

Inherent Stochastic Robustness of Model Predictive Control to Large and Infrequent Disturbances

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Abstract—We introduce a new class of large, infrequent disturbances to complement the small, persistent disturbances typically considered in robustness analysis. This new class of disturbances includes discrete disturbances that become pertinent when considering discrete actuators and production scheduling in control problems. To properly account for the infrequent nature of these disturbances, we define a stochastic form of robustness. Under suitable assumptions, we prove that certain closed-loop systems subject to large, infrequent disturbances admit an SISS-Lyapunov function and are robust in this stochastic context. We apply these results to economic model predictive control (MPC) with a strictly dissipative nominal system and stage cost, which includes tracking MPC as a special case, and prove that economic MPC is robust to large, infrequent disturbances. Without dissipativity assumptions, we define and establish robust asymptotic performance for economic MPC. We present a simple tracking problem to illustrate the results of this work, and a production scheduling (economic MPC) problem, to demonstrate the relevance of this analysis to practical applications.

Index Terms—Model predictive control, optimal control, robust stability, stochastic systems, time varying systems.

I. INTRODUCTION

IN THE absence of disturbances, suitable model predictive control (MPC) formulations ensure nominal stability [29, Ch. 2]. For practical implementation, however, MPC must also be robust to disturbances. Inherent robustness of nominal MPC is typically characterized by robust asymptotic stability of the closed-loop system and these robustness results assume that the size of the disturbance is bounded by some sufficiently small value [1], [13]. For many process control applications, this assumption is adequate to address the model mismatch, measurement noise, and small perturbations anticipated.

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While robust or stochastic MPC algorithms can be used to directly consider these disturbances, the inherent robustness of nominal MPC is often sufficient for implementation. Inherent robustness refers to the robustness of MPC obtained solely from feedback and without any constraint tightening, worst case analysis, or stochastic optimization techniques. Grimm *et al.* [13] demonstrate that for linear systems with convex constraints, MPC is inherently robust. Nonlinear MPC is also inherently robust if the optimal value function satisfies certain continuity assumptions [27]. These results extend to systems with compact input constraints, discrete-valued inputs, and discontinuous optimal value functions if state constraints, aside from the terminal constraint, are removed [1], [39]. More general conditions on inherent robustness, input-to-state stability (ISS), and their relation to Lyapunov functions are given in Grimm *et al.* [14] and Lazar *et al.* [22].

Recently, theoretical results for MPC have been extended to discrete actuators [28] and state-space models have been developed to formulate production scheduling as an economic MPC problem [15], [31], [32]. With discrete actuators and scheduling problems, we must now consider discrete disturbances such as task delays or breakdowns in equipment. While the transition from continuous to discrete disturbances does not directly affect the analysis performed in previous work, the argument that a discrete disturbance can be made “sufficiently small” is not applicable. For example, breakdowns considered in the scheduling of a production facility are not, and should not be considered, small disturbances. Furthermore, if we treat these disturbances with the current theory, we must bound the worst deterministic performance possible, e.g., the entire facility is broken or delayed at every sample time. In practice, however, these “large” disturbances are also infrequent, e.g., a production line breaks or is delayed with some probability not equal to one. Thus, a deterministic bound, if admitted by the underlying system, is far too conservative and offers little insight for a system subject to these types of disturbances.

We refer to this class of disturbances as *large*, because the disturbances are bounded away from zero and cannot be considered “sufficiently small,” and *infrequent*, because the probability that these disturbances occur is small. This description applies to a variety of disturbances such as faults, missing measurements, communication failures, breakdowns, large delays, and large price/demand spikes in economic applications. To understand the system’s robustness to this class of disturbances,

we exploit the infrequent nature of these disturbances and propose a stochastic form of robustness for large and infrequent disturbances.

Many forms of stochastic stability and robustness are already present in control theory. Originating in the 1960's [21], the notion of stochastic stability for nonlinear systems, i.e., global asymptotic stability in probability (GASiP), was refined more recently in Florchinger [10]. Subsequently, this analysis was used to design feedback controllers for stochastic systems [5], [6], [26]. Teel and co-workers constructed and established stronger definitions of global asymptotic stability, rigorously demonstrated that stochastic Lyapunov functions provide uniform convergence, and developed converse Lyapunov theorems for this stochastic definition of stability [34]–[36]. Analogous to ISS for deterministic systems, stochastic input-to-state stability (SISS) was also defined [19], [24], [33], [37]. Over the past decade, this SISS framework has been used in the analysis and control of continuous-time [16], [17], [38], [41] and discrete-time [8] nonlinear stochastic systems. These works assume that the impact of the stochastic disturbance vanishes once the state of the system reaches the origin. However, many applications of interest violate this assumption. For the current work, the SISS framework is significantly modified to address large, infrequent disturbances that do not vanish once the state reaches the origin.

We summarize the subsequent sections as follows. In Section II, we introduce MPC and the closed-loop stochastic system. In Section III, we characterize the class of large and infrequent disturbances addressed in this article and limit the maximum size of these large disturbances through a few assumptions. In Section IV, we present a motivating example and define robust asymptotic stability in probability (RASiP). We then define an SISS-Lyapunov function and establish that any closed-loop system that admits an SISS-Lyapunov function is RASiP. Furthermore, we establish that the assumptions presented in Sections II and III are sufficient to guarantee that the closed-loop stochastic system admits an SISS-Lyapunov function and is therefore RASiP. In Section V, we focus on economic MPC. We define and establish robust asymptotic performance in expectation for this closed-loop system subject to large, infrequent disturbances by assuming a specific cost evolution bound for the perturbed system. With additional nominal dissipativity and weak controllability assumptions, we prove that economic MPC is RASiP. In Section VI, we demonstrate the results and implications of this analysis with a production scheduling, i.e., economic MPC, example.

Notation: Let $\mathbb{I}$ denote integers, $\mathbb{R}$ denote reals, and subscripts on these sets denote restrictions (e.g., $\mathbb{I}_{\geq 0}$ for nonnegative integers). The set $\mathbb{T} \subseteq \mathbb{I}_{\geq 0}$ denotes discrete time points. The function $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is in class $\mathcal{PD}$ if it is continuous, $\rho(s) > 0$ for all $s > 0$, and $\rho(0) = 0$. The function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. The function $\alpha(\cdot)$ is in class $\mathcal{K}_{\infty}$ if it is in class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is in class $\mathcal{KL}$ if for fixed k the function $\beta(\cdot, k)$ is in class $\mathcal{K}$ and for fixed s the function $\beta(s, \cdot)$ is nonincreasing and $\lim_{k \rightarrow \infty} \beta(s, k) = 0$. Let $\|\cdot\|$ denote vector norm if applied to a vector and absolute value if applied to a scalar. We denote distance to a set $\mathcal{A}$ as $|x|_{\mathcal{A}} = \inf_{a \in \mathcal{A}} |x - a|$. Let $\oplus$ denote set addition. Sequences are denoted in bold face

and subscripts indicate the range (e.g., $\mathbf{w}_k$ indicates the sequence of w 's from $w(t)$ to $w(k-1)$). Let $I_S(x)$ denote the indicator function for a set S , i.e., $I_S(x) = 1$ if $x \in S$ and $I_S(x) = 0$ otherwise. Let $\Pr(A)$ denote the probability of event A .

II. PROBLEM FORMULATION AND PRELIMINARIES

We consider a discrete, time-varying system of the form

$$x^+ = f(x, u, w, t) \quad (1)$$

defined for the continuous function $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \times \mathbb{T} \rightarrow \mathbb{X}$, state $x \in \mathbb{X} \subseteq \mathbb{R}^n$, input $u \in \mathbb{U} \subseteq \mathbb{R}^m$, and disturbance $w \in \mathbb{W} \subseteq \mathbb{R}^p$, at the discrete time index $t \in \mathbb{T}$. The successor state at $t+1$ is denoted by x^+ . The system is subject to time-varying constraints $(\mathbb{Z}(t))_{t \in \mathbb{T}}$, such that at time $t \in \mathbb{T}$, $(x, u) \in \mathbb{Z}(t) \subseteq \mathbb{X} \times \mathbb{U}$.

We consider a nominal MPC problem with a horizon $N \in \mathbb{I}_{\geq 1}$, stage cost $\ell(\cdot, t) : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$, terminal constraints $\mathbb{X}_f(t) \subseteq \mathbb{X}$, and terminal cost $V_f(\cdot, t) : \mathbb{X}_f(t) \rightarrow \mathbb{R}$ for all $t \in \mathbb{T}$. The nominal system is described by

$$x^+ = f(x, u, 0, t). \quad (2)$$

For the current state $x \in \mathbb{X}$ and input sequence $\mathbf{u} := (u(t), u(t+1), \dots, u(t+N-1)) \in \mathbb{U}^N$ at time t , the function $\hat{\phi}(k; x, \mathbf{u}, t) \in \mathbb{X}$ denotes the open-loop state solution to the nominal system (2) at time $k \in \mathbb{I}_{[t, t+N]}$. We define the set of admissible inputs (4), admissible states (5), and objective function (6) by

$$\begin{aligned} \mathbb{Z}_N(t) &:= \{(x, \mathbf{u}) \mid (x(k), u(k)) \in \mathbb{Z}(k) \\ &\quad \forall k \in \mathbb{I}_{[t, t+N-1]} \\ &\quad x(t+N) \in \mathbb{X}_f(t+N)\} \end{aligned} \quad (3)$$

$$\mathcal{U}_N(x, t) := \{\mathbf{u} \mid (x, \mathbf{u}) \in \mathbb{Z}_N(t)\} \quad (4)$$

$$\mathcal{X}_N(t) := \{x \in \mathbb{X}(t) \mid \mathcal{U}_N(x, t) \neq \emptyset\} \quad (5)$$

$$\begin{aligned} V_N(x, \mathbf{u}, t) &:= \sum_{k=t}^{t+N-1} \ell(x(k), u(k), k) \\ &\quad + V_f(x(t+N), t+N) \end{aligned} \quad (6)$$

in which $x(k) := \hat{\phi}(k; x, \mathbf{u}, t)$.

The optimal control problem for $x \in \mathcal{X}_N(t)$ at time t is defined as

$$V_N^0(x, t) := \min_{\mathbf{u} \in \mathcal{U}_N(x, t)} V_N(x, \mathbf{u}, t) \quad (7)$$

and the optimal input trajectory is defined as $\mathbf{u}^0(x, t)$. The MPC control law $\kappa_N(x, t) := u^0(t; x, t)$ is defined as the first input in $\mathbf{u}^0(x, t)$.¹ For the controlled system, the state evolves according to

$$x^+ = f_c(x, w, t) := f(x, \kappa_N(x, t), w, t). \quad (8)$$

Note that, even if $f(\cdot)$ is continuous, $f_c(\cdot)$ may be discontinuous in x since $\kappa_N(\cdot)$ may be discontinuous. We define the

¹If there are multiple solutions to the optimization problem, we assume that some selection rule is applied, such that $\kappa_N(\cdot)$ is a single-valued mapping.

solution to (8) at time $k \geq t$ given the initial condition x at time t and the disturbance sequence $\mathbf{w}_k := (w_t, \dots, w_{k-1})$ as $\phi(k; x, \mathbf{w}_k, t) \in \mathbb{X}$.

We now introduce a probabilistic description of the disturbance w . We assume that the random variables w_t for all $t \in \mathbb{T}$ are independent and identically distributed (i.i.d.) with the probability measure $\mu : \mathcal{B}(\mathbb{W}) \rightarrow [0, 1]$ in which $\mathcal{B}(\mathbb{W})$ denotes the Borel field of the set $\mathbb{W}$. For the sequence of random variables $\mathbf{w}_k$ and a measurable function $g : \mathbb{W}^{k-t} \rightarrow \mathbb{R}$, we define probability that $g(\mathbf{w}_k) \in S$ for a measurable set S with the following Lebesgue integral.

$$\begin{aligned} \Pr(g(\mathbf{w}_k) \in S) \\ := \int_{\mathbb{W}^{k-t}} I_S(g((\omega_t, \dots, \omega_{k-1}))) d\mu(\omega_t) \cdots d\mu(\omega_{k-1}) \end{aligned}$$

in which $t \in \mathbb{T}$ and $k \in \mathbb{I}_{\geq t}$. For a sequence of random variables $\mathbf{w}_k$ and a measurable function $g : \mathbb{W}^{k-t} \rightarrow \mathbb{R}$, we define expected value with the following Lebesgue integral

$$\mathbb{E}[g(\mathbf{w}_k)] := \int_{\mathbb{W}^{k-t}} g((\omega_t, \dots, \omega_{k-1})) d\mu(\omega_t) \cdots d\mu(\omega_{k-1})$$

in which $t \in \mathbb{T}$ and $k \in \mathbb{I}_{\geq t}$. Furthermore, for a measurable function $g : \mathbb{X} \rightarrow \mathbb{R}$ and $x(k) := \phi(k; x, \mathbf{w}_k, t)$ we define conditional expected value as

$$\begin{aligned} \mathbb{E}_{|x(k-1)}[g(x(k))] &= \mathbb{E}_{|x}[g(f_c(x, w, k-1))] \\ &= \int_{\mathbb{W}} g(f_c(x, \omega, k-1)) d\mu(\omega). \end{aligned}$$

Remark 1: To ensure that stochastic properties of interest (e.g., probability and expected value) are well defined for the closed-loop stochastic system, we require that $\phi(\cdot)$ is a measurable function. For discontinuous control laws, and therefore discontinuous $f_c(\cdot)$, measurability is not guaranteed. Fortunately, optimization-based control laws, under suitable regularity conditions, are Borel measurable functions and thereby guarantee measurability of $\phi(\cdot)$ [12, Proposition 4, Remark 3]. Thus, we implicitly assume for the rest of this article that all stochastic properties are indeed well defined.

We define positive invariance and robust positive invariance for the time-varying system.

Definition 1 (Positive invariance): The sequence $(\mathcal{X}(t))_{t \in \mathbb{T}}$ is positive invariant for the nominal system $x^+ = f_c(x, 0, t)$, if $x(t) \in \mathcal{X}(t)$ implies $x^+ \in \mathcal{X}(t+1)$ for all $t \in \mathbb{T}$.

Definition 2 (Robust positive invariance): The sequence $(\mathcal{X}(t))_{t \in \mathbb{T}}$ is robustly positive invariant for the perturbed system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$, if $x(t) \in \mathcal{X}(t)$ implies $x^+ \in \mathcal{X}(t+1)$ for all $w \in \mathbb{W}$ and $t \in \mathbb{T}$.

We consider the following assumption for the nominal system.

Assumption 1 (Nominal system properties): The function $f_c(\cdot)$ is locally bounded.² The sequence $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ is positive invariant for $x^+ = f_c(x, 0, t)$. There exists a Lyapunov function

²A function $f : X \rightarrow Y$ is locally bounded if, for every $x_0 \in X$, there exists a neighborhood $\mathcal{N}$ of x_0 , such that $f(\mathcal{N})$ is bounded, i.e., for some $M > 0$, we have $|f(x)| \leq M$ for all $x \in \mathcal{N}$.

$V(\cdot, t) : \mathcal{X}_N(t) \rightarrow \mathbb{R}_{\geq 0}$ and functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_{\infty}$ that satisfy

$$\alpha_1(|x|) \leq V(x, t) \leq \alpha_2(|x|) \quad (9)$$

$$V(f_c(x, 0, t), t+1) \leq V(x, t) - \alpha_3(|x|) \quad (10)$$

for all $x \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$, i.e., the origin is asymptotically stable on the sets $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ for the nominal system.

Assumption 1 is satisfied for tracking MPC and economic MPC with suitable dissipativity assumptions [7], [29, Ch. 2.8]. In fact, with properly constructed terminal conditions, the optimal value of the MPC problem satisfies Assumption 1 with $V(x, t) := V_N^0(x, t)$. The assumption of nominal asymptotic stability is minimal and achieved by any acceptable control algorithm.

III. LARGE AND INFREQUENT DISTURBANCES

A. Characterization

In this section, we characterize the class of large and infrequent disturbances addressed in the subsequent analysis. We introduce this class of disturbances by discussing them in contrast to the class of small and persistent disturbances addressed by the deterministic definition of inherent robustness, i.e., robust asymptotic stability.

Definition 3 (Robust asymptotic stability): Suppose that the sequence $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ is robustly positive invariant for the system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ in which $\varepsilon_0 := \sup_{k \geq t} |w(k)|$. The origin of the system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ is robustly asymptotically stable if there exists $\delta_0 > 0$, $\beta(\cdot) \in \mathcal{KL}$, and $\gamma(\cdot) \in \mathcal{K}$, such that

$$|\phi(k; x, \mathbf{w}_k, t)| \leq \beta(|x|, k-t) + \gamma(\varepsilon_0) \quad (11)$$

for all $\varepsilon_0 \leq \delta_0$, $x \in \mathcal{X}_N(t)$, $t \in \mathbb{T}$, and $k \in \mathbb{I}_{\geq t}$.

Note that ε_0 is defined as the maximum size (Euclidean norm) of all disturbances in the trajectory. Thus, robust asymptotic stability guarantees that *there exists* a $\delta_0 > 0$, such that for disturbances of sufficiently small size ($\varepsilon_0 \leq \delta_0$), the closed-loop system is ISS, i.e., the bound in (11) holds. Nominal MPC is robust in this deterministic context for sufficiently small disturbances [1].

We characterize the set of disturbances addressed by this definition as $\mathbb{W}_0$ with $|\mathbb{W}_0| := \sup_{w \in \mathbb{W}_0} |w| \leq \delta_0$ in which we have chosen the scalar δ_0 from the deterministic definition of inherent robustness. Therefore, (11) holds for any realization of $\mathbf{w}_k$, such that $w_k \in \mathbb{W}_0$ for all $k \in \mathbb{I}_{\geq t}$. This deterministic definition is a strong form of robustness in that *any* probability distribution on the set $\mathbb{W}_0$ is permitted.

In this article, we are interested in the following question. If the disturbance w is *not small*, i.e., $w \notin \mathbb{W}_0$, what happens to the robustness of the nominal MPC controller? To address this question, we introduce a set of *large* disturbances and assign a probability to this set. Let $\mathbb{W}_1$ be, such that $\inf_{w \in \mathbb{W}_1} |w|$ is strictly great than zero, i.e., we have bounded the large disturbances “away from zero.” We denote the probability that the disturbance takes a value in this set as $\varepsilon := \Pr(w \in \mathbb{W}_1)$. Furthermore, we can define $\mathbb{W}_1$, such that the intersection of $\mathbb{W}_0$ and $\mathbb{W}_1$ is empty.

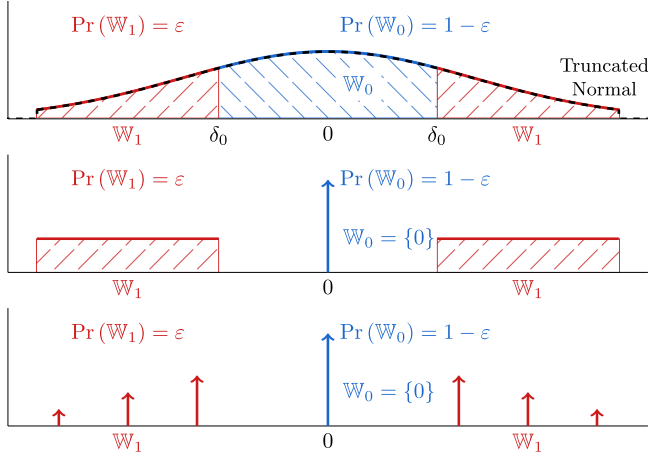


Fig. 1. Three probability distributions for the disturbance w depicting the small, frequent disturbances ($\mathbb{W}_0$) and large, infrequent disturbances ($\mathbb{W}_1$).

The disturbances in $\mathbb{W}_0$ are small. The disturbances in $\mathbb{W}_1$ are large. In particular, we note that $\mathbb{W}_1$ includes discrete-valued disturbances that may not be included in $\mathbb{W}_0$.

For example, consider the truncated normal distribution in the top plot of Fig. 1. Although the probability distribution of w admits large disturbances that exceed δ_0 , these disturbances are infrequent in that $\Pr(w \in \mathbb{W}_1) = \varepsilon$ is small. This description also applies to many other probability distributions as shown in the middle and bottom plots of Fig. 1.

We restrict the maximum size of $\mathbb{W}_1$ through assumptions introduced later in this section, e.g., the disturbance cannot be so large as to render the MPC optimization problem infeasible. However, neither the nominal MPC controller nor sometimes any controller will have deterministic robustness to the large disturbances included in $\mathbb{W}_1$. But the question remains: Is there *any* form of inherent robustness for the nominal MPC controller outside of what is already established for the set $\mathbb{W}_0$?

We show that the answer to this question is yes, provided that the large disturbances are also *infrequent*. This assumption is reasonable as it models many kinds of large disturbances such as faults, missing measurements, communication failures, breakdowns, large delays, and large price/demand spikes in economic applications. If these kinds of large faults are sufficiently frequent in nature, indeed no controller is expected to be robust in any sense against them. To model the infrequent nature of these disturbances we introduce the small parameter $\delta > 0$ and require that $\varepsilon = \Pr(w \in \mathbb{W}_1) \leq \delta$. So these disturbances are large in magnitude but infrequent in terms of sampling them as random variables. Furthermore, if $\mathbb{W}_0 \cap \mathbb{W}_1 = \emptyset$, we also have $\Pr(w \in \mathbb{W}_0) = 1 - \varepsilon$, and we recover the usual small disturbance case in the limit $\varepsilon \rightarrow 0$.

Note that we have assumed that the random variables are i.i.d. and therefore the value of ε is not time-varying. We can, however, extend the subsequent results to time-varying probability distributions by introducing a time-varying ε , i.e., $\varepsilon(k)$.

For clarity in this introductory article, we shall restrict attention to the case $\mathbb{W}_0 = \{0\}$, e.g., the middle and bottom plots of Fig. 1, so that we have only two possibilities: The

nominal behavior occurs with probability $1 - \varepsilon$ and the large disturbance occurs with probability ε . We then analyze what kind of robustness, if any, can we expect from the nominal MPC controller in this situation.

B. Assumptions

As noted in the previous subsection, we consider the case of only large disturbances and nominal behavior.

Assumption 2 (Only large disturbances): The disturbance set satisfies $\mathbb{W} = \mathbb{W}_0 \cup \mathbb{W}_1$ and we restrict $\mathbb{W}_0 = \{0\}$.

Although we classify these disturbances as large, we do not allow disturbances of *arbitrary* size. We restrict the size of $\mathbb{W}_1$ through the following assumptions. We begin by requiring that the MPC problem remains well defined for the closed-loop trajectory.

Assumption 3 (Recursive feasibility): The sequence $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ is robustly positive invariant for the system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$, i.e., the optimal control problem is recursively feasible.

Note that assuming recursive feasibility for MPC is sometimes inappropriate. MPC requires solving an optimization problem in real time, potentially subject to state constraints, and therefore a feasible solution is not always guaranteed. In Allan *et al.* [1], recursive feasibility is proven for the structure of the MPC formulation and by constraining the disturbance size to be sufficiently small. For large disturbances, we lose this capability and must be careful about enforcing constraints in the optimization problem. For a problem with state and input constraints, there exists a sufficiently large disturbance that creates an infeasible optimization problem. In general, if we want to consider large disturbances, the control algorithm must be recursively feasible by design.

For production scheduling applications of MPC, Assumption 3 is often reasonable. The input and state constraints only exist to enforce realistic decisions, e.g., no negative inventory. Thus, disturbances cannot force violations of these constraints. Furthermore, sufficiently long horizons and reasonable terminal conditions in scheduling problems ensure that any current state of the plant can be driven to the terminal set in N moves.

MPC implementations without state or terminal constraints, i.e., $\mathbb{X}(t) = \mathbb{X}_f(t) = \mathbb{R}^n$, easily satisfy Assumption 3. These MPC formulations have seen significant theoretical progress in recent years and, with suitable dissipativity assumptions and terminal costs, can ensure nominal (practical) asymptotic stability [11], [23], [40]. Therefore, there already exists a significant class of MPC implementations that satisfy both Assumptions 1 and 3.

In addition to feasibility, we also require a bound on the Lyapunov function increase due to a disturbance.

Assumption 4 (Maximum Lyapunov function increase): For the perturbed system $x^+ = f_c(x, w, t)$, there exist finite $b_1, b_2 \in \mathbb{R}_{\geq 0}$, such that the functions $V(\cdot)$ and $\alpha_3(\cdot)$ in Assumption 1 satisfy

$$V(f_c(x, w, t), t + 1) \leq V(x, t) + b_1 \alpha_3(|x|) + b_2 \quad (12)$$

for all $x \in \mathcal{X}_N(t)$, $w \in \mathbb{W}_1$, and $t \in \mathbb{T}$.

Note that this bound is significantly weaker than the typical ISS-Lyapunov function bound for a disturbance. The Lyapunov function value may increase based on the size of $|x|$ and some finite constants. In fact, the increase between t and $t + 1$ may even *grow* as $|x|$ increases for an equivalent disturbance. This growth, however, is limited by the size of the nominal Lyapunov function decrease, i.e., $\alpha_3(|x|)$. As we show in subsequent examples, there exists a class of systems and disturbances that satisfy Assumptions 1–4 and are *not* robust in the usual deterministic context.

Although Assumption 4 is difficult to verify for a time-varying Lyapunov function, this assumption provides an important limit on the class of disturbances admitted by this article. For example, Assumption 4 excludes systems such as $x^+ = x \tanh(x) + w$ in which the rate of nominal convergence decreases as the state of the system grows. The disturbance, however, influences the state at the same rate (additive) as the state grows.

If the composite set $\mathcal{X}_N := \bigcup_{t \in \mathbb{T}} \mathcal{X}_N(t)$ is bounded, Assumption 4 is satisfied for $b_1 = 0$ and some large, finite $b_2 > 0$. However, a bounded $\mathcal{X}_N$ that satisfies Assumption 3 may not exist for certain systems and disturbances. We note an important case in which Assumption 4 is satisfied without assuming that $\mathcal{X}_N$ is bounded.

Lemma 1: Let Assumptions 1 and 3 hold with $a, c_1, c_2, c_3 > 0$, such that $\alpha_1(s) = c_1 s^a$, $\alpha_2(s) = c_2 s^a$, and $\alpha_3(s) = c_3 s^a$, i.e., the nominal system is exponentially stable. If there exists $e_1, e_2 \geq 0$, such that

$$|f(x, u, w, t) - f(x, u, 0, t)| \leq e_1|x| + e_2 \quad (13)$$

for all $(x, u) \in \mathbb{Z}(t)$, $t \in \mathbb{T}$, and $w \in \mathbb{W}_1$, then Assumption 4 holds.

Proof: Let the $x \in \mathcal{X}_N(t)$ be the state and $w \in \mathbb{W}_1$ be the disturbance at time t . We have from (9) the bound $V(f_c(x, w, t), t + 1) \leq c_2|f_c(x, w, t)|^a$. We apply the triangle inequality and (13).

$$\begin{aligned} |f_c(x, w, t)|^a &\leq (|f_c(x, 0, t)| \\ &\quad + |f_c(x, w, t) - f_c(x, 0, t)|)^a \\ &\leq (|f_c(x, 0, t)| + e_1|x| + e_2)^a \\ &\leq 2^a|f_c(x, 0, t)|^a + (4e_1|x|)^a + (4e_2)^a. \end{aligned} \quad (14)$$

Using both (9) and (10) gives

$$\begin{aligned} |f_c(x, 0, t)|^a &\leq \frac{1}{c_1} V(f_c(x, 0, t), t + 1) \\ &\leq \frac{1}{c_1} V(x, t) - \frac{c_3}{c_1} |x|^a \leq \frac{c_2 - c_3}{c_1} |x|^a. \end{aligned} \quad (15)$$

We substitute (15) into (14) to get

$$\begin{aligned} V(f_c(x, w, t), t + 1) \\ \leq c_2 2^a \frac{c_2 - c_3}{c_1} |x|^a + c_2 (4e_1)^a |x|^a + c_2 (4e_2)^a. \end{aligned}$$

We substitute $|x|^a \leq \alpha_3(|x|)/c_1$ into this equation to give

$$V(f_c(x, w, t), t + 1) \leq b_1 \alpha_3(|x|) + b_2$$

in which $b_1 := (c_2/c_1)(2^a(c_2 - c_3)/c_1 + (4e_1)^a)$ and $b_2 := c_2(4e_2)^a$. Thus, Assumption 4 is satisfied.

Remark 2: MPC formulations with suitable terminal conditions and quadratic costs produce exponentially stable nominal systems [29, Ch. 2.4]. Unfortunately, constructing a bounded terminal set and cost that satisfy Assumption 3 is not necessarily possible with input constraints. Ideally, we want a global quadratic control Lyapunov function for the terminal cost, but constructing this function is only obvious for stable linear systems with input constraints [29, Ch. 2.5.3].

IV. ROBUSTNESS TO LARGE AND INFREQUENT DISTURBANCES

A. Motivating Example

We begin this section with a motivating example. Consider the scalar system $x^+ = x + u + 2w$ with $x \in \mathbb{R}$ and $u \in [-1, 1]$. We define the state cost $\ell(x, u) = x^2 + u^2$, terminal cost $V_f(x) = x^2$, and omit the terminal constraint, i.e., $\mathbb{X}_f = \mathbb{R}$. For an MPC problem with a horizon of $N = 2$, the optimal control law is $\kappa_N(x) := -\text{sat}(3x/5)$ [29, p. 104]. To streamline notation, we denote $x(k) := \phi(k; x(0), \mathbf{w}_k, 0)$.

If w is assumed to be a continuous random variable with $|w| < 0.5$, then the closed-loop system is ISS and therefore robust in the usual deterministic sense. However, if $w \in \mathbb{W} := \{0, 1\}$, i.e., $\mathbb{W}_1 := \{1\}$ is a large (discrete-valued) disturbance, then there exists a worst-case scenario in which $w = 1$ at every time and the system moves further away from $x = 0$ at each step. The system is not robust to this discrete-valued disturbance in the usual deterministic sense.

However, we can establish that this system satisfies Assumptions 1–4 for $w \in \mathbb{W} := \{0, 1\}$. Choose the Lyapunov function $V(x) = x^2$. We define

$$\alpha_3(|x|) := \begin{cases} \frac{21}{25}|x|^2 & ; |x| < \frac{5}{3} \\ 2|x| - 1 & ; |x| \geq \frac{5}{3} \end{cases}$$

to satisfy Assumption 1. If $w = 1$

$$V(x^+) \leq \begin{cases} (\frac{2}{5}|x| + 2)^2 & ; |x| < \frac{5}{3} \\ (|x| + 1)^2 & ; |x| \geq \frac{5}{3} \end{cases}$$

and we can bound this piecewise function by $V(x^+) \leq x^2 + \alpha_3(|x|) + \frac{64}{9}$. Therefore, Assumption 4 is satisfied with $b_1 = 1$, $b_2 = \frac{64}{9}$. Assumption 3 is satisfied because $\kappa_N(x)$ is defined for all $x \in \mathbb{R}$.

We now consider that this large (discrete-valued) disturbance is *infrequent* in that $\Pr(w = 1) = \varepsilon$ for some $\varepsilon \in (0, 1)$. We then conduct a simulation study of this system starting from $x(0) = 30$ for $\varepsilon = 0.4$. The results for 50 trials are plotted in Fig. 2. Note that each individual trial does not have a deterministic upper bound for $|x(k)|$ as $k \rightarrow \infty$. Given a sufficient number of time steps, the probability that $|x(k)|$ violates any finite bound is nonzero.

Instead, we propose a stochastic metric of robustness that captures the infrequent nature of the disturbance better than a deterministic metric. We define the p -confidence trajectory $c_p(|x(k)|)$ at each k as the minimum bound for $|x(k)|$ that holds for at least $p\%$ of trials for the system. We then evaluate the 95%-confidence trajectory of this closed-loop system for 1000

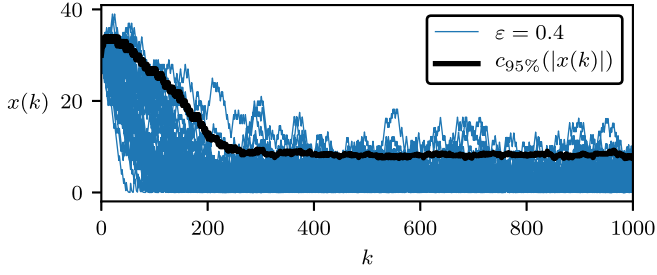


Fig. 2. Trajectories for 50 trials of the system $x^+ = x + \kappa_N(x) + 2w$ in which $\Pr(w = 1) = 0.4$. The 95% confidence bound ($c_{95\%}(|x(k)|)$) of the closed-loop trajectory for 1000 trials is plotted in black.

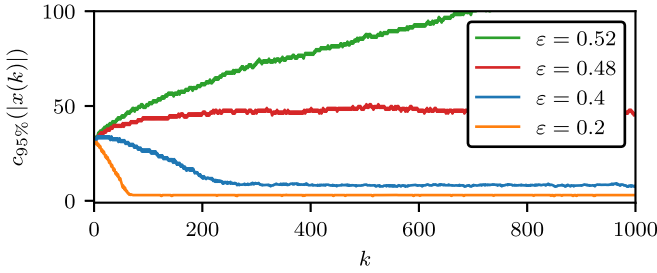


Fig. 3. 95% confidence bound of the closed-loop trajectory for 1000 trials of the system $x^+ = x + \kappa_N(x) + 2w$ with $\Pr(w = 1) = \epsilon$ for multiple values of ϵ .

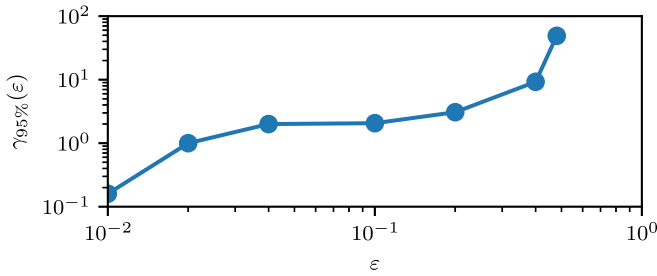


Fig. 4. Plot of $\hat{\gamma}_{95\%}(\epsilon)$. We evaluate $\hat{\gamma}_{95\%}(\epsilon)$ as the maximum value of $c_{95\%}(|x(k)|)$ for $k \in [800, 1000]$.

trials and plot the resulting trajectory in Fig. 2. We observe that this 95%-confidence trajectory converges to a region around $x = 0$ as $k \rightarrow \infty$ and appears to admit a finite upper bound for all k .

In Fig. 3, we consider multiple values of ϵ for which $\Pr(w = 1) = \epsilon$. For $\epsilon < 0.5$, all 95%-confidence trajectories exhibit similar behavior to $\epsilon = 0.4$ and converge to some region around $x = 0$ as $k \rightarrow \infty$. For $\epsilon < 0.5$, we denote the maximum value of $c_{95\%}(|x(k)|)$ for the last 200 time points as $\hat{\gamma}_{95\%}(\epsilon)$ and plot this bound in Fig. 4. Note that $\hat{\gamma}_{95\%}(\epsilon)$ increases with increasing ϵ and approaches zero as $\epsilon \rightarrow 0$.

All of these observations suggest that an ISS-type bound for the 95%-confidence trajectory is a reasonable characterization of this systems behavior. Thus, we postulate that there exists $\beta_{95\%}(\cdot) \in \mathcal{KL}$ and $\gamma_{95\%}(\cdot) \in \mathcal{K}$, such that

$$c_{95\%}(|x(k)|) \leq \beta_{95\%}(|x(0)|, k) + \gamma_{95\%}(\epsilon) \quad (16)$$

for all $x(0) \in \mathbb{R}$, $k \in \mathbb{I}_{\geq 0}$, and $\epsilon < 0.5$. Equivalently, we can write (16) as

$$\Pr(|x(k)| \leq \beta_{95\%}(|x(0)|, k) + \gamma_{95\%}(\epsilon)) \geq 0.95 \quad (17)$$

i.e., for each $k \in \mathbb{I}_{\geq 0}$ the closed-loop system satisfies this ISS-type bound with a probability of 95%.

This ISS-type bound, however, is not expected to hold for disturbances that occur with sufficient frequency. We note that for $\epsilon = 0.52$, the 95%-confidence trajectories in Fig. 3 diverges as $k \rightarrow \infty$ and therefore does not admit any ISS-type bound. Thus, we presume that there exists some $\delta < 0.5$, such that if $\epsilon > \delta$, the closed-loop system does not admit a stochastic ISS-type bound.

B. Robust Asymptotic Stability in Probability

The robustness of MPC is typically characterized through ISS in which the “input” to the closed-loop system is the disturbance. These robustness results are based on the size of disturbance and apply only for disturbances of sufficiently small size [1], [13]. Thus, a closed-loop system, and by extension the control law, is deemed *robustly asymptotically stable* if there exists some nonzero margin $\delta_0 > 0$, such that the closed-loop system is ISS for all disturbances that satisfying $|w| \leq \delta_0$ [1].

For large, infrequent disturbances, we instead consider the probability of the disturbance occurring, not the vector norm of the disturbance, as the description of the disturbance “size” and assume that the set $\mathbb{W}_1$ is fixed. The stochastic (infrequent) nature of these disturbances implies that a stochastic description of robustness is appropriate. Specifically, we leverage the concept of SISS and modify this definition for large, infrequent disturbances. We define RASiP for large, infrequent disturbances as follows.

Definition 4 (RASiP for large, infrequent disturbances): Suppose that the sequence $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ is robustly positive invariant for the system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ and we denote $\epsilon = \Pr(w \in \mathbb{W}_1)$. The origin of the closed-loop system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ is RASiP if there exists $\delta \in (0, 1]$ and for each $p \in [0, 1)$ there exists $\beta_p(\cdot) \in \mathcal{KL}$ and $\gamma_p(\cdot) \in \mathcal{K}$, such that the closed-loop system satisfies

$$\Pr(|\phi(k; x, \mathbf{w}_k, t)| \leq \beta_p(|x|, k - t) + \gamma_p(\epsilon)) \geq p \quad (18)$$

for all $\epsilon \in [0, \delta]$, $x \in \mathcal{X}_N(t)$, $t \in \mathbb{T}$, and $k \in \mathbb{I}_{\geq t}$.

RASiP implies that for sufficiently infrequent disturbances, i.e., $\epsilon \leq \delta$, we can construct the bound in (18) for any confidence level $p \in (0, 1)$. The bound inside the probability statement in (18), analogous to the standard ISS bound, contains two components. The first is a function $\beta_p(\cdot) \in \mathcal{KL}$ that captures the effect of the initial condition and the decay of this effect as $k \rightarrow \infty$. The second is a function $\gamma_p(\cdot) \in \mathcal{K}$ that captures the effect of the disturbance, through the value of ϵ , that persists for all $k \in \mathbb{I}_{\geq t}$.

Note that the definition of RASiP for large, infrequent disturbances captures the behavior observed in the motivating example. Indeed, (18) with $p = 0.95$ is identical to the ISS-type bound proposed in (17). As observed in the motivating example, however, we do not expect the closed-loop system to satisfy (18)

for large ε , i.e., sufficiently frequent disturbances. Analogous to the definition of robust asymptotic stability, we only require that there exists some nonzero margin $\delta > 0$, such that the closed-loop system satisfies (18) for $\varepsilon \leq \delta$.

The physical significance of RASiP is best illustrated with the motivating example. Consider the behavior in Fig. 3. For $\varepsilon < 0.5$ the closed-loop system can, on average, recover from the disturbance before another disturbance occurs and at each $k \in \mathbb{I}_{\geq 0}$, the closed-loop trajectory satisfies the ISS-type bound with 95% probability. For $\varepsilon = 0.52$, however, the 95%-confidence trajectory *diverges* as $k \rightarrow \infty$. Thus, we cannot define a finite bound, with any confidence level, as $k \rightarrow \infty$. For the motivating example, ε must exceed 0.5 for this behavior to occur. However, if there exists no margin of robustness, i.e., the closed-loop system is not RASiP, then the confidence trajectory diverges for a large disturbance that occurs with *any nonzero probability*. By contrast, a closed-loop system that is RASiP can recover from large disturbances provided the probability of that disturbance occurring is small.

Remark 3: As noted in Kozin [18], there are nuanced differences in stochastic stability definitions that have significant implications. In particular, including the condition $\forall k \in \mathbb{I}_{\geq t}$ inside the probability statement in (18) is a stronger property than placing it *outside*. Since the effect of the random variable on the process does not vanish as $x \rightarrow 0$, we can establish RASiP only when $\forall k \in \mathbb{I}_{\geq t}$ is outside the probability statement.

C. SISS-Lyapunov Functions and Main Results

To establish RASiP for large, infrequent disturbances, we define an SISS-Lyapunov function.

Definition 5 (SISS-Lyapunov Function): The function $V(\cdot, t) : \mathcal{X}_N(t) \rightarrow \mathbb{R}_{\geq 0}$ is an SISS-Lyapunov function on the robust positive invariant sets $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ for the system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ in which $\varepsilon = \Pr(w \in \mathbb{W}_1)$ if there exists $\delta \in (0, 1]$, $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_4(\cdot) \in \mathcal{K}_\infty$, $\sigma(\cdot) \in \mathcal{K}$, such that

$$\alpha_1(|x|) \leq V(x, t) \leq \alpha_2(|x|) \quad (19)$$

$$\mathbb{E}_{|x} [V(f_c(x, w, t), t+1)] \leq V(x, t) - \alpha_4(|x|) + \sigma(\varepsilon) \quad (20)$$

for all $\varepsilon \in [0, \delta]$, $x \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$.

In comparison to a typical ISS-Lyapunov function, we note two differences. First, the Lyapunov function evolution inequality is no longer a deterministic bound and considers the expected value of the subsequent Lyapunov function. This bound on conditional expectation is typical of other versions of SISS-Lyapunov functions in the literature [8], [19], [33], [35]. Second, the input to the $\mathcal{K}$ function $\sigma(\cdot)$ is the probability of the disturbance occurring, not the size of the disturbance. This type of input is distinct from other SISS-Lyapunov functions in the literature. Subsequently, we use this SISS-Lyapunov function to establish RASiP for a perturbed closed-loop system. To establish this result, however, we also require a stochastic Lyapunov function and the following proposition.

Definition 6 (Stochastic Lyapunov Function): The function $V(\cdot, t) : \mathcal{X}_N(t) \rightarrow \mathbb{R}_{\geq 0}$ is said to be a stochastic Lyapunov function on the robustly positive invariant sets $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ for the

system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ if there exists $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty$ and $\rho(\cdot) \in \mathcal{PD}$ satisfying

$$\alpha_1(|x|) \leq V(x, t) \leq \alpha_2(|x|)$$

$$\mathbb{E}_{|x} [V(f_c(x, w, t), t+1)] \leq V(x, t) - \rho(|x|)$$

for all $x \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$.

Proposition 2: If the system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ admits a stochastic Lyapunov function on the robustly positive invariant sets $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$, then the origin is ASiP, i.e., for each $p \in [0, 1)$, there exists $\beta_p(\cdot) \in \mathcal{KL}$, such that

$$\Pr(|\phi(k; x, \mathbf{w}_k, t)| \leq \beta_p(|x|, k-t)) \geq p$$

for all $x \in \mathcal{X}_N(t)$, $t \in \mathbb{T}$, and $k \in \mathbb{I}_{\geq t}$.

There are many versions of Proposition 2 and associated proofs throughout stochastic stability literature since the first example in Kushner [20]. For a proof of this specific version of this proposition, see [25]. Using Proposition 2, we establish the following result.

Proposition 3: If a system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ admits an SISS-Lyapunov function on the robustly positive invariant sets $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$, then the origin is RASiP.

Proof: We use an approach analogous to Theorem 4.2 in Krstic and Deng [19]. Choose $t \in \mathbb{T}$, $x \in \mathcal{X}_N(t)$ and let $x(k) := \phi(k; x, \mathbf{w}_k, t)$ for all $k \in \mathbb{I}_{\geq t}$. Define $\alpha_5 := \alpha_4 \circ \alpha_2^{-1}$. We note that $\alpha_4(s) \leq \alpha_2(s)$ because $\mathbb{E}[V(\cdot)] \geq 0$ and (20) must hold for $\varepsilon = 0$. Therefore, $0 \leq V(x, t) - \alpha_4(|x|) \leq \alpha_2(|x|) - \alpha_4(|x|)$. Since $\alpha_4(s) \leq \alpha_2(s)$, we also know that $\alpha_5(s) \leq s$ for $s \geq 0$ because $\alpha_4(\alpha_2^{-1}(s)) \leq \alpha_2(\alpha_2^{-1}(s)) = s$. Let $\tau \in \mathbb{I}_{\geq t}$ denote the (stopping) time at which the trajectory first enters the set defined by $V(x(k), k) \leq \tilde{\gamma}(\varepsilon) := 2\alpha_5^{-1}(\sigma(\varepsilon))$, i.e., $\tau = \inf\{k \geq t \mid V(x(k), k) \leq \tilde{\gamma}(\varepsilon)\}$.

For $k \in \mathbb{I}_{[t, \tau)}$, we have that $V(x(k), k) \geq 2\alpha_5^{-1}(\sigma(\varepsilon))$ and

$$\begin{aligned} \mathbb{E}_{|x(k)} [V(x(k+1), k+1)] &\leq V(x(k), k) - \alpha_4(\alpha_2^{-1}(V(x(k), k))) + \sigma(\varepsilon) \\ &\leq V(x(k), k) - \alpha_5(V(x(k), k)) + \alpha_5(V(x(k), k)/2) \\ &= V(x(k), k) - \rho(V(x(k), k)) \end{aligned} \quad (21)$$

in which $\rho(s) := \alpha_5(s) - \alpha_5(s/2) \in \mathcal{PD}$. Thus, for $k \in \mathbb{I}_{[t, \tau)}$, $V(\cdot)$ is a stochastic Lyapunov function for the closed-loop system. By Proposition 2 and (21), the system is ASiP for $k \in \mathbb{I}_{[t, \tau)}$, i.e., for each $p \in [0, 1)$ there exists $\beta_p(\cdot) \in \mathcal{KL}$, such that

$$\Pr(|x(k)| \leq \beta_p(|x|, k-t)) \geq p \quad (22)$$

for all $k \in \mathbb{I}_{[t, \tau)}$.

For $k \in \mathbb{I}_{\geq \tau}$, we proceed by induction. We assume that $\mathbb{E}[V(x(k), k)] \leq \tilde{\gamma}(\varepsilon)$ for some $k \in \mathbb{I}_{\geq \tau}$. However, the deterministic value of $V(x(k), k)$ is unknown. If $V(x(k), k) > \tilde{\gamma}(\varepsilon)$, we have

$$\mathbb{E}_{|x(k)} [V(x(k+1), k+1)] \leq V(x(k), k)$$

from (21) and the fact that $\rho(\cdot) \geq 0$. Therefore

$$\mathbb{E} [\mathbb{E}_{|x(k)} [V(x(k+1), k+1)]] \leq \mathbb{E} [V(x(k), k)] \leq \tilde{\gamma}(\varepsilon).$$

If $\tilde{\gamma}(\varepsilon)/2 \leq V(x(k), k) \leq \tilde{\gamma}(\varepsilon)$, we have

$$\begin{aligned} \mathbb{E}_{|x(k)}[V(x(k+1), k+1)] &\leq V(x(k), k) - \alpha_5(V(x(k), k)) + \sigma(\varepsilon) \\ &\leq V(x(k), k) - \alpha_5(\tilde{\gamma}(\varepsilon)/2) + \sigma(\varepsilon) \\ &= V(x(k), k) \leq \tilde{\gamma}(\varepsilon). \end{aligned}$$

If $V(x(k), k) \leq \tilde{\gamma}(\varepsilon)/2$, we have

$$\begin{aligned} \mathbb{E}_{|x(k)}[V(x(k+1), k+1)] &\leq V(x(k), k) - \alpha_5(V(x(k), k)) + \sigma(\varepsilon) \\ &\leq \tilde{\gamma}(\varepsilon)/2 + \sigma(\varepsilon) \\ &\leq \tilde{\gamma}(\varepsilon)/2 + \alpha_5(\tilde{\gamma}(\varepsilon)/2) \leq \tilde{\gamma}(\varepsilon) \end{aligned}$$

in which the last inequality is because $\alpha_5(s) \leq s$. Therefore, if $V(x(k), k) \leq \tilde{\gamma}(\varepsilon)$, we have

$$\mathbb{E}[\mathbb{E}_{|x(k)}[V(x(k+1), k+1)]] \leq \mathbb{E}[\tilde{\gamma}(\varepsilon)] \leq \tilde{\gamma}(\varepsilon).$$

Thus, we have that $\mathbb{E}[V(x(k), k)] \leq \tilde{\gamma}(\varepsilon)$ implies $\mathbb{E}[V(x(k+1), k+1)] \leq \tilde{\gamma}(\varepsilon)$. By definition, $\mathbb{E}[V(x(\tau), \tau)] \leq \mathbb{E}[\tilde{\gamma}(\varepsilon)] \leq \tilde{\gamma}(\varepsilon)$. So by induction, we have

$$\mathbb{E}[V(x(k), k)] \leq \tilde{\gamma}(\varepsilon)$$

for all $k \in \mathbb{I}_{\geq \tau}$. By applying Markov's inequality for each $p \in [0, 1]$ we have

$$\Pr\left(V(x(k), k) \leq \frac{\tilde{\gamma}(\varepsilon)}{1-p}\right) \geq p.$$

Define $\gamma_p(\varepsilon) := \alpha_1^{-1}(\tilde{\gamma}(\varepsilon)/(1-p))$ and since $|x(k)| \leq \alpha_1^{-1}(V(x(k), k))$ we have

$$\begin{aligned} \Pr(|x(k)| \leq \gamma_p(\varepsilon)) &\geq \Pr\left(\alpha_1^{-1}(V(x(k), k)) \leq \alpha_1^{-1}\left(\frac{\tilde{\gamma}(\varepsilon)}{1-p}\right)\right) \geq p \quad (23) \end{aligned}$$

for all $k \in \mathbb{I}_{\geq \tau}$. Finally, we combine (22) and (23) as follows:

$$\Pr(|x(k)| \leq \max\{\beta_p(|x|, k-t), \gamma_p(\varepsilon)\}) \geq p$$

which holds for all $k \in \mathbb{I}_{\geq t}$.

We can now establish the main result of this work.

Theorem 4 (Robustness to large, infrequent disturbances): Let Assumptions 1–4 hold. Then the origin is RASiP on the sets $(\mathcal{X}_N(t))_{t \in \mathbb{T}}$ for the system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$.

Proof: We consider the evolution of the system with and without a disturbance. Choose $x \in \mathcal{X}_N(t)$ at time t and denote $x^+ = f_c(x, w, t)$. If there is no disturbance, then the standard Lyapunov function decrease applies from (10). If $w \in \mathbb{W}_1$, then we apply the Lyapunov function bound from (12). We combine these two cases using the indicator function for $\mathbb{W}_1$ so that $\mathbb{E}[I_{\mathbb{W}_1}(w)] = \Pr(w \in \mathbb{W}_1) = \varepsilon$ and

$$\begin{aligned} V(x^+, t+1) &\leq V(x, t) - (1 - I_{\mathbb{W}_1}(w))\alpha_3(|x|) \\ &\quad + I_{\mathbb{W}_1}(w)(b_1\alpha_3(|x|) + b_2). \end{aligned}$$

Taking the expected value and combining terms give

$$\mathbb{E}_{|x}[V(x^+, t+1)] \leq V(x, t) - (1 - \varepsilon - b_1\varepsilon)\alpha_3(|x|) + \varepsilon b_2.$$

We choose $\delta \in (0, 1]$, such that $\delta < 1/(1 + b_1)$. Therefore

$$\mathbb{E}_{|x}[V(x^+, t+1)] \leq V(x, t) - \alpha_4(|x|) + \sigma(\varepsilon) \quad (24)$$

for all $\varepsilon \in [0, \delta]$ in which $\alpha_4(s) := (1 - (1 + b_1)\delta)\alpha_3(s)$ and $\sigma(\varepsilon) := b_2\varepsilon$. Note that $\alpha_4(\cdot), \sigma(\cdot) \in \mathcal{K}_\infty$. Thus, $V(\cdot)$ is an SISS-Lyapunov function and by Proposition 3 the proof is complete.

V. TIME-VARYING ECONOMIC MPC

A. Robust Performance

In economic MPC, the stage cost is not necessarily a positive definite tracking objective and asymptotic stability is not guaranteed by the problem formulation. In some economic MPC applications, stability is secondary to closed-loop economic performance. Thus, for nominal economic MPC, an important result is the average asymptotic performance bound presented by Angeli *et al.* [2, Th. 1]. This result ensures that at long time the performance, defined as the sum of stage costs, of economic MPC is no worse than the reference trajectory used to construct the terminal region. We consider the following assumptions for the economic MPC problem similar to Risbeck and Rawlings [30].

Assumption 5: The functions $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ are continuous. The reference trajectory $(\mathbf{x}_r, \mathbf{u}_r)$ satisfies $x_r(t+1) = f(x_r(t), u_r(t), 0, t)$.

Assumption 6: For each $t \in \mathbb{T}$, the set $\mathbb{Z}(t)$ is closed and $(x_r(t), u_r(t)) \in \mathbb{Z}(t)$. The set $\mathbb{U}$ is compact.

We define the shifted stage cost as $\bar{\ell}(x, u, t) := \ell(x, u, t) - \ell(x_r(t), u_r(t), t)$. We also define

$$\bar{V}_N^0(x, t) := V_N^0(x, t) - \sum_{k=t}^{N+t-1} \ell(x_r(k), u_r(k), k).$$

Assumption 7: For each $t \in \mathbb{T}$ and $x \in \mathbb{X}_f(t)$, the set

$$\kappa_f(x, t) := \{u \in \mathbb{U}(t) \mid x^+ := f(x, u, 0, t) \in \mathbb{X}_f(t+1),$$

$$V_f(x^+, t+1) \leq V_f(x, t) - \bar{\ell}(x, u, t)\}$$

is nonempty and $V_f(x_r(t), t) = 0$.

Assumption 8: The functions $\bar{\ell}(x, u, t)$ and $V_f(x, t)$ are uniformly bounded from below for $(x, u) \in \mathbb{Z}(t)$ and $x \in \mathbb{X}_f(t)$, respectively, for all $t \in \mathbb{T}$.

Risbeck and Rawlings [30] establish the following result.

Theorem 5 (Asymptotic performance of economic MPC): If Assumptions 5–8 hold, then the nominal system $x^+ = f_c(x, 0, t)$ satisfies

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=t}^{t+T-1} \bar{\ell}(x(k), u(k), k) \leq 0$$

in which $x(k) = \phi(k; x(t), 0, t)$ and $u(k) = \kappa_N(x(k), k)$, for all $x(t) \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$.

Proof: See Risbeck and Rawlings [30, Appendix A]

We seek an analogous performance result for the perturbed system. We require the following assumption.

Assumption 9: For the perturbed system $x^+ = f_c(x, w, t)$, there exists finite $b_1, b_2 \in \mathbb{R}_{\geq 0}$, such that

$$\bar{V}_N^0(f_c(x, w, t), t+1) \leq \bar{V}_N^0(x, t) + b_1|\bar{\ell}(x, \kappa_N(x, t), t)| + b_2$$

for all $x \in \mathcal{X}_N(t)$, $w \in \mathbb{W}_1$, and $t \in \mathbb{T}$.

Note the similarities to Assumption 4. The cost relative to the reference trajectory may increase based on the stage cost at the current state and some arbitrary constants. We observe an analogous requirement: The cost increase due to a disturbance cannot grow faster than the cost decrease without a disturbance. Furthermore, if $\bar{\ell}(x, u, t) \geq \alpha_3(|x - x_r(t)|)$, e.g., is a tracking cost, Assumption 4 for $V(x, t) := \bar{V}_N^0(x, t)$ implies Assumption 9. With this assumption we establish the following result.

Theorem 6 (Robust asymptotic performance of economic MPC): Let Assumptions 2, 3, and 5–9 hold. Then for the closed-loop system $x^+ = f_c(x, w, t)$; $w \in \mathbb{W}$ in which $\varepsilon = \Pr(w \in \mathbb{W}_1)$ there exists $\delta \in (0, 1]$ and $\bar{\gamma}(\cdot) \in \mathcal{K}$, such that

$$\limsup_{T \rightarrow \infty} \mathbb{E}_{|x} \left[\frac{1}{T} \sum_{k=t}^{t+T-1} \bar{\ell}(x(k), u(k), k) \right] \leq \bar{\gamma}(\varepsilon) \quad (25)$$

in which $x(k) = \phi(k; x, \mathbf{w}_k, t)$ and $u(k) = \kappa_N(x(k), k)$, for all $\varepsilon \in [0, \delta]$, $x \in \mathcal{X}_N(t)$, and $t \in \mathbb{T}$.

Proof: Choose $x \in \mathcal{X}_N(t)$, $t \in \mathbb{T}$ and denote $x^+ = f_c(x, w, t)$. If there is no disturbance ($w = 0$), then the standard cost decrease applies (See Risbeck and Rawlings [30, Appendix A]), i.e.,

$$\bar{V}_N^0(x^+, t+1) \leq \bar{V}_N^0(x, t) - \bar{\ell}(x, \kappa_N(x, t), t).$$

We combine this bound with Assumption 9 using the indicator function of $\mathbb{W}_1$ (same as Theorem 4), giving

$$\begin{aligned} \bar{V}_N^0(x^+, t+1) &\leq \bar{V}_N^0(x, t) - (1 - I_{\mathbb{W}_1}(w))\bar{\ell}(x, \kappa_N(x, t), t) \\ &\quad + I_{\mathbb{W}_1}(w) (b_1 \bar{\ell}(x, \kappa_N(x, t), t) + b_2). \end{aligned}$$

From Assumption 8, there exists $m \in \mathbb{R}$, such that $\bar{\ell}(x, u, t) \geq m$ for all $(x, u) \in \mathbb{Z}(t)$ and $t \in \mathbb{T}$. Therefore, $|\bar{\ell}(x, u, t)| \leq \bar{\ell}(x, u, t) + 2|m|$. Taking the expected value and combining terms gives

$$\begin{aligned} \mathbb{E}_{|x}[\bar{V}_N^0(x^+, t+1)] - \bar{V}_N^0(x, t) \\ \leq -(1 - \varepsilon - b_1\varepsilon)\bar{\ell}(x, \kappa_N(x, t), t) + b_3\varepsilon \end{aligned}$$

in which $b_3 := b_2 + 2b_1|m|$. We choose $\delta < 1/(1 + b_1)$ and note $\delta \in (0, 1]$, which gives

$$\begin{aligned} \mathbb{E}_{|x}[\bar{V}_N^0(x^+, t+1)] - \bar{V}_N^0(x, t) \\ \leq -b_4\bar{\ell}(x, \kappa_N(x, t), t) + b_3\varepsilon \end{aligned} \quad (26)$$

with $b_4 := (1 - (1 + b_1)\delta)$. Note that the choice of x and t was arbitrary. and thus. (26) holds for any $x \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$.

From $x(t) \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$, we define the closed-loop trajectory as $x(k) := \phi(k; x(t), \mathbf{w}_k, t)$ and the input $u(k) := \kappa_N(x(k), k)$ for all $k \in \mathbb{I}_{\geq t}$. By (26) and the properties of iterated expectations [9, p. 35] we may write

$$\begin{aligned} \mathbb{E}_{|x(t)}[\bar{V}_N^0(x(k+1), k+1)] - \mathbb{E}_{|x(t)}[\bar{V}_N^0(x(k), k)] \\ \leq -b_4\mathbb{E}_{|x(t)}[\bar{\ell}(x(k), u(k), k)] + b_3\varepsilon \end{aligned}$$

for all $k \in \mathbb{I}_{\geq t}$. We take the sum from t to $t + T - 1$ with $T \in \mathbb{I}_{\geq 1}$, divide by T , and rearrange

$$\begin{aligned} \mathbb{E}_{|x(t)} \left[\frac{b_4}{T} \sum_{k=t}^{T+t-1} \bar{\ell}(x(k), u(k), k) \right] \\ \leq \frac{\bar{V}_N^0(x(t), t) - \mathbb{E}_{|x}[\bar{V}_N^0(x(T), T)]}{T} + b_3\varepsilon. \end{aligned}$$

By Assumption 8, there exists some finite $M \in \mathbb{R}$, such that $\bar{V}_N^0(x(T), T) \geq M$ and we have

$$\begin{aligned} \mathbb{E}_{|x(t)} \left[\frac{1}{T} \sum_{k=t}^{T+t-1} \bar{\ell}(x(k), u(k), k) \right] \\ \leq \frac{\bar{V}_N^0(x(t), t) - M}{b_4 T} + \bar{\gamma}(\varepsilon) \end{aligned}$$

with $\bar{\gamma}(\varepsilon) := \frac{b_3}{b_4}\varepsilon \in \mathcal{K}$. If we take the limsup of this equation as $T \rightarrow \infty$ the initial cost and M vanish and we have (25).

As the probability of the disturbance occurring becomes larger, the expected average cost can become worse than the reference trajectory. Conversely, as $\varepsilon \rightarrow 0$ we return to the nominal guarantee for the system (in expected value). We note similarities of the left-hand side to the results in Chatterjee and Lygeros [4] for stochastic MPC and Bayer *et al.* [3] for robust economic MPC. However, the nominal MPC algorithm presented in this article does not require any stochastic information about the disturbance and yet still provides a stochastic form of robustness to these disturbances.

Remark 4: The assumptions for Theorem 6 admit systems with integrality constraints³ on inputs, discontinuous optimal value functions (even along the reference trajectory), and state constraints provided they do not affect Assumption 3. We also require no assumptions of dissipativity. As such, the conditions for this result are mild and the performance guarantee is particularly relevant to production scheduling applications.

Satisfying Assumptions 3 and 7 simultaneously, however, may require long horizons and modifications to the problem formulation. In addition, verifying Assumption 9 is nontrivial. Again, if $\mathcal{X}_N$ is bounded, Assumption 9 holds for $b_1 = 0$ and a sufficiently large value of b_2 . However, a bounded $\mathcal{X}_N$ that satisfies Assumption 3 may not exist for certain systems. Therefore, we consider a set of conditions for which Assumption 9 holds without a bounded $\mathcal{X}_N$.

Lemma 7: Let Assumptions 3, 5, 6, and 8 hold. Assume there exists a set $\mathcal{A} \subseteq \mathbb{X}$, such that:

- 1) there exists $e_1, e_2 \geq 0$ satisfying

$$|f(x, u, w, t)|_{\mathcal{A}} \leq e_1|x|_{\mathcal{A}} + e_2 \quad (27)$$

for all $(x, u) \in \mathbb{Z}(t)$, $w \in \mathbb{W}$, and $t \in \mathbb{T}$;

- 2) there exists $c_1, c_2, a > 0$ and $d_1, d_2 \geq 0$ satisfying

$$c_1|x|_{\mathcal{A}}^a - d_1 \leq |\bar{\ell}(x, u, t)| \leq c_2|x|_{\mathcal{A}}^a + d_2$$

$$V_f(x, t) \leq c_2|x|_{\mathcal{A}}^a + d_2$$

³Constraints that require a variable to be an integer, e.g., $u \in \{0, 1\}$.

for all $(x, u) \in \mathbb{Z}(t)$ and $x \in \mathbb{X}_f(t)$, respectively, for all $t \in \mathbb{T}$.

Then Assumption 9 holds.

Proof: For all $x \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$, we have from (27) that for all $k \in \mathbb{I}_{[t, N]}$,

$$|\phi(k; x, \mathbf{u}, t)|_{\mathcal{A}} \leq e_1^{k-t} |x|_{\mathcal{A}} + e_2 \sum_{i=0}^{k-t} e_1^i \leq e_3 |x|_{\mathcal{A}} + e_4$$

in which $e_3 := \max\{e_1, e_1^N\}$ and $e_4 := Ne_3e_2$. We bound the optimal value as follows:

$$\begin{aligned} \bar{V}_N^0(x, t) &\leq \sum_{k=t}^{t+N-1} \bar{\ell}(x(k), u(k), k) + V_f(x(t+N), t+N) \\ &\leq \sum_{k=t}^{t+N} c_2 |x(k)|_{\mathcal{A}}^a + d_2 \\ &\leq Nc_2(e_3|x|_{\mathcal{A}} + e_4)^a + Nd_2 \end{aligned}$$

with $x(k) := \hat{\phi}(k, x, \mathbf{u}^0(x, t), t)$. Choose $x \in \mathcal{X}_N(t)$, $w \in \mathbb{W}$, and $t \in \mathbb{T}$. Define $x^+ := f_c(x, w, t)$. We bound the optimal value at x^+ and $t+1$ as follows:

$$\begin{aligned} \bar{V}_N^0(x^+, t+1) &\leq Nc_2(e_3|x^+|_{\mathcal{A}} + e_4)^a + Nd_2 \\ &\leq Nc_2(e_3e_1|x|_{\mathcal{A}} + e_3e_2 + e_4)^a + Nd_2 \\ &\leq Nc_2(2e_3e_1)^a |x|_{\mathcal{A}}^a + d_3 \end{aligned}$$

in which $d_3 := Nc_2(2e_3e_2 + 2e_4)^a + Nd_2$. Then, we use the lower bound on the stage cost

$$\bar{V}_N^0(x^+, t+1) \leq b_1 |\bar{\ell}(x, \kappa_N(x, t), t)| + b_1 d_1 + d_3$$

with $b_1 := Nc_2(2e_3e_1)^a / c_1$. From Assumption 8, we know that there exists some $M \in \mathbb{R}$ satisfying $M \leq \bar{V}_N^0(x, t)$. Thus

$$\bar{V}_N^0(x^+, t+1) \leq \bar{V}_N^0(x, t) + b_1 |\bar{\ell}(x, \kappa_N(x, t), t)| + b_2$$

with $b_2 := \max\{b_1 d_1 + d_3 - M, 0\}$.

The cost bound requirements for Lemma 7 are analogous to the exponential cost bounds employed in Lemma 1. However, the constants $d_1, d_2 \geq 0$ can be used to address the cost of all bounded modes of the state. We can use the set $\mathcal{A}$ to remove or ignore modes of the state that incur bounded or zero cost from the power-law bounds of condition 2). Thus, the power-law bounds in condition 2) need to apply to only the unbounded modes of the state that incur unbounded cost on the set $\mathbb{X}$. If $\bar{\ell}(x, u, t)$ and $V_f(x, t)$ are linear or quadratic functions of the unbounded modes of x these cost bounds hold. Thus, the bound in (27) requires that the size of unbounded modes in the successor state are bounded by an affine function of the unbounded modes in the current state. In particular, the bounds in Lemma 7 apply to linear systems with linear cost functions, which are a common feature in closed-loop scheduling problems [32].

B. Robust Stability

If robust stability is desirable, we require the system to be strictly dissipative. We define time-varying dissipativity with respect to the reference trajectory $\mathbf{x}_r$ as follows.

Definition 7 (Strict dissipativity): The time-varying system $x^+ = f(x, u, 0, t)$ is strictly dissipative with respect to the reference trajectory $\mathbf{x}_r$ and supply rate $s(x, u, t) : \mathbb{X} \times \mathbb{U} \times \mathbb{T} \rightarrow \mathbb{R}$ if there exists a storage function $\lambda(x, t) : \mathbb{X} \times \mathbb{T} \rightarrow \mathbb{R}$ satisfying $\lambda(x_r(t), t) = 0$ and $\tilde{\alpha}_1(\cdot) \in \mathcal{K}_\infty$, such that

$$\lambda(f(x, u, 0, t), t+1) - \lambda(x, t) \leq s(x, u, t) - \tilde{\alpha}_1(|x - x_r(t)|)$$

for all $(x, u) \in \mathbb{Z}(t)$ and $t \in \mathbb{T}$.

Assumption 10 (Strict dissipativity): The nominal system (2) is strictly dissipative with respect to the reference trajectory $\mathbf{x}_r$ and the supply rate $s(x, u, t) = \ell(x, u, t) - \ell(x_r(t), u_r(t), t)$.

Assumption 11 (Uniform weak controllability): There exists $\tilde{\alpha}_2(\cdot) \in \mathcal{K}_\infty$, such that the optimal value function and storage function satisfy $\bar{V}_N^0(x, t) + \lambda(x, t) \leq \tilde{\alpha}_2(|x - x_r(t)|)$ for all $x \in \mathcal{X}_N(t)$ and $t \in \mathbb{T}$.

These assumptions are typically required, in some form, to ensure nominal stability of economic MPC. Note that the optimization problem we are solving in economic MPC is stated in (7) and does not require any knowledge of the storage function $\lambda(\cdot)$. Next, we redefine the system of interest in terms of deviation from the reference trajectory. We define the deviation variables $y(t) := x(t) - x_r(t)$, sets $\mathcal{Y}_N(t) := \mathcal{X}_N(t) \oplus \{-x_r(t)\}$, and system

$$y^+ = \bar{f}_c(y, w, t) := f_c(y + x_r(t), w, t) - x_r(t+1) \quad (28)$$

for all $t \in \mathbb{T}$. Thus, if $y(t) = 0$, then $x(t) = x_r(t)$. Furthermore, we define the rotated optimal value function as

$$\tilde{V}_N^0(y, t) := \bar{V}_N^0(y + x_r(t), t) + \lambda(y + x_r(t), t). \quad (29)$$

Using these assumptions, we state a nominal stability theorem modified from Risbeck and Rawlings [30, Th. 2].

Theorem 8: Let Assumptions 5–7, 10, and 11 hold. Then the sets $(\mathcal{Y}_N(t))_{t \in \mathbb{T}}$ and nominal system $y^+ = \bar{f}_c(y, 0, t)$ satisfy Assumption 1 with $\tilde{V}_N^0(z, t)$ as the Lyapunov function, $\alpha_1(s) = \alpha_3(s) = \tilde{\alpha}_1(s)$, and $\alpha_2(s) = \tilde{\alpha}_2(s)$, i.e., the origin of the deviation system is asymptotically stable on the sets $(\mathcal{Y}_N(t))_{t \in \mathbb{T}}$ for the system $y^+ = \bar{f}_c(y, 0, t)$.

Proof: See Risbeck and Rawlings [30, Appendix B].

Since our Lyapunov function for this system is now $\tilde{V}_N^0(x, t)$, Assumption 4 must hold for $V(x, t) := \tilde{V}_N^0(x, t)$. As an alternative, we may use the following assumption.

Assumption 12: There exists $a, c_1, c_2 > 0$, such that $\tilde{\alpha}_1(s)$ and $\tilde{\alpha}_2(s)$ from Assumptions 10 and 11 satisfy $c_1 s^a \leq \tilde{\alpha}_1(s)$ and $c_2 s^a \geq \tilde{\alpha}_2(s)$. Furthermore, there exists $e_1, e_2 \geq 0$, such that $|f(x, u, w, t) - f(x, u, 0, t)| \leq e_1 |x| + e_2$ for all $(x, u) \in \mathbb{Z}(t)$, $t \in \mathbb{T}$, and $w \in \mathbb{W}$.

Remark 5: If $\bar{\ell}(x, u, t)$ is a tracking cost with respect to the reference trajectory $\mathbf{x}_r$, then we satisfy Assumption 10 with $\lambda(\cdot) = 0$. Thus, tracking MPC is a special case of (dissipative) economic MPC. More detail on the nominal stability of tracking MPC can be found in [29, Ch. 2]. Furthermore, if we use quadratic stage and terminal tracking costs we satisfy the exponential cost function bounds required by Assumption 12.

The reference trajectory for the closed-loop system is robustly stable in probability under these assumptions.

Theorem 9 (Robustness of economic MPC to large, infrequent disturbances): Let Assumptions 2, 3, 5–7, and 10–12 hold. Then

the origin is RASiP on the sets $(\mathcal{Y}_N(t))_{t \in \mathbb{T}}$ for the system $y^+ = \bar{f}_c(y, w, t)$; $w \in \mathbb{W}$.

Proof: From Theorem 8, Assumption 1 is satisfied with $\alpha_1(s) = \alpha_3(s) = c_1 s^a$ and $\alpha_2(s) = c_2 s^a$. From Assumption 12 and Lemma 1 we can establish that Assumption 4 is satisfied. Therefore, applying Theorem 4 completes the proof.

VI. PRODUCTION SCHEDULING EXAMPLE

We consider a simple scheduling example to illustrate the relevance of this analysis approach to an industrial application. The goal is to meet demand of the product 1 (P1) by converting raw material (assumed to be in abundant supply) to P1 through task 1 (T1) carried out on a single available unit. Task 1 may have a batch size between 5 and 16 kgs of P1 and has a nominal processing time of 2 h. The demand for P1 is 4 kgs per hour and storing P1 costs \$10/(kg/hr). If the demand is not met, the facility accumulates backlog that must be offset at later times. The penalty for maintaining backlog is \$100/(kg/hr).

To model this system, we use a state space scheduling model developed by Subramanian *et al.* [32]. We define a binary decision variable W that is unity if T1 starts at time t . We also define the continuous input B that represents the batch size. To track these decisions in the state of the system, we lift W and B with the state variables $\bar{W}_n, \bar{B}_n$ for $n \in \{0, 1, 2\}$. The value of n represents the number of hours the task has progressed (e.g., at $n = 2$, the task is complete). We also consider disturbances in the form of 1-h delays (Y). Note that this is an inherently discrete-valued (large) and infrequent disturbance in this model, i.e., $Y \in \{0, 1\}$.⁴

The dynamics for this part of the system are represented in the following equations:

$$\begin{aligned}\bar{W}_0^+ &= (W + \bar{W}_0) Y \\ \bar{W}_1^+ &= (W + \bar{W}_0) (1 - Y) + \bar{W}_1 Y \\ \bar{W}_2^+ &= \bar{W}_1 (1 - Y) \\ \bar{B}_0^+ &= (B + \bar{B}_0) Y \\ \bar{B}_1^+ &= (B + \bar{B}_0) (1 - Y) + \bar{B}_1 Y \\ \bar{B}_2^+ &= \bar{B}_1 (1 - Y).\end{aligned}$$

Note that if $Y = 1$, the progress of the task does not move forward. Inventory and backlog (unmet demand) of P1, S , and U , respectively, are integrators influenced by the batch size of task 1 ending, shipments to meet demand (H), and demand $\xi(t)$. We also allow up to 1 kg of backlog to be outsourced or canceled each hour with the decision variable C

$$\begin{aligned}S^+ &= S + \bar{B}_2 - H \\ U^+ &= U + \xi(t) - C - H.\end{aligned}$$

In general, demand varies with time, but we have $\xi(t) = 4$.

Next, we require certain constraints to enforce the realism of this scheduling model. Specifically, $U \geq 0$, $S \in [0, 20]$, $W \in$

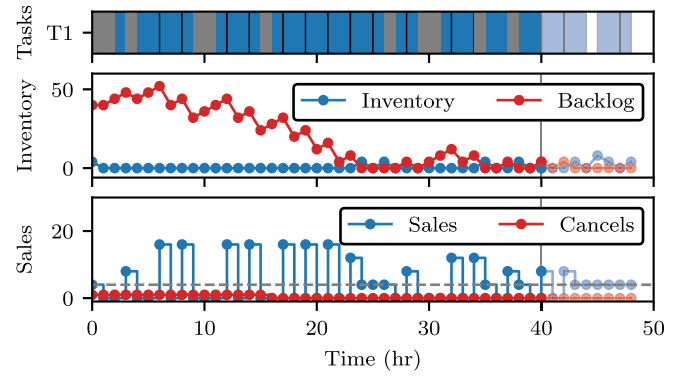


Fig. 5. Closed-loop trajectory for production facility. The top figure is a Gantt chart with blue blocks representing T1 being executed and gray blocks representing 1-h task delays. The inventory, backlog, sales, and canceled orders at each hour are shown in the lower plots. The control horizon, or schedule, for the next 8 h is shown in faded colors.

$\{0, 1\}$, $C \in [0, 1]$, $H \in [0, 20]$, and the constraint $\bar{W}_0 + \bar{W}_1 + W \leq 1$ prevents T1 from being run twice at the same time. We also constrain B to be within the batch size constraints if T1 is starting and zero otherwise, i.e., $5W \leq B \leq 16W$.

Now we have a discrete-time, state-space representation of the system with $x = [\bar{W}_0, \bar{W}_1, \bar{W}_2, \bar{B}_0, \bar{B}_1, \bar{B}_2, S, U]^T$, $u = [W, B, C, H]^T$, $w = [Y]$ and a dynamic evolution equation, $x^+ = f(x, u, w, t)$. We have state and input constraints $(x, u) \in \mathbb{Z}$ that also enforce discreteness of W . We define the stage cost as $\ell(x, u, t) := 10(S) + 100(U) + 800(C)$.

In the nominal case, the facility can meet the demand while operating at 50% capacity. For an 8-h horizon, the optimal periodic solution to this scheduling problem is to run T1 every two hours at a batch size of 8 kg. Demand is met every hour and an inventory of 4 kg is retained every other hour. For economic MPC, we consider an 8-h horizon and use this periodic solution as the reference trajectory. We enforce an exact terminal constraint for all state variables except backlog. For backlog, the terminal region includes any nonnegative real number, $\mathbb{R}_{\geq 0}$, and we use $V_f(x) = 900(U) + 100(U^2)$.

Note that by allowing backlog to take any nonnegative value, the set $\mathcal{X}_N(t)$ is now robustly positive invariant and Assumption 3 is satisfied. Assumptions 5, 6, and 8 are satisfied by the problem setup. Assumption 7 is satisfied by the terminal cost and constraint combination.⁵ We can establish that Assumption 9 is satisfied by using Lemma 7 and the integrator dynamics of backlog.

Clearly, if a one hour task delay occurs every hour, no P1 is ever produced. A more realistic scenario includes task delays that occur infrequently. Hence, we restrict $\Pr(Y = 1) = \varepsilon$ and consider 100 trials for multiple values of ε . Each trial starts with an initial backlog of 40 kgs.

An example closed-loop trajectory is shown in Fig. 5 for $\varepsilon = 0.3$. As the closed-loop trajectory evolves, the backlog decreases despite a few task delays. Once the backlog reaches a region around zero, infrequent task delays force the backlog to increase

⁴The discrete-time representation of the scheduling model results in discrete-valued delays.

⁵We establish this fact using the terminal control law $\kappa_f(x, t) := u_r(t) + [0, 0, \min\{U, 1\}, 0]^T$.

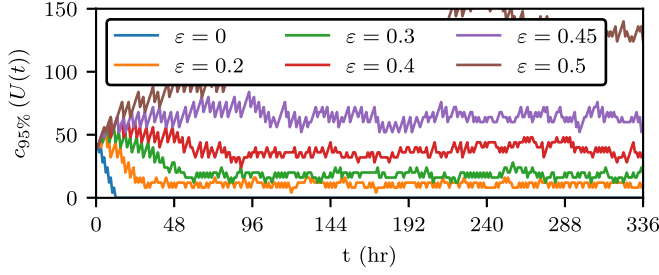


Fig. 6. Sample confidence interval for the closed-loop trajectory of 100 trials with different values of $\Pr(Y = 1) = \varepsilon$.

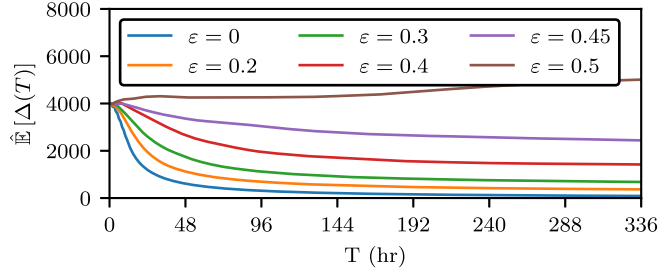


Fig. 7. Sample average $\Delta(T)$ for the closed-loop trajectory of 100 trials with different values of $\Pr(Y = 1) = \varepsilon$.

but, on average, the facility can recover before another delay occurs.

In Fig. 6, the backlog 95% confidence bound for 100 trials is plotted for multiple values of ε . For $\varepsilon \leq 0.45$, we observe convergence of the confidence bound to a region around zero. The size of this region increases with increasing ε . For $\varepsilon = 0.5$, the sample average backlog diverges and we presume that $0.45 < \delta < 0.5$. Again, the qualitative results of our simulation correspond exactly to the theory for large, infrequent disturbances. However, we do not claim that this system is strictly dissipative, and this result does not hold for all state variables.

In general, dissipativity of the stage cost is not a requirement or achieved for closed-loop scheduling formulations. Instead, the main concern is closed-loop economic performance. To quantify this performance, we define the average cost from 0 to T as follows:

$$\Delta(T) := \frac{1}{T} \sum_{k=0}^{T-1} \bar{\ell}(x(k), u(k), k).$$

Note that $\Delta(T)$ is identical to the average cost considered in Theorem 6. In Fig. 7, we plot the sample average of $\Delta(T)$ as a function of T for different values of ε . As T increases, the sample average of $\Delta(T)$ decays toward some nonnegative constant specific to each value of ε . As ε is increased, this constant increases, until at $\varepsilon = 0.5$ the sample average diverges. In short, the system exhibits behavior identical to Theorem 6. As expected from the previous discussion for backlog, we presume $0.45 < \delta < 0.5$.

VII. CONCLUSION

In this article, we developed robustness results for closed-loop systems subject to a class of large, infrequent disturbances not

typically considered in robustness analysis. In particular, we considered disturbances sufficiently large to exclude nominal robustness results, but, due to their infrequent nature, still admit stochastic descriptions of robustness. For recursively feasible and nominally stabilizing (economic) MPC controllers, we demonstrate that a specific upper bound on the (rotated) optimal value increase of the perturbed system is sufficient to guarantee robust asymptotic stability in probability for large, infrequent disturbances. In the absence of dissipative stage costs, we establish a performance bound in expectation for economic MPC subject to large, infrequent disturbances. We demonstrated the relevance of this analysis to practical problems through an application of economic MPC to production scheduling.

We emphasize that these stochastic robustness results are achieved by *nominal* MPC formulations without the need to include stochastic information in the optimization problem. Although stochastic MPC implementations may improve upon the robustness achieved by nominal MPC, in the form of larger δ and smaller $\gamma_p(\cdot)$, the results in this work suggest that nominal MPC formulations provide adequate stochastic robustness for large but sufficiently infrequent disturbances. Indeed, the inherent robustness of nominal MPC to this new class of disturbances may be sufficient in many applications.

There exist numerous future directions for this work. First, we intend to combine these results with typical robustness results for small, persistent disturbances and develop a more comprehensive theory of robustness for MPC. An additional future direction is to expand the set of systems and disturbances that are guaranteed to satisfy Assumptions 4 and 9 without bounded $\mathcal{X}_N$. We presented two examples that satisfy these assumptions, despite not satisfying the requirements of Lemma 1 or 7. There are likely additional versions of Lemma 1 and 7, particularly for neutrally stable systems with input constraints. To consider large, infrequent disturbances, Assumption 3 requires that the optimal control problem remains feasible subject to any realization of these large disturbances. In practice, this requires that the optimal control problem be *designed* for recursive feasibility subject to these disturbances. We accomplish this goal in the production scheduling example by careful selection of state constraints and terminal conditions. Thus, an important avenue of research involves ensuring recursive feasibility of MPC applied to practical problems, particularly through well-designed terminal conditions.

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