Nonlinear Stochastic Model Predictive Control: Existence, Measurability, and Stochastic Asymptotic Stability

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***Abstract*—In this article, we establish a collection of new theoretical properties for nonlinear stochastic model predictive control (SMPC). Based on the concept of stochastic input-to-state stability (SISS), we define robust asymptotic stability in expectation (RASiE) and establish that nonlinear SMPC renders the origin of the closed-loop system RASiE. Moreover, we establish several new foundational results that have not been addressed in previous research. Specifically, we verify that, under basic regularity assumptions, a solution to the SMPC optimization problem exists and the closed-loop trajectory is Borel measurable thereby guaranteeing that all relevant stochastic properties of the closed-loop system are indeed well-defined. We present a numerical example to demonstrate the nonintuitive behavior that can arise from nonlinear SMPC.**

***Index Terms*—Stability of nonlinear systems, stochastic systems, stochastic model predictive control (SMPC), stochastic optimal control.**

I. Introduction

Model predictive control (MPC), under appropriate assumptions, guarantees asymptotic stability of the nominal closed-loop system, i.e., the system model is perfect and no disturbances occur [32, ch 2.]. In practice, however, disturbances and model mismatch are pervasive, and therefore, some degree of robustness is required for successful industrial implementation. Although nominal MPC does not consider uncertainty explicitly in the problem formulation, the *inherent* robustness afforded by feedback is often sufficient for successful implementation of MPC. The inherent robustness of MPC has been established over the past two decades [1], [11], [17], [27], [41].

Stochastic MPC (SMPC) aims to improve upon the inherent robustness of nominal MPC by including a probabilistic description of uncertainty in the optimal control problem. In general, the SMPC optimization problem minimizes the expected value of the cost function subject to both deterministic and probabilistic constraints on the state variables [7], [21], [25]. By defining the optimal control problem based on stochastic properties of the system, the complexity of the optimization problem and closed-loop analysis increases significantly relative to the nominal MPC problem. Consequently, the majority of results for SMPC are restricted to linear systems. While there is an extensive body of literature devoted to approximating and solving SMPC and stochastic optimal control problems (see [25]), we focus this article on the closed-loop properties of SMPC.

Primbs and Sung [30] consider linear systems with multiplicative uncertainty such that the effect of the disturbance vanishes at the origin. By assuming that the terminal cost is a *global* stochastic Lyapunov function, the authors establish that the origin is asymptotically stable with probability one for the closed-loop system. Similarly, we can established stability in expectation of SMPC for unconstrained linear systems with additive disturbances [28]. For constrained linear SMPC subject to bounded disturbances, Cannon, Kouvaritakis, and coauthors show that we can construct a terminal set and cost that ensure recursive feasibility and stability in expectation of the closed-loop system [3]–[5], [13]. Lorenzen *et al.* [19] propose a less restrictive constraint tightening approach and establish that linear SMPC asymptotically stabilizes, with probability one, the minimal robust positively invariant set for the system. Similar results are established in other subsequent papers for modified SMPC algorithms [12], [35].

For *nonlinear* SMPC, Chatterjee and Lygeros [6] establish, for unconstrained nonlinear systems, that the expected value of the optimal cost along the closed-loop trajectory is bounded if the terminal cost is a *global* stochastic Lyapunov function. Mayne and Falugi [22] extend these results to address constrained nonlinear systems subject to bounded, stochastic disturbances, and with terminal constraints, require the terminal cost to be only a *local* stochastic Lyapunov function. Under certain viability and stochastic controllability assumptions, nonlinear SMPC without terminal conditions is also stabilizing, but these assumptions are difficult to verify for nonlinear systems [20].

Given its early stage of development, there are naturally many limitations to the current theory of nonlinear SMPC. All results for the convergence of the optimal cost, and therefore, closed-loop state, are restricted to systems that admit an *exponential* decrease. Although useful for quadratic stage costs, these bounds do not admit a general class of nonlinear stage costs. As we

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show in the subsequent sections, the extension of results for an exponential optimal cost decrease to an asymptotic optimal cost decrease is nontrivial. Furthermore, the established properties for nonlinear SMPC are limited to stability in expected value for the closed-loop system. If we assume that the disturbance vanishes at the origin, we can recover asymptotic stability in probability, but this assumption is quite restrictive and omits even linear systems with additive disturbances. To the best of our knowledge, there is currently no result establishing any notion of (robust) *asymptotic* stability of nonlinear SMPC without requiring the disturbance to vanish at the origin.

Although the original stochastic stability results developed by Kushner [15], [16] are frequently cited in the SMPC literature, there is a noticeable gap between the stability results available for SMPC and the current state of the nonlinear stochastic stability theory. The concepts of stochastic Lyapunov functions and global asymptotic stability in probability (GASiP) were refined significantly by Florchinger [8]. Teel and coworkers advanced these results further by establishing that stochastic Lyapunov functions ensure a stronger definition of GASiP that requires *uniform* convergence and developed converse Lyapunov theorems for this stochastic definition of stability [10], [37]–[39]. Analogous to input-to-state stability (ISS) for deterministic systems, stochastic input-to-state stability (SISS) was also defined and established using SISS-Lyapunov functions in [14], [18], [36], and [40]. These works, however, assume that the effect of the stochastic disturbance vanishes once the state of the system reaches the origin (i.e., a multiplicative disturbance) and typically require the closed-loop system to be continuous (i.e., defined by a continuous function). SMPC applications, by contrast, often consider additive disturbances that do not vanish at the origin. Furthermore, SMPC may define a discontinuous control law, and therefore, produces a discontinuous closed-loop system. Consequently, the results from the stochastic stability theory are not applicable in their current form to the nonlinear SMPC problem.

For nominal MPC under basic regularity assumptions, we can establish that the optimal control problem is well-defined, i.e., the minimum is attained [32, Prop. 2.4]. For nonlinear SMPC, there are no analogous results in the literature. The authors either assume explicitly that the "minimization problem is well-defined" or omit this discussion. Furthermore, the expected value of the closed-loop system is used in SMPC analysis without establishing that such a property is in fact well-defined. Although both of these basic properties may be considered likely to hold for the SMPC problem, glossing over the existence question entirely is not ideal. By contrast, these properties are well-established in the field of stochastic optimal control [2].

In this article, we establish and refine several foundational properties for nonlinear SMPC that are currently absent from the literature. We begin by establishing that, under basic regularity assumptions, the nonlinear SMPC optimization problem and all relevant stochastic properties (e.g., expected value of the optimal cost) are well-defined for the closed-loop system. We then define a type of SISS-Lyapunov function similar to the definition used in the nonlinear stochastic stability theory. We establish that nonlinear SMPC, under suitable assumptions, admits an SISS-Lyapunov function and renders the origin of the closed-loop system robustly asymptotically stable in expectation (RASiE). We conclude with an example that illustrates the implications of this analysis and demonstrates the nonintuitive closed-loop behavior that nonlinear SMPC may exhibit.

Preliminary results can be found in [24]. This article, however, is a more thorough investigation of these topics. Notably, we address stochastic asymptotic stability instead of stochastic exponential stability, which for stochastic Lyapunov functions is a nontrivial extension. We also discuss the fundamental properties of existence and measurability in greater detail and provide proofs of these properties in the Appendix. In addition, we derive performance guarantees for economic applications of SMPC that do not rely on a lower bound for the stage cost (see Theorem 12) and provide a detailed discussion of these results in comparison to previous results for SMPC.

Notation: Let $\mathbb{I}$ and $\mathbb{R}$ denote the integers and reals. Let superscripts and subscripts denote dimensions and restrictions (e.g., $\mathbb{R}^n_{\geq 0}$ denotes nonnegative reals of dimension n). Let $|\cdot|$ denote the Euclidean norm. For a closed set $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S := \min_{y \in S} |x - y|$ denotes the Euclidean point-to-set distance. Let $I_S(x)$ denote the indicator function for a set S, i.e., $I_S(x) = 1$ for all $x \in S$ and $I_S(x) = 0$ otherwise. A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous if and only if the set $\{x \in \mathbb{R}^n : f(x) \leq y\}$ is closed for every $y \in \mathbb{R}$. The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}_\infty$ if $\alpha(\cdot) \in \mathcal{K}$ and unbounded, i.e., $\lim_{s \to \infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{KL}$ if for every $k \in \mathbb{I}_{\geq 0}$, the function $\beta(\cdot, k)$ is in class $\mathcal{K}$, and for fixed $s \in \mathbb{R}_{\geq 0}$, the function $\beta(s, \cdot)$ is nonincreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

Let $\mathcal{P}(\Omega)$ denote the power set and $\mathcal{B}(\Omega)$ denote the Borel field of some set Ω. A set $F \subset \mathbb{R}^n$ is Borel measurable if $F \in \mathcal{B}(\mathbb{R}^n)$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Borel measurable if for each open set $O \subset \mathbb{R}^m$, the set $f^{-1}(O) := \{x \in \mathbb{R}^n : f(x) \in O\}$ is Borel measurable, i.e., $f^{-1}(O) \in \mathcal{B}(\mathbb{R}^n)$. For two metric spaces X and Y, a set-valued mapping denoted $S : X \rightrightarrows Y$ is the assignment of each $x \in X$ to a set $S(x) \subseteq Y$. A set-valued mapping $S : X \rightrightarrows Y$ is Borel measurable if for every open set $O \subseteq Y$, the set $S^{-1}(O) := \{x \in X : S(x) \cap O \neq \emptyset\}$ is Borel measurable, i.e., $S^{-1}(O) \in \mathcal{B}(X)$ [33].

II. Stochastic Model Predictive Control (SMPC)

A. Stochastic System

We consider the following discrete-time system:

$$x^+ = f(x, u, w) \quad f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to \mathbb{X} \tag{1}$$

in which $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u \in \mathbb{U} \subseteq \mathbb{R}^m$ is the controlled input, $w \in \mathbb{W} \subseteq \mathbb{R}^p$ is a disturbance (random variable), and x^+ denotes the successor state.

Let $(\Omega, \mathcal{F}, P)$ be a probability space for the sequence $\mathbf{w}_\infty : \Omega \to \mathbb{W}^\infty$ of random variables, i.e., $\mathbf{w}_\infty := \{w(i)\}_{i=0}^\infty$ for $w(i) : \Omega \to \mathbb{W}$. We define the subsequence $\mathbf{w}_i : \Omega \to \mathbb{W}^i$ as $\mathbf{w}_i := (w(0), \dots, w(i-1))$. Let $(\mathcal{F}_0, \mathcal{F}_1, \dots)$ denote the natural filtration of the sequence $\mathbf{w}_\infty$, that is, $\mathcal{F}_i \subset \mathcal{F}$ is all sets of the form $\{\omega \in \Omega : \mathbf{w}_i(\omega) \in F\}$ for $F \in \mathcal{B}(\mathbb{W}^i)$. We also define the expected value of the Borel measurable function $g : \mathbb{W}^i \to \mathbb{R}$

as

$$\mathbb{E}\left[g(\mathbf{w}_i)\right] := \int_\Omega g(\mathbf{w}_i(\omega))dP(\omega).$$

We make the following standard assumption for the disturbance in SMPC.

Assumption 1 (Disturbance): The disturbances $w(i) : \Omega \to \mathbb{W}$ are independent and identically distributed (i.i.d.) in time, zero mean ($\mathbb{E}[w(i)] = 0 \forall i \in \mathbb{I}_{\geq 0}$), random variables. The support $\mathbb{W}$ is compact and contains the origin. Each random variable has a known and equivalent probability measure $\mu : \mathcal{B}(\mathbb{W}) \to [0,1]$ defined such that $\mu(F) = P(\{\omega \in \Omega : w(i;\omega) \in F\})$ for all $F \in \mathcal{B}(\mathbb{W})$ and $i \in \mathbb{I}_{\geq 0}$. The second moment of $w(i)$ is finite and the covariance matrix is defined as $\Sigma = \mathbb{E}[w(i)w(i)^\top]$ for all $i \in \mathbb{I}_{\geq 0}$.

Note that, as is typical with SMPC formulations, we require bounded $\mathbb{W}$ to allow for compact input (and state) constraints and ensure that the compact terminal set is robustly positive invariant [22]. Requiring bounded $\mathbb{W}$ is a strong assumption and a limitation of this approach.

For the i.i.d. random variables $(w(i), w(i+1), \ldots, w(i+N-1))$ and $N \in \mathbb{I}_{\geq 1}$, their joint distribution measure $\mu^N : \mathcal{B}(\mathbb{W}^N) \to [0,1]$ is defined as

$$\mu^N(F) = \mu(F_i)\mu(F_{i+1})\ldots\mu(F_{i+N-1})$$

for all $F = (F_i, F_{i+1}, \ldots, F_{i+N-1}) \in \mathcal{B}(\mathbb{W}^N)$. We define conditional expected value such that

$$\begin{aligned}&\mathbb{E}\left[g(\mathbf{w}_{i+1}) \mid \mathcal{F}_i\right](\omega)\\ &:= \int_{\mathbb{W}} g(w(0;\omega), \ldots, w(i-1;\omega), w)d\mu(w)\end{aligned}$$

for any Borel measurable function $g : \mathbb{W}^{i+1} \to \mathbb{R}$ and all $i \in \mathbb{I}_{\geq 0}$.

B. Nonlinear SMPC Formulation

In the following sections, we drop the term "nonlinear" and refer to all nonlinear SMPC as simply SMPC, unless we are discussing key distinctions between linear and nonlinear SMPC. Although it seems convenient to use the same probability space defined for the stochastic system to discuss the SMPC formulation, we separate the stochastic system and the *stochastic model* of that system used for SMPC. This compartmentalization allows us to discuss the SMPC problem, at least initially, as a time-invariant problem instead of one conditioned on the filtrations $\mathcal{F}_k$ of the stochastic system.

Instead of selecting a trajectory of inputs $\mathbf{u}$, we intend to solve for a trajectory of control policies. We define the policy $\pi : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$ in which $x \in \mathbb{X}$ is the current state of the system and $v \in \mathbb{V} \subseteq \mathbb{R}^q$ are the parameters in the control policy. Thus, the resulting system of interest is defined as

$$x^+ = f_\pi(x, v, w) = f(x, \pi(x, v), w). \tag{2}$$

We use $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ to denote the state solution of (2) at time k, given the initial condition x, the trajectory of control policy parameters $\mathbf{v} = (v(0), v(1), \ldots, v(N-1))$, and the disturbance sequence $\mathbf{w} \in \mathbb{W}^N$.

We consider the case of hard input and state constraints, i.e., $(x, u) \in \mathbb{Z}_h \subseteq \mathbb{X} \times \mathbb{U}$. In addition, we allow (one step ahead) probabilistic constraints on the state defined as

$$\Pr\left(f(x, u, w) \in \tilde{\mathbb{X}}\right) \geq 1 - \varepsilon \tag{3}$$

for a set $\tilde{\mathbb{X}} \subseteq \mathbb{R}^n$ and constant $\varepsilon \in [0,1]$. We observe, however, that this method to represent probabilistic constraints appears inconsistent with other constraints. Thus, we reformulate this constraint using the function

$$G(x, u) := 1 - \int_{\mathbb{W}} I_{\tilde{\mathbb{X}}}\left(f(x, u, w)\right) d\mu(w)$$

and the constraint set $\tilde{\mathbb{Z}}_\varepsilon := \{(x, u) : G(x, u) \leq \varepsilon\}$. Note that (x, u) satisfy (3) if and only if $(x, u) \in \tilde{\mathbb{Z}}_\varepsilon$. Then, we define the combined hard and probabilistic constraints as

$$(x, u) \in \mathbb{Z} := \mathbb{Z}_h \cap \tilde{\mathbb{Z}}_\varepsilon.$$

We note that calculating or approximating $\tilde{\mathbb{Z}}_\varepsilon$ is a difficult and important research problem that we obscure with this reformulation. However, this reformulation is useful for the analysis of the SMPC problem. We subsequently establish that $\mathbb{Z}$ is in fact *closed* under basic regularity assumptions.

For SMPC with a horizon of $N \in \mathbb{I}_{\geq 1}$, the mixed constraint $\mathbb{Z}$, and an additional terminal constraint $\mathbb{X}_f \subseteq \mathbb{X}$, we have the set of admissible $(x, \mathbf{v})$ pairs defined as

$$\begin{aligned}\mathcal{Z}_N := \{&(x, \mathbf{v}) \in \mathbb{X} \times \mathbb{V}^N :\\ &(x(k), \pi(x(k), v(k))) \in \mathbb{Z} \quad \forall \mathbf{w} \in \mathbb{W}^N,\ k \in \mathbb{I}_{[0,N-1]}\\ &x(N) \in \mathbb{X}_f \quad \forall \mathbf{w} \in \mathbb{W}^N\}\end{aligned}$$

in which $x(k) = \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$. From this set, we define the set of admissible parameter trajectories given $x \in \mathbb{X}$ as $\mathcal{V}_N(x) := \{\mathbf{v} \in \mathbb{V}^N : (x, \mathbf{v}) \in \mathcal{Z}_N\}$ and the set of admissible initial states as $\mathcal{X}_N := \{x \in \mathbb{X} : \mathcal{V}_N(x) \neq \emptyset\}$.

Remark 1: The constraint (3) is enforced in $\mathcal{Z}_N$ for all $k \in \mathbb{I}_{[0,N-1]}$. These one-step ahead constraints, i.e., constraining $\Pr(x(k+1) \in \tilde{\mathbb{X}})$ given $x(k)$, are more restrictive than the multistep ahead constraints sometimes used in SMPC, i.e., constraining $\Pr(x(k+1) \in \tilde{\mathbb{X}})$ given $x(0)$. Using these one-step ahead constraints, we can ensure recursive feasibility of the optimization problem and guarantee that the closed-loop trajectory satisfies $\Pr(x(k) \in \tilde{\mathbb{X}}) \leq 1 - \varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$.

To characterize the performance of each feasible parameter and state trajectory, we define a stage cost $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ and a corresponding terminal cost $V_f : \mathbb{X} \to \mathbb{R}_{\geq 0}$ for the SMPC problem. With these costs, we define the function

$$J_N(x, \mathbf{v}, \mathbf{w}) = \sum_{k=0}^{N-1} \ell(x(k), \pi(x(k), v(k))) + V_f(x(N))$$

in which $x(k) := \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$, $(x, \mathbf{v}) \in \mathcal{Z}_N$, and $\mathbf{w} \in \mathbb{W}^N$. For SMPC, we define the cost function based on the expected value of $J_N(\cdot)$, i.e., we define

$$V_N(x, \mathbf{v}) := \int_{\mathbb{W}^N} J_N(x, \mathbf{v}, \mathbf{w})d\mu^N(\mathbf{w}).$$

The SMPC problem for any $x \in \mathcal{X}_N$ is defined as

$$\mathbb{P}_N(x): \ V_N^0(x) = \min_{\mathbf{v} \in \mathcal{V}_N(x)} V_N(x, \mathbf{v}) \tag{4}$$

and the optimal solutions for a given initial state are defined by the set-valued mapping $\mathbf{v}^0 : \mathcal{X}_N \rightrightarrows \mathbb{V}^N$ such that $\mathbf{v}^0(x) := \arg\min_{\mathbf{v} \in \mathcal{V}_N(x)} V_N(x, \mathbf{v})$. Note that $\mathbf{v}^0(x)$ is a set-valued mapping because there may be multiple solutions to $\mathbb{P}_N(x)$.

C. Assumptions

We require some basic regularity assumptions for the SMPC problem.

Assumption 2 (Continuity of system and cost): The functions $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to \mathbb{X}$, $\pi : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, and $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$ are continuous. The function $\ell(x, u)$ is lower bounded for all $(x, u) \in \mathbb{Z}_h$. Furthermore, we have that $f(0, 0, 0) = 0$, $\ell(0, 0) = 0$, and $V_f(0) = 0$.

Assumption 3 (Properties of constraint sets): The sets $\mathbb{Z}_h$ and $\tilde{\mathbb{X}}$ are closed and contain the origin. The sets $\mathbb{U}, \mathbb{V}, \mathbb{X}_f \subseteq \mathbb{X}$ are compact and contain the origin.

Note that we require $\mathbb{V}$, in addition to $\mathbb{U}$, to be compact. Since we intend to optimize over the set $\mathcal{V}_N(x)$ for a nonlinear (potentially noncoercive) function, compactness of $\mathbb{V}$ is required to ensure $\mathbb{P}_N$ is well-defined.

To ensure that the stochastic optimal control problem remains (robustly) recursively feasible and achieves certain performance objectives, we require a terminal control law assumption analogous to the one used in nominal MPC.

Assumption 4 (Terminal control law): There exists a continuous terminal control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that for all $x \in \mathbb{X}_f$

$$f(x, \kappa_f(x), w) \in \mathbb{X}_f \quad \forall w \in \mathbb{W} \tag{5}$$

$$V_f(f(x, \kappa_f(x), 0)) \leq V_f(x) - \ell(x, \kappa_f(x)). \tag{6}$$

Furthermore, $(x, \kappa_f(x)) \in \mathbb{Z}_h$, $\mathbb{X}_f \subseteq \tilde{\mathbb{X}}$, and $\pi(x, 0) = \kappa_f(x)$ for all $x \in \mathbb{X}_f$.

In contrast to [22, Assumption 5.2], we require $V_f(\cdot)$ to satisfy a cost decrease condition for only the *nominal* system, i.e., $w = 0$. We also note that by assuming $\mathbb{X}_f \subseteq \tilde{\mathbb{X}}$, and that $\tilde{\mathbb{X}}$ is robustly positive invariant, we ensure that the probabilistic constraint is satisfied with probability one for all $x \in \mathbb{X}_f$. The main difference between the terminal set assumptions for stochastic and nominal MPC is that we explicitly require robust positive invariance of the terminal set through (5) for the SMPC problem. However, robust positive invariance of the terminal region is often a requirement to establish the inherent robustness of nominal MPC [1]. Thus, none of the requirements in Assumption 4 are particularly strong relative to the nominal MPC problem.

To construct this terminal set and cost, we typically define a terminal control law based on the linear–quadratic regulator (LQR) solution of the linearized system (at the origin) . We then define the terminal cost based on the infinite horizon LQR cost and the terminal set as a level set of the terminal cost (see [32, Sec. 2.5.5]). This terminal set is robustly positive invariant for sufficiently small disturbances. For SMPC, we can further modify this terminal set to be robustly positive invariant for the specific support $\mathbb{W}$. For sufficiently large $\mathbb{W}$, however, a robustly positive invariant terminal set may not exist for any terminal control law. Furthermore, verifying that Assumption 4 holds for a nonlinear system and a specific support $\mathbb{W}$ is nontrivial.

If we are considering a tracking problem, we require the following assumption. Note that the lower bound on the stage cost and requirement of $\mathbb{X}_f$ to contain the origin in its interior are equivalent to the typical requirements to establish asymptotic stability of nominal MPC [32, Prop. 2.16].

Assumption 5 (Tracking cost bounds): There exists $\alpha_\ell(\cdot) \in \mathcal{K}_\infty$ such that $\ell(x, u) \geq \alpha_\ell(|x|)$ for all $(x, u) \in \mathbb{Z}_h$. Furthermore, $\mathbb{X}_f$ contains the origin in its interior and $\mathcal{X}_N$ is bounded.

Remark 2: Requiring bounded $\mathcal{X}_N$ is a minor restriction. Assumption 3 combined with the requirement that $f^{-1}(X) = \{(x, u) \in \mathbb{X} \times \mathbb{U} : f(x, u, 0) \in X\}$ is bounded for any bounded X ensures that $\mathcal{X}_N$ is bounded. For a proof and further discussion, see [32, Prop. 2.10(d)] and note that since $0 \in \mathbb{W}$, the set of feasible initial states for nominal MPC (i.e., $\mathbf{w} = 0$) is a superset of $\mathcal{X}_N$ for SMPC.

III. Basic Properties of SMPC

Before proceeding to any stability guarantees for SMPC, we begin by discussing some essential properties required to properly analyze the closed-loop stochastic system generated by SMPC. Namely, we establish that a solution to $\mathbb{P}_N$ exists and verify that relevant stochastic properties (i.e., expected value) for the closed-loop system are well-defined. We note that presenting expected value as a Lebesgue integral allows us to establish many useful properties for SMPC that may remain unclear with typical SMPC notation. All the proofs of results in the section are reported in the Appendix and further details can be found in [23].

For nominal MPC, the analogs of Assumptions 2 and 3 are sufficient to guarantee that the minimization problem is well-defined for all $x \in \mathcal{X}_N$, i.e., the minimum is attained [32, Prop. 2.4]. For SMPC, we demonstrate the same property for the minimization problem defined as $\mathbb{P}_N(x)$ for all $x \in \mathcal{X}_N$. We begin with the following results for the sets $\mathbb{Z}$ and $\mathcal{Z}_N$.

Lemma 1: Let Assumptions 1–3 hold. Then, the sets $\mathbb{Z}$ and $\mathcal{Z}_N$ are closed.

Thus, despite the fact that these constraints are defined by a stochastic nonlinear system, we can still establish that relevant sets are indeed *closed* under basic regularity assumptions. Note that if we directly define $\mathbb{Z}$ through an approximation of $\tilde{\mathbb{Z}}_\varepsilon$, this approximation must be a closed set. Using Lemma 1, we establish that solutions to the SMPC optimization problem exist.

Proposition 2 (Existence of minima): Let Assumptions 1–3 hold. Then, for each $x \in \mathcal{X}_N$, the function $V_N(x, \cdot) : \mathbb{V}^N \to \mathbb{R}$ is continuous, the set $\mathcal{V}_N(x)$ is compact, and a solution to $\mathbb{P}_N(x)$ exists.

To properly discuss expected value or probability of stochastic systems, we must first establish that such properties are indeed well-defined. We define the control law mapping for SMPC as $K_N(x) := \pi(x, v^0(0; x))$ in which $v^0(0; x)$ is defined as the first parameter vector in $\mathbf{v}^0(x)$. Note that if there are multiple solutions to $\mathbb{P}_N(x)$, $K_N(x)$ may be a set-valued mapping instead of a single-valued function. Nonetheless, we typically assume there exists some selection rule that defines a single-valued control law $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ such that $\kappa_N(x) \in K_N(x)$ for all

$x \in \mathcal{X}_N$. The resulting closed-loop stochastic system is

$$x^+ = f_{cl}(x, w) := f(x, \kappa_N(x), w). \tag{7}$$

We denote the solution to (7) at time $k \in \mathbb{I}_{\geq 0}$ given the initial condition x and disturbance sequence $\mathbf{w}_k = (w(0), \ldots, w(k-1))$ as $\phi(k; x, \mathbf{w}_k)$.

We note, however, that continuity of $\pi(\cdot)$ does not imply continuity or measurability of $\kappa_N(\cdot)$, since $v^0(0; x)$ is not necessarily continuous or Borel measurable. Furthermore, if $\kappa_N(\cdot)$ is not Borel measurable, the system defined in (7) may produce a nonmeasurable closed-loop system, i.e., $\phi(k; x, \mathbf{w}_k)$ is not measurable w.r.t. $\mathbf{w}_k \in \mathbb{W}^k$. For a nonmeasurable stochastic system with uncountable set Ω, Lebesgue integrals are undefined and all stochastic properties of the system based on these integrals (e.g., expected value) are also undefined.

Fortunately, the regularity conditions presented in Assumptions 2 and 3 are sufficient to ensure that the control law mapping $K_N(x)$ and the optimal cost function are Borel measurable. We note that the proof of the subsequent proposition relies on the excellent work of [2, Prop. 7.33] on the measurability of stochastic optimal control problems.

Proposition 3: Let Assumptions 1–3 hold. Then, the function $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ is lower semicontinuous (Borel measurable), the set $\mathcal{X}_N$ is closed, and the set-valued mapping $\mathbf{v}^0 : \mathcal{X}_N \rightrightarrows \mathbb{V}^N$ is Borel measurable. Furthermore, the optimal control law mapping $K_N(x) : \mathcal{X}_N \rightrightarrows \mathbb{U}$, defined as $K_N(x) := \pi(x, v^0(0; x))$ is Borel measurable.

If $\mathbf{v}^0(x)$ is a single-valued mapping, i.e., there is a unique minimizer for the optimization problem for each $x \in \mathcal{X}_N$, then the $K_N(x)$ is single-valued and $\kappa_N(x) = K_N(x)$, is a single-valued, Borel measurable function. If instead, $\mathbf{v}^0(x)$ is a set-valued mapping, i.e., there are multiple solutions for the optimization problem for some $x \in \mathcal{X}_N$, then $K_N(x)$ may be a set-valued mapping as well. In this case, we apply a selection rule to define the single-valued control law $\kappa_N : \mathcal{X}_N \to \mathbb{U}$.

In theory, we could select an exotic selection rule that produces a nonmeasurable function $\kappa_N(x)$ from the Borel measurable set-valued mapping $K_N(x)$ (see [23, Appendix A]). We postulate that unintentionally constructing such a selection rule for a real system is unlikely. To avoid any potential issues, we make the following standing assumption.

Standing Assumption 1: We have chosen a Borel measurable selection rule $\Psi : (\mathcal{P}(\mathbb{U}) \setminus \emptyset) \to \mathbb{U}$ such that $\Psi(A) \in A$ for every $A \in (\mathcal{P}(\mathbb{U}) \setminus \emptyset)$ and defined $\kappa_N(x) := \Psi(K_N(x))$.[1]

With this assumption, we ensure that $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ is indeed a Borel measurable control law. This fact, combined with the continuity of $f(\cdot)$, guarantees that $f_{cl} : \mathbb{X} \times \mathbb{W} \to \mathbb{X}$ is also a Borel measurable function. However, before addressing the measurability of $\phi(k, x, \mathbf{w}_k)$, we must first establish that $\phi(k, x, \mathbf{w}_k)$ is well-defined for the iteration in (7). If $\phi(k; x, \mathbf{w}_k) \notin \mathcal{X}_N$, then $\kappa_N(\phi(k; x, \mathbf{w}_k))$ is not defined and the iteration is no longer defined. We establish that $\phi(\cdot)$ is well-defined with the following definition and lemma. We note that Lemma 4 requires Assumption 4 to ensure robust positive invariance of $\mathcal{X}_N$.

[1] [2, Lem. 7.18] guarantee that such a selection rule exists for any compact set $\mathbb{U}$.

Definition 1 (Robust positive invariance): The set $\mathcal{X}$ is said to be robustly positive invariant for the system $x^+ = F(x, w)$, $w \in \mathbb{W}$ if $x \in \mathcal{X}$ implies $x^+ \in \mathcal{X}$.

Lemma 4: Let Assumptions 1–4 hold. Then, $\mathcal{X}_N$ is robustly positive invariant for the system $x^+ = f_{cl}(x, w), w \in \mathbb{W}$ and the function $\phi(k; x, \mathbf{w}_k)$ is well-defined for all $x \in \mathcal{X}_N, \mathbf{w}_k \in \mathbb{W}^k$, and $k \in \mathbb{I}_{\geq 0}$.

Stochastic properties of interest are defined by Lebesgue integrals of lower bounded, Borel measurable functions of $\phi(k; x, \mathbf{w}_k)$. Therefore, if $\phi(k; x, \mathbf{w}_k)$ is Borel measurable for all $k \in \mathbb{I}_{\geq 0}$, then all the stochastic properties of interest are well-defined.

Proposition 5: Let Assumptions 1–4 hold. Then, the functions $\phi(k; x, \mathbf{w}_k(\omega))$ for all $k \in \mathbb{I}_{\geq 0}$ are Borel measurable w.r.t. the measure space $(\Omega, \mathcal{F}, P)$. Furthermore, the integral

$$\int_\Omega g(\phi(k; x, \mathbf{w}_k(\omega)))dP(\omega)$$

is well-defined for all $x \in \mathcal{X}_N, k \in \mathbb{I}_{\geq 0}$, and any lower bounded, Borel measurable function $g : \mathcal{X}_N \to \mathbb{R}$. Note that the functions $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ and $\ell(\cdot, \kappa_N(\cdot)) : \mathcal{X}_N \to \mathbb{R}$ are lower bounded and Borel measurable.

The proof of Proposition 5 is based on [10, Prop. 4]. Note that we require Assumption 4 only to ensure that $\mathcal{X}_N$ is robustly positive invariant. Any SMPC (or MPC) algorithm that can ensure robust positive invariant $\mathcal{X}_N$ and satisfies Assumptions 1–3 retains the properties of Proposition 5 as well. With this result in hand, we next present relevant stochastic properties of the closed-loop system.

IV. Optimal Cost Decrease in Expectation

We now establish that the optimal cost for SMPC satisfies a cost decrease inequality along the closed-loop trajectory based on the stage cost and the disturbance. We find the following results (based on [29, Lem. 14]) useful in the subsequent analysis.

Lemma 6: If $\alpha(\cdot) \in \mathcal{K}$, then for any $b \in \mathbb{R}_{\geq 0}$, there exists $\alpha_v(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_v(\cdot)$ is convex and $\alpha_v(s) \leq \alpha(s)$ for all $s \in [0, b]$.

Proof: We define

$$\alpha_v(s) := \frac{1}{b} \int_0^s \alpha(r)dr.$$

Immediately, we have that $\alpha_v(\cdot)$ is strictly increasing and unbounded as $s \to \infty$ since $\alpha(s) > 0$ for all $s > 0$. Since $\alpha(r)$ is continuous, we have that $\alpha_v(s)$ is continuous as well [34, Th. 6.20]. Thus, $\alpha_v \in \mathcal{K}_\infty$. The derivative of $\alpha_v(\cdot)$, i.e., $\frac{d\alpha_v}{ds}(s) = \alpha(s)/b$, is strictly increasing, and therefore, $\alpha_v(\cdot)$ is a convex function. Furthermore, we have

$$\alpha_v(s) = \frac{1}{b} \int_0^s \alpha(r)dr \leq \frac{1}{b} \int_0^s \alpha(s)dr = \frac{s}{b}\alpha(s) \leq \alpha(s)$$

for all $s \in [0, b]$. □

Corollary 7: If $\alpha(\cdot) \in \mathcal{K}_\infty$, then for any $b \in \mathbb{R}_{\geq 0}$, there exists $\alpha_c(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_c(\cdot)$ is concave and $\alpha(s) \leq \alpha_c(s)$ for all $s \in [0, b]$.

Proof: Note that the inverse of a $\mathcal{K}_\infty$ function is also a $\mathcal{K}_\infty$ function. We use Lemma 6 to construct a convex function $\alpha_v(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_v(r) \leq \alpha_2^{-1}(r)$ for all $r \in [0, \alpha_2^{-1}(b)]$.

Therefore, $\alpha_c(s) = \alpha_v^{-1}(s) \geq \alpha_2(s)$ for all $s \in [0, b]$. The inverse of a continuous, strictly increasing, and convex function is concave, and therefore, $\alpha_c(\cdot) \in \mathcal{K}_\infty$ and is a concave function. □

We also use a technical result from [1, Prop. 20].

Proposition 8: Let $C \subseteq D \subseteq \mathbb{R}^n$ with C compact and D closed. If $f : D \to \mathbb{R}^n$ is continuous, there exists $\alpha(\cdot) \in \mathcal{K}_\infty$ such that for all $x \in C$ and $y \in D$, we have that $|f(x) - f(y)| \leq \alpha(|x - y|)$.

We begin with the following result for the terminal region.

Lemma 9: Let Assumptions 1–4 hold. Then, there exists $\sigma(\cdot) \in \mathcal{K}$ such that

$$\int_{\mathbb{W}} V_f(f(x, \kappa_f(x), w)) d\mu(w) \leq V_f(x) - \ell(x, \kappa_f(x)) + \sigma\left(\text{tr}(\Sigma)\right)$$

for all $x \in \mathbb{X}_f$.

Proof: Since $V_f(\cdot)$, $f(\cdot)$, and $\kappa_f(\cdot)$ are continuous and $\mathbb{X}_f$ and $\mathbb{W}$ are bounded, we have from Proposition 8 that there exists $\alpha(\cdot) \in \mathcal{K}_\infty$ such that

$$|V_f(f(x, \kappa_f(x), w)) - V_f(f(x, \kappa_f(x), 0))| \leq \alpha(|w|)$$

for all $x \in \mathbb{X}_f$ and $w \in \mathbb{W}$. We can combine this inequality with (6) to give

$$V_f(f(x, \kappa_f(x), w)) \leq V_f(x) - \ell(x, \kappa_f(x)) + \alpha(|w|)$$

for all $x \in \mathbb{X}_f$ and $w \in \mathbb{W}$. Then, we apply Corollary 7 to construct a concave function $\alpha_c(\cdot) \in \mathcal{K}_\infty$ such that $\alpha(|w|) \leq \alpha_c(|w|)$ for all $w \in \mathbb{W}$ since $\mathbb{W}$ is bounded. We evaluate the Lebesgue integral of both sides of the inequality with respect to the probability space $(\mathbb{W}, \mathcal{B}(\mathbb{W}), \mu)$ and apply Jensen's inequality to give

$$\int_{\mathbb{W}} V_f(f(x, \kappa_f(x), w)) d\mu(w) \leq V_f(x) - \ell(x, \kappa_f(x)) + \alpha_c\left(\mathbb{E}\left[|w|\right]\right). \tag{8}$$

From Jensen's inequality, we can write $\mathbb{E}[|w|]^2 \leq \mathbb{E}[|w|^2] = \text{tr}(\Sigma)$. We define $\sigma(s) := \alpha_c(s^{1/2})$ and note that $\sigma(\cdot) \in \mathcal{K}$ because $\alpha_c(\cdot)$ and $s^{1/2}$ are $\mathcal{K}$-functions. Thus, we have that $\alpha_c(\mathbb{E}[|w|]) = \sigma(\mathbb{E}[|w|]^2) \leq \sigma(\text{tr}(\Sigma))$ and substitute this inequality into (8) to complete the proof. □

This result is similar to ISS results for continuous Lyapunov functions. However, the application to stochastic systems presented here is novel. In particular, the direct relation between the bound in Lemma 9 and the variance of the disturbance (tr(Σ)) is, to the best of our knowledge, entirely new for nonlinear SMPC. Typically, for nonlinear SMPC, the term $\sigma(\text{tr}(\Sigma))$ is treated as a fixed constant and any connection to the probability distribution is ignored [6], [20], [22]. A more familiar result is achieved using a common choice of the disturbance model and terminal cost function.

Lemma 10: Let Assumptions 1–4 hold with $f(x, u, w) := g(x, u) + w$ and $V_f(x) := x'Px$ for positive semidefinite P. Then, for all $x \in \mathbb{X}_f$, we have

$$\int_{\mathbb{W}} V_f(f(x, \kappa_f(x), w)) d\mu(w) \leq V_f(x) - \ell(x, \kappa_f(x)) + \text{tr}(P\Sigma).$$

The proof of Lemma 10 is simple, and therefore, omitted. The term tr($P\Sigma$) appears in the exact same form for the *linear* SMPC problem as well [19]. We also note that for this system and terminal cost, the bound tr($P\Sigma$) is often the tightest bound possible that is also independent of x (e.g., consider any terminal control law such that $x = 0$ and $\kappa_f(0) = 0$).

In either case, the implications of this bound are clear: the distribution of w, specifically the variance of w, determines the size of this bound. As tr(Σ) $\to 0$, i.e., the variance of w approaches zero, we know that $\sigma(\text{tr}(\Sigma)) \to 0$ and we recover the nominal cost decrease condition for the terminal region. Analogous to the nominal MPC problem, we now extend this result in the terminal region to the entire set $\mathcal{X}_N$ for the optimal cost function $V_N^0(\cdot)$.

Proposition 11: Let Assumptions 1–4 hold. Then, the set $\mathcal{X}_N$ is robustly positive invariant for the system $x^+ = f_{cl}(x, w), w \in \mathbb{W}$ and there exists $\sigma(\cdot) \in \mathcal{K}$ such that

$$\int_{\mathbb{W}} V_N^0(f_{cl}(x, w)) d\mu(w) \leq V_N^0(x) - \ell(x, \kappa_N(x)) + \sigma(\text{tr}(\Sigma))$$

for all $x \in \mathcal{X}_N$.

Proof: If $x \in \mathcal{X}_N$, we have that for $\mathbf{v}^0 \in \mathbf{v}^0(x)$ and all $\mathbf{w} := (w(0), w(1), \ldots, w(N-1)) \in \mathbb{W}^N$, $x(N, \mathbf{w}) = \hat{\phi}(N; x, \mathbf{v}^0, \mathbf{w}) \in \mathbb{X}_f$ and

$$f(x(N, \mathbf{w}), \kappa_f(x(N, \mathbf{w})), w(N)) \in \mathbb{X}_f$$

for all $w(N) \in \mathbb{W}$ by Assumption 4. Thus, the candidate trajectory

$$\tilde{\mathbf{v}}^+ = \left(v^0(1), v^0(2), \ldots, v^0(N-1), 0\right)$$

satisfies $\tilde{\mathbf{v}}^+ \in \mathcal{V}_N(x^+)$ for $x^+ = f(x, \kappa_N(x), w(0))$ and all $w(0) \in \mathbb{W}$. Since $\mathcal{V}_N(x^+)$ is nonempty, $x^+ \in \mathcal{X}_N$, and $\mathcal{X}_N$ is robustly positive invariant. Letting

$$\tilde{\mathbf{w}}^+ = (w(1), w(2), \ldots, w(N-1), w(N))$$

and using the definition of $J_N(\cdot)$, we obtain

$$J_N(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}^+) = J_N(x, \mathbf{v}^0, \mathbf{w}) - \ell(x, \kappa_N(x)) + \eta(x(N, \mathbf{w}), w(N)) \tag{9}$$

in which

$$\eta(x, w) := -V_f(x) + \ell(x, \kappa_f(x)) + V_f(f(x, \kappa_f(x), w)).$$

From Lemma 9 and the fact that $x(N, \mathbf{w}) \in \mathbb{X}_f$, we have that

$$\int_{\mathbb{W}^{N+1}} \eta(x(N, \mathbf{w}), w(N)) d\mu^N(\mathbf{w}) d\mu(w(N)) \leq \sigma\left(\text{tr}(\Sigma)\right).$$

We also have the following equality:

$$\int_{\mathbb{W}^{N+1}} J_N(x, \mathbf{v}^0, \mathbf{w}) d\mu^N(\mathbf{w}) d\mu(w(N)) = V_N^0(x).$$

And by optimality, we have that

$$V_N^0(x^+) \leq \int_{\mathbb{W}^N} J_N(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}^+) d\mu(w(1)) \ldots d\mu(w(N)).$$

We combine these inequalities with (9) to give

$$\int_{\mathbb{W}} V_N^0(x^+)d\mu(w(0))$$
$$\leq \int_{\mathbb{W}^{N+1}} J_N(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}^+)d\mu^N(\mathbf{w})d\mu(w(N))$$
$$\leq V_N^0(x) - \ell(x, \kappa_N(x)) + \sigma\left(\text{tr}(\Sigma)\right).$$

Substitute $x^+ = f(x, \kappa_N(x), w)$ and let $w = w(0)$ to complete the proof. □

We emphasize that Proposition 11 is valid only if the probability distribution (and therefore, Σ) used in the SMPC optimization problem is identical to the true probability distribution of the system. If the distribution of the underlying stochastic system is not identical to the distribution used in the disturbance model, these results do not necessarily hold.

We may apply the cost decrease of Proposition 11 to establish the following result.

Theorem 12: Let Assumptions 1–4 hold. Then, for the closed-loop system $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$, the set $\mathcal{X}_N$ is robustly positive invariant and there exists $\sigma(\cdot) \in \mathcal{K}$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}\left[\ell(x(k), u(k))\right] \leq \sigma(\text{tr}(\Sigma)) \tag{10}$$

for all $x \in \mathcal{X}_N$ in which $x(k) := \phi(k; x, \mathbf{w}_k)$ and $u(k) := \kappa_N(x(k))$.

Proof: From Lemma 4, we know that $\mathcal{X}_N$ is robustly positive invariant for the closed-loop system. From Proposition 5, we know that all subsequent integrals are well-defined.

For any $x \in \mathcal{X}_N$, let $x(k) := \phi(k; x, \mathbf{w}_k)$ and $u(k) := \kappa_N(x(k))$. From Proposition 11 and the definition of conditional expectation, we have that

$$\mathbb{E}\left[V_N^0(x(k+1)) \mid \mathcal{F}_k\right] \leq V_N^0(x(k))$$
$$- \ell(x(k), u(k)) + \sigma\left(\text{tr}(\Sigma)\right).$$

By the law of total expectation, we have

$$\mathbb{E}\left[V_N^0(x(k+1))\right] \leq \mathbb{E}\left[V_N^0(x(k))\right] - \mathbb{E}\left[\ell(x(k), u(k))\right]$$
$$+ \sigma\left(\text{tr}(\Sigma)\right).$$

We take the summation of each side from $k = 0$ to $T - 1$ and divide by T to give

$$\frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}\left[\ell(x(k), u(k))\right] \leq \sigma\left(\text{tr}(\Sigma)\right)$$
$$+ \frac{V_N^0(x) - \mathbb{E}\left[V_N^0(x(T))\right]}{T}.$$

Since $\ell(\cdot)$ is lower bounded, we have that there exists $M \in \mathbb{R}$ such that $V_N(x, \mathbf{v}) \geq M$ for all $(x, \mathbf{v}) \in \mathcal{Z}_N$. Therefore, $-\mathbb{E}_{|x}[V_N^0(x(T))] \leq -M$ as well. We apply this bound and evaluate the $\limsup_{T\to\infty}$ of each side to give (10). □

Note that $\limsup$ is used instead of $\lim$, as the limit may not exist. From Lemma 10, we can replace $\sigma(\text{tr}(\Sigma))$ with $\text{tr}(P\Sigma)$ if we have an additive disturbance model and a quadratic terminal cost function.

V. Stochastic Asymptotic Stability of SMPC

In this section, we specialize SMPC to tracking problems (with Assumption 5) and establish RASiE for the closed-loop system.

A. Robust Asymptotic Stability in Expectation (RASiE)

Definition 2 (RASiE): The origin is RASiE for the stochastic system $x^+ = f_{\text{cl}}(x, w)$, $w \in \mathbb{W}$ on the robustly positive invariant set $\mathcal{X}_N$ if there exist $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that the closed-loop trajectory satisfies

$$\mathbb{E}\left[|\phi(k; x, \mathbf{w}_k)|\right] \leq \beta(|x|, k) + \gamma\left(\text{tr}(\Sigma)\right) \tag{11}$$

for all $x \in \mathcal{X}_N$ and $k \in \mathbb{I}_{\geq 0}$.

In contrast to robust asymptotic stability or ISS for deterministic closed-loop systems, RASiE bounds the expected value of the norm of the closed-loop state based on the initial condition x and variance of the disturbance, i.e., $\text{tr}(\Sigma)$. Clearly, this upper bound implies the typical notion of stability in expectation that [22] establish. We note, however, that RASiE also ensures that the effect of the initial condition $|x|$ on the upper bound decays toward zero as $k \to \infty$.

To establish that a closed-loop system satisfies this condition, we use an SISS-Lyapunov function similar to the SISS-Lyapunov functions in the nonlinear stochastic stability theory. Note that we do not require continuity of $f_{\text{cl}}(\cdot)$ or an exponential cost decrease. Furthermore, $\sigma_2(\cdot)$ and $\sigma_3(\cdot)$ are both functions of $\text{tr}(\Sigma)$.

Definition 3 (SISS-Lyapunov Function): The Borel measurable function $V : \mathcal{X}_N \to \mathbb{R}_{\geq 0}$ is an SISS-Lyapunov function on the robustly positive invariant set $\mathcal{X}_N$ for the stochastic system $x^+ = f_{\text{cl}}(x, w)$, $w \in \mathbb{W}$, if there exist $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\sigma_2(\cdot), \sigma_3(\cdot) \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) + \sigma_2\left(\text{tr}(\Sigma)\right)$$
$$\int_{\mathbb{W}} V(f_{\text{cl}}(x, w))d\mu(w) \leq V(x) - \alpha_3(|x|) + \sigma_3\left(\text{tr}(\Sigma)\right)$$

for all $x \in \mathcal{X}_N$.

We can establish the following result for SISS-Lyapunov functions.

Proposition 13: If a system $x^+ = f_{\text{cl}}(x, w)$, $w \in \mathbb{W}$ admits an SISS-Lyapunov function on the robustly positive invariant and bounded set $\mathcal{X}_N$, then the origin is RASiE.

Proof: To streamline notation, we define $d := \text{tr}(\Sigma)$ and note that d is a constant for the stochastic system. Define $\alpha_4(s) := \alpha_3(\alpha_2^{-1}(s/2))$ and note that $\alpha_4(\cdot) \in \mathcal{K}_\infty$ and $\alpha_4(s) \leq s$ for all $s \in \mathbb{R}_{\geq 0}$ because $\alpha_3(s) \leq \alpha_2(s)$ for all $s \in \mathbb{R}_{\geq 0}$.[2] We have the following inequality:

$$\alpha_4(V(x)) \leq \alpha_4(\alpha_2(|x|) + \sigma_2(d))$$
$$\leq \alpha_4(2\alpha_2(|x|)) + \alpha_4(2\sigma_2(d))$$
$$= \alpha_3(|x|) + \alpha_4(2\sigma_2(d)).$$

[2] If this inequality does not hold, we simply construct a new $\alpha_2(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_3(s) \leq \alpha_2(s)$ for all $s \in \mathbb{R}_{\geq 0}$.

By rearranging, we have $-\alpha_3(|x|) \leq -\alpha_4(V(x)) + \alpha_4(2\sigma_2(d))$.

Choose $x \in \mathcal{X}_N$ and let $x(k) := \phi(k; x, \mathbf{w}_k)$ for all $k \in \mathbb{I}_{\geq 0}$. Then, we have

$$\mathbb{E}\left[V(x(k+1)) \mid \mathcal{F}_k\right] \leq V(x(k)) - \alpha_4(V(x(k))) + \sigma_4(d)$$

in which $\sigma_4(d) := \alpha_4(2\sigma_2(d)) + \sigma_3(d)$ and $\sigma_4(\cdot) \in \mathcal{K}$. Since $\mathcal{X}_N$ is compact and $V(x) \leq \alpha_2(|x|) + \sigma_2(d)$, there exists $b \geq 0$ such that $V(x) \leq b$ for all $x \in \mathcal{X}_N$. From Lemma 6, we can construct $\alpha_v \in \mathcal{K}_\infty$ such that $\alpha_v(\cdot)$ is convex and $\alpha_v(V(x)) \leq \alpha_4(V(x))$ for all $x \in \mathcal{X}_N$. Therefore, we can replace $\alpha_4(\cdot)$ with $\alpha_v(\cdot)$, apply the law of total expectation, and use Jensen's inequality to give

$$\mathbb{E}\left[V(x(k+1))\right] \leq \mathbb{E}\left[V(x(k))\right] - \alpha_v(\mathbb{E}\left[V(x(k))\right]) + \sigma_4(d).$$

Define $\tilde{\gamma}(d) := 2\max\{\alpha_v^{-1}(\sigma_4(d)), \sigma_4(d)\}$ and note that $\tilde{\gamma}(\cdot) \in \mathcal{K}$. If $\mathbb{E}[V(x(k))] \leq \tilde{\gamma}(d)/2$, then

$$\begin{aligned}\mathbb{E}\left[V(x(k+1))\right] &\leq \tilde{\gamma}(d)/2 + \sigma_4(d)\\ &\leq \tilde{\gamma}(d)/2 + \tilde{\gamma}(d)/2 = \tilde{\gamma}(d).\end{aligned}$$

If $\tilde{\gamma}(d)/2 \leq \mathbb{E}[V(x(k))] \leq \tilde{\gamma}(d)$, then

$$\begin{aligned}\mathbb{E}\left[V(x(k+1))\right] &\leq \mathbb{E}\left[V(x(k))\right] - \alpha_v\left(\tilde{\gamma}(d)/2\right) + \sigma_4(d)\\ &\leq \mathbb{E}\left[V(x(k))\right] \leq \tilde{\gamma}(d).\end{aligned}$$

Thus, for $\mathbb{E}[V(x(k))] \leq \tilde{\gamma}(d)$, we know that $\mathbb{E}[V(x(k+1))] \leq \tilde{\gamma}(d)$.

If $\mathbb{E}[V(x(k))] \geq \tilde{\gamma}(d)$, we have

$$\begin{aligned}&\mathbb{E}\left[V(x(k+1))\right]\\ &\leq \mathbb{E}\left[V(x(k))\right] - \alpha_v(\mathbb{E}\left[V(x(k))\right]) + \alpha_v(\mathbb{E}\left[V(x(k))\right]/2)\\ &\leq \lambda_1\left(\mathbb{E}\left[V(x(k))\right]\right)\end{aligned}$$

in which $\lambda_1(s) := s - \alpha_v(s) + \alpha_v(s/2)$. We have that $\lambda_1(\cdot)$ is continuous, $\lambda_1(0) = 0$, and $\lambda_1(s) < s$ for all $s > 0$. By the same process used in [32, Th. B.15], we construct $\lambda(\cdot) \in \mathcal{K}_\infty$ such that $\lambda_1(s) \leq \lambda(s) < s$ for $s > 0$. Thus, we have

$$\mathbb{E}[V(x(k+1))] \leq \lambda(\mathbb{E}[V(x(k))]). \tag{12}$$

Repeated application of the aforementioned equation and the fact that $\mathbb{E}[V(x(0))] = V(x)$ gives

$$\mathbb{E}[V(x(k+1))] \leq \tilde{\beta}(V(x), k) := \lambda^k(V(x))$$

in which $\lambda^k(\cdot)$ is the composition of $\lambda(\cdot)$ with itself k times. Using the same approach as [32, Th. B.15], we conclude that $\tilde{\beta}(\cdot) \in \mathcal{KL}$. We have that $\mathbb{E}[V(x(k))] \leq \tilde{\beta}(V(x), k)$ if $\mathbb{E}[V(x(k))] \geq \tilde{\gamma}(d)$ and $\mathbb{E}[V(x(k+1))] \leq \tilde{\gamma}(d)$ if $\mathbb{E}[V(x(k))] \leq \tilde{\gamma}(d)$. Therefore, for all $x \in \mathcal{X}_N$, we have

$$\mathbb{E}\left[V(x(k))\right] \leq \max\{\tilde{\beta}(V(x), k), \tilde{\gamma}(d)\}.$$

Using Lemma 6 and the fact that $\mathcal{X}_N$ is compact, we can construct a convex function $\alpha_{1,v}(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_{1,v}(|x|) \leq \alpha_1(|x|) \leq V(x)$ for all $x \in \mathcal{X}_N$. Thus, we may apply Jensen's inequality to give

$$\alpha_{1,v}(\mathbb{E}\left[|x(k)|\right]) \leq \mathbb{E}\left[\alpha_{1,v}(|x(k)|)\right] \leq \mathbb{E}\left[V(x(k))\right].$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\left[|x(k)|\right] &\leq \max\left\{\alpha_{1,v}^{-1}\left(\tilde{\beta}(V(x), k)\right), \alpha_{1,v}^{-1}\left(\tilde{\gamma}(d)\right)\right\}\\ &\leq \beta_1(V(x), k) + \alpha_{1,v}^{-1}\left(\tilde{\gamma}(d)\right)\end{aligned}$$

in which $\beta_1(\cdot) := \alpha_{1,v}^{-1} \circ \tilde{\beta}(\cdot) \in \mathcal{KL}$. We use the upper bound for $V(x)$ to give

$$\begin{aligned}\mathbb{E}\left[|x(k)|\right] &\leq \beta_1(\alpha_2(|x|) + \sigma_2(d), k) + \alpha_{1,v}^{-1}\left(\tilde{\gamma}(d)\right)\\ &\leq \beta_1(2\alpha_2(|x|), k) + \beta_1(2\sigma_2(d), k) + \alpha_{1,v}^{-1}\left(\tilde{\gamma}(d)\right)\\ &\leq \beta(|x|, k) + \gamma(d)\end{aligned}$$

in which $\beta(s, k) := \beta_1(2\alpha_2(s), k) \in \mathcal{KL}$, and $\gamma(d) := \beta_1(2\sigma_2(d), 0) + \alpha_{1,v}^{-1}(\tilde{\gamma}(d)) \in \mathcal{K}$. □

B. RASiE of SMPC

Analogous to the stability analysis for nominal MPC, we intend to use the optimal cost of the SMPC problem as an SISS-Lyapunov function. We already established the desired cost decrease bound in Proposition 11 and the stage cost bound from Assumption 5 provides a lower bound for the optimal cost function. Therefore, we focus on constructing the upper bound for $V_N^0(\cdot)$.

Lemma 14: Let Assumptions 1–5 hold. Then, there exist $\alpha_2(\cdot) \in \mathcal{K}_\infty$ and $\sigma_2(\cdot) \in \mathcal{K}$ such that $V_N^0(x) \leq \alpha_2(|x|) + \sigma_2(\mathrm{tr}(\Sigma))$ for all $x \in \mathcal{X}_N$.

Proof: We choose $x \in \mathbb{X}_f$ and consider the trajectory generated by repeated application of the terminal control law, i.e., $x(k) := \hat{\phi}(k; x, \mathbf{0}, \mathbf{w})$ since $\pi(x, 0) = \kappa_f(x)$. The set $\mathbb{X}_f$ is robustly positive invariant for this control law due to Assumption 4, and therefore, $x(k) \in \mathbb{X}_f$ for all $k \in \mathbb{I}_{\geq 0}$. We define $d := \mathrm{tr}(\Sigma)$. From Assumption 4 and Proposition 11, we have for all $k \in \mathbb{I}_{[0,N-1]}$ that

$$\begin{aligned}&\int_{\mathbb{W}^N} \left(V_f(x(k+1)) - V_f(x(k))\right) d\mu^N(\mathbf{w})\\ &\quad\leq -\int_{\mathbb{W}^N} \ell(x(k), \kappa_f(x(k))) d\mu^N(\mathbf{w}) + \sigma(d).\end{aligned}$$

We sum both sides of the inequality from $k = 0$ to $k = N-1$ to give

$$\begin{aligned}&\int_{\mathbb{W}^N} \left(V_f(x(N)) - V_f(x(0))\right) d\mu^N(\mathbf{w})\\ &\quad\leq -\int_{\mathbb{W}^N} \sum_{k=0}^{N-1} \ell(x(k), \kappa_f(x(k))) d\mu^N(\mathbf{w}) + N\sigma(d).\end{aligned}$$

By rearranging and substituting in the definition of $J_N(\cdot)$ and $x(0) = x$, we have

$$\int_{\mathbb{W}^N} J_N(x, 0, \mathbf{w}) d\mu^N(\mathbf{w}) \leq V_f(x) + N\sigma(d)$$

for all $x \in \mathbb{X}_f$. By optimality, we know that $V_N^0(x) \leq V_f(x) + N\sigma(d)$. From Assumption 2 and [31, Prop. 14], there exists $\alpha_f(\cdot) \in \mathcal{K}$ such that $V_f(x) \leq \alpha_f(|x|)$, and therefore, $V_N^0(x) \leq \alpha_f(|x|) + N\sigma(d)$ for all $x \in \mathbb{X}_f$.

We now establish that $V_N^0(x)$ is locally bounded on $\mathcal{X}_N$. Let X be any arbitrary compact subset of $\mathcal{X}_N$. The function $J_N : \mathbb{X} \times \mathbb{V}^N \times \mathbb{W}^N$ is continuous, and therefore, has an upper and lower bound on the compact set $X \times \mathbb{V}^N \times \mathbb{W}^N$. Since $\mathcal{V}_N(x) \subseteq \mathbb{V}^N$ for all $x \in \mathcal{X}_N$, $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ must satisfy the same upper and lower bound on X. Therefore, $V_N^0(\cdot)$ is locally bounded on $\mathcal{X}_N$.

To extend this upper bound, we define a new function $W(x) := \max\{V_N^0(x) - N\sigma(d), 0\}$ and note that $W(x) \geq 0$ and $W(x) \leq \alpha_f(|x|)$ for all $x \in \mathbb{X}_f$. Since $\mathbb{X}_f$ contains the origin in its interior, $W(x)$ is continuous at $x = 0$. Furthermore, we know that $W(0) = 0$, $\mathcal{X}_N$ is closed (Proposition 3), and $W(x)$ is locally bounded (since $V_N^0(x)$ is locally bounded). Therefore, [31, Prop. 14] applies and there exists $\alpha_2(\cdot) \in \mathcal{K}_\infty$ such that $W(x) \leq \alpha_2(|x|)$ for all $x \in \mathcal{X}_N$. We have that

$$V_N^0(x) - N\sigma(d) \leq W(x) \leq \alpha_2(|x|)$$

and we define $\sigma_2(d) = N\sigma(d)$ to give the desired upper bound. □

Note that this upper bound increases with the horizon length of the optimization problem. As an alternative to Lemma 14, we may assume that $V_N^0(\cdot)$ is continuous at the origin. With this alternate assumption, we can find $\alpha_2(\cdot) \in \mathcal{K}_\infty$ such that

$$V_N^0(x) \leq \alpha_2(|x|) + V_N^0(0).$$

Note, however, that $V_N^0(0)$ is not necessarily zero for SMPC if the stage cost is positive definite (as required by Assumption 5). Only in specific situations, e.g., multiplicative disturbance models, is $V_N^0(0) = 0$. Furthermore, we expect the value of $V_N^0(0)$ to also increase with increasing N similar to the bound we derived in Lemma 14. We propose, however, that this increase with the horizon length is not a weakness of the analysis approach, but an underlying characteristic of SMPC, particularly, for nonlinear systems.

Next, we establish the main result of this article.

Theorem 15: Let Assumptions 1–5 hold. Then, the origin is RASiE for the stochastic system $x^+ = f_{cl}(x, w), w \in \mathbb{W}$ on the robustly positive invariant set $\mathcal{X}_N$.

Proof: We establish this result by showing that $V_N^0(x)$ is an SISS-Lyapunov function. From Assumption 5, we have that $\alpha_\ell(|x|) \leq \ell(x, u) \leq V_N^0(x)$. From Lemma 14, we have the upper bound. From Proposition 11, we have the cost decrease condition. Thus, we apply Proposition 13 to complete the proof. □

C. Discussion

Although similar to the results of [20] and [22], we note a few key differences. Most significantly, the proof of Theorem 15 does not require an exponential decrease in the expected value of the optimal cost along the closed-loop trajectory. Instead, we use Jensen's inequality to move the expected value operator *within* the $\mathcal{K}$-functions. We also note that RASiE provides a specific upper bound for the expected value of the norm of the state. In the definition of RASiE, the effect of the initial condition on the upper bound asymptotically (and uniformly) decreases to zero as $k \to \infty$. The remaining term is independent of k and depends on the distribution of the disturbance w (i.e., $\text{tr}(\Sigma)$). In particular, if $\text{tr}(\Sigma) = 0$, i.e., a nominal MPC algorithm applied to a nominal closed-loop system, we recover the asymptotic stability result typical of nominal MPC.

We note the (intentional) similarity between this result and the results in [1] for the inherent robustness of nominal MPC. There is, however, a key distinction; the result for SMPC requires that we have exact knowledge of the disturbance that is affecting the system. Consequently, if the plant follows the model in (1), but with a disturbance probability distribution other than what is assumed in the SMPC optimization problem, then the bound in (11) does not necessarily hold for the variance of either probability distribution. For example, if we design an SMPC algorithm assuming $\text{tr}(\Sigma) = 1$ and the disturbances actual distribution has a value of $\text{tr}(\Sigma) = 0.5$, it is not clear how the bound in (11) changes. The robustness of nonlinear SMPC to unmodeled or incorrectly modeled disturbances is an open question.

Goulart and Kerrigan [9] establish that for linear systems with additive disturbances, SMPC produces an ISS closed-loop system. Thus, if the true system experiences $w = 0$, SMPC stabilizes the origin regardless of the probability distribution used in constructing the SMPC controller. But this result relies on many properties of the linear problem that do not extend to the nonlinear case (e.g., convexity and optimality of the terminal control law). For nonlinear SMPC, if we assume that $\text{tr}(\Sigma) > 0$ and the closed-loop system is in fact nominal (i.e., $w = 0$ and $\text{tr}(\Sigma) = 0$), the resulting stochastic controller may not stabilize the origin.

Even if we have exact knowledge of the disturbance distribution, there are still clear differences between the strength of results for nonlinear and linear stochastic MPC. Lorenzen *et al.* [19] establish that linear SMPC, under suitable assumptions, stabilizes the terminal region with probability one. For nonlinear SMPC, however, this property simply does not hold (see subsequent example). Indeed, nonlinear SMPC may converge in expected value to a point outside of the terminal region even if we initiate the system within the terminal region. In the subsequent section, we demonstrate these characteristics of nonlinear SMPC through an example.

VI. Example

Consider the following nonlinear discrete-time system:

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1^2 x_2 + x_1(1 - x_2) \\ 0.9x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w \\ 0 \end{bmatrix}$$

in which the probability distribution of the disturbance is given by $\text{Pr}(w = 0.5) = \text{Pr}(w = -0.5) = 0.25$ and $\text{Pr}(w = 0) = 0.5$. We consider the constraints

$$\begin{bmatrix} -3 \\ -0.1 \end{bmatrix} \leq x \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -20 \\ -0.1 \end{bmatrix} \leq u \leq \begin{bmatrix} 20 \\ 0.1 \end{bmatrix}$$

and stage cost $\ell(x, u) = x'Qx + u'Ru$ with $Q = R = \text{diag}([10, 0.1])$.

To construct the terminal cost/constraint, we linearize at $(x_s, u_s) = (0, 0)$ and find the LQR cost P and gain K assuming the inflated stage cost $Q_{\text{LQR}} = 1.1Q$ and $R_{\text{LQR}} = R$. We define the terminal cost as $V_f(x) := x'Px$ and terminal control law as

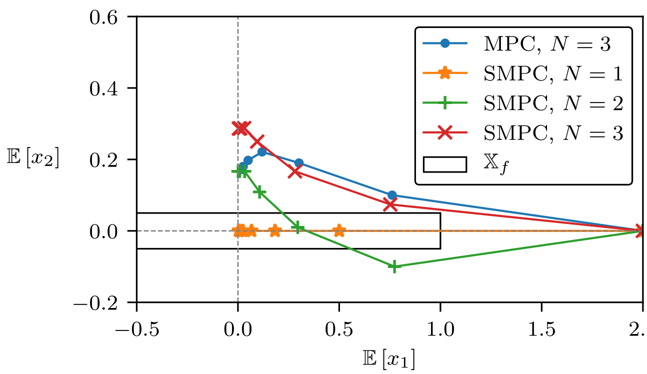

Fig. 1. Expected value of each state variable, i.e., $\mathbb{E}[x_1]$ and $\mathbb{E}[x_2]$ for each closed-loop stochastic trajectory. The points for each line correspond to different time steps starting with $x(0) = [2, 0]'$.

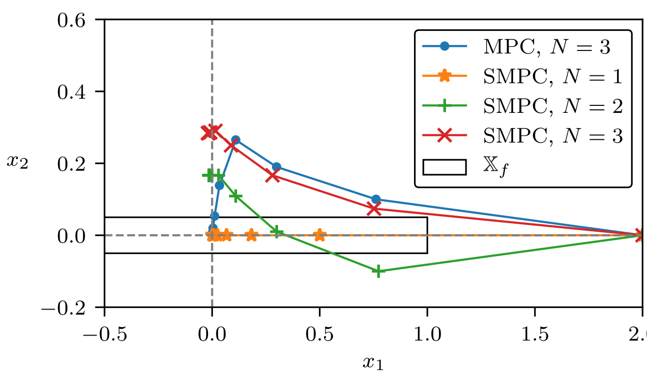

Fig. 2. State for each closed-loop trajectory if $\mathbf{w} = \mathbf{0}$. The points for each line correspond to different time steps starting with $x(0) = [2, 0]'$.

$\kappa_f(x) = Kx$. We select the terminal constraint

$$\begin{bmatrix} -1 \\ -0.1 \end{bmatrix} \leq x \leq \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}.$$

We confirm both requirements of Assumption 4 for this terminal region. We parameterize the control policy as $\pi(x, v) := Kx + v$ and select $\mathbb{V}$ such that for all $(x, u) \in \mathbb{Z}$, there exists $v \in \mathbb{V}$ such that $\pi(x, v) = u$. Note that Assumptions 1–3 and 5 are also satisfied by this formulation.

We simulate the closed-loop response of this stochastic system subject to nominal MPC and SMPC controllers with varying horizon lengths. We initialize the system at $x(0) = [2, 0]'$. Because the disturbance can take only three possible values, all expected values in the subsequent plots are calculated exactly through a scenario tree approach and are not based on sample averages.

We plot the expected value of the state in Fig. 1. We note that SMPC with a horizon length of $N = 1$ drives the expected value of the state to the origin. However, as we increase the horizon length of SMPC, the value of $\mathbb{E}[x_2]$ leaves the terminal region and increases with the increasing horizon length. Nonlinear SMPC with $N \geq 2$ does not stabilize the terminal region (with probability one) as we might expect for linear SMPC or ISS nonlinear stochastic systems [19], [26]. In Fig. 2, we plot the closed-loop trajectory for each algorithm if the realized system is in fact nominal (i.e., $\mathbf{w} = \mathbf{0}$). For nonlinear SMPC with $N \geq 2$, the origin and/or the terminal set are in fact *not* asymptotically stable for the nominal closed-loop system.

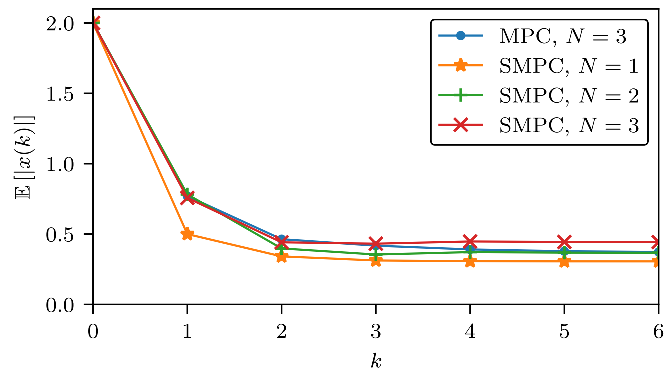

Fig. 3. Expected value of the norm of the state, i.e., $\mathbb{E}[|x(k)|]$, for each closed-loop stochastic system.

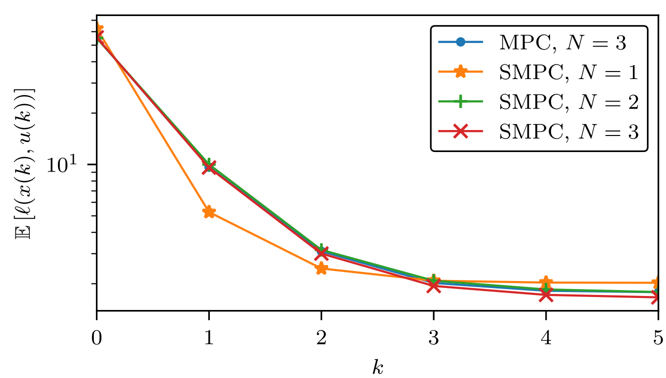

Fig. 4. Expected value of the stage cost, i.e., $\mathbb{E}[\ell(x(k), u(k))]$, for each closed-loop stochastic system.

In Fig. 3, we plot the values of $\mathbb{E}[|x(k)|]$ for the closed-loop stochastic system. We observe for all values of N that the SMPC algorithm produces results consistent with Theorem 15. However, we note that the value of $\mathbb{E}[|x(k)|]$ as $k \to \infty$ *increases* with increasing N. Thus, the dependence of $\sigma_2(\cdot)$ in Lemma 14 on the horizon length N appears to indicate an underlying characteristic of nonlinear SMPC and is not necessarily a shortcoming of the analysis approach used in this article.

In Fig. 4, we plot the expected value of the stage cost at each iteration of the closed-loop system. We observe that the performance of SMPC, in terms of the stage cost, improves with increasing horizon length. Thus, driving the system toward larger values of x_2 improves the expected cost of the closed-loop system after an initial transient. The reason for this behavior is that larger values of x_2 produce a system that is more robust to the disturbance. Since we apply a significantly larger cost to x_1, u_1 than x_2, u_2, using larger values of x_2 to guard against the disturbance is advantageous from a stochastic perspective. Therefore, the closed-loop behavior of SMPC for $N \geq 2$, although undesirable from the perspective of a tracking problem, is appropriate based on the chosen stage costs. Choosing different values for Q and R can result in significantly different behavior of the closed-loop system. For example, selecting $Q = R = I$ results in significantly smaller values of $\mathbb{E}[|x(k)|]$ as $k \to \infty$ and $\mathbb{E}[x(k)]$ enters and remains in the terminal region.

VII. Conclusion

In this article, we established several basic properties of the nonlinear SMPC problem including existence of optimal solutions and measurability of the closed-loop trajectory. We used an SISS-Lyapunov function to establish that SMPC, under suitable assumptions, is robustly asymptotically stable in expectation. In particular, we established that the bound on expected value of the closed-loop state is directly related to the variance of the disturbance. Thus, as the variance approaches zero, we recover the nominal asymptotic stability guarantee typical of MPC. This result is informative, but is also significantly weaker than the extent of results available for linear SMPC. We present an example to illustrate that nonlinear SMPC, at least under this set of assumptions, does not guarantee that the closed-loop system is ISS or that the state converges to the terminal region. Furthermore, increasing the horizon length for nonlinear SMPC, may actually increase the value of $\mathbb{E}[|x(k)|]$ as $k \to \infty$.

If economic performance objectives are used and tracking/stability is not required, SMPC appears to offer an excellent method to include available stochastic information in the optimization problem. If tracking/stability of a setpoint (origin) is desired, nonlinear SMPC may produce nonintuitive, and in some cases undesirable, closed-loop systems despite satisfaction of reasonable assumptions. This nonintuitive behavior, however, is primarily the result of optimizing a stochastic property of the tracking cost function, i.e., $\mathbb{E}[J_N(\cdot)]$, and not the probabilistic constraints. Therefore, using stochastic information to construct the set $\mathcal{Z}_N$ and optimizing over a nominal cost function, i.e., $J_N(x, \mathbf{v}, \mathbf{0})$, may offer a desirable compromise between the benefits of SMPC and the more intuitive behavior of nominal MPC.

Appendix

Here, we present the proofs omitted from Section III. Throughout the Appendix, we find the following technical result useful.

Lemma 16: Let $f : X \times S \to \mathbb{R}$ be a Borel measurable function defined for $X \subseteq \mathbb{R}^n$ and the probability space (S, Σ, μ). Then, the function $F : X \to \mathbb{R}$ defined by the Lebesgue integral

$$F(x) := \int_S f(x, s) d\mu(s)$$

satisfies the following:

1) if $f(x, s)$ is lower bounded and lower semicontinuous w.r.t. $x \in X$, then $F(x)$ is lower semicontinuous;
2) if $f(x, s)$ is continuous w.r.t. $x \in X$ and uniformly bounded for all $(x, s) \in X \times S$, then $F(x)$ is finite and continuous.

Proof: Fix $x \in X$ and let $(x_n)_{n=1}^{\infty}$ be any sequence of real numbers that converges to x, i.e., $\lim_{n\to\infty} x_n = x$. We define the corresponding sequence of functions $(f_n)_{n=1}^{\infty}$ such that $f_n(s) := f(x_n, s)$ for all $s \in S$.

1) If $f(x, s)$ is lower semicontinuous w.r.t. x, we have that $\liminf_{n\to\infty} f_n(s) \geq f(x, s)$. If $f(\cdot)$ is nonnegative, we apply Fatou's Lemma to give

$$\begin{aligned}\liminf_{n\to\infty} F(x_n) &= \liminf_{n\to\infty} \int_S f_n(s) d\mu(s) \\ &\geq \int_S \liminf_{n\to\infty} f_n(s) d\mu(s) \\ &\geq \int_S f(x, s) d\mu(s) = F(x).\end{aligned}$$

Since the choice of $x \in X$ and the sequence $(x_n)_{n=1}^{\infty}$ was arbitrary, we have that

$$\liminf_{t\to x} F(t) \geq F(x)$$

and therefore, $F(x)$ is lower semicontinuous.

If $f(\cdot)$ is lower bounded, we define $c \in \mathbb{R}$, such that $f(x, s) \geq c$ for all $(x, s) \in X \times S$. Next, we define $h(x, s) := f(x, s) - c$ and note that $h(\cdot)$ is nonnegative and lower semicontinuous because $f(\cdot)$ is lower semicontinuous. Thus

$$H(x) := \int_S h(x, s) d\mu(s)$$

is lower semicontinuous and $F(x) = c + H(x)$ is also lower semicontinuous.[3]

2) If instead $f(x, s)$ is continuous w.r.t. x, we know that $\lim_{n\to\infty} f_n(s) = f(x, s)$. If $f(x, s)$ is also uniformly bounded, we have from the dominated convergence theorem that $F(x)$ is finite and

$$\begin{aligned}\lim_{n\to\infty} F(x_n) &= \lim_{n\to\infty} \int_S f_n(s) d\mu(s) \\ &= \int_S \lim_{n\to\infty} f_n(s) d\mu(s) = \int_S f(x, s) d\mu(s) = F(x).\end{aligned}$$

Since the choice of $x \in X$ and the sequence $(x_n)_{n=1}^{\infty}$ was arbitrary, we have that $F(x)$ is continuous. □

Proof of Lemma 1: We begin by establishing that $G : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ is a lower semicontinuous function. Since $\tilde{\mathbb{X}}$ is closed, we know that $I_{\tilde{\mathbb{X}}} : \mathbb{X} \to \{0, 1\}$ is upper semicontinuous. Therefore, the function $h : \mathbb{X} \to \{0, 1\}$ defined as $h(x) := 1 - I_{\tilde{\mathbb{X}}}(x)$ is lower semicontinuous. Since $f(\cdot)$ is continuous, the composition $g(x, u, w) := h(f(x, u, w))$ is lower semicontinuous as well. We have that

$$G(x, u) = \int_{\mathbb{W}} g(x, u, w) d\mu(w)$$

and since $g(\cdot)$ is lower semicontinuous, we have from Lemma 16 that $G(\cdot)$ is lower semicontinuous.

Since $G(\cdot)$ is lower semicontinuous, the set $\tilde{\mathbb{Z}}_\varepsilon = \{(x, u) : G(x, u) \leq \varepsilon\}$ is closed for all $\varepsilon \in [0, 1]$ by the definition of a lower semicontinuous function. Therefore, $\mathbb{Z} := \mathbb{Z}_h \cap \tilde{\mathbb{Z}}_\varepsilon$ is the intersection of two closed sets and is also closed.

From Assumption 3, we also know that $\mathbb{X}$ and $\mathbb{V}$ are closed sets. We define the set-valued mapping $\mathcal{Z}_N : \mathbb{W}^N \rightrightarrows \mathbb{X} \times \mathbb{V}^N$

[3]We can define $F(x) = c + H(x)$, because (S, Σ, μ) is a probability space, i.e., $\int_S c d\mu(s) = c$.

such that

$$\mathcal{Z}_N(\mathbf{w}) := \{(x, \mathbf{v}) \in \mathbb{X} \times \mathbb{V}^N : \eta_k(x, \mathbf{v}, \mathbf{w}) \leq 0 \quad \forall k \in \mathbb{I}_{[0,N]}\}$$

in which

$$\eta_k(x, \mathbf{v}, \mathbf{w}) := \left|(\hat{\phi}(k; x, \mathbf{v}, \mathbf{w}), \pi(\hat{\phi}(k; x, \mathbf{v}, \mathbf{w}), v(k))\right|_{\mathbb{Z}}$$

for all $k \in \mathbb{I}_{[0,N-1]}$ and $\eta_N(x, \mathbf{v}, \mathbf{w}) := |\hat{\phi}(N; x, \mathbf{v}, \mathbf{w})|_{\mathbb{X}_f}$. Since $f(\cdot)$ and $\pi(\cdot)$ are continuous functions, so is their composition. For each k, $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ is the composition of a finite number of continuous functions and is, therefore, continuous [32, Prop. 2.1]. Since $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ and point-to-set distance for the closed sets $\mathbb{Z}$ and $\mathbb{X}_f$ are continuous functions, $\eta_k(\cdot)$ is also continuous for each $k \in \mathbb{I}_{[0,N]}$. The inequality $\eta_k(x, \mathbf{v}, \mathbf{w}) \leq 0$, therefore, defines a closed set for each $k \in \mathbb{I}_{[0,N]}$. Thus, for each $\mathbf{w} \in \mathbb{W}^N$, the set $\mathcal{Z}_N(\mathbf{w})$ is the intersection of a finite number of closed sets and is, therefore, closed. By the definition of $\mathcal{Z}_N$, we have that

$$\mathcal{Z}_N = \bigcap_{\mathbf{w} \in \mathbb{W}^N} \mathcal{Z}_N(\mathbf{w}).$$

Since the intersection of an arbitrary collection of closed sets is a closed set, $\mathcal{Z}_N$ is a closed set. □

Proof of Proposition 2: From the previous proof, we know that for each k, $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ is continuous. Thus, $J_N(x, \mathbf{v}, \mathbf{w})$ is a continuous function since it is the composition of a finite number of continuous functions. For each $x \in \mathcal{X}_N$, we have that $J_N(x, \cdot) : \mathbb{V}^N \times \mathbb{W}^N \to \mathbb{R}$ is continuous and uniformly bounded because $\mathbb{V}$ and $\mathbb{W}$ are compact. Thus, from Lemma 16, we know that for each $x \in \mathcal{X}_N$, the function $V_N(x, \cdot) : \mathbb{V}^N \to \mathbb{R}$ is continuous.

From Lemma 1, we know that $\mathcal{Z}_N$ is closed and the function $|(x, \mathbf{v})|_{\mathcal{Z}_N}$ is continuous. Therefore, the set $\mathcal{V}_N(x) = \{\mathbf{v} \in \mathbb{V}^N : |(x, \mathbf{v})|_{\mathcal{Z}_N} \leq 0\}$ is closed for any $x \in \mathcal{X}_N \subseteq \mathbb{X}$. Since $\mathcal{V}_N(x) \subseteq \mathbb{V}^N$ and $\mathbb{V}$ is bounded, we know that $\mathcal{V}_N(x)$ is also bounded. Thus, $\mathcal{V}_N(x)$ is compact.

For each $x \in \mathcal{X}_N$, the function $V_N(x, \cdot)$ is continuous and $\mathcal{V}_N(x)$ is compact. By Weierstrass's theorem, a solution to $\mathbb{P}_N(x)$ exists for all $x \in \mathcal{X}_N$ [32, Prop. A.7]. □

Proof of Proposition 3: From Assumption 3, we have that $\mathbb{X}$ is closed and $\mathbb{V}^N$ is compact. From Lemma 1, we have that $\mathcal{Z}_N$ is closed. We know that $J_N(x, \mathbf{v}, \mathbf{w})$ is continuous and lower bounded (because the stage and terminal costs are lower bounded). Since continuity implies lower semicontinuity, by Lemma 16, we know that the function $V_N : \mathcal{Z}_N \to \mathbb{R}$ is lower semicontinuous and lower bounded. Thus, from [2, Prop. 7.33], we have that $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ is lower semicontinuous and the mapping $\mathbf{v}^0 : \mathcal{X}_N \rightrightarrows \mathbb{V}^N$ is Borel measurable. We define $K_N : \mathcal{X}_N \rightrightarrows \mathbb{V}$ such that $K_N(x) = \{h(x, \mathbf{v}) : \mathbf{v} \in \mathbf{v}^0(x)\}$ in which $h(x, \mathbf{v}) := \pi(x, v(0))$. Since $h(\cdot)$ is a continuous function, $K_N : \mathcal{X}_N \rightrightarrows \mathbb{U}$ is also Borel measurable. □

Proof of Lemma 4: From Proposition 11, we have that $\mathcal{X}_N$ is robustly positive invariant for the system $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$. Since $x^+ \in \mathcal{X}_N$, we know that $\kappa_N(x^+)$ is well-defined and the subsequent iteration $f_{cl}(x^+, w^+)$, $w^+ \in \mathbb{W}$ is defined as well. By induction, we can establish that $\phi(k; x, \mathbf{w}_k) \in \mathcal{X}_N$ is well-defined for all $x \in \mathcal{X}_N$, $\mathbf{w}_k \in \mathbb{W}^k$, and $k \in \mathbb{I}_{\geq 0}$. □

Proof of Proposition 5: Adapted from [10, Prop. 4]. From Proposition 3 and Standing Assumption 1, we have that $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ is Borel measurable. Since $f(\cdot)$ is continuous, $f_c(x, w) = f(x, \kappa_N(x), w)$ is Borel measurable. From Lemma 4, we know that $\phi(k; x, \mathbf{w}_k)$ is well-defined for all $x \in \mathcal{X}_N$, $\mathbf{w}_k \in \mathbb{W}^k$, and $k \in \mathbb{I}_{\geq 0}$.

We proceed by induction. For some $k \in \mathbb{I}_{\geq 0}$ let $\phi(k; x, \mathbf{w}_k)$ be Borel measurable. Then,

$$\phi(k+1; x, \mathbf{w}_{k+1}) = f_c(\phi(k; x, \mathbf{w}_k), w(k))$$

is also Borel measurable. Since $\phi(1; x, \mathbf{w}_1) = f_c(x, w(0))$ is Borel measurable, we have that for all $k \in \mathbb{I}_{\geq 0}$, $\phi(k; x, \mathbf{w}_k)$ is Borel measurable. By definition, $\mathbf{w}_k(\omega)$ is measurable w.r.t. $\omega \in \Omega$, and therefore, $\phi(k; x, \mathbf{w}_k(\omega))$ is also Borel measurable w.r.t. $\omega \in \Omega$.

For lower bounded, real-valued, Borel measurable functions, Lebesgue integrals are well-defined. From Assumption 2, we have that $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ and $V_N : \mathbb{X} \times \mathbb{V}^N \to \mathbb{R}$ are lower bounded and continuous. Thus, $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ is lower bounded as well. From Proposition 3 and Standing Assumption 1, we know that $V_N^0(\cdot)$ and $\kappa_N(\cdot)$ are Borel measurable. Therefore, $\ell(x, \kappa_N(x))$ and $V_N^0(x)$ are lower bounded and Borel measurable functions. □

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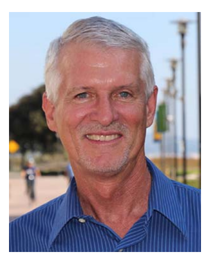

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