. Thus Step 1 is completed in the case when b, σ have bounded derivatives of all orders.

To complete Step 1, assume that $b \in C^\infty(\mathcal{X};\mathbf{R}^{\mathfrak{m}})$ and $\sigma \in C^\infty(\mathcal{X}; M_{\mathfrak{m}\times r})$ are not necessarily bounded and x_t is nonexplosive. For the sequence $\{U_n\}$ above, note that the restrictions of b, σ to U_n can be extended to functions on $\mathbf{R}^{\mathfrak{m}}$ which are bounded with bounded derivatives all orders. Setting $\tau_n = \inf\{t > 0 : \tilde{x}_t \notin U_n\}$, we have shown above that if $\tilde{x}_t^n := \tilde{x}_{t\wedge\tau_n}$, then $(t,x) \mapsto \tilde{u}_n(t,x) := \mathbf{E}_x\psi(\tilde{x}_t^n)$ satisfies $\tilde{u}_n \in C^\infty((0,\infty)\times U_n)$ and $\partial_t\tilde{u}_n = L\tilde{u}_n$ on $(0,\infty)\times U_n$ in the classical sense. However, since ψ is bounded, then $\tilde{u}_n$ is also uniformly bounded in n and by (4.10), $\tilde{u}_n \to u$ pointwise on $(0,\infty)\times U$. Step 1 now follows after applying Lemma 4.2.

Proof of Step 2. Let $\chi \in C_0^\infty(\mathbf{R}^{\mathfrak{m}}; [0,\infty))$ with $\int_{\mathbf{R}^{\mathfrak{m}}} \chi(x)\,dx = 1$. Let $\chi_\epsilon(x) = \epsilon^{-\mathfrak{m}}\chi(\epsilon^{-1}x)$ and extend $\tilde{p}_t(x,y)$ to be zero for all $(t,x,y) \notin (0,\infty)\times U \times U$. For any $(t,x,y) \in (0,\infty)\times \mathbf{R}^{\mathfrak{m}} \times \mathbf{R}^{\mathfrak{m}}$, define

$$u_\epsilon(t,x,y) := \int_{\mathbf{R}^{\mathfrak{m}}} \tilde{p}_t(x,z)\chi_\epsilon(y-z)\,dz. \tag{4.11}$$

Consider any bounded open $V \subset U$ with $\overline{V} \subset U$. For each fixed $y \in \overline{V}$, there is $\epsilon_0(y) > 0$ small enough such that $z \mapsto \chi_\epsilon(y-z) \in C_0^\infty$ is compactly supported in U for all $\epsilon \in (0, \epsilon_0(y)]$. It follows by compactness of $\overline{V}$, that we may choose $\epsilon_0 > 0$ such that $z \mapsto \chi_\epsilon(y-z)$ is compactly supported in U for all $y \in \overline{V}$ and all $\epsilon \in (0,\epsilon_0]$. Thus, for any $\epsilon \in (0,\epsilon_0]$ and $(t,x) \in (0,\infty)\times U$, the formula (4.11) makes sense and $y \mapsto u_\epsilon(t,x,y) \in C^\infty(V)$. Also, by Step 1, for each $y \in V$ and $\epsilon \in (0,\epsilon_0]$, $(t,x) \mapsto u_\epsilon(t,x,y) \in C^\infty((0,\infty)\times U)$ and $\partial_t u_\epsilon(t,x,y) = L_x u_\epsilon(t,x,y)$ on $(0,\infty)\times U$ in the classical sense. To conclude the result, it suffices to show for any $\varphi \in C_0((0,\infty)\times U \times V)$ with φ supported in $K \subset (0,\infty)\times U \times V$ compact, that

$$\lim_{\epsilon\to 0} \int_K \varphi(t,x,y)[u_\epsilon(t,x,y) - \tilde{p}_t(x,y)]dx\,dy\,dt = 0\,.$$

Indeed, simply replace φ in the above formula by $(\partial_t - L_x)^*\varphi$, where φ is smooth with compact support.

First, since $\int_{\mathbf{R}^{\mathfrak{m}}} \chi(z)\,dz = 1$, a standard substitution gives for all $\epsilon \in (0,\epsilon_0]$ and $y \in V$

$$u_\epsilon(t,x,y) - \tilde{p}_t(x,y) = \int_{\mathbf{R}^{\mathfrak{m}}} [\tilde{p}_t(x,y-\epsilon z) - \tilde{p}_t(x,y)]\chi(z)\,dz.$$

Next, let $\hat{p}_t(x,y) = \tilde{p}_t(x,y)$ on K and $\hat{p}_t(x,y) = 0$ otherwise. By Tonelli's Theorem, $(t,x,y) \mapsto \hat{p}_t(x,y) \in L^1(\mathbf{R}\times(\mathbf{R}^{\mathfrak{m}})^2)$. Moreover,

$$\begin{aligned}&\left|\int_K \varphi(t,x,y)[u_\epsilon(t,x,y) - \tilde{p}_t(x,y)]\,dx\,dy\,dt\right| \\ &\le \|\varphi\|_{L^\infty} \int_{\mathbf{R}\times(\mathbf{R}^{\mathfrak{m}})^3} |\hat{p}_t(x,y-\epsilon z) - \hat{p}_t(x,y)|\chi(z)dz\,dx\,dy\,dt.\end{aligned}$$

Since $\hat{p}_t \in L^1(\mathbf{R}\times(\mathbf{R}^{\mathfrak{m}})^2)$, for any $\delta > 0$ there is $\psi \in C_0^\infty((\mathbf{R}^{\mathfrak{m}})^3)$ within L^1 distance δ of $\hat{p}_t$. Thus, by translation invariance and Fubini's Theorem

$$\begin{aligned}&\left|\int_K \varphi(t,x,y)[u_\epsilon(t,x,y) - \tilde{p}_t(x,y)]\,dx\,dy\,dt\right| \\ &\le 2\delta\|\varphi\|_{L^\infty} + \|\varphi\|_{L^\infty} \int_{(\mathbf{R}^{\mathfrak{m}})^4} |\psi(t,x,y-\epsilon z) - \psi(t,x,y)|\chi(z)dz\,dx\,dy\,dt.\end{aligned}$$

Since ψ is compactly supported, passing $\epsilon \to 0$, using the Dominated Convergence Theorem, and then $\delta \to 0$ finishes the proof of Theorem 4.3(ii). □

4.3. Interior smoothness of functionals of the first positive exit time

In this section we analyze quantities of the form

$$\mathbf{E}_x \varphi(\tau), \ x \in U, \tag{4.12}$$

for various functions φ and τ as in (2.2). The first two choices

$$\varphi(x) = \mathbf{1}\{|x| < \infty\} \quad \text{and} \quad \varphi(x) = x,$$

that respectively yield in (4.12)

$$\mathbf{P}_x\{\tau < \infty\} \qquad \text{and} \qquad \mathbf{E}_x \tau \tag{4.13}$$

are of significant importance for analyzing recurrence and transience in Section 7 below. Throughout, we assume the parabolic Hörmander condition **(PH)** for simplicity, which in particular implies that **(L1)**–**(L4)** are satisfied.

Proposition 4.7. *Suppose that the process x_t is nonexplosive as in Definition 2.1 and that conditions **(U00)** and **(PH)** are satisfied. For any $(t,x) \in (0,\infty) \times U$, define*

$$u_1(t,x) = \mathbf{P}_x\{\tau > t\} \qquad \text{and} \qquad u_2(x) = \mathbf{P}_x\{\tau = \infty\}. \tag{4.14}$$

Then:

(i) $u_1 \in C^\infty((0,\infty)\times U)$ and $\partial_t u_1 = L u_1$ on $(0,\infty)\times U$ in the classical sense;
(ii) $u_2 \in C^\infty(U)$ and $L u_2 = 0$ on U in the classical sense.

Proof. For (i), observe that for any $x \in U$, $u_1(t,x) = \mathbf{P}_x\{\tau > t\} = \tilde{\mathcal{P}}_t(x,U)$ where $\tilde{\mathcal{P}}_t(x,\cdot)$ is the transition kernel of the stopped process $\tilde{x}_t = x_{t\wedge\tau_0}$. Let $K \subset U$ be compact and by Theorem 4.3 $\tilde{\mathcal{P}}_t(x,K) = \int_K \tilde{p}_t(x,y)\,dy$. Furthermore, by Theorem 4.3 and the Dominated Convergence Theorem, it follows that $(t,x) \mapsto \tilde{\mathcal{P}}_t(x,K)$ is $C^\infty((0,\infty)\times U)$ and $\partial_t \tilde{\mathcal{P}}_t(x,K) = L\tilde{\mathcal{P}}_t(x,K)$ on $(0,\infty)\times U$ in the classical sense. To obtain the same result for $\tilde{\mathcal{P}}_t(x,U)$, let $K_n \subset U$ be a sequence of compact sets with $K_n \uparrow U$ as $n \to \infty$. Then, by the Monotone Convergence Theorem, the sequence $(\tilde{\mathcal{P}}_t(x,K_n))_n$ is uniformly bounded and converges pointwise to $\tilde{\mathcal{P}}_t(x,U)$. The assertion (i) follows after applying Lemma 4.2.

To obtain (ii), first observe that

$$u_2(x) = \mathbf{P}_x\{\tau = \infty\} = \lim_{t\to\infty} \tilde{\mathcal{P}}_t(x,U).$$

Thus we seek to apply an argument similar to that in Lemma 4.2. To this end, note that

$$\bar{u}(t,x) := \int_t^{t+1} \tilde{\mathcal{P}}_s(x,U)\,ds \to u_2(x) \quad \text{pointwise as} \quad t \to \infty$$

and that $\bar{u}(t,x)$ is uniformly bounded on $[0,\infty)\times U$. Thus, for any function $\varphi \in C_0^\infty(U)$ and any $t>0$, using integration by parts and the Fubini-Tonelli Theorem we have

$$\begin{aligned} 0 &= \int_t^{t+1}\int_U (L-\partial_s)\tilde{\mathcal{P}}_s(x,U)\varphi(x)\,dx\,ds \\ &= \int_U \bar{u}(t,x)L^*\varphi(x)\,dx - \int_U [\tilde{\mathcal{P}}_{t+1}(x,U)-\tilde{\mathcal{P}}_t(x,U)]\varphi(x)\,dx. \end{aligned}$$

By passing $t\to\infty$ and using the Dominated Convergence Theorem, u_2 solves $Lu_2=0$ in the sense of distributions on U. Part (ii) follows by hypoellipticity of L on U. □

We next state and prove a regularity result for $\mathbf{E}_x\tau$; that is, for $\varphi(x)=x$ in (4.12).

Proposition 4.8. *Suppose that the process x_t is nonexplosive as in Definition 2.1 and that conditions **(U00)** and **(PH)** are satisfied. Suppose, furthermore, that $\mathbf{P}_x\{\tau<\infty\}=1$ for all $x\in U$.*

(i) If $U_0\subset U$ is bounded open with $\overline{U_0}\subset U$, then $v_0(x):=\mathbf{E}_x\tau_{U_0}$ is bounded on $\overline{U_0}$. Furthermore, $v_0\in C^\infty(U_0)$ and $Lv_0=-1$ on U_0 in the classical sense.

(ii) Let $U_0\subset U$ be an open set and suppose that $v(x):=\mathbf{E}_x\tau$ is finite on a dense set $D\subset U_0$. Then, $v\in C^\infty(U_0)$ and v solves $Lv=-1$ on U_0 in the classical sense.

Remark 4.9. The technical hypothesis in (ii) that v is finite on a dense set and not everywhere is needed below in Section 7. We defer the proof of Proposition 4.8(ii) until after we prove a hypoelliptic Harnack inequality (Theorem 5.5).

Proof of Proposition 4.8(i). Let $\sigma=\tau_{U_0}$ and note that $\mathbf{P}_x\{\tau<\infty\}=1$ implies that for all $x\in\overline{U_0}$, there exists $t>0,\alpha\in(0,1/2)$ such that $\mathbf{P}_x\{\sigma>t\}\le(1-2\alpha)$. By Proposition 4.7(i), there exists $\epsilon=\epsilon(x)>0$ such that $\mathbf{P}_y\{\sigma>t\}\le(1-\alpha)$ for all $y\in B_\epsilon(x)$. Using compactness of $\overline{U_0}$ and taking a finite subcover, it follows that there exists $t_*>0,\alpha_*\in(0,1)$ such that $\mathbf{P}_y\{\sigma>t_*\}\le(1-\alpha_*)$ for all $y\in\overline{U_0}$. Since τ is almost surely finite, so is σ by $U_0\subset U$ and path continuity. Thus, for any $x\in\overline{U_0}$ we have

$$\begin{aligned} \mathbf{E}_x\sigma &= \sum_{m=1}^\infty \mathbf{E}_x\sigma\mathbf{1}\{\sigma\in[(m-1)t_*,mt_*)\} \\ &\le 3t_* + \sum_{m=3}^\infty mt_*\mathbf{P}_x\{\sigma>(m-2)t_*\}. \end{aligned} \tag{4.15}$$

By the Markov property, for $x\in\overline{U_0}$ and $m\in\mathbf{N}$ we have

$$\begin{aligned} \mathbf{P}_x\{\sigma>mt_*\} &= \mathbf{E}_x[\mathbf{E}_x\mathbf{1}\{\sigma>mt_*\}|\mathcal{F}_{(m-1)t_*}] \\ &= \mathbf{E}_x\mathbf{1}\{\sigma>(m-1)t_*\}\mathbf{P}_{x_{(m-1)t_*}}\{\sigma>t_*\} \end{aligned}$$

$$\leq (1-\alpha_*)\mathbf{P}_x\{\sigma > (m-1)t_*\} \leq (1-\alpha_*)^m$$

and (4.15) implies for all $x \in \overline{U_0}$

$$\mathbf{E}_x\sigma \leq 3t_* + \sum_{m=3}^{\infty} mt_*(1-\alpha_*)^{m-2},$$

which is finite and bounded independently of $x \in \overline{U_0}$.

Next, in order to show that $v_0 \in C^\infty(U_0)$ and satisfies $Lv_0 = -1$ on U_0 in the classical sense, observe that

$$v_0(x) = \mathbf{E}_x\sigma = \int_0^\infty \mathbf{P}_x\{\sigma > t\}\,dt =: \int_0^\infty u(t,x)\,dt.$$

Proposition 4.7 yields $u \in C^\infty((0,\infty)\times U_0)$ and $Lu = \partial_t u$ on U_0 in the classical sense. Thus if $v_n(x) = \int_{1/n}^n u(t,x)\,dt$ for $n \in \mathbf{N}$, it follows that $v_n \in C^\infty(U_0)$ and for all $x \in U_0$

$$\begin{aligned} Lv_n(x) = \int_{\frac{1}{n}}^n Lu(t,x)\,dt &= \int_{\frac{1}{n}}^n \partial_t u(t,x)\,dt \\ &= u(n,x) - u(1/n,x) \\ &= \mathbf{P}_x\{\sigma > n\} - \mathbf{P}_x\{\sigma > 1/n\} \to -1 \end{aligned}$$

as $n \to \infty$ in the sense of distributions on U_0. Since the sequence v_n is uniformly bounded by bounded v_0 and $v_n \to v_0$ pointwise on U_0, the result follows from Lemma 4.2. □

We next consider the expression (4.12) for more general $\varphi : [0,\infty) \to \mathbf{R}$. First we need an auxiliary lemma, which is a representation result for the expected value of certain random variables.

Lemma 4.10. *Suppose that the process x_t is nonexplosive as in Definition 2.1 and the condition **(U00)** and is met. Let $\varphi : [0,\infty) \to \mathbf{R}$ be C^1, strictly increasing and satisfy $\varphi(t) \to \infty$ as $t \to \infty$. Then*

$$\mathbf{E}_x\varphi(\tau) = \varphi(0) + \int_0^\infty \varphi'(t)\mathbf{P}_x\{\tau > t\}\,dt. \tag{4.16}$$

Proof. Note that by shifting the formula (4.16), it suffices to assume that $\varphi(0) = 0$. Since φ is C^1, strictly increasing with $\varphi(t) \to \infty$ as $t \to \infty$, φ has a C^1 inverse φ^{-1} mapping $[0,\infty)$ onto $[0,\infty)$. Moreover, φ^{-1} is strictly increasing on $[0,\infty)$. It thus follows that $\varphi^{-1}(t) \to \infty$ as $t \to \infty$. Then,

$$\begin{aligned} \mathbf{E}_x(\varphi(\tau)\wedge N) = \int_0^N \mathbf{P}_x\{\varphi(\tau) > t\}\,dt &= \int_0^N \mathbf{P}_x\{\tau > \varphi^{-1}(t)\}\,dt \\ &= \int_0^{\varphi^{-1}(N)} \varphi'(t)\mathbf{P}_x\{\tau > t\}\,dt. \end{aligned}$$

Passing $N \to \infty$ and using the Monotone Convergence Theorem finishes the proof. □

Next, we state and prove our main result for general φ.

Proposition 4.11. *Suppose that the process x_t is nonexplosive as in Definition 2.1 and conditions **(U00)** and **(PH)** are satisfied. Let $\varphi \in C^2([0,\infty);\mathbf{R})$ be strictly increasing with $\varphi(t) \to \infty$ as $t \to \infty$ with derivative $\varphi' \in C^1([0,\infty);\mathbf{R})$ which is strictly increasing with $\varphi'(t) \to \infty$ as $t \to \infty$. Suppose furthermore that*

$$v_1(x) := \mathbf{E}_x \varphi(\tau) \qquad \text{and} \qquad v_2(x) = \mathbf{E}_x \varphi'(\tau)$$

are bounded on compact subsets of U. Then $Lv_1 = -v_2$ on U in the sense of distributions.

Remark 4.12. Compare the equation $Lv_1 = -v_2$ with $Lv = -1$ when $\varphi(x) = x$. Although such a φ does not satisfy the hypotheses above, we still have the analogous conclusion.

Proof of Proposition 4.11. By Lemma 4.10, we have

$$v_1(x) = \varphi(0) + \int_0^\infty \varphi'(t)\mathbf{P}_x\{\tau > t\}\, dt,$$
$$v_2(x) = \varphi'(0) + \int_0^\infty \varphi''(t)\mathbf{P}_x\{\tau > t\}\, dt.$$

In the latter formula, simply apply Lemma 4.10 to φ' instead of φ. Consider the sequence of functions on U given by

$$v_{1,n}(x) := \varphi(0) + \int_{1/n}^n \varphi'(t)\mathbf{P}_x\{\tau > t\}\, dt\,.$$

Observe that by Proposition 4.7, $v_{1,n} \in C^\infty(U)$ and

$$\begin{aligned} Lv_{1,n}(x) &= \int_{1/n}^n \varphi'(t) L_x(\mathbf{P}_x\{\tau > t\})\, dt \\ &= \int_{1/n}^n \varphi'(t)\partial_t(\mathbf{P}_x\{\tau > t\})\, dt \\ &= \varphi'(n)\mathbf{P}_x\{\tau > n\} - \varphi'(1/n)\mathbf{P}_x\{\tau > 1/n\} - \int_{1/n}^n \varphi''(t)\mathbf{P}_x\{\tau > t\}\, dt\,. \end{aligned}$$

Since for any $x \in U$ we have $\mathbf{E}_x\varphi'(\tau) < \infty$, it follows that

$$\lim_{n\to\infty} \varphi'(n)\mathbf{P}_x\{\tau > n\} \le \lim_{n\to\infty} \mathbf{E}_x\varphi'(\tau)\mathbf{1}\{\tau \ge n\} = 0 \tag{4.17}$$

on U. Furthermore, since $\mathbf{P}_x\{\tau = 0\} = 0$ for any $x \in U$, it also follows that on U

$$\lim_{n\to\infty} \varphi'(1/n)\mathbf{P}_x\{\tau > 1/n\} = \varphi'(0)\mathbf{P}_x\{\tau > 0\} = \varphi'(0). \tag{4.18}$$

In addition, φ' is strictly increasing, and therefore $\varphi'' \ge 0$ and by the monotone convergence theorem, $Lv_{1,n} \to -v_2$ pointwise as $n \to \infty$ on U. Since v_1, v_2 are bounded on compact subsets of U and $v_{1,n} \to v_1$ as $n \to \infty$, Lemma 4.2 implies $Lv_1 = -v_2$ on U in the sense of distributions. □

Let us note some more specific examples and results that follow as an immediate consequence of the previous results.

Corollary 4.13. *Suppose that x_t is nonexplosive as in Definition 2.1 and conditions **(U00)** and **(PH)** are satisfied. Then:*

(i) If for some $k \in \mathbf{N}$, $k \geq 2$, $v_1^k(x) := \mathbf{E}_x \tau^k$ is bounded on compact subsets of U, then $v_1^k \in C^\infty(U)$ and $Lv_1^k = -k\mathbf{E}_x \tau^{k-1}$ on U in the classical sense.

(ii) Suppose that for some $\delta > 0$, $w_\delta(x) := \mathbf{E}_x e^{\delta\tau}$ is bounded on compact subsets of U. Then $w_\delta \in C^\infty(U)$ and $Lw_\delta = -\delta w_\delta$.

Proof. For (i), set $\varphi(t) = t^k$ and note that $v_1^k(x)$ and $v_2^k(x) := k\mathbf{E}_x \tau^{k-1}$ are both bounded on compact subsets of U. By Proposition 4.11, $Lv_1^k = -v_2^k$ on U in the sense of distributions. By induction and Proposition 4.8, $v_2^k \in C^\infty(U)$ so that $v_1^k \in C^\infty(U)$ and $Lv_1^k = -v_2^k$ in the classical sense.

For (ii), set $\varphi(t) = e^{\delta t}$. Then, Proposition 4.11 implies $(L + \delta)w_\delta = 0$ in the sense of distributions. However, $L + \delta$ is hypoelliptic on U, so that $w_\delta \in C^\infty(U)$ and $Lw_\delta = -\delta w_\delta$ in the classical sense. □

5. Green's functions and Bony's Harnack inequality

In this section, we explore *Green's functions* in the hypoelliptic setting for an open set U. As a consequence, we generalize Bony's form of the Harnack inequality [9].

Let $\beta > 0$ be a constant and suppose $f \in C^\infty(U) \cap B(\overline{U})$. Often (see [9]) one refers to the *Green's operator* G_β as the 'mapping' $f \mapsto v$, where v is the 'unique' solution of the Poisson problem

$$\begin{cases} (L - \beta)v = -f & \text{on } U\,, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{5.1}$$

When $\beta = 0$ and L is hypoelliptic on U, the uniqueness of solutions of (5.1) heavily depends on the structure of the diffusion x_t driven by L near ∂U. Indeed, as mentioned in the introduction, the parts of the boundary that cannot be attained by the underlying stochastic process are sources of nonuniqueness. This is also the case for (5.1) when $\beta > 0$. Here, we employ stochastic methods to define and deduce properties of $G_\beta f$ without needing to assume uniqueness.

Formally, our 'best guess' of solution of (5.1) would be

$$v(x) = \mathbf{E}_x \int_0^\tau f(x_s) e^{-\beta s}\, ds\,, \tag{5.2}$$

where $\tau = \inf\{t > 0 : x_t \notin U\}$. The expression (5.2) is motivated by a formal application of Itô's formula to the function $e^{-\beta t} v(x_t)$, where v is a presumed classical solution of (5.1). Even though it is not clear nor necessarily true that v solves (5.1), v in (5.2) is well-defined for any function $f \in B(\overline{U})$. Note that v in (5.2) is well-defined even if $\tau = \infty$, due to the presence of the exponentially decaying factor $e^{-\beta s}$.

Remark 5.1. Under further conditions on the stopping time τ, one can define G_0 or even sometimes $G_{-\beta}$ for $\beta > 0$ small enough. Here, for simplicity, we keep β positive to avoid any further complexities of assumptions on τ. Additionally, under similar assumptions, one can replace β by a spatially dependent function $\beta = \beta(x) \in C(\overline{U})$ to arrive at the Feynman-Kac formula

$$\mathbf{E}_x \int_0^\tau f(x_s) e^{-\int_0^s \beta(x_v)\, dv}\, ds, \tag{5.3}$$

provided (5.3) makes sense.

5.1. Definition and properties of G_β

Given the preliminary remarks above, we now define G_β.

Definition 5.2. Suppose that the process x_t is nonexplosive as in Definition 2.1 and suppose **(U00)** is satisfied. For any $\beta > 0$ and any $f : \overline{U} \to [0, \infty)$ measurable, define $G_\beta f : U \to [0, +\infty]$ by

$$G_\beta f(x) := \mathbf{E}_x \int_0^\tau f(x_s) e^{-\beta s}\, ds. \tag{5.4}$$

If $f : \overline{U} \to \mathbf{R}$ is measurable and $G_\beta |f|(x) < \infty$ for all $x \in U$, we define

$$G_\beta f(x) := G_\beta f_+(x) - G_\beta f_-(x) = \mathbf{E}_x \int_0^\tau f(x_s) e^{-\beta s}\, ds \tag{5.5}$$

where f_+, f_- denote the positive and negative parts of f, respectively. We call G_β the *Green's operator of order* $\beta > 0$. Let

$$\mathcal{D}_\beta := \{f : \overline{U} \to \mathbf{R} \text{ measurable} : x \mapsto G_\beta |f|(x) \text{ is bounded on compacts in } U\}$$

be the *domain* of G_β.

Remark 5.3. Observe that the expression (5.5) is better behaved compared with the formal stochastic representation corresponding to the usual Poisson problem (i.e. with $\beta = 0$) due to the presence of the exponentially decaying factor $e^{-\beta s}$.

Often it is convenient to express the operator $G_\beta f$ as a traditional integral operator, meaning that for any 'reasonable' function f on $\overline{U}$ and any $x \in U$:

$$G_\beta f(x) = \int_{\overline{U}} f(y) g_\beta(x, y)\, dy \tag{5.6}$$

for some kernel g_β to be referred to as the *Green's function* associated to G_β. However, we must be careful as there are subtleties in defining $g_\beta(x, y)$ for $y \in \partial U$ and for $x = y$ in U.

To obtain a workable expression for g_β, under the assumptions that x_t is nonexplosive and condition **(U00)** is satisfied, we recall the stopped process

$\tilde{x}_t = x_{t\wedge\tau_0}$ from (2.10) is a Markov process distributed on $\overline{U}$, where $\tau_0 = \inf\{t \geq 0 : x_t \notin U\}$. We also recall its transition kernel $\tilde{\mathcal{P}}_t(x, \cdot)$ which is defined on Borel subsets of $\overline{U}$. Note that, for any $x \in U$ and any $f \in \mathcal{D}_\beta$, the Dominated Convergence theorem implies

$$\begin{aligned}\mathbf{E}_x \int_0^{\tau_0} f(\tilde{x}_s)\mathbf{1}_U(\tilde{x}_s)e^{-\beta s}\,ds &= \lim_{\delta\downarrow 0} \mathbf{E}_x \int_0^{(\tau_0-\delta)\vee 0} f(\tilde{x}_s)\mathbf{1}_U(\tilde{x}_s)e^{-\beta s}\,ds \\ &= \lim_{\delta\downarrow 0} \mathbf{E}_x \int_0^{(\tau_0-\delta)\vee 0} f(\tilde{x}_s)e^{-\beta s}\,ds \\ &= \mathbf{E}_x \int_0^{\tau_0} f(\tilde{x}_s)e^{-\beta s}\,ds.\end{aligned}$$

Hence, for any $f \in \mathcal{D}_\beta$ and $x \in U$, we have by Fubini's theorem

$$\begin{aligned}G_\beta f(x) &= \mathbf{E}_x \int_0^{\tau_0} f(\tilde{x}_s)e^{-\beta s}\,ds = \mathbf{E}_x \int_0^{\infty} \mathbf{1}_U(\tilde{x}_s)f(\tilde{x}_s)e^{-\beta s}\,ds \\ &= \int_0^\infty \int_U f(y)\tilde{\mathcal{P}}_s(x,dy)e^{-\beta s}\,ds = \int_U f(y)\int_0^\infty \tilde{\mathcal{P}}_s(x,dy)e^{-\beta s}\,ds \qquad (5.7) \\ &=: \int_U f(y)g_{\beta,x}(dy)\,,\end{aligned}$$

where for each $x \in U$, $g_{\beta,x}$ is a finite Borel measure on U given by

$$g_{\beta,x}(A) = \int_0^\infty \tilde{\mathcal{P}}_s(x, A\cap U)e^{-\beta s}\,ds. \qquad (5.8)$$

In particular, the purported Green's 'function' $g_{\beta,x}$, $x \in U$, does not charge the boundary ∂U. In order to show that $g_{\beta,x}$ has a density with respect to Lebesgue measure on U, we prove the following result:

Theorem 5.4. *Let $\beta > 0$, and suppose that x_t is nonexplosive and that conditions **(U00)** and **(PH)** are satisfied. Then:*

- *(i) For every $x \in U$, the finite measure $g_{\beta,x}$ on Borel subsets of U given by (5.8) is absolutely continuous with respect to Lebesgue measure on U with density $g_\beta(x,y)$.*
- *(ii) The mapping $(x,y) \mapsto g_\beta(x,y) \in C^\infty(U\times U\setminus \mathrm{Diag})$, where $\mathrm{Diag} = \{(x,x) : x \in U\}$. Furthermore, for fixed $x \in U$, $(L_y^* - \beta)g_\beta(x,y) = -\delta_x$ in the sense of distributions on U and, for fixed $y \in U$, $(L_x - \beta)g_\beta(x,y) = -\delta_y$ in the sense of distributions on U.*
- *(iii) For any $f \in \mathcal{D}_\beta \cap C(U)$, $(L_x - \beta)G_\beta f = -f$ on U in the sense of distributions.*
- *(iv) If $u \in C^2(U;[0,\infty))$ satisfies $Lu = 0$ on U, then $u \geq \beta G_\beta u$ on U.*

Proof. First, we show (i). By Theorem 4.3, for $s > 0$ and $x \in U$, the measure $\tilde{\mathcal{P}}_s(x, \cdot \cap U)$ is absolutely continuous with respect to Lebesgue measure on U with density $\tilde{p}_t(x,y)$. Furthermore, $(t,x,y) \mapsto \tilde{p}_t(x,y) : (0,\infty)\times U \times U \mapsto \mathbf{R}$ is C^∞ and $\tilde{p}_t$ satisfies both the forward and backward Kolmogorov equations

in the classical sense as in Theorem 4.3(i), (ii). Thus, if $|A \cap U| = 0$, then $\tilde{\mathcal{P}}_s(x, A \cap U) = 0$ for any $s > 0$ and by (5.8) one has $g_{\beta,x}(A) = 0$. Hence, by the Radon-Nikodym theorem, for each $x \in U$, $g_{\beta,x}$ has density $y \mapsto g_\beta(x, y)$ which belongs to $L^1(U)$. This concludes (i).

Next, we establish parts (ii) and (iii) simultaneously. Note that for any $f \in C_0^\infty(U)$, we have for any $x \in U$,

$$G_\beta((L-\beta)f)(x) = \int_U (L-\beta)f(y)g_\beta(x,y)dy. \tag{5.9}$$

On the other hand, because f vanishes outside of a compact set in U, Dynkin's formula (2.8) applied to $f(x_{t\wedge\tau_0})e^{-\beta(t\wedge\tau_0)}$ gives

$$\mathbf{E}_x f(x_{t\wedge\tau_0})e^{-\beta(t\wedge\tau_0)} = f(x) + \mathbf{E}_x \int_0^{t\wedge\tau_0} (L-\beta)f(x_s)e^{-\beta s}ds. \tag{5.10}$$

Since f is compactly supported in U and $\beta > 0$, after passing $t \to \infty$ we have

$$\mathbf{E}_x f(x_{t\wedge\tau_0})e^{-\beta(t\wedge\tau_0)} \to 0.$$

By (5.5) and (5.10),

$$G_\beta(L-\beta)f(x) = -f(x) \tag{5.11}$$

on U for all $f \in C_0^\infty(U)$. By combining (5.9) and (5.11) we obtain

$$(L_y^* - \beta)g_\beta(x,\cdot) = -\delta_x \quad \text{on } U \tag{5.12}$$

in the sense of distributions. Hence, by hypoellipticity of $L^* - \beta$, for each $x \in U$, the mapping $y \mapsto g_\beta(x, y) \in C^\infty(U \setminus \{x\})$.

To obtain regularity in the x variable, for any $f \in C(U) \cap B(U) \subset \mathcal{D}_\beta$, consider the sequence of approximations

$$G_\beta^n f(x) := \int_{1/n}^n \int_U f(y)\tilde{p}_s(x,y)\,dy\,e^{-\beta s}\,ds =: \int_U f(y)g_\beta^n(x,y)\,dy$$

and let $h \in C_0^\infty(U)$. Employing Theorem 4.3 and the Dominated Convergence Theorem (in order to interchange the integral and derivatives) it follows that

$$\begin{aligned}\int_U h(x)(L-\beta)_x G_\beta^n f(x)\,dx &= \int_U \int_U h(x)f(y) \int_{1/n}^n \partial_t(\tilde{p}_t(x,y)e^{-\beta t})\,dt\,dy\,dx \\ &= \int_U h(x)[e^{-\beta n}\mathbf{E}_x f(\tilde{x}_n) - e^{-\beta/n}\mathbf{E}_x f(\tilde{x}_{1/n})]\,dx.\end{aligned}$$

Passing $n \to \infty$ and using the Dominated Convergence Theorem, we find that G_β satisfies

$$(L-\beta)_x G_\beta f(x) = -f(x) \tag{5.13}$$

in the sense of distributions on U. More generally, if $f \in \mathcal{D}_\beta \cap C(U)$, then for every $N \in \mathbf{N}$ define $f_N := f \wedge N \in C(U) \cap B(U)$. Then $(L-\beta)_x G_\beta f_N = -f_N$

in the sense of distributions on U. Passing $N \to \infty$, it follows by Lemma 4.2 and definition of $\mathcal{D}_\beta$, that $(L-\beta)_x G_\beta f = -f$ in the sense of distributions on U. This proves (iii).

To finish the rest of (ii), let $f, h \in C_0^\infty(U)$ and note that by (5.13)

$$\int_{U\times U} [(L-\beta)_x^* h](x) g_\beta(x,y) f(y)\, dydx = -\int_U h(x) f(x)\, dx. \tag{5.14}$$

Since f and h are arbitrary,

$$(L-\beta)_x g_\beta(x,y) = -\delta_y(x) \tag{5.15}$$

on U in the sense of distributions, and therefore the hypoellipticity of $L-\beta$ implies that $x \mapsto g_\beta(x,y)$ is $C^\infty(U \setminus \{y\})$, as desired.

Finally, for part (iv), for $n \geq 1$ let U_n be a sequence of bounded open sets with $U_n \uparrow U$ and $\overline{U}_n \subset U$. Set $\tau_n = \inf\{t \geq 0 \,:\, x_t \notin U_n\}$. Then, Dynkin's formula (2.8), $Lu = 0$, and $u \geq 0$ imply for any $x \in U$,

$$0 \leq \mathbf{E}_x e^{-\beta t\wedge\tau_0\wedge\tau_n} u(x_{t\wedge\tau_0\wedge\tau_n}) = u(x) - \beta \mathbf{E}_x \int_0^{t\wedge\tau_0\wedge\tau_n} u(x_s) e^{-\beta s}\, ds.$$

Rearranging and passing $n \to \infty$ and $t \to \infty$ gives the result by the Monotone Convergence Theorem. □

We now apply Theorem 5.4 to obtain a version of Harnack's inequality originally due to Bony [9]. Our assumptions are weaker as we do not assume the uniqueness of the solution of (5.1). Before we proceed, let

$$\mathcal{H}_U = \{u \in C^2(U;[0,\infty)) \,:\, Lu = 0 \text{ on } U\}. \tag{5.16}$$

Of course, if L is hypoelliptic on U, then any distribution u with $Lu = 0$ on U must belong to $C^\infty(U)$ and satisfy $Lu = 0$ on U in the classical sense.

Theorem 5.5 (Harnack inequality)**.** *Suppose that x_t is nonexplosive and that conditions **(U00)** and **(PH)** are satisfied. Consider any compact set $K \subset U$, any set $D \subset U$ which is dense in U and any multiindex α. Then, there exist points $y_1, y_2, \ldots, y_k \in D$ and a constant $c > 0$ such that the following inequality*

$$\sup_{x\in K} |D^\alpha u(x)| \leq c[u(y_1) + u(y_2) + \cdots + u(y_k)] \tag{5.17}$$

is satisfied for all $u \in \mathcal{H}_U$.

Proof. From this point in the paper, the argument is a slight modification of the proof of [9, Lemma 7.1]. Fix any multi-index α. Let $K \subset U$ be compact and $x_0 \in K$. We show that there exist an open neighborhood $V \subset U$ of x_0, an element $y \in D$, and a constant $c > 0$ such that

$$u(y) \geq c \sup_{x\in V} |D^\alpha u(x)| \quad \text{for all } u \in \mathcal{H}_U. \tag{5.18}$$

The result then follows from (5.18) using compactness of K. By Theorem 5.4, fixing $\beta > 0$, we obtain for all $y \in U$ and all $u \in \mathcal{H}_U$

$$u(y) \geq \beta \int_U u(z) g_\beta(y,z)\, dz.$$

We next claim that there exists $y \in D \setminus \{x_0\}$ such that $g_\beta(y, x_0) > 0$. If not, by Theorem 5.4, $g_\beta(\cdot, x_0) = 0$ on $U \setminus \{x_0\}$, a contradiction to $(L-\beta)_y g_\beta(y, x_0) = -\delta_{x_0}(x)$ on U in the sense of distributions (see (5.15)). Thus fix $y \in D \setminus \{x_0\}$ so that $g_\beta(y, x_0) > 0$ and by continuity, choose disjoint neighborhoods W of y and X of x_0 and a constant $c > 0$ such that

$$g_\beta(w, x) \geq c \quad \text{for all } (w, x) \in W \times X.$$

Nonnegativity of g_β then implies for fixed $y \in D$

$$u(y) \geq \beta c \int_X u(x)\, dx$$

for all $u \in \mathcal{H}_U$. In order to bound the integral $c\int_X u(x)\, dx = c\|u\|_{L^1(X)}$ from below, we bootstrap (2.16) to obtain for any $s > 0$ and $t < 0$ and any open neighborhood V of x_0 with $\overline{V} \subseteq X$, the existence of a constant $C_{s,t}$ depending only on s, t and V, X such that

$$\|u\|_{H^s(V)} \leq C_{s,t} \|u\|_{H^t(X)} \quad \text{for all } u \in \mathcal{H}_U.$$

For sufficiently negative t we have $L^1 \hookrightarrow H^t$, and there exists $C > 0$ independent of u such that

$$\sup_{x \in V} |D^\alpha u(x)| \leq C \int_X u(x)\, dx \leq \frac{C}{\beta c} u(y). \square$$

Given the previous result, we return to the proof of Proposition 4.8(ii).

Proof of Proposition 4.8(ii). Take a sequence of bounded open sets U_n, $n \geq 1$, with $\overline{U_n} \subset U$ and $U_n \uparrow U$ as $n \to \infty$. By Proposition 4.8(i), for any $n \geq 1$, if $\tau_n := \tau_{U_n}$, then $w_n(x) := \mathbf{E}_x \tau_n$ is bounded on $\overline{U_n}$ and $C^\infty(U_n)$ with $Lw_n = -1$ on U_n in the classical sense. In addition, since $U_n \subset U$, we have $w_n \leq v$. Let $x \in U_0$ and fix $\delta > 0$ such that $\overline{B_\delta(x)} \subset U_0$. Fix $m \in \mathbf{N}$ such that $U_m \supset \overline{B_\delta(x)}$. Then, for each $n \geq m$, the function $z_n = w_n - w_m$ satisfies $Lz_n = 0$ on $B_\delta(x)$ and for any $y \in D$, $0 \leq z_n(y) \leq v(y) < \infty$. By Theorem 5.5, z_n has bounded derivatives of all orders independently of n on $\overline{B_\delta(x)}$. Since m is fixed, $w_n = z_n + w_m$ has has bounded derivatives of all orders independently of n on $\overline{B_\delta(x)}$. Then, w_n converges to a function $\overline{v}$, uniformly on compact subsets of U_0.

The assertion (ii) follows from Lemma 4.2 once we show that $v = \overline{v}$ on U_0. Since $U_n \uparrow U$, then $\tau_n \uparrow \tau$ and $v = \overline{v}$ follows from the monotone convergence theorem. This finishes the proof of (ii). $\square$

6. General results on the formal stochastic solution

6.1. Preliminary remarks

Throughout this section, we assume that $U \subset \mathbf{R}^{\mathfrak{m}}$ is nonempty, open set with nonempty complement U^c. We return to one of our main goals: satisfaction of (1.2) by the formal stochastic representation, which for the Poisson equation is given by (provided it makes sense)

$$v_{\text{stoc},0}(x) = \mathbf{E}_x \int_0^{\tau} f(x_s)\, ds = \int_0^{\tau_0} f(\tilde{x}_s)\, ds, \;\; x \in U, \tag{6.1}$$

where $\tilde{x}_t = x_{t\wedge\tau_0}$ is the process stopped at the boundary ∂U. For example if $f \equiv 1$, $v(x) := \mathbf{E}_x\tau = \mathbf{E}_x\tau_0 = \mathbf{E}_x \int_0^{\tau_0} 1\, ds$, $x \in U$ and as long as the hypotheses of Proposition 4.8(ii) are satisfied, $v \in C^\infty(U)$ and $Lv = -1$ on U in the classical sense. Thus, we expect that under analogous assumptions, $v_{\text{stoc},0} \in C^\infty(U)$ and $Lv_{\text{stoc},0} = -f$ in the classical sense on U.

Additionally, if $g \in C_0^\infty(\mathcal{X})$ for the formal stochastic solution of the Dirichlet problem

$$u_{\text{stoc},0}(x) = \mathbf{E}_x g(x_\tau), \tag{6.2}$$

Dynkin's formula applied to g yields for any $x \in U$, $t \geq 0$

$$u_{\text{stoc},0}(x,t) := \mathbf{E}_x g(x_{\tau\wedge t}) = g(x) + \mathbf{E}_x \int_0^{\tau\wedge t} Lg(x_s)\, ds. \tag{6.3}$$

Thus by formally passing $t \to \infty$, rearranging and using (6.1), one has that $u_{\text{stoc},0} - g$ satisfies $L(u-g) = Lg$, and interior smoothness and $Lu_{\text{stoc},0} = 0$ on U in the classical sense follows. Then, by an approximation argument and Lemma 4.2, the same properties follow for more general g.

6.2. Interior smoothness for $v_{stoc,0}$ and $u_{stoc,0}$

We begin with the interior smoothness for $v_{\text{stoc},0}$. Suppose the process x_t is nonexplosive as in Definition 2.1 and that condition **(U00)** holds. For any measurable $f : \overline{U} \to [0,\infty)$, define $G_0 f : U \to [0,\infty]$ by (cf. (5.4))

$$G_0 f(x) = \mathbf{E}_x \int_0^{\tau} f(x_s)\, ds. \tag{6.4}$$

For any $f : \overline{U} \to \mathbf{R}$ measurable with $G_0|f|(x) < \infty$ for all $x \in U$, we let (cf. (5.5))

$$G_0 f(x) := G_0 f_+(x) - G_0 f_-(x) = \mathbf{E}_x \int_0^{\tau} f(x_s)\, ds \tag{6.5}$$

and

$$\begin{aligned}\mathcal{D}_0(U) = \{f : \overline{U} \to \mathbf{R} \text{ measurable} : x \mapsto G_0|f|(x) \\ \text{is bounded on compact sets in } U\}.\end{aligned} \tag{6.6}$$

Theorem 6.1 (Interior Smoothness for $v_{\text{stoc},0}$)**.** *Let $f \in C(U) \cap \mathcal{D}_0(U)$. Assume that x_t is nonexplosive as in Definition 2.1 and that conditions **(U00)** and **(PH)** are satisfied. Suppose, moreover $\mathbf{P}_x\{\tau < \infty\} = 1$ for all $x \in U$. Then*

$$Lv_{stoc,0} = -f \quad on \quad U \tag{6.7}$$

in the sense of distributions. Also, if $f \in C^\infty(U) \cap \mathcal{D}_0(U)$, then $v_{stoc,0} \in C^\infty(U)$ and $Lv_{stoc,0} = -f$ on U in the classical sense.

Proof. Suppose $f \in C(U) \cap \mathcal{D}_0(U)$. Let U_n be bounded, open set with $\overline{U}_n \subset U$ and $U_n \uparrow U$ as $n \to \infty$. For $n, k \in \mathbf{N}$ define

$$v_{n,k}(x) := \int_{1/k}^{k} \int_{U_n} \tilde{p}_t(x,y) f(y)\, dy\, dt.$$

By Theorem 4.3, $v_{n,k} \in C^\infty(U)$. Fubini's Theorem then gives

$$\begin{aligned} Lv_{n,k}(x) &= \int_{\frac{1}{k}}^{k} \int_{U_n} L_x \tilde{p}_t(x,y) f(y)\, dy\, dt \\ &= \int_{\frac{1}{k}}^{k} \int_{U_n} \partial_t \tilde{p}_t(x,y) f(y)\, dy\, dt \\ &= \mathbf{E}_x \mathbf{1}_{U_n}(\tilde{x}_k) f(\tilde{x}_k) - \mathbf{E}_x \mathbf{1}_{U_n}(\tilde{x}_{1/k}) f(\tilde{x}_{1/k}) \to -\mathbf{1}_{U_n}(x) f(x) \end{aligned}$$

as $k \to \infty$ in the sense of distributions on U, where we used that f is bounded, continuous on U_n and for all $x \in U$, $\mathbf{E}_x \mathbf{1}_{U_n}(\tilde{x}_k) \le 1 - \mathbf{P}\{\tilde{x}_k \in \partial U\} \to 0$ as $k \to \infty$ by $\mathbf{P}_x\{\tau < \infty\} = 1$.

Since f is bounded on U_n, by the Dominated Convergence theorem $v_{n,k} \to v_n$ as $k \to \infty$, where

$$v_n(x) := \int_0^\infty \int_{U_n} \tilde{p}_t(x,y) f(y)\, dy\, dt = \mathbf{E}_x \int_0^\tau \mathbf{1}_{U_n}(\tilde{x}_s) f(\tilde{x}_s)\, ds.$$

Then, by Lemma 4.2, $Lv_n = -f$ on U_n in the sense of distributions. Splitting f into positive and negative parts (using $f \in \mathcal{D}_0(U)$) and using the Monotone Convergence Theorem, we obtain that v_n, which is locally uniformly bounded on U, converges pointwise to $v_{\text{stoc},0}$ as $n \to \infty$. By employing Lemma 4.2 again, we obtain the desired result. □

Given the previous result, we next investigate the interior smoothness of $u_{\text{stoc},0}$.

Theorem 6.2 (Interior smoothness for $u_{\text{stoc},0}$)**.** *Assume that x_t is nonexplosive as in Definition 2.1 and that conditions **(U00)** and **(PH)** are satisfied. Suppose $\mathbf{P}_x\{\tau < \infty\} = 1$ for all $x \in U$ and $g \in C(\mathcal{X})$ is such that $u_{stoc,0}(x) = \mathbf{E}_x g(x_\tau)$ is bounded on compact subsets of U. Then, $u_{stoc,0} \in C^\infty(U)$ and $Lu_{stoc,0} = 0$ on U in the classical sense.*

Remark 6.3. Note that by the Tietze Extension Theorem we can replace the assumption $g \in C(\mathcal{X})$ with g being continuous only on a neighborhood of ∂U.

Proof. First let $U_0 \subset U$ be bounded, open with $\overline{U_0} \subset U$. By Proposition 4.8(i), $v_0(x) := \mathbf{E}_x \tau_{U_0}$ is bounded on U_0. Thus, for any open V_0 with $V_0 \supset \overline{U}_0$ and any $g \in C^2(V_0)$, the boundedness of U_0 implies $Lg \in C(U_0) \cap \mathcal{D}_0(U_0)$ (with respect to τ_{U_0}). By Theorem 6.1, the function

$$v_0(x) = \mathbf{E}_x \int_0^{\tau_{U_0}} Lg(x_s)\, ds \tag{6.8}$$

satisfies $Lv_0 = -Lg$ on U_0 in the sense of distributions. For $u_0(x) := \mathbf{E}_x g(x_{\tau_{U_0}})$, Dynkin's formula, the Dominated Convergence Theorem, and $\mathbf{E}_x \tau_{U_0} < \infty$ imply

$$\begin{aligned} u_0(x) = \lim_{t\to\infty} \mathbf{E}_x g(x_{\tau_{U_0}\wedge t}) &= g(x) + \lim_{t\to\infty} \mathbf{E}_x \int_0^{\tau_{U_0}\wedge t} Lg(x_s)\, ds \\ &= g(x) + \mathbf{E}_x \int_0^{\tau_{U_0}} Lg(x_s)\, ds. \end{aligned}$$

Hence, by (6.8) and $Lv_0 = -Lg$,

$$Lu_0 = Lg - Lg = 0 \quad \text{on} \quad U_0$$

in the sense of distributions. Since L is hypoelliptic, $u_0 \in C^\infty(U_0)$ and $Lu_0 = 0$ on U_0 in the classical sense.

Next assume g is merely continuous and supported on a bounded neighborhood W_0 of the boundary ∂U_0, and without loss of generality assume $g \equiv 0$ on W_0^c. Fix any $\psi \in C_0^\infty(\mathbf{R}^{\mathfrak{m}}; [0,\infty))$ with $\int_{\mathbf{R}^{\mathfrak{m}}} \psi\, dx = 1$, denote $\psi_\epsilon(x) = \epsilon^{-\mathfrak{m}} \psi(\epsilon^{-1}x)$ and set

$$g_\epsilon(x) = \int_{\mathbf{R}^{\mathfrak{m}}} g(y)\psi_\epsilon(x-y)\, dy \in C_0^\infty(\mathbf{R}^{\mathfrak{m}}) \qquad \text{and} \qquad u_\epsilon(x) = \mathbf{E}_x g_\epsilon(x_{\tau_{U_0}}). \tag{6.9}$$

Since g_ϵ is smooth, by the first part of the proof, $u_\epsilon \in C^\infty(U_0)$ and $Lu_\epsilon = 0$ on U_0 in the classical sense. Furthermore, g is bounded, and therefore u_ϵ is uniformly bounded and converges pointwise to $u_0(x) = \mathbf{E}_x g(x_{\tau_{U_0}})$ on U_0 as $\epsilon \to 0$. It follows by Lemma 4.2 and hypoellipticity of L that $u_0 \in C^\infty(U_0)$ and $Lu_0 = 0$ on U_0 in the classical sense.

Finally, let U_n, $n \in \mathbf{N}$, be a sequence of bounded open sets with $\overline{U_n} \subset U$ and $U_n \uparrow U$. Suppose that $g \in C(\mathcal{X}; [0,\infty))$ is nonnegative and by assumption $u_+(x) := \mathbf{E}_x g(x_\tau)$ is bounded on compact subsets of U. Set $\tau_n = \inf\{t > 0 : x_t \notin U_n\}$ and note that we already proved that $u_{n,N}(x) := \mathbf{E}_x[g(x_{\tau_n}) \wedge N] \in C^\infty(U_n)$ with $Lu_{n,N} = 0$ on U_n in the classical sense. After passing $n \to \infty$, Lemma 4.2, the Dominated Convergence Theorem, and $\tau_n \to \tau$ imply that $u_{\infty,N}(x) := \mathbf{E}_x[g(x_\tau) \wedge N]$ satisfies $u_{\infty,N} \in C^\infty(U)$ with $Lu_{\infty,N} = 0$ on U in the classical sense. Passing $N \to \infty$ and again applying Lemma 4.2 and Monotone convergence theorem, we find that $u_+ \in C^\infty(U)$ with $Lu_+ = 0$ on U in the classical sense. The result follows after decomposing g into positive and negative parts, $g = g_+ - g_-$. □

6.3. Boundary behavior

Note that under the hypotheses of Theorem 6.1 and Theorem 6.2, the formal stochastic solution u_{stoc} belongs to (1.4) is $C^\infty(U)$ and u_{stoc} is also classical solution of $Lu_{\text{stoc}} = -f$ on U, provided $f \in C^\infty(U)$. The next natural problem is to determine satisfaction of the boundary condition in (1.2). We explore it first for the Dirichlet part $u_{\text{stoc},0}$.

Theorem 6.4. *Fix $x_* \in \partial U$, $g \in C(\mathcal{X})$ and assume x_t is nonexplosive as in Definition 2.1. Assume that condition **(U00)** is satisfied and $\mathbf{P}_x\{\tau < \infty\} = 1$ for all $x \in U$. If, furthermore, conditions **(UID(g, x_*))** and **(CE(x_*))** are satisfied and x_* is regular as in Definition 3.4, then $u_{stoc,0}(x) \to g(x_*)$ as $x \to x_*, x \in U$.*

Remark 6.5. Note that if U is bounded, by extending b, σ to $\mathbf{R}^{\mathrm{m}}$ so that x_t is nonexplosive as in Remark 2.2, conditions **(UID(g, x_*))** and **(CE(x_*))** are satisfied for every $x_* \in \partial U$. Indeed, g is continuous on $\mathbf{R}^{\mathrm{m}}$ and bounded on $\overline{U}$. Moreover, one can set $\mathcal{X}_n = B_n$ so that for all n large enough $\mathbf{P}_x\{\tau_{\mathcal{X}_n} < \tau\} = 0$ for all $x \in U$.

Remark 6.6. As discussed above, checking that a given point $x_* \in \partial U$ is regular can be challenging. See [11, 39, 16] for some criteria. However, if it makes sense to slightly modify the set U, then regularity of x_* can be easier to verify. See Theorem 3.12 for further information.

Proof of Theorem 6.4. By **(UID(g, x_*))**, choose $\delta > 0$ so that $\mathcal{G}_{g,\delta}(x_*)$ as in (2.19) is uniformly integrable. For $|x - x_*| < \delta$ and $n \in \mathbf{N}$, write

$$u_{\text{stoc},0}(x) - g(x_*) = \mathbf{E}_x[(g(x_\tau) - g(x_*))\mathbf{1}\{x_\tau \notin \overline{\mathcal{X}_n}\}] + \mathbf{E}_x[(g(x_\tau) - g(x_*))\mathbf{1}\{x_\tau \in \overline{\mathcal{X}_n}\}] =: T_1(x,n) + T_2(x,n).$$

Fix $\epsilon > 0$ and we claim that there exists $n \in \mathbf{N}$ and $\delta_1 \in (0, \delta]$ such that $|T_1(x,n)| < \epsilon/2$ for all $|x - x_*| < \delta_1, x \in U$. Since the family $\mathcal{G}_{g,\delta}(x_*)$ is uniformly integrable, there is $\epsilon' > 0$ be such that

$$\epsilon' < \frac{\epsilon}{4(|g(x_*)| + 1)}$$

and whenever $\mathbf{P}\{A\} < \epsilon'$ we have $\mathbf{E}|X|\mathbf{1}_A < \frac{\epsilon}{4}$ for all $X \in \mathcal{G}_{g,\delta}(x_*)$. However, by **(CE($x_*$))**, there exists $n \in \mathbf{N}, \delta_1 \in (0, \delta]$ such that $|x - x_*| < \delta_1$, $x \in U$ implies

$$\mathbf{P}_x\{x_\tau \notin \overline{\mathcal{X}_n}\} \le \mathbf{P}_x\{\tau_{\mathcal{X}_n} < \tau\} < \epsilon'.$$

Hence, for such n and δ_1, uniformly integrability gives $|T_1(x,n)| < \frac{\epsilon}{2}$ for all $|x - x_*| < \delta_1, x \in U$, establishing the claim.

We next claim that for this choice of n, there exists $\delta_2 \in (0, \delta_1]$ such that $|T_2(x,n)| < \frac{\epsilon}{2}$, thus finishing the proof of the result. Indeed, since g is continuous

on the compact set $\overline{\mathcal{X}_n}$, there is a Lipschitz function $h \in C(\overline{\mathcal{X}_n})$ such that $\|g-h\|_{L^\infty(\overline{\mathcal{X}_n})} < \frac{\epsilon}{8}$. Hence, for any $\epsilon_* > 0$ we have

$$\begin{aligned}
|T_2(x,n)| &\le \frac{\epsilon}{4} + \mathbf{E}_x[(h(x_\tau) - h(x_*))\mathbf{1}\{x_\tau \in \overline{\mathcal{X}_n}\}] \\
&\le \frac{\epsilon}{4} + C_1\mathbf{E}|x_\tau(x) - x_*| \wedge C_2\mathbf{1}\{\tau < \epsilon_*\} + 2\|h\|_{L^\infty}\mathbf{P}_x\{\tau \ge \epsilon_*\} \\
&\le \frac{\epsilon}{4} + C_1\mathbf{E}\sup_{s\in[0,\epsilon_*]}|x_s - x_*| \wedge C_2 + 2\|h\|_{L^\infty}\mathbf{P}_x\{\tau \ge \epsilon_*\} \\
&=: \frac{\epsilon}{4} + T_2^1 + T_2^2\,,
\end{aligned}$$

where $C_1, C_2 > 0$ are constants depending only on n and h. To estimate T_2^1, observe that

$$\begin{aligned}
T_2^1 &= C_1\mathbf{E}\sup_{s\in[0,\epsilon_*]}|x_s(x) - x_*| \wedge C_2 \\
&\le C_1\big(\mathbf{E}\sup_{s\in[0,\epsilon_*]}|x_s(x) - x_s(x_*)|^2 \wedge C_2^2\big)^{1/2} \\
&\quad + C_1\big(\mathbf{E}\sup_{s\in[0,\epsilon_*]}|x_s(x_*) - x_*|^2 \wedge C_2^2\big)^{1/2}
\end{aligned}$$

where the last inequality follows by triangle and Cauchy-Schwarz inequalities. Using Lemma 4.1(i) and path continuity of x_t, there exits $\delta_2 \in (0, \delta_1]$ and $\epsilon_* > 0$ small enough so that $T_2^1 \le \epsilon/8$ for all x with $|x - x_*| < \delta_2$. For T_2^2, Lemma 4.1(ii) and the regularity of x_*, ensure that $T_2^2 < \epsilon/8$ if $\delta_2 > 0$ is sufficiently small. This finishes the proof of the result. □

Remark 6.7. Note that if U us bounded, then **(CE(x_*))** is trivially satisfied by choosing n large such that $U \subset \mathcal{X}_n$. Here, we verify **(CE(x_*))** for unbounded U if $x \mapsto \mathbf{E}_x\tau$ is bounded on $B_{\delta_2}(x_*) \cap U$ for some $\delta_2 > 0$ and hypotheses of Lemma 2.4 are satisfied.

For any $t > 0$, we have

$$\mathbf{P}_x\{\tau_{\mathcal{X}_n} < \tau\} \le \mathbf{P}_x\{\tau \ge t\} + \mathbf{P}_x\{\tau_{\mathcal{X}_n} \le t\}.$$

Fix any $\delta_1 > 0$. Then, Chebychev's inequality gives for all $x \in B_\delta(x_*) \cap U$ that

$$\mathbf{P}_x\{\tau \ge t\} \le t^{-1}\mathbf{E}_x\tau \le C/t\,.$$

Fix $t > 0$ large enough independent of x, such that $\mathbf{P}_x\{\tau \ge t\} < \frac{\delta_1}{2}$. As in the proof of Lemma 2.4, we have

$$w_n e^{-Ct}\mathbf{P}_x\{\tau_{\mathcal{X}_n} \le t\} \le w(x) + D\,.$$

with $w_n \to \infty$ as $n \to \infty$. For already fixed $t > 0$, there is $n \in \mathbf{N}$ large enough so that $\mathbf{P}_x\{\tau_{\mathcal{X}_n} \le t\} < \frac{\delta_1}{2}$ for each $x \in B_{\delta_2}(x_*)$ and (2.21) is satisfied.

Next, we derive sufficient conditions on the Poisson part $v_{\text{stoc},0}$ of u_{stoc} that ensure $v_{\text{stoc},0}(x) \to 0$ as $x \to x_* \in \partial U$, $x \in U$.

Theorem 6.8. *Suppose x_t is nonexplosive as in Definition 2.1, $x_* \in \partial U$ is regular as in Definition 3.4 and condition **(U00)** is satisfied. Suppose, furthermore, that conditions **(UIP(f, x_*))** and **(CE(x_*))** are satisfied for some $f : U \to \mathbf{R}$ which is measurable and bounded on $\mathcal{X}_n \cap U$ for all n. If for every $x \in U$, $\mathbf{P}_x\{\tau < \infty\} = 1$, then $v_{stoc,0}(x) \to 0$ as $x \to x_*$, $x \in U$.*

Proof. Let $\epsilon > 0$ and by **(UIP(f, x_*))** there is $\delta_1 > 0$ such that for all $x \in U$ with $|x - x_*| < \delta_1$ the family $\mathcal{G}^f_{\delta_1}(x_*)$ in (2.20) is uniformly integrable. For simplicity, set $Y_t = \int_0^t f(x_s)\,ds$. Then, since f is bounded on $\mathcal{X}_n \cap U$, for any $\delta > 0$

$$\begin{aligned}|v_{\mathrm{stoc},0}(x)| &\le |\mathbf{E}_x Y_\tau \mathbf{1}\{\tau_{\mathcal{X}_n} < \tau\}| + |\mathbf{E}_x Y_\tau \mathbf{1}\{\tau_{\mathcal{X}_n} \ge \tau\}| \\ &\le |\mathbf{E}_x Y_\tau \mathbf{1}\{\tau_{\mathcal{X}_n} < \tau\}| + |\mathbf{E}_x Y_{\tau\wedge\delta} \mathbf{1}\{\tau_{\mathcal{X}_n} \ge \tau, \tau \le \delta\}| + |\mathbf{E}_x Y_\tau \mathbf{1}\{\tau > \delta\}| \\ &\le |\mathbf{E}_x Y_\tau \mathbf{1}\{\tau_{\mathcal{X}_n} < \tau\}| + \delta \|f\|_{L^\infty(\mathcal{X}_n \cap U)} + |\mathbf{E}_x Y_\tau \mathbf{1}\{\tau > \delta\}|.\end{aligned}$$

By uniform integrability, there exists $\epsilon' > 0$ such that $\mathbf{P}\{A\} < \epsilon'$ implies $\mathbf{E}_x Y \mathbf{1}_A < \epsilon/3$ for all $|x - x_*| < \delta_1$. By **(CE(x_*))**, there are $n \in \mathbf{N}, \delta_2 \in (0, \delta_1]$ such that $\mathbf{P}_x\{\tau_{\mathcal{X}_n} < \tau\} < \epsilon'$ for all $|x - x_*| < \delta_2$, $x \in U$. For this choice of n, let

$$\delta = \frac{\epsilon}{3(\|f\|_{L^\infty(\mathcal{X}_n \cap U)} + 1)}.$$

By making $\delta_2 > 0$ smaller if necessary, by Lemma 4.1(ii) we can ensure that $\mathbf{P}_x\{\tau > \delta\} < \epsilon'$ for all $|x - x_*| < \delta_2$, $x \in U$. The result follows since for $|x - x_*| < \delta_2$, $x \in U$, we have $|v_{\mathrm{stoc},0}(x)| < \epsilon$ and $\epsilon > 0$ is arbitrary. □

We can now combine the previous results and relate them back to the original problem (1.2).

Corollary 6.9. *Suppose x_t is nonexplosive as in Definition 2.1, $\mathbf{P}_x\{\tau < \infty\} = 1$ for all $x \in U$, and condition **(U00)** is satisfied. Let $g \in C(\mathcal{X})$ and $f \in C^\infty(U) \cap \mathcal{D}_0(U)$ be such that $u_{stoc,0}$ is bounded on compact subsets of U and $f \in B(\mathcal{X}_n \cap U)$ for all n. If U is boundary regular for x_t and the conditions **(UID(g))**, **(UIP(f))**, and **(CE)** are satisfied, then u_{stoc} is a classical solution of (1.2). If U is furthermore assumed to be bounded, then u_{stoc} is the unique classical solution of (1.2).*

Proof. The only assertion we have left to prove is uniqueness when U is bounded.

Let $\{U_n\}$ be a sequence of bounded open subsets of U with $\overline{U}_n \subset U_{n+1}$ and $U = \cup_n U_n$. If u is a classical solution of (1.2), then u is smooth on U_n for each n, and by Dynkin's formula (2.8) we have for any $x \in U$ and $n \in \mathbf{N}$ large enough

$$\mathbf{E}_x u(x_{t\wedge\tau_{U_n}}) = u(x) + \mathbf{E}_x \int_0^{t\wedge\tau_{U_n}} Lu(x_s)\,ds = u(x) - \mathbf{E}_x \int_0^{t\wedge\tau_{U_n}} f(x_s)\,ds,$$

and therefore

$$u(x) = \mathbf{E}_x u(x_{t\wedge\tau_{U_n}}) + \mathbf{E}_x \int_0^{t\wedge\tau_{U_n}} f(x_s)\,ds.$$

Since u is continuous on the compact set $\overline{U}$, u is bounded on U. Also, since $g \in C(\mathcal{X})$ and $f \in C^\infty(U) \cap \mathcal{D}_0(U)$, passing $t \to \infty$ and then $n \to \infty$, boundary regularity and the Dominated Convergence Theorem imply that $u(x) = u_{\text{stoc}}(x)$. □

7. The transience and recurrence dichotomy for degenerate diffusions

The goal of this section is to carefully establish the dichotomy between transience and recurrence for degenerate diffusions by adapting the classic cycle constructions of Khasminskii [35], which was carried out in the setting of elliptic diffusions on Euclidean space. Note that this has been done previously using the language of invariant control sets as in [37]. However, we found a gap in the arguments in [37] that we could not fix in an obvious way (see Remark 7.22 below). Moreover, some regularity claims in [37] could not be verified without calculations analogous to ones in the previous sections. Although it is known that there are alternative, probabilistic paths which circumnavigate these issues (we refer, in particular, to the work of Harris [30], the survey paper of Baxendale [3] and the work of Meyn-Tweedie [50]), here we establish the dichotomy using classical ideas of Khasminskii and regularity properties established above. Note that this approach traces back to Maruyama and Tanaka [48] and Watanabe [63] in the case of a one-dimensional, elliptic diffusion. We also refer to the works [1, 27] which we found helpful.

7.1. Nice diffusions

In this subsection, we briefly introduce the structural assumptions we make on the diffusion x_t satisfying (1.3) in this section. Recall that if x_t is nonexplosive as in Definition 2.1, then $x_t \in \mathcal{X}$ for all $t \geq 0$ and any initial condition $x_0 = x \in \mathcal{X}$.

In order to formulate our results, we need a notion of *irreducibility* of x_t as introduced in the following definition.

Definition 7.1. Suppose that x_t is nonexplosive as in Definition 2.1. We call x_t *irreducible* if for any $x, y \in \mathcal{X}$ and $\delta > 0$ we have $\mathbf{P}_y\{\tau_{B_\delta(x)^c} < \infty\} > 0$.

Note that irreducibility means that, for all $x, y \in \mathcal{X}$, the process started at $y \in \mathcal{X}$ enters an arbitrarily small neighborhood of $x \in \mathcal{X}$ with positive probability. Thus, the process can transition between arbitrarily small neighborhoods of any two points in $\mathcal{X}$.

Remark 7.2. Comparing terminology, if x_t is irreducible in the sense of Definition 7.1, then $\mathcal{X}$ is the unique invariant control set of x_t as in [37]. Certainly, the methods used below can be applied in more general settings, e.g. if there is more than one invariant control set or if the process x_t eventually enters an invariant control set from a larger set to not return to other parts of space. For our purposes, one irreducible set $\mathcal{X}$ is sufficient.

We are ready to introduce the assumptions we impose on the diffusion x_t in this section.

Definition 7.3. We say that the diffusion x_t is *nice* if the following conditions are met:

(i) x_t is nonexplosive as in Definition 2.1;
(ii) The generator L of x_t is satisfies the parabolic Hörmander condition on $\mathcal{X}$ as in Definition 2.12;
(iii) x_t is irreducible as in Definition 7.1.

One key property of a nice diffusion employed below is that the process leaves bounded sets in $\mathcal{X}$ sufficiently fast.

Proposition 7.4. *Suppose that x_t is a nice diffusion and $U \subset \mathcal{X}$ is nonempty, open and bounded with $\overline{U} \subset \mathcal{X}$. Then there exists $\delta > 0$ such that*

$$\sup_{x \in \overline{U}} \mathbf{E}_x e^{\delta \tau_{\overline{U}}} < \infty. \tag{7.1}$$

Proof. The proof of this result follows a similar reasoning used in the proof of Proposition 4.8(i). Let $z \in \mathcal{X} \setminus \overline{U}$ and fix $\epsilon > 0$ such that $B_\epsilon(z) \subset \mathcal{X} \setminus \overline{U}$. By irreducibility of x_t, for all $x \in \overline{U}$ there exists $t = t(x) > 0$ and $\alpha = \alpha(x) \in (0,1)$ such that

$$\mathbf{P}_x\{\tau_{B_\epsilon(z)^c} > t\} \le 1 - \alpha.$$

Applying Proposition 4.7(i) along with the parabolic Hörmander condition on $\mathcal{X}$ and compactness of $\overline{U}$, there exists $t_* > 0$ and $\alpha_* \in (0,1)$ independent of x so that

$$\mathbf{P}_x\{\tau_{\overline{U}} > t_*\} \le \mathbf{P}_x\{\tau_{B_\epsilon(z)^c} > t_*\} \le 1 - \alpha_* \quad \text{for all} \quad x \in \overline{U}. \tag{7.2}$$

Following the proof of Proposition 4.8(i), the Markov property implies that

$$\mathbf{P}_x\{\tau_{\overline{U}} \ge m t_*\} \le (1 - \alpha_*)^{m-1}$$

for all $m \in \mathbf{N}$ and all $x \in \overline{U}$. Hence $\tau_{\overline{U}} < \infty$, $\mathbf{P}_x$-almost surely. Furthermore, choosing $\delta = \delta(t_*, \alpha_*) > 0$ small enough so that $e^{\delta t_*}(1 - \alpha_*) < 1$, it follows that for any $x \in \overline{U}$:

$$\begin{aligned}
\mathbf{E}_x e^{\delta \tau_{\overline{U}}} = \sum_{m=1}^{\infty} \mathbf{E}_x e^{\delta \tau_{\overline{U}}} \mathbf{1}\{(m-1)t_* \le \tau_{\overline{U}} < m t_*\} &\le \sum_{m=1}^{\infty} e^{\delta m t_*} \mathbf{P}_x\{\tau_{\overline{U}} \ge (m-1)t_*\} \\
&\le (1 - \alpha_*)^{-2} \sum_{m=1}^{\infty} (e^{\delta t_*}(1 - \alpha_*))^m \\
&< \infty. \square
\end{aligned}$$

Remark 7.5. Yet another way to rephrase the conclusion of Proposition 7.4 is that the process x_t exits any bounded domain in $\mathcal{X}$ in *logarithmic time* or *exponentially fast* on average.

7.2. Recurrence and transience for nice diffusions

We start with the definition of transience and recurrence.

Definition 7.6. Suppose the diffusion x_t is nice. We say that a point $x \in \mathcal{X}$ is *recurrent* if for any $\delta > 0$ and any $y \in \mathcal{X} \setminus B_\delta(x)$

$$\mathbf{P}_y\{\tau_{B_\delta(x)^c} < \infty\} = 1. \tag{7.3}$$

Otherwise, we say $x \in \mathcal{X}$ is *transient*.

Our next goal is to prove that points in $\mathcal{X}$ are either all recurrent, in which case we call x_t *recurrent*, or all transient, in which case we call x_t *transient*. Thus, the dichotomy between transience and recurrence is established in the following proposition. Afterwards, we establish further properties of transience and recurrence.

Proposition 7.7. *Assume x_t is nice. If $x \in \mathcal{X}$ is recurrent, then all points in $\mathcal{X}$ are recurrent. Consequently, either all points in $\mathcal{X}$ are recurrent or all points in $\mathcal{X}$ are transient.*

Proof. Suppose $x \in \mathcal{X}$ is recurrent and let $y \in \mathcal{X}$ with $x \neq y$. We show that y is also recurrent. Suppose that $\delta > 0$ is any positive real number such that $x \notin B_{2\delta}(y)$ and $\overline{B_{2\delta}(y)} \subset \mathcal{X}$. By irreducibility in Definition 7.1 and path continuity (if $x_t \in B_\delta(y)$ for some t, then the inclusion holds for rational t), there exists $t_* > 0$ such that

$$2\alpha := P_{t_*}(x, B_\delta(y)) > 0. \tag{7.4}$$

Since L satisfies the parabolic Hörmander condition on $\mathcal{X}$, $w \mapsto P_{t_*}(w, B_\delta(y))$ is continuous at x by Remark 4.6 and Theorem 4.3. In particular, there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap B_{2\delta}(y) = \emptyset$, $\overline{B_\epsilon(x)} \subset \mathcal{X}$ and $P_{t_*}(w, B_\delta(y)) \geq \alpha$ for all $w \in \overline{B_\epsilon(x)}$. Define stopping times $\sigma_j, j = 0, 1, \ldots$, inductively as follows:

$$\begin{aligned}
\sigma_1 &= \inf\{t \geq 0 \,:\, x_t \in B_\epsilon(x)\},\\
\sigma_2 &= \sigma_1 + t_*\\
\sigma_3 &= \inf\{t \geq \sigma_2 \,:\, x_t \in B_\epsilon(x)\}\\
&\vdots \quad \vdots\\
\sigma_{2k} &= \sigma_{2k-1} + t_*\\
\sigma_{2k+1} &= \inf\{t \geq \sigma_{2k} \,:\, x_t \in B_\epsilon(x)\}.
\end{aligned}$$

for $k \geq 2$. Then for all $j \geq 1$, the stopping time σ_j is almost surely finite since x_t is nice and x is recurrent. Next, by the strong Markov property we have for each $j \geq 0$:

$$\begin{aligned}
\mathbf{P}_w\{\tau_{B_\delta(y)^c} > \sigma_{2j+2}\} &= \mathbf{E}_w\mathbf{E}_w[\mathbf{1}\{\tau_{B_\delta(y)^c} > \sigma_{2j+2}\}|\mathcal{F}_{\sigma_{2j+1}}]\\
&= \mathbf{E}_w\mathbf{1}\{\tau_{B_\delta(y)^c} > \sigma_{2j+1}\}\mathbf{E}_{x_{\sigma_{2j+1}}}\mathbf{1}\{\tau_{B_\delta(y)^c} > t_*\}
\end{aligned}$$

$$\leq (1-\alpha)\mathbf{P}_w\{\tau_{B_\delta(y)^c} > \sigma_{2j}\}.$$

Thus, by induction, $\mathbf{P}_w\{\tau_{B_\delta(y)^c} > \sigma_{2j}\} \leq (1-\alpha)^j$ for all $j \geq 0$. Finally, the Borel-Cantelli Lemma implies $\mathbf{P}_w\{\tau_{B_\delta(y)^c} < \infty\} = 1$ for all $w \in \mathcal{X}$, and therefore y is recurrent. □

We also have the following corollary of Proposition 7.7.

Corollary 7.8. *Suppose x_t is nice. If x_t is recurrent, then for any $x, y \in \mathcal{X}$ and any open set $U_y \subset \mathcal{X}$ containing y:*

$$\mathbf{P}_x\{\omega : \exists\, s_j(\omega) \in (0,\infty) \uparrow \infty \text{ for which } x_{s_j} \in U_y \text{ for all } j\} = 1.$$

Remark 7.9. Corollary 7.8 states that if x_t is recurrent, then for all $x \in \mathcal{X}$, almost surely the process started from x visits infinitely often any neighborhood of any $y \in \mathcal{X}$.

Proof of Corollary 7.8. Let $x, y \in \mathcal{X}$ and $U_y \subset \mathcal{X}$ be open with $y \in U_y$. Fix $z \in \mathcal{X}$ with $z \neq x$ and $z \neq y$. Choose $\delta > 0$ such that $B_\delta(y) \subset U_y$, $B_\delta(z) \cap B_\delta(x) = \emptyset$, $B_\delta(z) \cap B_\delta(y) = \emptyset$ and $B_\delta(z) \subset \mathcal{X}$. Define stopping times $\sigma_i, i = 0, 1, \ldots$ inductively by

$$\begin{aligned}
\sigma_0 &= 0;\\
\sigma_1 &= \inf\{t \geq \sigma_0 : x_t \in B_\delta(y)\}\\
\sigma_2 &= \inf\{t \geq \sigma_1 + 1 : x_t \in B_\delta(z)\}\\
&\vdots \quad \vdots\\
\sigma_{2k+1} &= \inf\{t \geq \sigma_{2k} : x_t \in B_\delta(y)\}\\
\sigma_{2k+2} &= \inf\{t \geq \sigma_{2k+1} + 1 : x_t \in B_\delta(z)\}.
\end{aligned}$$

By recurrence, σ_k, $k \geq 0$ is almost surely finite. The result follows by setting $s_j = \sigma_{2j+1}$. □

The next proposition further explores implications of transience for a nice diffusion x_t.

Proposition 7.10. *Suppose x_t is nice and transient. For any $x, y \in \mathcal{X}$ there exists $\delta > 0$ small enough such that $\overline{B_\delta(y)} \subset \mathcal{X}$ and*

$$\mathbf{P}_x\{\omega : \exists\, t_0(\omega) \in [0,\infty) \text{ such that } x_t(\omega) \notin B_\delta(y)\ \forall t \geq t_0(\omega)\} = 1. \tag{7.5}$$

Proof. Since x_t is transient, there exist $\delta_1 > 0$, $y \in \mathcal{X}$, and $z \in \mathcal{X} \setminus B_{2\delta_1}(y)$ such that

$$\mathbf{P}_z\{\tau_{B_{\delta_1}(y)^c} = \infty\} =: 2\alpha > 0. \tag{7.6}$$

By Proposition 4.7, there exists $\epsilon > 0$ such that $\overline{B_\epsilon(z)} \subset \mathcal{X}$ and

$$\mathbf{P}_w\{\tau_{B_{\delta_1}(y)^c} = \infty\} \geq \alpha$$

for all $w \in B_\epsilon(z)$. In order to obtain a contradiction, for every $\delta > 0$ define

$$E_\delta(x) := \{\omega \,:\, \exists\, s_j(\omega) \in (0,\infty) \uparrow \infty \text{ such that } x_{s_j}(\omega) \in B_\delta(y), x_0 = x\} \quad (7.7)$$

and assume there is $x \in \mathcal{X}$ such that for each $\delta > 0$

$$\mathbf{P}\{E_\delta(x)\} > 0. \quad (7.8)$$

Note that for each $t \geq 0$, one has $E_\delta(x) = E_\delta(x_t(x))$, where $x_t(x)$ is the process with $x_0(x) = x$. Next, we claim that, on the set $E_\delta(x)$ for all $\delta > 0$ small enough, the process x_t almost surely enters $\overline{B_{\epsilon/2}(z)}$.

To prove the claim, first observe that since $\mathcal{X}$ is irreducible,

$$\mathbf{P}_y\{\tau_{B_{\epsilon/2}(z)^c} < \infty\} > 0.$$

Thus, by path continuity (cf. the proof of Proposition 7.7) there exists $t_* > 0$ such that $P_{t_*}(y, B_{\epsilon/2}(z)) =: 2a > 0$. Since $w \mapsto P_{t_*}(w, B_{\epsilon/2}(z))$ is continuous, there exists $\delta \in (0, \delta_1)$ such that $\overline{B_\delta(y)} \subset \mathcal{X}$ and $P_{t_*}(w, B_{\epsilon/2}(z)) \geq a$ for all $w \in \overline{B_\delta(y)}$. Fix such $\delta \in (0, \delta_1)$ and inductively define stopping times ζ_i, $i = 1, 2, \ldots$, by

$$\begin{aligned}
\zeta_1 &= \inf\{t \geq 0 \,:\, x_t \in B_\delta(y)\}\,,\\
\zeta_2 &= \zeta_1 + t_*\,,\\
&\vdots \quad \vdots\\
\zeta_{2k+1} &= \inf\{t \geq \zeta_{2k} \,:\, x_t \in B_\delta(y)\}\,,\\
\zeta_{2k+2} &= \zeta_{2k+1} + t_*.
\end{aligned}$$

By the definition of $E_\delta(x)$, the stopping times ζ_i are almost surely finite on $E_\delta(x)$. Then, the strong Markov property (cf. the proof of Proposition 7.7) yields

$$\mathbf{P}_x\{\tau_{B_{\epsilon/2}(z)^c} > \zeta_{2j} \,|\, E_\delta(x)\} \leq (1-a)^j \quad (7.9)$$

for any j. Thus the Borel-Cantelli lemma implies that $\mathbf{P}_x\{\tau_{B_{\epsilon/2}(z)^c} < \infty | E_\delta(x)\} = 1$, establishing the claim.

Next, define stopping times σ'_i, $i = 1, 2, \ldots$, by

$$\begin{aligned}
\sigma'_1 &= \inf\{t \geq 0 \,:\, x_t \in B_\delta(y)\}\,,\\
\sigma'_2 &= \inf\{t \geq \sigma'_1 \,:\, x_t \in B_{\epsilon/2}(z)\}\,,\\
&\vdots \quad \vdots\\
\sigma'_{2k+1} &= \inf\{t \geq \sigma'_{2k} \,:\, x_t \in B_\delta(y)\}\,,\\
\sigma'_{2k+2} &= \inf\{t \geq \sigma'_{2k+1} \,:\, x_t \in B_{\epsilon/2}(z)\}.
\end{aligned}$$

By a similar argument to the one used above, it also follows that on the event $E_\delta(x)$, σ'_j is finite almost surely for all $j \geq 1$. Observe that $\mathbf{P}(\sigma'_j < \infty) \geq$

$\mathbf{P}(E_\delta(x)) \geq c$, where c is independent of j. However, by the strong Markov property and iteration, if $j \geq 2$ we have

$$\begin{aligned}\mathbf{P}_x\{\sigma'_{2j} < \infty\} &= \mathbf{E}_x\mathbf{E}_x[\mathbf{1}\{\sigma'_{2j} < \infty\} \,|\, \mathcal{F}_{\sigma'_{2j-2}}] \\ &= \mathbf{E}_x\mathbf{1}\{\sigma'_{2j-2} < \infty\}\mathbf{P}_{x_{\sigma'_{2j-2}}}\{\sigma'_2 < \infty\} \leq (1-\alpha)\mathbf{P}_x\{\sigma'_{2j-2} < \infty\} \\ &\leq (1-\alpha)^{j-1}.\end{aligned}$$

Thus Borel-Cantelli implies that $\sigma'_{2j} < \infty$ for only finitely many j, a contradiction. □

As an immediate consequence, we have the following corollary.

Corollary 7.11. *If x_t is transient, then for any compact set $K \subset \mathcal{X}$ and $x \in \mathcal{X}$*

$$\mathbf{P}_x\{\omega : \exists\, t_0^K(\omega) \in [0,\infty) \text{ such that } x_t(\omega) \notin K \ \forall t \geq t_0^K(\omega)\} = 1$$

and

$$\lim_{t\to\infty} \mathbf{P}_x\{x_t \in K\} = 0\,.$$

Proof. Fix $x \in \mathcal{X}$ and for any $y \in K$ fix $\delta_y > 0$ and $t_0(\omega) = t_{0,y}(\omega)$ such that the conclusion of Proposition 7.10 holds true. From the open cover $\{B_{\delta(y)}(y)\}_{y\in K}$ of K choose a finite subcover and define t_0^K to be the maximum of $t_{0,y}$ in this finite subcover. For the second conclusion, we note that

$$\mathbf{P}_x\{x_t \in K\} \leq \mathbf{P}_x\{t_0^K(\omega) > t\} \to 0$$

as $t \to \infty$, where t_0^K is as in the first assertion. □

7.3. Invariant measures

A central interest in the theory of stochastic differential equations is the large-time behavior of the process x_t. In particular, we are interested in the relationship between recurrence, transience and the existence of *invariant measures*. Such measures are the random analogues of equilibrium points of deterministic ordinary differential equations.

To introduce invariant measures, throughout this section we again assume x_t is a nice diffusion. In particular, x_t is a nonexplosive process on $\mathcal{X}$, and consequently the process x_t is Markov with Markov semigroup $(\mathcal{P}_t)_{t\geq 0}$. Recalling that $\mathcal{B}$ denotes the Borel sigma algebra of subsets of $\mathcal{X}$, we call a positive, $\mathcal{B}$-measure μ an *invariant measure*, if $\mu\mathcal{P}_t = \mu$ for all $t \geq 0$, where $\mu\mathcal{P}_t$ was defined in (2.6). An invariant measure μ with $\mu(\mathcal{X}) = 1$ is called an *invariant probability measure*.

Remark 7.12. Observe that if μ is an invariant probability measure, the equality $\mu\mathcal{P}_t = \mu$ for all $t \geq 0$ means that the process x_t with initial distribution μ has the distribution μ for all times $t \geq 0$. In other words, the statistics remain invariant under the dynamics.

Remark 7.13. It is common in the literature to implicitly assume that an invariant measure is an invariant probability measure. However, below we need to distinguish between invariant measures which are probability distributions and those which are not.

We first show that a nice diffusion x_t which is transient cannot have an invariant probability measure.

Corollary 7.14. *If x_t is nice diffusion which is transient, then x_t cannot have an invariant probability measure.*

Proof. Suppose to the contrary that there exists an invariant probability measure μ. Then, there exists a compact set $K \subset \mathbf{R}^{\mathrm{m}}$ with $K \subset \mathcal{X}$ and $\mu(K) > 0$. Since μ is invariant,

$$0 < \mu(K) = \int_{\mathcal{X}} \mu(dx) P_t(x, K) \qquad \text{for all } t \geq 0.$$

Using the Bounded Convergence Theorem and Corollary 7.11, it follows that

$$\int_{\mathcal{X}} \mu(dx) P_t(x, K) \to 0$$

as $t \to \infty$. Hence, $\mu(K) = 0$, a contradiction. □

On the other hand, when x_t is recurrent, one can always construct a σ-finite invariant measure using an embedded Markov chain via cycles. We provide details below, but first we prove an auxiliary result that allows us to further categorize recurrence.

Proposition 7.15. *Let x_t be a nice diffusion. If $U \subset \mathcal{X}$ is a bounded, nonempty, open set with $\overline{U} \subset \mathcal{X}$ and $\mathbf{E}_x \tau_{U^c} < \infty$ for all $x \in \mathcal{X}$, then:*

(i) x_t is recurrent;
(ii) $\mathbf{E}_x \tau_{V^c} < \infty$ for all $x \in \mathcal{X}$ and any nonempty, open set $V \subset \mathcal{X}$ with $\overline{V} \subset \mathcal{X}$

Remark 7.16. The argument is similar to previous cycle constructions, except that one has to control expected values rather than probabilities.

Proof of Proposition 7.15. We prove both conclusions simultaneously. By making V smaller, we may suppose without loss of generality that $V \subset \overline{V} \subset \mathcal{X}$ is an open ball. Fixing $x \in \mathcal{X}$, our goal is to show that $\mathbf{E}_x \tau_{\overline{V}^c} < \infty$.

Let $\mathcal{X}_k \subset \mathcal{X}$ be an increasing sequence of bounded open sets with $\overline{\mathcal{X}_k} \subset \mathcal{X}_{k+1}$ and $\cup_k \mathcal{X}_k = \mathcal{X}$. By compactness, there is $k_0 \in \mathbf{N}$ large enough so that $\mathcal{X}_k \supset \overline{U} \cup \overline{V} \cup \{x\}$ for each $k \geq k_0$. Since x_t is nice, for each fixed $y \in \overline{U}$ there is $\ell = \ell(y) \geq k_0 + 1$ such that $u_\ell(y) := \mathbf{P}_y\{\tau_{\overline{V}^c} < \tau_{\mathcal{X}_\ell}\} > 0$. Note that for each $k \geq k_0$ and $z \in Q := \mathcal{X}_k \setminus \overline{V}$ one has $\mathbf{P}_z\{\tau_Q < \infty\} = 1$. Indeed, by Proposition 7.4, x_t almost surely leaves $\mathcal{X}_k$ in finite time. Setting $g = 1$ on ∂V and $g = 0$ on $\partial \mathcal{X}_\ell$, we have $u_\ell(y) = \mathbf{E}_y g(x_{\tau_Q})$ and by Theorem 6.2, $u_\ell(w) > 0$ for all $w \in B_\epsilon(y)$ and some $\epsilon > 0$. Since $\ell \mapsto u_\ell(x)$ is nondecreasing for any

$x \in \mathcal{X}_{k_0+1}$, the compactness implies the existence of $k^* \geq k_0 + 1$ and $q > 0$ such that

$$\inf_{y \in \overline{U}} \mathbf{P}_y\{\tau_{\overline{V}^c} < \tau_{\mathcal{X}_{k^*}}\} \geq q\,.$$

Next, define stopping times η_i, $i = -2, 0, 1, 2 \ldots$, as follows: $\eta_{-2} = 0$ and

$$\begin{aligned}
\eta_0 &= \inf\{t \geq 0 \,:\, x_t \in \overline{U}\},\\
\eta_1 &= \inf\{t \geq \eta_0 \,:\, x_t \in \partial\mathcal{X}_{k^*}\},\\
\eta_2 &= \inf\{t \geq \eta_1 \,:\, x_t \in \overline{U}\},\\
&\vdots \quad \vdots\\
\eta_{2i+1} &= \inf\{t \geq \eta_{2i} \,:\, x_t \in \partial\mathcal{X}_{k^*}\},\\
\eta_{2i+2} &= \inf\{t \geq \eta_{2i+1} \,:\, x_t \in \overline{U}\}.
\end{aligned}$$

As in the proof of Proposition 7.7, using Proposition 7.4 one can show that each η_j is almost surely finite and $\mathbf{P}_x\{\tau_{\overline{V}^c} \geq \eta_{2N}\} \leq (1-q)^{N-1}$. Then, by Borel-Cantelli, $\tau_{\overline{V}^c} < \eta_{2N}$, for $\mathbf{P}_x$-almost surely for some bounded random index N. The conclusion in (i) follows as x was arbitrary.

To prove (ii), note that

$$\mathbf{E}_x\tau_{\overline{V}^c} = \sum_{j=0}^{\infty} \mathbf{E}_x\tau_{\overline{V}^c}\mathbf{1}_{[\eta_{2(j-1)},\eta_{2j})}(\tau_{\overline{V}^c}) \leq \sum_{j=0}^{\infty} \mathbf{E}_x\eta_{2j}\mathbf{1}_{[\eta_{2(j-1)},\eta_{2j})}(\tau_{\overline{V}^c}) \tag{7.10}$$

and if $E_\ell := \{\omega \,:\, x_t \in \overline{V} \ \text{ for some } \ t \in [\eta_{2(\ell-1)}, \eta_{2\ell})\}$, then

$$\{\omega \,:\, \tau_{\overline{V}^c} \in [\eta_{2(j-1)}, \eta_{2j})\} = \left[\bigcap_{\ell=1}^{j-1} E_\ell^c\right] \cap E_j. \tag{7.11}$$

Define $\alpha = \sup_{y\in\overline{U}} \mathbf{E}_y\eta_2$. We next claim that $\alpha < \infty$. Indeed, for any $y \in \overline{U}$, we have $\eta_0 = 0$. Moreover, by the strong Markov property,

$$\begin{aligned}
\mathbf{E}_y\eta_2 = \mathbf{E}_y\eta_1 + \mathbf{E}_y\mathbf{E}_y[\eta_2 - \eta_1|\mathcal{F}_{\eta_1}] &= \mathbf{E}_y\eta_1 + \mathbf{E}_y\mathbf{E}_{x_{\eta_1}}\tau_{\overline{U}^c}\\
&\leq \mathbf{E}_y\eta_1 + \sup_{z\in\partial\mathcal{X}_{k^*}} \mathbf{E}_z\tau_{\overline{U}^c}.
\end{aligned}$$

Applying Proposition 7.4 with $U = \mathcal{X}_{k^*}$, $\sup_{y\in\overline{U}} \eta_1 < \infty$. Also, by assumption $\mathbf{E}_z\tau_{\overline{U}^c} < \infty$ for all $z \in \mathcal{X}$. Applying Proposition 4.8(ii) (with U replaced by $\mathcal{X} \setminus \overline{U}$) and $x_t \in \mathcal{X}$ for each $t \geq 0$, yield that $z \mapsto \mathbf{E}_z\tau_{\overline{U}^c}$ is smooth on $\mathcal{X} \setminus \overline{U}$. In particular, $z \mapsto \mathbf{E}_z\tau_{\overline{U}^c}$ is bounded on the compact set $\partial\mathcal{X}_{k^*}$, and the claim follows.

In addition, for our fixed $x \in \mathcal{X}_{k_*}$, one has $\mathbf{E}_x\eta_0 < \infty$, and consequently by Proposition 7.4 and the arguments above

$$\beta := \mathbf{E}_x\eta_2 = \mathbf{E}_x\eta_0 + \mathbf{E}_x[\eta_1 - \eta_0] + \mathbf{E}_x\mathbf{E}_x[\eta_2 - \eta_1|\mathcal{F}_{\eta_1}] \tag{7.12}$$

$$\leq \mathbf{E}_x \eta_0 + \sup_{y\in\overline{U}} \mathbf{E}_y \tau_{\mathcal{X}_{k_*}} + \sup_{y\in\partial\mathcal{X}_{k_*}} \mathbf{E}_y \tau_{\overline{U}^c} < \infty. \tag{7.13}$$

We next claim that for $j \geq 2$ the following estimate holds:

$$\mathbf{E}_x \eta_{2j} \mathbf{1}_{[\eta_{2(j-1)},\eta_{2j})}(\tau_{\overline{V}^c}) \leq (1-q)^{j-2}\beta + (j-2)(1-q)^{j-2}\alpha + (1-q)^{j-1}\alpha. \tag{7.14}$$

We proceed by induction. If $j = 2$, then

$$\begin{aligned}\mathbf{E}_x \eta_4 \mathbf{1}_{[\eta_2,\eta_4)}(\tau_{\overline{V}^c}) &= \mathbf{E}_x \mathbf{E}_x[\eta_4 \mathbf{1}_{E_1^c\cap E_2}|\mathcal{F}_{\eta_2}] \\ &\leq \mathbf{E}_x \eta_2 + \mathbf{E}_x \mathbf{1}_{E_1^c} \mathbf{E}_{x_{\eta_2}} \eta_2 \leq \beta + (1-q)\alpha,\end{aligned}$$

which establishes the base case of the induction argument. Suppose (7.14) holds for some $j \geq 2$. Then,

$$\begin{aligned}\mathbf{E}_x \eta_{2(j+1)} \mathbf{1}_{[\eta_{2j},\eta_{2(j+1)})}(\tau_{\overline{V}^c}) &= \mathbf{E}_x[\mathbf{E}_x \eta_{2(j+1)} \mathbf{1}_{\bigcap_{\ell=1}^j E_\ell^c \cap E_{j+1}}|\mathcal{F}_{\eta_2}] \\ &= \mathbf{E}_x \eta_2 \mathbf{P}_x\left(\bigcap_{\ell=1}^j E_\ell^c \cap E_{j+1}|\mathcal{F}_{\eta_2}\right) \\ &\quad + \mathbf{E}_x[\mathbf{E}_x(\eta_{2(j+1)} - \eta_2)\mathbf{1}_{\bigcap_{\ell=1}^j E_\ell^c\cap E_{j+1}}|\mathcal{F}_{\eta_2}] \\ &\leq \beta(1-q)^{j-1} + (1-q)\sup_{y\in\overline{U}} \mathbf{E}_y \eta_{2j} \mathbf{1}_{\bigcap_{\ell=1}^{j-1} E_i^c\cap E_j} \\ &\leq \beta(1-q)^{j-1} + (j-1)(1-q)^{j-1}\alpha + (1-q)^j\alpha.\end{aligned}$$

This finishes the proof of (7.14). Then, (ii) follows from (7.10). □

The previous result gives rise to the following definition.

Definition 7.17. Suppose x_t is a nice diffusion and assume there exists a bounded, nonempty open set $U \subset \mathcal{X}$ with $\overline{U} \subset \mathcal{X}$ such that $\mathbf{E}_y \tau_{U^c} < \infty$ for all $y \in \mathcal{X}$. Then, we call x_t *positive recurrent*. Otherwise, if x_t is recurrent but not positive recurrent, we call x_t *null recurrent*.

Note that by Proposition 7.15, positive recurrence immediately implies recurrence. Next, we show that if x_t is positive recurrent, it has an invariant probability measure. The following result can be found in a number of references, see for example [37, 56]. We provide most of the details for completeness.

Proposition 7.18. *Suppose x_t is nice. If x_t is recurrent, then there exists a σ-finite invariant measure. If x_t is positive recurrent, then there exists an invariant probability measure.*

Proof. Fix open balls U, V with $\overline{U} \subseteq V \subseteq \overline{V} \subseteq \mathcal{X}$. Denote $\Gamma_1 = \partial V$ and $\Gamma_2 = \partial U$ and introduce stopping times σ_i, $i = 0, 1, \ldots$ defined by

$$\begin{aligned}\sigma_0 &= 0\,, \\ \sigma_1 &= \inf\{t \geq 0 \,:\, x_t \in \Gamma_1\}\,, \\ \sigma_2 &= \inf\{t \geq \sigma_1 \,:\, x_t \in \Gamma_2\}\,,\end{aligned}$$

$$\begin{aligned}\vdots \quad &\vdots \\ \sigma_{2k+1} &= \inf\{t \geq \sigma_{2k} \,:\, x_t \in \Gamma_1\}\,, \\ \sigma_{2k+2} &= \inf\{t \geq \sigma_{2k+1} \,:\, x_t \in \Gamma_2\}\,.\end{aligned}$$

Since x_t is a nice recurrent diffusion, each of the stopping times σ_i is almost surely finite. We can thus define a discrete-time Markov chain $\{X_n\}_{n\geq 0}$ on Γ_2 by $X_0 = x \in \Gamma_2$ and $X_n = x_{\sigma_{2n}}$, $n \geq 1$. Because $\{X_n\}$ has compact state space Γ_2, it possesses an invariant probability measure ν supported on Γ_2 by the Krylov-Bogolyubov Theorem.[1] Then, ν induces a measure μ on $\mathcal{B}$ by

$$\mu(B) = \int_{\Gamma_2} \nu(dx)\mathbf{E}_x \int_0^{\sigma_2} \mathbf{1}_B(x_s)\,ds =: \int_{\Gamma_2} \nu(dx)\mathbf{E}_x\sigma^B\,, \tag{7.15}$$

where σ^B is the total time spent by the process x_t in B during one 'cycle' $[0, \sigma_2]$. The calculations in [56, starting on p. 31] yield that μ is an invariant measure for x_t.

To prove that μ is σ-finite, we show that for any compact set $K \subset \mathcal{X}$

$$\sup_{x\in\Gamma_2} \mathbf{E}_x\sigma^K < \infty\,,$$

where

$$\sigma^K = \int_0^{\sigma_2} \mathbf{1}_K(x_s)\,ds = \int_0^{\sigma_1} \mathbf{1}_K(x_s)\,ds + \int_{\sigma_1}^{\sigma_2} \mathbf{1}_K(x_s)\,ds =: \sigma_1^K + \sigma_2^K\,. \tag{7.16}$$

Without loss of generality, by making K larger, we can assume $\overline{U} \cup \overline{V} \subset K$.

First observe that

$$\sup_{x\in\Gamma_2} \mathbf{E}_x\sigma^K \leq \sup_{x\in\overline{U}} \mathbf{E}_x\sigma_1 + \sup_{y\in\Gamma_1} \mathbf{E}_y \int_0^{\tau_{\overline{U}^c}} \mathbf{1}\{x_s \in K\}\,ds.$$

Using Proposition 7.4, $\sup_{x\in\overline{U}} \mathbf{E}_x\sigma_1 < \infty$. For the other term, note that since $\Gamma_1 \subset K$ we have

$$\sup_{y\in\Gamma_1} \mathbf{E}_y \int_0^{\tau_{\overline{U}^c}} \mathbf{1}\{x_s \in K\}\,ds \leq \sup_{y\in K} \mathbf{E}_y \int_0^{\tau_{\overline{U}^c}} \mathbf{1}\{x_s \in K\}\,ds.$$

Following the arguments in [37], set $\eta = \int_0^{\tau_{\overline{U}^c}} \mathbf{1}\{x_s \in K\}\,ds$ and let $A(t) = \{\eta \geq t\}$. By definition of η, note that $\eta \leq \tau_{\overline{U}^c}$. Hence

$$\mathbf{P}_y\{A(t)\} \leq \mathbf{P}_y\{\tau_{\overline{U}^c} \geq t\} \leq \mathbf{P}_y\{\tau_{\overline{U}^c} > t/2\}.$$

Thus using the fact that x_t is recurrent, applying Proposition 4.7(i) and compactness of K we deduce the existence of $\alpha < 1$ and large $t_0 > 0$ such that

$$\mathbf{P}_y\{A(t_0)\} \leq \mathbf{P}_y\{\tau_{\overline{U}^c} \geq t_0\} \leq \mathbf{P}_y\{\tau_{\overline{U}^c} > t_0/2\} \leq \alpha \tag{7.17}$$

[1] Here, if $Q(x, dy)$ denotes the one-step transition kernel of X_n, a probability measure is a invariant if $\mu Q = \mu$.

for all $y \in K$. Fix such t_0 and define

$$\eta_K(t) = \inf\{w \geq 0 \,:\, \textstyle\int_0^w \mathbf{1}_K(x_s)\, ds = t + t_0\}.$$

Then, by path continuity, $x_{\eta_K(t)} \in K$. Hence, by the strong Markov property and (7.17)

$$\begin{aligned}\mathbf{P}_y\{A(2(t+t_0))\} = \mathbf{E}_y \mathbf{1}\{\eta \geq 2(t+t_0)\} &= \mathbf{E}_y \mathbf{1}\{\eta \geq t+t_0\} \mathbf{E}_{x_{\eta_K(t)}} \mathbf{1}\{\eta \geq t+t_0\} \\ &\leq \alpha^2\end{aligned}$$

Repeating the process inductively yields $\mathbf{P}_y\{A(n(t+t_0))\} \leq \alpha^n$ for all $y \in K$. Finite expectation of η, hence sigma finiteness of μ, now follows.

To prove that μ is a finite measure if x_t is positive recurrent, note that σ_1 is almost surely finite and $\mu(\mathcal{X}) = \int_{\Gamma_2} \nu(dx)\mathbf{E}_x\sigma_2$. By the strong Markov property, for any $x \in \Gamma_2$ we have

$$\mathbf{E}_x\sigma_2 = \mathbf{E}_x\sigma_1 + \mathbf{E}_x(\sigma_2 - \sigma_1) = \mathbf{E}_x\sigma_1 + \mathbf{E}_x\mathbf{E}_{x_{\sigma_1}}\tau_{\overline{U}^c} \leq \sup_{x\in\Gamma_2} \mathbf{E}_x\sigma_1 + \sup_{y\in\Gamma_1} \mathbf{E}_y\tau_{\overline{U}^c}.$$

By Proposition 7.4, $\sup_{x\in\Gamma_2} \mathbf{E}_x\sigma_2 < \infty$. By positive recurrence $\mathbf{E}_y\tau_{\overline{U}^c} < \infty$ for each $y \in \mathcal{X}$, and by Proposition 4.8(i), we obtain $\sup_{y\in\Gamma_1} \mathbf{E}_y\tau_{\overline{U}^c} < \infty$, and the finiteness of μ follows. Thus, μ can be normalized to an invariant probability measure, and the proof is complete. □

Proposition 7.19. *Suppose x_t is a nice diffusion. If x_t has an invariant probability measure $\tilde{\mu}$, then μ is unique and defined (cf. (7.15)) by*

$$\tilde{\mu}(B) = \frac{1}{N}\int_{\Gamma_2} \nu(dx)\mathbf{E}_x\sigma^B \qquad \text{for any } B \in \mathcal{B}\,, \tag{7.18}$$

where $N = \int_{\Gamma_2} \nu(dx)\mathbf{E}_x\sigma^{\mathcal{X}} < \infty$, and ν, Γ_2 and σ^A are as in the proof of Proposition 7.18. In addition, $\tilde{\mu}(B) > 0$ for any nonempty open $B \subset \mathcal{X}$.

Proof. Suppose $\hat{\mu}$ is an invariant probability measure and fix a nonempty, open set $B \subset \mathcal{X}$. We may assume without loss of generality that B is bounded. We first claim that $\hat{\mu}(B) > 0$. Since any invariant probability measure is a solution of $L^*\hat{\mu} = 0$ on $\mathcal{X}$ in the sense of distributions, by hypoellipticity of L^* on $\mathcal{X}$, $\hat{\mu}$ has a continuous probability density $\hat{\rho}$ with respect to Lebesgue measure on $\mathcal{X}$. Since $\hat{\mu}(\mathcal{X}) = 1$, then $\hat{\rho} \geq c > 0$ on some bounded open set $W \subset \mathcal{X}$ and some constant $c > 0$. By Definition 7.3(iii) and path continuity, for any $x \in W$ there exists $t > 0$ with $\mathcal{P}_t(x, B) > 0$. By applying Remark 4.6 to Theorem 4.3, $x \mapsto \mathcal{P}_t(x, B)$ is continuous, and therefore $\mathcal{P}_t(x, B) \geq c' > 0$ for all x in an open subset W' of W. Then,

$$\hat{\mu}(B) = \int_{\mathcal{X}} \hat{\mu}(dx)\mathcal{P}_t(x, B) \geq \int_{W'} \hat{\mu}(dx)\mathcal{P}_t(x, B) \geq cc'|W'| > 0\,,$$

where $|W'|$ is the Lebesgue measure of W'. This finishes the proof of the claim.

If there is more than one invariant probability measure, then by standard arguments we can choose two distinct ergodic invariant measures, which are in particular mutually singular. But we showed by the claim above that $\mathcal{X}$ belongs to the support of any invariant probability measure, and therefore such measure is unique.

Since x_t has an invariant probability measure $\hat{\mu}$, Corollary 7.14 implies that x_t is recurrent and by Proposition 7.18, μ in (7.15) is σ-finite.

Since $\hat{\mu}$ is the unique invariant probability measure, it is ergodic and we proved that the support of $\hat{\mu}$ is $\mathcal{X}$. Then, by Birkhoff's Ergodic Theorem, for any compact set $K \subset \mathcal{X}$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_K(x_t)\, dt = \hat{\mu}(K) \tag{7.19}$$

for almost every $x \in \operatorname{supp}(\hat{\mu}) = \mathcal{X}$. Hence, by the invariance of μ, for any compact set $K \subset \mathcal{X}$ and $t \geq 0$, we have

$$\mu(K) = \int_{\mathcal{X}} \mathbf{E}_x \mathbf{1}_K(x_t)\mu(dx) = \lim_{T\to\infty} \int_{\mathcal{X}} \mu(dx)\mathbf{E}_x \frac{1}{T} \int_0^T \mathbf{1}_K(x_s)\, ds \geq \mu(\mathcal{X})\hat{\mu}(K)\,,$$

where the last inequality follows from Fatou's lemma and (7.19). Since there exists a compact $K \subseteq \mathcal{X}$ so that $\hat{\mu}(K) > 0$ and by σ-finiteness $\mu(K) < \infty$ we have shown that $\mu(\mathcal{X}) < \infty$.

After normalization, μ, and therefore $\tilde{\mu}$, is a well defined invariant probability measure and, by uniqueness, $\tilde{\mu} = \hat{\mu}$, as desired. □

Remark 7.20. Note that one can choose arbitrarily the sets U, V, $\Gamma_1 = \partial V$ and $\Gamma_2 = \partial U$ in the definition of μ in (7.15), as long as $\overline{U} \subseteq V \subseteq \overline{V} \subseteq \mathcal{X}$ and U, V are nonempty bounded, open with smooth boundaries Γ_1, Γ_2, respectively. By Proposition 7.19, different choices of U and V induce the same measure μ up to a normalization constant.

The final result in this section establishes that in our context, the existence of an invariant probability measure implies that x_t is positive recurrent.

Theorem 7.21. *Suppose x_t is a nice diffusion. Then, x_t has an invariant probability measure if and only if x_t is positive recurrent.*

Proof. If x_t is positive recurrent, then there is an invariant probability measure, which is moreover unique, according to Proposition 7.18 and Proposition 7.19.

Conversely, if x_t has an invariant probability measure $\tilde{\mu}$, then it is unique and given by (7.18). Fix an open, nonempty, bounded set $V \subset \mathcal{X}$ with $\overline{V} \subset \mathcal{X}$. It suffices to show that $\mathbf{E}_x \tau_{\overline{V}^c} < \infty$ for all x in a dense set $D \subset \mathcal{X} \setminus \overline{V}$, because then recurrence combined with Proposition 4.8(ii) implies $\mathbf{E}_x \tau_{\overline{V}^c} < \infty$ for all $x \in \mathcal{X} \setminus \overline{V}$ and the result follows.

Fix $x \in \mathcal{X} \setminus \overline{V}$ and $\epsilon > 0$ such that $B_\epsilon(x) \subset \mathcal{X} \setminus \overline{V}$. By hypothesis and (7.15), $\tilde{\mu}(\mathcal{X}) = 1 < \infty$, which implies $\mathbf{E}_y \sigma_2 < \infty$, for every $y \in \Gamma_2' \subset \Gamma_2$, where $\nu(\Gamma_2') = 1$. By Proposition 7.19 one has $\tilde{\mu}(B_{\epsilon/2}(x)) > 0$, and there exists $x_* \in \Gamma_2'$

for which $\mathbf{P}_{x_*}\{\tau_{B^c_{\epsilon/2}(x)} < \sigma_2\} > 0$. Denoting for simplicity $\eta = \tau_{B^c_{\epsilon/2}(x)}$, the strong Markov property implies

$$\begin{aligned}\infty > \mathbf{E}_{x_*}\sigma_2 &\geq \mathbf{E}_{x_*}\mathbf{1}\{\eta < \sigma_2\}\sigma_2 \\ &= \mathbf{E}_{x_*}\eta\mathbf{1}\{\eta < \sigma_2\} + \mathbf{E}_{x_*}\mathbf{1}\{\eta < \sigma_2\}\mathbf{E}_{x_*}[(\sigma_2 - \eta)|\mathcal{F}_\eta] \geq \mathbf{E}_{x_*}\mathbf{1}\{\eta < \sigma_2\}\mathbf{E}_{x_\eta}\tau_{\overline{V}^c}.\end{aligned}$$

Since $\mathbf{P}_{x_*}\{\eta < \sigma_2\} > 0$, there exists $y \in B_\epsilon(x)$ for which $\mathbf{E}_y\tau_{\overline{V}^c} < \infty$. This finishes the proof. □

Remark 7.22. We indicate a gap in the proof of [37, Lemma 3.7] and note some missing details in [37, Lemma 4.4].

The result [37, Lemma 3.7] is crucially used to establish the dichotomy for transient and recurrent points (see [37, Theorem 3.2]). In our notation, the context of the argument is as follows. It is assumed that for *fixed* x, $\int_0^\infty \mathcal{P}_t(x, V)\,dt < \infty$ for some open neighborhood V of x. It is then claimed that

$$\limsup_{y\to x}\int_0^\infty \mathcal{P}_t(y, V)\,dt \leq \int_0^\infty \limsup_{y\to x}\mathcal{P}_t(y, V)\,dt \leq \int_0^\infty \mathcal{P}_t(x, V)\,dt < \infty.$$

Thus it then follows that there exists an open neighborhood of x where the integral is finite. To the best of our knowledge, the limit-integral exchange was not justified and we could not find a simple solution. In essence, our fix of the arguments in [37] were presented this section.

In addition, there are missing details in the proof of [37, Lemma 4.4]. Indeed it is claimed that Bony's form of the Harnack inequality applies without the assumptions made in Bony's original paper [9, Theorem 7.1]. Although we have seen that this is indeed true as claimed (see Theorem 5.5), it requires some nontrivial arguments like those presented in this paper. Similarly, the claim in [37, Lemma 4.4] is effectively Proposition 4.8.

Acknowledgments

We acknowledge fruitful conversations on the topic of this paper with Nathan Glatt-Holtz.

References

[1] Azéma, J., Kaplan-Duflo, M. and Revuz, D. (1966). Récurrence fine des processus de Markov. *Ann. Inst. H. Poincaré Probab. Statist.* **2**, 185–220. MR0199889

[2] Bally, V. and Kohatsu-Higa, A. (2010). Lower bounds for densities of Asian type stochastic differential equations. *J. Func. Anal.* **258**, 3134–3164. MR2595738

[3] Baxendale, P. (2011). T.E. Harris's contributions to recurrent Markov processes and stochastic flows. *Ann. Probab.* **39** 417–428. MR2789501

[4] Bec, J., Celani, A., Cencini, M., and Musacchio, S. (2005). Clustering and collisions of heavy particles in random smooth flows. *Phys. Fluids* **17**, 11pp. MR2155067
[5] Bec, J., Cencini, M., and Hillerbrand, R. (2007). Clustering of heavy particles in random self-similar flow. *Phys. Rev. E* **75**.
[6] Bedrossian, J. and Liss, K. (2020). Quantitative spectral gaps and uniform lower bounds in the small noise limit for Markov semigroups generated by hypoelliptic stochastic differential equations, preprint arXiv:2007.13297 MR4408018
[7] Birrell, J., Herzog, D.P. and Wehr, J. (2012). The transition from ergodic to explosive behavior in a family of stochastic differential equations. *Stochastic Process. Appl.* **122**, 1519–1539. MR2914761
[8] Bonfiglioli, A., Lanconelli, E., and Uguzzoni, F. (2007). Stratified Lie Groups and Potential Theory for their Sub-Laplacians. Springer Science & Business Media. MR2363343
[9] Bony, J.-M. (1969). Principe du maximum, inégalité de Harnack et unicité du probleme de Cauchy pour les opérateurs elliptiques dégénérés. *Ann. Inst. Fourier* **19**, 277–304. MR0262881
[10] Camrud, E., Gordina, M., Herzog, D. P., and Stoltz, G. (2022). Weighted L^2-contractivity of Langevin dynamics with singular potentials. *Nonlinearity* **35**, 998–1035. MR4373993
[11] Carfagnini, M. and Gordina, M. (2022). Dirichlet sub-Laplacians on homogeneous Carnot groups: spectral properties, asymptotics, and heat content, preprint arXiv:2202.10410
[12] Capogna, L., Garofalo, N., and Nhieu, D.-M. (2008). *Mutual absolute continuity of harmonic and surface measures for Hörmander type operators. Perspectives in partial differential equations, harmonic analysis and applications*, 49–100, Proc. Sympos. Pure Math., **79**. MR2500489
[13] Cibelli, G., Polidoro, S. and Rossi, F. (2019). Sharp estimates for Geman–Yor processes and applications to arithmetic average Asian options. *J. Math. Pures Appl.* **129**, 87–130. MR3998791
[14] Coti Zelati, M. and Hairer, M. (2021). A noise-induced transition in the Lorenz system. *Comm. Math. Phys.* **383**, 2243–2274. MR4244269
[15] Cuneo, N., Eckmann, J.-P., Hairer, M., and Rey-Bellet, L. (2018). Non-equilibrium steady states for networks of oscillators. *Electron. J. Probab.* **23**, 1–28. MR3814249
[16] Carfagnini, M., Földes, J., and Herzog, D.P. (2022). A functional law of the iterated logarithm for weakly hypoelliptic diffusions at time zero. textitStochastic Process. Appl. **149**, 188–223. MR4410039
[17] Eberle, A., Guillin, A., and Zimmer, R. (2019). Couplings and quantitative contraction rates for Langevin dynamics. *Ann. Probab.* **47**, 1982–2010. MR3980913
[18] E, W. and Mattingly, J.C. (2001). Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation. *Comm. Pure Appl. Math.* **54**, 1386–1402. MR1846802
[19] Földes, J., Glatt-Holtz, N. E., and Herzog, D. P. (2021). Sensitivity

of steady states in a degenerately damped stochastic Lorenz system. *Stoch. Dyn.* **21**, 32 pp. MR4342756

[20] Folland, G.B. and Stein, E.M. (1974). Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.* **27**, 429–522. MR0367477

[21] Friedman, A. (1975). *Stochastic Differential Equations and Applications*, **1**, Academic Press. MR0494490

[22] Garofalo, N. and Segala, F. (1990). Estimates of the fundamental solution and Wiener's criterion for the heat equation on the Heisenberg group. *Indiana Univ. Math. J.* **39**, 1155–1196. MR1087188

[23] Gawędzki, K., Herzog, D.P. and Wehr, J. (2011). Ergodic properties of a model for turbulent dispersion of inertial particles. *Comm. Math. Phys.* **308**, 49–80. MR2842970

[24] Getoor, R.K. (1980). *Transience and recurrence of Markov processes. Séminaire de Probabilités XIV 1978/79*, 397–409, Springer. MR0580144

[25] Geller, D. (2014). *Analytic Pseudodifferential Operators for the Heisenberg Group and Local Solvability*, **37**, Princeton University Press. MR1030277

[26] Giacomin, G, Olla, S., Saada, E., Spohn, H. and Stoltz, G. (2019). *Stochastic Dynamics Out of Equilibrium*, Springer. MR3986060

[27] Gilbarg, D. and Trudinger, N. S. (2015). *Elliptic Partial Differential Equations of Second Order*, **224**, Springer. MR1421821

[28] Gromov, M. (1996). *Carnot-Carathéodory Spaces Seen from Within, Sub-Riemannian Geometry*, Springer, 79–323. MR1421823

[29] Hairer, M. and Mattingly, J. C. (2011) *Yet another look at Harris' ergodic theorem for Markov chains, Seminar on Stochastic Analysis, Random Fields and Applications VI*, Springer, 109–117. MR2857021

[30] Harris, T.E. (1956). *The existence of stationary measures for certain Markov processes, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954-1955, vol. II*, University of California Press, Berkeley-Los Angeles, Calif., 113–124. MR0084889

[31] Herzog, D. P. and Mattingly, J. C. (2019). Ergodicity and Lyapunov functions for Langevin dynamics with singular potentials. *Comm. Pure Appl. Math.* **72**, 2231–2255. MR3998640

[32] Herzog, D.P. and Nguyen, H. D. (2021). Stability and invariant measure asymptotics in a model for heavy particles in rough turbulent flows, *preprint arXiv:2104.08629*

[33] Hörmander, L. (1967). Hypoelliptic second order differential equations. *Acta Math.* **119**, 147–171. MR0222474

[34] Jerison, D.S. (1981). The Dirichlet problem for the Kohn Laplacian on the Heisenberg group, I. *J. Func. Anal.* **43**, 97–142. MR0639800

[35] Khasminskii, R. (1960). Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations. *Theory Probab. Appl.* **5**, 179–196. MR0133871

[36] Khasminskii, R. (2011). *Stochastic Stability of Differential Equations*, **66**, Springer Science & Business Media. MR2894052

[37] Kliemann, W. (1987). Recurrence and invariant measures for degenerate diffusions. *Ann. Probab.* **15**, 690–707. MR0885138
[38] Kogoj, A.E. and Lanconelli, E. (2004). An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations. *Mediterr. J. Math.* **1**, 51–80. MR2088032
[39] Kogoj, A.E. (2017). On the Dirichlet problem for hypoelliptic evolution equations: Perron–Wiener solution and a cone-type criterion. *J. Differential Equations* **262**, 1524–1539. MR3582202
[40] Kogoj, A.E. and Polidoro, S. (2016). Harnack inequality for hypoelliptic second order partial differential operators. *Potential Anal.* **45**, 545–555. MR3554403
[41] Kolmogorov, A.N. (1934). Zufällige Bewegungen. *Ann. of Math.* **35**, 116–117. MR1503147
[42] Kusuoka, S. and Stroock, D. (1985). Applications of the Malliavin calculus, Part II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math* **32**, 1–76. MR0783181
[43] Kusuoka, S. and Stroock, D. (1987). Applications of the Malliavin calculus, Part III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math* **34**, 391–442. MR0914027
[44] Lachal, A. (1997). Local asymptotic classes for the successive primitives of Brownian motion. *Ann. Probab.* **25**, 1712–1734. MR1487433
[45] Lanconelli, E., Tralli, G., and Uguzzoni, F. (2017). Wiener-type tests from a two-sided Gaussian bound. *Ann. Mat. Pura Appl.* **196**, 217–244. MR3600865
[46] Leimkuhler, B., Sachs, M., and Stoltz, G. (2020). Hypocoercivity properties of adaptive Langevin dynamics, *SIAM J. Appl. Math.* **80**, 1197–1222. MR4099815
[47] Ma, Z.M. and Röckner, M. (1992). *Introduction to the Theory of (Nonsymmetric) Dirichlet Forms*, Universitext, Springer-Verlag, Berlin. MR1214375
[48] Maruyama, G. and Tanaka, H. (1957). Some properties of one-dimensional diffusion processes. *Mem. Fac. Sci. Kyushu Univ. A 11*, 117–141. MR0097128
[49] Meyn, S. P. and Tweedie, R. L. (1993). Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25**, 518–548. MR1234295
[50] Meyn, S. P. and Tweedie, R. L. (2009). *Markov Chains and Stochastic Stability*, Second Edition, Springer Science & Business Media. MR2509253
[51] Negrini, P. and Scornazzani, V. (1987). Wiener criterion for a class of degenerate elliptic operators. *J. Differential Equations* **66**, 151–164. MR0871992
[52] Oksendal, B. (2013). *Stochastic Differential Equations: An Introduction with Applications*, Springer Science & Business Media. MR2001996
[53] Pigato, P. (2018). Tube estimates for diffusion processes under a weak Hörmander condition. *Ann. Inst. Henri Poincaré Probab. Stat.* **54**, 299–342. MR3765891
[54] Pigato, P. (2022). Density estimates and short-time asymptotics for

a hypoelliptic diffusion process. *Stochastic Process. Appl.* **145**, 117–142. MR4356685

[55] Ramaswamy, S. (1997). Dirichlet problem for some hypoelliptic operators. *Proc. Indian Acad. Sci. Math. Sci.* **107**, 405–409. MR1484375

[56] Rey-Bellet, L. (2006). *Ergodic properties of Markov processes*, Open quantum systems II, 1–39, Springer. MR2248986

[57] Romero, C. (1992). *Potential theory for the Kohn Laplacian on the Heisenberg group*, Ph.D. Thesis, ProQuest LLC, Ann Arbor, MI. MR2686874

[58] Romito, M. (2004). Ergodicity of the finite dimensional approximation of the 3D Navier–Stokes equations forced by a degenerate noise. *J. Stat. Phys.* **114**, 155–177. MR2032128

[59] Ruzhansky, M. and Suragan, D. (2017). Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups. *Adv. Math.* **308**, 483–528. MR3600064

[60] Schütt, C. and Werner, E. (2003). *Polytopes with vertices chosen randomly from the boundary of a convex body*, Geometric aspects of functional analysis, Springer, 241–422. MR2083401

[61] Leliévre, T., Stoltz, G., and Rousset, M. (2010). *Free Energy Computations: A Mathematical Perspective*, World Scientific. MR2681239

[62] Stroock, D. (2008). *Partial Differential Equations for Probabilists*, Cambridge Univ. Press. MR2410225

[63] Watanabe, H. and Motoo, M. (1958). Ergodic property of recurrent diffusion processes. *J. Math. Soc. Japan* **10**, 272–286. MR101570