

Advances in Mixed-Integer Model Predictive Control

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Abstract—Mixed-integer model predictive control (MPC) allows for discrete-valued control decisions, which are a common feature of industrial processes, to be optimized in combination with the continuous-valued control decisions typically considered in MPC. By choosing these decisions comprehensively and optimally, mixed-integer MPC can produce a significant improvement in process performance. In this work, we present a selection of the theoretical and computational advances in the field of mixed-integer MPC. First, we review the means by which theoretical results for standard MPC with continuous actuators are easily extendable to mixed-integer MPC. We then discuss additional theoretical results that address the robustness of mixed-integer MPC to both (i) small, persistent and (ii) large, infrequent disturbances. Next, we review the computational methods available to solve online these mixed-integer MPC problems, and discuss a few algorithms to approximate and solve the resulting mixed-integer nonlinear optimization problems by leveraging the structure of the MPC problem. We conclude with a tutorial example of mixed-integer MPC.

I. INTRODUCTION

Discrete-valued actuators, such as on/off decisions, are pervasive in large-scale industrial processes. Yet the early MPC literature focused entirely on continuous actuators, while leaving these discrete decisions to other automation methods such as heuristics [1], [2]. This preference in both industrial practice and MPC literature was driven primarily by the computational complexity incurred when including integer constraints in the optimization problem. However, advances in mixed-integer optimization, and computational power in general, allow these discrete decisions to be included directly in many MPC optimization problems.

By including discrete decisions directly in the optimization problem, mixed-integer MPC can produce significant improvements in the operation of these systems by treating “low-level” continuous actuators (e.g., valves) and “high-level” discrete decisions (e.g., on/off decisions) in a single framework. There are a range of potential applications of mixed-integer MPC including heating, ventilation, and air conditioning (HVAC) [3], [4], path planning [5], traffic control [6], and production scheduling [7]. In fact, mixed-integer MPC is currently *deployed* to optimize and control the Stanford campus heating and cooling systems [8] and has been applied to a solar thermal climate system at Karlsruhe University of Applied Sciences [9].

While systems with both integer and continuous variables are sometimes referred to as “hybrid” systems, we reserve the term hybrid to describe a specific class of systems that possess both discrete and continuous time *dynamics*,

as defined in [10]. These hybrid systems pose a number of control challenges and subsume the complications of adding integer constraints to the MPC problem [11]. We therefore restrict mixed-integer MPC to nonlinear discrete-time systems with mixed integer and continuous inputs. Note that this description does include piecewise affine systems [12], nonlinear switched systems [13], [14], and controllers that switch modes [15], provided that switches between dynamic models are determined directly by a controlled input to the system. In the following sections, we highlight a selection of the theoretical and computational advances in the field of mixed-integer MPC.

II. MIXED-INTEGER MPC

We consider discrete-time systems of the form

$$x^+ = f(x, u) \quad (1)$$

in which $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and input, and $x^+ \in \mathbb{R}^n$ denotes the subsequent state. For a prediction horizon of $N \in \mathbb{I}_{\geq 1}$, we use $\hat{\phi}(k; x, \mathbf{u})$ to denote the state solution of (1) at time $k \in \mathbb{I}_{0:N}$, given the initial condition x and the control trajectory $\mathbf{u} = (u(0), u(1), \dots, u(N-1))$.

One of the key strengths of MPC is the ability to enforce state and input constraints of the form $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Note that the set $\mathbb{U}$ is general enough to enforce integer constraints on a subset of the inputs. Thus, the formulation discussed here already includes mixed-integer MPC as a specific case.

We also use the terminal constraint $\mathbb{X}_f \subseteq \mathbb{X}$ to ensure nominal stability of the controller. Using these constraints, we define the following sets to characterize the admissible input trajectories and initial states for the MPC optimization problem.

$$\begin{aligned} \mathcal{U}_N(x) &:= \{\mathbf{u} : u(k) \in \mathbb{U}, \hat{\phi}(k; x, \mathbf{u}) \in \mathbb{X} \forall k \in \mathbb{I}_{0:N-1}, \\ &\quad \hat{\phi}(N; x, \mathbf{u}) \in \mathbb{X}_f\} \\ \mathcal{X}_N &:= \{x : \mathcal{U}_N(x) \neq \emptyset\} \end{aligned}$$

The stage cost is defined as $\ell : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and a corresponding terminal cost is defined as $V_f : \mathbb{R}^n \rightarrow \mathbb{R}$. The cost function is then

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(\hat{\phi}(k; x, \mathbf{u}), u(k)) + V_f(\hat{\phi}(N; x, \mathbf{u})) \quad (2)$$

The MPC optimization problem is defined as

$$\mathbb{P}_N(x) : V_N^0(x) = \min_{\mathbf{u} \in \mathcal{U}_N(x)} V_N(x, \mathbf{u}) \quad (3)$$

and the optimal solutions for a given initial state $x \in \mathcal{X}_N$ are denoted $\mathbf{u}^0(x) := \arg \min_{\mathbf{u} \in \mathcal{U}_N(x)} V_N(x, \mathbf{u})$. We define

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the control law as $\kappa_N(x) := u^0(0; x)$ in which $u^0(0; x)$ is the first input in $\mathbf{u}^0(x)$.¹ The resulting closed-loop system is

$$x^+ = f(x, \kappa_N(x)) \quad (4)$$

A. Nominal stability and performance

The first property to address for MPC is the nominal stability of a specified steady state, which is usually chosen to be the origin. There are various results on the stability of MPC with discrete decisions. Bemporad and Morari [16] establish convergence to the origin (but not Lyapunov stability) via certain restrictions on the stage cost. Picasso et al. [17] establish asymptotic stability for open-loop stable systems and boundedness for open-loop unstable systems. Baotic et al. [18] establish asymptotic stability for piecewise affine systems with a control law derived from an infinite-horizon optimization problem. By embedding a hybrid Lyapunov function within the optimal control problem, Di Cairano et al. [19] ensure asymptotic stability. More generally, [20] establish that much (if not all) of the theory already available for MPC is easily extendable to mixed-integer MPC.

To demonstrate the ease with which we can extend MPC theory to include discrete actuators, we consider the following standard assumptions for MPC [21, s. 2].

Assumption 1. The functions $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, and $V_f : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous and $f(0, 0) = 0$, $\ell(0, 0) = 0$, $V_f(0) = 0$. The functions $\ell(\cdot)$ and $V_f(\cdot)$ are lower bounded on $\mathbb{X} \times \mathbb{U}$ and $\mathbb{X}_f$, respectively.

Assumption 2. The set $\mathbb{U}$ is compact and contains the origin. The sets $\mathbb{X}$ and $\mathbb{X}_f$ are closed and contain the origin.

Assumption 3. There exists a terminal control law $\kappa_f : \mathbb{X}_f \rightarrow \mathbb{U}$ such that for all $x \in \mathbb{X}_f$, $f(x, \kappa_f(x)) \in \mathbb{X}_f$ and

$$V_f(f(x, \kappa_f(x))) \leq V_f(x) - \ell(x, \kappa_f(x))$$

Assumption 4. There exist $a, c_1, c_2 > 0$ such that

$$\begin{aligned} \ell(x, u) &\geq c_1 |x|^a & \forall (x, u) \in \mathbb{X} \times \mathbb{U} \\ V_f(x) &\leq c_2 |x|^a & \forall x \in \mathbb{X}_f \end{aligned}$$

Furthermore, $\mathbb{X}_f$ contains the origin in its interior.

Under these assumptions, we may establish the exponential stability of nominal MPC.

Theorem 5. *Under Assumptions 1-4, the origin of the closed-loop system (4) is exponentially stable in any compact subset of $\mathcal{X}_N$.*

In addition to the set-point (or reference trajectory) tracking problems that typically characterize process control, economic MPC allows the user to define the stage cost to directly represent an economic or performance metric for the system [22], [23], [24]. The key generalization made by economic MPC is that the stage cost may not be positive definite w.r.t. $x \in \mathbb{X}$ (Assumption 4 does not hold). If we

assume that the system is still strictly dissipative w.r.t. the stage cost and the steady state (a weaker condition that is difficult to verify), we can still establish exponential stability of the nominal system. However, for many emerging applications of economic MPC with discrete actuators, such as HVAC central plant optimization and production scheduling, economic performance is more important than exponential stability. In these cases, the following theoretical result is of more interest.

Theorem 6. *Under Assumptions 1-3, the closed-loop system (4) satisfies*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} \ell(x(k), u(k)) \leq 0$$

in which $x(k)$ and $u(k)$ are the closed-loop state and input trajectories for any initial condition $x(0) \in \mathcal{X}_N$.

Theorem 6 ensures that after an initial transient period the performance of the closed-loop system is no worse than that of the prescribed steady state, i.e., $(x, u) = (0, 0)$, used to construct the terminal cost and constraint in Assumption 3. Both Theorems 5 and 6 can be extended to time-varying or periodic systems, stage costs, and constraints [25].

We emphasize that none of these assumptions preclude enforcing integer-valued constraints on the inputs through the choice of $\mathbb{U}$. Assumption 2 does not require that the set $\mathbb{U}$ contain the origin in its interior, as is common in the early MPC literature [2]. Thus, $\mathbb{U}$ is already flexible enough to represent integer-valued constraints on the inputs. This fact was first articulated in [20] and led to the following notion.

Folk Theorem 1. *Any result that holds for standard MPC also holds for MPC with discrete actuators.*

In this case, “standard” MPC implies MPC without discrete-valued input decisions. Five years later, this observation remains valid, and there has been a significant expansion of the theoretical results and practical applications of mixed-integer MPC. This result, however, does not mean that nothing changes when discrete actuators are added to the MPC problem. By allowing input constraints to be active at the origin, the terminal ingredients (Assumption 3) must be carefully constructed (see [26] for such a construction). Furthermore, the feasible set $\mathcal{X}_N$ for mixed-integer MPC problems with many integer constraints may consist of many disconnected and nonconvex regions.

The extension from standard MPC to mixed-integer MPC is not effortless. Instead, the key observation in [20] is that the hurdles to including integer constraints in MPC problems are manageable with the current set of theoretical and computational tools available. Moreover, there have been multiple theoretical and computational advances in the field of mixed-integer MPC in the last decade. On the theoretical side, we focus on the results characterizing and establishing the robustness of mixed-integer MPC. We then discuss the computational challenges and methods proposed for solving these mixed-integer MPC problems in online applications.

¹If there are multiple solutions to $\mathbb{P}_N(x)$, we assume that some selection rule is applied to produce a single-valued control law. All subsequent results hold for any such selection rule.

III. INHERENT ROBUSTNESS

A. Small and persistent disturbances

In addition to nominal stability, a control method must ensure some margin of robustness to disturbances for successful implementation. Thus, we consider the plant dynamics

$$x^+ = f(x, \kappa_N(x)) + w \quad (5)$$

in which $w \in \mathbb{W} \subseteq \mathbb{R}^p$ is a disturbance affecting the system. We use $\phi(k; x, \mathbf{w}_k)$ to denote the closed-loop state trajectory generated by (5) at time $k \in \mathbb{I}_{\geq 0}$, given the initial condition x and disturbance sequence $\mathbf{w}_k = (w(0), w(1), \dots, w(k-1))$. We also define $\|\mathbf{w}_k\| := \max_{i \in \mathbb{I}_{0:k-1}} |w(i)|$. The function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$.

To discuss robustness, we first define robust positive invariance and robust exponential stability (RES).

Definition 7 (Robust positive invariance). A set $\mathcal{S}$ is robustly positive invariant for the closed-loop system (5) if $x \in \mathcal{S}$ implies $x^+ \in \mathcal{S}$ for all $w \in \mathbb{W}$.

Definition 8 (Robust exponential stability). The origin of the closed-loop system (5) is robustly exponentially stable (RES) in a robustly positive invariant set $\mathcal{S}$ if there exist $\rho > 0$, $\lambda \in (0, 1)$, and $\gamma(\cdot) \in \mathcal{K}$ such that

$$|\phi(k; x, \mathbf{w}_k)| \leq \lambda^k \rho |x| + \gamma(\|\mathbf{w}_k\|)$$

for all $x \in \mathcal{S}$, $k \in \mathbb{I}_{\geq 0}$, and $\mathbf{w}_k \in \mathbb{W}^k$.

While different researchers may use slightly modified versions of these definitions (e.g., an extension to robust asymptotic stability using $\mathcal{KL}$ functions), the general notion of robustness for MPC remains the same: For a compact subset of the feasible set ($\mathcal{S} := \mathcal{X}_N \cap \{x : V_N^0(x) \leq \tau\}$ and $\tau > 0$), there exists a nonzero margin of robustness ($\delta > 0$) such that $\mathcal{S}$ is robustly positive invariant, the optimization problem remains feasible (because $\mathcal{S} \subseteq \mathcal{X}_N$), and the origin is RES for all sufficiently small disturbances ($\mathbb{W} \subseteq \{w : |w| \leq \delta\}$). For the nominal MPC formulation defined in this work, we often refer to this property as *inherent* robustness, as we do not consider disturbances directly in problem formulation as done in stochastic or robust MPC formulations. Instead, nominal MPC relies on feedback to address these disturbances. This inherent robustness is often sufficient in industrial applications.

In closed-loop analysis of MPC, the optimal cost function $V_N^0(x)$ is (almost) always used as a Lyapunov function for the closed-loop system. If $V_N^0(x)$ is continuous, which is true for a linear dynamic model and convex constraints, inherent robustness follows immediately from the analysis used to establish Theorem 5. For nonlinear systems or, in particular, integer-valued constraints, the optimal cost function for MPC is not necessarily continuous and establishing the inherent robustness of MPC is a nontrivial endeavor.

In a significant contribution, Yu et al. [27] establish that nonlinear MPC is inherently robust to sufficiently small additive disturbances without the requirement that $V_N^0(x)$ is continuous. The authors require that hard state constraints

are omitted ($\mathbb{X} = \mathbb{R}^n$) to ensure recursive feasibility of the optimization problem. Instead, these state constraints can be included in the stage cost as exact penalties [28]. A terminal cost function and constraint are constructed via the linear-quadratic regulator (LQR) solution of the linearized system. The authors, however, require that the origin lies in the interior of $\mathbb{U}$ and thereby exclude integer constraints.

Leveraging some of these results, [29] establish that suboptimal MPC, i.e., an MPC algorithm that does not require optimal solutions to the proposed optimal control problem, is inherently robust without requiring $\mathbb{U}$ to have an interior. Thus, suboptimal and optimal MPC with integer decision variables are inherently robust. Given the computational burden of nonlinear and mixed-integer optimization, the ability to use potentially suboptimal solutions is particularly important for the online implementation of MPC.

B. Large and infrequent disturbances

By extending MPC theory to include problems with discrete decisions, “higher-level” scheduling or planning problems are now within the purview of MPC theory and application [7], [30], [31]. These additional applications of MPC, however, introduce their own unique complications. With discrete decisions and, in particular, scheduling problems, we must now consider the possibility of discrete disturbances such as delays or breakdowns in equipment. Typically, we cannot treat these discrete disturbances as sufficiently small and bounding the worst deterministic trajectory possible, e.g., all units are broken, leads to an excessively conservative and uninformative bound. Instead, we leverage the fact that these disturbances, although large, are also infrequent, e.g., a breakdown occurs with small probability. We refer to this class of disturbances as *large and infrequent*.

We define this class of disturbances in contrast to the small and persistent disturbances considered in the previous section. We denote the set of sufficiently small disturbances as $\mathbb{W}_0$ with $\sup_{w \in \mathbb{W}_0} |w| \leq \delta_0$ in which $\delta_0 > 0$ is the margin of robustness discussed in the previous section. Large (discrete) disturbances are then defined by the set $\mathbb{W}_1$ such that $\inf_{w \in \mathbb{W}_1} |w| > 0$, i.e., the disturbances are bounded away from zero. We assume that $\mathbb{W}_1$ and $\mathbb{W}_0$ are disjoint and define $\varepsilon := \Pr(\varepsilon \in \mathbb{W}_1)$. Note that $\mathbb{W}_1$ includes the discrete-valued disturbances that may not be included in $\mathbb{W}_0$. This class also includes large disturbances such as faults, communication failures, and large price/demand spikes in economic applications. In this work, we restrict our attention to systems in which only large disturbances or nominal behavior occur, i.e., $\mathbb{W}_0 = 0$ and $\mathbb{W} := \mathbb{W}_1 \cup \{0\}$. For this class of disturbances, we consider a different, stochastic notion of robustness.

Definition 9 (Robust exponential stability in expectation). The origin of the closed-loop system (5) with $\varepsilon := \Pr(w \in \mathbb{W}_1)$ is robustly exponentially stable in expectation (RESiE) in a robustly positive invariant set $\mathcal{S}$ for some $\delta > 0$ if there exist $\rho > 0$, $\lambda \in (0, 1)$, and $\gamma(\cdot) \in \mathcal{K}$ such that

$$\mathbb{E} [|\phi(k; x, \mathbf{w}_k)|] \leq \lambda^k \rho |x| + \gamma(\varepsilon) \quad (6)$$

for all $x \in \mathcal{S}$, $\varepsilon \in [0, \delta]$, and $k \in \mathbb{I}_{\geq 0}$.

As shown in [32], MPC is RESiE to this class of large disturbances, under specific assumptions, provided these disturbances are sufficiently infrequent, i.e., there exists $\delta > 0$ such that the origin is RESiE in $\mathcal{X}_N$ for all $\varepsilon \leq \delta$. These results can be extended to asymptotic stability and confidence (probabilistic) bounds for the state trajectory [33]. Furthermore, we can establish for economic MPC problems (without Assumption 4) that MPC is robust to these large and infrequent disturbances in an economic context: For sufficiently infrequent disturbances, $\varepsilon \leq \delta$, there exists $\hat{\gamma}(\cdot) \in \mathcal{K}$ such that the closed-loop system satisfies:

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} [\ell(x(k), u(k))] \leq \hat{\gamma}(\varepsilon) \quad (7)$$

for $x(k) = \phi(k; x, \mathbf{w}_k)$, $u(k) = \kappa_N(x(k))$, and all $x \in \mathcal{X}_N$.

In either case, these definitions of robustness bound a stochastic property of the closed-loop system (expected value) based on a stochastic property of the disturbance ($\varepsilon = \Pr(w \in \mathbb{W}_1)$). These bounds imply that the closed-loop system can, on average, recover from a large disturbance before another disturbance occurs provided the disturbances are sufficiently infrequent (unlikely). The economic form of robustness in (7) is particularly relevant to production scheduling problems [34].

IV. COMPUTATION

One of the key complications introduced by the addition of discrete decisions to the MPC problem, is the transformation of the original optimization problem with continuous decisions (linear, quadratic, or nonlinear) to its corresponding mixed-integer counterpart. The presence of these integer constraints renders the optimization problem nonconvex and requires solution methods that subsume typical linear, quadratic, or nonlinear solvers. As the field of mixed-integer optimization is large and not the primary focus of this paper, we do not attempt to review contributions to this field in detail. For more information, we suggest a few review articles on mixed-integer linear programming (MILP) [35], [36] and mixed-integer nonlinear programming (MINLP) [37], [38], [39]. Instead, we offer a general overview of these different mixed-integer optimization problems in the context of optimal control and focus on a few contributions that specifically address mixed-integer optimal control problems.

In general, these mixed-integer problems are solved by iterating between solving the continuous relaxation of the mixed-integer problem, i.e., the optimization problem without integer constraints, and a branch and bound combinatorial optimization problem. Thus, the properties and structure of this continuous relaxation are significant to the solution method and algorithmic efficiency. Mixed-integer linear and quadratic programs (MILPs/MIQPs) can often be efficiently solved, at significant scale, through available optimization packages such as Gurobi. Open-source packages, such as Cbc/Bonmin, are also available. Thus, mixed-integer MPC problems with linear system models ($f(x, u) = Ax + Bu$),

linear or quadratic stage and terminal costs (e.g., $\ell(x, u) = x'Qx + u'Ru$), and linear constraints aside from the integer constraints (e.g., $\mathbb{X} := \{x : Ax \leq b\}$), are readily solvable with available optimization methods. In fact, the optimal control package CasADi already includes functionality to enforce integer constraints and solves these problems with Bonmin [40]. As MIQPs are particularly important to MPC applications (linear systems with quadratic costs), there are also multiple methods to improve computational efficiency of the MIQP by exploiting the structure of the MPC problem and using warm starts for the optimizer [41], [42], [43], [44].

For nonlinear systems, however, the continuous relaxation of the mixed-integer MPC problem is a nonlinear and typically nonconvex optimization problem. Although significant improvements have been made in the field of mixed-integer nonlinear optimization, these global optimization solvers are often too slow for online implementation and therefore MPC. Nonetheless, mixed-integer MPC for nonlinear systems is still tractable. Suboptimal, but feasible, solutions to the MINLP that can be computed within the available sample time are sufficient in many applications of MPC.

One strategy to solve these MINLPs with more computationally attractive methods is through piecewise affine approximations. By approximating nonlinear system models, costs, and constraints via piecewise affine functions, we can reduce the original MINLP to a MILP with additional integer (binary) variables for these piecewise regions [45], [4]. The accuracy of this approximation improves with the number of regions considered in the piecewise approximation, but so does the computational complexity of the resulting MILP.

Another option is to approximately solve the MINLP by leveraging the structure of the optimal control problem. One such method is the combinatorial integral approximation (CIA) [46]. The motivation for this approximation is that the state of the system, if the binary inputs enter the dynamic system linearly, is influenced more by the integral (average) of the control inputs than their pointwise values.

For the CIA algorithm, the integer variables are assumed, without loss of generality, to be binary variables and we divide the input trajectory $\mathbf{u} = (\mathbf{u}_c, \mathbf{u}_b)$ into continuous ($\mathbf{u}_c$) and binary variables ($\mathbf{u}_b \in \mathbf{B}$). We use the specialized norm:

$$\|\mathbf{u}_b\|_{CIA} := \max_{i, k' \leq N} \left| \sum_{k=0}^{k'-1} u_{b,i}(k) \right|$$

The CIA algorithm is characterized by three steps:

- 1) Solve the relaxed MINLP, i.e., an NLP, to find a solution $\mathbf{u}_c^*$ and $\mathbf{u}_b^*$, possibly with $\mathbf{u}_b^* \notin \mathbf{B}$.
- 2) Solve for a binary trajectory $\mathbf{u}_b^{**} := \arg \min_{\mathbf{u}_b \in \mathbf{B}} \|\mathbf{u}_b - \mathbf{u}_b^*\|_{CIA}$, which is an MILP.
- 3) Solve the MINLP with a fixed binary trajectory $\mathbf{u}_b^{**}$, i.e., an NLP, to find the continuous inputs $\mathbf{u}_c^{**}$.

The trajectory of inputs $(\mathbf{u}_c^{**}, \mathbf{u}_b^{**})$ is feasible for the original MINLP, but typically not optimal. Further details are available in [47] and there is an open-source tool, pycombina, for solving the CIA problem [48].

Versions of both the piecewise approximation and CIA algorithm have been successfully deployed, in real time, on

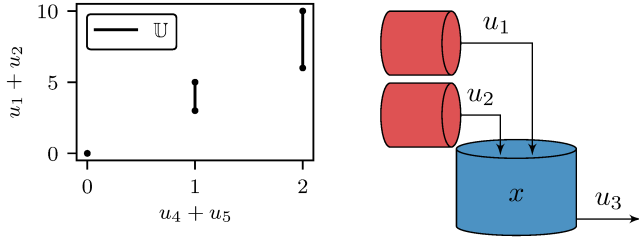


Fig. 1. Left: Input constraint set ($\mathbb{U}$) in the example. Note $\mathbb{U}$ is disconnected and does not have an interior. Right: Illustration of the example.

industrial size applications [8], [9]. While there is still room for improvement in MINLP solvers, the current capabilities of optimization solvers appear to be within the requirements for deploying mixed-integer MPC in industrial practice.

V. TUTORIAL EXAMPLE

We consider a small scheduling and inventory control example to demonstrate the potential applications of mixed-integer MPC and illustrate some of the robustness results discussed in this work. Two batch reactors deliver product to a single holding tank from which product may be withdrawn at the start of each hour. The goal of the controller is to maintain the level in the tank at 10 while withdrawing 8 every hour. The discrete-time dynamic model is

$$x^+ = x + u_1(1 + w_1) + u_2(1 + w_2) + u_3$$

in which $x \in [0, 20]$ is the tank level, u_1, u_2 are the volume of product produced by reactors 1 and 2, and $u_3 \in [0, 10]$ is the outlet flow/delivery rate from the tank. To enforce minimum capacity requirements on each reactor, we use the binary variables u_4, u_5 to denote if reactors 1 and 2, respectively, are “on.” The input constraints are shown in Figure 1. We consider the disturbances $w_1, w_2 \in \mathbb{R}$ that represent a proportional loss/gain of material production, i.e., if $w_1 = -0.1$ the reactor 1 produces only 90% of the intended production level. We also require $u_3 \leq x$ to ensure that we do not withdraw more material than is available at the start of each time step.

The steady state target is $x_{ss} = 10$, $u_{ss} = [4, 4, 8, 1, 1]'$ and we define the stage cost $\ell(x, u) = |x - x_{ss}|^2 + |u - u_{ss}|^2$. To construct the terminal cost, we fix u_1, u_2, u_4, u_5 at their steady state values and use the LQR solution with u_3 as the only available input. The LQR solution P is used as the terminal cost $V_f(x) := x'Px$ and the terminal constraint is defined as the level set $\mathbb{X}_f := \{x : x'Px \leq 1\}$. We choose $N = 6$ to ensure robust recursive feasibility. Note that the open-loop optimization problem is an MIQP. We use CasADi and MPCTools with the MINLP solver Bonmin to formulate and solve the open-loop optimization problem [40], [49].

We consider two options for the disturbance: (i) $w_1, w_2 \in [-0.1, 0.1]$ (sampled uniformly) is a small persistent disturbance that represents up to a 10% loss/gain in production or (ii) $w_1, w_2 \in \{0, -1\}$ is a large infrequent disturbance that represents a reactor breakdown for a given time step with $\Pr(w_1 = -1) = \Pr(w_2 = -1) = 0.1$. We simulate the

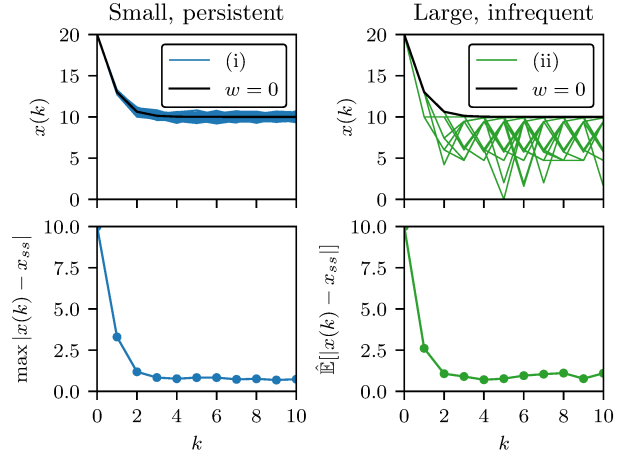


Fig. 2. Top left and right are closed-loop trajectories for 100 simulations of (i) small persistent and (ii) large infrequent disturbances. Bottom left is the maximum distance over all disturbance realizations of the closed-loop trajectory to the setpoint at each k . Bottom right is the sample average distance of the closed-loop trajectory to the setpoint at each k .

closed-loop trajectory for either small persistent or large infrequent disturbances for 100 realizations of the disturbance trajectory and plot the state trajectories in Figure 2. For small persistent disturbances, we plot the maximum distance of the closed-loop trajectory to the setpoint at each k in the bottom left of Figure 2. For large infrequent disturbances, we plot the sample average distance of the closed-loop trajectory to the setpoint at each k in the bottom right of Figure 2. These plots are consistent with RES and RESiE, respectively.

VI. CONCLUSIONS

Mixed-integer MPC is both theoretically sound and computationally viable. Potential applications of mixed-integer MPC range from traditional process control problems to higher-level scheduling and planning problems. Many of the theoretical results presented can be extended to time-varying systems [25], [33] and the improvements in optimization algorithms and computational power continue to expand the scope of problems that can be addressed with mixed-integer MPC. Potential directions for future research include: additional methods to approximate/solve mixed-integer nonlinear MPC problems, the robustness of economic mixed-integer MPC to small persistent disturbances, and novel applications of mixed-integer MPC.

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