HYBRIDIZATION AND POSTPROCESSING IN FINITE ELEMENT EXTERIOR CALCULUS

GERARD AWANOU, MAURICE FABIEN, JOHNNY GUZMÁN, AND ARI STERN

Abstract. We hybridize the methods of finite element exterior calculus for the Hodge–Laplace problem on differential k-forms in $\mathbb{R}^n$. In the cases $k = 0$ and $k = n$, we recover well-known primal and mixed hybrid methods for the scalar Poisson equation, while for $0 < k < n$, we obtain new hybrid finite element methods, including methods for the vector Poisson equation in $n = 2$ and $n = 3$ dimensions. We also generalize Stenberg postprocessing from $k = n$ to arbitrary k, proving new superconvergence estimates. Finally, we discuss how this hybridization framework may be extended to include nonconforming and hybridizable discontinuous Galerkin methods.

1. Introduction

Finite element exterior calculus (FEEC) is a powerful framework that unifies the analysis of several families of conforming finite element methods for problems involving Laplace-type operators (Arnold, Falk, and Winther [4, 5], Arnold [2]). These include the classic "continuous Galerkin" Lagrange finite element method and the Raviart–Thomas (RT) [39] and Brezzi–Douglas–Marini (BDM) [8] mixed methods for the scalar Poisson equation, as well as mixed methods based on Nédélec elements [34, 35] for the 2- and 3-dimensional vector Poisson equation. In FEEC, these are all seen as finite element methods for the Hodge–Laplace operator on differential k-forms in $\mathbb{R}^n$, where scalar fields are identified with 0- and n-forms and vector fields with 1- and $(n-1)$-forms.

In this paper, we hybridize FEEC for arbitrary dimension n and form degree k. That is, we construct hybrid finite element methods using discontinuous spaces of differential forms, enforcing continuity and boundary conditions using Lagrange multipliers on the element boundaries. The solutions agree with those of the original, non-hybrid FEEC methods, and the Lagrange multipliers are seen to correspond to weak tangential and normal traces. This hybrid formulation enables static condensation: since only the Lagrange multipliers are globally coupled, the remaining internal degrees of freedom can be eliminated using an efficient local procedure, and the resulting Schur complement system can be substantially smaller than the original one. We also present a generalization of Stenberg postprocessing [41], which for $0 < k < n$ is shown to give new improved estimates.

The special cases $k = 0$ and $k = n$ are shown to recover known results on hybridization and postprocessing for the scalar Poisson equation. In particular, the case $k = n$ corresponds to the hybridized RT [3] and BDM [8] methods, and the postprocessing procedure is precisely that of Stenberg [41]. The case $k = 0$ corresponds to the more recent hybridization of the continuous Galerkin method by Cockburn, Gopalakrishnan, and Wang [20].

The hybrid and postprocessing schemes in the remaining cases $0 < k < n$ are new and, to the best of our knowledge, have not appeared in the literature even for the vector Poisson equation when $n = 2$ or $n = 3$. In particular, the hybridization of Nédélec edge elements is different from that in Cockburn and Gopalakrishnan [18]: here, the Lagrange multipliers are simply traces of standard elements, rather than living in a space of "jumps." We expect these new methods to be especially useful in computational electromagnetics, where Nédélec elements are ubiquitous and the differential forms point of view has provided significant insight (cf. Hiptmair [27]).

While we restrict our attention primarily to hybrid methods for conforming simplicial meshes, we remark that the framework developed here has the potential to be applied to other types of domain decomposition methods, including methods on cubical meshes, nonconforming meshes, mortar methods, etc. We also discuss briefly how the unified hybridization framework of Cockburn, Gopalakrishnan, and Lazarov [19], which includes hybridizable discontinuous Galerkin (HDG) methods, may also be generalized to the Hodge–Laplace problem for $0 < k < n$.

1.1. **Why hybridize?** There are several theoretical and practical benefits of hybridization:

- *additional information about solutions*: The Lagrange multiplier functions often correspond to weak boundary traces of solution components, even though the numerical solution may not be regular enough for a trace to exist in the usual sense (e.g., the trace of an L^2 function or normal derivative of an H^1 function).
- *static condensation*: Degrees of freedom for discontinuous function spaces can be locally eliminated. The resulting Schur complement only involves boundary degrees of freedom for the Lagrange multipliers, so it can be substantially smaller than the original global problem.
- *local postprocessing and superconvergence*: The numerical solution may be efficiently "post-processed" by using the boundary traces to solve a local problem on each element, resulting in an improved approximation compared to the original solution.

Seminal work on hybridization of mixed finite element methods was done by Fraeijs de Veubeke [22]. For the scalar Poisson equation, the RT method was hybridized in this manner by Arnold and Brezzi [3], who introduced the notion of postprocessing. Hybridization and postprocessing were also discussed in the original paper introducing the BDM method [8], and an interesting characterization of the Lagrange multipliers for the hybridized RT and BDM methods appears in Cockburn and Gopalakrishnan [17]. A refined local postprocessing procedure for mixed methods, which can be applied with or without hybridization, was given by Stenberg [41]; see also Gastaldi and Nochetto [23], who discovered this independently (cf. [23, eqs. 4.14–4.15]), as well as Bramble and Xu [7].

More recently, Cockburn, Gopalakrishnan, and Wang [20] hybridized the continuous Galerkin method, using an approach similar to the "three-field domain decomposition method" of Brezzi and Marini [9], and showed that static condensation yields the same condensed system as that obtained by the original, non-hybrid static condensation procedure of Guyan [26]. Even more recently, Cockburn, Gopalakrishnan, and Lazarov [19] introduced an important unified hybridization framework that includes the above methods, as well as nonconforming and HDG methods, for the scalar Poisson equation. A survey of historical and recent developments appears in Cockburn [16].

1.2. **Organization of the paper.** The paper is organized as follows:

- Section 2 recalls the basic machinery and terminology of differential forms, the Hodge–Laplace problem, and FEEC. This includes a discussion of tangential and normal traces, which play an important role throughout the paper.
- Section 3 presents a domain decomposition of the Hodge–Laplace problem. The variational form of this problem involves broken spaces of differential forms, along with boundary traces that act as Lagrange multipliers enforcing interelement continuity and boundary conditions.
- Section 4 develops hybrid finite element methods for the Hodge–Laplace problem, based on the domain-decomposed variational principle from the previous section. We prove that these are hybridized versions of the FEEC methods, show how static condensation can be used to reduce the size of the global system, and develop error estimates for the hybrid variables.
- Section 5 generalizes the postprocessing procedure of Stenberg [41] from $k = n$ to arbitrary k. This procedure only uses the statically condensed variables, so it can be applied immediately after solving the condensed system, or it can be applied to solutions obtained by ordinary finite element methods without hybridization. In addition to known superconvergence results for $k = n$, we give new improved error estimates for $k < n$.
- Section 6 gives concrete illustrations of the hybrid and postprocessing methods when $n = 3$, using the language of vector calculus and classic families of finite elements.
- Section 7 presents numerical experiments, confirming the error estimates of Sections 4 and 5.
- Finally, Section 8 presents an extension of the framework of Cockburn, Gopalakrishnan, and Lazarov [19], whereas the previous sections only address conforming methods. This lays the groundwork for hybridization of nonconforming and discontinuous Galerkin methods for FEEC, although we postpone the analysis of such methods for future work.

2. Background: differential forms and finite element exterior calculus

In this section, we quickly recall the exterior calculus of differential forms, the Hodge–Laplace problem, and FEEC, in order to lay the foundation and fix the notation for the subsequent sections. We refer to Arnold, Falk, and Winther [4, 5], Arnold [2], and references therein for a comprehensive treatment. We also discuss tangential and normal traces of differential forms, which will play an important role in domain decomposition and hybridization. Our treatment of these traces follows that in Weck [42] (see also Kurz and Auchmann [30]), which extended work of Buffa and Ciarlet [11, 12], Buffa, Costabel, and Sheen [13] for vector fields in $\mathbb{R}^3$.

2.1. **Exterior calculus of differential forms.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and denote by $\Lambda^k(\Omega)$ the space of smooth differential k-forms on Ω, where $k = 0, \dots, n$. We assume that the reader is familiar with the following basic operations of exterior calculus:

- the wedge product $\wedge\colon \Lambda^k(\Omega) \times \Lambda^\ell(\Omega) \to \Lambda^{k+\ell}(\Omega)$,
- the (Euclidean) Hodge star isomorphism $\star\colon \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$,
- the exterior derivative $\mathrm{d}\colon \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$,
- the codifferential $\delta \coloneqq (-1)^k \star^{-1} \mathrm{d} \star\colon \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$,
- the Hodge–Laplace operator $L \coloneqq \mathrm{d}\delta + \delta\mathrm{d}\colon \Lambda^k(\Omega) \to \Lambda^k(\Omega)$.

These are graded operators, but we suppress the form degree for notational simplicity, e.g., writing d rather than d^k. From the Leibniz rule for d and definition of δ, we have the important identity

$$\mathrm{d}(\tau \wedge \star v) = \mathrm{d}\tau \wedge \star v - \tau \wedge \star \delta v, \tag{1}$$

where $\tau \in \Lambda^{k-1}(\Omega)$ and $v \in \Lambda^k(\Omega)$.

The Hilbert space $L^2\Lambda^k(\Omega)$ is the completion of $\Lambda^k(\Omega)$ with respect to the L^2 inner product $(v, w)_\Omega \coloneqq \int_\Omega v \wedge \star w$, whose associated norm is denoted $\|\cdot\|_\Omega$. Taking d in the sense of distributions allows it to be extended to a closed, densely defined operator with domain

$$H\Lambda^k(\Omega) \coloneqq \big\{v \in L^2\Lambda^k(\Omega) : \mathrm{d}v \in L^2\Lambda^{k+1}(\Omega)\big\},$$

which is itself a Hilbert space with the graph inner product $(v, w)_{H\Lambda^k(\Omega)} \coloneqq (v, w)_\Omega + (\mathrm{d}v, \mathrm{d}w)_\Omega$. The subspace $\mathring{H}\Lambda^k(\Omega) \subset H\Lambda^k(\Omega)$ is defined to be the closure of $C_0^\infty\Lambda^k(\Omega)$, the space of smooth k-forms with compact support in Ω. Likewise, δ may be extended to a closed, densely defined operator with domain

$$H^*\Lambda^k(\Omega) \coloneqq \big\{v \in L^2\Lambda^k(\Omega) : \delta v \in L^2\Lambda^{k-1}(\Omega)\big\} = \star H\Lambda^{n-k}(\Omega),$$

which is a Hilbert space with the graph inner product $(v, w)_{H^*\Lambda^k(\Omega)} \coloneqq (v, w)_\Omega + (\delta v, \delta w)_\Omega$, and the subspace $\mathring{H}^*\Lambda^k(\Omega) = \star\mathring{H}\Lambda^{n-k}(\Omega) \subset H^*\Lambda^k(\Omega)$ is the closure of $C_0^\infty\Lambda^k(\Omega)$.

2.2. **Tangential and normal traces.** The restriction of a differential form to the boundary $\partial\Omega$ is encoded in a pair of differential forms on $\partial\Omega$, called the tangential trace and normal trace. This is analogous to decomposing a vector field into its tangential and normal components at the boundary.

We begin with the case of smooth differential forms, where the boundary $\partial\Omega$ is also smooth. The *trace map* $\mathrm{tr}\colon \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ is defined to be the pullback of k-forms by the inclusion $\partial\Omega \hookrightarrow \Omega$, i.e., $\mathrm{tr}\, v \in \Lambda^k(\partial\Omega)$ is just the restriction of $v \in \Lambda^k(\Omega)$ to vectors tangent to the boundary. Denote the Hodge star on $\partial\Omega$ by $\widehat{\star}$ and the associated L^2 inner product by $\langle\cdot,\cdot\rangle_{\partial\Omega}$.

Definition 2.1 (tangential and normal traces)**.** Given $v \in \Lambda^k(\Omega)$,

$$v^{\tan} \coloneqq \mathrm{tr}\, v \in \Lambda^k(\partial\Omega), \qquad v^{\mathrm{nor}} \coloneqq \widehat{\star}^{-1}\, \mathrm{tr} \star v \in \Lambda^{k-1}(\partial\Omega).$$

These definitions allow a particularly elegant expression of the integration by parts formula for differential forms. The following result is standard, but the proof is short and illuminates the definition of the normal trace.

k	proxy field	tangential trace	normal trace
0	$\varphi \in C^\infty(\Omega)$	$\varphi\|_{\partial\Omega}$	0
1	$v \in C^\infty(\Omega, \mathbb{R}^3)$	$v\|_{\partial\Omega} - (v \cdot \widehat{n})\widehat{n}$	$v \cdot \widehat{n}$
2	$w \in C^\infty(\Omega, \mathbb{R}^3)$	$(w \cdot \widehat{n})\widehat{n}$	$w \times \widehat{n}$
3	$\psi \in C^\infty(\Omega)$	0	$\psi\widehat{n}$

TABLE 1. Tangential and normal traces of differential forms on $\Omega \subset \mathbb{R}^3$, in terms of scalar and vector proxy fields.

Proposition 2.2. *If $\tau \in \Lambda^{k-1}(\Omega)$ and $v \in \Lambda^k(\Omega)$, then we have the integration by parts formula*

$$\langle \tau^{\tan}, v^{\mathrm{nor}} \rangle_{\partial\Omega} = (\mathrm{d}\tau, v)_\Omega - (\tau, \delta v)_\Omega. \tag{2}$$

Proof. Using the definitions of $\tau^{\tan}$ and v^{nor}, we calculate

$$\langle \tau^{\tan}, v^{\mathrm{nor}} \rangle_{\partial\Omega} = \int_{\partial\Omega} \tau^{\tan} \wedge \widehat{\star} v^{\mathrm{nor}} = \int_{\partial\Omega} \operatorname{tr}\tau \wedge \operatorname{tr}\star v = \int_{\partial\Omega} \operatorname{tr}(\tau \wedge \star v) = \int_\Omega \mathrm{d}(\tau \wedge \star v),$$

where the last step uses Stokes' theorem. Applying (1) completes the proof. □

An equivalent description of tangential and normal traces uses the outer unit normal vector field $\widehat{n}$ and its associated 1-form $\widehat{n}^\flat = \widehat{n}_i \, \mathrm{d}x^i$. Letting $\iota_{\widehat{n}}$ denote the interior product (or contraction) with $\widehat{n}$, the Leibniz rule for this operator gives the identity

$$v|_{\partial\Omega} = \iota_{\widehat{n}}(\widehat{n}^\flat \wedge v) + \widehat{n}^\flat \wedge (\iota_{\widehat{n}} v).$$

We may then identify $v^{\tan}$ with the k-form $\iota_{\widehat{n}}(\widehat{n}^\flat \wedge v)$ and v^{nor} with the $(k-1)$-form $\iota_{\widehat{n}} v$. When $\Omega \subset \mathbb{R}^3$, the correspondence of these traces to scalar and vector proxy fields is given in Table 1, using the proxy operations for $\iota_{\widehat{n}}$ and $\widehat{n}^\flat \wedge$, and (2) recovers the familiar integration by parts formulas of vector calculus.

Weck [42] showed that it is possible to extend the tangential and normal traces so that a weak version of (2) holds for $\tau \in H\Lambda^{k-1}(\Omega)$ and $v \in H^*\Lambda^k(\Omega)$, where $\partial\Omega$ is only assumed to be Lipschitz. We denote the trace spaces in which $\tau^{\tan}$ and v^{nor} live by $\widehat{H}\Lambda^{k-1,\tan}(\partial\Omega)$ and $\widehat{H}^*\Lambda^{k-1,\mathrm{nor}}(\partial\Omega)$, respectively. These are generally subspaces of $H^{-1/2}\Lambda^{k-1}(\partial\Omega)$, but not necessarily of $L^2\Lambda^{k-1}(\partial\Omega)$, so $\langle \cdot, \cdot \rangle_{\partial\Omega}$ should be interpreted as a duality pairing extending the L^2 inner product on $\partial\Omega$ [42, Theorem 8]. See Kurz and Auchmann [30] for an excellent account of Weck's results and some concrete applications to electromagnetics. Mitrea, Mitrea, and Shaw [33] obtain comparable results by extending the alternative approach using $\iota_{\widehat{n}}$ and $\widehat{n}^\flat \wedge$ described above.

The definitions of $\widehat{H}\Lambda^{k-1,\tan}(\partial\Omega)$ and $\widehat{H}^*\Lambda^{k-1,\mathrm{nor}}(\partial\Omega)$ are somewhat technical, but thankfully, we may make use of [42, Theorems 5 and 7], which give isomorphisms

$$\widehat{H}\Lambda^{k-1,\tan}(\partial\Omega) \cong H\Lambda^{k-1}(\Omega)/\mathring{H}\Lambda^{k-1}(\Omega), \qquad \widehat{H}^*\Lambda^{k-1,\mathrm{nor}}(\partial\Omega) \cong H^*\Lambda^k(\Omega)/\mathring{H}^*\Lambda^k(\Omega). \tag{3}$$

Therefore, we may treat the trace spaces as quotient spaces, equipped with the quotient norms

$$\|\widehat{\tau}^{\tan}\|_{\tan,\partial\Omega} \coloneqq \inf\big\{\|\tau\|_{H\Lambda^{k-1}(\Omega)} : \tau^{\tan} = \widehat{\tau}^{\tan}\big\}, \qquad \|\widehat{v}^{\mathrm{nor}}\|_{\mathrm{nor},\partial\Omega} \coloneqq \inf\big\{\|v\|_{H^*\Lambda^k(\Omega)} : v^{\mathrm{nor}} = \widehat{v}^{\mathrm{nor}}\big\}.$$

These generalize the "minimum energy extension" quotient norms discussed in Carstensen, Demkowicz, and Gopalakrishnan [14, Section 2] for H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$ traces in $\mathbb{R}^3$. The next result, relating these norms to the duality pairing, is a straightforward generalization of [14, Lemma 2.2].

Lemma 2.3. *For all $\widehat{\tau}^{\tan} \in \widehat{H}\Lambda^{k-1,\tan}(\partial\Omega)$ and $\widehat{v}^{\nor} \in \widehat{H}^*\Lambda^{k-1,\nor}(\partial\Omega)$, we have the equalities*

$$\|\widehat{\tau}^{\tan}\|_{\tan,\partial\Omega} = \sup_{\widehat{v}^{\nor}\neq 0} \frac{\langle \widehat{\tau}^{\tan}, \widehat{v}^{\nor}\rangle_{\partial\Omega}}{\|\widehat{v}\|_{\nor,\partial\Omega}}, \qquad \|\widehat{v}^{\nor}\|_{\nor,\partial\Omega} = \sup_{\widehat{\tau}^{\tan}\neq 0} \frac{\langle \widehat{\tau}^{\tan}, \widehat{v}^{\nor}\rangle_{\partial\Omega}}{\|\widehat{\tau}\|_{\tan,\partial\Omega}}.$$

That is, the duality isomorphisms $\widehat{\tau}^{\tan} \mapsto \langle \widehat{\tau}^{\tan}, \cdot\rangle_{\partial\Omega}$ and $\widehat{v}^{\nor} \mapsto \langle \cdot, \widehat{v}^{\nor}\rangle_{\partial\Omega}$ are isometries.

Proof. Given $\widehat{\tau}^{\tan}$, the Riesz representation theorem gives a unique $w \in H^*\Lambda^k(\Omega)$ such that

$$(w, v)_\Omega + (\delta w, \delta v)_\Omega = \langle \widehat{\tau}^{\tan}, v^{\nor}\rangle_{\partial\Omega}, \quad \forall v \in H^*\Lambda^k(\Omega),$$

so $w + \mathrm{d}\delta w = 0$ with $(-\delta w)^{\tan} = \widehat{\tau}^{\tan}$. Taking $\tau = -\delta w \in H\Lambda^{k-1}(\Omega)$, we have $\tau + \delta\mathrm{d}\tau = 0$ with $\tau^{\tan} = \widehat{\tau}^{\tan}$, so $(\tau, \phi)_\Omega + (\mathrm{d}\tau, \mathrm{d}\phi)_\Omega = 0$ for all $\phi \in \mathring{H}\Lambda^{k-1}(\Omega)$. This is precisely the variational problem satisfied uniquely by the minimum-$H\Lambda$-norm extension of $\widehat{\tau}^{\tan}$, so τ is this extension and $\|\widehat{\tau}^{\tan}\|_{\tan,\partial\Omega} = \|\tau\|_{H\Lambda^{k-1}(\Omega)}$. Since $\tau = -\delta w$ and $\mathrm{d}\tau = w$, we have $\|\tau\|_{H\Lambda^{k-1}(\Omega)} = \|w\|_{H^*\Lambda^k(\Omega)}$, and

$$\|\widehat{\tau}^{\tan}\|_{\tan,\partial\Omega} = \|w\|_{H^*\Lambda^k(\Omega)} = \sup_{\substack{v\in H^*\Lambda^k(\Omega),\\ v\neq 0}} \frac{(w, v)_\Omega + (\delta w, \delta v)_\Omega}{\|v\|_{H^*\Lambda^k(\Omega)}} = \sup_{\substack{v\in H^*\Lambda^k(\Omega),\\ v\neq 0}} \frac{\langle \widehat{\tau}^{\tan}, v^{\nor}\rangle_{\partial\Omega}}{\|v\|_{H^*\Lambda^k(\Omega)}}.$$

For any $v^{\nor} = \widehat{v}^{\nor}$, the denominator is minimized when $\|v\|_{H^*\Lambda^k(\Omega)} = \|\widehat{v}^{\nor}\|_{\nor,\partial\Omega}$, so the first equality follows. The second equality is proved similarly. □

Remark 2.4. As an immediate consequence of the isomorphisms (3), we have

$$\mathring{H}\Lambda^k(\Omega) = \big\{v \in H\Lambda^k(\Omega) : v^{\tan} = 0\big\}, \qquad \mathring{H}^*\Lambda^k(\Omega) = \big\{v \in H^*\Lambda^k(\Omega) : v^{\nor} = 0\big\}.$$

More generally, any closed extension of $\mathrm{d}\colon C_0^\infty\Lambda^k(\Omega) \to C_0^\infty\Lambda^{k+1}(\Omega)$ resulting in a Hilbert complex $\mathring{H}\Lambda^k(\Omega) \subset V^k \subset H\Lambda^k(\Omega)$ is called a choice of *ideal boundary conditions*, cf. Brüning and Lesch [10]. For example, one may take a suitably nice decomposition of $\partial\Omega$ into two pieces, $\Gamma^{\tan}$ and $\Gamma^{\nor}$, and let $V^k := \big\{v \in H\Lambda^k(\Omega) : v^{\tan}|_{\Gamma^{\tan}} = 0\big\}$. For an analysis of these mixed boundary conditions (including what qualifies as a "suitably nice decomposition"), see Jakab, Mitrea, and Mitrea [29], Gol'dshtein, Mitrea, and Mitrea [25].

2.3. **The Hodge decomposition and Poincaré inequality.** Although much of the following analysis applies to more general Hilbert complexes, we focus our attention on

$$0 \to H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}} H\Lambda^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} H\Lambda^n(\Omega) \to 0.$$

The operators d satisfy a compactness property, as shown by Picard [37], and in particular they are Fredholm and thus have closed range. Define

$$\mathfrak{B}^k := \big\{\mathrm{d}\tau : \tau \in H\Lambda^{k-1}(\Omega)\big\}, \qquad \mathfrak{Z}^k := \big\{v \in H\Lambda^k(\Omega) : \mathrm{d}v = 0\big\}, \qquad \mathfrak{H}^k := \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp},$$

which are the subspaces of *exact*, *closed*, and *harmonic* k-forms in $L^2\Lambda^k(\Omega)$. It follows that

$$L^2\Lambda^k(\Omega) = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp},$$

which is an L^2-orthogonal decomposition called the *Hodge decomposition*. By Banach's closed range theorem and the adjointness of d and δ, we may also write

$$\mathfrak{B}^{k\perp} = \big\{v \in \mathring{H}^*\Lambda^k(\Omega) : \delta v = 0\big\} =: \mathring{\mathfrak{Z}}^*_k, \qquad \mathfrak{Z}^{k\perp} = \big\{\delta\eta : \eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)\big\} =: \mathring{\mathfrak{B}}^*_k,$$

called *coclosed* and *coexact* k-forms. This implies

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathring{\mathfrak{Z}}^*_k = \big\{v \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) : \mathrm{d}v = 0,\, \delta v = 0\big\},$$

which is an equivalent characterization of harmonic forms.

Finally, since d is an $H\Lambda$-bounded isomorphism between $H\Lambda^k(\Omega) \cap \mathfrak{Z}^{k\perp}$ and $\mathfrak{B}^{k+1}$, Banach's bounded inverse theorem implies that there exists a constant $c_P(\Omega)$ such that

$$\|v\|_\Omega \le c_P(\Omega)\|\mathrm{d}v\|_\Omega, \quad \forall v \in H\Lambda^k(\Omega) \cap \mathfrak{Z}^{k\perp},$$

which is called the *Poincaré inequality*. Note that Arnold, Falk, and Winther [5], Arnold [2] write the Poincaré inequality differently, using the $\|\cdot\|_{H\Lambda^k(\Omega)}$ norm, so that the constant is $\sqrt{1+c_P(\Omega)^2}$. However, the form we have chosen is more convenient for scaling arguments that we will apply later.

2.4. **The Hodge–Laplace problem.** Recall the Hodge–Laplace operator $L \coloneqq \mathrm{d}\delta + \delta\mathrm{d}$ on k-forms, which we can now interpret in a weak sense. Given $f \in L^2\Lambda^k(\Omega)$, we wish to solve the following problem: Find $u \in \mathfrak{H}^{k\perp}$, $p \in \mathfrak{H}^k$, such that

$$\begin{aligned} Lu + p &= f \quad \text{in } \Omega, \\ u^{\mathrm{nor}} = 0,\ (\mathrm{d}u)^{\mathrm{nor}} &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

The solution gives the Hodge decomposition $f = \mathrm{d}\sigma + p + \delta\rho$, where $\sigma = \delta u$ and $\rho = \mathrm{d}u$.

FEEC is based on the following mixed formulation of the Hodge–Laplace problem: Find $\sigma \in H\Lambda^{k-1}(\Omega)$, $u \in H\Lambda^k(\Omega)$, $p \in \mathfrak{H}^k$ such that

$$(\sigma,\tau)_\Omega - (u,\mathrm{d}\tau)_\Omega = 0, \qquad \forall \tau \in H\Lambda^{k-1}(\Omega), \tag{4a}$$

$$(\mathrm{d}\sigma, v)_\Omega + (\mathrm{d}u,\mathrm{d}v)_\Omega + (p,v)_\Omega = (f,v)_\Omega, \quad \forall v \in H\Lambda^k(\Omega), \tag{4b}$$

$$(u,q)_\Omega = 0, \qquad \forall q \in \mathfrak{H}^k, \tag{4c}$$

where both boundary conditions are natural. More generally, nonvanishing natural boundary conditions may be imposed by adding $\langle\cdot,\cdot\rangle_{\partial\Omega}$ terms on the right-hand side. The well-posedness of this mixed formulation is proved in Arnold, Falk, and Winther [4, Theorem 7.2] and generalized to abstract Hilbert complexes in Arnold, Falk, and Winther [5, Theorem 3.2].

Remark 2.5. Instead of natural boundary conditions, one may impose essential boundary conditions $\sigma^{\tan} = 0$ and $u^{\tan} = 0$ by taking the test and trial functions from $\mathring{H}\Lambda^{k-1}(\Omega)$, $\mathring{H}\Lambda^k(\Omega)$, $\mathring{\mathfrak{H}}^k$, cf. [5, Section 6.2]. This may be generalized to nonvanishing $\sigma^{\tan}$ and $u^{\tan}$ via a standard extension argument. We may also impose other ideal boundary conditions $\mathring{H}\Lambda(\Omega) \subset V \subset H\Lambda(\Omega)$, as discussed in Remark 2.4. For example, mixed boundary conditions are essential for $\sigma^{\tan}$, $u^{\tan}$ on $\Gamma^{\tan}$ and natural for u^{nor}, $(\mathrm{d}u)^{\mathrm{nor}}$ on Γ^{nor}.

2.5. **Finite element exterior calculus.** Just as the Galerkin method approximates problems on infinite-dimensional Hilbert spaces by restricting to finite-dimensional subspaces, FEEC approximates problems on infinite-dimensional Hilbert *complexes* by restricting to finite-dimensional *subcomplexes*.

A subcomplex $V_h \subset H\Lambda(\Omega)$ is a sequence of (here, finite-dimensional) subspaces $V_h^k \subset H\Lambda^k(\Omega)$ that is closed with respect to d, i.e., $\mathrm{d}V_h^k \subset V_h^{k+1}$. Just as in Section 2.3, we have subspaces

$$\mathfrak{B}_h^k \coloneqq \{\mathrm{d}\tau_h : \tau_h \in V_h^{k-1}\}, \qquad \mathfrak{Z}_h^k \coloneqq \{v_h \in V_h^k : \mathrm{d}v_h = 0\}, \qquad \mathfrak{H}_h^k \coloneqq \mathfrak{Z}_h^k \cap \mathfrak{B}_h^{k\perp},$$

along with a discrete Hodge decomposition $V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp}$ and discrete Poincaré inequality. Note that the subcomplex assumption implies $\mathfrak{B}_h^k \subset \mathfrak{B}^k$ and $\mathfrak{Z}_h^k \subset \mathfrak{Z}^k$, although in general $\mathfrak{H}_h^k \not\subset \mathfrak{H}^k$ and $\mathfrak{Z}_h^{k\perp} \not\subset \mathfrak{Z}^{k\perp}$. An additional key assumption in the analysis (but not implementation) of FEEC is the existence of *bounded commuting projections* $\pi_h^k \colon H\Lambda^k(\Omega) \to V_h^k$, which among other uses gives control of the discrete Poincaré constant in terms of $c_P(\Omega)$.

In FEEC, one then approximates the Hodge–Laplace problem (4) by the following finite-dimensional variational problem: Find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, $p_h \in \mathfrak{H}_h^k$ such that

$$(\sigma_h,\tau_h)_\Omega - (u_h,\mathrm{d}\tau_h)_\Omega = 0, \qquad \forall \tau_h \in V_h^{k-1}, \tag{5a}$$

$$(\mathrm{d}\sigma_h, v_h)_\Omega + (\mathrm{d}u_h,\mathrm{d}v_h)_\Omega + (p_h,v_h)_\Omega = (f,v_h)_\Omega, \quad \forall v_h \in V_h^k, \tag{5b}$$

$$(u_h,q_h)_\Omega = 0, \qquad \forall q_h \in \mathfrak{H}_h^k. \tag{5c}$$

Arnold, Falk, and Winther [4, 5] establish stability and convergence for this problem, proving quasi-optimal error estimates in the $H\Lambda$-norm and improved L^2-error estimates under additional

regularity assumptions using the aforementioned compactness property. (In [5], much of this analysis takes place in the setting of abstract Hilbert complexes.) As in Remark 2.5, we may instead take essential boundary conditions for $\sigma_h^{\tan}$ and $u_h^{\tan}$. Licht [32] has recently extended the analysis of FEEC to mixed boundary conditions, including the construction of bounded commuting projections.

One more essential ingredient of FEEC is the construction of finite elements for the spaces V_h^k. Suppose that $\Omega \subset \mathbb{R}^n$ is polyhedral, and let $\mathcal{T}_h$ be a triangulation of Ω by n-simplices $K \in \mathcal{T}_h$. Arnold, Falk, and Winther [4, 5] construct two families of piecewise-polynomial differential forms, called $\mathcal{P}_r\Lambda$ and $\mathcal{P}_r^-\Lambda$, which we will sometimes refer to collectively as $\mathcal{P}_r^\pm\Lambda$. Arnold, Falk, and Winther [4, 5] show that any of the pairs of spaces

$$V_h^{k-1} = \mathcal{P}_{r+1}^\pm \Lambda^{k-1}(\mathcal{T}_h), \qquad V_h^k = \left\{ \begin{array}{c} \mathcal{P}_r\Lambda^k(\mathcal{T}_h) \text{ (if } r \geq 1) \\ \text{or} \\ \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h) \end{array} \right\}, \tag{6}$$

results in a subcomplex for the problem (5) satisfying the needed analytical assumptions.

3. Domain decomposition of the Hodge–Laplace problem

This section presents a domain decomposition of the Hodge–Laplace problem, where $\Omega \subset \mathbb{R}^n$ is partitioned into non-overlapping Lipschitz subdomains $K \in \mathcal{T}_h$. This will be the foundation for the hybrid methods in Section 4, where Ω is polyhedral and $K \in \mathcal{T}_h$ are elements of a conforming mesh. However, the results of this section also apply to more general types of domain decomposition.

3.1. Decomposition of Hilbert complexes of differential forms.

Define the broken spaces

$$H\Lambda^k(\mathcal{T}_h) \coloneqq \prod_{K\in\mathcal{T}_h} H\Lambda^k(K), \qquad H^*\Lambda^k(\mathcal{T}_h) \coloneqq \prod_{K\in\mathcal{T}_h} H^*\Lambda^k(K).$$

As product spaces, these naturally inherit the inner products

$$(\cdot,\cdot)_{\mathcal{T}_h} \coloneqq \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_K, \quad (\cdot,\cdot)_{H\Lambda^k(\mathcal{T}_h)} \coloneqq \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_{H\Lambda^k(K)}, \quad (\cdot,\cdot)_{H^*\Lambda^k(\mathcal{T}_h)} \coloneqq \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_{H^*\Lambda^k(K)}.$$

We can then define $\mathrm{d}\colon H\Lambda^k(\mathcal{T}_h) \to H\Lambda^{k+1}(\mathcal{T}_h)$ to be $\mathrm{d}|_{H\Lambda^k(K)}$ on each $K \in \mathcal{T}_h$, and likewise for $\delta\colon H^*\Lambda^k(\mathcal{T}_h) \to H^*\Lambda^{k-1}(\mathcal{T}_h)$. These broken Hilbert complexes are simply the $H\Lambda$ and $H^*\Lambda$ complexes for the disjoint union $\bigsqcup_{K\in\mathcal{T}_h} K$.

For these broken spaces, we can define tangential and normal traces on $\partial\mathcal{T}_h \coloneqq \bigsqcup_{K\in\mathcal{T}_h} \partial K$ by taking the trace on ∂K for each $K \in \mathcal{T}_h$. Defining the pairing $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} \coloneqq \sum_{K\in\mathcal{T}_h}\langle\cdot,\cdot\rangle_{\partial K}$, we immediately get the integration by parts formula

$$\langle \tau^{\tan}, v^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = (\mathrm{d}\tau, v)_{\mathcal{T}_h} - (\tau, \delta v)_{\mathcal{T}_h}, \quad \forall \tau \in H\Lambda^{k-1}(\mathcal{T}_h),\ v \in H^*\Lambda^k(\mathcal{T}_h),$$

simply by summing the integration by parts formulas for each $K \in \mathcal{T}_h$. Note that, if $e = \partial K^+ \cap \partial K^-$ is the interface between $K^\pm \in \mathcal{T}_h$, then e appears twice in the disjoint union $\partial\mathcal{T}_h$: once as part of ∂K^+, and a second time as part of ∂K^-. The traces of broken differential forms can therefore be seen as "double valued," since there is no continuity imposed at interfaces between subdomains.

There are natural inclusions $H\Lambda^k(\Omega) \hookrightarrow H\Lambda^k(\mathcal{T}_h)$ and $H^*\Lambda^k(\Omega) \hookrightarrow H^*\Lambda^k(\mathcal{T}_h)$, which are defined by restriction to each $K \in \mathcal{T}_h$. The next result characterizes these subspaces of unbroken differential forms, generalizing some classic results on domain decomposition of H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$ spaces (cf. Propositions 2.1.1–2.1.3 of Boffi, Brezzi, and Fortin [6]). In a weak sense, it says that unbroken differential forms are precisely those with "single valued" tangential or normal traces.

Proposition 3.1. *If $\mathcal{T}_h$ is a decomposition of Ω into Lipschitz subdomains, then*

$$\begin{aligned}
H\Lambda^k(\Omega) &= \big\{v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\nor}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)\big\},\\
\mathring{H}\Lambda^k(\Omega) &= \big\{v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\nor}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \eta \in H^*\Lambda^{k+1}(\Omega)\big\},\\
H^*\Lambda^k(\Omega) &= \big\{v \in H^*\Lambda^k(\mathcal{T}_h) : \langle \tau^{\tan}, v^{\nor}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \tau \in \mathring{H}\Lambda^{k-1}(\Omega)\big\},\\
\mathring{H}^*\Lambda^k(\Omega) &= \big\{v \in H^*\Lambda^k(\mathcal{T}_h) : \langle \tau^{\tan}, v^{\nor}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \tau \in H\Lambda^{k-1}(\Omega)\big\}.
\end{aligned}$$

Proof. These four identities are proved using essentially the same argument, so we give only a proof of the first. If $v \in H\Lambda^k(\Omega)$, then for all $\eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)$,

$$\langle v^{\tan}, \eta^{\nor}\rangle_{\partial\mathcal{T}_h} = (\mathrm{d}v, \eta)_{\mathcal{T}_h} - (v, \delta\eta)_{\mathcal{T}_h} = (\mathrm{d}v, \eta)_\Omega - (v, \delta\eta)_\Omega = \langle v^{\tan}, \eta^{\nor}\rangle_{\partial\Omega} = 0.$$

Conversely, suppose that $v \in H\Lambda^k(\mathcal{T}_h) \subset L^2\Lambda^k(\mathcal{T}_h) \cong L^2\Lambda^k(\Omega)$ satisfies $\langle v^{\tan}, \eta^{\nor}\rangle_{\partial\mathcal{T}_h} = 0$ for all $\eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)$. Then, using integration by parts and Cauchy–Schwarz,

$$(v, \delta\eta)_\Omega = (v, \delta\eta)_{\mathcal{T}_h} = (\mathrm{d}v, \eta)_{\mathcal{T}_h} \le \|\mathrm{d}v\|_{\mathcal{T}_h}\|\eta\|_{\mathcal{T}_h} = \|\mathrm{d}v\|_{\mathcal{T}_h}\|\eta\|_\Omega.$$

In particular, this holds for $\eta \in C_0^\infty\Lambda^{k+1}(\Omega)$, implying $\mathrm{d}v \in L^2\Lambda^{k+1}(\Omega)$ and hence $v \in H\Lambda^k(\Omega)$. □

3.2. **Decomposition of the Hodge–Laplace problem.** For each $K \in \mathcal{T}_h$, observe that σ and u solve the local problem

$$\begin{aligned}
(\sigma, \tau)_K - (u, \mathrm{d}\tau)_K &= 0, && \forall \tau \in \mathring{H}\Lambda^{k-1}(K),\\
(\mathrm{d}\sigma, v)_K + (\mathrm{d}u, \mathrm{d}v)_K &= (f - p, v)_K, && \forall v \in \mathring{H}\Lambda^k(K),
\end{aligned}$$

with essential boundary conditions $\sigma^{\tan}$ and $u^{\tan}$. However, if the space of *local harmonic forms* $\mathring{\mathfrak{H}}^k(K)$ is nontrivial, then this local problem is not well-posed.[1] Therefore, we include an additional local variable $\overline{p} \in \mathring{\mathfrak{H}}^k(K)$ and solve

$$(\sigma, \tau)_K - (u, \mathrm{d}\tau)_K = 0, \qquad \forall \tau \in \mathring{H}\Lambda^{k-1}(K), \tag{7a}$$

$$(\mathrm{d}\sigma, v)_K + (\mathrm{d}u, \mathrm{d}v)_K + (\overline{p}, v)_K = (f - p, v)_K, \quad \forall v \in \mathring{H}\Lambda^k(K), \tag{7b}$$

$$(u, \overline{q})_K = (\overline{u}, \overline{q})_K, \qquad \forall \overline{q} \in \mathring{\mathfrak{H}}^k(K), \tag{7c}$$

where $\overline{u}$ is the projection of u onto $\mathring{\mathfrak{H}}^k(K)$. Following Remark 2.5, these local solvers are well-posed for any right-hand side and tangential traces $\sigma^{\tan}$, $u^{\tan}$.

We now allow the tangential traces $\widehat{\sigma}^{\tan}$, $\widehat{u}^{\tan}$ to be independent variables and impose the constraints $\sigma^{\tan} = \widehat{\sigma}^{\tan}$, $u^{\tan} = \widehat{u}^{\tan}$ using Lagrange multipliers $\widehat{u}^{\nor}$, $\widehat{\rho}^{\nor}$, which will turn out to be the normal traces of u and $\rho = \mathrm{d}u$. Define the spaces

$$\begin{aligned}
W^k &\coloneqq H\Lambda^k(\mathcal{T}_h), & \overline{\mathfrak{H}}^k &\coloneqq \prod_{K\in\mathcal{T}_h} \mathring{\mathfrak{H}}^k(K),\\
\widehat{W}^{k,\nor} &\coloneqq \big\{\eta^{\nor} : \eta \in H^*\Lambda^{k+1}(\mathcal{T}_h)\big\}, & \widehat{V}^{k,\tan} &\coloneqq \big\{v^{\tan} : v \in H\Lambda^k(\Omega)\big\}.
\end{aligned}$$

Note that $\widehat{V}^{k,\tan}$ consists of "single valued" traces from the unbroken space $H\Lambda^k(\Omega)$, whereas the other three spaces contain broken k-forms. Consider the variational problem: Find

$$\begin{aligned}
&\text{(local variables)} && \sigma \in W^{k-1}, && u \in W^k, && \overline{p} \in \overline{\mathfrak{H}}^k, && \widehat{u}^{\nor} \in \widehat{W}^{k-1,\nor}, && \widehat{\rho}^{\nor} \in \widehat{W}^{k,\nor},\\
&\text{(global variables)} && && p \in \mathfrak{H}^k, && \overline{u} \in \overline{\mathfrak{H}}^k, && \widehat{\sigma}^{\tan} \in \widehat{V}^{k-1,\tan}, && \widehat{u}^{\tan} \in \widehat{V}^{k,\tan},
\end{aligned}$$

[1] When $K \in \mathcal{T}_h$ are contractible (e.g., simplices in a triangulation), this is only an issue for $k = n$, where $\mathring{\mathfrak{H}}^n(K) \cong \mathbb{R}$.

satisfying

$$(\sigma,\tau)_{\mathcal{T}_h} - (u,\mathrm{d}\tau)_{\mathcal{T}_h} + \langle \widehat{u}^{\mathrm{nor}}, \tau^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \tau \in W^{k-1}, \tag{8a}$$

$$(\mathrm{d}\sigma, v)_{\mathcal{T}_h} + (\mathrm{d}u,\mathrm{d}v)_{\mathcal{T}_h} + (\overline{p}+p, v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = (f,v)_{\mathcal{T}_h}, \qquad \forall v \in W^k, \tag{8b}$$

$$(\overline{u}-u,\overline{q})_{\mathcal{T}_h} = 0, \qquad \forall \overline{q} \in \overline{\mathfrak{H}}^k, \tag{8c}$$

$$\langle \widehat{\sigma}^{\mathrm{tan}} - \sigma^{\mathrm{tan}}, \widehat{v}^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{v}^{\mathrm{nor}} \in \widehat{W}^{k-1,\mathrm{nor}}, \tag{8d}$$

$$\langle \widehat{u}^{\mathrm{tan}} - u^{\mathrm{tan}}, \widehat{\eta}^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{\eta}^{\mathrm{nor}} \in \widehat{W}^{k,\mathrm{nor}}, \tag{8e}$$

$$(u,q)_{\mathcal{T}_h} = 0, \qquad \forall q \in \mathfrak{H}^k, \tag{8f}$$

$$(\overline{p},\overline{v})_{\mathcal{T}_h} = 0, \qquad \forall \overline{v} \in \overline{\mathfrak{H}}^k, \tag{8g}$$

$$\langle \widehat{u}^{\mathrm{nor}}, \widehat{\tau}^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{\tau}^{\mathrm{tan}} \in \widehat{V}^{k-1,\mathrm{tan}}, \tag{8h}$$

$$\langle \widehat{\rho}^{\mathrm{nor}}, \widehat{v}^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{v}^{\mathrm{tan}} \in \widehat{V}^{k,\mathrm{tan}}. \tag{8i}$$

Given values for the global variables, notice that (8a)–(8e) simply amounts to solving the local problem (7) on each $K \in \mathcal{T}_h$.

We now prove that this is indeed a domain decomposition of the Hodge–Laplace problem (4), which in particular implies well-posedness of (8). A more general proof of well-posedness, where the right-hand side of (8) is allowed to be arbitrary, will be given in Section 3.3.

Theorem 3.2. *The following are equivalent:*

- $(\sigma, u, \overline{p}, \widehat{u}^{\mathrm{nor}}, \widehat{\rho}^{\mathrm{nor}}, p, \overline{u}, \widehat{\sigma}^{\mathrm{tan}}, \widehat{u}^{\mathrm{tan}})$ *is a solution to* (8).
- (σ, u, p) *is a solution to* (4)*, and furthermore,* $\overline{p} = 0$*,* $\widehat{u}^{\mathrm{nor}} = u^{\mathrm{nor}}$*,* $\widehat{\rho}^{\mathrm{nor}} = (\mathrm{d}u)^{\mathrm{nor}}$*,* $\overline{u}$ *is the projection of* u *onto* $\overline{\mathfrak{H}}^k$*,* $\widehat{\sigma}^{\mathrm{tan}} = \sigma^{\mathrm{tan}}$*, and* $\widehat{u}^{\mathrm{tan}} = u^{\mathrm{tan}}$*.*

Proof. Suppose we have a solution to (8). The claimed equalities are immediate from the variational problem, so it remains only to show that (σ, u, p) solves (4). Since $\sigma^{\mathrm{tan}} = \widehat{\sigma}^{\mathrm{tan}}$ and $u^{\mathrm{tan}} = \widehat{u}^{\mathrm{tan}}$, Proposition 3.1 implies that $\sigma \in H\Lambda^{k-1}(\Omega)$ and $u \in H\Lambda^k(\Omega)$. Therefore, taking test functions $\tau \in H\Lambda^{k-1}(\Omega)$ and $v \in H\Lambda^k(\Omega)$ in (8a)–(8b), the normal trace terms vanish by (8h)–(8i), and we obtain (4a)–(4b). Finally, (8f) is the same as (4c), which proves the forward direction.

Conversely, given a solution (σ, u, p) to (4), it is immediate that (8a)–(8g) hold. For the remaining two equations, first observe that combining (4a) and (8a) gives $\langle \widehat{u}^{\mathrm{nor}}, \tau^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0$ for $\tau \in H\Lambda^{k-1}(\Omega)$, which implies (8h). Similarly, combining (4b) and (8b) gives $\langle \widehat{\rho}^{\mathrm{nor}}, v^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0$ for $v \in H\Lambda^k(\Omega)$, which implies (8i). □

For the last step of the proof, we could instead have used that (4a) gives $u \in \mathring{H}^*\Lambda^k(\Omega)$ and (4b) gives $\mathrm{d}u \in \mathring{H}^*\Lambda^{k+1}(\Omega)$, applying Proposition 3.1 to conclude that their normal traces satisfy (8h)–(8i). However, as we will see, the variational argument above generalizes more readily to the hybridization of FEEC in Section 4.

Remark 3.3. Although the domain decomposition is presented above for $H\Lambda(\Omega)$ with natural boundary conditions on $\partial\Omega$, it is easily generalized to $\mathring{H}\Lambda(\Omega)$ or other ideal boundary conditions $\mathring{H}\Lambda(\Omega) \subset V \subset H\Lambda(\Omega)$, as in Remark 2.5. In this case, the broken spaces are unchanged, and we take the unbroken tangential traces and harmonic forms to be those from the complex V.

We note two special cases that recover known methods for the scalar Poisson equation:

- When $k = 0$, the only nontrivial fields are u, $\widehat{\rho}^{\mathrm{nor}}$, p, and $\widehat{u}^{\mathrm{nor}}$, and the Neumann problem on Ω is decomposed into local Dirichlet problems on $K \in \mathcal{T}_h$. If $V = \mathring{H}\Lambda(\Omega)$, so that $\partial\Omega$ also has Dirichlet conditions, then p is trivial, and we recover the "three-field domain decomposition method" of Brezzi and Marini [9]. This decomposition is the foundation for the hybridized continuous Galerkin method of Cockburn, Gopalakrishnan, and Wang [20].

- When $k = n$, the mixed formulation of the Dirichlet problem on Ω is decomposed into local Neumann problems on $K \in \mathcal{T}_h$. Assuming the subdomains are connected, the local harmonic variables $\overline{u}$ and $\overline{p}$ are piecewise constant, and we recover the domain decomposition appearing in Cockburn [16, Section 5.1], used for hybridization with local Neumann solvers.

3.3. **Saddle point formulation and well-posedness.** Define the bilinear forms

$$\begin{aligned} a\big((\sigma,u,\overline{p},\widehat{u}^{\mathrm{nor}},\widehat{\rho}^{\mathrm{nor}}),(\tau,v,\overline{q},\widehat{v}^{\mathrm{nor}},\widehat{\eta}^{\mathrm{nor}})\big) &\coloneqq -(\sigma,\tau)_{\mathcal{T}_h} + (u,\mathrm{d}\tau)_{\mathcal{T}_h} - \langle \widehat{u}^{\mathrm{nor}}, \tau^{\tan}\rangle_{\partial\mathcal{T}_h} \\ &\quad +(\mathrm{d}\sigma, v)_{\mathcal{T}_h} + (\mathrm{d}u,\mathrm{d}v)_{\mathcal{T}_h} + (\overline{p},v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v^{\tan}\rangle_{\partial\mathcal{T}_h} \\ &\quad +(u,\overline{q})_{\mathcal{T}_h} - \langle \sigma^{\tan}, \widehat{v}^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} - \langle u^{\tan}, \widehat{\eta}^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h}, \\ b\big((\tau,v,\overline{q},\widehat{v}^{\mathrm{nor}},\widehat{\eta}^{\mathrm{nor}}),(q,\overline{v},\widehat{\tau}^{\tan},\widehat{v}^{\tan})\big) &\coloneqq (v,q)_{\mathcal{T}_h} - (\overline{q},\overline{v})_{\mathcal{T}_h} + \langle \widehat{v}^{\mathrm{nor}}, \widehat{\tau}^{\tan}\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\eta}^{\mathrm{nor}}, \widehat{v}^{\tan}\rangle_{\partial\mathcal{T}_h}, \end{aligned}$$

where we have chosen the signs so that $a(\cdot,\cdot)$ is symmetric. Then the domain-decomposed Hodge–Laplace problem (8) becomes a particular instance of the saddle-point problem

$$a(x,x') + b(x',y) = F(x'), \quad \forall x' \in X, \tag{9a}$$

$$b(x,y') = G(y'), \quad \forall y' \in Y. \tag{9b}$$

Here, X is the space of local variables and Y is the space of global variables, so $a(\cdot,\cdot)$ corresponds to the local solvers and $b(\cdot,\cdot)$ to the coupling between local and global variables. This saddle point formulation will also be useful for describing the procedure of static condensation in Section 4.2.

Theorem 3.4. *The problem* (9) *is well-posed.*

Proof. It suffices to show that $b(\cdot,\cdot)$ satisfies a single inf-sup condition, meaning that the map $x \mapsto b(x,\cdot)$ is surjective, and that $a(\cdot,\cdot)$ satisfies a double inf-sup condition on the kernel of this map, cf. Boffi, Brezzi, and Fortin [6, Theorem 4.2.3].

Let q, $\overline{v}$, $\widehat{\tau}^{\tan}$, and $\widehat{v}^{\tan}$ be arbitrary. For the first two terms appearing in $b(\cdot,\cdot)$, we have

$$\|q\|_{\mathcal{T}_h} = \sup_{v\neq 0} \frac{(v,q)_{\mathcal{T}_h}}{\|v\|_{\mathcal{T}_h}}, \qquad \|\overline{v}\|_{\mathcal{T}_h} = \sup_{\overline{q}\neq 0} \frac{-(\overline{q},\overline{v})_{\mathcal{T}_h}}{\|\overline{q}\|_{\mathcal{T}_h}},$$

attained at $v = q$ and $\overline{q} = -\overline{v}$ when these are nonzero. Applying Lemma 2.3 to each $K \in \mathcal{T}_h$ gives

$$\|\widehat{\tau}^{\tan}\|_{\tan,\partial\mathcal{T}_h} = \sup_{\widehat{v}^{\mathrm{nor}}\neq 0} \frac{\langle \widehat{\tau}^{\tan}, \widehat{v}^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h}}{\|\widehat{v}^{\mathrm{nor}}\|_{\mathrm{nor},\partial\mathcal{T}_h}}, \qquad \|\widehat{v}^{\tan}\|_{\tan,\partial\mathcal{T}_h} = \sup_{\widehat{\eta}^{\mathrm{nor}}\neq 0} \frac{\langle \widehat{v}^{\tan}, \widehat{\eta}^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h}}{\|\widehat{\eta}^{\mathrm{nor}}\|_{\mathrm{nor},\partial\mathcal{T}_h}},$$

which proves the inf-sup condition for $b(\cdot,\cdot)$. It remains to show that $a(\cdot,\cdot)$ satisfies an inf-sup condition on the kernel of $x \mapsto b(x,\cdot)$. On this kernel, we have

$$u, v \perp \mathfrak{H}^k, \qquad \overline{q}, \overline{p} = 0, \qquad \widehat{u}^{\mathrm{nor}}, \widehat{v}^{\mathrm{nor}} \perp \widehat{V}^{k-1,\tan}, \qquad \widehat{\rho}^{\mathrm{nor}}, \widehat{\eta}^{\mathrm{nor}} \perp \widehat{V}^{k,\tan},$$

and we may further separate $a(\cdot,\cdot)$ into a pair of bilinear forms

$$\begin{aligned} \alpha\big((\sigma,u),(\tau,v)\big) &= -(\sigma,\tau)_{\mathcal{T}_h} + (u,\mathrm{d}\tau)_{\mathcal{T}_h} + (\mathrm{d}\sigma,v)_{\mathcal{T}_h} + (\mathrm{d}u,\mathrm{d}v)_{\mathcal{T}_h}, \\ \beta\big((\tau,v),(\widehat{v}^{\mathrm{nor}},\widehat{\eta}^{\mathrm{nor}})\big) &= -\langle \widehat{v}^{\mathrm{nor}}, \tau^{\tan}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\eta}^{\mathrm{nor}}, v^{\tan}\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

The inf-sup condition for $\beta(\cdot,\cdot)$ holds by another application of Lemma 2.3 on each $K \in \mathcal{T}_h$. Finally, using Proposition 3.1, the kernel of $\xi \mapsto \beta(\xi,\cdot)$ is precisely $H\Lambda^{k-1}(\Omega) \times \mathfrak{H}^{k\perp}$, so the inf-sup condition for $\alpha(\cdot,\cdot)$ on the kernel is just that for the non-domain-decomposed Hodge–Laplace problem, cf. Arnold, Falk, and Winther [5, Theorem 3.2]. □

4. Hybrid methods and static condensation

In this section, we present a hybridization of the FEEC methods of Section 2.5 for the Hodge–Laplace problem, based on the domain-decomposed variational principle (8). We then perform static condensation of these methods, using the local solvers to efficiently reduce the system to a smaller one involving only the global variables. This condensed system is shown to be as small or smaller than that for standard FEEC without hybridization, and we prove an explicit formula for the number of reduced degrees of freedom. Finally, we prove error estimates for the hybrid variables, which approximate tangential and normal traces.

4.1. Hybridized FEEC methods.

For each $K \in \mathcal{T}_h$, let $W_h(K) \subset H\Lambda(K)$ be a finite-dimensional subcomplex, so that

$$W_h \coloneqq \prod_{K\in\mathcal{T}_h} W_h(K), \qquad V_h \coloneqq V \cap W_h,$$

are respectively subcomplexes of $W = H\Lambda(\mathcal{T}_h)$ and $V = H\Lambda(\Omega)$.[2] Let $\overline{\mathfrak{H}}_h^k \coloneqq \prod_{K\in\mathcal{T}_h} \mathring{\mathfrak{H}}_h^k(K)$, where $\mathring{\mathfrak{H}}_h^k(K)$ is the space of local harmonic k-forms in $\mathring{W}_h^k(K)$, and let $\mathfrak{H}_h^k$ be the space of global harmonic k-forms in V_h^k. Next, we define broken and unbroken tangential traces,

$$\widehat{W}_h^{k,\tan} \coloneqq \{v_h^{\tan} : v_h \in W_h^k\}, \qquad \widehat{V}_h^{k,\tan} \coloneqq \{v_h^{\tan} : v_h \in V_h^k\} = \widehat{V}^{k,\tan} \cap \widehat{W}_h^{k,\tan},$$

and take $\widehat{W}_h^{k,\mathrm{nor}} \coloneqq (\widehat{W}_h^{k,\tan})^*$. Since $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h}$ is a duality pairing, we use this same notation for the pairing of $\widehat{W}_h^{k,\tan}$ with its dual space $\widehat{W}_h^{k,\mathrm{nor}}$.

Example 4.1 (decomposition of $\mathcal{P}_r^{\pm}\Lambda$ elements)**.** If $\mathcal{T}_h$ is a conforming simplicial mesh and $W_h^k(K) = \mathcal{P}_r^{\pm}\Lambda^k(K)$ for each $K \in \mathcal{T}_h$, then $V_h^k = \mathcal{P}_r^{\pm}\Lambda^k(\mathcal{T}_h)$. Since simplices are contractible, the local harmonic forms are trivial for $k < n$ and piecewise constants for $k = n$, and the global harmonic forms $\mathfrak{H}_h^k$ are as in Section 2.5.

For each $K \in \mathcal{T}_h$, the broken trace space $\widehat{W}_h^{k,\tan}$ contains tangential traces of $\mathcal{P}_r^{\pm}\Lambda^k(K)$, so the degrees of freedom are just those living on ∂K. Since this is a broken space, the degrees of freedom need not match on interior facets $e = \partial K^+ \cap \partial K^-$. By contrast, $\widehat{V}_h^{k,\tan}$ contains tangential traces from the unbroken space $\mathcal{P}_r^{\pm}\Lambda^k(\mathcal{T}_h)$, so the degrees of freedom are single-valued. Finally, we can use duality to identify $\widehat{W}_h^{k,\mathrm{nor}}$ with the degrees of freedom for $\widehat{W}_h^{k,\tan}$. Since these tangential traces are piecewise polynomial and thus in $L^2(\partial\mathcal{T}_h)$, for implementation we may simply take $\widehat{W}_h^{k,\mathrm{nor}} = \widehat{W}_h^{k,\tan}$ where $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h}$ is the L^2 inner product.

Now that we have defined these finite-dimensional subspaces, we may consider the following finite-dimensional version of the domain-decomposed variational problem (8): Find

$$\begin{array}{lccccccc}
\text{(local variables)} & \sigma_h \in W_h^{k-1}, & u_h \in W_h^k, & \overline{p}_h \in \overline{\mathfrak{H}}_h^k, & \widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}, & \widehat{p}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}, \\
\text{(global variables)} & & p_h \in \mathfrak{H}_h^k, & \overline{u}_h \in \overline{\mathfrak{H}}_h^k, & \widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, & \widehat{u}_h^{\tan} \in \widehat{V}_h^{k,\tan},
\end{array}$$

[2] As in Remark 3.3, the arguments readily generalize to $V = \mathring{H}\Lambda(\Omega)$ or other choices of ideal boundary conditions.

satisfying

$$(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \mathrm{d}\tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\tan} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \tau_h \in W_h^{k-1}, \tag{10a}$$

$$(\mathrm{d}\sigma_h, v_h)_{\mathcal{T}_h} + (\mathrm{d}u_h, \mathrm{d}v_h)_{\mathcal{T}_h} + (\overline{p}_h + p_h, v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h^{\tan} \rangle_{\partial \mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}, \quad \forall v_h \in W_h^k, \tag{10b}$$

$$(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} = 0, \qquad \forall \overline{q}_h \in \overline{\mathfrak{H}}_h^k, \tag{10c}$$

$$\langle \widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}, \tag{10d}$$

$$\langle \widehat{u}_h^{\tan} - u_h^{\tan}, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}, \tag{10e}$$

$$(u_h, q_h)_{\mathcal{T}_h} = 0, \qquad \forall q_h \in \mathfrak{H}_h^k, \tag{10f}$$

$$(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} = 0, \qquad \forall \overline{v}_h \in \overline{\mathfrak{H}}_h^k, \tag{10g}$$

$$\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, \tag{10h}$$

$$\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{k,\tan}. \tag{10i}$$

Given values for the global variables, (10a)–(10e) amounts to solving the local FEEC problems

$$(\sigma_h, \tau_h)_K - (u_h, \mathrm{d}\tau_h)_K = 0, \qquad \forall \tau_h \in \mathring{W}_h^{k-1}(K), \tag{11a}$$

$$(\mathrm{d}\sigma_h, v_h)_K + (\mathrm{d}u_h, \mathrm{d}v_h)_K + (\overline{p}_h, v_h)_K = (f - p_h, v_h)_K, \quad \forall v_h \in \mathring{W}_h^k(K), \tag{11b}$$

$$(u_h, \overline{q}_h)_K = (\overline{u}_h, \overline{q}_h)_K, \qquad \forall \overline{q}_h \in \mathring{\mathfrak{H}}_h^k(K), \tag{11c}$$

with essential tangential boundary conditions $\sigma_h^{\tan} = \widehat{\sigma}_h^{\tan}$ and $u_h^{\tan} = \widehat{u}_h^{\tan}$.

The following result shows that this is indeed a hybridization of the global FEEC problem (5), which in particular implies well-posedness of (10). The proof is quite similar to Theorem 3.2, but there are two important distinctions. First, $\widehat{u}_h^{\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}}$ generally *do not* equal the normal traces of u_h and $\rho_h = \mathrm{d}u_h$, except weakly, in a Galerkin sense. Furthermore, a crucial role is played by the specific choice of broken tangential and normal trace spaces above, particularly the fact that they are in duality with respect to $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}$.

Theorem 4.2. *The following are equivalent:*

- $(\sigma_h, u_h, \overline{p}_h, \widehat{u}_h^{\mathrm{nor}}, \widehat{\rho}_h^{\mathrm{nor}}, p_h, \overline{u}_h, \widehat{\sigma}_h^{\tan}, \widehat{u}_h^{\tan})$ *is a solution to* (10).
- (σ_h, u_h, p_h) *is a solution to* (5)*, and furthermore,* $\overline{p}_h = 0$*,* $\widehat{u}_h^{\mathrm{nor}}$ *and* $\widehat{\rho}_h^{\mathrm{nor}}$ *are uniquely determined by* (10a)–(10b)*,* $\overline{u}_h$ *is the projection of* u_h *onto* $\overline{\mathfrak{H}}_h^k$*,* $\widehat{\sigma}_h^{\tan} = \sigma_h^{\tan}$*, and* $\widehat{u}_h^{\tan} = u_h^{\tan}$.

Proof. Suppose we have a solution to (10). The claimed equalities are immediate from the variational problem, with uniqueness of the broken tangential and normal traces following from the fact that these spaces are in duality with respect to $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}$, so it remains only to show that (σ_h, u_h, p_h) solves (5). Since $\sigma_h^{\tan} = \widehat{\sigma}_h^{\tan}$ and $u_h^{\tan} = \widehat{u}_h^{\tan}$, Proposition 3.1 implies that $\sigma_h \in V_h^{k-1}$ and $u_h \in V_h^k$. Taking $\tau_h \in V_h^{k-1}$ and $v_h \in V_h^k$ in (10a)–(10b), the normal trace terms vanish by (10h)–(10i), and we obtain (5a)–(5b). Finally, (10f) is the same as (5c), which proves the forward direction.

Conversely, given a solution (σ_h, u_h, p_h) to (5), it is immediate that (10a)–(10g) hold, again using the fact that $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}$ is a dual pairing to get uniqueness of the broken tangential and normal traces. For the remaining two equations, first observe that combining (5a) and (10a) gives $\langle \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\tan} \rangle_{\partial \mathcal{T}_h} = 0$ for $\tau_h \in V_h^{k-1}$, which implies (10h). Similarly, combining (5b) and (10b) gives $\langle \widehat{\rho}_h^{\mathrm{nor}}, v_h^{\tan} \rangle_{\partial \mathcal{T}_h} = 0$ for $v_h \in V_h^k$, which implies (10i). □

4.2. **Static condensation.** We next perform static condensation of the hybridized FEEC method (10), eliminating the local variables using the local solvers (11) and thereby obtaining a condensed system involving only the global variables. We present the condensed system both in a matrix-free

variational form and as a matrix Schur complement, and we prove that this system is as small or smaller than the standard FEEC method (5) without hybridization.

As we did in Section 3.3 for the infinite-dimensional problem, we may write the hybridized FEEC method (10) as a saddle point problem,

$$a(x_h, x_h') + b(x_h', y_h) = F(x_h'), \quad \forall x_h' \in X_h, \tag{12a}$$

$$b(x_h, y_h') = G(y_h'), \quad \forall y_h' \in Y_h. \tag{12b}$$

Since the local FEEC solvers (11) corresponding to $a(\cdot,\cdot)$ are well-posed, for any given F and y_h we can write the solution to (12a) as $x_h = \mathsf{X}_F + \mathsf{X}_{y_h}$, where

$$a(\mathsf{X}_F, x_h') = F(x_h'), \qquad a(\mathsf{X}_{y_h}, x_h') = -b(x_h', y_h), \qquad \forall x_h' \in X_h.$$

This is an efficient local computation that may be done element-by-element in parallel. Substituting this into (12b) gives a reduced problem involving only the global variables: Find $y_h \in Y_h$ satisfying

$$b(\mathsf{X}_{y_h}, y_h') = G(y_h') - b(\mathsf{X}_F, y_h'), \quad \forall y_h' \in Y_h. \tag{13}$$

This procedure of eliminating variables using local solvers is known as *static condensation*. Once the condensed system has been solved for the global variables, the local variables may be recovered element-by-element, if desired, using the local solvers. Furthermore, we may use linearity to separate the influence of the individual components, computing $\mathsf{X}_F = \mathsf{X}_f$ and $\mathsf{X}_{y_h} = \mathsf{X}_{p_h} + \mathsf{X}_{\overline{u}_h} + \mathsf{X}_{\widehat{\sigma}_h^{\tan}} + \mathsf{X}_{\widehat{u}_h^{\tan}}$.

Given a finite element basis, (12) may also be written in the block-matrix form

$$\begin{bmatrix} A & B^T \\ B & \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} F_h \\ G_h \end{bmatrix}.$$

Since the matrix A corresponds to the local solvers (11), it has a block-diagonal structure, with blocks corresponding to each $K \in \mathcal{T}_h$, and can therefore be inverted efficiently block-by-block. Given F and y_h, we can locally solve

$$A\mathsf{X}_F = F_h, \qquad A\mathsf{X}_{y_h} = -B^T y_h \quad \Longrightarrow \quad x_h = \mathsf{X}_F + \mathsf{X}_{y_h} = A^{-1}F_h - A^{-1}B^T y_h.$$

Substituting this expression into $Bx_h = G_h$ gives the condensed system

$$-BA^{-1}B^T y_h = G_h - BA^{-1}F_h,$$

which is the matrix representation of the condensed variational problem (13). Here, the condensed stiffness matrix $-BA^{-1}B^T$ is precisely the *Schur complement* of the original stiffness matrix $\left[\begin{smallmatrix} A & B^T \\ B & \end{smallmatrix}\right]$.

Remark 4.3. The classical static condensation technique of Guyan [26] did not use hybridization, but simply partitioned the matrix system into blocks corresponding to internal and facet degrees of freedom, then applied the Schur complement approach above to eliminate the interior degrees of freedom. A similar approach has been applied to edge elements for Maxwell's equations, as discussed in the survey by Ledger and Morgan [31, Section 4.5]. The discovery of the relationship between Guyan's static condensation and hybridization is more recent, cf. Cockburn [16].

The next result proves that in full generality—without assumptions on the topology of $K \in \mathcal{T}_h$ or the elements used—the condensed system (13) on $Y_h = \mathfrak{H}_h^k \times \overline{\mathfrak{H}}_h^k \times \widehat{V}_h^{k-1,\tan} \times \widehat{V}_h^{k,\tan}$ is as small or smaller than the standard FEEC system (5) on $V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ without hybridization. Since the space $\mathfrak{H}_h^k$ appears in both systems, it suffices to compare $\dim \overline{\mathfrak{H}}_h^k + \dim \widehat{V}_h^{k-1,\tan} + \dim \widehat{V}_h^{k,\tan}$ (condensed) with $\dim V_h^{k-1} + \dim V_h^k$ (standard FEEC).

Theorem 4.4. *We have the equality*

$$\begin{aligned} &(\dim V_h^{k-1} + \dim V_h^k) - (\dim \overline{\mathfrak{H}}_h^k + \dim \widehat{V}_h^{k-1,\tan} + \dim \widehat{V}_h^{k,\tan}) \\ &\qquad = \sum_{K \in \mathcal{T}_h} \big(\dim \mathring{W}_h^{k-1}(K) + \dim \mathring{\mathfrak{B}}_h^k(K) + \dim \mathring{\mathfrak{Z}}_h^{k\perp}(K)\big). \end{aligned} \tag{14}$$

Consequently, the size of the hybridized and condensed FEEC system (13) *is always less than or equal to that of the standard FEEC system* (5)*, with equality if and only if* $\mathring{W}_h^{k-1}(K)$ *is trivial and* $\mathring{W}_h^k(K) = \mathring{\mathfrak{H}}_h^k(K)$ *for all* $K \in \mathcal{T}_h$*.*

Proof. By definition, $\widehat{V}_h^{k,\tan}$ is the image of V_h^k under the tangential trace map. Therefore, the rank-nullity theorem implies that their dimensions differ by the dimension of the kernel, i.e.,

$$\dim V_h^k - \dim \widehat{V}_h^{k,\tan} = \dim\{v_h \in V_h^k : v_h^{\tan} = 0\} = \dim \prod_{K\in\mathcal{T}_h} \mathring{W}_h^k(K) = \sum_{K\in\mathcal{T}_h} \dim \mathring{W}_h^k(K).$$

Applying the discrete Hodge decomposition to each $\mathring{W}_h^k(K)$ and using $\overline{\mathfrak{H}}_h^k \coloneqq \prod_{K\in\mathcal{T}_h} \mathring{\mathfrak{H}}_h^k(K)$ gives

$$\sum_{K\in\mathcal{T}_h} \dim \mathring{W}_h^k(K) = \dim \overline{\mathfrak{H}}_h^k + \sum_{K\in\mathcal{T}_h} \big(\dim \mathring{\mathfrak{B}}_h^k(K) + \dim \mathring{\mathfrak{Z}}_h^{k\perp}(K)\big).$$

Combining this with the previous expression and the corresponding one for $\dim V_h^{k-1} - \dim \widehat{V}_h^{k-1,\tan}$ implies (14), which completes the proof. □

We now give an explicit count of the reduced degrees of freedom when $\mathcal{T}_h$ is a simplicial mesh and $\mathcal{P}_r^{\pm}\Lambda$ elements are used. Arnold, Falk, and Winther [4, Sections 4.5–4.6] show that for $r \geq 1$,

$$\dim \mathring{\mathcal{P}}_r\Lambda^k(K) = \binom{r-1}{n-k}\binom{r+k}{k}, \qquad \dim \mathring{\mathcal{P}}_r^-\Lambda^k(K) = \binom{n}{k}\binom{r+k-1}{n},$$

with the convention that $\binom{a}{b} = 0$ when $b < 0$ or $b > a$. Applying these formulas to the stable pairs of spaces for FEEC given in (6), we get

$$\dim \mathring{\mathcal{P}}_{r+1}\Lambda^{k-1}(K) = \binom{r}{n-k+1}\binom{r+k}{k-1}, \qquad \dim \mathring{\mathcal{P}}_r\Lambda^k(K) = \binom{r-1}{n-k}\binom{r+k}{k} \quad (\text{if } r \geq 1),$$
$$\dim \mathring{\mathcal{P}}_{r+1}^-\Lambda^{k-1}(K) = \binom{n}{k-1}\binom{r+k-1}{n}, \quad \dim \mathring{\mathcal{P}}_{r+1}^-\Lambda^k(K) = \binom{n}{k}\binom{r+k}{n}.$$

For each $K \in \mathcal{T}_h$, these formulas count the number of internal degrees of freedom, which are precisely the ones eliminated by static condensation.

Since simplices are contractible, the local harmonic spaces are trivial, except for $\mathring{\mathfrak{H}}_h^n(K) \cong \mathbb{R}$. When $k = n$, static condensation *introduces* one global degree of freedom per simplex, so in this case, the number of degrees of freedom is reduced if and only if $r \geq 1$. When $r = 0$ (i.e., the lowest-order RT and BDM methods), the degrees of freedom for u_h are simply replaced by those for $\overline{u}_h$.

By checking when the spaces above have dimension greater than zero, we immediately obtain the following corollary to Theorem 4.4.

Corollary 4.5. *Let* $\mathcal{T}_h$ *be a simplicial mesh and* V_h^{k-1}*,* V_h^k *be one of the stable pairs in* (6)*. The hybridized and condensed FEEC system* (13) *is strictly smaller than the standard FEEC system* (5) *if and only if* $r \geq 1$ *and either*

- $V_h^k = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ *with* $r \geq n-k+1$*, or*
- $V_h^k = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h)$ *with* $r \geq n-k$*.*

4.3. **Error estimates for the hybrid variables.** Let $\{\mathcal{T}_h\}$ be a shape-regular (but not necessarily quasi-uniform) family of simplicial meshes of Ω, where h_K denotes the diameter of $K \in \mathcal{T}_h$ and $h \coloneqq \max_{K\in\mathcal{T}_h} h_K$. We assume again that V_h^{k-1}, V_h^k is one of the stable pairs (6). Error estimates are already known for σ, u, p (Arnold, Falk, and Winther [4, 5]), and for $\overline{u}$ when $k = n$ (Douglas and Roberts [21], Brezzi, Douglas, and Marini [8]), so it only remains to prove estimates for the tangential and normal traces.

The tangential traces are straightforward, since $\widehat{\sigma}_h^{\tan} = \sigma_h^{\tan}$ and $\widehat{u}_h^{\tan} = u_h^{\tan}$. We introduce a scaled version of the tangential trace norm from Section 2.2,

$$|||\widehat{\tau}^{\tan}|||^2_{\tan,\partial K} := \inf\{\|\tau\|_K^2 + h_K^2\|\mathrm{d}\tau\|_K^2 : \tau^{\tan} = \widehat{\tau}^{\tan}\},$$

and denote $|||\cdot|||^2_{\tan,\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}|||\cdot|||^2_{\tan,\partial K}$. It is an easy consequence that the errors for $\sigma_h^{\tan}$ and $u_h^{\tan}$ are controlled by those for σ_h and u_h, which we now state as a proposition.

Proposition 4.6. *For each $K \in \mathcal{T}_h$, we have*

$$|||\sigma^{\tan} - \sigma_h^{\tan}|||^2_{\tan,\partial K} \le \|\sigma - \sigma_h\|_K^2 + h_K^2\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_K^2,$$
$$|||u^{\tan} - u_h^{\tan}|||^2_{\tan,\partial K} \le \|u - u_h\|_K^2 + h_K^2\big\|\mathrm{d}(u - u_h)\big\|_K^2.$$

Consequently,

$$|||\sigma^{\tan} - \sigma_h^{\tan}|||^2_{\tan,\partial\mathcal{T}_h} \le \|\sigma - \sigma_h\|_\Omega^2 + h^2\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_\Omega^2,$$
$$|||u^{\tan} - u_h^{\tan}|||^2_{\tan,\partial\mathcal{T}_h} \le \|u - u_h\|_\Omega^2 + h^2\big\|\mathrm{d}(u - u_h)\big\|_\Omega^2.$$

Proof. The first pair of inequalities follows immediately from the fact that the scaled tangential trace norm is an infimum, and the second pair follows by summing over $K \in \mathcal{T}_h$. □

Given sufficient elliptic regularity, the estimates of Arnold, Falk, and Winther [5] now imply

$$|||\sigma^{\tan} - \sigma_h^{\tan}|||_{\tan,\partial\mathcal{T}_h} \lesssim \begin{cases} h^{r+2}\|f\|_{r+1,\Omega}, & \text{if } V_h^{k-1} = \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}_h), \\ h^{r+1}\|f\|_{r,\Omega}, & \text{if } V_h^{k-1} = \mathcal{P}^-_{r+1}\Lambda^{k-1}(\mathcal{T}_h), \end{cases}$$
$$|||u^{\tan} - u_h^{\tan}|||_{\tan,\partial\mathcal{T}_h} \lesssim \begin{cases} h\|f\|_{\Omega}, & \text{if } V_h^{k} = \mathcal{P}_1^-\Lambda^{k}(\mathcal{T}_h), \\ h^{r+1}\|f\|_{r-1,\Omega}, & \text{otherwise,} \end{cases}$$

which is the optimal order allowed by the polynomial degree of the tangential traces.

We next give estimates for the normal traces, generalizing an argument of Arnold and Brezzi [3] for the hybridized RT method. Recall that $\widehat{u}_h^{\mathrm{nor}} \in (\widehat{W}_h^{k-1,\tan})^*$ and $\widehat{\rho}_h^{\mathrm{nor}} \in (\widehat{W}_h^{k,\tan})^*$, so we compare them to the natural projections $\widehat{P}_h u^{\mathrm{nor}} \in (\widehat{W}_h^{k-1,\tan})^*$ and $\widehat{P}_h \rho^{\mathrm{nor}} \in (\widehat{W}_h^{k,\tan})^*$ defined by

$$\langle \widehat{P}_h u^{\mathrm{nor}}, \widehat{\tau}_h^{\tan}\rangle_{\partial\mathcal{T}_h} = \langle u^{\mathrm{nor}}, \widehat{\tau}_h^{\tan}\rangle_{\partial\mathcal{T}_h}, \quad \forall \widehat{\tau}_h \in \widehat{W}_h^{k-1,\tan},$$
$$\langle \widehat{P}_h \rho^{\mathrm{nor}}, \widehat{v}_h^{\tan}\rangle_{\partial\mathcal{T}_h} = \langle \rho^{\mathrm{nor}}, \widehat{v}_h^{\tan}\rangle_{\partial\mathcal{T}_h}, \quad \forall \widehat{v}_h \in \widehat{W}_h^{k,\tan}.$$

If we simply identify $\widehat{u}_h^{\mathrm{nor}}$ with the corresponding element of $\widehat{W}_h^{k-1,\tan} \subset L^2\Lambda^{k-1}(\partial\mathcal{T}_h)$, we generally *do not* observe convergence to the unprojected u^{nor}, and likewise for $\widehat{\rho}_h^{\mathrm{nor}}$ and ρ^{nor}. The reason is that the identification of $\widehat{u}_h^{\mathrm{nor}}$ with an element of $L^2\Lambda^{k-1}(\partial\mathcal{T}_h)$ is only unique up to the annihilator $(\widehat{W}_h^{k-1,\tan})^\perp$. Therefore, we should really measure the L^2 error after quotienting by the annihilator, which is equivalent to taking the projections above. We define the scaled L^2 norm $|||\cdot|||_{\partial K} := h_K^{1/2}\|\cdot\|_{\partial K}$ and denote $|||\cdot|||^2_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}|||\cdot|||^2_{\partial K}$.

Theorem 4.7. *For each $K \in \mathcal{T}_h$, we have*

$$|||\widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}|||_{\partial K} \lesssim \|P_h u - u_h\|_K + h_K\|\sigma - \sigma_h\|_K,$$
$$|||\widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}|||_{\partial K} \lesssim \big\|P_h\mathrm{d}(u - u_h)\big\|_K + h_K\Big(\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_K + \|p - p_h\|_K\Big),$$

where P_h denotes L^2 projection onto W_h. Consequently,

$$|||\widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}|||_{\partial\mathcal{T}_h} \lesssim \|P_h u - u_h\|_{\mathcal{T}_h} + h\|\sigma - \sigma_h\|_\Omega,$$
$$|||\widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}|||_{\partial\mathcal{T}_h} \lesssim \big\|P_h\mathrm{d}(u - u_h)\big\|_{\mathcal{T}_h} + h\Big(\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_\Omega + \|p - p_h\|_\Omega\Big).$$

Proof. A scaling argument shows that each $\widehat{\tau}_h^{\tan} \in \widehat{W}_h^{k-1,\tan}(\partial K)$ has an extension $\tau_h \in W_h^{k-1}(K)$ with $\tau_h^{\tan} = \widehat{\tau}_h^{\tan}$ such that

$$\|\tau_h\|_K + h_K \|\mathrm{d}\tau_h\|_K \lesssim |||\widehat{\tau}_h^{\tan}|||_{\partial K}.$$

Therefore, subtracting (10a) from (8a), we get

$$\begin{aligned} h_K \langle \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial K} &= h_K \langle u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\tan} \rangle_{\partial K} \\ &= h_K \Big[-(\sigma - \sigma_h, \tau_h)_K + (u - u_h, \mathrm{d}\tau_h)_K \Big] \\ &= h_K \Big[-(\sigma - \sigma_h, \tau_h)_K + (P_h u - u_h, \mathrm{d}\tau_h)_K \Big] \\ &\leq \Big(h_K \|\sigma - \sigma_h\|_K + \|P_h u - u_h\|_K \Big) \Big(\|\tau_h\|_K + h_K \|\mathrm{d}\tau_h\|_K \Big) \\ &\lesssim \Big(h_K \|\sigma - \sigma_h\|_K + \|P_h u - u_h\|_K \Big) |||\widehat{\tau}_h^{\tan}|||_{\partial K}. \end{aligned}$$

Since $\langle \cdot, \cdot \rangle_{\partial K}$ agrees with the L^2 inner product,

$$|||\widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}|||_{\partial K} = h_K^{1/2} \sup_{\widehat{\tau}_h^{\tan} \neq 0} \frac{\langle \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial K}}{|||\widehat{\tau}_h^{\tan}|||_{\partial K}} = \sup_{\widehat{\tau}_h^{\tan} \neq 0} \frac{h_K \langle \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial K}}{|||\widehat{\tau}_h^{\tan}|||_{\partial K}},$$

which completes the proof of the first estimate. The estimate for $|||\widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}|||_{\partial K}$ is obtained similarly, and the $|||\cdot|||_{\partial \mathcal{T}_h}$ estimates again follow immediately from the $|||\cdot|||_{\partial K}$ estimates. □

For $k < n$, we generally cannot improve on $\|P_h u - u_h\|_{\mathcal{T}_h} \leq \|u - u_h\|_\Omega$, so assuming sufficient elliptic regularity and applying the estimates from Arnold, Falk, and Winther [5] gives

$$|||\widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}|||_{\partial \mathcal{T}_h} \lesssim \begin{cases} h \|f\|_\Omega, & \text{if } V_h^k = \mathcal{P}_1^- \Lambda^k(\mathcal{T}_h), \\ h^{r+1} \|f\|_{r-1,\Omega}, & \text{otherwise,} \end{cases}$$

i.e., the convergence rate is the same as that for $u_h \to u$. When $k = n$, however, $\|P_h u - u_h\|_{\mathcal{T}_h}$ famously superconverges for the RT and BDM methods (Douglas and Roberts [21], Arnold and Brezzi [3], Brezzi, Douglas, and Marini [8]). In this case, we recover the superconvergence results of [3, 8] for the Lagrange multipliers:

$$|||\widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}|||_{\partial \mathcal{T}_h} \lesssim \begin{cases} h^2 \|f\|_{1,\Omega}, & \text{if } r = 0, \\ h^{r+3} \|f\|_{r+1,\Omega}, & \text{if } r \geq 1,\ V_h^{n-1} = \mathcal{P}_{r+1} \Lambda^{n-1}(\mathcal{T}_h), \\ h^{r+2} \|f\|_{r,\Omega}, & \text{if } r \geq 1,\ V_h^{n-1} = \mathcal{P}_{r+1}^- \Lambda^{n-1}(\mathcal{T}_h). \end{cases}$$

From the perspective of FEEC, this occurs since $W_h^n = V_h^n = \mathfrak{B}_h^n$, so $\|P_h u - u_h\|_{\mathcal{T}_h} = \big\|P_{\mathfrak{B}_h}(u - u_h)\big\|_\Omega$, which superconverges according to [5, Lemma 3.13]. On the other hand, when $k < n$, the error is dominated by the nonvanishing $\mathfrak{Z}_h^{k\perp}$ component [5, Lemma 3.16], so there is no improvement.

Similarly, when $k < n - 1$, we generally cannot do better than $\big\|P_h \mathrm{d}(u - u_h)\big\|_{\mathcal{T}_h} \leq \big\|\mathrm{d}(u - u_h)\big\|_\Omega$, so assuming sufficient elliptic regularity,

$$|||\widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}|||_{\partial \mathcal{T}_h} \lesssim \begin{cases} h^{r+1} \|f\|_{r,\Omega}, & \text{if } V_h^k = \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{T}_h), \\ h^r \|f\|_{r-1,\Omega}, & \text{if } V_h^k = \mathcal{P}_r \Lambda^k(\mathcal{T}_h), \end{cases}$$

and the convergence rate is the same as that for $\mathrm{d}u_h \to \mathrm{d}u$. However, when $k = n - 1$, we obtain superconvergence as a consequence of the following lemma (which holds for all k, not just $k = n - 1$).

Lemma 4.8. *The FEEC solution* (5) *satisfies* $\big\|P_{\mathfrak{B}_h} \mathrm{d}(u - u_h)\big\|_\Omega \lesssim h \Big(\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_\Omega + \|p - p_h\|_\Omega \Big)$.

Proof. The argument is similar to [5, Lemma 3.15]. Let $v_h \in \mathfrak{Z}_h^{k\perp}$ be such that $\mathrm{d}v_h = P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)$, and take $v = P_{\mathfrak{Z}^\perp}v_h$. Since v is orthogonal to $\mathrm{d}(\sigma-\sigma_h)$ and $p-p_h$, subtracting (5b) from (4b) gives

$$\begin{aligned}\big\|P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)\big\|_\Omega^2 &= \big(\mathrm{d}(u-u_h), \mathrm{d}v_h\big)_\Omega \\ &= \big(\mathrm{d}(\sigma-\sigma_h) + (p-p_h), v-v_h\big)_\Omega \\ &\le \Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_\Omega + \|p-p_h\|_\Omega\Big)\|v-v_h\|_\Omega \\ &\lesssim h\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_\Omega + \|p-p_h\|_\Omega\Big)\big\|P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)\big\|_\Omega.\end{aligned}$$

The last step uses [5, Lemma 3.12], which says that $\|v-v_h\|_\Omega \lesssim h\|\mathrm{d}v_h\|_\Omega$. □

Corollary 4.9. *For $k = n-1$, we have the improved estimate*

$$|||\widehat{P}_h\rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}|||_{\partial\mathcal{T}_h} \lesssim h\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_\Omega + \|p-p_h\|_\Omega\Big).$$

In particular, when $f \in \mathring{\mathfrak{B}}^*_{n-1}$, we have $\widehat{\rho}_h^{\mathrm{nor}} = \widehat{P}_h\rho^{\mathrm{nor}}$ exactly.

Proof. Since $\big\|P_h\mathrm{d}(u-u_h)\big\|_{\mathcal{T}_h} = \big\|P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)\big\|_\Omega$ when $k = n-1$, the improved estimate is immediate from Theorem 4.7 and Lemma 4.8. In particular, σ and p vanish when $f \in \mathring{\mathfrak{B}}^*_{n-1}$, so in that case the left-hand side is identically zero. □

Assuming sufficient elliptic regularity, this gives the superconvergent rates

$$|||\widehat{P}_h\rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}|||_{\partial\mathcal{T}_h} \lesssim \begin{cases} 0, & \text{if } f \in \mathring{\mathfrak{B}}^*_{n-1}, \\ h^{r+2}\|f\|_{r+1,\Omega}, & \text{otherwise.}\end{cases}$$

5. Postprocessing

In this section, we introduce a local postprocessing procedure, which generalizes that of Stenberg [41] from $k = n$ to arbitrary k. We develop new error estimates for the postprocessed solution when $k < n$; in particular, postprocessing gives a superconvergent approximation ρ_h^* to $\mathrm{d}u$ for $k = n-1$, and $\delta\rho_h^*$ is an improved approximation to $\delta\mathrm{d}u$ for all k. Finally, we discuss how this analysis corresponds to that of Stenberg [41] in the case $k = n$, giving superconvergence of u_h^* to u.

5.1. The postprocessing procedure.

To motivate the proposed procedure, recall that the exact local solver (7) corresponds to solving $Lu + \overline{p} = f - p$ such that $P_{\overline{\mathfrak{H}}}u = \overline{u}$, with tangential boundary conditions given by $\widehat{\sigma}^{\tan}$ and $\widehat{u}^{\tan}$. Instead of writing this as a variational problem on the $\mathring{H}\Lambda(K)$ complex, we can equivalently write it on the $H^*\Lambda(K)$ complex as

$$(\rho,\eta)_K - (u,\delta\eta)_K = \langle\widehat{u}^{\tan},\eta^{\mathrm{nor}}\rangle_{\partial K}, \qquad \forall\eta \in H^*\Lambda^{k+1}(K), \tag{15a}$$

$$(\delta\rho,v)_K + (\delta u,\delta v)_K + (\overline{p},v)_K = (f-p,v)_K - \langle\widehat{\sigma}^{\tan},v^{\mathrm{nor}}\rangle_{\partial K}, \quad \forall v \in H^*\Lambda^k(K), \tag{15b}$$

$$(u,\overline{q})_K = (\overline{u},\overline{q})_K, \qquad \forall\overline{q} \in \mathring{\mathfrak{H}}^k(K), \tag{15c}$$

where the tangential boundary conditions are now natural rather than essential. As before, we have $\sigma = \delta u$ and $\rho = \mathrm{d}u$

The postprocessing procedure is based on approximating (15) on a finite-dimensional subcomplex $W_h^*(K) \subset H^*\Lambda(K)$, meaning $\delta W_h^{*k+1}(K) \subset W_h^{*k}(K)$. Since $\star H^*\Lambda^k(K) = H\Lambda^{n-k}(K)$, an equivalent condition is that $\star W_h^*(K) \subset H\Lambda(K)$ is a subcomplex. Moreover, $\pi_h\colon H\Lambda(K) \to \star W_h^*(K)$ is a bounded commuting projection if and only if $\star^{-1}\pi_h\star\colon H^*\Lambda(K) \to W_h^*(K)$ is. For a simplicial mesh,

we may therefore take

$$\star W_h^{*k+1}(K) = \mathcal{P}^{\pm}_{r^*+1}\Lambda^{n-k-1}(K), \qquad \star W_h^{*k}(K) = \left\{ \begin{array}{c} \mathcal{P}_{r^*}\Lambda^{n-k}(K) \text{ (if } r^* \geq 1) \\ \text{or} \\ \mathcal{P}^{-}_{r^*+1}\Lambda^{n-k}(K) \end{array} \right\}.$$

This is just the Hodge dual of the stable pairs (6) with k replaced by $n-k$ and r by r^*, so all of the results of Arnold, Falk, and Winther [5] apply immediately to the dual problem. We write the discrete Hodge decomposition for this complex as

$$W_h^{*k}(K) = \mathfrak{B}_h^{*k}(K) \oplus \mathfrak{H}_h^{*k}(K) \oplus \mathfrak{Z}_h^{*k\perp}(K).$$

When K is contractible (e.g., a simplex), we have $\mathfrak{H}_h^{*k}(K) = \mathring{\mathfrak{H}}^k(K)$, which is $\cong \mathbb{R}$ for $k = n$ and trivial otherwise.

We are now ready to define the postprocessing procedure on $K \in \mathcal{T}_h$: Find $\rho_h^* \in W_h^{*k+1}(K)$, $u_h^* \in W_h^{*k}(K)$, $\overline{p}_h^* \in \mathfrak{H}_h^{*k}(K)$ such that

$$(\rho_h^*, \eta_h)_K - (u_h^*, \delta\eta_h)_K = \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta_h \in W_h^{*k+1}(K), \tag{16a}$$

$$(\delta\rho_h^*, v_h)_K + (\delta u_h^*, \delta v_h)_K + (\overline{p}_h^*, v_h)_K = (f - p_h, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v_h \in W_h^{*k}(K), \tag{16b}$$

$$(u_h^*, \overline{q}_h)_K = (\overline{u}_h, \overline{q}_h)_K, \qquad \forall \overline{q}_h \in \mathfrak{H}_h^{*k}(K). \tag{16c}$$

Remark 5.1. The right-hand side only depends on the global variables p_h, $\overline{u}_h$, $\widehat{\sigma}_h^{\tan}$, $\widehat{u}_h^{\tan}$. Therefore, after we solve the statically condensed problem (13), this procedure can be used as an alternative to the local solvers (11) for recovering approximations to the local variables on $K \in \mathcal{T}_h$.

We can also apply postprocessing if FEEC is implemented using (5), without hybridization, since $\overline{u}_h = P_{\overline{\mathfrak{H}}_h} u_h$, $\widehat{\sigma}_h^{\tan} = \sigma_h^{\tan}$, and $\widehat{u}_h^{\tan} = u_h^{\tan}$. In the simplicial case, since $\mathfrak{H}_h^{*k}(K) = \mathring{\mathfrak{H}}_h^k(K)$, we can simply replace $\overline{u}_h$ by u_h on the right-hand side of (16c) without projecting.

Note that, while the original solution variables are tangentially continuous between elements, the postprocessed solution variables generally do not have any tangential or normal continuity, i.e., they are neither $H\Lambda(\Omega)$- nor $H^*\Lambda(\Omega)$-conforming.

Example 5.2 (Stenberg postprocessing)**.** When $k = n$ and $\mathcal{T}_h$ is a simplicial mesh, the space $W_h^{*n+1}(K)$ is trivial, $W_h^{*n}(K) \cong \mathcal{P}_{r^*}(K)$, and $\mathfrak{H}_h^{*n}(K) \cong \mathbb{R}$. Therefore, (16) becomes

$$\begin{aligned} (\operatorname{grad} u_h^*, \operatorname{grad} v_h)_K + (\overline{p}_h^*, v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \widehat{n} \rangle_{\partial K}, \quad \forall v_h \in \mathcal{P}_{r^*}(K), \\ (u_h^*, \overline{q}_h)_K &= (\overline{u}_h, \overline{q}_h)_K, \qquad \forall \overline{q}_h \in \mathbb{R}, \end{aligned}$$

which coincides with Stenberg [41] postprocessing for the RT and BDM methods. Stenberg also considered a second form of postprocessing with $\overline{p}_h^*, \overline{q}_h \in \mathcal{P}_r(K)$, but we do not consider that here.

5.2. **Error estimates for $k < n$.** We now analyze this postprocessing procedure when, as before, $\{\mathcal{T}_h\}$ is a shape-regular family of simplicial meshes of Ω. We wish to determine the accuracy of the solution to the postprocessing problem (16), compared to that obtained using the local solvers (11).

The $k = n$ case has already been analyzed by Stenberg [41], so we restrict our attention to $k < n$. Since the local harmonic spaces are trivial, the exact solver (15) simplifies to

$$(\rho, \eta)_K - (u, \delta\eta)_K = \langle \widehat{u}^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta \in H^*\Lambda^{k+1}(K), \tag{17a}$$

$$(\delta\rho, v)_K + (\delta u, \delta v)_K = (f - p, v)_K - \langle \widehat{\sigma}^{\tan}, v^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v \in H^*\Lambda^k(K), \tag{17b}$$

and the postprocessing problem (16) simplifies to

$$(\rho_h^*, \eta_h)_K - (u_h^*, \delta\eta_h)_K = \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta_h \in W_h^{*k+1}(K), \tag{18a}$$

$$(\delta\rho_h^*, v_h)_K + (\delta u_h^*, \delta v_h)_K = (f - p_h, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v_h \in W_h^{*k}(K). \tag{18b}$$

To aid in the analysis, we introduce the intermediate approximation $\widetilde{\rho}_h \in W_h^{*k+1}(K)$, $\widetilde{u}_h \in W_h^{*k}(K)$ such that

$$(\widetilde{\rho}_h, \eta_h)_K - (\widetilde{u}_h, \delta\eta_h)_K = \langle \widehat{u}^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta_h \in W_h^{*k+1}(K), \tag{19a}$$

$$(\delta\widetilde{\rho}_h, v_h)_K + (\delta\widetilde{u}_h, \delta v_h)_K = (f - p, v_h)_K - \langle \widehat{\sigma}^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v_h \in W_h^{*k}(K), \tag{19b}$$

where the global variables on the right-hand side are the same as those in the exact solution (17). Note that (19) is just the FEEC approximation of (17) on the subcomplex $W_h^*(K) \subset H^*\Lambda(K)$, so the results of Arnold, Falk, and Winther [5] immediately give us estimates for $\rho - \widetilde{\rho}_h$ and $u - \widetilde{u}_h$. It therefore remains to analyze the difference between (18) and (19).

As in [5], we assume that the exact solution satisfies an elliptic regularity estimate of the form

$$\|u\|_{t+2,\Omega} + \|p\|_{t+2,\Omega} + \|\mathrm{d}u\|_{t+1,\Omega} + \|\sigma\|_{t+1,\Omega} + \|\mathrm{d}\sigma\|_{t,\Omega} \lesssim \|f\|_{t,\Omega},$$

for $0 \le t \le t_{\max}$, where $\|\cdot\|_{t,\Omega}$ denotes the H^t norm on Ω. We will frequently invoke [5, Theorem 3.11], which gives L^2 error estimates for the FEEC solution in terms of the best approximation allowed by the regularity of the exact solution and the polynomial degree of the finite element spaces. These estimates will be applied both to the original FEEC approximation (5) on V_h and to the intermediate approximation (19) on $W_h^*(K)$.

We want the postprocessed solution to be at least as good as the standard FEEC solution obtained from the local solvers (11). The following assumptions ensure that r^* is large enough for the $W_h^*(K)$ complex to approximate the exact solution as well as $W_h(K)$ does. If $f \perp \mathfrak{B}^k$, then $\sigma = 0$, so it is enough for $W_h^{*k}(K)$ to contain the same total space of polynomials as $W_h^k(K)$, i.e., $r^* \ge r$. Otherwise, in order to approximate $\sigma \neq 0$, we also need the stronger condition that $W_h^{*k-1}(K)$ contains the same total space of polynomials as $W_h^{k-1}(K)$.

Assumption A. Assume that we are in one of the following three cases:

(1) $f \perp \mathfrak{B}^k$ and $r^* \ge r$.

(2) $W_h^{k-1}(K) = \mathcal{P}_{r+1}\Lambda^{k-1}(K)$ and $\star W_h^{*k}(K) = \begin{cases} \mathcal{P}_{r^*}\Lambda^{n-k}(K), & r^* \ge r+2, \\ \mathcal{P}^-_{r^*+1}\Lambda^{n-k}(K), & r^* \ge r+1. \end{cases}$

(3) $W_h^{k-1}(K) = \mathcal{P}^-_{r+1}\Lambda^{k-1}(K)$ and $\star W_h^{*k}(K) = \begin{cases} \mathcal{P}_{r^*}\Lambda^{n-k}(K), & r^* \ge r+1, \\ \mathcal{P}^-_{r^*+1}\Lambda^{n-k}(K), & r^* \ge r. \end{cases}$

Our first result shows that $\delta\rho_h^*$ gives an improved approximation of $\delta\rho = \delta\mathrm{d}u$, compared to $\delta\mathrm{d}u_h$. In particular, when $f = \delta\rho \in \mathring{\mathfrak{B}}_k^*$, we can obtain an arbitrarily good approximation by taking the postprocessing degree r^* large enough.

Theorem 5.3. *For each $K \in \mathcal{T}_h$ and $0 \le s \le t_{\max}$, we have*

$$\big\|\delta(\rho - \widetilde{\rho}_h)\big\|_K \lesssim h_K^s \|f\|_{s,K}, \quad \text{if } s \le r^* + 1,$$
$$\big\|\delta(\widetilde{\rho}_h - \rho_h^*)\big\|_K \le \big\|\mathrm{d}(\sigma - \sigma_h)\big\|_K + \|p - p_h\|_K.$$

Consequently, if Assumption A holds, then

$$\big\|\delta(\rho - \rho_h^*)\big\|_{\mathcal{T}_h} \lesssim h^s \|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r^*+1, & f \in \mathring{\mathfrak{B}}_k^*, \\ s \le r+1, & \text{otherwise.} \end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11] applied to the problem (19). Next, subtracting (18b) from (19b) with $v_h \in \mathfrak{B}_h^{*k}(K)$ gives

$$\begin{aligned} \big(\delta(\widetilde{\rho}_h - \rho_h^*), v_h\big)_K &= (p_h - p, v_h)_K + \langle \widehat{\sigma}_h^{\tan} - \widehat{\sigma}^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K} \\ &= (p_h - p, v_h)_K + \big(\mathrm{d}(\sigma_h - \sigma), v_h\big)_K \\ &\le \Big(\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_K + \|p - p_h\|_K\Big)\|v_h\|_K, \end{aligned}$$

and taking $v_h = \delta(\widetilde{\rho}_h - \rho_h^*)$ implies the second estimate. Finally, summing over $K \in \mathcal{T}_h$ and applying [5, Theorem 3.11] once more gives

$$\big\|\delta(\widetilde{\rho}_h - \rho_h^*)\big\|_{\mathcal{T}_h} \lesssim \begin{cases} 0, & \text{if } f \in \mathring{\mathfrak{B}}_k^*, \\ h^s \|f\|_{s,\Omega}, & \text{if } s \le r+1, \text{ otherwise,} \end{cases}$$

so the last estimate follows by Assumption A and the triangle inequality. □

The next result says that, generically, δu_h^* approximates $\sigma = \delta u$ as well as σ_h does, but no better. In the case $f \in \mathring{\mathfrak{B}}_k^*$, when $\sigma = \sigma_h = 0$, we can make δu_h^* arbitrarily small by taking r^* large enough.

Theorem 5.4. *For each $K \in \mathcal{T}_h$ and $0 \le s \le t_{\max}$, we have*

$$\big\|\delta(u - \widetilde{u}_h)\big\|_K \lesssim h_K^{s+1} \|f\|_{s,K}, \quad \text{if } \begin{cases} s \le r^*+1, & f \in \mathring{\mathfrak{B}}_k^*, \\ s \le r^*, & \star W_h^{*k}(K) = \mathcal{P}^-_{r^*+1}\Lambda^{n-k}(K), \\ s \le r^*-1, & \star W_h^{*k}(K) = \mathcal{P}_{r^*}\Lambda^{n-k}(K), \end{cases}$$

$$\big\|\delta(\widetilde{u}_h - u_h^*)\big\|_K \lesssim \|\sigma - \sigma_h\|_K + h_K\Big(\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_K + \|p - p_h\|_K\Big).$$

Consequently, if Assumption A holds, then

$$\big\|\delta(u - u_h^*)\big\|_{\mathcal{T}_h} \lesssim h^{s+1}\|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r^*+1, & f \in \mathring{\mathfrak{B}}_k^*, \\ s \le r+1, & V_h^{k-1} = \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}_h), \\ s \le r, & V_h^{k-1} = \mathcal{P}^-_{r+1}\Lambda^{k-1}(\mathcal{T}_h). \end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11]. Next, subtracting (18b) from (19b) with $v_h \in \mathfrak{Z}_h^{*k\perp}(K)$ gives

$$\begin{aligned} \big(\delta(\widetilde{u}_h - u_h^*), \delta v_h\big)_K &= (p_h - p, v_h)_K + \langle \widehat{\sigma}_h^{\tan} - \widehat{\sigma}^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K} \\ &= (p_h - p, v_h)_K + \big(\mathrm{d}(\sigma_h - \sigma), v_h\big)_K - (\sigma_h - \sigma, \delta v_h)_K \\ &\lesssim \Big[\|\sigma - \sigma_h\|_K + h_K\Big(\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_K + \|p - p_h\|_K\Big)\Big]\|\delta v_h\|_K. \end{aligned}$$

In the last step, we have applied Cauchy–Schwarz and the Poincaré inequality with scaling, which says that $\|v_h\|_K \lesssim h_K \|\delta v_h\|_K$. Taking v_h such that $\delta v_h = \delta(\widetilde{u}_h - u_h^*)$ implies the second estimate. Finally, summing over $K \in \mathcal{T}_h$ and applying [5, Theorem 3.11] gives

$$\big\|\delta(\widetilde{u}_h - u_h^*)\big\|_{\mathcal{T}_h} \lesssim \begin{cases} 0, & \text{if } f \in \mathring{\mathfrak{B}}_k^*, \\ h^{s+1}\|f\|_{s,\Omega}, & \text{otherwise, if } \begin{cases} s \le r+1, & V_h^{k-1} = \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}_h), \\ s \le r, & V_h^{k-1} = \mathcal{P}^-_{r+1}\Lambda^{k-1}(\mathcal{T}_h), \end{cases} \end{cases}$$

so the last estimate follows by Assumption A and the triangle inequality. □

Thus far, we have been able to avoid dealing with the error term $\widehat{u}^{\tan} - \widehat{u}_h^{\tan}$, which dominates the postprocessing error, preventing improved convergence of the $\mathfrak{B}_h^*(K)$ components. There is one special exception, however: when $k = n-1$, the space $\mathfrak{B}_h^{*n}(K)$ is trivial, so there is no error in this component of ρ_h^*. In this case, we will see that ρ_h^* is an improved estimate compared to $\mathrm{d}u_h$. Since $\mathfrak{H}_h^{*n}(K) \cong \mathbb{R}$ is nontrivial, though, we need to control the $\overline{\mathfrak{H}}^n$ component of the error, which we will do with the aid of the following lemma.

Lemma 5.5. *If $k = n-1$ and $\eta_h \in \overline{\mathfrak{H}}^n$, then*

$$\langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} \lesssim h\Big(\big\|\mathrm{d}(\sigma - \sigma_h)\big\|_\Omega + \|p - p_h\|_\Omega\Big)\|\eta_h\|_\Omega.$$

In particular, if $f \in \mathring{\mathfrak{B}}_{n-1}^*$, then $\int_{\partial K} \operatorname{tr}(u - u_h) = 0$ for all $K \in \mathcal{T}_h$.

Proof. Since η_h is piecewise constant, $\langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = \big(\mathrm{d}(u-u_h), \eta_h\big)_{\mathcal{T}_h}$. Piecewise constants are in $V_h^n = \mathfrak{B}_h^n$, so the estimate follows by Lemma 4.8. In particular, σ and p vanish when $f \in \mathring{\mathfrak{B}}_{n-1}^*$, so in that case the left-hand side is identically zero. □

Remark 5.6. This generalizes the well-known property that, when $n = 1$ and $k = 0$, the continuous Galerkin solution equals the exact solution at nodes.

We now show that ρ_h^* approximates $\rho = \mathrm{d}u$ as well as $\mathrm{d}u_h$ does, but no better when $k < n-1$. However, when $k = n-1$, we get an improved estimate, and when $f \in \mathring{\mathfrak{B}}_{n-1}^*$, we can obtain an arbitrarily good approximation by taking r^* large enough.

Theorem 5.7. *For each $K \in \mathcal{T}_h$ and $0 \le s \le t_{\max}$,*

$$\|\rho - \widetilde{\rho}_h\|_K \lesssim h_K^{s+1} \|f\|_{s,K}, \quad \text{if } \begin{cases} s \le r^*+1, & \star W_h^{*k+1}(K) = \mathcal{P}_{r^*+1}\Lambda^{n-k-1}(K), \\ s \le r^*, & \star W_h^{*k+1}(K) = \mathcal{P}^-_{r^*+1}\Lambda^{n-k-1}(K), \end{cases}$$

$$\|\widetilde{\rho}_h - \rho_h^*\|_K \lesssim \big\|\mathrm{d}(u-u_h)\big\|_K + h_K\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_K + \|p-p_h\|_K\Big).$$

Consequently, if Assumption A holds, then

$$\|\rho - \rho_h^*\|_{\mathcal{T}_h} \lesssim h^{s+1}\|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r+1, & f \perp \mathring{\mathfrak{B}}_k^*, \\ s \le r, & V_h^k = \mathcal{P}^-_{r+1}\Lambda^k(\mathcal{T}_h) \\ s \le r-1, & V_h^k = \mathcal{P}_r\Lambda^k(\mathcal{T}_h). \end{cases}$$

In the case $k = n-1$, this estimate may be improved to

$$\|\rho - \rho_h^*\|_{\mathcal{T}_h} \lesssim h^{s+1}\|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r^*+1, & f \in \mathring{\mathfrak{B}}_{n-1}^*, \ \star W_h^{*k+1}(K) = \mathcal{P}_{r^*+1}\Lambda^{n-k-1}(K), \\ s \le r^*, & f \in \mathring{\mathfrak{B}}_{n-1}^*, \ \star W_h^{*k+1}(K) = \mathcal{P}^-_{r^*+1}\Lambda^{n-k-1}(K), \\ s \le r+1, & \textit{otherwise.} \end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11]. Next, subtracting (18a) from (19a) with $\eta_h \in \mathfrak{Z}_h^{*k+1}(K)$ gives

$$(\widetilde{\rho}_h - \rho_h^*, \eta_h)_K = \langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K} = \big(\mathrm{d}(u-u_h), \eta_h\big)_K \le \big\|\mathrm{d}(u-u_h)\big\|_K \|\eta_h\|_K,$$

which implies

$$\big\|P_{\mathfrak{Z}_h^*(K)}(\widetilde{\rho}_h - \rho_h^*)\big\|_K \le \big\|\mathrm{d}(u-u_h)\big\|_K.$$

Furthermore, by the Poincaré inequality and Theorem 5.3,

$$\big\|P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{\rho}_h - \rho_h^*)\big\|_K \lesssim h_K \big\|\delta(\widetilde{\rho}_h - \rho_h^*)\big\|_K \le h_K\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_K + \|p-p_h\|_K\Big),$$

so the second estimate follows by the Hodge decomposition and triangle inequality. Summing over $K \in \mathcal{T}_h$ and applying [5, Theorem 3.11] gives

$$\|\widetilde{\rho}_h - \rho_h^*\|_{\mathcal{T}_h} \lesssim h^{s+1}\|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r+1, & f \perp \mathring{\mathfrak{B}}_k^*, \\ s \le r, & V_h^k = \mathcal{P}^-_{r+1}\Lambda^k(\mathcal{T}_h), \\ s \le r-1, & V_h^k = \mathcal{P}_r\Lambda^k(\mathcal{T}_h), \end{cases}$$

so the third estimate follows by Assumption A and the triangle inequality.

Finally, consider the special case $k = n-1$. Taking $\eta_h \in \overline{\mathfrak{H}}^n$ and applying Lemma 5.5 gives

$$(\widetilde{\rho}_h - \rho_h^*, \eta_h)_{\mathcal{T}_h} = \langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} \lesssim h\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_\Omega + \|p-p_h\|_\Omega\Big)\|\eta_h\|_\Omega,$$

and therefore,

$$\big\|P_{\overline{\mathfrak{H}}}(\widetilde{\rho}_h - \rho_h^*)\big\|_{\mathcal{T}_h} \lesssim h\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_\Omega + \|p-p_h\|_\Omega\Big).$$

Note that this eliminates the $\big\|\mathrm{d}(u-u_h)\big\|_\Omega$ term that appears in the $k<n-1$ case. Hence,

$$\|\widetilde{\rho}_h-\rho_h^*\|_{\mathcal{T}_h}\lesssim h\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_\Omega+\|p-p_h\|_\Omega\Big)\lesssim\begin{cases}0, & \text{if } f\in\mathring{\mathfrak{B}}^*_{n-1},\\ h^{s+1}\|f\|_{s,\Omega}, & \text{if } s\le r+1, \text{ otherwise,}\end{cases}$$

and the improved estimate follows. □

Finally, we show that u_h^* approximates u as well as u_h does, but no better.

Theorem 5.8. *For each $K\in\mathcal{T}_h$ and $0\le s\le t_{\max}$,*

$$\|u-\widetilde{u}_h\|_K\lesssim\begin{cases}h_K\|f\|_K, & \text{if } \star W_h^{*k}=\mathcal{P}_1^-\Lambda^{n-k}(K),\\ h_K^{s+2}\|f\|_{s,K}, & \text{if } s\le r^*-1, \text{ otherwise,}\end{cases}$$

$$\|\widetilde{u}_h-u_h^*\|_K\lesssim\|u-u_h\|_K+h_K\Big(\big\|\mathrm{d}(u-u_h)\big\|_K+\|\sigma-\sigma_h\|_K\Big)+h_K^2\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_K+\|p-p_h\|_K\Big).$$

Consequently, if Assumption A holds, then

$$\|u-u_h^*\|_{\mathcal{T}_h}\lesssim\begin{cases}h\|f\|_\Omega, & \text{if } V_h^k=\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h),\\ h^{s+2}\|f\|_{s,\Omega}, & \text{if } s\le r-1, \text{ otherwise.}\end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11]. Next, subtracting (18a) from (19a) with $\eta_h\in\mathfrak{Z}_h^{*k+1\perp}(K)$ gives

$$\begin{aligned}(\widetilde{u}_h-u_h^*,\delta\eta_h)_K&=(\widetilde{\rho}_h-\rho_h^*,\eta_h)_K-\langle\widehat{u}^{\tan}-\widehat{u}_h^{\tan},\eta_h^{\mathrm{nor}}\rangle_{\partial K}\\&=\big(P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{\rho}_h-\rho_h^*),\eta_h\big)_K-\big(\mathrm{d}(u-u_h),\eta_h\big)_K+(u-u_h,\delta\eta_h)_K\\&\lesssim\Big(\|u-u_h\|_K+h_K\big\|\mathrm{d}(u-u_h)\big\|_K+h_K^2\big\|\delta(\widetilde{\rho}_h-\rho_h^*)\big\|_K\Big)\|\delta\eta_h\|_K,\end{aligned}$$

by Cauchy–Schwarz and the Poincaré inequality. With Theorem 5.3, this implies

$$\big\|P_{\mathfrak{B}_h^*(K)}(\widetilde{u}_h-u_h^*)\big\|_K\lesssim\|u-u_h\|_K+h_K\big\|\mathrm{d}(u-u_h)\big\|_K+h_K^2\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_K+\|p-p_h\|_K\Big).$$

Furthermore, by the Poincaré inequality and Theorem 5.4,

$$\big\|P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{u}_h-u_h^*)\big\|_K\lesssim h_K\|\sigma-\sigma_h\|_K+h_K^2\Big(\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_K+\|p-p_h\|_K\Big),$$

so the second estimate follows by the Hodge decomposition and triangle inequality. Finally, summing over $K\in\mathcal{T}_h$ and applying [5, Theorem 3.11] gives

$$\|\widetilde{u}_h-u_h^*\|_{\mathcal{T}_h}\lesssim\begin{cases}h\|f\|_\Omega, & \text{if } V_h^k=\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h),\\ h^{s+2}\|f\|_{s,\Omega}, & \text{if } s\le r-1, \text{ otherwise,}\end{cases}$$

so the last estimate follows by Assumption A and the triangle inequality. □

5.3. **Remarks on the case $k=n$.** Although the case $k=n$ has already been analyzed by Stenberg [41], we now briefly describe this analysis from the FEEC viewpoint, relating it to the techniques developed in this section. In this case, the postprocessing procedure (16) becomes

$$\begin{aligned}(\delta u_h^*,\delta v_h)_K+(\overline{p}_h^*,v_h)_K&=(f,v_h)_K-\langle\widehat{\sigma}_h^{\tan},v_h^{\mathrm{nor}}\rangle_{\partial K}, && \forall v_h\in W_h^{*n}(K),\\(u_h^*,\overline{q}_h)_K&=(u_h,\overline{q}_h)_K, && \forall\overline{q}_h\in\mathfrak{H}_h^{*n}(K),\end{aligned}$$

and the intermediate approximation is given by

$$\begin{aligned}(\delta\widetilde{u}_h,\delta v_h)_K+(\widetilde{p}_h,v_h)_K&=(f,v_h)_K-\langle\widehat{\sigma}^{\tan},v_h^{\mathrm{nor}}\rangle_{\partial K}, && \forall v_h\in W_h^{*n}(K),\\(\widetilde{u}_h,\overline{q}_h)_K&=(u,\overline{q}_h)_K, && \forall\overline{q}_h\in\mathfrak{H}_h^{*n}(K).\end{aligned}$$

The argument in Theorem 5.4 still works, so applying the Poincaré inequality gives

$$\big\|P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{u}_h-u_h^*)\big\|_K\lesssim h_K\|\sigma-\sigma_h\|_K+h_K^2\big\|\mathrm{d}(\sigma-\sigma_h)\big\|_K.$$

Furthermore, since $\overline{\mathfrak{H}}^n$ consists of piecewise constants, which are in $V_h^n = \mathfrak{B}_h^n$, we have

$$\left\|P_{\overline{\mathfrak{H}}}(\widetilde{u}_h - u_h^*)\right\|_K = \left\|P_{\overline{\mathfrak{H}}}(u - u_h)\right\|_K \leq \left\|P_{\mathfrak{B}_h}(u - u_h)\right\|_K.$$

Summing over $K \in \mathcal{T}_h$ and applying [5, Lemma 3.13] implies

$$\|\widetilde{u}_h - u_h^*\|_{\mathcal{T}_h} \lesssim \begin{cases} h^{s+1}\|f\|_{s,\Omega}, & \text{if } s \leq 1,\ V_h^n = \mathcal{P}_1^-\Lambda^n(\mathcal{T}_h), \\ h^{s+2}\|f\|_{s,\Omega}, & \text{otherwise, if } \begin{cases} s \leq r+1, & V_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}_h), \\ s \leq r, & V_h^{n-1} = \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}_h), \end{cases} \end{cases}$$

so by Assumption A and the triangle inequality, this same estimate holds for $\|u - u_h^*\|_{\mathcal{T}_h}$. This is precisely the improved estimate in Stenberg [41, Theorem 2.2], by essentially the same proof.

6. Illustration of the methods in $n = 3$ dimensions

We now give a concrete illustration of the hybridization and postprocessing schemes in $n = 3$ dimensions, using scalar and vector proxy fields and the familiar operations of vector calculus. Let $\mathcal{T}_h$ be a simplicial triangulation of a bounded, polyhedral domain $\Omega \subset \mathbb{R}^3$. For simplicity, we also assume that Ω is contractible, so that $\mathfrak{H}^0 \cong \mathbb{R}$ and $\mathfrak{H}^k$ is trivial for $k = 1, 2, 3$.

Let V_h be a stable subcomplex of

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\operatorname{grad}} H(\operatorname{curl};\Omega) \xrightarrow{\operatorname{curl}} H(\operatorname{div};\Omega) \xrightarrow{\operatorname{div}} L^2(\Omega) \longrightarrow 0,$$

containing continuous Lagrange elements, Nédélec edge and face elements, and discontinuous Lagrange elements. Let W_h be the corresponding "broken" complex, with $W_h^k(K) = V_h^k|_K$ for $K \in \mathcal{T}_h$. Using the scalar and vector proxies for tangential traces in Table 1, we have

$$\begin{aligned} \widehat{V}_h^{0,\tan} &= \left\{v_h|_{\partial\mathcal{T}_h} : v_h \in V_h^0\right\}, & \widehat{W}_h^{0,\mathrm{nor}} = \widehat{W}_h^{0,\tan} &= \left\{v_h|_{\partial\mathcal{T}_h} : v_h \in W_h^0\right\}, \\ \widehat{V}_h^{1,\tan} &= \left\{v_h|_{\partial\mathcal{T}_h} - (v_h\cdot\widehat{n})\widehat{n} : v_h \in V_h^1\right\}, & \widehat{W}_h^{1,\mathrm{nor}} = \widehat{W}_h^{1,\tan} &= \left\{v_h|_{\partial\mathcal{T}_h} - (v_h\cdot\widehat{n})\widehat{n} : v_h \in W_h^1\right\}, \\ \widehat{V}_h^{2,\tan} &= \left\{(v_h\cdot\widehat{n})\widehat{n} : v_h \in V_h^2\right\}, & \widehat{W}_h^{2,\mathrm{nor}} = \widehat{W}_h^{2,\tan} &= \left\{(v_h\cdot\widehat{n})\widehat{n} : v_h \in W_h^2\right\}, \end{aligned}$$

whose degrees of freedom are just those of V_h^k and W_h^k living on $\partial\mathcal{T}_h$.

For postprocessing on $K \in \mathcal{T}_h$, let $W_h^*(K)$ be a stable subcomplex of

$$0 \longleftarrow L^2(\Omega) \xleftarrow{-\operatorname{div}} H(\operatorname{div};\Omega) \xleftarrow{\operatorname{curl}} H(\operatorname{curl};\Omega) \xleftarrow{-\operatorname{grad}} H^1(\Omega) \longleftarrow 0,$$

whose normal traces have scalar and vector proxies given in Table 1.

6.1. **The case $k = 0$.** The hybrid method is: Find $u_h \in W_h^0$, $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{W}_h^{0,\mathrm{nor}}$, $p_h \in \mathbb{R}$, $\widehat{u}_h^{\tan} \in \widehat{V}_h^{0,\tan}$ such that

$$\begin{aligned} (\operatorname{grad} u_h, \operatorname{grad} v_h)_{\mathcal{T}_h} + (p_h, v_h)_{\mathcal{T}_h} - \langle\widehat{\rho}_h^{\mathrm{nor}}, v_h\rangle_{\partial\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, & \forall v_h &\in W_h^0, \\ \langle\widehat{u}_h^{\tan} - u_h, \widehat{\eta}_h^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} &= 0, & \forall \widehat{\eta}_h^{\mathrm{nor}} &\in \widehat{W}_h^{0,\mathrm{nor}}, \\ (u_h, q_h)_{\mathcal{T}_h} &= 0, & \forall q_h &\in \mathbb{R}, \\ \langle\widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan}\rangle_{\partial\mathcal{T}_h} &= 0, & \forall \widehat{v}_h^{\tan} &\in \widehat{V}_h^{0,\tan}, \end{aligned}$$

which is the hybridized continuous Galerkin method of Cockburn, Gopalakrishnan, and Wang [20] for the Neumann problem. The postprocessing scheme on $K \in \mathcal{T}_h$ is: Find $\rho_h^* \in W_h^{*1}(K)$, $u_h^* \in W_h^{*0}(K)$ such that

$$\begin{aligned} (\rho_h^*, \eta_h)_K + (u_h^*, \operatorname{div}\eta_h)_K &= \langle\widehat{u}_h^{\tan}, \eta_h\cdot\widehat{n}\rangle_{\partial K}, & \forall \eta_h &\in W_h^{*1}(K), \\ -(\operatorname{div}\rho_h^*, v_h)_K &= (f - p_h, v_h)_K, & \forall v_h &\in W_h^{*0}(K). \end{aligned}$$

6.2. **The case $k = 1$.** The hybrid method is: Find $\sigma_h \in W_h^0$, $u_h \in W_h^1$, $\widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{0,\mathrm{nor}}$, $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{W}_h^{1,\mathrm{nor}}$, $\widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{0,\tan}$, $\widehat{u}_h^{\tan} \in \widehat{V}_h^{1,\tan}$ such that

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{grad} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^0, \\
(\operatorname{grad} \sigma_h, v_h)_{\mathcal{T}_h} + (\operatorname{curl} u_h, \operatorname{curl} v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^1, \\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{0,\mathrm{nor}}, \\
\langle \widehat{u}_h^{\tan} - u_h, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{1,\mathrm{nor}}, \\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{0,\tan}, \\
\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{1,\tan}.
\end{aligned}$$

The postprocessing scheme on $K \in \mathcal{T}_h$ is: Find $\rho_h^* \in W_h^{*2}(K)$, $u_h^* \in W_h^{*1}(K)$ such that

$$\begin{aligned}
(\rho_h^*, \eta_h)_K - (u_h^*, \operatorname{curl} \eta_h)_K &= \langle \widehat{u}_h^{\tan}, \eta_h \times \widehat{n} \rangle_{\partial K}, && \forall \eta_h \in W_h^{*2}(K), \\
(\operatorname{curl} \rho_h^*, v_h)_K + (\operatorname{div} u_h^*, \operatorname{div} v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \cdot \widehat{n} \rangle_{\partial K}, && \forall v_h \in W_h^{*1}(K).
\end{aligned}$$

6.3. **The case $k = 2$.** The hybrid method is: Find $\sigma_h \in W_h^1$, $u_h \in W_h^2$, $\widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{1,\mathrm{nor}}$, $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{W}_h^{2,\mathrm{nor}}$, $\widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{1,\tan}$, $\widehat{u}_h^{\tan} \in \widehat{V}_h^{2,\tan}$ such that

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{curl} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^1, \\
(\operatorname{curl} \sigma_h, v_h)_{\mathcal{T}_h} + (\operatorname{div} u_h, \operatorname{div} v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^2, \\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{1,\mathrm{nor}}, \\
\langle \widehat{u}_h^{\tan} - u_h, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{2,\mathrm{nor}}, \\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{1,\tan}, \\
\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{2,\tan},
\end{aligned}$$

The postprocessing scheme on $K \in \mathcal{T}_h$ is: Find $\rho_h^* \in W_h^{*3}(K)$, $u_h^* \in W_h^{*2}(K)$ such that

$$\begin{aligned}
(\rho_h^*, \eta_h)_K + (u_h^*, \operatorname{grad} \eta_h)_K &= \langle \widehat{u}_h^{\tan}, \eta_h \widehat{n} \rangle_{\partial K}, && \forall \eta_h \in W_h^{*3}(K), \\
-(\operatorname{grad} \rho_h^*, v_h)_K + (\operatorname{curl} u_h^*, \operatorname{curl} v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \times \widehat{n} \rangle_{\partial K}, && \forall v_h \in W_h^{*2}(K).
\end{aligned}$$

6.4. **The case $k = 3$.** The hybrid method is: Find $\sigma_h \in W_h^2$, $u_h \in W_h^3$, $\overline{p}_h \in \mathbb{R}^{\mathcal{T}_h}$, $\widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{2,\mathrm{nor}}$, $\overline{u}_h \in \mathbb{R}^{\mathcal{T}_h}$, $\widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{2,\tan}$ such that

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{div} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^2, \\
(\operatorname{div} \sigma_h, v_h)_{\mathcal{T}_h} + (\overline{p}_h, v_h)_{\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^3, \\
(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{q}_h \in \mathbb{R}^{\mathcal{T}_h}, \\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{2,\mathrm{nor}}, \\
(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{v}_h \in \mathbb{R}^{\mathcal{T}_h}, \\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{2,\tan},
\end{aligned}$$

which is the alternative hybridization of the RT and BDM methods in Cockburn [16, Section 5] using local Neumann solvers; its solution coincides with the classic hybridized RT and BDM methods of Arnold and Brezzi [3], Brezzi, Douglas, and Marini [8] using local Dirichlet solvers. The

r	N	$\|\|\|\widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}\|\|\|_{\partial\mathcal{T}_h}$		$\|u - u_h^*\|_{\mathcal{T}_h}$		$\left\|\delta(u - u_h^*)\right\|_{\partial\mathcal{T}_h}$		$\|\|\|\widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}\|\|\|_{\partial\mathcal{T}_h}$		$\|\rho - \rho_h^*\|_{\mathcal{T}_h}$		$\left\|\delta(\rho - \rho_h^*)\right\|_{\mathcal{T}_h}$	
0	1	2.06e-01	—	5.03e-01	—	5.87e-01	—	7.87e-01	—	1.38e+00	—	6.24e+00	—
	2	4.67e-01	-1.2	5.27e-01	-0.1	6.32e-01	-0.1	1.56e+00	-1.0	1.27e+00	0.1	4.70e+00	0.4
	4	3.13e-01	0.6	2.91e-01	0.9	1.95e-01	1.7	9.38e-01	0.7	6.83e-01	0.9	2.47e+00	0.9
	8	1.80e-01	0.8	1.51e-01	0.9	5.35e-02	1.9	5.02e-01	0.9	3.49e-01	1.0	1.28e+00	1.0
	16	9.42e-02	0.9	7.66e-02	1.0	1.38e-02	2.0	2.57e-01	1.0	1.76e-01	1.0	6.46e-01	1.0
1	1	1.99e-01	—	3.33e-01	—	3.69e-01	—	1.65e+00	—	1.02e+00	—	3.62e+00	—
	2	1.76e-01	0.2	8.45e-02	2.0	4.97e-02	2.9	8.09e-01	1.0	2.81e-01	1.9	7.46e-01	2.3
	4	5.82e-02	1.6	2.56e-02	1.7	7.93e-03	2.6	2.44e-01	1.7	7.70e-02	1.9	2.16e-01	1.8
	8	1.60e-02	1.9	6.84e-03	1.9	1.06e-03	2.9	6.47e-02	1.9	1.98e-02	2.0	5.71e-02	1.9
	16	4.14e-03	1.9	1.75e-03	2.0	1.36e-04	3.0	1.66e-02	2.0	5.01e-03	2.0	1.46e-02	2.0
2	1	1.09e-01	—	5.68e-02	—	2.01e-02	—	5.14e-01	—	2.10e-01	—	5.65e-01	—
	2	4.61e-02	1.2	1.19e-02	2.3	4.46e-03	2.2	2.32e-01	1.1	5.06e-02	2.1	1.09e-01	2.4
	4	7.16e-03	2.7	1.52e-03	3.0	2.84e-04	4.0	3.52e-02	2.7	6.67e-03	2.9	1.20e-02	3.2
	8	9.68e-04	2.9	1.92e-04	3.0	1.77e-05	4.0	4.71e-03	2.9	8.41e-04	3.0	1.44e-03	3.1
	16	1.25e-04	3.0	2.42e-05	3.0	1.11e-06	4.0	6.05e-04	3.0	1.05e-04	3.0	1.78e-04	3.0

TABLE 2. Errors and rates for a manufactured solution with $n = 3$, $k = 1$, using hybridization with $\mathcal{P}^-_{r+1}\Lambda^0 \cong \mathtt{CG}_{r+1}$ and $\mathcal{P}^-_{r+1}\Lambda^1 \cong \mathtt{N1E}_{r+1}$ elements and local postprocessing with broken $\star\mathcal{P}^-_{r+2}\Lambda^1 \cong \mathtt{N1E}_{r+2}$ and $\star\mathcal{P}^-_{r+2}\Lambda^2 \cong \mathtt{N1F}_{r+2}$ elements. Since $k < n - 1$, we get improved convergence of $\delta\rho_h^*$ but not $\widehat{\rho}_h^{\mathrm{nor}}$ or ρ_h^*.

postprocessing scheme on $K \in \mathcal{T}_h$ is exactly that of Stenberg [41]: Find $u_h^* \in W_h^{*3}(K)$, $\overline{p}_h \in \mathbb{R}$ such that

$$\begin{aligned}(\operatorname{grad} u_h^*, \operatorname{grad} v_h)_K + (\overline{p}_h^*, v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \widehat{n}\rangle_{\partial K}, && \forall v_h \in W_h^{*3}(K),\\ (u_h^*, \overline{q}_h)_K &= (\overline{u}_h, \overline{q}_h)_K, && \forall \overline{q}_h \in \mathbb{R}.\end{aligned}$$

7. Numerical experiments

In this section, we present numerical experiments in $n = 3$ dimensions that illustrate and confirm the foregoing theory. We omit the cases $k = 0$ and $k = n$, since these correspond to known methods for the scalar Poisson equation whose properties are already well understood. The remaining cases correspond to hybridization and postprocessing methods for the vector Poisson equation.

For the sake of brevity, we present only numerical experiments using $\mathcal{P}^-_{r+1}\Lambda$ elements with $\star\mathcal{P}^-_{r^*+1}\Lambda$ postprocessing, where r^* is chosen optimally according to Assumption A, and where f has nonvanishing components in both $\mathfrak{B}^k$ and $\mathring{\mathfrak{B}}^*_k$. Errors and rates are shown only for the normal traces and postprocessed solution components, since the convergence behavior of the remaining variables follows from previous work. We have conducted many additional numerical experiments, which all conform with the theoretical results.

All computations have been carried out using the Firedrake finite element library [38] (version 0.13.0+3719.g8e730839), and a Firedrake component called Slate [24] was used to implement the local solvers for static condensation and postprocessing.

7.1. **Test problems.** On the unit cube $\Omega = [0, 1]^3$, a structured tetrahedral mesh $\mathcal{T}_h$ is formed by partitioning Ω into $N \times N \times N$ cubes, each of which is divided into six tetrahedra. As in Section 6, we identify $H\Lambda(\Omega)$ and $H^*\Lambda(\Omega)$ with the complexes of scalar and vector proxy fields. We use the "method of manufactured solutions" by choosing a smooth u satisfying the boundary conditions, taking $f = -\Delta u$, and applying the numerical method to this f. For $k = 1$, we choose

$$u(x, y, z) = \begin{bmatrix}\sin(\pi x)\\ \sin(\pi y)\\ \sin(\pi z)\end{bmatrix} + \begin{bmatrix}\sin(\pi x)\cos(\pi y)\\ -\cos(\pi x)\sin(\pi y)\\ 0\end{bmatrix},$$

r	N	$\|\|\|\widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}\|\|\|_{\partial\mathcal{T}_h}$		$\|u - u_h^*\|_{\mathcal{T}_h}$		$\|\delta(u - u_h^*)\|_{\partial\mathcal{T}_h}$		$\|\|\|\widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}}\|\|\|_{\partial\mathcal{T}_h}$		$\|\rho - \rho_h^*\|_{\mathcal{T}_h}$		$\|\delta(\rho - \rho_h^*)\|_{\mathcal{T}_h}$	
0	1	7.19e-01	—	6.97e-01	—	2.14e+00	—	1.28e+00	—	1.97e+00	—	1.31e+01	—
	2	5.98e-01	0.3	5.07e-01	0.5	1.75e+00	0.3	1.17e+00	0.1	7.57e-01	1.4	9.79e+00	0.4
	4	3.30e-01	0.9	2.62e-01	0.9	9.51e-01	0.9	3.98e-01	1.6	2.19e-01	1.8	5.32e+00	0.9
	8	1.74e-01	0.9	1.33e-01	1.0	4.90e-01	1.0	1.15e-01	1.8	5.78e-02	1.9	2.72e+00	1.0
	16	8.84e-02	1.0	6.66e-02	1.0	2.47e-01	1.0	3.02e-02	1.9	1.47e-02	2.0	1.37e+00	1.0
1	1	6.54e-01	—	5.55e-01	—	1.72e+00	—	2.97e+00	—	9.23e-01	—	9.40e+00	—
	2	2.83e-01	1.2	1.59e-01	1.8	4.69e-01	1.9	3.97e-01	2.9	2.21e-01	2.1	3.66e+00	1.4
	4	8.64e-02	1.7	4.26e-02	1.9	1.24e-01	1.9	5.31e-02	2.9	3.07e-02	2.8	1.01e+00	1.9
	8	2.32e-02	1.9	1.09e-02	2.0	3.15e-02	2.0	6.79e-03	3.0	3.94e-03	3.0	2.59e-01	2.0
	16	5.97e-03	2.0	2.74e-03	2.0	7.93e-03	2.0	8.60e-04	3.0	4.97e-04	3.0	6.51e-02	2.0
2	1	2.44e-01	—	2.25e-01	—	6.60e-01	—	3.66e-01	—	6.63e-01	—	6.18e+00	—
	2	8.47e-02	1.5	4.16e-02	2.4	1.02e-01	2.7	8.26e-02	2.1	5.40e-02	3.6	1.08e+00	2.5
	4	1.29e-02	2.7	5.71e-03	2.9	1.33e-02	2.9	5.51e-03	3.9	3.75e-03	3.8	1.50e-01	2.8
	8	1.73e-03	2.9	7.31e-04	3.0	1.68e-03	3.0	3.54e-04	4.0	2.41e-04	4.0	1.92e-02	3.0
	16	2.22e-04	3.0	9.20e-05	3.0	2.11e-04	3.0	2.24e-05	4.0	1.51e-05	4.0	2.42e-03	3.0

TABLE 3. Errors and rates for a manufactured solution with $n = 3$, $k = 2$, using hybridization with $\mathcal{P}^-_{r+1}\Lambda^1 \cong \mathtt{N1E}_{r+1}$ and $\mathcal{P}^-_{r+1}\Lambda^2 \cong \mathtt{N1F}_{r+1}$ elements and local postprocessing with broken $\star\mathcal{P}^-_{r+1}\Lambda^0 \cong \mathtt{CG}_{r+1}$ and $\star\mathcal{P}^-_{r+1}\Lambda^1 \cong \mathtt{N1E}_{r+1}$ elements. Since $k = n - 1$, we get superconvergence of $\widehat{\rho}_h^{\mathrm{nor}}$ and ρ_h^* as compared with $\rho_h = \mathrm{d}u_h$.

where the first term is in $\mathfrak{B}^1$ and the second is in $\mathring{\mathfrak{B}}_1^*$. For $k = 2$, we choose

$$u(x, y, z) = \begin{bmatrix} \sin(\pi y)\sin(\pi z) \\ \sin(\pi x)\sin(\pi z) \\ \sin(\pi x)\sin(\pi y) \end{bmatrix} + \begin{bmatrix} \cos(\pi x)\sin(\pi y)\sin(\pi z) \\ \sin(\pi x)\cos(\pi y)\sin(\pi z) \\ \sin(\pi x)\sin(\pi y)\cos(\pi z) \end{bmatrix},$$

where the first term is in $\mathfrak{B}^2$ and the second is in $\mathring{\mathfrak{B}}_2^*$.

7.2. **Results.** Table 2 shows the errors and rates for the $k = 1$ problem, using $\mathcal{P}^-_{r+1}\Lambda$ elements and $\star\mathcal{P}^-_{r+2}\Lambda$ postprocessing. (Since $\mathcal{P}^-_{r+1}\Lambda^0 \cong \mathcal{P}_{r+1}\Lambda^0$, the minimum degree satisfying Assumption A is $r^* = r + 1$.) Table 3 shows the errors and rates for the $k = 2$ problem, using $\mathcal{P}^-_{r+1}\Lambda$ elements and $\star\mathcal{P}^-_{r+1}\Lambda$ postprocessing. For clarity, the captions describe the elements both in FEEC notation and in terms of their classical scalar and vector proxies. Adopting the Unified Form Language (UFL) [1] notation used by Firedrake, we denote Lagrange finite elements by `CG`, Nédélec $H(\mathrm{curl})$ edge elements of the first kind by `N1E`, and Nédélec $H(\mathrm{div})$ face elements of the first kind by `N1F`.

These results match the error estimates in Sections 4.3 and 5.2. Specifically, when $k = 1 < n - 1$, we do not get superconvergence of $\widehat{\rho}_h^{\mathrm{nor}}$ or ρ_h^*: both converge with the same rate $\mathcal{O}(h^{r+1})$ as $\rho_h = \mathrm{d}u_h$. However, $\delta\rho_h^*$ converges with improved rate $\mathcal{O}(h^{r+1})$, compared with $\mathcal{O}(h^r)$ for $\delta\rho_h$. On the other hand, when $k = 2 = n - 1$, we see that $\widehat{\rho}_h^{\mathrm{nor}}$ and ρ_h^* both superconverge with rate $\mathcal{O}(h^{r+2})$.

8. A view toward HDG methods for finite element exterior calculus

In this last section, we briefly present an even more general approach to domain decomposition and hybrid methods for the Hodge–Laplace problem. This includes hybridization of the conforming FEEC methods we have discussed so far, as well as nonconforming and HDG methods. In the cases $k = 0$ and $k = n$, we recover the unified hybridization framework of Cockburn, Gopalakrishnan, and Lazarov [19] for the scalar Poisson equation. When $n = 3$, the cases $k = 1$ and $k = 2$ include some recently proposed HDG methods for the vector Poisson equation and Maxwell's equations. Although we lay out the framework here, we postpone a detailed discussion and analysis of these methods for future work.

8.1. **Variational principle.** To motivate the variational principle for these more general methods, we begin with a new formulation of the exact local solvers for the Hodge–Laplace problem. Given $\widehat{\sigma}^{\tan}$, $\widehat{u}^{\tan}$ on ∂K, $\overline{u} \in \mathring{\mathfrak{H}}^k(K)$, and $p \in \mathfrak{H}^k$, observe that the exact solution satisfies

$$\begin{aligned}
(\sigma,\tau)_K - (u,\mathrm{d}\tau)_K + \langle u^{\mathrm{nor}}, \tau^{\tan}\rangle_{\partial K} &= 0, && \forall \tau \in H\Lambda^{k-1}(K) \cap H^*\Lambda^{k-1}(K),\\
(\sigma,\delta v)_K + (\rho,\mathrm{d}v)_K + (\overline{p},v)_K - \langle \rho^{\mathrm{nor}}, v^{\tan}\rangle_{\partial K} &= (f-p,v)_K - \langle \widehat{\sigma}^{\tan}, v^{\mathrm{nor}}\rangle_{\partial K}, \\
&&& \forall v \in H\Lambda^{k}(K) \cap H^*\Lambda^{k}(K),\\
(\rho,\eta)_K - (u,\delta\eta)_K &= \langle \widehat{u}^{\tan}, \eta^{\mathrm{nor}}\rangle_{\partial K}, && \forall \eta \in H\Lambda^{k+1}(K) \cap H^*\Lambda^{k+1}(K),\\
(u,\overline{q})_K &= (\overline{u},\overline{q})_K, && \forall \overline{q} \in \mathring{\mathfrak{H}}^k(K).
\end{aligned}$$

Here, both d and δ are taken weakly, as they are only applied to test functions.

Now, suppose we choose finite element spaces $W_h^k(K) \subset H\Lambda^k(K) \cap H^*\Lambda^k(K)$ for each $K \in \mathcal{T}_h$, giving the broken space $W_h^k \coloneqq \prod_{K\in\mathcal{T}_h} W_h^k(K)$, and likewise for $W_h^{k\pm1}$. Suppose we also choose unbroken spaces $\widehat{V}_h^{k-1,\tan} \subset \widehat{V}^{k-1,\tan}$ and $\widehat{V}_h^{k,\tan} \subset \widehat{V}^{k,\tan}$, which do not necessarily correspond to tangential traces of W_h^{k-1} and W_h^k. Then we consider the variational problem: Find

$$\begin{array}{llllll}
\text{(local variables)} & \sigma_h \in W_h^{k-1}, & u_h \in W_h^k, & \rho_h \in W_h^{k+1}, & \overline{p}_h \in \overline{\mathfrak{H}}_h^k,\\
\text{(global variables)} & p_h \in \mathfrak{H}_h^k, & \overline{u}_h \in \overline{\mathfrak{H}}_h^k, & \widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, & \widehat{u}_h^{\tan} \in \widehat{V}_h^{k,\tan},
\end{array}$$

satisfying

$$\begin{aligned}
&(\sigma_h,\tau_h)_{\mathcal{T}_h} - (u_h,\mathrm{d}\tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\tan}\rangle_{\partial\mathcal{T}_h} = 0, && \forall \tau_h \in W_h^{k-1}, && \text{(20a)}\\
&(\sigma_h,\delta v_h)_{\mathcal{T}_h} + (\rho_h,\mathrm{d}v_h)_{\mathcal{T}_h} + (\overline{p}_h + p_h, v_h)_{\mathcal{T}_h}\\
&\qquad + \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h^{\tan}\rangle_{\partial\mathcal{T}_h} = (f,v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^k, && \text{(20b)}\\
&(\rho_h,\eta_h)_{\mathcal{T}_h} - (u_h,\delta\eta_h)_{\mathcal{T}_h} - \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0, && \forall \eta_h \in W_h^{k+1}, && \text{(20c)}\\
&(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} = 0, && \forall \overline{q}_h \in \overline{\mathfrak{H}}_h^k, && \text{(20d)}\\
&(u_h, q_h)_{\mathcal{T}_h} = 0, && \forall q_h \in \mathfrak{H}_h^k, && \text{(20e)}\\
&(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} = 0, && \forall \overline{v}_h \in \overline{\mathfrak{H}}_h^k, && \text{(20f)}\\
&\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan}\rangle_{\partial\mathcal{T}_h} = 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, && \text{(20g)}\\
&\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan}\rangle_{\partial\mathcal{T}_h} = 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{k,\tan}. && \text{(20h)}
\end{aligned}$$

To complete the specification of the problem, one must define the approximate normal traces $\widehat{u}_h^{\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}}$, which play the same role as the "numerical flux" does in [19]. The discrete harmonic spaces $\overline{\mathfrak{H}}_h^k$ and $\mathfrak{H}_h^k$ are then defined so that the local and global solvers have unique solutions.

Remark 8.1. For the scalar Poisson equation, we recover the unified hybridization framework of [19]. If $k = 0$, then in terms of scalar and vector proxies, (20) simplifies to

$$\begin{aligned}
(\rho_h, \operatorname{grad} v_h)_{\mathcal{T}_h} + (p_h, v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h\rangle_{\partial\mathcal{T}_h} &= (f,v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^0,\\
(\rho_h,\eta_h)_{\mathcal{T}_h} + (u_h, \operatorname{div}\eta_h)_{\mathcal{T}_h} - \langle \widehat{u}_h^{\tan}, \eta_h\cdot\widehat{n}\rangle_{\partial\mathcal{T}_h} &= 0, && \forall \eta_h \in W_h^1,\\
(u_h, q_h)_{\mathcal{T}_h} &= 0, && \forall q_h \in \mathfrak{H}_h^0,\\
\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan}\rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{0,\tan},
\end{aligned}$$

which gives the methods of [19] for the Neumann problem, using local Dirichlet solvers. Alternatively, if $k = n$, and each $K \in \mathcal{T}_h$ is connected (e.g., simplicial), then (20) becomes

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{div} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^{n-1}, \\
-(\sigma_h, \operatorname{grad} v_h)_{\mathcal{T}_h} + (\overline{p}_h, v_h)_{\mathcal{T}_h} + \langle \widehat{\sigma}_h^{\tan}, v_h \widehat{n} \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^n, \\
(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{q}_h \in \mathbb{R}^{\mathcal{T}_h}, \\
(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{v}_h \in \mathbb{R}^{\mathcal{T}_h}, \\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan},
\end{aligned}$$

which is the alternative hybridization of Cockburn [16, Section 5] using local Neumann solvers.

8.2. **Examples of methods.** Different choices of the finite element spaces and approximate normal traces in (20) yield different families of methods. We now discuss a few specific examples.

8.2.1. *The hybridized FEEC methods.* Suppose we choose the spaces W_h and $\widehat{V}_h$ as in Section 4. We then define $\widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}$ to be new unknown variables, which are determined by augmenting (20) by the equations

$$\langle \widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \quad \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}, \tag{10d}$$

$$\langle \widehat{u}_h^{\tan} - u_h^{\tan}, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \quad \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}. \tag{10e}$$

Using these, (20b) and (20c) become equivalent to (10b) and $\rho_h = \mathrm{d} u_h$, respectively. Hence, the variational problem is equivalent to (10), so we recover the hybridized FEEC methods of Section 4.

8.2.2. *Mixed and nonconforming hybrid methods.* Suppose we take $\widehat{u}_h^{\mathrm{nor}} = u_h^{\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}} = \rho_h^{\mathrm{nor}}$. Then, using integration by parts, (20) simplifies to

$$\begin{aligned}
(\delta u_h, \delta v_h)_{\mathcal{T}_h} + (\delta \rho_h, v_h)_{\mathcal{T}_h} + (\overline{p}_h + p_h, v_h)_{\mathcal{T}_h} + \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^k, \\
(\rho_h, \eta_h)_{\mathcal{T}_h} - (u_h, \delta \eta_h)_{\mathcal{T}_h} - \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \eta_h \in W_h^{k+1}, \\
(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{q}_h \in \overline{\mathfrak{H}}_h^k, \\
(u_h, q_h)_{\mathcal{T}_h} &= 0, && \forall q_h \in \mathfrak{H}_h^k, \\
(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{v}_h \in \overline{\mathfrak{H}}_h^k, \\
\langle u_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, \\
\langle \rho_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{k,\tan},
\end{aligned}$$

and $\sigma_h = \delta u_h$. When $k = 0$, we obtain mixed hybrid methods for the Neumann problem using local Dirichlet solvers, including the classic hybridized RT and BDM methods [3, 8]. When $k = n$, we obtain primal hybrid methods for the Dirichlet problem using local Neumann solvers, including the nonconforming hybrid method of Raviart and Thomas [40].

8.2.3. *Hybridizable discontinuous Galerkin methods.* Suppose we take

$$\widehat{u}_h^{\mathrm{nor}} = u_h^{\mathrm{nor}} - \lambda(\widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}), \qquad \widehat{\rho}_h^{\mathrm{nor}} = \rho_h^{\mathrm{nor}} + \mu(\widehat{u}_h^{\tan} - u_h^{\tan}),$$

where λ and μ are penalty functions on $\partial \mathcal{T}_h$. Section 8.2.2 corresponds to the case $\lambda = \mu = 0$, while the hybridized FEEC methods of Section 4 can be seen as the limiting case $\lambda, \mu \to \infty$.

When $k = 0$, (20) becomes the hybrid local discontinuous Galerkin (LDG-H) method of [19], while $k = n$ gives the alternative implementation of [16, Section 5] using local Neumann solvers. For the vector Poisson equation when $n = 2$ or $n = 3$, (20) corresponds to the recent HDG methods of Nguyen, Peraire, and Cockburn [36], Chen, Qiu, Shi, and Solano [15], which have been applied to Maxwell's equations. Since the initial appearance of the current manuscript as a preprint, Hong, Li,

and Xu [28] have analyzed several methods of this type for general k and n within the extended Galerkin (XG) framework.

Finally, a different family of HDG methods may be constructed by taking

$$\widehat{u}_h^{\mathrm{nor}} = u_h^{\mathrm{nor}} - \lambda\big(\widehat{\sigma}_h^{\tan} - (\delta u_h)^{\tan}\big), \qquad \widehat{\rho}_h^{\mathrm{nor}} = \mathrm{d}u_h^{\mathrm{nor}} + \mu(\widehat{u}_h^{\tan} - u_h^{\tan}),$$

which generalizes the hybrid interior penalty (IP-H) method of [19].

Acknowledgments

Gerard Awanou, Johnny Guzman, and Ari Stern would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program "Geometry, compatibility and structure preservation in computational differential equations," when work on this paper was undertaken. This program was supported by EPSRC grant number EP/R014604/1.

Gerard Awanou was supported by NSF grant DMS-1720276, Johnny Guzman by NSF grants DMS-1620100 and DMS-1913083, and Ari Stern by NSF grant DMS-1913272.

Finally, Ari Stern wishes to thank Mary Barker for assisting with preliminary numerical experiments and Colin Cotter, Thomas Gibson, and David Ham for help with Firedrake.

References

[1] M. S. Alnæs, A. Logg, K. B. Ølgaard, M. E. Rognes, and G. N. Wells, *Unified form language: a domain-specific language for weak formulations and partial differential equations*, ACM Trans. Math. Software, 40 (2014), pp. Art. 9, 37.

[2] D. N. Arnold, *Finite element exterior calculus*, vol. 93 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018.

[3] D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 7–32.

[4] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer., 15 (2006), pp. 1–155.

[5] ———, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. Amer. Math. Soc. (N.S.), 47 (2010), pp. 281–354.

[6] D. Boffi, F. Brezzi, and M. Fortin, *Mixed finite element methods and applications*, vol. 44 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2013.

[7] J. H. Bramble and J. Xu, *A local post-processing technique for improving the accuracy in mixed finite-element approximations*, SIAM J. Numer. Anal., 26 (1989), pp. 1267–1275.

[8] F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[9] F. Brezzi and L. D. Marini, *A three-field domain decomposition method*, in Domain decomposition methods in science and engineering (Como, 1992), vol. 157 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1994, pp. 27–34.

[10] J. Brüning and M. Lesch, *Hilbert complexes*, J. Funct. Anal., 108 (1992), pp. 88–132.

[11] A. Buffa and P. Ciarlet, Jr., *On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra*, Math. Methods Appl. Sci., 24 (2001), pp. 9–30.

[12] ———, *On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications*, Math. Methods Appl. Sci., 24 (2001), pp. 31–48.

[13] A. Buffa, M. Costabel, and D. Sheen, *On traces for* $\mathbf{H}(\mathbf{curl}, \Omega)$ *in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[14] C. Carstensen, L. Demkowicz, and J. Gopalakrishnan, *Breaking spaces and forms for the DPG method and applications including Maxwell equations*, Comput. Math. Appl., 72 (2016), pp. 494–522.

[15] H. Chen, W. Qiu, K. Shi, and M. Solano, *A superconvergent HDG method for the Maxwell equations*, J. Sci. Comput., 70 (2017), pp. 1010–1029.

[16] B. Cockburn, *Static condensation, hybridization, and the devising of the HDG methods*, in Building bridges: connections and challenges in modern approaches to numerical partial differential equations, vol. 114 of Lect. Notes Comput. Sci. Eng., Springer, [Cham], 2016, pp. 129–177.

[17] B. Cockburn and J. Gopalakrishnan, *A characterization of hybridized mixed methods for second order elliptic problems*, SIAM J. Numer. Anal., 42 (2004), pp. 283–301.

[18] ———, *Incompressible finite elements via hybridization. II. The Stokes system in three space dimensions*, SIAM J. Numer. Anal., 43 (2005), pp. 1651–1672.

[19] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems*, SIAM J. Numer. Anal., 47 (2009), pp. 1319–1365.
[20] B. Cockburn, J. Gopalakrishnan, and H. Wang, *Locally conservative fluxes for the continuous Galerkin method*, SIAM J. Numer. Anal., 45 (2007), pp. 1742–1776.
[21] J. Douglas, Jr. and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.
[22] B. Fraeijs de Veubeke, *Displacement and equilibrium models in the finite element method by B. Fraeijs de Veubeke, Chapter 9, pages 145–197 of stress analysis, Edited by O. C. Zienkiewicz and G. S. Holister, Published by John Wiley & Sons, 1965*, Internat. J. Numer. Methods Engrg., 52 (2001), pp. 287–342. Edited by O. C. Zienkiewicz and G. S. Holister and with introductory remarks by Zienkiewicz.
[23] L. Gastaldi and R. H. Nochetto, *Sharp maximum norm error estimates for general mixed finite element approximations to second order elliptic equations*, RAIRO Modél. Math. Anal. Numér., 23 (1989), pp. 103–128.
[24] T. H. Gibson, L. Mitchell, D. A. Ham, and C. J. Cotter, *Slate: extending Firedrake's domain-specific abstraction to hybridized solvers for geoscience and beyond*, Geosci. Model Dev., 13 (2020), pp. 735–761.
[25] V. Gol'dshtein, I. Mitrea, and M. Mitrea, *Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds*, J. Math. Sci. (N.Y.), 172 (2011), pp. 347–400.
[26] R. Guyan, *Reduction of stiffness and mass matrices*, AIAA J., 3 (1965), p. 380.
[27] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numer., 11 (2002), pp. 237–339.
[28] Q. Hong, Y. Li, and J. Xu, *An extended Galerkin analysis in finite element exterior calculus*, 2021. arXiv:2101.09735 [math.NA].
[29] T. Jakab, I. Mitrea, and M. Mitrea, *On the regularity of differential forms satisfying mixed boundary conditions in a class of Lipschitz domains*, Indiana Univ. Math. J., 58 (2009), pp. 2043–2071.
[30] S. Kurz and B. Auchmann, *Differential forms and boundary integral equations for Maxwell-type problems*, in Fast boundary element methods in engineering and industrial applications, vol. 63 of Lect. Notes Appl. Comput. Mech., Springer, Heidelberg, 2012, pp. 1–62.
[31] P. D. Ledger and K. Morgan, *The application of the hp-finite element method to electromagnetic problems*, Arch. Comput. Methods Engrg., 12 (2005), pp. 235–302.
[32] M. W. Licht, *Smoothed projections and mixed boundary conditions*, Math. Comp., 88 (2019), pp. 607–635.
[33] D. Mitrea, M. Mitrea, and M.-C. Shaw, *Traces of differential forms on Lipschitz domains, the boundary de Rham complex, and Hodge decompositions*, Indiana Univ. Math. J., 57 (2008), pp. 2061–2095.
[34] J.-C. Nédélec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.
[35] ———, *A new family of mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 50 (1986), pp. 57–81.
[36] N. C. Nguyen, J. Peraire, and B. Cockburn, *Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell's equations*, J. Comput. Phys., 230 (2011), pp. 7151–7175.
[37] R. Picard, *An elementary proof for a compact imbedding result in generalized electromagnetic theory*, Math. Z., 187 (1984), pp. 151–164.
[38] F. Rathgeber, D. A. Ham, L. Mitchell, M. Lange, F. Luporini, A. T. T. McRae, G.-T. Bercea, G. R. Markall, and P. H. J. Kelly, *Firedrake: automating the finite element method by composing abstractions*, ACM Trans. Math. Software, 43 (2017), pp. Art. 24, 27.
[39] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.
[40] ———, *Primal hybrid finite element methods for 2nd order elliptic equations*, Math. Comp., 31 (1977), pp. 391–413.
[41] R. Stenberg, *Postprocessing schemes for some mixed finite elements*, RAIRO Modél. Math. Anal. Numér., 25 (1991), pp. 151–167.
[42] N. Weck, *Traces of differential forms on Lipschitz boundaries*, Analysis (Munich), 24 (2004), pp. 147–169.

Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago
Email address: awanou@uic.edu

Division of Applied Mathematics, Brown University
Email address: fabien@brown.edu
Email address: johnny_guzman@brown.edu

Department of Mathematics and Statistics, Washington University in St. Louis
Email address: stern@wustl.edu