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A Suboptimal Economic Model Predictive Control Algorithm for Large and Infrequent Disturbances

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***Abstract*—We present a suboptimal economic model predictive control (MPC) algorithm that combines the strengths of two common suboptimal MPC approaches, one based on a warm start and one based on an optimality gap. This algorithm is specifically designed to address a class of large and infrequent disturbances that are relevant when considering discrete actuators and production scheduling in control problems. We establish that this algorithm provides the same nominal performance guarantee as optimal economic MPC and is inherently robust, in an economic context, to large and infrequent disturbances. This inherent robustness guarantee is not attained by either individual algorithm. We conclude with a small production scheduling example to demonstrate the benefits of the proposed algorithm for a practical application.**

***Keywords*—Suboptimal model predictive control, Stochastic systems, Stability of nonlinear systems, Constrained control**

I. Introduction

In economic model predictive control (MPC) formulations, the controller optimizes a general performance metric that is not necessarily related to a specific steady-state setpoint for the process. There are multiple economic MPC formulations developed to achieve desired closed-loop properties such as performance guarantees and/or asymptotic stability [3–5, 9]. By extending the theoretical results of economic MPC to include discrete-valued actuators [15], we can also cast high-level production planning and scheduling problems as economic MPC problems [6, 18, 21].

For problems with discrete-valued actuators, and particularly production scheduling problems, we must also consider discrete-valued, large, and infrequent disturbances (e.g., task delays or breakdowns in equipment) that are not typically discussed in control theory. For economic MPC applications without strict dissipativity assumptions, [11] establish that optimal economic MPC provides an economic form of inherent robustness to large and infrequent disturbances. These results, however, assume that an optimal solution to the MPC problem is used to define the control law. Unfortunately, solving large-scale and industrially relevant mixed-integer optimization problems to optimality is often intractable. Thus, algorithms that ensure desired properties, such as nominal performance and robustness guarantees, for the closed-loop system without the need for optimal solutions to the economic MPC problem are desirable.

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The approaches to suboptimal MPC are divided into two main categories that we term: warm-start suboptimal MPC and optimality-gap suboptimal MPC. Warm-start suboptimal MPC, first presented in [20], requires that a feasible initial control sequence (warm start), based on the previous open-loop trajectory computed by the MPC controller, is used to initialize the optimizer. The optimizer must then compute a control sequence no worse (in terms of the objective function) than the warm start. This type of suboptimal MPC algorithm renders the origin nominally asymptotically stable and is inherently robust to sufficiently small disturbances [1, 13, 20]. A critical requirement of warm-start suboptimal MPC is that disturbances are sufficiently small to ensure that the warm start input trajectory is feasible. The large disturbances considered in this work, however, typically render the warm start infeasible. For example, a delay or breakdown in a scheduling problem may render the previously computed schedule infeasible.

In optimality-gap suboptimal MPC, first described in [8], robustness to disturbances is demonstrated under the condition that the objective value of the computed solution is within a specific gap of the global optimum. This type of suboptimal MPC algorithm also renders the origin robust to disturbances [8, 14], but requires *global* optimization solvers that report bounds on optimality gaps. For large scale nonconvex nonlinear problems, these global solvers are often too slow to use in real time applications. Fortunately, the mixed-integer linear/quadratic programs (MILP/MIQPs) that are common in many economic MPC applications with discrete actuators can be solved with efficient global optimization solvers that report bounds on global optimality gaps of the computed solutions. Nominal asymptotic stability for this suboptimal MPC algorithm is also established. This result, however, requires a positive definite stage cost and does not extend to the nominal performance guarantee that is typically established for optimal economic MPC.

If the economic MPC problem is strictly dissipative, and therefore asymptotically stabilizes a steady-state target, the suboptimal algorithms and results for tracking MPC are expected to extend (with some adjustments) to economic MPC. However, for many applications of economic MPC, such as production scheduling, economic performance is more important than asymptotic stability of a steady state and strict dissipativity does not hold. Risbeck and Rawlings [17] discuss a warm-start suboptimal economic MPC algorithm that provides a nominal performance guarantee. Robustness for this suboptimal economic MPC algorithm is not discussed.

In the subsequent sections, we present a suboptimal economic MPC algorithm designed specifically to ensure inherent robustness to these large and infrequent disturbances. The key feature of this algorithm is that it combines the strengths of warm-start and optimality-gap suboptimal MPC to achieve both nominal performance and robustness guarantees for large disturbances. This robustness guarantee is not attained by solely applying warm-start or optimality-gap suboptimal MPC.

II. Preliminaries and Problem Formulation

Let $\mathbb{I}$ denote integers, $\mathbb{R}$ denote reals, and subscripts/superscripts on these sets denote restrictions/dimensions (e.g., $\mathbb{R}^n_{\geq 0}$ for n-dimensional nonnegative reals). We use $|\cdot|$ to denote Euclidean norm.

A. Optimal economic MPC formulation

We consider time-varying, discrete-time systems of the form

$$x^+ = f(x,u,w,t) \quad f : \mathbb{X}\times\mathbb{U}\times\mathbb{W}\times\mathbb{T}\to\mathbb{X} \tag{1}$$

defined for the state $x\in\mathbb{X}\subseteq\mathbb{R}^n$, input $u\in\mathbb{U}\subseteq\mathbb{R}^m$, and disturbance $w\in\mathbb{W}\subseteq\mathbb{R}^p$, at the discrete-time index $t\in\mathbb{T}:=\mathbb{I}_{\geq 0}$. The successor state at $t+1$ is denoted x^+. The system is subject to time-varying constraints $(\mathbb{Z}(t))_{t\in\mathbb{T}}$ such that at time $t\in\mathbb{T}$, $(x,u)\in\mathbb{Z}(t)\subseteq\mathbb{X}\times\mathbb{U}$. For more discussion of the following MPC formulation see [17] or [16, s. 2.4.5].

We consider a nominal MPC problem with a horizon $N\in\mathbb{I}_{\geq 1}$, stage cost $\ell(\cdot,t):\mathbb{X}\times\mathbb{U}\to\mathbb{R}$, terminal constraints $\mathbb{X}_f(t)\subseteq\mathbb{X}$, and terminal cost $V_f(\cdot,t):\mathbb{X}\to\mathbb{R}$, defined for all $t\in\mathbb{T}$. The nominal system is

$$x^+ = f(x,u,0,t) \tag{2}$$

For the current state $x\in\mathbb{X}$ and input sequence $\mathbf{u}:=(u(t),u(t+1),\ldots,u(t_N-1))\in\mathbb{U}^N$ at time t, the function $\hat{\phi}(k;x,\mathbf{u},t)$ denotes the state of (2) at time $k\in\mathbb{I}_{[t,t+N-1]}$. We define the set of admissible initial state and input trajectory pairs $\mathcal{Z}_N(t)$, the set of admissible input trajectories $\mathcal{U}_N(x,t)$, and the set of feasible initial states $\mathcal{X}_N(t)$ as

$$\begin{aligned}\mathcal{Z}_N(t) := \{(x,\mathbf{u})\in\mathbb{X}\times\mathbb{U}^N : \\ (x(k),u(k))\in\mathbb{Z}(k) \quad \forall k\in\mathbb{I}_{[t,t+N-1]} \\ x(t+N)\in\mathbb{X}_f(t+N)\} \\ \mathcal{U}_N(x,t) := \{\mathbf{u}\in\mathbb{U} : (x,\mathbf{u})\in\mathcal{Z}_N(t)\} \\ \mathcal{X}_N(t) := \{x\in\mathbb{X} : \exists\mathbf{u}\in\mathcal{U}_N(x,t)\}\end{aligned}$$

in which $x(k):=\hat{\phi}(k;x,\mathbf{u},t)$. We define the cost function

$$V_N(x,\mathbf{u},t) := \sum_{k=t}^{t+N-1}\ell(x(k),u(k),k) + V_f(x(t+N),t+N)$$

in which $x(k):=\hat{\phi}(k;x,\mathbf{u},t)$.

The optimal control problem for $x\in\mathcal{X}_N(t)$ at time $t\in\mathbb{T}$ is

$$\mathbb{P}_N(x,t) : V_N^0(x,t) = \min_{\mathbf{u}\in\mathcal{U}_N(x,t)} V_N(x,\mathbf{u},t) \tag{3}$$

and $\mathbf{u}^0(x,t)$ denotes the optimal input trajectory. We denote the optimal control law as $\kappa_N^0(x,t) := u^0(t;x,t)$ in which $u^0(t;x,t)$ is the first input vector in $\mathbf{u}^0(x,t)$.

We consider a time-varying reference trajectory $(\mathbf{x}_r,\mathbf{u}_r)$ for the MPC controller.

Assumption 1. The reference trajectory $(\mathbf{x}_r,\mathbf{u}_r)$ satisfies $x_r(t+1)=f(x_r(t),u_r(t),0,t)$ and $(x_r(t),u_r(t))\in\mathbb{Z}(t)$ for all $t\in\mathbb{T}$.

The performance of this reference trajectory, in terms of the stage cost $\ell(\cdot)$, is also important because all subsequent theoretical guarantees are established relative to this reference trajectory. Therefore, this reference trajectory is often chosen as the solution to a (finite horizon) periodic optimization problem with the same dynamical model, constraints, and cost function as the MPC problem.

We define the shifted stage cost as $\bar{\ell}(x,u,t):=\ell(x,u,t)-\ell(x_r(t),u_r(t),t)$. We also define the shifted optimal cost as

$$\bar{V}_N^0(x,t) := V_N^0(x,t) - \sum_{k=t}^{t+N-1}\ell(x_r(k),u_r(k),k) \tag{4}$$

Assumption 2. The model $f:\mathbb{X}\times\mathbb{U}\times\mathbb{W}\times\mathbb{T}\to\mathbb{X}$, stage cost $\ell(\cdot,t):\mathbb{X}\times\mathbb{U}\to\mathbb{R}$, and terminal cost $V_f(\cdot,t):\mathbb{X}\to\mathbb{R}$ are continuous. The functions $\bar{\ell}(x,u,t)$ and $V_f(x,t)$ are bounded from below for $(x,u)\in\mathbb{Z}(t)$ and $x\in\mathbb{X}_f(t)$, respectively, uniformly for all $t\in\mathbb{T}$.

Assumption 3. For each $t\in\mathbb{T}$, the sets $\mathbb{Z}(t)$ and $\mathbb{X}_f(t)$ are closed. The set $\mathbb{U}$ is compact.

Assumption 4. There exists a terminal control law, $\kappa_f(\cdot,t):\mathbb{X}_f(t)\to\mathbb{U}$ such that $(x,\kappa_f(x,t))\in\mathbb{Z}(t)$,

$$f(x,\kappa_f(x,t),0,t)\in\mathbb{X}_f(t+1)$$
$$V_f(f(x,\kappa_f(x,t),0,t))\leq V_f(x,t)-\bar{\ell}(x,\kappa_f(x,t),t)$$

for all $x\in\mathbb{X}_f(t)$ and $t\in\mathbb{T}$. Furthermore, $x_r(t)\in\mathbb{X}_f(t)$ and $V_f(x_r(t),t)=0$ for all $t\in\mathbb{T}$.

Assumption 4 can be satisfied by, e.g., the terminal equality constraint $\mathbb{X}_f(t):=\{x_r(t)\}$. To construct a larger terminal region, one can extend the methods in [2] to time-varying systems or use approaches discussed in [12, 17] for specific applications of economic MPC.

B. A hybrid suboptimal MPC algorithm

We propose a hybrid algorithm that combines features of both the warm-start and optimality-gap algorithms to address large and infrequent disturbances. Specifically, we require that the solution computed by the hybrid algorithm achieves an optimality gap less than some specified and fixed constant $\rho\geq 0$. In addition, if the warm start generated by extending the previous input trajectory is feasible, we also require the computed solution to perform better than this warm start. We define this algorithm in the following paragraphs.

We first define the warm start by extending the previous schedule with the terminal control law, i.e.,

$$\zeta(x,\mathbf{u},t) := (u(t+1),\ldots,u(t+N-1),\kappa_f(x(t+N),t+N))$$

in which $x(t+N) = \hat{\phi}(t+N; x, \mathbf{u}, t)$. Let $\tilde{\mathbf{u}}^+ := \zeta(x, \mathbf{u}, t)$ denote the warm start at $t+1$ given the state x and computed input trajectory $\mathbf{u}$ at time t. We define the set of admissible inputs for warm-start suboptimal MPC with the warm start $\tilde{\mathbf{u}}$ at time $t \in \mathbb{T}$ as

$$\check{\mathcal{U}}_N^w(x, \tilde{\mathbf{u}}, t) := \{\mathbf{u} : \mathbf{u} \in \mathcal{U}_N(x,t),\ V_N(x,\mathbf{u},t) \leq V_N(x, \tilde{\mathbf{u}}, t)\}$$

Note, however, that the set $\check{\mathcal{U}}_N^w(x, \tilde{\mathbf{u}}, t)$ may be empty if $\tilde{\mathbf{u}} \notin \mathcal{U}_N(x,t)$. We also define the set of admissible inputs with an optimality gap less than $\rho \geq 0$ as

$$\check{\mathcal{U}}_N^g(x,t) := \{\mathbf{u} : \mathbf{u} \in \mathcal{U}_N(x,t),\ V_N(x,\mathbf{u},t) \leq V_N^0(x,t) + \rho\}$$

Thus, the hybrid algorithm is defined by the following set of admissible inputs.

$$\check{\mathcal{U}}_N(x, \tilde{\mathbf{u}}, t) := \begin{cases} \check{\mathcal{U}}_N^w(x, \tilde{\mathbf{u}}, t) \cap \check{\mathcal{U}}_N^g(x,t) & ;\ \tilde{\mathbf{u}} \in \mathcal{U}_N(x,t) \\ \check{\mathcal{U}}_N^g(x,t) & ;\ \tilde{\mathbf{u}} \notin \mathcal{U}_N(x,t) \end{cases}$$

Note that any input computed by the hybrid algorithm must satisfy the required optimality gap, i.e., $\mathbf{u} \in \check{\mathcal{U}}_N^g(x,t)$, even if $\tilde{\mathbf{u}}$ is feasible. We summarize this suboptimal MPC algorithm as follows.

Algorithm 1. Obtain the initial state $x \in \mathcal{X}_N$ and any initial warm start $\tilde{\mathbf{u}} \in \mathbb{U}^N$. Then repeat

1) Obtain the current estimate of the state x.
2) Compute any $\mathbf{u} \in \check{\mathcal{U}}_N(x, \tilde{\mathbf{u}}, t)$ via a global optimization solver.
   a) If $\tilde{\mathbf{u}}$ is feasible, compute $\mathbf{u} \in \check{\mathcal{U}}_N^w(x, \tilde{\mathbf{u}}, t) \cap \check{\mathcal{U}}_N^g(x,t)$.
   b) If $\tilde{\mathbf{u}}$ is not feasible, compute $\mathbf{u} \in \check{\mathcal{U}}_N^g(x,t)$.
3) Inject the first element of the input sequence $\mathbf{u}$.
4) Compute the next warm start $\tilde{\mathbf{u}}^+ = \zeta(x, \mathbf{u}, t)$.

The suboptimal control law $\kappa_N(x, \tilde{\mathbf{u}}, t)$ is a function of the warm start, which is itself a function of the previous state and input. We therefore find it convenient to define the extended state $z := (x, \tilde{\mathbf{u}})$. The extended state evolves according to

$$z^+ \in H(z, w, t) := \left\{ \begin{pmatrix} f(x, u(t), w, t) \\ \zeta(x, \mathbf{u}, t) \end{pmatrix} :\ \mathbf{u} \in \check{\mathcal{U}}_N(z,t) \right\} \tag{5}$$

in which $u(t)$ is the first element of $\mathbf{u}$. We use $\psi(k; z, \mathbf{w}_k, t)$ to denote any solution of (5) with initial extended state $z \in \mathcal{Z}_N$ at time t and the disturbance sequence $\mathbf{w}_k := (w(t), \ldots, w(k-1))$. We use $\phi_x(k; z, \mathbf{w}_k, t)$ and $\phi_u(k; z, \mathbf{w}_k, t)$ to denote the corresponding x and u trajectory, respectively. We also use $\phi_{\mathbf{u}}(k; z, \mathbf{w}_k, t)$ to denote the computed open-loop control trajectory $\mathbf{u}$ at each time $k \geq t$. Note that these trajectories denote a *selection* from the set of potential solutions for the closed-loop system defined by (5). All subsequent results are then established for any selection from the set of potential feasible solutions.

We assume the random variables $w(t)$ are independent and identically distributed (i.i.d.) in time with probability measure $\mu : \mathcal{B}(\mathbb{W}) \to [0,1]$ in which $\mathcal{B}(\mathbb{W})$ denote the Borel field of the set $\mathbb{W}$. For the sequence of random variables $\mathbf{w}_k$ and measurable function $g : \mathbb{W}^{k-t} \to \mathbb{R}$, we define expected value with the following Lebesgue integral.

$$\mathbb{E}\left[g(\mathbf{w}_k)\right] := \int_{\mathbb{W}^{k-t}} g\left((\omega_t, \ldots, \omega_{k-1})\right) d\mu(\omega_t) \cdots d\mu(\omega_{k-1})$$

We use $\mathbb{E}_{|z(t)}[\cdot]$ to denote the expected value conditioned on $z(t)$. Let $\Pr(w \in W)$ denote the probability that w is in the set $W \subseteq \mathbb{R}^p$.

III. Nominal Performance

We now establish the following nominal performance guarantee for Algorithm 1. This result and the associated proof are based on Remark 1 and Theorem 1 in [17] and rely on the warm-start component of Algorithm 1.

Theorem 1. *Let Assumptions 1-4 hold. Then, starting from any $z(t) \in \mathcal{Z}_N(t)$ and $t \in \mathbb{T}$, we have that*

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{k=t}^{t+T-1} \bar{\ell}(x(k), u(k), k) \leq 0 \tag{6}$$

for the nominal closed-loop evolution in (5) in which $x(k) = \phi_x(k; z, \mathbf{0}, t)$ and $u(k) = \phi_u(k; z, \mathbf{0}, t)$.

Proof. From Assumption 4, we have that $\mathcal{U}_N(x,t)$ is invariant under the update $\zeta(\cdot)$. Thus, $\tilde{\mathbf{u}}^+ = \zeta(x, \mathbf{u}) \in \mathcal{U}_N(x^+, t+1)$ for $x^+ = f(x, u(t), 0, t)$, and $z^+ \in \mathcal{Z}_N(t+1)$ for any $z \in \mathcal{Z}_N(t)$ and $t \in \mathbb{T}$. Since $\tilde{\mathbf{u}}$ is a feasible warm start, we have that the computed input trajectory $\mathbf{u}$ satisfies $\mathbf{u} \in \check{\mathcal{U}}_N^w(z,t)$ for all $z \in \mathcal{Z}_N(t)$ and $t \in \mathbb{T}$.

Given the selected control trajectory $\mathbf{u} \in \check{\mathcal{U}}_N(z,t)$, let u be the first element of $\mathbf{u}$, $x_f := \hat{\phi}(N; x, \mathbf{u}, t)$, $u_f = \kappa_f(x_f, t+N)$, and $x_f^+ = f(x_f, u_f, 0, t)$. From the update $\zeta(\cdot)$ and Assumption 4, we have that

$$\begin{aligned} \bar{V}_N(z^+, t+1) &= \bar{V}_N(x, \mathbf{u}, t) - \bar{\ell}(x,u,t) - V_f(x_f, t+N) \\ &\quad + \bar{\ell}(x_f, u_f, t+N) + V_f(x_f^+, t+N+1) \\ &\leq \bar{V}_N(x, \mathbf{u}, t) - \bar{\ell}(x,u,t) \\ &\leq \bar{V}_N(z,t) - \bar{\ell}(x,u,t) \end{aligned} \tag{7}$$

in which the last inequality holds because $\mathbf{u} \in \check{\mathcal{U}}_N^w(z,t)$.

Note that (7) holds for all subsequent time $k \geq t$. Thus, we rearrange and sum this bound for T time steps to give

$$\sum_{k=t}^{t+T-1} \bar{\ell}(x(k), u(k), k) \leq \bar{V}_N(z(t), t) - \bar{V}_N(z(t+T), t+T)$$

in which $z(k) = \psi(k; z, \mathbf{0}, t)$, $x(k) = \phi_x(k; z, \mathbf{0}, t)$, and $u(k) = \phi_u(k; z, \mathbf{0}, t)$. By Assumption 2, there exists some $c \in \mathbb{R}$ such that $\bar{V}_N(z(t+T), t+T) \geq c$. We use this bound in the previous equation and divide by T to give

$$\frac{1}{T} \sum_{k=t}^{t+T-1} \bar{\ell}(x(k), u(k), k) \leq \frac{\bar{V}_N(z(t), t) - c}{T} \tag{8}$$

We take the $\limsup$ of (8) for $T \to \infty$ to give the desired bound. $\square$

Since the warm start remains feasible for the nominal system, we always use the warm-start suboptimal MPC method and thereby achieve the desired cost decrease condition in

(7) and nominal performance guarantee. We are interested, however, in a class of large disturbances that typically render the warm start infeasible. Thus, the optimality-gap algorithm is added to warm-start suboptimal MPC specifically to address this class of disturbances.

We note that if optimality-gap suboptimal MPC algorithm is used without the addition of the warm-start algorithm, the bound in Theorem 1 is weakened based on the value of the optimality gap ρ. Without the requirement that $\mathbf{u} \in \check{\mathcal{U}}_N^w(z,t)$, (7) is now

$$\bar{V}_N(z^+, t+1) \leq \bar{V}_N(x, \mathbf{u}, t) - \bar{\ell}(x, u, t)$$

For all $\mathbf{u} \in \check{\mathcal{U}}_N^g(x,t)$, the best bound we can construct is

$$\bar{V}_N(x, \mathbf{u}, t) \leq \bar{V}_N(z, t) + \rho$$

because we may have $\bar{V}_N(z,t) = \bar{V}_N^0(x,t)$, i.e., the warm-start is optimal. We therefore have

$$\bar{V}_N(z^+, t+1) \leq \bar{V}_N(z, t) - \bar{\ell}(x, u, t) + \rho \qquad (9)$$

If we apply the remaining steps in the Proof of Theorem 1 to (9), we observe that the right hand side of the inequality in (6) is replaced by ρ.

IV. Robustness to large and infrequent disturbances

An important class of disturbances for economic MPC problems with discrete actuators, such as scheduling problems, are large and infrequent disturbances. We consider the same class of large and infrequent disturbances addressed in [11]. We define these disturbances by discussing them in contrast to the class of small persistent disturbances typically considered in robustness analysis. We denote the set of small persistent disturbances as $\mathbb{W}_0$ with $\sup_{w \in \mathbb{W}_0} |w| \leq \delta_0$ in which $\delta_0 > 0$ is sufficiently small. Large disturbances are then defined by the set $\mathbb{W}_1$ such that $\inf_{w \in \mathbb{W}_1} |w| > \delta_0$. We denote the probability that the disturbance takes a value in this set as $\varepsilon := \Pr(w \in \mathbb{W}_1)$. The disturbances in $\mathbb{W}_0$ are small. The disturbances in $\mathbb{W}_1$ are large and include discrete-valued disturbances that may not be included in $\mathbb{W}_0$. As shown in [11], MPC is inherently robust to this class of large disturbances provided these disturbances are sufficiently *infrequent*, i.e., $\varepsilon < \delta$ for some sufficiently small $\delta > 0$. This description includes many kinds of large disturbances such as faults, communications failures, breakdowns, large delays, and large price/demand spikes in economic applications.

In [11], however, all results are derived for the optimal control law, i.e., these results require that the MPC problem is solved to optimality in the allotted computation time. We establish in this section that Algorithm 1 is robust in the same economic context presented in [11, Theorem 6].

A. Assumptions

We consider the case of only large disturbances and nominal behavior.

Assumption 5. The disturbance set satisfies $\mathbb{W} = \mathbb{W}_0 \cup \mathbb{W}_1$ and we restrict $\mathbb{W}_0 = \{0\}$.

Although we consider these disturbances to be large, we do not allow disturbances of arbitrary size. If we want to consider large disturbances, the control algorithm must be recursively feasible by design.

Assumption 6. If $x \in \mathcal{X}_N(t)$, then $f(x,u,w,t) \in \mathcal{X}_N(t+1)$ for all $(x,u) \in \mathbb{Z}(t)$, $w \in \mathbb{W}$, and $t \in \mathbb{T}$, i.e., the sets $(\mathcal{X}_N(t))_{t\in\mathbb{T}}$ are robustly positive invariant.

In addition, we require a bound on the cost increase due to a disturbance.

Assumption 7. There exist $b_1, b_2 \in \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned}\bar{V}_N^0(f(x, \kappa_N^0(x,t), w, t), t+1) \\ \leq \bar{V}_N^0(x,t) + b_1|\bar{\ell}(x, \kappa_N^0(x,t), t)| + b_2\end{aligned}$$

for all $x \in \mathcal{X}_N(t)$, $w \in \mathbb{W}_1$, and $t \in \mathbb{T}$.

Note that we require this bound for only the *optimal* control law. A further discussion of these assumptions is available in [11]. For production scheduling problems, Assumptions 6 and 7 are satisfied with a careful choice of the terminal constraint and cost [12]. Under specific conditions, Assumption 7 can be verified without explicit knowledge of the optimal cost function or control law [11, Lemma 7].

In addition to the assumptions required in [11], we also require that the stage cost satisfies the following condition.

Assumption 8. There exists $d \geq 0$ such that

$$|\ell(x, u_1, t) - \ell(x, u_2, t)| \leq d$$

for all $(x, u_1) \in \mathbb{Z}(t)$, $(x, u_2) \in \mathbb{Z}(t)$, and $t \in \mathbb{T}$.

This assumption is satisfied for all linear stage costs and quadratic stage costs that do not include any bilinear combinations of the unbounded modes of x. Since many applications of economic MPC use linear or quadratic stage costs, this assumption still admits many relevant problems. Assumption 8 is needed to address the fact that Assumption 7 applies for only the optimal control law.

B. Main result

With these assumptions, we can established that the proposed hybrid suboptimal MPC algorithm is economically robust to large and infrequent disturbances.

Theorem 2. *Let Assumptions 1-8 hold. Then for the closed-loop system evolution in (5) in which $\varepsilon := \Pr(w \in \mathbb{W}_1)$, there exist $\delta \in (0,1]$ and $\bar{\gamma} > 0$ such that for all initial $z \in \mathcal{X}_N(t) \times \mathbb{U}^N$, $t \in \mathbb{T}$, and $\varepsilon \in [0, \delta]$ we have that*

$$\limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{k=t}^{t+T-1} \bar{\ell}(x(k), u(k), k)\right] \leq \bar{\gamma}\varepsilon \qquad (10)$$

in which $x(k) = \phi_x(k; x, \mathbf{w}_k, t)$ and $u(k) = \phi_u(k; x, \mathbf{w}_k, t)$.

Proof. Choose any $z \in \mathcal{X}_N(t) \times \mathbb{U}^N$ and $t \in \mathbb{T}$. Choose an input trajectory based on Algorithm 1, i.e., $\mathbf{u} \in \check{\mathcal{U}}_N(z,t)$, and denote the first input of this trajectory as u.

If $w = 0$, we have that $x^+ = f(x,u,0,t)$ and $\tilde{\mathbf{u}}^+ = \zeta(x,\mathbf{u},t)$ is a feasible control trajectory from x^+ because of

Assumption 4, i.e, $z^+ = (x^+, \tilde{\mathbf{u}}^+) \in \mathcal{Z}_N(t+1)$. Thus, we can apply the same approach as in the proof of Theorem 1 to conclude that

$$\bar{V}_N(z^+, t+1) \leq \bar{V}_N(x, \mathbf{u}, t) - \bar{\ell}(x, u, t)$$

We denote the input trajectory selected at $t+1$ as $\mathbf{u}^+ \in \check{\mathcal{U}}_N(z^+, t+1)$ and note that $\mathbf{u}^+ \in \check{\mathcal{U}}_N^w(z^+, t+1)$ because $w = 0$ and therefore $\tilde{\mathbf{u}}^+$ is a feasible warm start. Thus, if $w = 0$, we have

$$\bar{V}_N(x^+, \mathbf{u}^+, t+1) \leq \bar{V}_N(x, \mathbf{u}, t) - \bar{\ell}(x, u, t)$$

If $w \in \mathbb{W}_1$, we have $x^+ = f(x, u, w, t)$ and note that $\tilde{\mathbf{u}}^+$ is not necessarily a feasible warm start for x^+. Nonetheless, we have that the subsequent computed input trajectory satisfies $\mathbf{u}^+ \in \check{\mathcal{U}}_N^g(x^+, t+1)$ and therefore

$$\bar{V}_N(x^+, \mathbf{u}^+, t+1) \leq V_N^0(x^+, t+1) + \rho$$

We combine this inequality with the inequality in Assumption 7 and by optimality we have

$$\bar{V}_N(x^+, \mathbf{u}^+, t+1) \leq \bar{V}_N(x, \mathbf{u}, t) + b_1|\bar{\ell}(x, \kappa_N^0(x,t), t)| + b_2 + \rho$$

From Assumption 2, there exists $m \in \mathbb{R}$ such that $\bar{\ell}(x, u, t) \geq m$ for all $(x, u) \in \mathbb{Z}$ and $t \in \mathbb{T}$. Therefore, $|\bar{\ell}(x, u, t)| \leq \bar{\ell}(x, u, t) + 2|m|$ and

$$\bar{V}_N(x^+, \mathbf{u}^+, t+1) \leq \bar{V}_N(x, \mathbf{u}, t) + b_1\bar{\ell}(x, \kappa_N^0(x,t), t) + \tilde{b}_2$$

in which $\tilde{b}_2 := b_2 + \rho + 2m$. We apply Assumption 8 to give

$$\bar{V}_N(x^+, \mathbf{u}^+, t+1) \leq \bar{V}_N(x, \mathbf{u}, t) + b_1\bar{\ell}(x, u, t) + b_3$$

in which $b_3 = b_1 d + \tilde{b}_2$.

To streamline notation, we define $y = (x, \mathbf{u})$ and $y^+ = (x^+, \mathbf{u}^+)$. Note that y represents the current state x and the *computed* input trajectory $\mathbf{u}$ (not the warm start $\tilde{\mathbf{u}}$). We then combine the bounds with and without the disturbance through the indicator function of $\mathbb{W}_1$.

$$\begin{aligned}\bar{V}_N(y^+, t+1) \leq \bar{V}_N(y, t) &- (1 - I_{\mathbb{W}_1}(w))\bar{\ell}(x, u, t) \\ &+ I_{\mathbb{W}_1}(w)(b_1\bar{\ell}(x, u, t) + b_3)\end{aligned}$$

in which $I_{\mathbb{W}_1}(w) = 1$ if $w \in \mathbb{W}_1$ and zero otherwise. Taking expected value and combining terms gives,

$$\mathbb{E}_{|z}\left[\bar{V}_N(y^+, t+1)\right] - \bar{V}_N(y, t) \leq -(1 - \varepsilon - b_1\varepsilon)\bar{\ell}(x, u, t) + b_3\varepsilon$$

We choose $\delta < 1/(1 + b_1)$ and note $\delta \in (0, 1]$, which gives

$$\mathbb{E}_{|z}\left[\bar{V}_N(y^+, t+1)\right] - \bar{V}_N(y, t) \leq -b_4\bar{\ell}(x, u, t) + b_3\varepsilon \quad (11)$$

with $b_4 := (1 - (1 + b_1)\delta) > 0$.

From $z(t) \in \mathcal{X}_N(t) \times \mathbb{U}^N$ and $t \in \mathbb{T}$, we denote the closed-loop trajectories $x(k) = \phi_x(k; z(t), \mathbf{w}_k, t)$ and $u(k) = \phi_u(k; z(t), \mathbf{w}_k, t)$. We also denote $\mathbf{u}(k) = \phi_{\mathbf{u}}(k; z(t), \mathbf{w}_k, t)$ and therefore $y(k) = (x(k), \mathbf{u}(k))$. By (11) and the properties of iterated expectations, we have

$$\begin{aligned}\mathbb{E}_{|z(t)}\left[\bar{V}_N(y(k+1), k+1)\right] &- \mathbb{E}_{|z(t)}\left[\bar{V}_N(y(k), k)\right] \\ &\leq -b_4\mathbb{E}_{|z(t)}\left[\bar{\ell}(x(k), u(k), k)\right] + b_3\varepsilon\end{aligned}$$

for all $k \in \mathbb{I}_{\geq t}$. We take the sum from t to $t + T - 1$ with $T \in \mathbb{I}_{\geq 1}$, divide by T, and rearrange to give

$$\begin{aligned}&b_4\mathbb{E}_{|z(t)}\left[\frac{1}{T}\sum_{k=t}^{T+t-1}\bar{\ell}(x(k), u(k), k)\right] \\ &\quad\leq \frac{\bar{V}_N(y(t), t) - \mathbb{E}_{|z(t)}\left[\bar{V}_N(y(t+T), t+T)\right]}{T} + b_3\varepsilon\end{aligned}$$

By Assumption 2, there exists $c \in \mathbb{R}$ such that $\bar{V}_N(y(t+T), t+T) \geq c$ and we have

$$\mathbb{E}_{|z(t)}\left[\frac{1}{T}\sum_{k=t}^{T+t-1}\bar{\ell}(x(k), u(k), k)\right] \leq \frac{\bar{V}_N(y(t), t) - c}{b_4 T} + \bar{\gamma}\varepsilon$$

in which $\bar{\gamma} := b_3/b_4$. We take the $\limsup$ of this inequality as $T \to \infty$ so that the initial cost and c vanish to give (10). $\square$

Theorem 2 ensures that the the closed-loop system is able to, on average, recover from large, but sufficiently infrequent disturbances. The calculated bound in (10), however, is often too conservative to provide useful quantitative information. Similar to Theorem 1, if optimality-gap suboptimal MPC is used without a warm start, the bound in Theorem 2 is weakened by the value of ρ. Specifically, the right hand side of (10) becomes $\bar{\gamma}\varepsilon + \rho$.

V. Production Scheduling Example

We consider a simple production scheduling example. The goal is to meet demand of product 1 (M1) by converting raw material (assumed to be in abundant supply) to M1 through task 1 (T1) carried out on unit 1 (U1). We can also produce product 2 (M2) with task 2 (T2) also carried out on U1. T1 and T2 have processing times of 2 and 3 hours, respectively, and a batch size between 10 and 20 kgs. We can store up to 100 kg of each product at a cost of \$1(/kg/hr). The demand for M1 is 50 kg every 6 hours. If demand is not met, the facility accumulates backlog that must be offset at later times. The penalty for maintaining backlog is \$50(/kg/hr). We can sell up to 5 kg of M2 for a profit of \$20(/kg) at any time. Thus, the optimal schedule is one that produces enough M1 to meet demand while producing and selling as much M2 as possible.

We model this system using the state-space scheduling model developed in [21]. We define the binary decision variables W_1, W_2 that are unity if T1, T2 start at time t. We also define the continuous inputs B_1, B_2 that represent the batch size assigned to T1, T2. To track these decisions in the state of the system, we lift W_i and B_i with the state variables $\bar{W}_i^n$, $\bar{B}_i^n$ for $n \in \{0, 1, \ldots, \tau_i\}$ in which τ_i is the processing time of task i. The value of n represents the progress of the task (e.g., at $n = 1$, the task is n/τ_i complete). We consider 1 hour delays on U1 (Y). For a one hour delay, $Y = 1$ and the active task's progression does not advance. Note that this disturbance is an inherently discrete-valued (large) disturbance, i.e., $Y \in \{0, 1\}$.

Inventory and backlog (unmet demand) are denoted S_1, S_2 and U_1. We also allow for up to 1 kg of M1 to be moved to a long-term storage facility at a cost of \$20(/kg). We denote this action D_1 and note that this action is used to construct a valid terminal cost for the scheduling problem [12]. The inventory and backlog dynamics are integrators influenced by the batch size of ending tasks ($\bar{B}_i^{\tau_i}$), shipments to long-term storage (D_1), shipments of M1 to meet demand (H_1), shipments of M2 for profit (V_2), and demand for M1 ($\xi_1(t)$). We also impose constraints to enforce one-task-per-unit requirements, batch size requirements, and appropriate upper/lower bounds on variables. The general equations for these dynamics and constraints can be found in [12] or [10, s. 6.2.1].

To streamline notation, we define each variable without subscripts to indicate a column vector containing the variable at each subscript, e.g., $\bar{W} := \left[(\bar{W}_i^n \; \forall i \in \{1,2\}, n \in \{0,\dots,\tau_i\})\right]'$. We define the state, input, and disturbance as

$$x = \left[\bar{W}, \bar{B}, S, U\right]', \;\; u = [W, B, V, H, D]', \;\; w = [Y]$$

and the dynamic evolution equation is then $x^+ = f(x,u,w,t)$. We have state and input constraints $(x,u) \in \mathbb{Z}$ that also enforce discreteness of W. We define the stage cost as

$$\ell(x,u,t) = S_1 + S_2 + 50U_1 - 20V_1 + 20D_1$$

Note that this stage cost satisfies Assumption 8.

We use the procedures detailed in [12] to construct a reference trajectory, terminal constraint, and terminal cost that satisfy the required assumptions. Specifically, we solve for an optimal 72 hour periodic schedule with the requirement that we *overproduce* M1 with a margin $\sigma = 0.1$ per hour. That is, we constrain $D_1 \geq \sigma$ and solve a 72 hour optimization problem with a periodic constraint on the state, i.e., $x(0) = x(72)$. This optimal periodic schedule is used as the reference trajectory and has the form

$$x_r(t) = \left[\bar{W}_r(t), \bar{B}_r(t), S_{r,1}(t), S_{r,2}(t), U_{r,1}(t)\right]'$$

From this reference trajectory, we define the parameter

$$\omega_1 = \min_{t \in \{0,1,\dots,71\}} 100 - S_{r,1}(t)$$

to be used in the construction of the terminal constraint. We also define $\Delta S_1(t) := S_1 - S_{r,1}(t)$ and $\Delta U_1(t) := U_1 - U_{r,1}(t)$. We then define the terminal constraint as

$$\begin{aligned}\mathbb{X}_f(t) := \{x \in \mathbb{X} : \bar{W} = \bar{W}_r(t), \bar{B} = \bar{B}_r(t),\\ \Delta S_1(t) \in [0,\omega_1], \Delta U_1(t) \geq 0, S_2 = S_{r,2}(t)\}\end{aligned}$$

We define the terminal control law as

$$\kappa_f(x,t) := \begin{bmatrix} W_r(t) \\ B_r(t) \\ V_{r,2}(t) \\ H_{r,1}(t) + \min\{\Delta U_1(t), \sigma\} \\ D_{r,1}(t) + \min\{\Delta S_1(t), 0.5\} - \min\{\Delta U_1(t), \sigma\} \end{bmatrix}$$

and the terminal cost as

$$V_f(x,t) := 21\Delta S_1(t) + (\Delta S_1(t))^2 + 30\Delta U_1(t) + 250(\Delta U_1(t))^2$$

We note that these terminal conditions satisfy Assumptions 4 and ensure that Assumption 6 holds for this problem. Furthermore, we can establish that Assumption 7 is satisfied [12].

Let ρ denote the allowed optimality gap and $\varepsilon := \Pr(Y = 1)$, i.e., the probability that a 1 hour delay occurs for Unit 1.

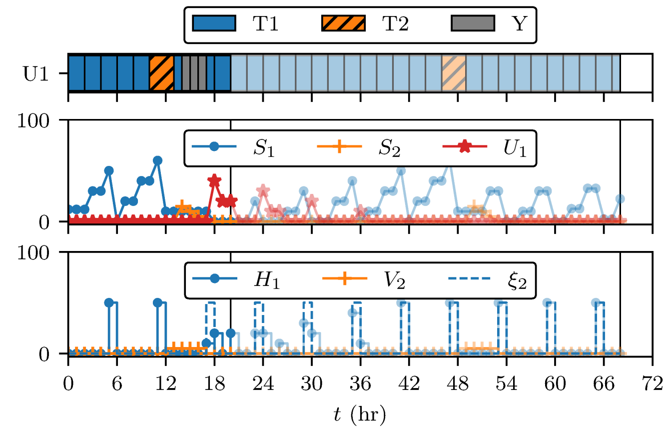

Fig. 1. An example closed-loop trajectory with $\varepsilon = 0.1$ and $\rho = 200$. The closed-loop schedule is drawn in solid colors and the open-loop schedule is drawn in faded colors. The top plot is a Gantt chart, the middle plot shows the inventory and backlog for each material, and the bottom plot shown the shipments of M1 to meet demand (H_1), sale of M2 (V_2), and demand for M1 (ξ_1).

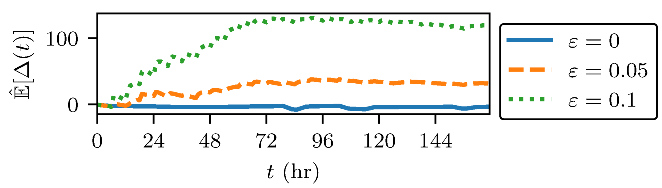

Fig. 2. The sample averages of $\Delta(t)$ for 30 realizations of the closed-loop trajectory for multiple values of ε and $\rho = 200$.

Note that the optimizer may report an optimality gap of greater than zero even if the current solution is, in fact, optimal. We use a 48 hour open-loop horizon ($N = 48$). We use Gurobi to solve these optimization problems [7]. In Figure 1, we plot an example closed-loop trajectory for $\rho = 200$ and $\varepsilon = 0.1$. The closed-loop trajectory is shown in the solid colors while the computed open-loop trajectory (schedule) is shown in the faded colors. Note that three delays occur (in a row) after $t = 14$ and cause the closed-loop trajectory to accumulate backlog. Furthermore, the warm start is infeasible after each of these disturbances occurs.

We simulate 30 realizations of the closed-loop trajectory for $\varepsilon \in \{0, 0.05, 0.1\}$ and optimality gaps of $\rho \in \{0, 50, 100, 200\}$.[1] We use the first state in the periodic reference as the initial state for each of these simulations. We define the running average economic cost for the closed-loop trajectory relative to the reference trajectory as $\Delta(t)$.

$$\Delta(t) := \frac{1}{1+t}\sum_{k=0}^{t} \bar{\ell}(x(k), u(k), k)$$

Note that $\Delta(t)$ for large t is the finite horizon approximation of the performance metric used in Theorems 1 and 2. We denote the sample average of $\Delta(t)$ as $\hat{\mathbb{E}}\left[\Delta(t)\right]$.

We plot $\hat{\mathbb{E}}\left[\Delta(t)\right]$ for multiple values of ε and $\rho = 200$ in Figure 2. For the nominal system ($\varepsilon = 0$), we see that the

[1]200 is approximately 25% of cost for the 72 hour periodic schedule.

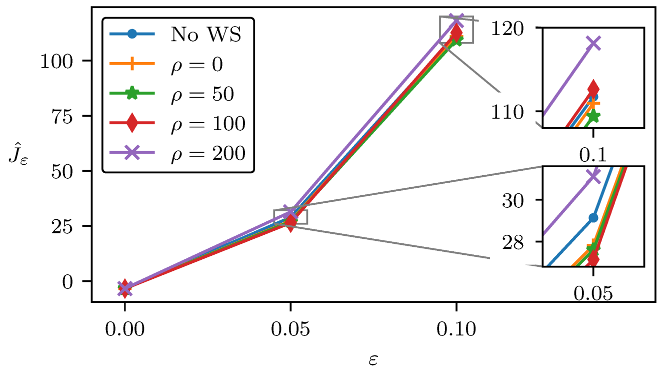

Fig. 3. The value of $\hat{J}_\varepsilon$ for multiple values of ε and different optimality gaps $\rho \in \{0, 50, 100, 200\}$ as well as optimal MPC without a warm start (labeled "No WS").

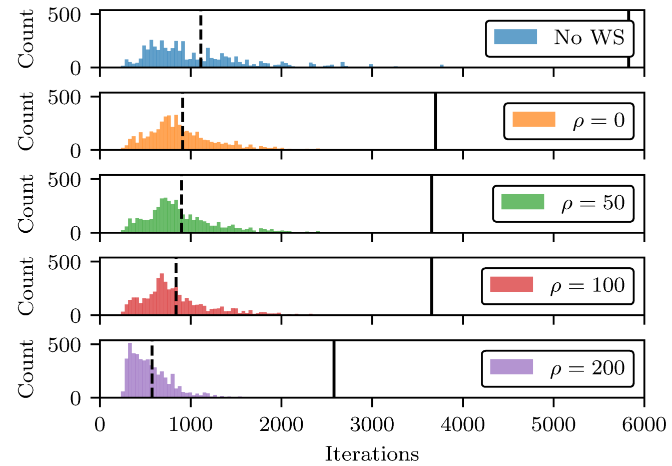

Fig. 4. Histograms of the number of iterations for each open-loop optimization problem and all 30 simulations. From top to bottom, we have the results for the algorithm without a warm start, $\rho = 0$, $\rho = 50$, $\rho = 100$, and $\rho = 200$. The black dashed line is the mean and the solid black line is the maximum number of iterations.

closed-loop trajectory is better than the reference trajectory ($\hat{\mathbb{E}}[\Delta(t)] < 0$). If we implemented only the warm start, without any additional optimization, we would expect $\hat{\mathbb{E}}[\Delta(t)] = 0$ as $t \to \infty$. Including even a small number of iterations of the optimization solver (until the guaranteed optimality gap is less than 200) is sufficient to improve the closed-loop trajectory even if the warm start is feasible. For $\varepsilon = 0.05, 0.1$, we see that the disturbances drive $\hat{\mathbb{E}}[\Delta(t)]$ away from zero. Both of these trajectories, however, appear to approach an asymptotic limit as $t \to \infty$.

We treat the value of $\hat{\mathbb{E}}[\Delta(t)]$ for $t = 164$ as an approximation of the infinite limit in Theorem 2 and denote this value as $\hat{J}_\varepsilon$. We plot the value of $\hat{J}_\varepsilon$ in Figure 3 for each value of $\rho \in \{0, 50, 100, 200\}$. We also plot the value of $\hat{J}_\varepsilon$ for the closed-loop system if no warm start is provided (labeled "No WS"). Note that the algorithms with and without a warm start with zero optimality gap may produce different results because the MIQP has multiple solutions. In fact, we observe that $\hat{J}_{0.05}$ is larger for the algorithm without a warm start than for the algorithms with a warm start and $\rho \leq 100$.

The results in Figure 3 are consistent with Theorem 2. The value of $\hat{J}_\varepsilon$ is less than zero for $\varepsilon = 0$ and increases with increasing ε. We also observe that, by increasing the value of ρ, the performance is nearly equivalent for $\varepsilon = 0$ and degrades slightly for $\varepsilon > 0$ (15% and 6.5% for $\varepsilon = 0.05, 0.1$, respectively, with $\rho = 200$). We emphasize, however, that an optimality gap of $\rho = 200$ is large for this example. A more reasonable gap of $\rho = 50, 100$ results in performance that is very close to the performance for $\rho = 0$. Note that an optimal solution to the finite horizon open-loop optimization problem does not guarantee superior closed-loop performance, e.g., the closed-loop performance of $\rho = 50$ is better than $\rho = 0$ at $\varepsilon = 0.1$.

In Figure 4, we plot the number of iterations required for each open-loop optimization problem and all 30 simulations for each algorithm considered. By providing the optimizer with a warm start, we observe a significant improvement in both the average (decrease of 27%) and maximum (decrease of 36%) number of iterations required. For this problem, increasing the optimality gap to $\rho = 50, 100$ does not significantly reduce the computational burden or produce a noticeable degradation in performance. For large-scale and more complex problems, however, this additional optimality gap can produce a significant difference in the computational burden [19]. If we further increase the optimality gap to $\rho = 200$, we observe another significant decrease in the average (decrease of 30%) and maximum (decrease of 30%) number of iterations compared to $\rho = 100$.

VI. Conclusions

We proposed a suboptimal economic MPC algorithm that combines the strengths of two common suboptimal MPC algorithms. We then established that this suboptimal MPC algorithm is economically robust to large and infrequent disturbances. For a small production scheduling example, we demonstrated that the proposed suboptimal MPC algorithm achieves similar (and in some cases superior) economic performance compared to optimal MPC (without a warm start) while reducing the average and maximum number of iterations required for the open-loop optimization problems. This algorithm is readily applicable to large-scale economic MPC problems that are solved as MILPs/MIQPs.

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