The Stochastic Robustness of Nominal and Stochastic Model Predictive Control

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***Abstract*—In this work, we establish and compare the stochastic and deterministic robustness properties achieved by nominal model predictive control (MPC), stochastic MPC (SMPC), and a proposed constraint-tightened MPC (CMPC) formulation, which represents an idealized version of tube-based MPC. We consider three definitions of robustness for nonlinear systems and bounded disturbances: robustly asymptotically stable (RAS), robustly asymptotically stable in expectation (RASiE), and RASiE with respect to the stage cost $\ell(\cdot)$ used in these MPC formulations (ℓ-RASiE). Via input-to-state stability (ISS) and stochastic ISS (SISS) Lyapunov functions, we establish that MPC, subject to sufficiently small disturbances, and CMPC ensure all three definitions of robustness without a stochastic objective function. While SMPC is RASiE and ℓ-RASiE, SMPC is not neccesarily RAS for nonlinear systems. Through a few simple examples, we illustrate the implications of these results and demonstrate that, depending on the definition of robustness considered, SMPC is not necessarily more robust than nominal MPC even if the disturbance model is exact.**

***Index Terms*—Model predictive control (MPC), stability of nonlinear systems, stochastic systems, stochastic optimal control.**

I. Introduction

In PRACTICE, a control algorithm must ensure some margin of robustness to disturbances. Nominal model predictive control (MPC) is known to be robustly asymptotically stable (RAS) with respect to (w.r.t.) sufficiently small disturbances [1], [11], [29], [39]. We use the term *inherent* robustness to describe this property of nominal MPC as this robustness is achieved through feedback, and disturbances are not explicitly considered in the problem formulation. This inherent robustness is often sufficient for successful implementation of MPC.

Stochastic MPC (SMPC) instead considers disturbances explicitly in the problem formulation. The SMPC problem is typically defined as a minimization of the expected value of the cost function, given a probabilistic description of the uncertainty in the system [8], [22], [28]. The optimization problem is often subject to deterministic and probabilistic state and input constraints. This stochastic optimization problem, however, is more computationally intensive and further complicates the closed-loop analysis relative to nominal MPC. In this article, we focus on the closed-loop properties of SMPC.

With appropriate terminal constraints and costs, linear SMPC ensures robust recursive feasibility and stability in expectation for the closed-loop system [2], [3], [4], [17]. If we assume that the effect of the disturbance vanishes at the origin, one can also establish that the origin of the closed-loop system is asymptotically stable in probability [31]. Lorenzen et al. [21] established that linear SMPC asymptotically stabilizes, with probability one, the minimal robust positively invariant set for the system. For *nonlinear* SMPC, Chatterjee and Lygeros [6] established, for unconstrained nonlinear systems, that the expected value of the optimal cost along the closed-loop trajectory is bounded if the terminal cost is a *global* stochastic Lyapunov function. Mayne and Falugi [23] extended these results to address constrained nonlinear systems subject to bounded, stochastic disturbances and, with terminal constraints, require the terminal cost to be only a *local* stochastic Lyapunov function. In [27], the authors established that SMPC renders the closed-loop system RAS in expectation (RASiE). Fundamental mathematical properties, such as the existence of solutions and the measurability of the closed-loop trajectory, are also established [26], [27]. In [27], however, there is no discussion of the theoretical properties of other MPC formulations or a comparison of these formulations.

Tube-based stochastic/robust MPC formulations offer a middle ground between nominal MPC and SMPC. These methods use information about the disturbance distribution and/or support to tighten constraints while retaining the computational and theoretical convenience afforded by a nominal objective function. These tube-based methods were proposed for linear systems subject to worst-case disturbances [7], [24] and extended to consider nonlinear systems [5], [20], [25]. In [4], a stochastic description of these disturbances was used to construct tubes that satisfy probabilistic (chance) constraints for the system. The notion of incremental stabilizability can also be used to tighten constraints without the need for complicated offline computations [16], [35].

Despite the obvious connection between these different MPC formulations, we are unaware of any rigorous comparison of the theoretical properties achieved by these different techniques.

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We attribute this limitation to the fact that closed-loop results for nominal MPC and SMPC are typically deterministic and stochastic, respectively. Thus, there are currently no precise characterizations of robustness that are comparable across these two methods.

In this article, we address this limitation and compare these MPC formulations via three definitions of deterministic and stochastic robustness for closed-loop nonlinear systems with bounded disturbances. The first, RAS, is based on input-to-state stability (ISS) [14] and is a definition of deterministic robustness. The second, RASiE, is influenced by nonlinear stochastic stability results [19] and the more recent definitions of stochastic input-to-state stability (SISS) [18], [37], [38].[1] RASiE is a definition of stochastic robustness that characterizes the behavior of a stochastic closed-loop system, in terms of the norm of the state, subject to different disturbance distributions. The third definition, RASiE w.r.t. the stage cost $\ell(\cdot)$ (ℓ-RASiE), extends the notion of stochastic robustness, which typically addresses the norm of the state, to a more general performance metric $\ell(\cdot)$. Since the MPC problem directly considers this stage cost $\ell(\cdot)$ in the optimization problem, ℓ-RASiE offers a more natural definition of stochastic robustness to compare across different MPC formulations.

The rest of this article is organized as follows. In Section II, we introduce three definitions of robustness for closed-loop systems: RAS, RASiE, and ℓ-RASiE. We then establish sufficient conditions for these definitions through ISS and SISS Lyapunov functions. In Section III, we establish that nominal MPC satisfies all of these definitions of robustness for sufficiently small disturbances. In Section IV, we establish that SMPC is RASiE and ℓ-RASiE, but is *not* necessarily RAS. In Section V, we propose a constraint-tightened MPC (CMPC) formulation that represents an idealized version of tube-based MPC, i.e., tightened constraints with a nominal objective function. We then establish that CMPC is RAS, RASiE, and ℓ-RASiE for the same disturbance set considered in SMPC. In Section VI, we consider several small examples to illustrate and compare the theoretical properties of these MPC formulations., Finally, Section VII concludes this article.

Notation: Let $\mathbb{I}$ and $\mathbb{R}$ denote the integers and reals, respectively. Let superscripts and subscripts denote dimension and restrictions (e.g., $\mathbb{R}^n_{\geq 0}$ denotes nonnegative reals of dimension n), respectively. Let $|\cdot|$ denote Euclidean norm. For a sequence $\mathbf{w}_k := (w(0), w(1), \ldots, w(k-1))$, we define $||\mathbf{w}_k|| := \max_{i \in \mathbb{I}_{[0,k-1]}} |w(i)|$. The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}_\infty$ if $\alpha(\cdot) \in \mathcal{K}$ and unbounded, i.e., $\lim_{s\to\infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{KL}$ if for every $k \in \mathbb{I}_{\geq 0}$ the function $\beta(\cdot, k)$ is in class $\mathcal{K}$ and for fixed $s \in \mathbb{R}_{\geq 0}$, the function $\beta(s, \cdot)$ is nonincreasing and $\lim_{k\to\infty} \beta(s, k) = 0$. Let $\mathcal{B}(\Omega)$ denote the Borel algebra of some set Ω. Let $\mathrm{tr}(A)$ denote the trace of a matrix A.

[1]These definitions, however, are distinct from the one used in this work. See [27] for a further discussion.

II. Robustness of Closed-Loop Systems

A. Closed-Loop Stochastic System and Preliminaries

We consider the following discrete-time system:

$$x^+ = f(x, u, w) \quad f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{W} \to \mathbb{R}^n \tag{1}$$

in which $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{U} \subseteq \mathbb{R}^m$ is the controlled input, $w \in \mathbb{W} \subseteq \mathbb{R}^q$ is a disturbance (random variable), and x^+ denotes the successor state. Let $(\Omega, \mathcal{F}, P)$ be the probability space for the sequence $\mathbf{w}_\infty : \Omega \to \mathbb{W}^\infty$ of random variables w, i.e., $\mathbf{w}_\infty := \{w(i)\}_{i=0}^\infty$ for $w(i) : \Omega \to \mathbb{W}$. We define the subsequence $\mathbf{w}_i : \Omega \to \mathbb{W}^i$ as $\mathbf{w}_i := (w(0), \ldots, w(i-1))$. We also define the expected value of a Borel measurable function $g : \mathbb{W}^i \to \mathbb{R}$ as the following Lebesgue integral:

$$\mathbb{E}\left[g(\mathbf{w}_i)\right] := \int_\Omega g(\mathbf{w}_i(\omega)) dP(\omega).$$

We make the following assumption for the disturbances.

Assumption 1 (Disturbances): The disturbances $w(i) : \Omega \to \mathbb{W}$ are independent and identically distributed (i.i.d.), zero mean ($\mathbb{E}[w(i)] = 0$), random variables. The support $\mathbb{W}$ is compact and contains the origin.

Given Assumption 1, each random variable has an equivalent probability measure that we denote $\mu : \mathcal{B}(\mathbb{W}) \to [0,1]$. This probability measure satisfies $\mu(F) = P(\{\omega \in \Omega : w(i;\omega) \in F\})$ for all $F \in \mathcal{B}(\mathbb{W})$ and $i \in \mathbb{I}_{\geq 0}$. We use $\mathcal{P}(\mathbb{W})$ to denote the collection of all possible probability measures $\mu(\cdot)$ on the support $\mathbb{W}$ that satisfy Assumption 1, i.e.,

$$\int_\mathbb{W} w d\mu(w) = 0 \quad \forall\, \mu(\cdot) \in \mathcal{P}(\mathbb{W}).$$

Since $\mathbb{W}$ is bounded, the second moment of w is finite. For any $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, we denote the covariance matrix of w as

$$\Sigma := \mathbb{E}\left[ww'\right] = \int_\mathbb{W} ww' d\mu(w).$$

Note that this covariance matrix is the same for all $i \in \mathbb{I}_{\geq 0}$ because of Assumption 1. For the i.i.d. random variables $(w(i), w(i+1), \ldots, w(i+N-1))$ and $N \in \mathbb{I}_{\geq 1}$, their joint distribution measure $\mu^N : \mathcal{B}(\mathbb{W}^N) \to [0,1]$ is defined as $\mu^N(F) = \mu(F_i)\mu(F_{i+1}) \ldots \mu(F_{i+N-1})$ for all $F = (F_i, F_{i+1}, \ldots, F_{i+N-1}) \in \mathcal{B}(\mathbb{W}^N)$.

Each MPC formulation uses a stage cost, i.e., performance metric, $\ell(\cdot)$ to define a feedback controller on a feasible set $\mathcal{X} \subseteq \mathbb{R}^n$. We consider the origin to be the steady-state target, without loss of generality, and consider the following regularity assumption.

Assumption 2 (Continuity of system and cost): The system $f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{W} \to \mathbb{R}^n$ and stage cost $\ell : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ are continuous and satisfy $f(0,0,0) = 0$, $\ell(0,0) = 0$.

Since we intend to analyze SMPC, we must also allow the control law to depend on the probability measure $\mu(\cdot)$, as this probability measure is included in the SMPC optimization problem. Thus, we define a generic control law $\kappa_\mu : \mathcal{X} \to \mathbb{U}$ for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. The resulting closed-loop system is then

$$x^+ = f(x, \kappa_\mu(x), w). \tag{2}$$

We use $\phi_\mu(k; x, \mathbf{w}_k)$ to denote the closed-loop state for (2) at time $k \in \mathbb{I}_{\geq 0}$, given the initial state $x \in \mathcal{X}$ and disturbance sequence $\mathbf{w}_k = (w(0), \ldots, w(k-1)) \in \mathbb{W}^k$. In addition to the closed-loop state trajectory $\phi_\mu(\cdot)$, we may also consider the performance of the system via the stage cost. Specifically, we investigate the closed-loop stage cost

$$\ell(x(k), \kappa_\mu(x(k)))$$

along the closed-loop trajectory $x(k) = \phi_\mu(k; x, \mathbf{w}_k)$. If we are discussing a control law for a specific $\mu(\cdot)$ or a control law that does not depend on the probability measure, e.g., nominal MPC, we may write the control law without this subscript as $\kappa(\cdot)$ and the corresponding closed-loop trajectory as $\phi(k; x, \mathbf{w}_k)$.

Since the control laws for MPC, SMPC, and CMPC are defined by solving an optimization problem at each time step, we require that these optimization problems remain feasible along the closed-loop trajectory to ensure that the control law remains well defined, i.e., we require that the optimization problem for each of these formulations is robustly recursively feasible. We characterize this property through robust positive invariance (RPI).

Definition 1 (RPI): A set $\mathcal{X}$ is RPI for the system $x^+ = f(x, \kappa_\mu(x), w)$, $w \in \mathbb{W}$ if $x^+ \in \mathcal{X}$ for all $x \in \mathcal{X}$, $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, and $w \in \mathbb{W}$.

If the feasible set of the optimization problem is RPI, then the optimal control problem is robustly recursively feasible. If the control law is independent of $\mu(\cdot)$, i.e., $\kappa_\mu(\cdot) = \kappa(\cdot)$, then this definition of RPI reduces to the standard definition of RPI as found in [33, Def. 3.7].

Remark 2: Although $\kappa_\mu : \mathcal{X} \to \mathbb{U}$ is not necessarily a continuous function, McAllister and Rawlings [26] established that, under basic regularity assumptions, SMPC, and by extension MPC and CMPC, produce a Borel measurable control law. Therefore, all relevant functions (e.g., $\phi_\mu(k; \cdot)$) are Borel measurable functions and the expected value is well defined for the closed-loop system.

B. Robust Asymptotic Stability

We define RAS as follows. Note that RAS applies to a specific control law, and we therefore consider $\kappa(\cdot)$ and $\phi(\cdot)$ without the subscript $\mu(\cdot)$.

Definition 3 (RAS): The origin of a system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$ is RAS in an RPI set $\mathcal{X}$ if there exist $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that

$$|\phi(k; x, \mathbf{w}_k)| \leq \beta(|x|, k) + \gamma(||\mathbf{w}_k||) \tag{3}$$

for all $x \in \mathcal{X}$, $\mathbf{w}_k \in \mathbb{W}^k$, and $k \in \mathbb{I}_{\geq 0}$.

The definition of RAS is based on the more general notion of ISS for discrete-time systems [14]. To establish RAS, we use an ISS Lyapunov function.

Definition 4 (ISS Lyapunov function): The function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is an ISS Lyapunov function for the system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$ in an RPI set $\mathcal{X}$ if there exist $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\sigma(\cdot) \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{4}$$

$$V(f(x, \kappa(x), w)) \leq V(x) - \alpha_3(|x|) + \sigma(|w|) \tag{5}$$

for all $x \in \mathcal{X}$ and $w \in \mathbb{W}$.

Proposition 5: If a system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$, admits an ISS Lyapunov function in an RPI set $\mathcal{X}$, then the origin is RAS in $\mathcal{X}$.

See [14, Lemma 3.5] for a proof of Proposition 5 for continuous ISS Lyapunov functions and [1, Prop. 19] for a more general proof for discontinuous ISS Lyapunov functions.

C. Robust Asymptotic Stability in Expectation

RAS is a strong property in that the bound in (3) is deterministic and holds for any realization of the disturbance trajectory ($\mathbf{w}_k \in \mathbb{W}^k$). Thus, RAS provides a bound for a deterministic property of the system based on a deterministic property of the disturbance trajectory. If we however use a stochastic representation for the disturbance, we can construct a similar bound for a stochastic property of the closed-loop system based on a stochastic property of the disturbance. Specifically, we define RASiE as a definition of stochastic robustness and derive some associated results in this section.

To motivate the following definition of stochastic robustness, we begin with a classic result for the stochastic linear-quadratic regulator (LQR). For the LQR problem, we have that $f(x, u, w) = Ax + Bu + w$ and $\ell(x, u) = x'Qx + u'Ru$ with $Q, R \succ 0$ and (A, B) stabilizable. We solve the infinite horizon stochastic optimal control problem through a discrete-time algebraic Riccati equation to yield the feedback controller $\kappa(x) = Kx$ and the Schur stable matrix $A_K = A + BK$, i.e., all eigenvalues of A_K are strictly inside the unit circle. Thus, the closed-loop system satisfies

$$\phi(k; x, \mathbf{w}_k) = A_K^k x + \sum_{i=0}^{k-1} A_K^{k-1-i} w(i).$$

Since A_K is Schur stable, there exists $c > 0$ and $\lambda \in (0, 1)$ such that

$$|\phi(k; x, \mathbf{w}_k)| \leq \lambda^k c|x| + c \sum_{i=0}^{k-1} \lambda^{k-1-i} |w(i)| \tag{6}$$

which leads to RAS by noting $\sum_{i=0}^{k-1} \lambda^{k-1-i} \leq 1/(1-\lambda)$. Alternatively, we can take the expected value of (6) and note that $\mathbb{E}[|w(i)|] \leq \sqrt{\text{tr}(\Sigma)}$ to give

$$\mathbb{E}\left[|\phi(k; x, \mathbf{w}_k)|\right] \leq \lambda^k c|x| + \frac{c}{1-\lambda} \sqrt{\text{tr}(\Sigma)}.$$

Often this bound is derived for a *single* probability distribution $\mu(\cdot)$ and corresponding variance Σ. But this bound in fact holds with the same constants c and λ for *all* $\mu(\cdot)$ and Σ such that $\mathbb{E}[w] = 0$. The influence of the probability distribution on this bound is captured entirely through the value of tr(Σ). Thus, we want to define a nonlinear extension of this bound with the same requirement: the upper bound depends on the probability distribution through only a function of tr(Σ) and this bound holds for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$ and corresponding Σ.

One of the convenient results in the stochastic LQR problem is that the feedback controller $\kappa(x) = Kx$ is independent of the probability distribution. For SMPC, however, the control law varies with the probability distribution, i.e., we have $\kappa_\mu(\cdot)$.[2] Thus, we define stochastic robustness such that the control law can vary with $\mu(\cdot)$ and thereby accommodate SMPC. We also require, per the previous discussion, that the upper bound in the definition of RASiE depends on the value of $\mu(\cdot)$ through only a function of $\text{tr}(\Sigma)$.

Definition 6 (RASiE): The origin of a closed-loop stochastic system $x^+ = f(x, \kappa_\mu(x), w)$, $w \in \mathbb{W}$ is RASiE in an RPI set $\mathcal{X}$ if there exist $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that

$$\mathbb{E}\left[|\phi_\mu(k; x, \mathbf{w}_k)|\right] \le \beta(|x|, k) + \gamma(\text{tr}(\Sigma)) \tag{7}$$

for all $x \in \mathcal{X}$, $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, and $k \in \mathbb{I}_{\ge 0}$.

Note that the upper bound depends on the probability measure through only the argument $\text{tr}(\Sigma)$. The functions $\beta(\cdot)$ and $\gamma(\cdot)$ are the same for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$.

Analogous to the ISS Lyapunov function, we define a stochastic ISS (SISS) Lyapunov function. We also allow this Lyapunov function to vary with $\mu(\cdot)$ as the optimal cost for SMPC varies with $\mu(\cdot)$ as well (see Section IV).

Definition 7 (SISS Lyapunov function): The measurable function $V_\mu : \mathcal{X} \to \mathbb{R}_{\ge 0}$, defined for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, is an SISS Lyapunov function for the system $x^+ = f(x, \kappa_\mu(x), w)$, $w \in \mathbb{W}$, in the RPI set $\mathcal{X}$ if there exist $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\sigma_2(\cdot), \sigma_3(\cdot) \in \mathcal{K}$ such that

$$\alpha_1(|x|) \le V_\mu(x) \le \alpha_2(|x|) + \sigma_2(\text{tr}(\Sigma)) \tag{8}$$

$$\int_{\mathbb{W}} V_\mu(f(x, \kappa_\mu(x), w)) d\mu(w) \le V_\mu(x) - \alpha_3(|x|) + \sigma_3(\text{tr}(\Sigma)) \tag{9}$$

for all $x \in \mathcal{X}$ and $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$.

In Definition 7, we allow the SISS Lyapunov function to vary with the probability measure $\mu(\cdot)$, but require that the $\mathcal{K}$-functions are the same for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. The bounds vary with the probability measure $\mu(\cdot)$ exclusively through the argument $\text{tr}(\Sigma)$ used in $\sigma_2(\cdot), \sigma_3(\cdot)$. Definition 7 is novel because we consider all possible values of $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Our previous definition of RASiE in [27], like the rest of SMPC literature, is derived for only a single $\mu(\cdot)$. However, the proof procedures for the results presented here for SMPC require only minor extensions to the proof procedures used in [27].

Proposition 8: If a system $x^+ = f(x, \kappa_\mu(x), w)$, $w \in \mathbb{W}$, admits an SISS Lyapunov function in an RPI and bounded set $\mathcal{X}$, then the origin is RASiE in $\mathcal{X}$.

Outline of Proof: For an arbitrary $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, we use the same approach as used in [27, Prop. 13] to construct the relevant $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ functions in (7) in the definition of RASiE. We then note that the construction of these functions relies on only the $\mathcal{K}$ and $\mathcal{K}_\infty$ functions used in bounds (8) and (9) and does not depend explicitly on the probability measure $\mu(\cdot)$. Since the $\mathcal{K}$ and $\mathcal{K}_\infty$ functions in (8) and (9) apply for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, the constructed $\beta(\cdot)$ and $\gamma(\cdot)$ also hold for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, thus meeting the requirement of RASiE in Definition 6. ■

We find the following results (based on [30, Lemma 14]) useful in the subsequent analysis. See [27, Cor. 7] for a proof.

Lemma 9: If $\alpha(\cdot) \in \mathcal{K}_\infty$, then for any $b \in \mathbb{R}_{\ge 0}$, there exists $\alpha_c(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_c(\cdot)$ is concave and $\alpha(s) \le \alpha_c(s)$ for all $s \in [0, b]$.

This lemma allows us to establish the following new result for a single control law $\kappa(\cdot)$.

Proposition 10: Let Assumption 1 hold. If a function $V : \mathcal{X} \to \mathbb{R}_{\ge 0}$ is an ISS Lyapunov function for the system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$, in an RPI set $\mathcal{X}$, then $V(\cdot)$ is also an SISS Lyapunov function for the system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$, in $\mathcal{X}$.

Proof: Since $V(\cdot)$ is an ISS Lyapunov function, there exist $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\sigma(\cdot) \in \mathcal{K}$ such that (4) and (5) hold. Immediately, we have that (8) holds for the same functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and any $\sigma_2(\cdot) \in \mathcal{K}$.

Since $\mathbb{W}$ is compact, there exists $b \ge 0$ such that $|w| \in [0, b]$ for all $w \in \mathbb{W}$. We define a function $\tilde{\sigma}(\cdot) \in \mathcal{K}_\infty$ such that $\sigma(s) \le \tilde{\sigma}(s)$ for all $s \in \mathbb{R}_{\ge 0}$, e.g., $\tilde{\sigma}(s) := \varepsilon s + \sigma(s)$ with $\varepsilon > 0$. We use Lemma 9 to construct a concave function $\sigma_c(\cdot) \in \mathcal{K}_\infty$ such that $\sigma(|w|) \le \tilde{\sigma}(|w|) \le \sigma_c(|w|)$ for all $w \in \mathbb{W}$. We apply this bound and Jensen's inequality to give

$$\begin{aligned}
&\int_{\mathbb{W}} V(f(x, \kappa(x), w)) d\mu(w) \\
&\quad \le V(x) - \alpha_3(|x|) + \int_{\mathbb{W}} \sigma(|w|) d\mu(w) \\
&\quad \le V(x) - \alpha_3(|x|) + \int_{\mathbb{W}} \sigma_c(|w|) d\mu(w) \\
&\quad \le V(x) - \alpha_3(|x|) + \sigma_c\left(\int_{\mathbb{W}} |w| d\mu(w)\right) \\
&\quad = V(x) - \alpha_3(|x|) + \sigma_c\left(\mathbb{E}[|w|]\right).
\end{aligned}$$

From Jensen's inequality, we can also write $\mathbb{E}[|w|]^2 \le \mathbb{E}[|w|^2] = \text{tr}(\Sigma)$. We define $\sigma_3(s) := \sigma_c(s^{1/2})$ and note that $\sigma_3(\cdot) \in \mathcal{K}$. Thus, we have that $\sigma_c(\mathbb{E}[|w|]) = \sigma_3(\mathbb{E}[|w|]^2) \le \sigma_3(\text{tr}(\Sigma))$ and

$$\int_{\mathbb{W}} V(f(x, \kappa(x), w)) d\mu(w) \le V(x) - \alpha_3(|x|) + \sigma_3\left(\text{tr}(\Sigma)\right).$$

Note that this bound holds for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Therefore, (9) holds for $\alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\sigma_3(\cdot) \in \mathcal{K}$, and $V_\mu(\cdot) = V(\cdot)$ is an SISS Lyapunov function.

The converse of Proposition 10, however, does not hold. For example, consider the scalar system $x^+ = (0.9 + w)x$ for $w \in \mathbb{W} := \{-0.2, 0, 0.2\}$ distributed such that $\mathbb{E}[w] = 0$. The system is not ISS because $w = 0.2$ produces an unstable system $x^+ = 1.1x$, and therefore, $|x(k)| \to \infty$ for some $w \in \mathbb{W}$. But, $V(x) = x^2$ is an SISS Lyapunov function. ■

Proposition 10 is both somewhat obvious and also, to the best of our knowledge, new. Furthermore, this result allows us to, for the first time, *directly compare the theoretical properties of nominal MPC and SMPC*. Specifically, we use Proposition 10 in the subsequent analysis of nominal MPC to establish that the

[2] The liquid-level control example in Section VI-B has this feature. For different $\mu(\cdot)$, the control law and closed-loop trajectory for SMPC are different.

ISS Lyapunov function typically derived for nominal MPC also confers RASiE. Moreover, Proposition 10 leads to the following corollary that establishes a significant connection between RAS and RASiE.

Corollary 11: Let Assumption 1 hold. If the origin of a stochastic system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$, is RAS in an RPI and bounded set $\mathcal{X}$, then the origin is also RASiE in $\mathcal{X}$.

Outline of proof: We use [12, Th. 2.3] to establish that RAS (ISS) implies the existence of an ISS Lyapunov function. Note that we must specialize the results in [12] to a robustly positive invariant set $\mathcal{X}$ and bounded disturbances, but this extension is minor. By Proposition 10, this ISS Lyapunov function is also an SISS Lyapunov function and by Proposition 8 the origin is RASiE. ■

D. Robustness W.R.T. the Stage Cost

For steady-state tracking applications of MPC, a standard requirement is that the stage cost is lower-bounded by a $\mathcal{K}_\infty$-function of $|x|$, as detailed in the following assumption.

Assumption 3 (Stage cost bound): There exists $\alpha_\ell(\cdot) \in \mathcal{K}_\infty$ such that $\alpha_\ell(|x|) \leq \ell(x, u)$ for all $(x, u) \in \mathbb{R}^n \times \mathbb{U}$.

This requirement ensures that if $\ell(x, u) \to 0$, then $|x| \to 0$, but allows for significant flexibility in selecting $\ell(\cdot)$. This flexibility is useful to tune the stage cost to reflect the importance of different state and input variables according to the problem of interest. This flexibility, however, also separates the objective of the MPC problem from the metric considered in the definition of RASiE. Thus, a reasonable metric for evaluating the robustness of MPC formulations is the one specifically prescribed to the MPC problem formulation: the stage cost. We therefore consider a definition of stochastic robustness w.r.t. this stage cost $\ell(\cdot)$ that we term RASiE w.r.t. the stage cost $\ell(\cdot)$ and abbreviate as ℓ-RASiE.

Definition 12 (ℓ-RASiE): The origin of a stochastic system $x^+ = f(x, \kappa_\mu(x), w)$, $w \in \mathbb{W}$, is said to be ℓ-RASiE w.r.t. the stage cost $\ell(x, \kappa_\mu(x))$ in an RPI set $\mathcal{X}$ if there exist $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that

$$\mathbb{E}\left[\ell(x(k), \kappa_\mu(x(k)))\right] \leq \beta(|x|, k) + \gamma(\mathrm{tr}(\Sigma)) \tag{10}$$

in which $x(k) = \phi_\mu(k; x, \mathbf{w}_k)$, for all $x \in \mathcal{X}$, $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, and $k \in \mathbb{I}_{\geq 0}$.

If we have that $\alpha_\ell(|x|) \leq \ell(x, u)$ for all $x \in \mathbb{R}^n$, and that $\mathcal{X}$ is bounded, ℓ-RASiE also implies RASiE. Furthermore, we note that this definition of robustness w.r.t. stage cost is stronger than the stage cost bound often derived for SMPC (see [27, Th. 12] or [6, Th. 6] for an example).

We now establish that an SISS Lyapunov function that also satisfies $\ell(x, \kappa_\mu(x)) \leq V_\mu(x)$ ensures that the origin is ℓ-RASiE. Since the Lyapunov function constructed for MPC is (almost) always based on the optimal cost function, requiring $\ell(x, \kappa_\mu(x)) \leq V_\mu(x)$ is minor.

Proposition 13: If a system $x^+ = f(x, \kappa_\mu(x), w)$, $w \in \mathbb{W}$, admits an SISS Lyapunov function $V_\mu : \mathcal{X} \to \mathbb{R}_{\geq 0}$ in an RPI and bounded set $\mathcal{X}$ that satisfies $\ell(x, \kappa_\mu(x)) \leq V_\mu(x)$ for all $x \in \mathcal{X}$ and $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, then the origin is ℓ-RASiE in $\mathcal{X}$.

Proof: Using the SISS Lyapunov function, we proceed with the same steps as in [27, Prop. 13], for an arbitrary $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, to give

$$\mathbb{E}\left[V_\mu(x(k))\right] \leq \max\{\tilde{\beta}(V_\mu(x), k), \tilde{\gamma}(\mathrm{tr}(\Sigma))\}$$

in which $x(k) = \phi_\mu(k; x, \mathbf{w}_k)$, $\tilde{\beta}(\cdot) \in \mathcal{KL}$, and $\tilde{\gamma}(\cdot) \in \mathcal{K}$. The construction of $\beta(\cdot)$ and $\gamma(\cdot)$ is similar to the standard approach for discrete-time ISS Lyapunov functions, as discussed in [14, Lemma 3.5]. We use the fact that $\ell(x, \kappa_\mu(x)) \leq V_\mu(x)$ to give

$$\mathbb{E}\left[\ell(x(k), \kappa_\mu(x(k)))\right] \leq \max\{\tilde{\beta}(V_\mu(x), k), \tilde{\gamma}(\mathrm{tr}(\Sigma))\}.$$

We use the upper bound for $V_\mu(x)$ to give

$$\begin{aligned}
&\mathbb{E}\left[\ell(x(k), \kappa_\mu(x(k)))\right] \\
&\quad \leq \tilde{\beta}\left(\alpha_2(|x|) + \sigma_2(\mathrm{tr}(\Sigma)), k\right) + \tilde{\gamma}(\mathrm{tr}(\Sigma)) \\
&\quad \leq \beta(|x|, k) + \gamma(\mathrm{tr}(\Sigma))
\end{aligned}$$

in which $\beta(s, k) := \tilde{\beta}(2\alpha_2(s), k) \in \mathcal{KL}$ and $\gamma(s) := \tilde{\beta}(2\sigma_2(s), 0) + \tilde{\gamma}(s) \in \mathcal{K}$. Note that the functions $\beta(\cdot)$ and $\gamma(\cdot)$ are constructed independently of $\mu(\cdot)$ and therefore apply for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. ■

Remark 14: We can consider additional performance metrics of the form $\ell_\mu : \mathcal{X} \to \mathbb{R}_{\geq 0}$ in the definition of ℓ-RASiE. If these metrics satisfy $\ell_\mu(x) \leq V_\mu(x)$, or, more generally, $\ell_\mu(x) \leq \alpha(|x|)$ for some $\alpha(\cdot) \in \mathcal{K}_\infty$, then the SISS Lyapunov function is sufficient to establish ℓ-RASiE w.r.t. this performance metric.

III. Nominal MPC

For nominal MPC, the disturbance is not explicitly considered in the optimization problem. Thus, the system model is

$$x^+ = f(x, u, 0). \tag{11}$$

For a prediction horizon $N \in \mathbb{I}_{\geq 1}$, we use $\hat{\phi}(k; x, \mathbf{u})$ to denote the open-loop state for (11) at time $k \in \mathbb{I}_{[0,N]}$, given the initial state x and the control trajectory $\mathbf{u} = (u(0), u(1), \ldots, u(N-1))$. For nominal MPC, we allow hard input constraints $u \in \mathbb{U} \subseteq \mathbb{R}^m$, but do not enforce hard state constraints. While satisfaction of hard state constraints is desirable, there is no guarantee that hard state constraints can be satisfied for a perturbed system. Instead, we assume that state constraints are converted to exact penalty functions that are included in the stage cost [15], [36], [40].

We do, however, use a terminal constraint $\mathbb{X}_f \subseteq \mathbb{R}^n$ and a define a terminal cost as $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. For MPC with a horizon of $N \in \mathbb{I}_{\geq 1}$, we define the set of admissible inputs, admissible states, and objective function, respectively, as

$$\begin{aligned}
\mathcal{U}(x) &:= \{\mathbf{u} \in \mathbb{U}^N : x(N) \in \mathbb{X}_f\} \\
\mathcal{X} &:= \{x : \mathcal{U}(x) \neq \emptyset\} \\
V(x, \mathbf{u}) &:= \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N))
\end{aligned}$$

in which $x(k) := \hat{\phi}(k; x, \mathbf{u})$. The nominal MPC problem for any $x \in \mathcal{X}$ is defined as

$$\mathbb{P}(x) : \ V^0(x) = \min_{\mathbf{u} \in \mathcal{U}(x)} V(x, \mathbf{u}) \tag{12}$$

and the optimal solutions for a given initial state are denoted $\mathbf{u}^0(x) := \arg\min_{\mathbf{u}\in\mathcal{U}(x)} V(x, \mathbf{u})$. Note that $\mathbf{u}^0(x)$ is a set-valued mapping because there may be multiple solutions to $\mathbb{P}(x)$. To streamline the following presentation, we assume that there exists some Borel measurable selection rule that defines a single-valued control law $\kappa : \mathcal{X} \to \mathbb{U}$ such that $\kappa(x) \in u^0(0; x)$ for all $x \in \mathcal{X}$, in which $u^0(0; x)$ is the set of first inputs in $\mathbf{u}^0(x)$. The resulting closed-loop system is then

$$x^+ = f(x, \kappa(x), w). \tag{13}$$

We use $\phi(k; x, \mathbf{w}_k)$ to denote the closed-loop state for (13) at time $k \in \mathbb{I}_{\geq 0}$ given the initial state $x \in \mathcal{X}$ and disturbance sequence $\mathbf{w}_k \in \mathbb{W}^k$.

We consider the following assumptions adapted from [1].

Assumption 4 (Properties of the constraint sets): The set $\mathbb{U}$ is compact and contains the origin. The set $\mathbb{X}_f$ is defined by $\mathbb{X}_f := \{x \in \mathbb{R}^n : V_f(x) \leq \tau\}$ for some $\tau > 0$.

Assumption 5 (Terminal ingredients): The function $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuous and satisfies $V_f(0) = 0$. There exists a terminal control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that for all $x \in \mathbb{X}_f$, $f(x, \kappa_f(x), 0) \in \mathbb{X}_f$ and

$$V_f(f(x, \kappa_f(x), 0)) \leq V_f(x) - \ell(x, \kappa_f(x)).$$

Under these assumptions, we can establish the following theorem for the robustness of nominal MPC.

Theorem 15 (MPC): Let Assumptions 1–5 hold. For every $\rho > 0$, there exists $\delta > 0$ such that for $\mathbb{W} \subseteq \{w \in \mathbb{R}^q : |w| \leq \delta\}$, the closed-loop system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$, and the set $\mathcal{S} := \{x \in \mathbb{R}^n : V^0(x) \leq \rho\} \cap \mathcal{X}$ we have the following.

i) The set $\mathcal{S}$ is RPI.
ii) The origin is RAS in the set $\mathcal{S}$.
iii) The origin is RASiE in the set $\mathcal{S}$.
iv) The origin is ℓ-RASiE in the set $\mathcal{S}$.

Outline of proof: Allan et al. [1] established (i) and (ii) for suboptimal MPC (and thereby optimal MPC) by using the cost function as an ISS Lyapunov function for the closed-loop system. (iii) By Proposition 10, this ISS Lyapunov function is also an SISS Lyapunov function and by Proposition 8, the origin is RASiE on the bounded and RPI set $\mathcal{S}$. (iv) Since the cost function is used as the ISS/SISS Lyapunov function, we also know that $\ell(x, \kappa(x)) \leq V(x)$ and by Proposition 13, the origin is ℓ-RASiE in the set $\mathcal{S}$. ■

Thus, nominal MPC, for sufficiently small disturbances ($|w| \leq \delta$), satisfies all of the definitions of deterministic and stochastic robustness in the set $\mathcal{S}$. Moreover, continuity of the optimal value function is not required for this result due to the choice of the terminal constraint as a sublevel set of an appropriate terminal cost (Assumptions 4 and 5). This robustness is achieved without directly considering the disturbance w in the optimization problem and is *inherent* to nominal MPC through feedback.

IV. Stochastic MPC

For SMPC, we consider the disturbance explicitly in the optimization problem and optimize over a set of potential control policy parameterizations. We first define the parameterized control policy $\pi : \mathbb{R}^n \times \mathbb{V} \to \mathbb{U}$ in which $x \in \mathbb{R}^n$ is the current state of the system and $v \in \mathbb{V} \subseteq \mathbb{R}^l$ are the parameters in the control policy (e.g., $\pi(x, v) = Kx + v$).[3] Thus, the resulting system of interest is defined as

$$x^+ = f(x, \pi(x, v), w). \tag{14}$$

We use $\hat{\phi}^s(k; x, \mathbf{v}, \mathbf{w})$ to denote the open-loop state for (14) at time k, given the initial state $x \in \mathbb{R}^n$, the trajectory of control policy parameters $\mathbf{v} = (v(0), v(1), \ldots, v(N-1))$, and disturbance trajectory $\mathbf{w} \in \mathbb{W}^N$.

We consider the case of hard input and state constraints, i.e., $(x, u) \in \mathbb{Z} \subseteq \mathbb{R}^n \times \mathbb{U}$. If we also consider (one-step-ahead) probabilistic, we can reformulate these constraints as $(x, u) \in \mathbb{Z}_\mu \subseteq \mathbb{Z}$ (see [27]), in which $\mathbb{Z}_\mu$ varies with $\mu(\cdot)$. To streamline the definitions and analysis in this initial work, however, we omit probabilistic constraints and therefore the feasible set for SMPC is independent of $\mu(\cdot)$. For a horizon $N \in \mathbb{I}_{\geq 1}$ and the terminal constraint $\mathbb{X}_f \subseteq \mathbb{R}^n$, we define the admissible parameter trajectories and feasible initial states as

$$\begin{aligned}\mathcal{V}(x) := \{\mathbf{v} \in \mathbb{V}^N :& \\ &(x(k), \pi(x(k), v(k))) \in \mathbb{Z} \ \forall \mathbf{w} \in \mathbb{W}^N, k \in \mathbb{I}_{[0,N-1]} \\ &x(N) \in \mathbb{X}_f \ \forall \mathbf{w} \in \mathbb{W}^N\} \\ \mathcal{X}^s := \{x : \mathcal{V}(x) \neq \emptyset\}&\end{aligned}$$

in which $x(k) = \hat{\phi}^s(k; x, \mathbf{v}, \mathbf{w})$.

We use the same stage and terminal cost defined for nominal MPC to define the function

$$J(x, \mathbf{v}, \mathbf{w}) = \sum_{k=0}^{N-1} \ell(x(k), \pi(x(k), v(k))) + V_f(x(N))$$

in which $x(k) = \hat{\phi}^s(k; x, \mathbf{v}, \mathbf{w})$. We define the SMPC cost function based on the expected value of $J(\cdot)$, i.e.,

$$V_\mu^s(x, \mathbf{v}) := \int_{\mathbb{W}^N} J(x, \mathbf{v}, \mathbf{w}) d\mu^N(\mathbf{w}).$$

The optimization problem for any $x \in \mathcal{X}^s$ is defined as

$$\mathbb{P}_\mu^s(x) : \ V_\mu^{s0}(x) = \min_{\mathbf{v} \in \mathcal{V}(x)} V_\mu^s(x, \mathbf{v}) \tag{15}$$

and the optimal solutions for a given initial state are denoted $\mathbf{v}_\mu^{s0}(x) := \arg\min_{\mathbf{v}\in\mathcal{V}(x)} V_\mu^s(x, \mathbf{v})$. Note that $\mathbf{v}_\mu^{s0}(x)$ is a set-valued mapping because there may be multiple solutions to $\mathbb{P}_\mu^s(x)$. As with nominal MPC, we assume that there exists some Borel measurable selection rule that defines a single-valued control law $\kappa_\mu^s : \mathcal{X}^s \to \mathbb{U}$ such that $\kappa_\mu^s(x) \in \{\pi(x, v) : v \in v_\mu^{s0}(0; x)\}$ for all $x \in \mathcal{X}^s$, in which $v_\mu^{s0}(0; x)$ is the set of first parameter vectors in $\mathbf{v}_\mu^{s0}(x)$. Both the optimal cost and control

[3]Nominal MPC formulations can use a similar control law parameterization to "prestabilize" the open-loop system and thereby ensure that the MPC optimization problem is well conditioned [13], [34].

law for SMPC depend on the probability measure $\mu(\cdot)$. The resulting closed-loop system is then

$$x^+ = f(x, \kappa_\mu^s(x), w). \tag{16}$$

We use $\phi_\mu^s(k; x, \mathbf{w}_k)$ to denote the closed-loop state for (16) at time $k \in \mathbb{I}_{\geq 0}$ given the initial state $x \in \mathcal{X}^s$, disturbance sequence $\mathbf{w}_k \in \mathbb{W}^k$, and probability measure $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$.

For SMPC formulations, we typically assume that the model of the disturbance support and distribution is identical to that of the underlying plant. We note, however, that this assumption is idealized and typically not satisfied for any practical implementation of SMPC. Nonetheless, we proceed under this assumption. The study of idealized SMPC is analogous to that of nominal stability properties for a control algorithm. The merit of this analysis is in establishing the best performance one can expect from SMPC to serve as a baseline. If the performance is not satisfactory under ideal conditions, then there is little incentive to study nonideal conditions.

For SMPC, we require some of the same assumptions already stated for MPC (specifically, Assumptions 1–3) as well as a few modified assumptions.

Assumption 6 (Properties of the constraint sets; SMPC): The set $\mathbb{Z}$ is closed and contains the origin. The sets $\mathbb{U}$, $\mathbb{X}_f$ are compact and contain the origin. The set $\mathbb{X}_f$ contains the origin in its interior. The set $\mathcal{X}^s$ is bounded.

Assumption 7 (Terminal ingredients; SMPC): The function $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuous and satisfies $V_f(0) = 0$. There exists a continuous terminal control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that for all $x \in \mathbb{X}_f$,

$$f(x, \kappa_f(x), w) \in \mathbb{X}_f \quad \forall w \in \mathbb{W}$$

$$V_f(f(x, \kappa_f(x), 0)) \leq V_f(x) - \ell(x, \kappa_f(x)).$$

Furthermore, $(x, \kappa_f(x)) \in \mathbb{Z}$ and $\pi(x, 0) = \kappa_f(x)$ for all $x \in \mathbb{X}_f$.

Assumption 8 (Parameterization): The set $\mathbb{V}$ is compact and contains the origin. The function $\pi : \mathbb{R}^n \times \mathbb{V} \to \mathbb{U}$ is continuous.

Assumption 8 sets some basic requirements for the control law parameterization while Assumptions 6 and 7 are the versions of Assumptions 4 and 5 adjusted for SMPC. Assumption 6 addresses the additional state constraints added to the SMPC problem while dropping the requirement that $\mathbb{X}_f$ is a level set of the terminal cost. Assumption 7 requires, in addition to the nominal cost decrease in the terminal region, that the terminal control law renders the terminal set RPI. This requirement ensures that the SMPC algorithm is robustly recursively feasible.

We note that Assumption 7 may seem significantly stronger than Assumption 5. As such, we present the following result to better understand the relationship between these two assumptions (see the Appendix for proof).

Lemma 16: Let Assumptions 2–5 hold and $\kappa_f(\cdot)$ be a continuous function. Then, there exists $\delta > 0$ such that for any $\mathbb{W} \subseteq \{w \in \mathbb{R}^q : |w| \leq \delta\}$, the terminal set $\mathbb{X}_f$ is RPI for the system $x^+ = f(x, \kappa_f(x), w)$, $w \in \mathbb{W}$.

Thus, the assumptions required for MPC are already sufficient to guarantee that the terminal control law renders $\mathbb{X}_f$ RPI for disturbances up to some size $\delta > 0$. Indeed, the terminal control law and set already in use for an MPC formulation may also be satisfactory for an SMPC formulation. Of course, sufficiently large disturbances may render the construction of a suitable terminal control law and terminal set either difficult or impossible if we consider nonlinear systems and/or input constraints. Assumption 7 also ensures that $\mathbb{X}_f \subseteq \mathcal{X}^s$, and therefore, $\mathcal{X}^s$ is not empty (See Lemma 22).

With these assumptions, we have the following result.

Theorem 17 (SMPC): Let Assumptions 1–3 and 6–8 hold with $\mathbb{W}$ and $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$ known exactly. For the closed-loop system $x^+ = f(x, \kappa_\mu^s(x), w)$, $w \in \mathbb{W}$, we have the following.

i) The set $\mathcal{X}^s$ is RPI.
ii) The origin is RASiE in the set $\mathcal{X}^s$.
iii) The origin is ℓ-RASiE in the set $\mathcal{X}^s$.

Proof: In [27, Prop. 11], we establish that the set $\mathcal{X}^s$ is RPI for arbitrary $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Since $\mathcal{X}^s$ is not a function of $\mu(\cdot)$, $\mathcal{X}^s$ is therefore RPI for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, i.e., (i) holds. In [27, Prop. 11], we also show that there exists $\sigma(\cdot) \in \mathcal{K}$ such that

$$\begin{aligned}\int_{\mathbb{W}} V_\mu^{s0}(f(x, \kappa_\mu^s(x), w)) d\mu(w) \\ \leq V_\mu^{s0}(x) - \ell(x, \kappa_\mu^s(x)) + \sigma(\text{tr}(\Sigma))\end{aligned} \tag{17}$$

for all $x \in \mathcal{X}^s$ and arbitrary $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Although not explicitly stated in [27], (17) in fact holds for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$ with the same function $\sigma(\cdot) \in \mathcal{K}$ since this $\mathcal{K}$-function is constructed independently of the probability measure $\mu(\cdot)$. The function $\sigma(\text{tr}(\Sigma))$ captures the effect of varying probability measures entirely through the argument tr(Σ). By using Assumption 3, we have that there exists $\alpha_\ell(\cdot) \in \mathcal{K}_\infty$ such that

$$\begin{aligned}\int_{\mathbb{W}} V_\mu^{s0}(f(x, \kappa_\mu^s(x), w)) d\mu(w) \\ \leq V_\mu^{s0}(x) - \alpha_\ell(|x|) + \sigma(\text{tr}(\Sigma))\end{aligned}$$

for all $x \in \mathcal{X}^s$ and $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Furthermore, we have that $\alpha_\ell(|x|) \leq \ell(x, \kappa_\mu^s(x)) \leq V_\mu^{s0}(x)$ for all $x \in \mathcal{X}^s$ and $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$.

We now construct the upper bound for $V_\mu(\cdot)$. From the proof of [27, Lemma 14], we have that

$$V_\mu^{s0}(x) \leq V_f(x) + N\sigma(\text{tr}(\Sigma)) \tag{18}$$

for all $x \in \mathbb{X}_f$ and arbitrary $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Since the functions $V_f(\cdot)$ and $\sigma(\cdot)$ are constructed independently of $\mu(\cdot)$, we know that (18) holds for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. We then define

$$W(x) := \max \left\{ \sup_{\mu \in \mathcal{P}(\mathbb{W})} \left(V_\mu^{s0}(x) - N\sigma(\text{tr}(\Sigma)) \right),\ 0 \right\}$$

and note that $0 \leq W(x) \leq V_f(x)$ for all $x \in \mathbb{X}_f$. Since $V_f(\cdot)$ is continuous, $W(0) = V_f(0) = 0$, and $\mathbb{X}_f$ contains the origin in its interior, we know that $W(x)$ is continuous at the origin.

We now establish that $W(x)$ is locally bounded. Let X be a compact subset of $\mathcal{X}^s$. The function $J : \mathbb{R}^n \times \mathbb{V}^N \times \mathbb{W}^N \to \mathbb{R}_{\geq 0}$ is a composition of a finite number of continuous functions and is therefore continuous. Thus, $J(\cdot)$ has an upper bound on the compact set $X \times \mathbb{V}^N \times \mathbb{W}^N$. Since $\mathcal{V}(x) \subseteq \mathbb{V}^N$ for all $x \in$

$\mathcal{X}^s$, $V_\mu^{s0} : \mathcal{X}^s \to \mathbb{R}_{\geq 0}$ must satisfy the same upper bound for all $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Thus, $W(x)$ must satisfy the same upper bound because $W(x) \leq \sup_{\mu \in \mathcal{P}(\mathbb{W})} V_\mu^{s0}(x)$. Since $0 \leq W(x)$ and the choice of X is arbitrary, $W(x)$ is locally bounded on $\mathcal{X}^s$.

Since $W(x)$ is locally bounded, satisfies $W(0) = 0$, and is continuous at $x = 0$, there exists $\alpha_2(\cdot) \in \mathcal{K}_\infty$ such that $W(x) \leq \alpha_2(|x|)$ for all $x \in \mathcal{X}^s$ [32, Prop. 14]. Furthermore, we have that

$$V_\mu^{s0}(x) - N\sigma(\mathrm{tr}(\Sigma)) \leq W(x) \leq \alpha_2(|x|)$$

for all $x \in \mathcal{X}^s$ and $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$. Thus, $V_\mu^{s0}(\cdot)$ is an SISS Lyapunov function and we apply Proposition 8 to establish (ii). Since we also have that $\ell(x, \kappa_\mu(x)) \leq V_\mu(x)$, we use Proposition 13 to establish (iii). ■

Note that, unlike nominal MPC, SMPC is *not* (necessarily) RAS in the set $\mathcal{X}^s$. For linear systems, quadratic costs, and specifically chosen control law parameterizations and terminal costs, Goulart and Kerrigan [9] established that SMPC is RAS, but this result relies on properties, such as convexity of the optimal cost, that do not extend to the nonlinear SMPC problem. We discuss the practical implications of this fact in Section VI.

V. CMPC

A key strength of SMPC is that the disturbances considered in the problem formulation provide a natural means to tighten the state and input constraints and ensure robust constraint satisfaction for the closed-loop system, i.e.,

$$(\phi_\mu^s(k; x, \mathbf{w}_k), \kappa_\mu^s(\phi_\mu^s(k; x, \mathbf{w}_k))) \in \mathbb{Z}$$

for all $x \in \mathcal{X}$, $\mu(\cdot) \in \mathcal{P}(\mathbb{W})$, $\mathbf{w}_k \in \mathbb{W}^k$, and $k \in \mathbb{I}_{\geq 0}$. In certain control problems, SMPC is used primarily for this purpose, and the stochastic objective function is not essential to the design goal. Tube-based MPC is particularly suited for these problems as it provides a middle ground between nominal and stochastic MPC. By using the stochastic/robust MPC framework to systematically tighten constraints, tube-based MPC ensures robust constraint satisfaction while retaining a nominal objective function. For nonlinear systems, these tube-based formulations use methods to (conservatively) tighten the state and input constraints offline and thereby reduce the online computational burden.

Since this article focuses on the closed-loop properties of these MPC algorithms, we consider a somewhat different problem than the typical tube-based MPC formulation. Specifically, we propose using the same control parameterization, disturbance support, and therefore set of admissible control parameterizations ($\mathcal{V}(x)$) as SMPC, but consider an objective function evaluated for only the nominal trajectory. We denote this formulation CMPC. This formulation, unlike tube-based MPC, does not lend itself to offline computation of the set $\mathcal{V}(\cdot)$ and therefore does not offer the same computational efficiencies as tube-based MPC. However, this formulation serves as an idealized version of tube-based MPC. That is, we tighten the constraints no more than necessary to ensure robust constraint satisfaction. Tube-based MPC formulations can be viewed as methods to conservatively approximate $\mathcal{V}(\cdot)$ offline (or with minimal online computation).

We define the CMPC optimization problem as

$$\mathbb{P}^c(x): \; V^{c0}(x) = \min_{\mathbf{v} \in \mathcal{V}(x)} J(x, \mathbf{v}, \mathbf{0}) \tag{19}$$

for any $x \in \mathcal{X}^s$ and the optimal solutions for a given initial state are denoted $\mathbf{v}^{c0}(x) := \arg\min_{\mathbf{v} \in \mathcal{V}(x)} J(x, \mathbf{v}, \mathbf{0})$.

Thus, we are using the disturbance support $\mathbb{W}$ to construct the tightened constraint set $\mathcal{V}(x)$, but we optimize over the *nominal* objective function ($\mathbf{w} = \mathbf{0}$). We use a state feedback parameterization, but disturbance feedback parameterizations are also used in tube-based MPC. These two parameterizations are equivalent for linear systems [10].

We use a Borel measurable selection rule to define the single-valued control law $\kappa^c : \mathcal{X}^s \to \mathbb{U}$ such that $\kappa^c(x) \in \{\pi(x, v) : v \in v^{c0}(0; x)\}$ for all $x \in \mathcal{X}^s$, in which $v^{c0}(0; x)$ is the set of first parameter vectors in $\mathbf{v}^{c0}(x)$. The resulting closed-loop system is then

$$x^+ = f(x, \kappa^c(x), w). \tag{20}$$

Note that the optimal cost and control law for CMPC do not depend on the probability measure $\mu(\cdot)$.

Theorem 18 (CMPC): Let Assumptions 1–3 and 6–8 hold with $\mathbb{W}$ known exactly. For the closed-loop system $x^+ = f(x, \kappa^c(x), w)$, $w \in \mathbb{W}$, we have the following.

i) The set $\mathcal{X}^s$ is RPI.
ii) The origin is RAS in the set $\mathcal{X}^s$.
iii) The origin is RASiE in the set $\mathcal{X}^s$.
iv) The origin is ℓ-RASiE in the set $\mathcal{X}^s$.

Proof: If $x \in \mathcal{X}^s$, we have that for $\mathbf{v}^0 \in \mathbf{v}^{c0}(x)$ and all $\mathbf{w} := (w(0), w(1), \ldots, w(N-1)) \in \mathbb{W}^N$, $x(N, \mathbf{w}) = \hat{\phi}^s(N; x, \mathbf{v}^0, \mathbf{w}) \in \mathbb{X}_f$ and

$$f(x(N, \mathbf{w}), \kappa_f(x(N, \mathbf{w})), w(N)) \in \mathbb{X}_f$$

for all $w(N) \in \mathbb{W}$ by Assumption 7. Thus, the trajectory

$$\tilde{\mathbf{v}}^+ = (v^0(1), v^0(2), \ldots, v^0(N-1), 0)$$

satisfies $\tilde{\mathbf{v}}^+ \in \mathcal{V}(x^+)$ for $x^+ = f(x, \kappa^c(x), w(0))$ and all $w(0) \in \mathbb{W}$. Since $\mathcal{V}(x^+) \neq \emptyset$, $x^+ \in \mathcal{X}^s$. So $\mathcal{X}^s$ is RPI and (i) holds. We also have from Assumption 7 that

$$J(f(x, \kappa^c(x), 0), \tilde{\mathbf{v}}^+, \mathbf{0}) \leq J(x, \mathbf{v}^0, \mathbf{0}) - \ell(x, \kappa^c(x)).$$

We have that $J(\cdot)$ is continuous and $\mathcal{X}^s$, $\mathbb{U}$, and $\mathbb{V}$ are compact. By [1, Prop. 20], there exists $\sigma(\cdot) \in \mathcal{K}$ such that

$$|J(f(x, u, w), \mathbf{v}, \mathbf{0}) - J(f(x, u, 0), \mathbf{v}, \mathbf{0})| \leq \sigma(|w|)$$

for all $(x, u, w) \in \mathcal{X}^s \times \mathbb{U} \times \mathbb{W}$ and $\mathbf{v} \in \mathbb{V}^N$. Thus, we have for all $x \in \mathcal{X}^s$ and $w \in \mathbb{W}$,

$$\begin{aligned} V^{c0}(x^+) &\leq J(f(x, \kappa^c(x), w), \tilde{\mathbf{v}}^+, \mathbf{0}) \\ &\leq J(f(x, \kappa^c(x), 0), \tilde{\mathbf{v}}^+, \mathbf{0}) + \sigma(|w|) \\ &\leq J(x, \mathbf{v}^0, \mathbf{0}) - \ell(x, \kappa^c(x)) + \sigma(|w|) \\ &\leq V^{c0}(x) - \alpha_\ell(|x|) + \sigma(|w|). \end{aligned}$$

By Assumption 3, we have that $\alpha_\ell(|x|) \leq \ell(x, \kappa^c(x)) \leq V^{c0}(x)$ for all $x \in \mathcal{X}^s$.

We now construct the upper bound for $V^{c0}(x)$. We choose $x \in \mathbb{X}_f$ and consider a nominal trajectory generated by repeated application of the terminal control law, denoted $x(k) := \hat{\phi}^s(k; x, \mathbf{0}, \mathbf{0})$ since $\pi(x, 0) = \kappa_f(x)$. The set $\mathbb{X}_f$ is RPI for this control law due to Assumption 7 and the fact that $0 \in \mathbb{W}$. Therefore, $\mathbf{0} \in \mathcal{V}(x)$. From Assumption 7, we have that

$$V_f(x(k+1)) - V_f(x(k)) \leq -\ell(x(k), \kappa_f(x(k)))$$

for all $k \in \mathbb{I}_{[0,N-1]}$. We sum both sides of the inequality from $k = 0$ to $k = N-1$ to give

$$V_f(x(N)) - V_f(x) \leq -\sum_{k=0}^{N-1} \ell(x(k), \kappa_f(x(k))).$$

By optimality and the definition of $J(\cdot)$, we have

$$\begin{aligned} V^{c0}(x) &\leq J(x, \mathbf{0}, \mathbf{0}) \\ &= \sum_{k=0}^{N-1} \ell(x(k), \kappa_f(x(k))) + V_f(x(N)) \\ &\leq V_f(x) \end{aligned}$$

for all $x \in \mathbb{X}_f$. Since $0 \leq V^{c0}(x)$, $V_f(\cdot)$ is continuous, $V_f(0) = 0$, and $\mathbb{X}_f$ contains the origin in its interior, we have that $V^{c0}(\cdot)$ is continuous at the origin.

The function $J(x, \mathbf{v}, \mathbf{0})$ is continuous and therefore has an upper bound on the compact set $X \times \mathbb{V}^N$ for any compact $X \subseteq \mathcal{X}^s$. Since $\mathcal{V}(x) \subseteq \mathbb{V}^N$ for all $x \in \mathcal{X}^s$, $V^{c0}(x)$ has the same upper bound on X. Since X is arbitrary, $V^{c0}(\cdot)$ is locally bounded on $\mathcal{X}^s$. Since $V^{c0}(\cdot)$ is continuous at the origin and locally bounded on $\mathcal{X}^s$, there exists $\alpha_2(\cdot) \in \mathcal{K}_\infty(\cdot)$ such that $V^{c0}(x) \leq \alpha_2(|x|)$ for all $x \in \mathcal{X}^s$ [32, Prop. 14].

Thus, $V^{c0}(\cdot)$ is an ISS Lyapunov function on the RPI set $\mathcal{X}^s$. Also, $V^{c0}(\cdot)$ is an SISS Lyapunov function by Proposition 10. Furthermore, we have that $\mathcal{X}^s$ is bounded (Assumption 6) and that $\ell(x, \kappa^c(x)) \leq V^{c0}(x)$. Thus, by Propositions 5, 8, and 13, we have (ii)–(iv). ■

An important property of CMPC is that, unlike SMPC, the origin is RAS. Furthermore, the probability measure of the disturbance is not required to solve the optimization problem (except to construct probabilistic constraints). Analogous to nominal MPC, we still satisfy the proposed definitions of stochastic robustness (RASiE and ℓ-RASiE), without requiring a stochastic objective function.

VI. Examples and Comparisons

With Theorems 15, 17, and 18 in hand, we make the following observation: nominal MPC, SMPC, and CMPC satisfy the same definitions of stochastic robustness. The next question to answer is then: *Which method is more robust?* Based on the definitions of stochastic robustness presented in this article, we consider three specific conjectures that characterize the notion that SMPC is more robust than nominal MPC.

Naive Conjecture 19: Let $f(\cdot)$, $\ell(\cdot)$, $V_f(\cdot)$, $\mathbb{U}$, $\mathbb{X}_f$, $\mu(\cdot)$, and $\mathbb{W}$ be the same for nominal MPC and SMPC. Then, the feasible set for SMPC, $\mathcal{X}^s$, is larger than the RPI set for nominal MPC, $\mathcal{S}$, for the same disturbance set $\mathbb{W}$, i.e., $\mathcal{S} \subseteq \mathcal{X}^s$.

Naive Conjecture 20: Let $f(\cdot)$, $\ell(\cdot)$, $V_f(\cdot)$, $\mathbb{U}$, $\mathbb{X}_f$, $\mu(\cdot)$, and $\mathbb{W}$ be the same for nominal MPC and SMPC. For any $x \in \mathcal{X}^s$,

$$\lim_{k\to\infty} \mathbb{E}\left[|\phi^s_\mu(k; x, \mathbf{w}_k)|\right] \leq \lim_{k\to\infty} \mathbb{E}\left[|\phi(k; x, \mathbf{w}_k)|\right]$$

if these limits exist, i.e., SMPC is better than MPC in terms of the expected norm of the closed-loop state (RASiE).

Conjecture 21: Let $f(\cdot)$, $\ell(\cdot)$, $V_f(\cdot)$, $\mathbb{U}$, $\mathbb{X}_f$, $\mu(\cdot)$, and $\mathbb{W}$ be the same for nominal MPC and SMPC. For any $x \in \mathcal{X}^s$,

$$\lim_{k\to\infty} \mathbb{E}\left[\ell(x^s(k), \kappa^s_\mu(x^s(k)))\right] \leq \lim_{k\to\infty} \mathbb{E}\left[\ell(x(k), \kappa(x(k)))\right]$$

if these limits exist in which $x^s(k) = \phi^s_\mu(k; x, \mathbf{w}_k)$ and $x(k) = \phi(k; x, \mathbf{w}_k)$, i.e., SMPC is better than MPC in terms the expected value of the closed-loop stage cost (ℓ-RASiE).

In the following sections, we use a few simple examples to investigate these conjectures and compare the strengths, weaknesses, and closed-loop behavior of MPC, SMPC, and CMPC. In particular, we demonstrate that Naive Conjectures 19 and 20 do not hold. Conjecture 21, however, is supported by the following examples.

A. RPI Sets

Naive Conjecture 19 frames the discussion of robustness based on the respective RPI sets for each control method. We note that CMPC and SMPC, due to their formulations, share the same RPI set $\mathcal{X}^s$. We begin with a comparison of three important sets for each of these problems.

Lemma 22: Let Assumptions 1, 2, and 6–8 hold. Let $f(\cdot)$, $\mathbb{U}$, $\mathbb{X}_f$, and $\mathbb{W}$ be the same for nominal MPC and SMPC. Then, $\mathbb{X}_f \subseteq \mathcal{X}^s \subseteq \mathcal{X}$.

Proof: For any $x \in \mathbb{X}_f$, we have that $\mathbf{0} \in \mathcal{V}(x)$ because Assumption 7 ensures that $\mathbb{X}_f$ is RPI for the system $x^+ = f(x, \kappa_f(x), w)$, $w \in \mathbb{W}$, and $(x, \kappa_f(x)) \in \mathbb{Z}$. So for any $x \in \mathbb{X}_f$, $\mathcal{V}(x) \neq \emptyset$ and therefore $x \in \mathcal{X}^s$ as well. Thus, $\mathbb{X}_f \subseteq \mathcal{X}^s$.

For any $x \in \mathcal{X}^s$ and $\mathbf{v} \in \mathcal{V}(x)$, we know that $\hat{\phi}^s(N; x, \mathbf{v}, \mathbf{0}) \in \mathbb{X}_f$ because $0 \in \mathbb{W}$ by Assumption 1. Thus, we can define $\mathbf{u} = (u(0), \dots, u(N-1))$ such that $u(k) = \pi(\hat{\phi}^s(k; x, \mathbf{v}, \mathbf{0}), v(k))$ and we have that $\hat{\phi}(k; x, \mathbf{u}) = \hat{\phi}^s(k; x, \mathbf{v}, \mathbf{0})$. Therefore, $\mathbf{u} \in \mathbb{U}^N$, $\hat{\phi}(N; x, \mathbf{u}) \in \mathbb{X}_f$, and $\mathbf{u} \in \mathcal{U}(x)$. So for any $x \in \mathcal{X}^s$, $\mathcal{U}(x) \neq \emptyset$ and therefore $x \in \mathcal{X}$ as well. Thus, $\mathcal{X}^s \subseteq \mathcal{X}$. ■

Naive Conjecture 19, however, compares the set $\mathcal{X}^s$ to the RPI set for nominal MPC from Theorem 15, i.e., the set $\mathcal{S}$, for an equivalent disturbance set $w \in \mathbb{W}$. By definition, $\mathcal{S} \subseteq \mathcal{X}$, but establishing the relative sizes of $\mathcal{S}$ and either $\mathbb{X}_f$ and $\mathcal{X}^s$ for a general nonlinear control problem is difficult.

Instead, we demonstrate a counter example to Naive Conjecture 19. Consider the scalar system $x^+ = x + u + w$ with $|u| \leq 2$ and $|w| \leq 1$. Choose the stage cost $\ell(x, u) = x^2 + u^2$, terminal cost $V_f(x) = 2x^2$, terminal constraints $\mathbb{X}_f := [-2, 2]$, and control law parameterization $\pi(x, v) = -x + v$. We have that $\mathcal{X} := \{x : |x| \leq 2 + 2N\}$ and $\mathcal{X}^s := \{x : |x| \leq 2 + N\}$ since SMPC must address the potential for a disturbance of $|w| = 1$ at each time step while still satisfying the terminal constraint. Thus, we have that $\mathbb{X}_f \subset \mathcal{X}^s \subset \mathcal{X}$ for all $N \geq 1$, in which these are strict subsets. For the disturbance of interest,

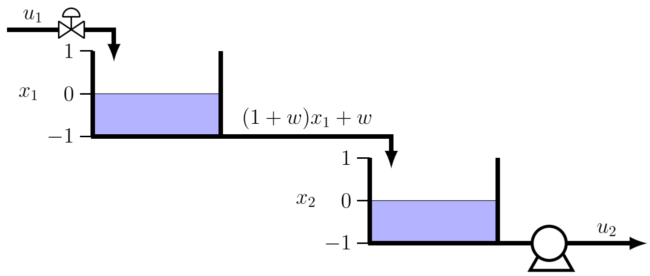

Fig. 1. Two tanks with gravity driven flow between tanks 1 and 2.

however, the entire feasible set $\mathcal{X}$ is RPI for the nominal MPC controller. Thus, we have

$$\mathcal{X}^s \subset \mathcal{S} = \mathcal{X}$$

in which $\mathcal{X}^s$ is a strict subset of $\mathcal{S}$, i.e., Naive Conjecture 19 does not hold. In the next section, we also establish a counter example to Naive Conjecture 20.

B. Liquid-Level Control

We consider a simple example with two tanks as shown in Fig. 1. The goal is to control the height of liquid in each tank via the inlet flow rate into tank 1 and the effluent flow rate from tank 2. Tank 1 drains into tank 2 by gravity at a rate proportional to the height of tank 1.

This proportionality constant is subject to uncertainty and w may take values in the set $\mathbb{W} := \{-0.3, 0, 0.3\}$ with the probability measure $\mu(\{0.3\}) = \mu(\{-0.3\}) = 0.35$ and $\mu(\{0\}) = 0.3$. We write the differential equations for the system in deviation variables as follows:

$$\frac{dx_1}{dt} = -(1+w)x_1 + u_1 - w$$
$$\frac{dx_2}{dt} = (1+w)x_1 - u_2 + w$$

in which x_1, x_2 are the tank heights and u_1, u_2 are the flow rates. The nominal system ($w = 0$) is linear, but the disturbance results in both an additive and multiplicative effect on the system in terms of deviation variables. Since the disturbance support is finite, we can discretize this differential equation (assuming a zero-order hold on the inputs and disturbance) exactly for all $w \in \mathbb{W}$. We evaluate the expected value of the objective function in the SMPC optimization problem by enumerating all possible disturbance trajectories. We similarly evaluate expected value of the closed-loop state and stage cost by simulating all possible disturbance trajectories.

We have the input constraints $u_1, u_2 \in [-1, 1]$. We define the stage cost as $\ell(x, u) := x'Qx + u'Ru$ with $Q := \mathrm{diag}([0.1, 20])$ and $R := \mathrm{diag}([0.1, 0.1])$. Note that we have selected penalties that strongly discourage any deviations in the height of the second tank. Nonetheless, this stage cost is positive definite and satisfies all the usual requirements for nominal MPC and SMPC.

We use the LQR cost P and gain K from the nominal system ($w = 0$) to define the terminal cost $V_f(x) := x'Px$ and control law parameterization $\pi(x, v) := Kx + v$. We define the terminal constraint as $\mathbb{X}_f := \{x : |x_1| \le 0.4,\ |x_2| \le 0.4\}$ and verify that this terminal constraint satisfies the required assumptions with the terminal control law $\kappa_f(x) := Kx$.[4]

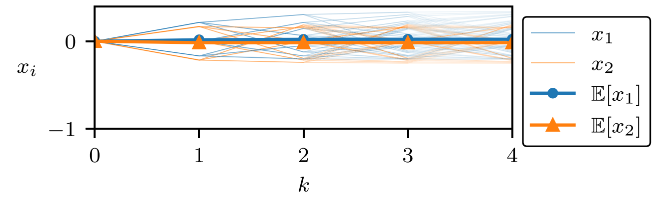

Fig. 2. Closed-loop trajectory for MPC for the liquid-level control problem.

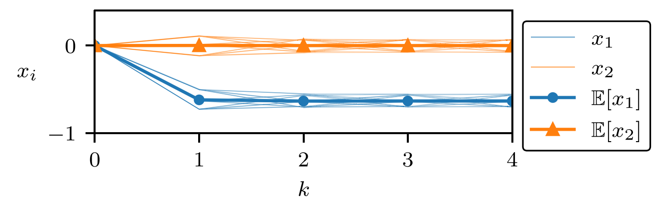

Fig. 3. Closed-loop trajectory for SMPC for the liquid-level control problem.

In Figs. 2 and 3, we plot the resulting closed-loop trajectories for each realization of the disturbance and the expected values of these trajectories for MPC and SMPC, respectively, with $N = 3$. Since deviations in x_2 are assigned a large cost, the SMPC controller decides to decrease the height of the first tank to minimize the effect of the disturbance on x_2. While there are clear benefits to this approach in terms of the expected stage cost of the system, the behavior is nonintuitive in terms of a typical tracking control problem. Indeed, SMPC drives the system away from the origin. The closed-loop trajectory for CMPC is identical to nominal MPC and therefore omitted.

We plot the expected value of the norm of the state and stage cost for the closed-loop trajectory of each controller in Fig. 4. As we may expect, SMPC achieves a lower expected stage cost as $k \to \infty$. The value of $\mathbb{E}[|x(k)|]$, however, is larger for SMPC than for MPC. By the end of the simulation at $k = 3$, the value $\mathbb{E}[|x(k)|]$ appears to be constant and we presume that the limit of $\mathbb{E}[|x(k)|]$ exists and is approximately the same as the value at $k = 3$. Thus, Naive Conjecture 20 does not hold.

As noted in the previous theoretical analysis, one significant distinction between MPC and SMPC (for nonlinear systems) is that SMPC does not guarantee robust asymptotic stability. We demonstrate the implications of this shortcoming by considering a nominal realization of the disturbance, i.e., $\mathbf{w}_k = \mathbf{0}$. We plot the nominal closed-loop trajectory for SMPC in Fig. 5. Despite the fact that no disturbance occurs, SMPC drives the system away from the origin and is therefore not RAS or nominally asymptotically stable.

[4]We have not chosen $\mathbb{X}_f$ as a level set of $V_f(\cdot)$ as required by Assumption 4. However, the nominal system is linear and the constraints are convex, and therefore, nominal MPC is RAS [11].

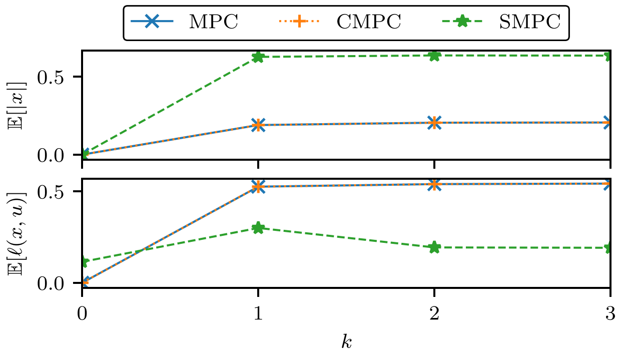

Fig. 4. Expected value of the norm of the state and stage cost for the closed-loop trajectory of each controller in the liquid-level control problem.

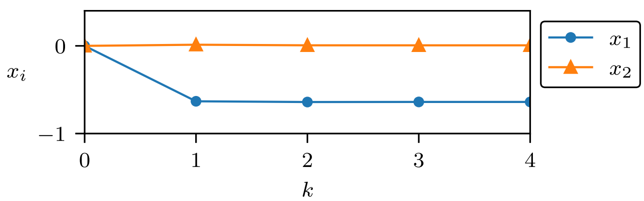

Fig. 5. Closed-loop trajectory for SMPC subject to a nominal realization of the disturbances, i.e., $\mathbf{w}_k = \mathbf{0}$, for the liquid-level control problem.

C. State Constraints

We now consider a two-state linear system to illustrate the benefits of the systematic constraint tightening procedure inherent to SMPC and CMPC. The system is described by

$$x^+ = Ax + Bu + Gw$$

$$A = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.95 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 0.1 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $u \in [-1, 1]$. We again assume a finite support for w with $\mu(\{0.05\}) = \mu(\{-0.05\}) = 0.35$ and $\mu(\{0\}) = 0.3$. We also consider the state constraint $|x_1| \leq 1$.

We use a quadratic stage cost with $Q = \text{diag}([1, 10])$ and $R = 0.1$. We use the LQR cost P and gain K for the unconstrained system to define the terminal cost $V_f(x) := x'Px$, terminal constraint $\mathbb{X}_f := \{x : x'Px \leq 1\}$, and terminal control law $\kappa_f(x) := Kx$. We define $\pi(x, v) = Kx + v$ for SMPC and CMPC and verify that this design satisfies all the required assumptions. We choose a horizon of $N = 4$.

Since SMPC and CMPC can guarantee robust state constraint satisfaction (if the disturbance support is accurate), we include the state constraint as a hard constraint in these optimization problems. For MPC, we instead convert the state constraint to a large violation penalty to ensure that the constraint is satisfied if possible while retaining robust recursive feasibility of the optimization problem. Specifically, we redefine the stage cost as

$$\ell(x, u) = x'Qx + u'Ru + \lambda |x_1|_{[-1,1]}$$

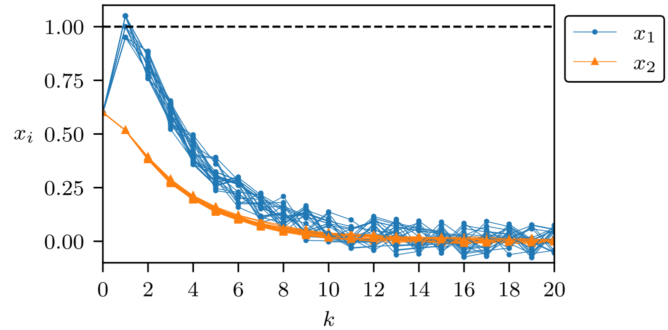

Fig. 6. Closed-loop trajectories for MPC subject to 30 different realizations of the disturbance sequence for the state constraint example.

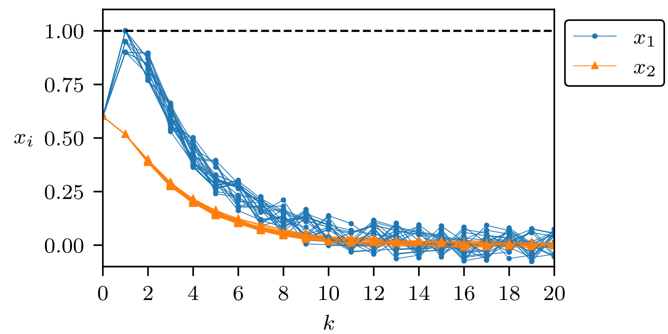

Fig. 7. Closed-loop trajectories for SMPC subject to 30 different realizations of the disturbance sequence for the state constraint example.

in which $\lambda \geq 0$ is a large violation penalty and $|x_1|_{[-1,1]} := \min\{|y - x_1| : y \in [-1, 1]\}$ denotes point-to-set distance. We find that $\lambda = 100$ is sufficient to ensure constraint satisfaction in the nominal optimization problem (when possible). We use this stage cost in the subsequent statistics for the closed-loop system.

In Figs. 6 and 7, we plot the closed-loop trajectory for 30 realizations of the disturbance (drawn from the known distribution) for MPC and SMPC, respectively. Both controllers initially drive x_1 away from the origin in the interest of minimizing the value of x_2 and therefore the stage cost. The main difference between these methods is that MPC drives the nominal value of $x_1(1)$ to the state constraint and therefore cannot ensure that this constraint is satisfied for the perturbed system. By contrast, SMPC leaves a buffer between the nominal value of $x_1(1)$ and the state constraint to ensure robust constraint satisfaction. For subsequent time steps, however, the two control methods produce similar trajectories.

We plot the performance of each method in terms of $\mathbb{E}[|x|]$ and $\mathbb{E}[\ell(x, \kappa(x))]$ in Fig. 8. We note that CMPC produces nearly identical performance to SMPC without the need for a stochastic objective. At $k = 1$, MPC violates the state constraint, and therefore, the closed-loop performance of MPC is inferior to SMPC or CMPC. But for $k \geq 2$, all of these controllers produce nearly equivalent performance. Once the state is inside the terminal region and state/input constraints are not active, the optimal controller for both the nominal and stochastic linear system is

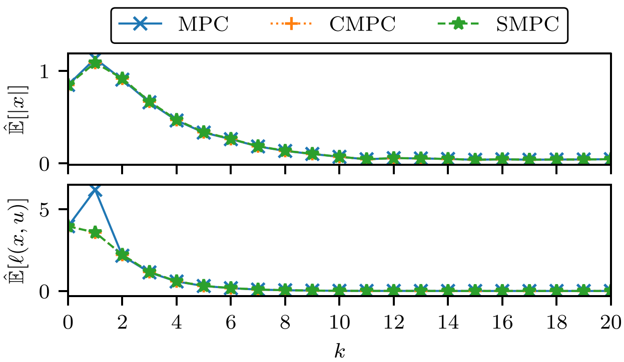

Fig. 8. Sample average closed-loop performance of each method subject to 30 different realizations of the disturbance sequence for the state constraint example.

the LQR feedback gain. Thus, MPC, SMPC, and CMPC all use the same control law within the terminal region and achieve the same closed-loop performance in this region.

VII. Conclusion

SMPC can find a superior operating point or trajectory than nominal MPC, in terms of the expected stage cost. Thus, Conjecture 21 is well motivated and is supported by these examples, although we were unable to prove (or disprove) this conjecture. Naive Conjectures 19 and 20, however, do not hold. Thus, the claim that SMPC is necessarily "more robust" than nominal MPC should be qualified. There are reasonable definitions of robustness for which MPC can outperform SMPC.

In summary, SMPC appears to offer a clear benefit for control applications that prioritize economic performance (i.e., stage cost minimization) over stability of a target steady state. If stability of a target steady state is prioritized; however, the benefits of SMPC are less obvious. Feedback is sufficient to ensure that MPC, without any knowledge of the disturbance model or distribution, achieves the same type of stochastic robustness afforded by SMPC, i.e., RASiE and ℓ-RASiE, for sufficiently small disturbances ($|w| \leq \delta$). If this nonzero margin of robustness ($\delta > 0$) is too small and/or robust state constraint satisfaction is required for safety-critical applications, constraint-tightening methods such as CMPC can be employed that also ensure RASiE and ℓ-RASiE without a stochastic objective function. Furthermore, SMPC does not ensure RAS of the origin, a property often seen as essential for a control algorithm. We also emphasize that these results address only idealized SMPC, in which the probability measure of the disturbance is known exactly. In practice, we do not have an exact disturbance model and the performance of SMPC may degrade relative to the idealized case.

Appendix

Proof of Proposition 16: Since $V_f(\cdot)$, $f(\cdot)$, and $\kappa_f(\cdot)$ are continuous and $\mathbb{X}_f$ and $\mathbb{U}$ are bounded, we have from [1, Prop. 20] that there exists $\sigma(\cdot) \in \mathcal{K}_\infty$ such that

$$|V_f(f(x,\kappa_f(x),w)) - V_f(f(x,\kappa_f(x),0))| \leq \sigma(|w|)$$

for all $x \in \mathbb{X}_f$ and $w \in \mathbb{W}$. We combine this bound with Assumption 5 to give

$$V_f(f(x,\kappa_f(x),w)) \leq V_f(x) - \ell(x,\kappa_f(x)) + \sigma(|w|)$$

for all $x \in \mathbb{X}_f$ and $w \in \mathbb{W}$. We apply the bound in Assumption 3 to give

$$V_f(f(x,\kappa_f(x),w)) \leq V_f(x) - \alpha_\ell(|x|) + \sigma(|w|)$$

for all $x \in \mathbb{X}_f$ and $w \in \mathbb{W}$. Furthermore, since $V_f(0) = 0$, then by [1, Prop. 20] there exists $\sigma_f(\cdot) \in \mathcal{K}_\infty$ such that

$$V_f(x) = |V_f(x) - V_f(0)| \leq \sigma_f(|x|).$$

Recall that $\mathbb{X}_f := \{x \in \mathbb{X} : V_f(x) \leq \tau\}$ for some $\tau > 0$. If $V_f(x) \geq \tau/2$, then we have $|x| \geq \sigma_f^{-1}(\tau/2)$ and therefore

$$V_f(f(x,\kappa_f(x),w)) \leq \tau - \alpha_\ell(\sigma_f^{-1}(\tau/2)) + \sigma(|w|).$$

If $V_f(x) < \tau/2$, we have that

$$V_f(f(x,\kappa_f(x),w)) \leq \tau/2 + \sigma(|w|).$$

Therefore, for all $x \in \mathbb{X}_f$, we have

$$V_f(f(x,\kappa_f(x),w)) \leq \tau - \gamma + \sigma(|w|)$$

in which $\gamma = \min\{\tau/2, \alpha_\ell(\sigma_f^{-1}(\tau/2))\}$. By bounding $|w| \leq \sigma^{-1}(\gamma) =: \delta$, we have that $V_f(f(x,\kappa_f(x),w)) \leq \tau$ for all $x \in \mathbb{X}_f$. Thus, for any $w \in \mathbb{W} \subseteq \{w \in \mathbb{R}^q : |w| \leq \delta\}$, we have that $x \in \mathbb{X}_f$ implies $f(x,\kappa_f(x),w) \in \mathbb{X}_f$. ■

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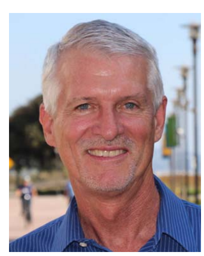

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