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On the Generalized Bykovskiĭ Presentation of Steinberg Modules

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We study presentations of the virtual dualizing modules of special linear groups of number rings, the Steinberg modules. Bykovskiĭ gave a presentation for the Steinberg modules of the integers, and our main result is a generalization of this to the Gaussian integers and the Eisenstein integers. We also show that this generalization does not give a presentation for the Steinberg modules of several Euclidean number rings.

1 Introduction

1.1 Cohomology

In this paper, we study the cohomology of special linear groups of number rings in large degrees. Let $\mathcal{O}_K$ denote the ring of integers in a number field K with r_1 real embeddings

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and r_2 pairs of complex conjugate embeddings. It follows from the work by Borel–Serre [3, Theorem 11.4.2] (also see e.g., Church–Farb–Putman [6, Section 1.4]) that

$$\nu_n \coloneqq \frac{r_1}{2}((n+1)n-2) + r_2(n^2-1) - n + 1$$

is the virtual cohomological dimension of $\mathrm{SL}_n(\mathcal{O}_K)$, and hence $H^i(\mathrm{SL}_n(\mathcal{O}_K);\mathbb{Q}) = 0$ for $i > \nu_n$. This does not mean that $H^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_K);\mathbb{Q}) \neq 0$, only that there is some twisted coefficient system where this group is nontrivial. We investigate the following question.

Question 1.1. For $\mathcal{O}_K$ a number ring, what is the largest i such that $H^i(\mathrm{SL}_n(\mathcal{O}_K);\mathbb{Q})$ is non-zero?

In particular, we seek better bounds on vanishing of rational cohomology than just the virtual cohomological dimension. See [5, 6, 8, 9, 11, 16–18, 23] for progress on this question as well as applications of this question to computations in algebraic K-theory. The main such results are Lee–Szczarba's theorem [17, Theorem 1.3] that $H^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_K);\mathbb{Q}) = 0$ for $n \geq 2$ and $\mathcal{O}_K$ a Euclidean domain, and Church–Putman's theorem [8, Theorem A] that $H^{\nu_n-1}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) = 0$ for $n \geq 3$. Our main theorem extends Church–Putman's result to two other number rings: the Gaussian integers $\mathbb{Z}[i]$ and the Eisenstein integers $\mathbb{Z}[\rho]$ with $\rho = \frac{1+\sqrt{-3}}{2}$ a 6th root of unity. These are Euclidean domains, with $\nu_n = n^2 - n$.

Theorem A. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Then

$$H^{\nu_n-1}(\mathrm{GL}_n(\mathcal{O}_K);\mathbb{Q}) = 0 \qquad \text{for } n \geq 2,$$
$$H^{\nu_n-1}(\mathrm{SL}_n(\mathcal{O}_K);\mathbb{Q}) = 0 \qquad \text{for } n \geq 3.$$

In fact, it suffices to only invert $(2n+1)!$, but we restrict to rational statements in the introduction.

1.2 Dualizing modules

Our strategy for proving Theorem A is to give a presentation for the virtual dualizing module of the groups $\mathrm{GL}_n(\mathcal{O}_K)$ and $\mathrm{SL}_n(\mathcal{O}_K)$ and use it to show that the $(\nu_n - 1)$st cohomology group vanishes.

Recall that the *Tits building* $\mathcal{T}_n(K)$ of a field K is the geometric realization of the poset of non-empty proper subspaces of the K-vector space K^n ordered by inclusion.

This poset is spherical of dimension $(n-2)$ [12, 25, 28] and its top reduced homology is called the *Steinberg module* and denoted $\mathrm{St}_n(K)$. The action of $\mathrm{GL}_n(\mathcal{O}_K)$ on $\mathcal{T}_n(K)$ gives this the structure of a $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O}_K)]$-module. Borel–Serre [3, §11] proved that for $\mathcal{O}_K$ the ring of integers in a number field K, $\mathrm{St}_n(K)$ is the virtual dualizing module of K. That is, there is a natural isomorphism

$$H^{\nu_n-i}(\mathrm{SL}_n(\mathcal{O}_K);\mathbb{Q}) \xrightarrow{\cong} H_i(\mathrm{SL}_n(\mathcal{O}_K);\mathrm{St}_n(K)\otimes\mathbb{Q}).$$

Thus, to show $H^{\nu_n-1}(\mathrm{SL}_n(\mathcal{O}_K);\mathbb{Q})=0$, it suffices to show $H_1(\mathrm{SL}_n(\mathcal{O}_K);\mathrm{St}_n(K)\otimes\mathbb{Q})=0$.

This will be done by finding a presentation of the relevant Steinberg modules. For $\mathcal{O}$ an integral domain, let $\mathrm{Byk}_n(\mathcal{O})$ denote the quotient of the free abelian group on symbols $[[\vec{v}_1,\dots,\vec{v}_n]]$, with $\vec{v}_1,\dots,\vec{v}_n$ an ordered basis of $\mathcal{O}^n$, by the following relations:

(1) $[[\vec{v}_1,\dots,\vec{v}_n]] = sgn(\sigma)[[\vec{v}_{\sigma(1)},\dots,\vec{v}_{\sigma(n)}]]$ for σ a permutation of $\{1,\dots,n\}$ and $sgn(\sigma)$ its sign.

(2) $[[\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n]] = [[u\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n]]$ for u a unit in $\mathcal{O}$.

(3) $[[\vec{v}_1,\vec{v}_2,\vec{v}_3,\dots,\vec{v}_n]] - [[\vec{v}_1+\vec{v}_2,\vec{v}_2,\vec{v}_3,\dots,\vec{v}_n]] + [[\vec{v}_1+\vec{v}_2,\vec{v}_1,\vec{v}_3,\dots,\vec{v}_n]] = 0.$

The symbols $[[\vec{v}_1,\dots,\vec{v}_n]]$ are sometimes called *modular symbols*. Letting $A\in\mathrm{GL}_n(\mathcal{O})$ act on a symbol $[[\vec{v}_1,\dots,\vec{v}_n]]$ by $[[A\vec{v}_1,\dots,A\vec{v}_n]]$ gives $\mathrm{Byk}_n(\mathcal{O})$ a $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O})]$-module structure.

Let K denote the field of fractions of $\mathcal{O}$. Given an ordered basis $\vec{v}_1,\dots,\vec{v}_n$, the subposet of subspaces of K^n that are spanned by a non-empty proper subset of $\vec{v}_1,\dots,\vec{v}_n$ is isomorphic to the barycentric subdivision of $\partial\Delta^{n-1}$ and thus realizes to an $(n-2)$-dimensional sphere with canonical orientation. These spheres are called *apartments* and sending $[[\vec{v}_1,\dots,\vec{v}_n]]$ to the fundamental class of the apartment gives a homomorphism of $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O})]$-modules

$$\mathrm{Byk}_n(\mathcal{O}) \longrightarrow \mathrm{St}_n(K).$$

Bykovskiĭ [4] proved that $\mathrm{Byk}_n(\mathbb{Z})\to\mathrm{St}_n(\mathbb{Q})$ is an isomorphism. If $\mathrm{Byk}_n(\mathcal{O})\to\mathrm{St}_n(K)$ is an isomorphism, we say *the generalized Bykovskiĭ presentation holds for* $\mathcal{O}$. That the Bykovskiĭ's presentation holds for $\mathbb{Z}$ is the key ingredient in Church–Putman's vanishing result for $H^{\nu_n-1}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q})$. We investigate the following question and give a partial answer:

Question 1.2. For which number rings does the generalized Bykovskiĭ presentation hold?

Theorem B. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers, and K its field of fractions. Then $\mathrm{Byk}_n(\mathcal{O}_K)\to\mathrm{St}_n(K)$ is an isomorphism of $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O}_K)]$-modules for all n.

Theorem A follows quickly from Theorem B. Surjectivity of the map $\mathrm{Byk}_n(\mathcal{O}) \to \mathrm{St}_n(K)$ follows from work of Ash–Rudolph [1, Theorem 4.1] whenever $\mathcal{O}$ is Euclidean. In fact, for number rings $\mathcal{O}_K$, the generalized Riemann hypothesis implies that $\mathrm{Byk}_n(\mathcal{O}_K) \to \mathrm{St}_n(K)$ is surjective if and only if $\mathcal{O}_K$ is Euclidean [23, Corollary 1.2]. One might think that the generalized Bykovskiĭ presentation holds for all Euclidean number rings, but this is not the case.

Theorem C. Let $\mathcal{O}_K$ be the ring of integers in $K = \mathbb{Q}(\sqrt{d})$. Assume $\mathcal{O}_K$ is a Euclidean domain that is not additively generated by units. Then the map $\mathrm{Byk}_n(\mathcal{O}_K) \to \mathrm{St}_n(K)$ is not injective for all $n \geq 2$.

Norm-Euclidean number rings satisfying the hypothesis of this theorem have been classified and are $\mathbb{Q}(\sqrt{d})$ for $d \in \{-11, -7, -2, 6, 7, 11, 17, 19, 33, 37, 41, 57, 73\}$; see Remark 3.29. Thus there are many examples of Euclidean number rings where the generalized Bykovskiĭ presentation fails. In fact, we give a more general result (Theorem 4.13) allowing the reader to possibly find more examples.

Remark 1.3. The main technical result is that certain simplicial complexes of "augmented partial frames" are highly connected. This has applications in a forthcoming paper [15] on homological stability for general linear groups of certain Euclidean domains.

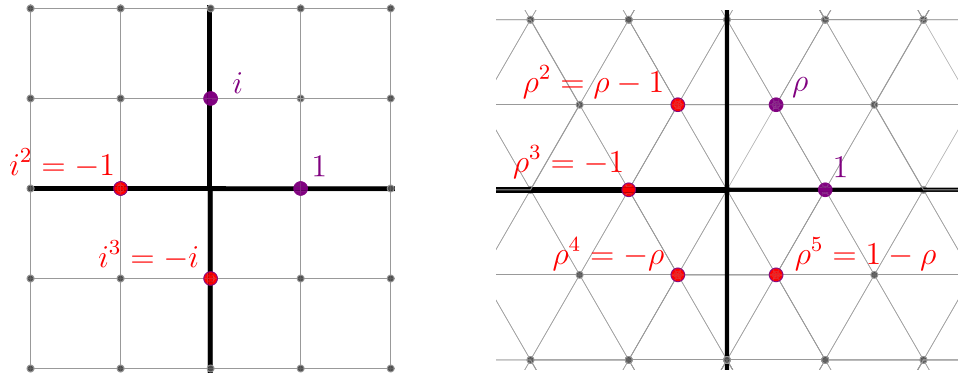
2 Elementary Properties of the Gaussian Integers and Eisenstein Integers

In this section, we establish some elementary properties of the Gaussian integers and Eisenstein integers. These properties are the primary reason that the proof of Theorem A in this paper only applies to these rings.

Notation 2.1. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. We will pick preferred ring generators for each ring,

$$\begin{aligned} \text{Gaussian integers}: &\quad i, \\ \text{Eisenstein integers}: &\quad \rho = e^{\frac{2\pi i}{6}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}. \end{aligned}$$

The latter is not the conventional choice of an additive generator for the Eisenstein integers, which is more typically $\rho^2 = e^{\frac{2\pi i}{3}}$. See Figure 1.



(A) The Gaussian integers. (B) The Eisenstein integers.

Fig. 1. The additive generators and the units in $\mathcal{O}_K$. (A) The Gaussian integers. (B) The Eisenstein integers.

With this notation, observe that the complex norm is given as follows,

$$\text{Gaussian integers}: \qquad |x+iy|^2 = x^2+y^2 \qquad (x,y\in\mathbb{R}),$$

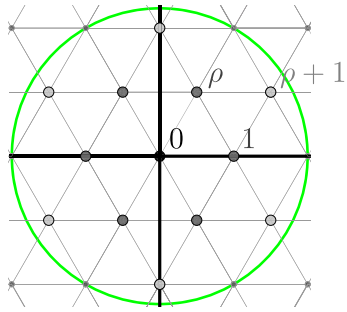
$$\text{Eisenstein integers}: \qquad |x+\rho y|^2 = \left|x+\frac{1}{2}(1+i\sqrt{3})y\right|^2 = x^2+xy+y^2 \qquad (x,y\in\mathbb{R}).$$

The complex norm is a Euclidean function for $\mathcal{O}_K$ the Gaussian integers or Eisenstein integers. That is, given $a,b\in\mathcal{O}_K$ with $|b|>0$, then there is a $q\in\mathcal{O}_K$ with $|a-qb|<|b|$. Moreover, this function is multiplicative in the sense that $|ab|=|a||b|$. We now prove some key lemmas.

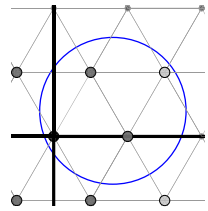
Lemma 2.2. Let $\mathcal{O}_K$ be the Gaussian integers or Eisenstein integers. Let $a,b\in\mathcal{O}_K$ with $|a|=|b|>0$. Then there is a unit u with $|a-ub|<|a|$.

Proof. Pick u such that the angle between a and ub is less than $\pi/3$. It is an elementary exercise in trigonometry to see that $|a-ub|<|a|$. ■

Convention 2.3. Let H be a group and S a set of generators. In this paper, the term *Cayley graph* for H with generators S means the undirected graph with vertices elements of H and an edge between h and g if and only if they differ by left multiplication by an element in S. In particular, if s and $s^{-1}\in S$, then the Cayley graph for H with generators S agrees with the Cayley graph for H with generators $S\setminus\{s^{-1}\}$.



(A) Eisenstein integers within complex distance 2 of the origin.



(B) An instance of a ball B_z containing 0 and $\rho+1$.

Fig. 2. (A) Eisenstein integers within complex distance 2 of the origin. (B) An instance of a ball B_z containing 0 and $\rho+1$.

Lemma 2.4. Let $\mathcal{O}_K$ be the Gaussian integers or Eisenstein integers. Let G be the Cayley graph of $\mathcal{O}_K$ with units as generators. Let $z \in \mathbb{C}$ and let G_z be the full subgraph of G on vertices x with $|x-z|<1$. Then G_z is connected.

Proof. Consider the open ball B_z of radius 1 centered on the point $z \in \mathbb{C}$. This ball must contain at least one element of $\mathcal{O}_K$, and without loss of generality we may assume it contains 0.

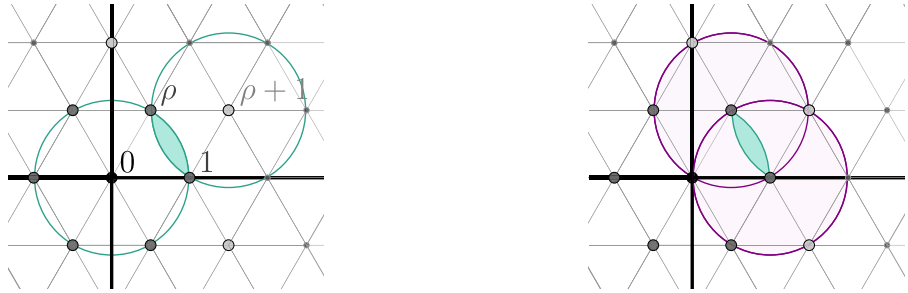
We will first consider the case where $\mathcal{O}_K$ is the Eisenstein integers. Any other element of $\mathcal{O}_K$ in B_z must have distance < 2 from the origin. There are only twelve such points, as shown in Figure 2a. Six of these (colored dark gray) are joined to 0 by an edge, and the other six (colored light gray) are distance 2 from the origin in the edge metric on the Cayley graph.

Up to symmetry, then, it suffices to assume that both 0 and $\rho+1$ are contained in B_z and show that either ρ or 1 must be contained in B_z. See Figure 2b. In this case, z must be contained in the intersection $B_0 \cap B_{\rho+1}$, as in Figure 5a. But this intersection is contained in the union $B_\rho \cup B_1$, as in Figure 5b. Hence $\rho \in B_z$ or $1 \in B_z$.

Next suppose $\mathcal{O}_K$ is the Gaussian integers. There are only eight points other than 0 that could be contained in the ball B_z, as in Figure 4.

It suffices to check that, if 0 and $i+1$ are contained in B_z, then so is one of i or 1. But $B_0 \cap B_{i+1} \subseteq B_i \cup B_1$, as shown in Figure 5.

The result follows. ∎



(A) The intersection $B_0 \cap B_{\rho+1}$. (B) The containment $B_0 \cap B_{\rho+1} \subseteq B_\rho \cup B_1$.

Fig. 3. (A) The intersection $B_0 \cap B_{\rho+1}$. (B) The containment $B_0 \cap B_{\rho+1} \subseteq B_\rho \cup B_1$.

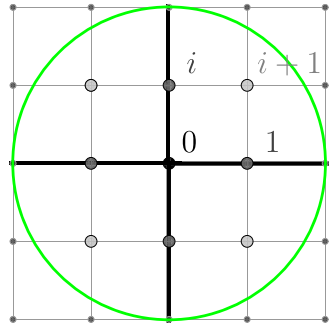
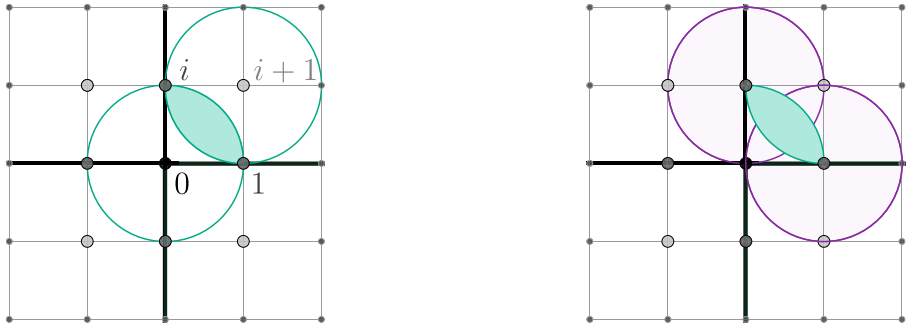


Fig. 4. Gaussian integers within complex distance 2 of the origin.



(A) The intersection $B_0 \cap B_{i+1}$. (B) The containment $B_0 \cap B_{i+1} \subseteq B_i \cup B_1$.

Fig. 5. Gaussian integers within complex distance 2 of the origin. (A) The intersection $B_0 \cap B_{i+1}$. (B) The containment $B_0 \cap B_{i+1} \subseteq B_i \cup B_1$.

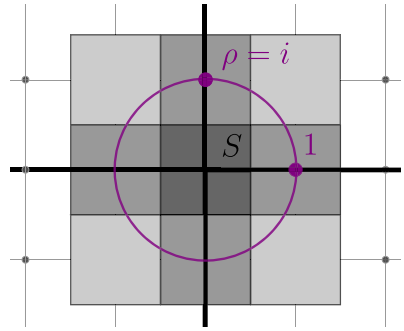


Fig. 6. The region S and its translates $S \pm i$, $S \pm 1$, $S \pm i \pm 1$.

Lemma 2.5. Let $\mathcal{O}_K$ be the Gaussian integers. If z_1 and z_2 are any complex numbers, then there exist $r_1, r_2 \in \mathcal{O}_K$ so that

$$|z_1 - r_1| < 1, \quad |z_2 - r_2| < 1, \quad \text{and} \quad |(z_1 - r_1) + (z_2 - r_2)| < 1.$$

Proof. Up to the addition of elements of $\mathcal{O}_K$, we may assume that z_1, z_2 are contained in the set

$$S = \left\{ z \in \mathbb{C} \;\middle|\; \frac{-1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}, \; \frac{-1}{2} \leq \mathrm{Im}(z) \leq \frac{1}{2} \right\},$$

the closure of the square fundamental domain for $\mathcal{O}_K$ centered around zero in Figure 6.

The sum $z_1 + z_2$ must have both real and imaginary parts in the interval $[-1, 1]$, so the sum is contained in S or in one of its eight translates shown in Figure 6. If $|z_1 + z_2| \leq 1$, then we are done, so suppose otherwise. Up to symmetry, we may consider two cases: $(z_1 + z_2) \in (S + 1)$ or $(z_1 + z_2) \in (S + 1 + i)$.

First suppose that $(z_1 + z_2) \in (S + 1)$. This means that at least one of z_1 and z_2 (say, z_1) must have real part at least $\frac{1}{4}$. Then we will replace z_1 by $(z_1 - 1)$. By assumption

$$\frac{-3}{4} \leq \mathrm{Re}(z_1 - 1) \leq \frac{-1}{2} \qquad \frac{-1}{2} \leq \mathrm{Im}(z_1 - 1) \leq \frac{1}{2}$$

and so $(z_1 - 1)$ lies in the rectangular region shown in Figure 7. The number $(z_1 - 1)$ is contained in the unit ball, as

$$\left(\frac{-3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 < 1;$$

see Figure 7. But now $((z_1 - 1) + z_2)$ is contained in S and therefore in the unit ball, so we have completed this case.

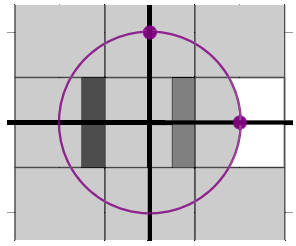
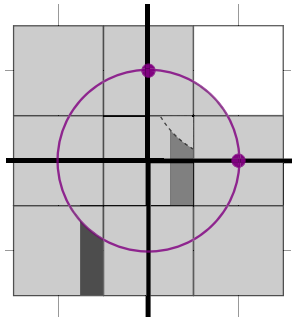
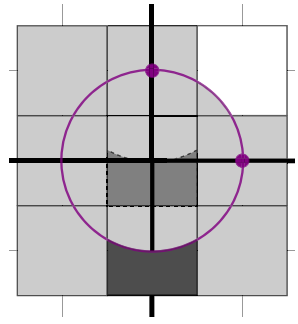


Fig. 7. The regions containing z_1 and $(z_1 - 1)$ are shaded gray and dark gray, respectively. The region containing $(z_1 + z_2)$ is white.



(A) The regions containing z_1 and $(z_1 - 1 - i)$ are shaded gray and dark gray, respectively. The region containing $(z_1 + z_2)$ is white.



(B) The regions containing z_2 and $(z_2 - i)$ are shaded gray and dark gray, respectively. The region containing $(z_1 + z_2)$ is white.

Fig. 8. (A) The regions containing z_1 and $(z_1 - 1 - i)$ are shaded gray and dark gray, respectively. The region containing $(z_1 + z_2)$ is white. (B) The regions containing z_2 and $(z_2 - i)$ are shaded gray and dark gray, respectively. The region containing $z_1 + z_2$ is white.

Now suppose that $(z_1 + z_2) \in (S + 1 + i)$. Again we may assume that z_1 has real part at least $\frac{1}{4}$, and again we know $|z_1 - 1| < 1$. If $|z_1 - 1 - i| < 1$, then we could replace z_1 by $(z_1 - 1 - i)$, and the sum $(z_1 - 1 - i) + z_2$ would be contained in S. So suppose $|z_1 - 1 - i| \geq 1$. See Figure 8a. In this case we will replace z_1 by $(z_1 - 1)$ and z_2 by $(z_2 - i)$, and then

$$(z_1 - 1) + (z_2 - i) \in S$$

is contained in the unit ball as desired. It remains to show that $|z_2 - i| < 1$.

Assume for contradiction that $|z_2 - i| \geq 1$, so z_2 is contained in the region shown in Figure 8b. This assumption implies in particular that

$$\mathrm{Im}(z_2) \leq 1 - \frac{\sqrt{3}}{2}.$$

The assumption that $|z_1 - 1 - i| \geq 1$ implies that

$$\mathrm{Im}(z_1) \leq 1 - \frac{\sqrt{7}}{4},$$

as we see in Figure 8a. But then

$$\mathrm{Im}(z_1 + z_2) \leq \left(1 - \frac{\sqrt{3}}{2}\right) + \left(1 - \frac{\sqrt{7}}{4}\right) < \frac{1}{2},$$

which contradicts the premise that $z_1 + z_2$ is contained in the region $(S + 1 + i)$. So we conclude that $|z_2 - i| < 1$, which concludes the proof. ■

Lemma 2.6. Let $\mathcal{O}_K$ be the Eisenstein integers. If z_1 and z_2 are any complex numbers, then there exist $r_1, r_2 \in \mathcal{O}_K$ so that

$$|z_1 - r_1| < 1, \quad |z_2 - r_2| < 1, \quad \text{and} \quad |(z_1 - r_1) + (z_2 - r_2)| < 1.$$

Proof. Consider the rectangular fundamental domains for $\mathcal{O}_K$ shown in Figure 9a. Up to the addition of elements of $\mathcal{O}_K$, we may assume that z_1 and z_2 are in (the closure of) the fundamental domain centered about zero,

$$S = \left\{ z \in \mathbb{C} \;\middle|\; \frac{-1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2},\ \frac{-\sqrt{3}}{4} \leq \mathrm{Im}(z) \leq \frac{\sqrt{3}}{4} \right\}.$$

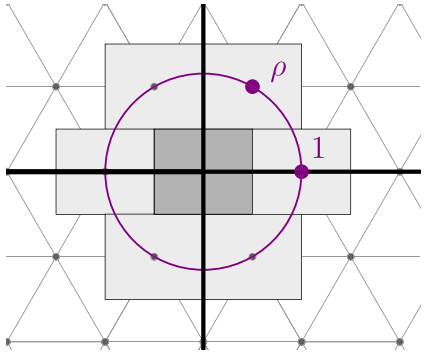
The region S is shaded medium gray in Figure 9a. Observe that $|z_1|, |z_2| < 1$.

By symmetry under reflection in the real and imaginary axes, we may assume that $\mathrm{Im}(z_1), \mathrm{Re}(z_1) \geq 0$. Thus z_1 is in the quadrant shaded dark gray in Figure 9b, and

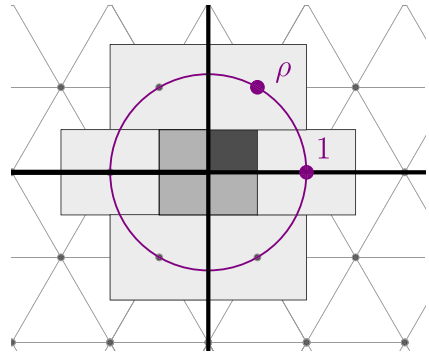
$$\frac{-1}{2} \leq \mathrm{Re}(z_1 + z_2) \leq 1, \qquad \frac{-\sqrt{3}}{4} \leq \mathrm{Im}(z_1 + z_2) \leq \frac{\sqrt{3}}{2}.$$

We deduce that the sum $(z_1 + z_2)$ must be contained in the rectangle shown in Figure 10a.

If $|z_1 + z_2| < 1$ then we are done, so assume otherwise. In this case, either $z_1 + z_2 = \rho - 1$, or $(z_1 + z_2)$ is in one of the regions A or B shown in Figure 10b.

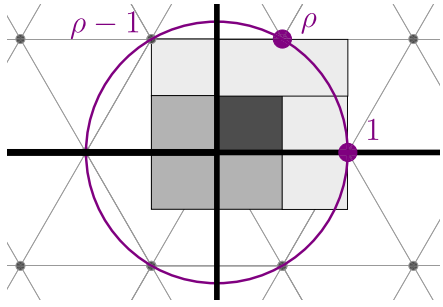


(A) The unit circle and a rectangular fundamental domain for $\mathcal{O}_K$ along with six of its translates.

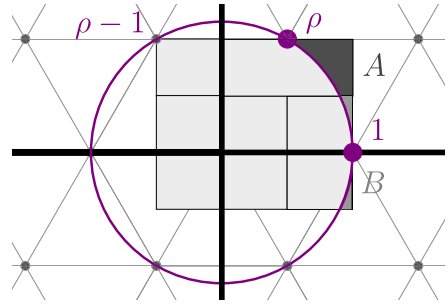


(B) We may assume z_1 is in the dark gray region.

Fig. 9. (A) The unit circle and a rectangular fundamental domain for $\mathcal{O}_K$ along with six of its translates. (B) We may assume z_1 is in the dark gray region.



(A) The sum $z_1 + z_2$ is contained in the rectangle



(B) The region A is shown in dark gray and B in medium gray

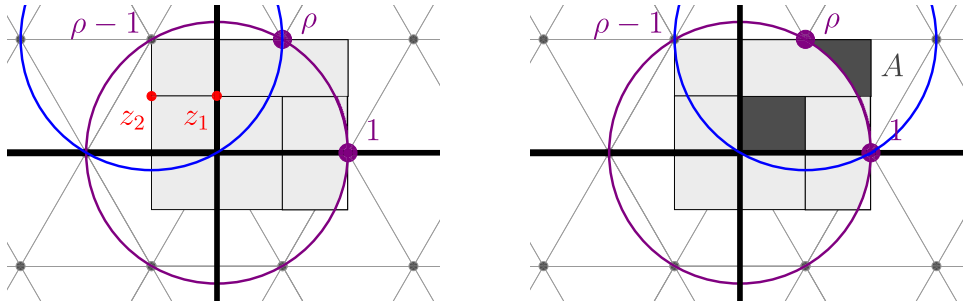
Fig. 10. (A) The sum $z_1 + z_2$ is contained in the rectangle. (B) The region A is shown in dark gray and B in medium gray.

If $z_1 + z_2 = \rho - 1$, then we must have the extremal values of z_1 and z_2 given by

$$z_1 = \left(\frac{\sqrt{3}}{4}\right)i \qquad \text{and} \quad z_2 = -\frac{1}{2} + \left(\frac{\sqrt{3}}{4}\right)i,$$

as shown in Figure 11a. In this case both z_1 and $z_1 + z_2$ are strictly within distance 1 of $\rho - 1$. We may then replace z_1 by $(z_1 - \rho + 1)$, so

$$|z_1 - \rho + 1| < 1 \qquad \text{and} \qquad |(z_1 - \rho + 1) + z_2| = 0 < 1.$$



(A) The case $z_1 + z_2 = \rho - 1$ (B) The case $z_1 + z_2 \in A$

Fig. 11. (A) The case $z_1 + z_2 = \rho - 1$. (B) The case $z_1 + z_2 \epsilon A$.

Next suppose $z_1 + z_2 \in A$. Necessarily $z_1 \neq 0$, otherwise $z_1 + z_2 = z_2$ would have norm strictly less than one. But then both z_1 and $z_1 + z_2$ are within strict distance 1 of ρ, as in Figure 11b. We may replace z_1 with $z_1 - \rho$, so

$$|z_1 - \rho| < 1 \qquad \text{and} \qquad |(z_1 - \rho) + z_2| < 1.$$

Finally, suppose that $z_1 + z_2 \in B$. It follows that

$$\mathrm{Re}(z_1 + z_2) \geq \frac{\sqrt{13}}{4} > \frac{1}{2}.$$

This implies that $\mathrm{Re}(z_i) \geq \frac{1}{4}$ for at least one choice of $i = 1, 2$, so z_i is contained in the dark gray region in Figure 12. But then both z_i and $z_1 + z_2$ are within distance strictly less than 1 of 1, as in Figure 12. We may replace z_i with $z_i - 1$, and

$$|z_i - 1| < 1 \qquad \text{and} \qquad |z_1 + z_2 - 1| < 1.$$

This concludes the proof. ■

3 Connectivity Results

Our results will be a consequence of connectivity/non-connectivity results for complexes of augmented partial frames. These complexes were introduced by Church–Putman [8] to give a topological proof of Bykovskiĭ's presentation of $\mathrm{St}_n(\mathbb{Q})$. The original proof used more algebraic methods [4].

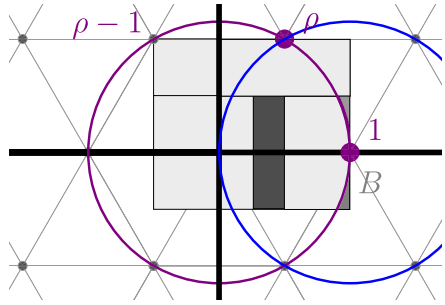


Fig. 12. The case $z_1 + z_2 \in B$. The region of S with real part at least $\frac{1}{4}$ is shaded dark gray.

3.1 Definitions and previously known results

We say that a simplicial complex is *d-spherical* if it is simultaneously d-dimensional and $(d-1)$-connected, in which case it is homotopy equivalent to a wedge of d-spheres. A simplicial complex is *Cohen–Macaulay* of dimension d if it is d-spherical and the link of every k-simplex is $(d-k-1)$-spherical. We follow the usual convention that (-1)-connected means non-empty. An example of a Cohen–Macaulay complex is the Tits building:

Definition 3.1. For a finite-dimensional vector space V over a field K, let $\mathcal{T}(V)$ denote the geometric realization of the poset of proper non-empty subspaces of V ordered by inclusion. When $V = K^n$, write $\mathcal{T}_n(K)$ for $\mathcal{T}(K^n)$ and call it the *nth Tits building* of K.

The following theorem seems to have first appeared in Solomon [28] for finite fields; see Garland [12, Theorem 2.2] and Quillen [25, Theorem 2] for the general case.

Theorem 3.2 (Solomon–Tits). For K a field, $\mathcal{T}_n(K)$ is Cohen–Macaulay of dimension $(n-2)$.

Since $\mathcal{T}_n(K)$ is $(n-2)$-spherical,

$$\mathrm{St}_n(K) := \widetilde{H}_{n-2}(\mathcal{T}_n(K))$$

is the only possible non-zero reduced homology group, called the nth *Steinberg module*. It may be helpful to remark that for $\mathcal{O}$ a Dedekind domain with field of fractions K, there is a natural bijection between summands of $\mathcal{O}^n$ and subspaces of K^n; see for example, Church–Farb–Putman [6, Lemma 2.3].

3.1.1 *Partial frames*

We now recall the definition of the complexes of (augmented) partial frames. The complex of partial frames is closely related to the complex of partial bases considered (for example) by Maazen [19] and van der Kallen [30] in the context of homological stability. See Church–Farb–Putman [6] and [23] for a discussion of the relationship between the complex of partial bases/frames and generators of Steinberg modules. From now on, we let $\mathcal{O}$ denote an integral domain.

Convention 3.3. In this paper, a *line* in $\mathcal{O}^n$ will mean a rank one free summand.

Definition 3.4. A vector $\vec{v} \in \mathcal{O}^n$ is called *primitive* if its span is a direct summand. In that case, we denote its span by v. Similarly if v is a line, we let $\vec{v}$ denote a primitive vector that spans that line. The vector $\vec{v}$ is well defined up to multiplication by a unit of $\mathcal{O}$.

Definition 3.5. A *partial frame* is an unordered collection of lines $v_0, \dots, v_p$ in $\mathcal{O}^n$ such that there are lines $v_{p+1}, \dots, v_{n-1}$ so that the natural map $v_0 \oplus \cdots \oplus v_{n-1} \to \mathcal{O}^n$ is an isomorphism.

Definition 3.6. Let M be a finite-rank free $\mathcal{O}$-module. The *complex of partial frames* $B(M)$ is the simplicial complex with p-simplices given by the set of partial frames in M of cardinality $p+1$. A simplex $\{w_0, \dots, w_q\}$ is a face of $\{v_0, \dots, v_p\}$ if and only if $\{w_0, \dots, w_q\} \subseteq \{v_0, \dots, v_p\}$. We write $B_n(\mathcal{O})$ for $B(\mathcal{O}^n)$.

We consider $\mathcal{O}^m$ as a submodule of $\mathcal{O}^{m+n}$ by the inclusion of the 1st m coordinates, and let $e_1, \dots, e_m$ denote the lines spanned by the standard basis vectors of $\mathcal{O}^m$. As in Church–Putman [8, Definition 4.1], we use the shorthand

$$B_n^m(\mathcal{O}) \coloneqq \mathrm{Link}_{B_{n+m}(\mathcal{O})}(e_1, \dots, e_m).$$

Observe that $B_n^0(\mathcal{O})$ is equal to $B_n(\mathcal{O})$.

Theorem 3.7. If $\mathcal{O}$ is a Euclidean domain, then the simplicial complexes $B_n^m(\mathcal{O})$ are Cohen–Macaulay of dimension $(n-1)$.

We prove this theorem using Maazen [19], though it can also be established by adapting the techniques of Church–Putman [8, Theorem 4.2], who prove it $\mathcal{O} = \mathbb{Z}$.

Proof of Theorem 3.7. Since the link of a p-simplex in $B_n^m(\mathcal{O})$ is isomorphic to $B_{n-p-1}^{m+p+1}(\mathcal{O})$, it suffices to show that the complexes $B_n^m(\mathcal{O})$ are $(n-2)$-connected. Let $B_n^m(\mathcal{O})_\bullet$ denote the semi-simplicial set of *ordered partial frames*. That is, $B_n^m(\mathcal{O})_p$ is the set of ordered $(p+1)$-tuples $(v_0,\dots,v_p)$ with underlying set $\{v_0,\dots,v_p\}$ a p-simplex of $B_n^m(\mathcal{O})$. The ith face map forgets the ith line. By [14, Lemma 3.16], it suffices to show $B_n^m(\mathcal{O})_\bullet$ is $(n-2)$-connected. Let $\mathrm{O}_n^m(\mathcal{O})_\bullet$ denote the semi-simplicial set of *ordered partial bases*. That is, $\mathrm{O}_n^m(\mathcal{O})_p$ is the set of ordered $(p+1)$-tuples $(\vec{v}_0,\dots,\vec{v}_p)$ with $\{v_0,\dots,v_p\}$ a p-simplex of $B_n^m(\mathcal{O})$. As before, the ith face map forgets the ith vector. Maazen [19, Section III.4, Theorem 4.2, and Corollary 4.5] proved that the barycentric subdivision of $||\mathrm{O}_n^m(\mathcal{O})_\bullet||$ is $(n-2)$-connected and hence $||\mathrm{O}_n^m(\mathcal{O})_\bullet||$ is $(n-2)$-connected. Consider the natural projections:

$$\begin{aligned} \mathrm{O}_n^m(\mathcal{O})_\bullet &\longrightarrow B_n^m(\mathcal{O})_\bullet \\ (\vec{v}_0,\vec{v}_1,\dots,\vec{v}_p) &\longmapsto (\mathrm{span}(\vec{v}_0),\mathrm{span}(\vec{v}_1),\dots,\mathrm{span}(\vec{v}_p)). \end{aligned}$$

By picking a representative $\vec{v}$ for all lines v, one can construct a splitting of this map; see [23, Proposition 2.13]. Thus, $||B_n^m(\mathcal{O})_\bullet||$ and hence $B_n^m(\mathcal{O})$ is $(n-2)$-connected. ■

3.1.2 *Augmented partial frames*

One of the innovations of Church–Putman [8] is the introduction of a simplicial complex of augmented partial frames, obtained by adding new "additive" simplices to $B_n^m(\mathcal{O})$. These correspond to certain relations in Steinberg modules.

Definition 3.8. An *augmented partial frame* is an unordered collection of lines $v_0,\dots,v_p$ such that, possibly after re-indexing, $v_1,\dots,v_p$ is a partial frame and there are units u_1,u_2 in $\mathcal{O}$ so that $\vec{v}_0 = u_1\vec{v}_1 + u_2\vec{v}_2$.

Definition 3.9. Let M be a finite-rank free $\mathcal{O}$-module. The *complex of augmented partial frames* $BA(M)$ is the simplicial complex with p-simplices given by the union of the set of partial frames in M of cardinality $p+1$ and the set of augmented partial frames in M of cardinality $p+1$. A set $w_0,\dots,w_q$ is a face of $v_0,\dots,v_p$ if and only if $\{w_0,\dots,w_q\} \subseteq \{v_0,\dots,v_p\}$. We generally write $BA_n(\mathcal{O})$ for $BA(\mathcal{O}^n)$.

We adapt the following notation from Church–Putman [8, Definitions 4.7 and 4.11].

Definition 3.10. Fix $n > 0$. Let $BA_n^m(\mathcal{O})$ be the full subcomplex of

$$\mathrm{Link}_{BA_{n+m}(\mathcal{O})}(e_1, \dots, e_m)$$

of simplices $v_0, \dots, v_p$ such that $v_i \not\subseteq \mathrm{span}(\vec{e}_1, \dots, \vec{e}_m)$ for all i.

Definition 3.11. For $\sigma = \{w_0, \dots, w_q\}$, let $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O})}(\sigma)$ denote the full subcomplex of $\mathrm{Link}_{BA_n^m(\mathcal{O})}(\sigma)$ of simplices $\{v_0, \dots, v_p\}$ such that for all i,

$$v_i \not\subseteq \mathrm{span}(\vec{e}_1, \dots, \vec{e}_m, \vec{w}_0, \dots, \vec{w}_q).$$

Observe that $BA_n^m(\mathcal{O})$ is equal to $\widehat{\mathrm{Link}}_{BA_{n+m}(\mathcal{O})}(e_1, \dots, e_m)$. The span of the lines in a p-simplex $\{v_0, \dots, v_p\}$ of $BA_n^m(\mathcal{O})$ has rank $p+1$ or p. The following definition, analogous to [8, Definition 4.9], describes the latter type of simplices:

Definition 3.12. Let $\sigma = \{v_0, \dots, v_p\}$ be a p-simplex of $BA_n^m(\mathcal{O})$.

(i) We say σ is an *internally additive* simplex if $\vec{v}_i = u_k\vec{v}_k + u_j\vec{v}_j$ for some i, j, k and some units u_k, u_j in $\mathcal{O}$.

(ii) We say σ is an *externally additive* simplex if $\vec{v}_i = u_k\vec{e}_k + u_j\vec{v}_j$ for some i, j, k and some units u_k, u_j in $\mathcal{O}$. More generally, if σ is a simplex in $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O})}(\{w_0, \dots, w_p\})$, we say that σ is an *externally additive* simplex if $\vec{v}_i = u_k\vec{v}_k + u\vec{w}$ for some units u_k, u in $\mathcal{O}$, and primitive vector $\vec{w}$ spanning $e_1, \dots, e_m, w_0, \dots, w_{p-1}$ or w_p.

(iii) We say that a simplex is *additive* if it is externally or internally additive.

We will need a subcomplex of certain links with control on the last coordinate:

Definition 3.13. Let $f\colon \mathcal{O}^{n+m} \to \mathcal{O}$ denote the projection onto the last coordinate. If $\mathcal{O}$ comes equipped with a preferred multiplicative Euclidean function $|-|$, we let $F(v) = |f(\vec{v})|$ for v a line spanned by $\vec{v}$. This is well defined since $|-|$ is multiplicative. If $\mathcal{O}$ is the Gaussian integers or Eisenstein integers, we will take $|-|$ to be the usual norm on the complex numbers, $|a+bi| = \sqrt{a^2+b^2}$. We will occasionally denote $|f(\vec{v})|$ by $F(\vec{v})$ instead of $F(v)$.

Definition 3.14. Let σ be a simplex of $BA_n^m(\mathcal{O})$. Then

$$\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O})}(\sigma)$$

denotes the full subcomplex of $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O})}(\sigma)$ consisting of lines v such that there is a vertex w in σ with $F(v) < F(w)$. Similarly define

$$\mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma) \qquad \text{and} \qquad \mathrm{Link}^{<}_{BA_n^m(\mathcal{O})}(\sigma)$$

for simplices σ in $B_n^m(\mathcal{O})$ or $BA_n^m(\mathcal{O})$, respectively.

We can deduce a connectivity result for the latter complex from that for $B_n^m(\mathcal{O})$.

Lemma 3.15. Let $\mathcal{O}$ be a Euclidean domain and let σ be a simplex in $B_n^m(\mathcal{O})$. Assume that $F(w) > 0$ for some vertex w in σ. Then $\mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma)$ is Cohen–Macaulay of dimension $(n - \dim(\sigma) - 2)$.

Proof. For a simplex τ in $\mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma)$,

$$\mathrm{Link}_{\mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma)}(\tau) = \mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma * \tau),$$

where $*$ denotes simplicial join; this uses the assumption that for all $v \in \tau$ we have $F(v) < F(w)$ for some $w \in \sigma$. Our goal is therefore to show that $\mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma)$ is $(n - \dim(\sigma) - 3)$-connected.

This result is proved in the case $\mathcal{O} = \mathbb{Z}$ in Church–Putman [8, Lemma 4.5], and their proof adapts readily to all Euclidean domains. For completeness, we sketch a proof here. Given Theorem 3.7, it suffices to show that there is a simplicial retraction

$$\pi : \mathrm{Link}_{B_n^m(\mathcal{O})}(\sigma) \longrightarrow \mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma).$$

Let x be a vertex of σ with $M = F(x)$ maximal among the vertices, and fix a representative vector $\vec{x}$. We define the map π on vertices of $\mathrm{Link}_{B_n^m(\mathcal{O})}(\sigma)$ as follows. For $v \in \mathrm{Link}_{B_n^m(\mathcal{O})}(\sigma)$, choose a representative $\vec{v} \in \mathcal{O}^{n+m}$, and let $q_v \in \mathcal{O}$ be a quotient of $f(\vec{v})$ on division by $f(\vec{x})$, in the sense of the Euclidean algorithm. If $F(\vec{v}) < F(\vec{x})$ we take $q_v = 0$. Then by construction

$$0 \leq F(\vec{v} - q_v\vec{x}) < F(\vec{x}) = M.$$

We can thus define $\pi(v)$ to be the line spanned by $(\vec{v} - q_v\vec{x})$. It is straightforward to verify that this map on vertices extends over simplices in $\mathrm{Link}_{B_n^m(\mathcal{O})}(\sigma)$, and fixes simplices in $\mathrm{Link}^{<}_{B_n^m(\mathcal{O})}(\sigma)$. ∎

3.2 New connectivity results

By modifying the proof of Church–Putman [8, Theorem C′], we will prove the following.

Theorem 3.16. For $\mathcal{O}_K$ the Gaussian integers or Eisenstein integers, the simplicial complexes $BA_n^m(\mathcal{O}_K)$ are Cohen–Macaulay of dimension n for all n and m satisfying $n \geq 1$ and $n + m \geq 2$.

Recall $BA_n(\mathcal{O}_K) = BA_n^0(\mathcal{O}_K)$ so the above theorem implies $BA_n(\mathcal{O}_K)$ is spherical.

3.2.1 *Low-dimensional cases*

Before we can prove that $BA_n^m(\mathcal{O}_K)$ is Cohen–Macaulay for general n and m, we will first need to study small values. The argument will be by induction, and the following will be the base case.

Lemma 3.17. Let $m \geq 1$. Then $BA_1^m(\mathcal{O})$ is connected if and only if $\mathcal{O}$ is additively generated by multiplicative units.

Proof. An $(m+1)$-simplex in $BA_{m+1}(\mathcal{O})$ containing $e_1, e_2, \ldots, e_m$ is an augmented frame for $\mathcal{O}^{m+1}$ of the form

$$\{e_1, e_2, \ldots, e_m, x, y\}$$

where $\{\vec{e}_1, \ldots, \vec{e}_m, \vec{x}\}$ is a basis for $\mathcal{O}^{m+1}$, and $\vec{y} = u_1\vec{x} + u_2\vec{e}_j$ or $\vec{y} = u_1\vec{e}_i + u_2\vec{e}_j$ for some i, j and some units $u_1, u_2 \in \mathcal{O}$. By definition $BA_1^m(\mathcal{O})$ is the subgraph of the edges $\{x, y\}$ on those lines x, y described above that are not contained in $\mathcal{O}^m$.

In particular, the vertices of this graph are the spans of vectors of the form $(x_1, \ldots, x_m, 1)$ with $x_i \in \mathcal{O}$; these vertices may be uniquely represented by a vector with $(m+1)$st coordinate equal to 1, and the other coordinates may be any values in $\mathcal{O}^m$. There is an edge from $(x_1, \ldots, x_m, 1)$ to $(y_1, \ldots, y_m, 1)$ if and only if there is an i such that $x_j = y_j$ for all $j \neq i$ and $x_i - y_i$ is a unit. See Figure 13. There is therefore a path from $(x_1, \ldots, x_m, 1)$ to $(y_1, \ldots, y_m, 1)$ if and only if each value $x_i - y_i$ is a sum of units. The claim follows. ∎

As illustrated in Figure 13, if $\mathcal{O}$ is additively generated by units, then $BA_1^m(\mathcal{O})$ is the Cayley graph for the additive group $\mathcal{O}^m$ associated to the generating set $\{u\vec{e}_i \mid u \text{ a unit}, i = 1, \ldots, \text{m}\}$.

Proposition 3.18. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Then $BA_2(\mathcal{O}_K)$ is Cohen–Macaulay of dimension 2.

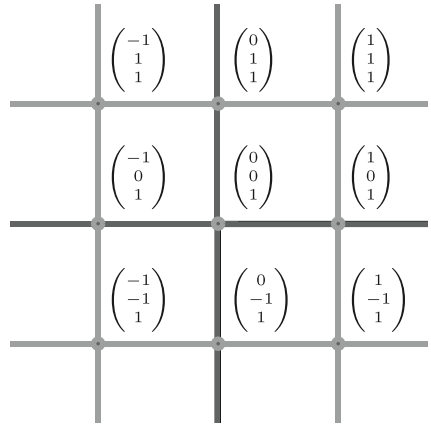


Fig. 13. Part of the complex $BA_1^2(\mathbb{Z})$.

Proof. For $BA_2(\mathcal{O}_K)$ to be Cohen–Macaulay of dimension 2, no condition is imposed on the link of 2-simplices. The link of a 1-simplex $\{v_1, v_2\}$ always contains the vertex $\operatorname{span}(\vec{v}_1 + \vec{v}_2)$, so is (-1)-connected. The link of a vertex v is isomorphic to $BA_1^1(\mathcal{O}_K)$, which is 0-connected by Lemma 3.17 since $\mathcal{O}_K$ is additively generated by units. Thus it remains to show that $BA_2(\mathcal{O}_K)$ is 1-connected.

First observe that $BA_2(\mathcal{O}_K)$ is 0-connected since its 1-skeleton is $B_2(\mathcal{O}_K)$, which is connected by Theorem 3.7. Now suppose $\phi\colon S^1 \to BA_2(\mathcal{O}_K)$ is a simplicial map with respect to some simplicial structure on S^1. Our goal is to show that ϕ is homotopic to a constant map.

Let

$$M := \max_{\text{vertices}} \max_{x \in S^1} \{F(\phi(x))\}.$$

Here F is as in Definition 3.13. If $M = 0$, then ϕ is the constant map at e_1. Hence assume that $M > 0$ and that ϕ is not constant. We will prove that ϕ can be homotoped to a map $\hat{\phi}$ having one less vertex mapping to a line w with $F(w) = M$. As it is not constant, by collapsing edges, we may assume that ϕ is simplex-wise injective. Pick a vertex $x \in S^1$ with $F(\phi(x)) = M$. Let x_1 and x_2 be the vertices adjacent to x, and let $\phi(x) = w$ and $\phi(x_i) = v_i$ denote the images in $BA_2(\mathcal{O}_K)$.

We will homotope ϕ to make the vertices adjacent to x have last coordinate $< M$, if they do not already. So suppose without loss of generality that $F(v_1) = M$. By Lemma 2.2, we can find a unit u with $F(\vec{v}_1 - u\vec{w}) < M$. Since we assume that ϕ is simplex-wise

injective, $v_1 \neq w$ and hence $\vec{v}_1 - u\vec{w} \neq 0$. Let v_1' denote the span of $\vec{v}_1 - u\vec{w}$ and note that $\{v_1, v_1', w\}$ forms a simplex in $BA_2(\mathcal{O}_K)$. Thus, there is a homotopy H_t with:

- $H_0 = \phi$,
- $H_t(y) = \phi(y)$ for $y \in S^1$ not in the interior of $\{x_1, x\}$,
- H_1 is a simplicial map from S^1 with the edge $\{x_1, x\}$ subdivided once and with middle vertex being mapped to v_1'.

Since $F(v_1') < M$, we see that ϕ is homotopic to a simplicial map where one of the vertices adjacent to x has last coordinate smaller than M. If necessary, we can also apply this procedure to v_2.

Thus, we may assume that the images of vertices adjacent to x are $v_1, v_2 \in \mathrm{Link}^{<}_{BA_2(\mathcal{O}_K)}(w)$. Pick representatives $\vec{v}_1$ and $\vec{w}$. The vertices of

$$\mathrm{Link}^{<}_{BA_2(\mathcal{O}_K)}(w)$$

are precisely the spans of vectors of the form $\vec{v}_1 + a\vec{w}$ with $a \in \mathcal{O}_K$ such that $F(\vec{v}_1 + a\vec{w}) < F(w) = M$, equivalently, such that

$$\left| \frac{f(\vec{v})}{f(\vec{w})} + a \right| < 1.$$

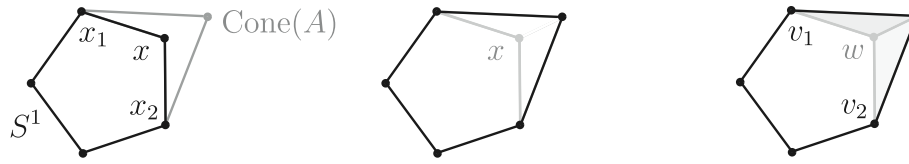
There is an edge between the span of $\vec{v}_1 + a\vec{w}$ and the span of $\vec{v}_1 + b\vec{w}$ if and only if $a - b$ is a unit. Thus, $\mathrm{Link}^{<}_{BA_2(\mathcal{O}_K)}(w)$ is isomorphic to the graph G_z of Lemma 2.4 for $z = -f(\vec{v})/f(\vec{w})$. Hence, that lemma implies it is connected. Let $A \cong S^0$ denote the set containing x_1 and x_2. Since the target is connected, the map

$$\phi|_A \longrightarrow \mathrm{Link}^{<}_{BA_2(\mathcal{O}_K)}(w)$$

is null-homotopic. Let

$$g\colon \mathrm{Cone}(A) \longrightarrow \mathrm{Link}^{<}_{BA_2(\mathcal{O}_K)}(w)$$

be a null-homotopy from a choice of simplicial complex structure on $\mathrm{Cone}(A)$; note that such a simplicial structure is just a subdivision of an interval. See Figure 14a.



(A) The cone on A (B) The complex $Z \simeq S^1$ (C) The image of $\hat{\phi}$

Fig. 14. Sample illustrations of the homotopy between ϕ and $\hat{\phi}$. (A) The cone on A. (B) The complex $Z \simeq S^1$. (C) The image of $\hat{\phi}$.

Let Z be S^1 with simplicial structure given by replacing $\{x_1, x\} \cup \{x, x_2\}$ with Cone(A), as in Figure 14b. Let $\hat{\phi}\colon Z \to BA_2(\mathcal{O}_K)$ be given by the formula:

$$\hat{\phi}(y) = \begin{cases} \phi(y) & \text{if} \quad y \in Z \setminus \text{Cone}(A), \\ g(y) & \text{if} \quad y \in \text{Cone}(A). \end{cases}$$

Observe that ϕ maps $\{x_1, x\} \cup \{x, x_2\}$ into $\text{Star}_{BA_2(\mathcal{O}_K)}(w)$, $\hat{\phi}$ maps Cone(A) into $\text{Star}_{BA_2(\mathcal{O}_K)}(w)$, and both maps agree on A. See Figure 14c. Thus, ϕ and $\hat{\phi}$ are homotopic. The new map $\hat{\phi}$ maps one fewer vertex to a vertex realizing the value M. Iterating this procedure produces a map with image contained in the star of e_2. ■

The following lemma is well known. See Section 4.1 for a review of some notation related to posets. When we refer to the connectivity of a poset we mean the connectivity of the geometric realization of its nerve, and similarly for maps between posets.

Lemma 3.19. Let $p\colon X \to Y$ be a map of simplicial complexes with Y Cohen–Macaulay of dimension n. Suppose for each simplex σ of Y the inverse image $p^{-1}(\sigma)$ is $(\dim(\sigma)-1)$-connected. Then X is $(n-1)$-connected.

Proof. Let $\mathsf{simp}(X)$ be the poset of simplices of X, ordered by inclusion, and similarly for $\mathsf{simp}(Y)$. Then p induces a functor $\mathsf{simp}(p)\colon \mathsf{simp}(X) \to \mathsf{simp}(Y)$ and there are homeomorphisms making the following diagram commute:

$$\begin{array}{ccc} |\mathsf{simp}(X)| & \xrightarrow{\cong} & X \\ \downarrow{\scriptstyle |\mathsf{simp}(p)|} & & \downarrow{\scriptstyle p} \\ |\mathsf{simp}(Y)| & \xrightarrow{\cong} & Y. \end{array}$$

These homeomorphisms are just the standard homeomorphisms between a simplicial complex and its barycentric subdivision. Thus it suffices to prove that $|\mathsf{simp}(p)|$ is n-connected. We now apply [31, Corollary 2.2] (also see [13, Theorem 4.1]): a map

$$f\colon \mathbf{X} \longrightarrow \mathbf{Y}$$

of posets is n-connected if there is a function $t\colon \mathbf{Y} \to \mathbb{Z}$ so that for all $y \in \mathbf{Y}$ we have

- $\mathbf{Y}_{>y} \coloneqq \{y' \in \mathbf{Y} \mid y' > y\}$ is $(n - t(y) - 2)$-connected,
- $f/y \coloneqq \{x \in \mathbf{X} \mid f(x) \leq y\}$ is $(t(y) - 1)$-connected.

Here we take n as in the hypothesis, and $t(\sigma) = \dim(\sigma)$. Then $\mathsf{simp}(p)/\sigma$ is $\mathsf{simp}(p^{-1}(\sigma))$, which by assumption is $(\dim(\sigma) - 1)$-connected. Similarly, $\mathsf{simp}(Y)_{>\sigma}$ is $\mathsf{simp}(\mathrm{Link}_Y(\sigma))$, which is $(n - \dim(\sigma) - 2)$-connected because Y is Cohen–Macaulay of dimension n. ■

Proposition 3.20. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Let w be a line in $\mathcal{O}_K^3$ with $F(w) > 0$. Then $\mathrm{Link}^{<}_{BA_3(\mathcal{O}_K)}(w)$ is 1-connected.

Proof. Let w_1, w_2 be lines with $\{w_1, w_2, w\}$ a simplex in $B_3(\mathcal{O}_K)$. Let $L\colon \mathcal{O}^3 \to w_1 \oplus w_2$ be given by

$$L(a\vec{w}_1 + b\vec{w}_2 + c\vec{w}) = a\vec{w}_1 + b\vec{w}_2.$$

If $a\vec{w}_1 + b\vec{w}_2 + c\vec{w}$ spans a line in the link of w, then $a\vec{w}_1 + b\vec{w}_2$ spans a line that only depends on the line spanned by $a\vec{w}_1 + b\vec{w}_2 + c\vec{w}$. We will show that L induces a simplicial map

$$p\colon \mathrm{Link}^{<}_{BA_3(\mathcal{O}_K)}(w) \longrightarrow BA(w_1 \oplus w_2)$$

given by sending the span of $a\vec{w}_1 + b\vec{w}_2 + c\vec{w}$ to the span of $a\vec{w}_1 + b\vec{w}_2$.

Claim: p is a simplicial map

Since the span of $a\vec{w}_1 + b\vec{w}_2 + c\vec{w}$ is a vertex of $\mathrm{Link}_{BA_3(\mathcal{O}_K)}(w)$, the span of $a\vec{w}_1 + b\vec{w}_2 + c\vec{w}$ is not w and so $a\vec{w}_1 + b\vec{w}_2$ is nonzero. In fact, $a\vec{w}_1 + b\vec{w}_2$ spans a line. Thus the formula for p produces a map on sets of vertices.

Now we check that it extends to a simplicial map, starting with 2-simplices. Let

$$\{v_0, v_1, v_2\} \in \mathrm{Link}^{<}_{BA_3(\mathcal{O}_K)}(w)$$

be an internally additive 2-simplex in the sense of Definition 3.12. Reorder and pick representatives so that $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$. Since $\{v_1, v_2\}$ is an edge of $\mathrm{Link}_{BA_3(\mathcal{O}_K)}(w)$ and

$\{v_0, v_1, v_2\}$ is internally additive, w, v_1, v_2 are a partial frame and thus $p(v_1), p(v_2)$ is also a partial frame. Since L is linear, $L(\vec{v}_0) = L(\vec{v}_1) + L(\vec{v}_2)$. Thus, $\{p(v_0), p(v_1), p(v_2)\}$ forms a simplex in $BA(w_1 \oplus w_2)$.

Next, let

$$\{v_0', v_1', v_2'\} \in \mathrm{Link}^{<}_{BA_3(\mathcal{O}_K)}(w)$$

be an externally additive 2-simplex. Pick representatives and reindex so that $\vec{v}_0' = \vec{v}_1' + \vec{w}$. Then $p(\vec{v}_0') = p(\vec{v}_1')$, and p maps $\{v_0', v_1', v_2'\}$ to $\{p(v_1'), p(v_2')\}$, which forms a 1-simplex in $BA(w_1 \oplus w_2)$. Since every simplex of $\mathrm{Link}^{<}_{BA_3(\mathcal{O}_K)}(w)$ is contained in one of these two types of simplices, we have checked that p is a simplicial map.

Proposition 3.18 implies that $BA(w_1 \oplus w_2)$ is Cohen–Macaulay of dimension 2. Thus, to apply Lemma 3.19 to p with $n = 2$, it suffices to show the fiber over a simplex $\sigma \subseteq BA(w_1 \oplus w_2)$ is $(\dim(\sigma) - 1)$-connected.

Claim: If $\dim(\sigma) = 0$, then $p^{-1}(\sigma)$ is (-1)-connected (in fact, connected)
Let $\sigma = \{v_0\}$. Fix a representative $\vec{v}_0$ spanning v_0 and $\vec{w}$ spanning w. Note that $p^{-1}(\{v_0\})$ has vertices of the form $\vec{v}_0 + a\vec{w}$ such that $a \in \mathcal{O}_K$ and $F(\vec{v}_0 + a\vec{w}) < F(w)$. There is an edge between $\vec{v}_0 + a\vec{w}$ and $\vec{v}_0 + b\vec{w}$ if and only if $a - b$ is a unit. The constraint $F(\vec{v}_0 + a\vec{w}) < F(w)$ is a constraint on a equivalent to the condition that

$$\left| a + \frac{f(\vec{v}_0)}{f(\vec{w})} \right| < 1.$$

Thus, $p^{-1}(\{v_0\})$ is isomorphic to a subgraph of the Cayley graph of $\mathcal{O}_K$ with units as generators. Specifically, $p^{-1}(\{v_0\})$ is the subgraph on those vertices contained in the open ball of radius 1 (in the complex metric) around the complex number $-f(\vec{v}_0)/f(\vec{w})$. This subgraph is 0-connected by Lemma 2.4 and hence is also (-1)-connected.

Claim: If $\dim(\sigma) = 1$, then $p^{-1}(\sigma)$ is connected (in fact, 1-connected)
Let $\sigma = \{v_0, v_1\}$. Then the vertices of $p^{-1}(\{v_0, v_1\})$ are lines spanned by vectors of the form

$$\vec{v}_0 + a\vec{w}, \quad \vec{v}_1 + b\vec{w}$$

with $a, b \in \mathcal{O}_K$ subject to the condition that $F(\vec{v}_0 + a\vec{w}), F(\vec{v}_1 + b\vec{w}) < F(w)$. The edges correspond to pairs of these vectors of the form

$$\{\vec{v}_0 + a\vec{w},\ \vec{v}_1 + b\vec{w}\}, \quad \{\vec{v}_0 + a\vec{w},\ \vec{v}_0 + (a+u)\vec{w}\}, \quad \{\vec{v}_1 + b\vec{w},\ \vec{v}_1 + (b+u)\vec{w}\}$$

with u a unit in $\mathcal{O}_K$, and the 2-simplices correspond to triples of these vectors

$$\{\vec{v}_0 + a\vec{w},\ \vec{v}_1 + b\vec{w},\ \vec{v}_0 + (a+u)\vec{w}\}, \quad \{\vec{v}_0 + a\vec{w},\ \vec{v}_1 + b\vec{w},\ \vec{v}_1 + (b+u)\vec{w}\},$$

with u a unit. Notably, $p^{-1}(\{v_0, v_1\})$ is the 2-skeleton of the join of $p^{-1}(\{v_0\})$ and $p^{-1}(\{v_1\})$. Since $p^{-1}(\{v_0\})$ and $p^{-1}(\{v_1\})$ are connected, $p^{-1}(\{v_0, v_1\})$ is 1-connected and hence 0-connected.

Claim: If $\dim(\sigma) = 2$**, then** $p^{-1}(\sigma)$ **is 1-connected**

Let $\sigma = \{v_0, v_1, v_2\}$, so σ is necessarily an augmented frame. Let

$$X = p^{-1}(\{v_0, v_1\}) \cup p^{-1}(\{v_0, v_2\}) \cup p^{-1}(\{v_1, v_2\}) \quad \subseteq \quad p^{-1}(\{v_0, v_1, v_2\}).$$

Since X contains the 1-skeleton of $p^{-1}(\{v_0, v_1, v_2\})$,

$$\pi_i(X, x_0) \longrightarrow \pi_i(p^{-1}(\{v_0, v_1, v_2\}), x_0)$$

is surjective for $i = 0, 1$ and all basepoints x_0. Observe that

$$p^{-1}(\{v_i, v_j\}) \cap p^{-1}(\{v_i, v_k\}) = p^{-1}(\{v_i\})$$

if $j \neq k$, and the inclusions of the intersection into each term is the inclusion of a subcomplex. This implies that X is connected and hence so is $p^{-1}(\{v_0, v_1, v_2\})$.

Our next goal is to pick basepoints $x_i \in p^{-1}(\{v_i\})$ such that $\{x_0, x_1, x_2\}$ forms a simplex in $p^{-1}(\{v_0, v_1, v_2\})$. Pick representatives for v_i and w such that $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$. As noted before, a representative for a line in $p^{-1}(\{v_i\})$ is a vector of the form $\vec{v}_i + r_i\vec{w}$ with $r_i \in \mathcal{O}_K$ such that

$$\left| r_i + \frac{f(\vec{v}_i)}{f(\vec{w})} \right| < 1.$$

By Lemma 2.5 and Lemma 2.6 applied to $z_1 = -\frac{f(\vec{v}_1)}{f(\vec{w})}$ and $z_2 = -\frac{f(\vec{v}_2)}{f(\vec{w})}$, we see that we can find r_0, r_1, r_2 with $r_0 = r_1 + r_2$ and with

$$x_i := \mathrm{span}(\vec{v}_i + r_i\vec{w}) \in p^{-1}(\{v_i\}) \quad \text{for } i = 0, 1 \quad \text{and } 2.$$

Since

$$(\vec{v}_0 + r_0\vec{w}) = (\vec{v}_1 + r_1\vec{w}) + (\vec{v}_2 + r_2\vec{w}),$$

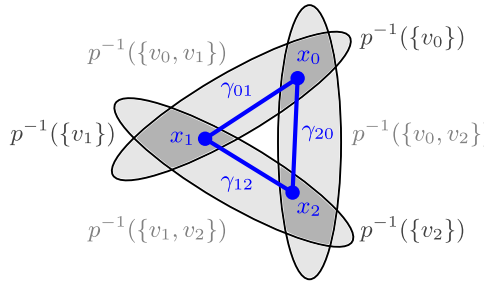


Fig. 15. A schematic of a generator of $\pi_1(X, x_0)$.

the lines $\{x_0, x_1, x_2\}$ form a simplex.

Since each $p^{-1}(\{v_i, v_j\})$ is 1-connected and each $p^{-1}(\{v_i\})$ is 0-connected, a groupoid version of the Seifert–van Kampen theorem (see e.g., May [20, Chapter 2, Section 7]) implies that $\pi_1(X, x_0)$ is generated by any loop that is a concatenation of

- a path γ_{01} in $p^{-1}(\{v_0, v_1\})$ from x_0 to x_1,
- a path γ_{12} in $p^{-1}(\{v_1, v_2\})$ from x_1 to x_2,
- a path γ_{20} in $p^{-1}(\{v_0, v_2\})$ from x_2 to x_0.

Pick each of these paths from x_i to x_j to be the edge from x_i to x_j. See Figure 15. Our specific choice of x_0, x_1, x_2 from the previous paragraph then implies that

$$\pi_1(X, x_0) \longrightarrow \pi_1(p^{-1}(\{v_0, v_1, v_2\}, x_0)$$

is the zero map since $\{x_0, x_1, x_2\}$ forms a simplex. Since $\pi_1(X, x_0) \to \pi_1(p^{-1}(\{v_0, v_1, v_2\}, x_0)$ is also surjective, this implies that $\pi_1(p^{-1}(\{v_0, v_1, v_2\}), x_1)$ is trivial. Since we already showed that $p^{-1}(\{v_0, v_1, v_2\})$ is connected, this completes the argument. ■

3.2.2 *General case*

Having completed these low-dimensional cases, we proceed to prove that the complex of augmented partial bases is spherical for general n and m.

Lemma 3.21. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Let w be a line with $F(w) \neq 0$. The inclusion

$$\iota\colon \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w) \longrightarrow \widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$$

admits a (not necessarily simplicial) retraction.

Lemma 3.21 is an analogue of Church–Putman [8, Proposition 4.17]. We first define the retraction π on the vertices of $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$. Unfortunately, this map on vertices does not extend to a simplicial map. There are certain 1- and 2-simplices σ, the *carrying* simplices, for which the images of the vertices do not span a simplex in $\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$, although every face of σ does map to a simplex. As in Church–Putman, we remedy this problem by changing the simplicial structure on $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$. Specifically, we subdivide each carrying simplex σ—leaving its boundary unaltered—and correspondingly subdivide each simplex that has σ as a face. We may then define the map π on the new simplicial structure. Although π does not respect the simplicial structure of $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$, it does define a topological retraction. We note that to prove [8, Proposition 4.17], Church–Putman subdivided carrying triangles by inserting a single vertex. In the case of the Gaussian integers or Eisenstein integers, this is not possible. Instead, we will use a more elaborate connectivity argument.

Proof of Lemma 3.21. Recall the functions f and F of Definition 3.13. For each vertex $v \in \widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$, pick a vertex $v^\pi \in \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$ such that:

(i) There are representatives $\vec{v}$ and $\vec{w}$ and $a \in \mathcal{O}_K$ such that v^π is the span of $\vec{v} + a\vec{w}$,

(ii) $F(v^\pi) < F(w)$,

(iii) $v^\pi = v$ if $F(v) < F(w)$.

Such an assignment exists because of the Euclidean algorithm. Specifically, we choose a so that $f(\vec{v} + a\vec{w})$ is a least residue of $f(\vec{v})$ modulo $f(\vec{w})$.

If the vectors $\vec{w}, \vec{v}_0, \vec{v}_1, \vec{v}_2, \ldots$ form a partial basis, then so too will the vectors

$$\vec{w},\ \ \vec{v}_0^{\,\pi} = \vec{v}_0 + a_0\vec{w},\ \ \vec{v}_1^{\,\pi} = \vec{v}_1 + a_1\vec{w},\ \ \vec{v}_2^{\,\pi} = \vec{v}_2 + a_2\vec{w}, \ldots$$

Unfortunately, if a triple of vectors $\vec{v}_0, \vec{v}_1, \vec{v}_2$ satisfies $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$, there is no reason that the same linear relation will hold amongst representatives of their images $\vec{v}_0^{\,\pi}, \vec{v}_1^{\,\pi}, \vec{v}_2^{\,\pi}$. (Lemma 2.5 and Lemma 2.6 imply that, for a fixed such triple $\vec{v}_0, \vec{v}_1, \vec{v}_2$, we could choose least residues to arrange that $\vec{v}_0^{\,\pi} = \vec{v}_1^{\,\pi} + \vec{v}_2^{\,\pi}$. However, there is no way to choose an image v^π for each line v to preserve all such linear relations simultaneously.) Similarly, if $\vec{v}_0 = \vec{v}_1 + \vec{w}$, the same relation need not hold amongst their images. Consequently, the assignment on vertices $v \mapsto v^\pi$ does not extend over simplices in $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$.

Following Church–Putman [8, Proof of Proposition 4.17], we call an internally additive 2-simplex $\sigma = \{v_0, v_1, v_2\}$ *carrying* if $\{v_0^\pi, v_1^\pi, v_2^\pi\}$ does not form a 2-simplex.

Let $\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)}(w)$ denote $\mathrm{Link}_{BA(v_0\oplus v_1\oplus w)}(w)\cap \widehat{\mathrm{Link}}^{<}_{BA^m_n(\mathcal{O}_K)}(w)$. Observe that the edges $\{v_0^\pi, v_1^\pi\}$, $\{v_1^\pi, v_2^\pi\}$, $\{v_2^\pi, v_0^\pi\}$ form simplices. Thus, the union of these three edges forms a loop in $\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)}(w)$, which we denote by γ_σ. We will use Proposition 3.20 to deduce that this loop is null homotopic in $\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)}(w)$ once we describe an isomorphism

$$\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)}(w)\cong \mathrm{Link}^{<}_{BA_3}(\tilde{w})$$

for some line $\tilde{w}\subset \mathcal{O}_K^3$. Let $g: v_0\oplus v_1\oplus w\to \mathcal{O}_K$ be the restriction of f. The image of g is a nonzero principal ideal of $\mathcal{O}_K$, say (a). Let $g': v_0\oplus v_1\oplus w\to \mathcal{O}_K$ be given by $g'(\vec{v}) = g(\vec{v})/a$. Note that $|g(\vec{v})| < |g(\vec{w})|$ if and only if $|g'(\vec{v})| < |g'(\vec{w})|$. Since g' is surjective, we may pick an isomorphism $\phi: v_0\oplus v_1\oplus w\to \mathcal{O}_K^3$, which identifies g' with projection onto the last coordinate. The isomorphism ϕ identifies $\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)}(w)$ with $\mathrm{Link}^{<}_{BA_3}(\tilde{w})$ for $\tilde{w}=\phi(w)$. Thus, γ_σ is null homotopic in $\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)}(w)$.

For each carrying internally additive 2-simplex $\sigma=\{v_0, v_1, v_2\}$, pick a simplicial map

$$H_\sigma: T(\sigma)\longrightarrow \mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)}(w)$$

from a triangulation $T(\sigma)$ of the standard 2-simplex, with γ_σ equal to the restriction of H_σ to the boundary of $T(\sigma)$. In particular, the triangulation $T(\sigma)$ does not subdivide the boundary triangle.

Similarly, call an externally additive 1-simplex $\sigma=\{v_0, v_1\}$ *carrying* if $\{v_0^\pi, v_1^\pi\}$ does not form a 1-simplex. The vertices v_0^π, v_1^π form a 0-sphere in $\mathrm{Link}^{<}_{BA(v_0\oplus w)}(w)$, which we denote by γ_σ. In the proof of Proposition 3.18, we identified $\mathrm{Link}^{<}_{BA(v_0\oplus w)}(w)$ with a graph of the form G_z. Thus, by Lemma 2.4, it is connected and so γ_σ is null homotopic. For each carrying externally additive 1-simplex $\sigma=\{v_0, v_1\}$, pick a simplicial map

$$H_\sigma: T(\sigma)\longrightarrow \mathrm{Link}^{<}_{BA(v_0\oplus w)}(w)$$

with $T(\sigma)$ a triangulation of the standard 1-simplex, and γ_σ equal to the restriction of H_σ to the boundary of $T(\sigma)$.

Call a simplex *carrying* if it is a carrying internally additive 2-simplex or a carrying externally additive 1-simplex. Let X be obtained from $\widehat{\mathrm{Link}}_{BA^m_n(\mathcal{O}_K)}(w)$ by

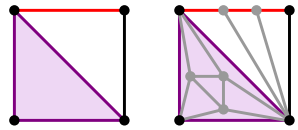


Fig. 16. Examples of a carrying 1-simplex (red) and a carrying 2-simplex (purple) before and after subdivision. Both subdivisions can be executed simultaneously.

replacing

$$\mathrm{Star}_{\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)}(\sigma) \qquad \text{with} \qquad \mathrm{Link}_{\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)}(\sigma) * T(\sigma)$$

for each carrying simplex σ. It makes sense to replace all of these simplices simultaneously because every simplex of $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$ contains at most one carrying subsimplex and because the subdivisions $T(\sigma)$ do not subdivide the boundary of σ. See Figure 16.

The space X is homeomorphic to $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$ but has extra vertices that we will use to construct our retraction.

Let vert denote the functor that sends a simplicial complex to its set of vertices. Then

$$\mathrm{vert}(X) = \mathrm{vert}\left(\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)\right) \cup \bigcup_{\sigma \text{ carrying}} \mathrm{vert}(T(\sigma)).$$

Note that this union is not a disjoint union as the vertices of γ_σ are vertices of $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$ and of $T(\sigma)$. Define

$$\pi : \mathrm{vert}(X) \longrightarrow \mathrm{vert}\left(\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)\right)$$

via the formula:

$$\pi(y) = \begin{cases} y^\pi & \text{if } \ y \ \text{ is a vertex of } \ \widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w), \\ H_\sigma(y) & \text{if } \ y \ \text{ is a vertex of } \ T(\sigma). \end{cases}$$

If y is a vertex of both $T(\sigma)$ and $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$, then $H_\sigma(y) = y^\pi$ by construction so π is a well-defined function on vertices. We now check that π induces a simplicial map. Let

$\tau = \{x_0, \ldots, x_p\} \subseteq X$ be a simplex. We will show that $\{\pi(x_0), \ldots, \pi(x_p)\}$ forms a simplex. We will consider the following cases:

Case: τ contains no interior vertices of any $T(\sigma)$

Since τ does not contain any vertices in the interior of any $T(\sigma)$, we can view $\tau = \{x_0, \ldots, x_p\}$ as a simplex in $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$. If τ is not additive, then by the definition of π its image $\pi(\tau) = \{x_0^\pi, \ldots, x_p^\pi\}$ is a non-additive simplex in $\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$.

Now assume τ is internally additive; the externally additive case is similar. Reorder and pick representatives so that $\vec{x}_0 = \vec{x}_1 + \vec{x}_2$. As in the previous paragraph, the vertices $\{x_1^\pi, \ldots, x_p^\pi\}$ span a non-additive simplex. We will now check that $\pi(\sigma) = \{x_0^\pi, x_1^\pi, \ldots, x_p^\pi\}$ forms a simplex by checking that $\{x_0^\pi, x_1^\pi, x_2^\pi\}$ is an additive simplex. Note that $\{x_0, x_1, x_2\}$ is not carrying since all of the carrying simplices have been subdivided. Thus $\{x_0^\pi, x_1^\pi, x_2^\pi\}$ forms a simplex. Observe that Condition (i) implies that the sum of the submodules $x_0^\pi, x_1^\pi, x_2^\pi$ satisfies

$$x_0^\pi + x_1^\pi + x_2^\pi + w = x_0 \oplus x_1 \oplus w.$$

Since the module $x_0^\pi + x_1^\pi + x_2^\pi + w$ is only rank 3, $\{x_0^\pi, x_1^\pi, x_2^\pi\}$ must be additive. Thus, $\pi(\sigma) = \{x_0^\pi, \ldots, x_p^\pi\}$ forms a simplex in $\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$.

Case: τ contains interior vertices of some $T(\sigma)$

Suppose that τ contains interior vertices of $T(\sigma)$ and that $\sigma = \{v_0, v_1, v_2\}$ is an internally additive carrying 2-simplex; the externally additive case is similar. Since any simplex of $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$ can contain at most one carrying subsimplex, we can decompose τ as the join $\tau = \alpha * \beta$ of simplices $\alpha, \beta \subset X$ such that

- $\alpha = T(\sigma) \cap \tau$,
- β contains no interior vertices of any $T(\sigma')$ for any carrying simplex σ'.

Note that $\pi(\alpha) = H_\sigma(\alpha)$ is a simplex of $\mathrm{Link}^{<}_{BA(v_0 \oplus v_1 \oplus w)(\mathcal{O}_K)}(w)$. Since the star of any simplex in $T(\sigma)$ is contained in

$$\mathrm{Link}_{\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)}(\sigma) * T(\sigma),$$

β is a simplex of

$$\mathrm{Link}_{\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)}(v_0, v_1, v_2) = \mathrm{Link}_{B_n^m(\mathcal{O}_K)}(v_0, v_1, w) = \mathrm{Link}_{B_n^m(\mathcal{O}_K)}(v_0^\pi, v_1^\pi, w).$$

There is a natural inclusion

$$\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)(\mathcal{O}_K)}(w) * \mathrm{Link}^{<}_{B_n^m(\mathcal{O}_K)}(v_0^\pi, v_1^\pi, w) \hookrightarrow \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$$

since simplices in $\mathrm{Link}^{<}_{BA(v_0\oplus v_1\oplus w)(\mathcal{O}_K)}(w)$ all arise from (possibly augmented) partial frames for $v_0 \oplus v_1 \oplus w = v_0^\pi \oplus v_1^\pi \oplus w$, whereas simplices in $\mathrm{Link}^{<}_{B_n^m(\mathcal{O}_K)}(v_0^\pi, v_1^\pi, w)$ all arise from non-augmented partial frames for a direct complement of $v_0^\pi \oplus v_1^\pi \oplus w$ in $\mathcal{O}_K^{n+m}$. In particular, $\pi(\sigma) = \pi(\alpha) * \pi(\beta)$ forms a simplex in $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$.

Having checked that π gives a simplicial map $X \to \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$, we use this to construct the desired retraction. Let Δ^d denote the standard d-simplex. Pick for each carrying simplex σ, pick a homeomorphism $h_\sigma : \Delta^{\dim(\sigma)} \to T(\sigma)$ that is simplicial on the boundary. They induce homeomorphisms

$$\mathrm{id} * h_\sigma : \mathrm{Star}_{\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)}(\sigma) \xrightarrow{\cong} \mathrm{Link}_{\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)}(\sigma) * T(\sigma)$$

and assemble to give a homeomorphism

$$h : \widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w) \longrightarrow X.$$

The map h is not simplicial. However, since simplices in $\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$ cannot be carrying by Condition(iii), the composition

$$\pi \circ h \circ \iota : \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w) \longrightarrow \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$$

is simplicial. Thus we can check that it is the identity by checking it is the identity on vertices. This follows from Condition(iii). Thus

$$\pi \circ h : \widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w) \longrightarrow \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$$

is a retraction of $\iota : \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w) \longrightarrow \widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$. ∎

Lemma 3.21 has the following corollary.

Corollary 3.22. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Let w be a line in $BA_m^n(\mathcal{O}_K)$ with $F(w) \neq 0$. If $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(w)$ is d-connected, so is $\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(w)$.

The following is a direct adaptation of Church–Putman [8, Proposition 4.14].

Lemma 3.23. Let $\mathcal{O}$ be a Euclidean domain. Let $n \geq 1$ and $m \geq 0$ such that $m+n \geq 2$. Assume that $BA_{n'}^{m'}(\mathcal{O})$ is Cohen–Macaulay of dimension n' for all $1 \leq n' < n$ and all $m' \geq 0$ satisfying $m'+n' = m+n$. Then for every p-simplex σ of $BA_n^m(\mathcal{O})$, the link $\mathrm{Link}_{BA_n^m(\mathcal{O})}(\sigma)$ is Cohen–Macaulay of dimension $(n-p-1)$.

Proof. The proof of Church–Putman [8, Proposition 4.14] applies without modification; we summarize it briefly. If a p-simplex σ is additive, then $\mathrm{Link}_{BA_n^m(\mathcal{O})}(\sigma) \cong B_{n-p}^{m+p}(\mathcal{O})$ is Cohen–Macaulay of dimension $(n-p-1)$ by Theorem 3.7.

Next, suppose we have a non-additive $(n-1)$-simplex $\sigma = \{v_1, \ldots, v_n\}$. Then $\mathrm{Link}_{BA_n^m(\mathcal{O})}(\sigma)$ contains the vertex corresponding to $\vec{v}_0 = \vec{e}_1 + \vec{v}_1$ (if $m \geq 1$) or $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$ (if $m = 0$), so $\mathrm{Link}_{BA_n^m(\mathcal{O})}(\sigma)$ is non-empty.

Finally, suppose we have a non-additive p-simplex σ with $p < n-1$. Then $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O})}(\sigma) \cong BA_{n-p-1}^{m+p+1}(\mathcal{O})$ is Cohen–Macaulay of dimension $(n-p-1)$ by assumption. Each vertex

$$v \in \mathrm{Link}_{BA_n^m(\mathcal{O})}(\sigma) \setminus \widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O})}(\sigma)$$

has $\mathrm{Link}_{\mathrm{Link}_{BA_n^m(\mathcal{O})}(\sigma)}(v) \cong B_{n-p-1}^{m+p+1}(\mathcal{O})$ contained in $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O})}(\sigma)$, and so the addition of each such vertex v has the effect of coning off a subcomplex of $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O})}(\sigma)$ that is Cohen–Macaulay of dimension $(n-p-2)$. The result follows by Church–Putman [8, Lemma 4.13]. ■

We now prove Theorem 3.16, which states that $BA_n^m(\mathcal{O}_K)$ is Cohen–Macaulay of dimension n for $\mathcal{O}_K$ the Gaussian integers or Eisenstein integers. Other than the proof of [8, Proposition 4.17], the proof of Church–Putman [8, Theorem C′] goes through without modification for all rings that are additively generated by multiplicative units and have a multiplicative Euclidean function with the property that if $|a| = |b| > 0$, then there is a unit u with $|a-ub| < |a|$. In place of [8, Proposition 4.17], we instead invoke our Corollary 3.22. That the ring is Euclidean is used throughout the proof [8, Theorem C′]; that it is additively generated by multiplicative units is used in the base case of the induction [8, page 21] and appears here as Lemma 3.17.

Recall that a combinatorial i-sphere is a simplicial complex that is PL-homeomorphic to the boundary of an $(i+1)$-simplex and a combinatorial i-disk is a simplicial complex that is PL-homeomorphic to an i-simplex. Moreover, links of p-simplices in the interior of combinatorial i-spheres and combinatorial i-disks

are combinatorial $(i-p-1)$-spheres. Given a simplicial complex X, the simplicial approximation theorem implies that we can represent every homotopy class of maps $S^i \to X$ by a simplicial map from a combinatorial i-sphere to X. It also implies that every null-homotopic simplicial map from a combinatorial i-sphere to X can be extended to a simplicial map from a combinatorial $(i+1)$-disk to X. Replacing the star of a p-simplex in a combinatorial i-sphere by a different $(i-p)$-disk (with the same combinatorial $(i-p-1)$-sphere as boundary) results in a combinatorial i-sphere again. We will use this fact while construct homotopies in the following proof. For a detailed introduction to the topic we refer the reader to Rourke–Sanderson [27].

Proof of Theorem 3.16. We summarize Church–Putman [8,Proof of Theorem C′]. The proof proceeds by induction on n and m. The base case is that $BA_1^m(\mathcal{O}_K)$ is connected for all $m \geq 1$, which was proven in Lemma 3.17. Now let $n \geq 1$ and $m \geq 0$ such that $m+n \geq 2$, and assume that $BA_{n'}^{m'}(\mathcal{O}_K)$ is Cohen–Macaulay of dimension n' for all $1 \leq n' < n$ with $2 \leq n'+m' \leq n+m$.

The links of simplices in $BA_n^m(\mathcal{O}_K)$ are appropriately highly connected by Lemma 3.23, so it suffices to prove that $BA_n^m(\mathcal{O}_K)$ is $(n-1)$-connected.

Let $\phi\colon S^i \to BA_n^m(\mathcal{O}_K)$ be a simplicial map from a combinatorial i-sphere for $i \leq n-1$. Let

$$M(\phi) := \max_{\text{vertices}} \sum_{x \in S^i} F(\phi(x))$$

where F is defined in Definition 3.13; the function M quantifies the "badness" of the map ϕ. Our goal is to homotope ϕ to reduce M. Then we can inductively homotope the map ϕ to a map ϕ' for which $\phi'(x)$ has $(m+n)^{th}$ coordinate zero for every vertex $x \in S^i$. The image of ϕ' is in the star of the vertex e_{m+n}, and so it can then be homotoped to the constant map at e_{m+n}.

Assume $M(\phi) = M > 0$. Following Church–Putman, we proceed in four steps. In the 1st step, we homotope ϕ so that for every simplex $\sigma \in S^i$ mapping to an additive simplex of $BA_n^m(\mathcal{O}_K)$ satisfies $F(\phi(x)) < M$ for all vertices $x \in \sigma$. We must achieve this homotopy without increasing $M(\phi)$. Choose σ of maximal dimension q among those simplices in S^i satisfying the following properties $(*)$:

- $\phi(\sigma) = \{v_0, \ldots, v_p\}$ is additive, say, $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$ for some generators of v_0, v_1, v_2,
- $F(\phi(x)) = M$ for some vertex $x \in \sigma$, and
- $F(\phi(x)) = M$ for every vertex $x \in \sigma$ with $\phi(x) \in \{v_3, \ldots, v_p\}$.

Note that $q \geq p$. By assumption of maximality,

$$\phi(\mathrm{Link}_{S^i}(\sigma)) \subseteq \mathrm{Link}^{<}_{BA^m_n(\mathcal{O}_K)}(\{v_0, v_1, \ldots, v_p\}) = \mathrm{Link}^{<}_{B^m_n(\mathcal{O}_K)}(\{v_1, \ldots, v_p\}).$$

For the last equality we have used the assumption that, possibly after re-indexing v_0, v_1, v_2, $F(v_i) = M$ for some $i = 1, \ldots, p$. The complex $\mathrm{Link}^{<}_{B^m_n(\mathcal{O}_K)}(\{v_1, \ldots, v_p\})$ is $(n-p-2)$-connected by Lemma 3.15. Thus, the restriction

$$S^{i-q-1} \cong \mathrm{Link}_{S^i}(\sigma) \longrightarrow \mathrm{Link}^{<}_{B^m_n(\mathcal{O}_K)}(\{v_1, \ldots, v_p\})$$

is null homotopic as $i-q-1 \leq n-p-2$. This implies there is a combinatorial $(i-q)$-disk D whose boundary is isomorphic to the combinatorial $(i-q-1)$-sphere $\mathrm{Link}_{S^i}(\sigma)$ and a map

$$g\colon D \longrightarrow \mathrm{Link}^{<}_{B^m_n(\mathcal{O}_K)}(\{v_1, \ldots, v_p\})$$

extending $\phi|_{\mathrm{Link}_{S^i}(\sigma)}$. Because g maps to the link of $\phi(\sigma)$, we may define the join of the maps

$$(\phi|_\sigma * g)\colon (\sigma * D) \to BA^m_n.$$

Let Z be the combinatorial i-sphere given by replacing $\mathrm{Star}_{S^i}(\sigma)$ in S^i with $D * \partial\sigma$. Let $\hat{\phi}\colon Z \to BA^m_n(\mathcal{O}_K)$ be given by the formula

$$\hat{\phi}(y) = \begin{cases} \phi(y) & \text{if} \quad y \in Z \setminus (\partial\sigma * D), \\ (\phi|_\sigma * g)(y) & \text{if} \quad y \in \partial\sigma * D. \end{cases}$$

Note that this map is continuous as ϕ and $(\phi|_\sigma * g)$ coincide on $\partial\sigma * \partial D \cong \partial\mathrm{Star}_{S^i}(\sigma)$. Moreover, observe that ϕ and $\hat{\phi}$ are homotopic through $\phi|_\sigma * g$ (extended by the constant homotopy outside of the star of σ). The new map $\hat{\phi}$ has one fewer maximal simplices satisfying $(*)$. A similar argument applies to externally additive simplices. Iterating this procedure produces the desired map.

In the 2nd step, Church–Putman homotope ϕ so that if vertices $x_1, x_2 \in S^i$ satisfy $\phi(x_1) = \phi(x_2) = v$ with $F(v) = M$, then x_1, x_2 are not joined by an edge. This new map must not increase $M(\phi)$ and must retain the properties achieved in Step 1. Choose a simplex σ of maximal dimension with the properties that $\phi|_\sigma$ is not injective and $F(\phi(x)) = M$ for every vertex $x \in \sigma$. Again let $p = \dim(\phi(\sigma))$. Then, using the properties achieved in Step 1, $\mathrm{Link}_{S^i}(\sigma)$ must map to the subcomplex

$$\mathrm{Link}^{<}_{B^m_n(\mathcal{O}_K)}(\phi(\sigma)) \subseteq \mathrm{Link}^{<}_{BA^m_n(\mathcal{O}_K)}(\phi(\sigma)).$$

This subcomplex is $(n-(p-1)-2)$–connected by Lemma 3.15, so again we can homotope ϕ to remove the simplex $\phi(\sigma)$ while preserving our desired properties.

In the 3rd step, Church–Putman further homotope the map ϕ so that it retains the properties from Steps 1 and 2 and has the additional property that, whenever vertices $x_1, x_2 \in S^i$ satisfy

$$F(\phi(x_1)) = F(\phi(x_2)) = M,$$

then x_1 and x_2 are not connected by an edge. Suppose $\{x_1, x_2\}$ is an edge violating this condition, with $\phi(x_1) = v_1$ and $\phi(x_2) = v_2$. Pick representatives $\vec{v}_1$ and $\vec{v}_2$. By Lemma 2.2, there is a unit u with $F(\vec{v}_1 - u\vec{v}_2) < F(\vec{v}_1)$. Let $v_0 = \mathrm{span}(\vec{v}_1 - u\vec{v}_2)$. By the property ensured in Step 2, $v_1 \neq v_2$. Thus $\vec{v}_1 - u\vec{v}_2 \neq \vec{0}$ and so v_0 is a line. Given the property ensured in Step 1, the image of $\mathrm{Link}_{S^i}(\{x_1, x_2\})$ is contained in $\mathrm{Link}_{BA_n^m(\mathcal{O}_K)}(\{v_0, v_1, v_2\})$. We can therefore homotope the map ϕ to map $\{x_1, x_2\}$ to the concatenation of $\{v_1, v_0\}$ with $\{v_0, v_2\}$. In particular, we replace the simplicial structure on

$$\mathrm{Star}_{S^i}(\{x_1, x_2\}) = \{x_1, x_2\} * \mathrm{Link}_{S^i}(\{x_1, x_2\})$$

with the join of the barycentric subdivision of the edge $\{x_1, x_2\}$ and $\mathrm{Link}_{S^i}(\{x_1, x_2\})$. This procedure removes the edge $\{v_1, v_2\}$ from the image of the map and preserves the properties from the previous steps.

In the final step, Church–Putman homotope the map ϕ to reduce $M(\phi)$. Let $x \in S^i$ be a vertex such that $\phi(x) = v$ with $F(v) = M$. The properties established in the previous steps ensure that

$$\phi(\mathrm{Link}_{S^i}(x)) \subseteq \widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(v).$$

The complex $\widehat{\mathrm{Link}}_{BA_n^m(\mathcal{O}_K)}(v) \cong BA_{n-1}^{m+1}(\mathcal{O}_K)$ is $(n-2)$–connected by inductive hypothesis, thus so is $\widehat{\mathrm{Link}}^{<}_{BA_n^m(\mathcal{O}_K)}(v)$ by Corollary 3.22. We can therefore homotope ϕ to remove v from its image while preserving the properties from previous steps. Iterating this final step will reduce $M(\phi)$ and complete the proof. ■

3.3 New non-connectivity results

In this subsection, we show that $BA_n(\mathcal{O})$ may not always be Cohen–Macaulay but is always highly connected. We begin with a general lemma about links in simplicial complexes.

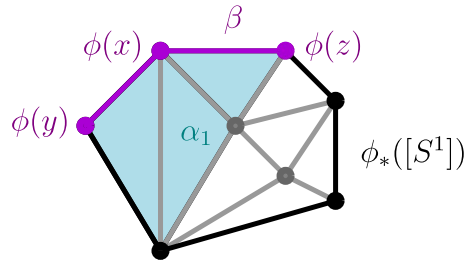


Fig. 17. An illustrative example of the chain α.

Lemma 3.24. Let X be a simplicial complex and fix a simplicial structure on S^1. Let x be a vertex of S^1 and let $y, z \in \mathrm{Link}_{S^1}(x)$. Let $\phi\colon S^1 \to X$ be a simplicial map such that no vertex other than x maps to $\phi(x)$. If $\phi_*([S^1]) = 0$ in $H_1(X)$, then $[\phi(y)] = [\phi(z)]$ in $\pi_0(\mathrm{Link}_X(\phi(x)))$.

Proof. Let C_* denote cellular chains. Suppose that $\phi_*([S^1]) = 0$, and let $\alpha \in C_2(X)$ be a chain such that

$$\partial(\alpha) = \phi_*([S^1]) \in C_1(X).$$

The chain α can be written as $\alpha_1 + \alpha_2$, with α_1 a sum of 2-simplices that have $\phi(x)$ as a vertex and α_2 a sum of 2-simplices that do not have $\phi(x)$ as a vertex. Let $\beta \in C_1(X)$ denote the chain associated to the simplicial path with vertices $\phi(y)$, $\phi(x)$, and $\phi(z)$. An instance of this complex is shown in Figure 17.

Each 1-simplex in the boundary of a 2-simplex appearing in α_1 either contains the vertex $\phi(x)$ or is contained in $\mathrm{Link}_X(\phi(x))$. By assumption on ϕ, the boundary $\partial(\alpha_1)$ must only pass through the vertex $\phi(x)$ once, and so it must be a sum of β and terms in the link of $\phi(x)$. Hence

$$\partial(\alpha_1) = \beta \in C_1(X, \mathrm{Link}_X(\phi(x))).$$

Thus, $[\beta]$ vanishes in $H_1(X, \mathrm{Link}_X(\phi(x)))$. This implies that its image $\delta([\beta])$ vanishes in $H_0(\mathrm{Link}_X(\phi(x)))$ under the connecting homomorphism δ in the long exact sequence of the pair $(X, \mathrm{Link}_X(\phi(x)))$. Since

$$\delta([\beta]) = [\phi(y)] - [\phi(z)],$$

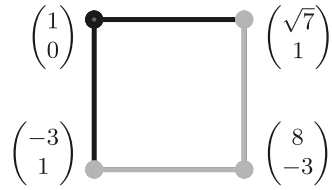


Fig. 18. A loop in $BA_2(\mathbb{Z}[\sqrt{7}])$ coming from a detour (marked in gray).

this implies that $\phi(y)$ and $\phi(z)$ are in the same path component. ■

Definition 3.25. We say that a ring $\mathcal{O}$ *has detours* if there are $r_1, r_2 \in \mathcal{O}$ such that

(i) $r_1 - r_2$ is not a sum of units.
(ii) There is a simplicial path in $B_2(\mathcal{O})$ from $\mathrm{span}\binom{r_1}{1}$ to $\mathrm{span}\binom{r_2}{1}$ that avoids $\mathrm{span}\binom{1}{0}$.

An example of a detour is given in Figure 18 for $\mathcal{O}_K = \mathbb{Z}[\sqrt{7}]$. Each vertex is labelled by a vector spanning the corresponding line.

Proposition 3.26. If $\mathcal{O}$ has detours, then $H_1(BA_2(\mathcal{O})) \neq 0$.

Proof. Let $v_1 = \mathrm{span}\binom{r_1}{1}$ and $v_2 = \mathrm{span}\binom{r_2}{1}$. We saw in the proof of Lemma 3.17 that two vertices $\mathrm{span}(x_1, \dots, x_m, 1)$ and $\mathrm{span}(y_1, \dots, y_m, 1)$ are in the same path component of $BA_1^m(\mathcal{O})$ if and only if $x_i - y_i$ is a sum of units for each i. It follows that v_1 and v_2 are not in the same path component of

$$BA_1^1(\mathcal{O}) = \widehat{\mathrm{Link}}_{BA_2(\mathcal{O})}(e_1) = \mathrm{Link}_{BA_2(\mathcal{O})}(e_1).$$

Consider the loop that is a concatenation of a detour from v_1 to v_2 with the path given by the three vertices v_1, e_1, and v_2. This loop is not zero in $H_1(BA_2(\mathcal{O}))$ by Lemma 3.24. ■

Before we give examples of rings with detours, we need the following lemma.

Lemma 3.27. The full subcomplex of $B_2(\mathbb{Z})$ minus the vertex e_1 is connected.

Proof. Let $v_1, v_2 \neq e_1$ be vertices in $B_2(\mathbb{Z})$. Since $B_2(\mathbb{Z})$ is connected [8, Theorem 4.2], we can find a simplicial path γ from v_1 to v_2. Suppose the path contains e_1. By removing

loops, we may assume that γ only passes through e_1 once. Let w_1 and w_2 be the vertices adjacent to e_1 in the path. Since $\mathrm{Link}_{BA_2(\mathbb{Z})}(e_1)$ is connected [8, Theorem C], we can find a path γ' from w_1 to w_2 in the link. Note that $\mathrm{Link}_{BA_2(\mathbb{Z})}(e_1) \subset B_2(\mathbb{Z})$. Let γ'' be γ with $\{w_1, e_1\} \cup \{e_1, w_2\}$ replaced with γ'. Observe γ'' gives a path from v_1 to v_2 that avoids e_1. ■

We now show that Euclidean quadratic number rings not generated by units have detours.

Proposition 3.28. Let $\mathcal{O}_K$ be the ring of integers in $K = \mathbb{Q}(\sqrt{d})$ for d squarefree. Assume that $\mathcal{O}_K$ is not generated by units but is Euclidean. Then $\mathcal{O}_K$ has detours.

Proof. Let $\delta = \sqrt{d}$ for $d \neq 1 \pmod 4$ and $\delta = \frac{1+\sqrt{d}}{2}$ for $d = 1 \pmod 4$ so that $\mathcal{O}_K = \mathbb{Z}[\delta]$. Since $\mathcal{O}_K$ is not generated by units, δ is not a sum of units. Thus, it suffices to find a path from $\binom{\delta}{1}$ to $\binom{0}{1}$ that avoids $\binom{1}{0}$.

We first consider the case $d > 0$. By Dirichlet's unit theorem, $\mathcal{O}_K$ has infinitely many units so there is a unit of the form $a + b\delta$ with $b \neq 0$, $a, b \in \mathbb{Z}$. Note that the lines spanned by the vectors

$$\left\{\binom{\delta}{1}, \binom{a}{-b}\right\}$$

form an edge in $B_2(\mathcal{O}_K)$. By Lemma 3.27, there is a path in $B_2(\mathbb{Z}) \subseteq B_2(\mathcal{O}_K)$ from $\binom{a}{-b}$ to $\binom{0}{1}$ that avoids $\binom{1}{0}$. The concatenation of this path with the previous edge is a detour.

Now assume $d < 0$. The only Euclidean quadratic imaginary number rings have $d = -1, -2, -3, -7$, and -11. For $d = -1$ and $d = -3$, these are generated by units while the other three rings are not generated by units (see e.g., Ashrafi–Vámos [2, Theorem 7]). Note that the units in the case $d = -2, -7$, or -11 are just ± 1 and so $(a + b\delta) - (c + d\delta)$ is not a sum of units whenever $b \neq d$. Unlike for real quadratic number rings where we had a conceptual construction of detours, in the imaginary case, will just exhibit an explicit detour for each ring.

Case: $d = -2$. The path with vertices spanned by the following vectors is a detour:

$$\binom{\delta}{1}, \binom{1}{-\delta}, \binom{0}{1}.$$

Case: $d = -7$**.** The path with vertices spanned by the following vectors is a detour:

$$\begin{pmatrix}\delta\\1\end{pmatrix},\begin{pmatrix}3-\delta\\-\delta\end{pmatrix},\begin{pmatrix}-1+2\delta\\1\end{pmatrix}.$$

Case: $d = -11$**.** The path with vertices spanned by the following vectors is a detour:

$$\begin{pmatrix}\delta\\1\end{pmatrix},\begin{pmatrix}2\\1-\delta\end{pmatrix},\begin{pmatrix}\delta\\2\end{pmatrix},\begin{pmatrix}1\\1-\delta\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}.$$

■

Remark 3.29. The quadratic norm-Euclidean number rings have been completely classified. They are the rings of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{d})$ with

$$d \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\};$$

see for example, Stark [29, Theorem 8.21]. Ashrafi–Vámos [2, Theorem 7] completely characterized which quadratic number rings are generated by units. When $d \not\equiv 1 \pmod 4$ and $d > 0$, $\mathcal{O}_K$ is generated by units if and only if d can be written as $d = a^2 \pm 1$ for some $a \in \mathbb{Z}$. When $d \equiv 1 \pmod 4$ and $d > 0$, the ring is generated by units if and only if d can be written as $d = a^2 \pm 4$ for some $a \in \mathbb{Z}$. For $d < 0$, the ring is generated by units if and only if $d \in \{-3, -1\}$. Thus, the norm-Euclidean number rings that are not generated by units are the rings of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{d})$ with

$$d \in \{-11, -7, -2, 6, 7, 11, 17, 19, 33, 37, 41, 57, 73\}.$$

On the other hand, there are Euclidean (but not norm-Euclidean) quadratic number rings that are not generated by units, such as the ring of integers in $\mathbb{Q}(\sqrt{69})$. Our results apply equally well to these rings.

We have just shown that it is not true that $BA_n(\mathcal{O})$ is spherical for all Euclidean domain. However, it is always highly connected.

Proposition 3.30. Let $\mathcal{O}$ be a Euclidean domain. Then $BA_n^m(\mathcal{O})$ is $(n-2)$-connected.

Proof. Since Theorem 3.7 says that $B_n^m(\mathcal{O})$ is $(n-2)$-connected, it suffices to show that $B_n^m(\mathcal{O}) \to BA_n^m(\mathcal{O})$ induces a surjection on free homotopy classes $[S^i, -]$ for $i \leq n-2$. We will in fact show that it is a surjection for $i \leq n-1$. Fix $i \leq n-1$ and let $\phi\colon S^i \to BA_n^m(\mathcal{O})$

be a simplicial map with respect to a combinatorial triangulation of S^i. Our goal is to show that ϕ is homotopic to a map to $B_n^m(\mathcal{O})$. Suppose that $\{v_0, v_1, v_2\}$ is an internally additive simplex in the image of ϕ. Then the link

$$\mathrm{Link}_{BA_n^m(\mathcal{O})}(\{v_0, v_1, v_2\}) \cong \mathrm{Link}_{B_n^m(\mathcal{O})}(\{v_1, v_2\}).$$

is $(n-4)$-connected by Theorem 3.7. As in the proof of [22, Lemma 2.49] and the 1st step of the proof of Theorem 3.16, we can homotope the map ϕ to avoid the simplex $\{v_0, v_1, v_2\}$ without introducing new additive simplices to its image. Iterating this procedure, and the analogous procedure for externally additive simplices, produces a map to $B_n^m(\mathcal{O})$. ■

4 Presentations of Steinberg Modules and Vanishing of Cohomology

In this section, we use our (non-)connectivity results to deduce the main theorems of the paper. We begin with a review of a useful tool: the spectral sequence associated to a map of posets, originally due to Quillen [26].

4.1 The map of posets spectral sequence

Let $\mathbf{Y}$ be a poset. Associated to $\mathbf{Y}$ is the simplicial complex $\Delta(\mathbf{Y})$ of non-degenerate simplices in the nerve of $\mathbf{Y}$. A p-simplex of $\Delta(\mathbf{Y})$ corresponds to a $(p+1)$-chain $y_0 < y_1 < \cdots < y_p$ of elements in $\mathbf{Y}$. The *dimension* of $\mathbf{Y}$ is defined to be the dimension of $\Delta(\mathbf{Y})$, and we let $|\mathbf{Y}|$ denote the geometric realization of $\Delta(\mathbf{Y})$. We note that, if Y is a simplicial complex and $\mathbf{Y}$ the corresponding poset of simplices under inclusion, then $\Delta(\mathbf{Y})$ is the barycentric subdivision of Y, and there is a homeomorphism $Y \cong |\mathbf{Y}|$.

For an element $y \in \mathbf{Y}$, recall we defined the subposets,

$$\mathbf{Y}_{\leq y} \coloneqq \{y' \in \mathbf{Y} \mid y' \leq y\} \qquad \text{and} \qquad \mathbf{Y}_{>y} \coloneqq \{y' \in \mathbf{Y} \mid y' > y\}.$$

Definition 4.1. Let $\mathbf{Y}$ be a poset. Let T be a functor from $\mathbf{Y}$ (viewed as a category) to the category $\underline{\mathrm{Ab}}$ of abelian groups. Define chain groups

$$C_p(\mathbf{Y}; T) \coloneqq \bigoplus_{y_0 < \cdots < y_p \in \mathbf{Y}} T(y_0)$$

with a differential $\sum_{i=0}^{p}(-1)^i d_i$, with the face maps d_i given by

$$d_i\colon \bigoplus_{y_0<\cdots<y_p} T(y_0) \longrightarrow \bigoplus_{y_0<\cdots<\hat{y}_i<\cdots<y_p} T(y_0) \qquad (1 \le i \le p)$$

$$d_0\colon \bigoplus_{y_0<\cdots<y_p} T(y_0) \longrightarrow \bigoplus_{y_1<\cdots<y_p} T(y_1),$$

defined as follows. For $i \neq 0$, the map d_i maps the summand indexed by $(y_0 < \cdots < y_p)$ to the summand indexed by $(y_0 < \cdots < \hat{y}_i < \cdots < y_p)$ and acts by the identity on the group $T(y_0)$. The map d_0 maps the summand indexed by $(y_0 < \cdots < y_p)$ to the summand indexed by $(y_1 < \cdots < y_p)$, and the map of abelian groups $T(y_0) \to T(y_1)$ is defined by applying T to the morphism $y_0 < y_1$ in $\mathbf{Y}$.

If $T = \mathbb{Z}$ is the constant functor with identity maps, then $H_*(\mathbf{Y};\mathbb{Z})$ is isomorphic to the usual homology $H_*(|\mathbf{Y}|)$. The following lemma is adapted from Charney [7, Lemma 1.3]. See also [23, Lemma 3.2]. Recall that the *height* of $y \in Y$ is by definition $\dim(\Delta(\mathbf{Y}_{\le y}))$.

Lemma 4.2. Suppose that $T\colon \mathbf{Y} \to \underline{\mathrm{Ab}}$ is a functor that is nonzero only on elements of height m. Then

$$H_p(\mathbf{Y};T) = \bigoplus_{\mathrm{height}(y_0)=m} \widetilde{H}_{p-1}(|\mathbf{Y}_{>y_0}|;T(y_0)).$$

Definition 4.3. Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a map of posets. For $y \in \mathbf{Y}$, define $f\backslash y$ to be the subposet of $\mathbf{X}$

$$f\backslash y \coloneqq \{x \in \mathbf{X} \mid f(x) \le y\}.$$

Consider a map of posets $f\colon \mathbf{X} \to \mathbf{Y}$, and fix a degree $q \in \mathbb{Z}_{\ge 0}$. Then there is a functor from the poset $\mathbf{Y}$ to $\underline{\mathrm{Ab}}$ that takes an object $y \in \mathbf{Y}$ to the abelian group $H_q(f\backslash y)$. With this functor, we may state the following theorem. The spectral sequence associated to a map $f\colon \mathbf{X} \to \mathbf{Y}$ of posets was introduced by Quillen [26, Section 7]; see also Charney [7, Section 1].

Theorem 4.4 (Quillen [26]). Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a map of posets. There is a strongly convergent spectral sequence

$$E^2_{p,q} = H_p\Big(\mathbf{Y};[y \mapsto H_q(f\backslash y)]\Big) \implies H_{p+q}(\mathbf{X}).$$

4.2 Generalized Bykovskiĭ presentations for the Gaussian integers and Eisenstein integers

In this subsection, we let $\mathcal{O}_K$ denote Gaussian integers or the Eisenstein integers and K its field of fractions. Our objective is to prove Theorem A and Theorem B. Recall that Theorem B is the statement that $\mathrm{Byk}_n(\mathcal{O}_K) \to \mathrm{St}_n(K)$ is an isomorphism for all n. We can deduce Theorem B from Theorem 3.16 using the same arguments that Church–Putman use to deduce [8, Theorem B] from [8, Theorem C]. We recall these arguments in the three lemmas below and the proof of Theorem B.

We make the following definition, as in Church–Putman [8, Proof of Theorem B].

Definition 4.5. For a Euclidean ring $\mathcal{O}$, we let $BA_n(\mathcal{O})'$ denote the subcomplex of $BA_n(\mathcal{O})$ consisting of simplices $\{v_0, v_1, \dots, v_p\}$ with $v_0 + v_1 + \cdots + v_p \subsetneq \mathcal{O}^n$.

Let sd denote the barycentric subdivision. This subcomplex is defined to give a map

$$\begin{aligned} \mathrm{span}\colon \mathrm{sd}(BA_n(\mathcal{O})') &\longrightarrow \mathcal{T}_n(K) \\ \{v_0, v_1, \dots, v_p\} &\longmapsto Kv_0 + Kv_1 + \cdots + Kv_p. \end{aligned}$$

This arises from a map of posets with domain $\mathsf{simp}(BA_n(\mathcal{O})')$ and target the poset defining the Tits building.

Lemma 4.6 (Following [8, Theorem B Step 3]). Let $\mathcal{O}_K$ be the Gaussian integers or Eisenstein integers. The map $\mathrm{span}\colon \mathrm{sd}(BA_n(\mathcal{O}_K)') \to \mathcal{T}_n(K)$ induces an isomorphism of $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O}_K)]$-modules $\widetilde{H}_{n-2}(BA_n(\mathcal{O}_K)') \xrightarrow{\cong} \widetilde{H}_{n-2}(\mathcal{T}_n(K))$.

We could prove this by quoting Church–Putman [8, Proposition 2.3] but will instead prove it using the spectral sequence of Theorem 4.4, as a warm-up for our proof of Theorem C.

Proof. When $n = 1$, both $BA_n(\mathcal{O}_K)'$ and $\mathcal{T}_n(K)$ are empty, so we may assume $n \geq 2$. We consider the spectral sequence of Theorem 4.4 associated to the functor span. Observe that, given a proper nonzero subspace $V \subsetneq K^n$,

$$\mathrm{span}\backslash V = \left\{ \{v_0, v_1, \dots, v_p\} \in \mathsf{simp}(BA_n(\mathcal{O}_K)') \;\middle|\; Kv_0 + Kv_1 + \cdots + Kv_p \subseteq V \right\} \cong BA(V \cap \mathcal{O}_K^n).$$

By Theorem 3.16 the complex $BA(V \cap \mathcal{O}_K^n)$ is Cohen–Macaulay of dimension $\dim(V)$ (for $\dim(V) \geq 2$) or dimension 0 (when $\dim(V) = 1$), so $H_q(\mathrm{span}\backslash V) = 0$ except possibly when

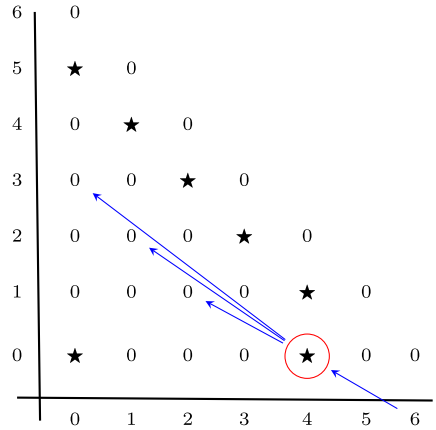


Fig. 19. The page $E^2_{p,q}$ when $n = 6$. There are no nonzero differentials to or from the term $E^r_{n-2,0}$ for $r \geq 2$.

$q = 0$ or $q = \dim(V)$. We can identify $\mathcal{T}_n(K)_{>V} \cong \mathcal{T}(K^n/V)$. Thus for $q > 0$, we find by Lemma 4.2 that

$$E^2_{p,q} \cong H_p\Big(\mathcal{T}_n(K); [V \mapsto H_q(\mathrm{span}\backslash V)]\Big)$$
$$\cong \bigoplus_{V \subseteq K^n,\, \dim(V)=q} \widetilde{H}_{p-1}\Big(\mathcal{T}(K^n/V); H_{\dim(V)}(BA(V \cap \mathcal{O}_K^n))\Big).$$

The building $\mathcal{T}(K^n/V)$ is spherical of dimension $\dim(K^n/V) - 2$, and so for $q > 0$, we conclude that $E^2_{p,q}$ vanishes unless $p-1 = n-\dim(V)-2$, equivalently, unless $p+q = n-1$. When $q = 0$,

$$E^2_{p,q} \cong H_p(\mathcal{T}_n(K)) = 0 \qquad \text{except when} \quad p = 0 \quad \text{or} \quad p = n-2.$$

The spectral sequence (see Figure 19) converges to $H_{p+q}(BA_n(\mathcal{O}_K)')$. The only nonzero E^2 term on the diagonal $p + q = n - 2$ is the term $E^2_{n-2,0} \cong H_{n-2}(\mathcal{T}_n(K))$, and this term admits no non-zero incoming or outgoing higher differentials. This gives the desired isomorphism. ■

Lemma 4.7 (Following [8, Theorem B Step 1]). Let $\mathcal{O}$ be an integral domain, then there is an isomorphism of $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O})]$-modules

$$\mathrm{Byk}_n(\mathcal{O}) \xrightarrow{\cong} H_{n-1}(BA_n(\mathcal{O}), BA_n(\mathcal{O})').$$

Proof. By construction, $BA_n(\mathcal{O})'$ contains all simplices of $BA_n(\mathcal{O})$ except for the n-simplices corresponding to augmented n-frames, and the $(n-1)$-simplices corresponding to non-augmented n-frames. From the exact sequence

$$C_n(BA_n(\mathcal{O}), BA_n(\mathcal{O})') \xrightarrow{\delta} C_{n-1}(BA_n(\mathcal{O}), BA_n(\mathcal{O})') \longrightarrow H_{n-1}(BA_n(\mathcal{O}), BA_n(\mathcal{O})') \longrightarrow 0$$

we see that $H_{n-1}(BA_n(\mathcal{O}), BA_n(\mathcal{O})')$ is the group generated by the simplices

$$\{\{v_1, \dots, v_n\} \mid v_1 \oplus \cdots \oplus v_n = \mathcal{O}^n\}$$

modulo relations of the form

$$\begin{aligned}&\delta\big(\{v_0, v_1, v_2, v_3 \dots, v_{n-1}\}\big) \qquad \text{with} \quad \vec{v_0} = \vec{v_1} + \vec{v_2} \\ &= \{v_1, v_2, v_3 \dots, v_{n-1}\} - \{v_0, v_2, v_3 \dots, v_{n-1}\} + \{v_0, v_1, v_3 \dots, v_{n-1}\} - 0 + 0 - \cdots \pm 0.\end{aligned}$$

This is precisely the presentation defining the group $\mathrm{Byk}_n(\mathcal{O})$. ∎

Lemma 4.8 (Following [8, Theorem B Step 2]). Let $\mathcal{O}_K$ be the Gaussian integers or Eisenstein integers. There is an isomorphism of $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O}_K)]$-modules

$$H_{n-1}(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)') \xrightarrow{\cong} \widetilde{H}_{n-2}(BA_n(\mathcal{O}_K)').$$

Proof. By Theorem 3.16, $BA_n(\mathcal{O}_K)$ is $(n-1)$–connected. Thus an isomorphism is given by the connecting homomorphism in the long exact sequence of the pair $(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)')$. ∎

Recall that Theorem B says that when $\mathcal{O}_K$ is the Gaussian integers or Eisenstein integers, the generalized Bykovskiĭ presentation holds.

Proof of Theorem B. Consider the maps

$$\mathrm{Byk}_n(\mathcal{O}_K) \underset{(*)}{\xrightarrow{\cong}} H_{n-1}(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)') \underset{(**)}{\xrightarrow{\cong}} \widetilde{H}_{n-2}(BA_n(\mathcal{O}_K)') \underset{(***)}{\xrightarrow{\cong}} \widetilde{H}_{n-2}(\mathcal{T}_n(K)),$$

the rightmost group being $\mathrm{St}_n(K)$ by definition. The map $(*)$ is an isomorphism by Lemma 4.7, the map $(**)$ is an isomorphism by Lemma 4.8, and the map $(***)$ is an isomorphism by Lemma 4.6. Thus the composite is an isomorphism of $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O}_K)]$-

modules. As in the proof of Church–Putman [8, Theorem B], this composite is the map described in Section 1.2. ■

Corollary 4.9. Let $\mathcal{O}_K$ be the Gaussian integers or Eisenstein integers, and K its field of fractions. Then $\mathrm{St}_n(K)$ admits a partial resolution by $\mathbb{Z}[\mathrm{GL}_n(\mathcal{O}_K)]$-modules

$$C_n(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)') \xrightarrow{\delta} C_{n-1}(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)') \longrightarrow \mathrm{St}_n(K) \longrightarrow 0.$$

Proof. By Theorem B, there is an isomorphism $\mathrm{Byk}_n(\mathcal{O}_K) \cong \mathrm{St}_n(K)$, and so the result follows from the proof of Lemma 4.7. ■

We will now use our resolution to show vanishing for group homology with coefficient in the Steinberg module.

Theorem 4.10. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers and let $\Bbbk$ be a ring with $n!$ and 3 invertible. Then

$$H_1(\mathrm{GL}_n(\mathcal{O}_K); \mathrm{St}_n(K) \otimes \Bbbk) = 0 \qquad \text{for } n \geq 2,$$
$$H_1(\mathrm{SL}_n(\mathcal{O}_K); \mathrm{St}_n(K) \otimes \Bbbk) = 0 \qquad \text{for } n \geq 3.$$

Proof. By Corollary 4.9 we have a partial resolution

$$C_n(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)'; \Bbbk) \xrightarrow{\partial} C_{n-1}(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)'; \Bbbk) \longrightarrow \mathrm{St}_n(K) \otimes \Bbbk \to 0$$

of $\mathrm{St}_n(K) \otimes \Bbbk$ by $\Bbbk[\mathrm{GL}_n(\mathcal{O}_K)]$-modules. These modules are flat, for example, by Church–Putman [8, Lemma 3.2]; its proof only requires that the orders of the stabilizers of simplices in $BA_n(\mathcal{O}_K)$ which are not in $BA_n(\mathcal{O}_K)'$, are invertible in the coefficients. There are two cases:

- If $\sigma = \{v_1, \dots, v_n\}$ is a basis of $\mathcal{O}_K^n$, it has stabilizer of order $|\mathcal{O}_K^\times|^n \cdot n!$.
- If $\sigma = \{v_0, \dots, v_n\}$ is an additive simplex spanning $\mathcal{O}_K^n$, it has stabilizer of order $6 \cdot |\mathcal{O}_K^\times|^{n-1} \cdot (n-2)!$.

Thus when G_n is $\mathrm{SL}_n(\mathcal{O}_K)$ or $\mathrm{GL}_n(\mathcal{O}_K)$, we may compute $H_*(G_n; \mathrm{St}_n(K) \otimes \Bbbk)$ by extending this partial resolution to a flat resolution, and taking the homology of the chain complex

$$\cdots \longrightarrow C_n(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)'; \Bbbk)_{G_n} \longrightarrow C_{n-1}(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)'; \Bbbk)_{G_n} \longrightarrow 0$$

obtained by passing to G_n-coinvariants. To show that $H_1(G_n; \mathrm{St}_n(K) \otimes \Bbbk) = 0$, then, it suffices to show that the coinvariants $C_n(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)'; \Bbbk)_{G_n}$ vanish.

Fix $n \geq 2$. The free $\Bbbk$-module $C_n(BA_n(\mathcal{O}_K), BA_n(\mathcal{O}_K)'; \Bbbk) = C_n(BA_n(\mathcal{O}_K); \Bbbk)$ is spanned by augmented frames $\{v_0, \ldots, v_n\}$ subject to the relation

$$\{v_0, \ldots, v_n\} = \mathrm{sgn}(\sigma)\{v_{\sigma(0)}, \ldots, v_{\sigma(n)}\},$$

with σ a permutation $\{1, \ldots, n\}$ and $\mathrm{sgn}(\sigma)$ its sign, and with G_n-action

$$g\{v_0, \ldots, v_n\} \coloneqq \{g(v_0), \ldots, g(v_n)\}, \qquad g \in G_n.$$

Consider an augmented basis $\{v_0, \ldots, v_n\}$, reorder, and pick representatives $\vec{v}_0, \ldots, \vec{v}_n$ such that $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$. Let $h \in \mathrm{GL}_n(\mathcal{O}_K)$ be the linear map defined by

$$h(\vec{v}_1) = \vec{v}_2, \qquad h(\vec{v}_2) = \vec{v}_1, \qquad h(\vec{v}_i) = \vec{v}_i \quad \text{for } i > 2.$$

Then $h(\{v_0, v_1, v_2, v_3, \ldots, v_n\}) = \{v_0, v_2, v_1, v_3, \ldots, v_n\} = -\{v_0, v_1, v_2, v_3, \ldots, v_n\}$. Since 2 is invertible in $\Bbbk$, this implies that generators of $C_n(BA_n(\mathcal{O}_K); \Bbbk)$ map to zero in $C_n(BA_n(\mathcal{O}_K); \Bbbk)_{\mathrm{GL}_n(\mathcal{O}_K)}$ and hence

$$C_n(BA_n(\mathcal{O}_K); \Bbbk)_{\mathrm{GL}_n(\mathcal{O}_K)} = 0 \qquad \text{for } n \geq 2.$$

The element h does not have determinant 1 and this is the reason we must assume $n \geq 3$ to show the $\mathrm{SL}_n(\mathcal{O}_K)$-coinvariants vanish. For $n \geq 3$, there is a linear map $\ell \in \mathrm{SL}_n(\mathcal{O}_K)$ satisfying

$$\ell(\vec{v}_1) = \vec{v}_2, \qquad \ell(\vec{v}_2) = \vec{v}_1, \qquad \ell(\vec{v}_3) = -\vec{v}_3, \qquad \ell(\vec{v}_i) = \vec{v}_i \quad \text{for } i > 3.$$

Again ℓ negates the generator $\{v_0, \ldots, v_n\}$, and we infer that

$$C_n(BA_n(\mathcal{O}_K); \Bbbk)_{\mathrm{SL}_n(\mathcal{O}_K)} \cong 0 \qquad \text{for } n \geq 3.$$

■

The following theorem implies Theorem A.

Theorem 4.11. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Let $\Bbbk$ be a ring with $(n+1)!$ invertible if n is congruent to 1 modulo 4 and with $(2n+1)!$ invertible

otherwise. Then

$$H^{\nu_n-1}(\mathrm{GL}_n(\mathcal{O}_K);\Bbbk)=0 \qquad \text{for } n\geq 2,$$
$$H^{\nu_n-1}(\mathrm{SL}_n(\mathcal{O}_K);\Bbbk)=0 \qquad \text{for } n\geq 3.$$

Proof. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Borel–Serre duality [3] applies not just with $\mathbb{Q}$-coefficients as stated in the introduction, but in fact their work implies that duality hold with any coefficients in which all torsion primes for the group are invertible. By [10, Lemma 3.9], the torsion primes of $\mathrm{GL}_n(\mathcal{O}_K)$ for $\mathcal{O}_K$ quadratic imaginary are bounded by $n+1$ if $n\equiv 1 \pmod 4$ and $2n+1$ otherwise. Thus, $H_1(\mathrm{SL}_n(\mathcal{O}_K);\mathrm{St}_n(K)\otimes\Bbbk)\cong H^{\nu_n-1}(\mathrm{SL}_n(\mathcal{O}_K);\Bbbk)$ as the orders of torsion elements of G_n are invertible in $\Bbbk$. Similarly, because K is quadratic imaginary, Putman–Studenmund [24, Theorem C and following paragraph] implies that

$$H^{\nu_n-1}(\mathrm{GL}_n(\mathcal{O}_K);\Bbbk)\xrightarrow{\cong} H_1(\mathrm{GL}_n(\mathcal{O}_K);\mathrm{St}_n(K)\otimes\Bbbk).$$

Theorem 4.10 completes the proof. ∎

Remark 4.12. Church–Putman [8, Theorem A] also gave a vanishing result for the twisted cohomology groups $H^{\nu_n-1}(\mathrm{SL}_n(\mathbb{Z});V_\lambda)$, where V_λ is the rational representation of $\mathrm{GL}_n(\mathbb{Q})$ with highest weight λ given by the partition $\lambda=(\lambda_1\geq\ldots\geq\lambda_n)$; it vanishes for $n\geq 3+||\lambda||$ with $||\lambda||=\sum_{i=1}^n(\lambda_i-\lambda_n)$. Their arguments are easily adapted to our situation:

$$H^{\nu_n-1}(\mathrm{SL}_n(\mathcal{O}_K);V_\lambda)=H^{\nu_n-1}(\mathrm{GL}_n(\mathcal{O}_K);V_\lambda)=0 \qquad \text{for } n\geq 3+||\lambda||,$$

where V_λ is now the rational representation of $\mathrm{GL}_n(K)$ with highest weight λ.

4.3 Examples of the failure of the generalized Bykovskiĭ presentation

In this subsection, we prove Theorem C, which gives examples of rings for which the Bykovskiĭ presentation does not hold. This will follow from the following more general theorem.

Theorem 4.13. Let $\mathcal{O}$ be a Euclidean domain with field of fractions K. If $\mathcal{O}$ has detours, then the map $\mathrm{Byk}_n(\mathcal{O})\to\mathrm{St}_n(K)$ is not injective for all $n\geq 2$.

Proof. Assume $\mathcal{O}$ has detours and is Euclidean.

The case $n = 2$. Recall from Lemma 4.7 that there is an isomorphism $H_1(BA_2(\mathcal{O}), BA_2(\mathcal{O})') \cong \mathrm{Byk}_2(\mathcal{O})$ of $\mathbb{Z}[\mathrm{GL}_2(\mathcal{O})]$-modules. Consider the long exact sequence of the pair $(BA_2(\mathcal{O}), BA_2(\mathcal{O})')$,

$$\begin{aligned} &H_1(BA_2(\mathcal{O})') \longrightarrow H_1(BA_2(\mathcal{O})) \longrightarrow H_1(BA_2(\mathcal{O}), BA_2(\mathcal{O})') \\ &\hookrightarrow H_0(BA_2(\mathcal{O})') \longrightarrow H_0(BA_2(\mathcal{O})) \longrightarrow H_0(BA_2(\mathcal{O}), BA_2(\mathcal{O})'). \end{aligned}$$

Proposition 3.26 implies that the group $H_1(BA_2(\mathcal{O}))$ is nonzero, and as $H_1(BA_2(\mathcal{O})') = 0$ the connecting homomorphism ∂ has nontrivial kernel. By the proof of Theorem B, the map $\mathrm{Byk}_2(\mathcal{O}) \to \mathrm{St}_2(K)$ factors as

$$\mathrm{Byk}_2(\mathcal{O}) \xrightarrow{\cong} H_1(BA_2(\mathcal{O}), BA_2(\mathcal{O})') \xrightarrow{\partial} \widetilde{H}_0(BA_2(\mathcal{O})') \longrightarrow \widetilde{H}_0(\mathcal{T}_n(K)) \coloneqq \mathrm{St}_2(K).$$

Since ∂ is not injective, the composite $\mathrm{Byk}_2(\mathcal{O}) \to \mathrm{St}_2(K)$ cannot be an isomorphism.

The case $n \geq 3$. Recall from Definition 4.5 that $BA_n(\mathcal{O})' \subseteq BA_n(\mathcal{O})$ is the subcomplex of all simplices $\{v_0, v_1, \ldots, v_p\}$ for which $v_0 + v_1 + \cdots + v_p$ is a proper summand of $\mathcal{O}^n$. There is a map

$$\mathrm{span}\colon \mathrm{sd}(BA_n(\mathcal{O})') \to \mathcal{T}_n(K)$$

and an associated strongly convergent spectral sequence

$$E^2_{p,q} = H_p\Big(\mathcal{T}_n(K); [V \mapsto H_q(\mathrm{span}\backslash V)]\Big) \Longrightarrow H_{p+q}(BA_n(\mathcal{O})'),$$

described in Theorem 4.4. We will verify that for $n \geq 3$ its E^2-page satisfies the following:

(i) For $q = 0$, the term $E^2_{p,0} = 0$ unless $p = 0$ or $p = n - 2$.

(iii) For $q = 1$, the term $E^2_{p,1} = 0$ unless $p = n - 3$.

(iii) For $q \geq 2$, the term $E^2_{p,q} = 0$ unless $(p + q)$ is equal to $(n - 1)$ or $(n - 2)$.

(iii) See Figure 20.

(iv) $E^2_{n-2,0} \cong H_{n-2}(\mathcal{T}_n(K))$.

(v) $E^2_{n-3,1} \cong \bigoplus_{\substack{V \subseteq K^n \\ \dim(V)=2}} \widetilde{H}_{n-4}(\mathcal{T}(K^n/V); H_1(BA(V))$.

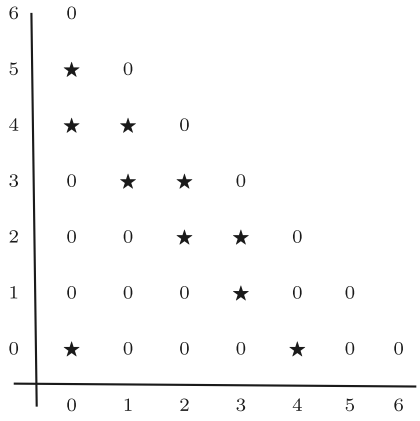


Fig. 20. The page $E^2_{p,q}$ when $n = 6$.

We first analyze the posets span$\backslash V$. For a proper nonzero subspace $V \subseteq K^n$, observe that

$$\text{span}\backslash V = \left\{ \{v_0, v_1, \ldots, v_p\} \in \mathsf{simp}(BA_n(\mathcal{O})') \;\middle|\; Kv_0 + Kv_1 + \cdots + Kv_p \subseteq V \right\} \cong BA(V \cap \mathcal{O}^n).$$

By abuse of notation, for $V \subseteq K^n$ we write $BA(V)$ to denote the complex $BA(V \cap \mathcal{O}^n)$. Proposition 3.30 states that the complex $BA(V)$ is $(\dim(V) - 2)$-connected, so

$$H_q(\text{span}\backslash V) = 0 \qquad \text{except possibly when } q = 0,\ \dim(V) - 1 \text{ or } \dim(V).$$

First consider the case $q = 0$. The complex $BA(V)$ is always connected; this follows from Proposition 3.30 when $\dim(V) \geq 2$ and because $BA(V)$ is a point when $\dim(V) = 1$. Thus when $q = 0$ we find that the functor $[V \mapsto H_0(BA(V)]$ is the trivial functor $\mathbb{Z}$. Then $E^2_{p,0} \cong H_p(\mathcal{T}_n(K))$. By the Solomon–Tits theorem (Theorem 3.2), $\mathcal{T}_n(K)$ is spherical of dimension $(n-2)$, so the term $E^2_{p,0} = 0$ unless $p = 0$ or $p = n-2$, properties (i) and (iv).

Let $q = 1$. Because $BA(V)$ is a point when $\dim(V) = 1$, the group $H_1(BA(V))$ can be nonzero only when $\dim(V) = 2$. Thus the functor $[V \mapsto H_1(\text{span}\backslash V)]$ is nonzero only on elements of a single height. Taking the quotient by V gives an isomorphism $\mathcal{T}_n(K)_{>V} \to \mathcal{T}(K^n/V)$, and so by Lemma 4.2,

$$E^2_{p,1} = \bigoplus_{\substack{V \subseteq K^n \\ \dim(V)=2}} \widetilde{H}_{p-1}(\mathcal{T}(K^n/V); H_1(BA(V)).$$

The Solomon–Tits theorem implies that $E^2_{p,1} = 0$ unless $p = n-3$, properties (ii) and (v).

Let $q \geq 2$. In order to apply Lemma 4.2 to the terms $E^2_{p,q}$, we will write the functors $[V \mapsto H_q(\mathrm{span}\backslash V) = H_q(BA(V))]$ as extensions of functors that are each nonzero only on elements of a single height. There is a short exact sequence of functors,

$$0 \longrightarrow H'_q \longrightarrow H_q(BA(-)) \longrightarrow H''_q \longrightarrow 0,$$

with functors H'_q, H''_q given by

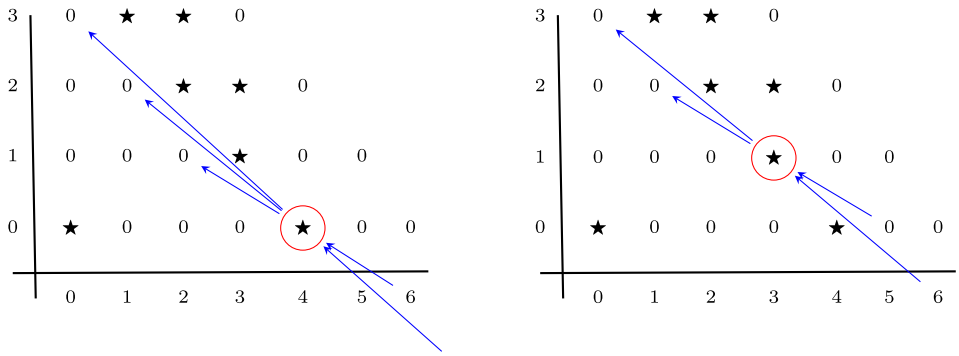
$$H'_q(V) := \begin{cases} H_q(BA(V)) & \text{if } \dim(V) = q+1, \\ 0 & \text{otherwise,} \end{cases} \qquad H''_q(V) := \begin{cases} H_q(BA(V)) & \text{if } \dim(V) = q, \\ 0 & \text{otherwise,} \end{cases}$$

and natural transformations between them given by

$$\begin{array}{ccccc} & & \dim(U) = q & & \dim(W) = q+1 \\ H'_q & & 0 & \longrightarrow & H_q(BA(W)) \\ \downarrow & & \downarrow & & \| \\ H_q(BA(-)) & & H_q(BA(U)) & \xrightarrow{(U \hookrightarrow W)_*} & H_q(BA(W)) \\ \downarrow & & \| & & \downarrow \\ H''_q & & H_q(BA(U)) & \longrightarrow & 0 \end{array}$$

We can then apply Lemma 4.2 to the terms in the associated long exact sequence on homology:

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ H_p(\mathcal{T}_n(K); H'_q) & = & \bigoplus_{\substack{W \subseteq K^n \\ \dim(W) = q+1}} \widetilde{H}_{p-1}(\mathcal{T}(K^n/W); H_q(BA(W)) \\ \downarrow & & \\ H_p\Big(\mathcal{T}_n(K), [V \mapsto H_q(BA(V))]\Big) & = & E^2_{p,q} \\ \downarrow & & \\ H_p(\mathcal{T}_n(K); H''_q) & = & \bigoplus_{\substack{U \subseteq K^n \\ \dim(U) = q}} \widetilde{H}_{p-1}(\mathcal{T}(K^n/U); H_q(BA(U)) \\ \downarrow & & \\ \vdots & & \end{array}$$



(A) The term $E^r_{n-2,0}$ (B) The term $E^r_{n-3,1}$

Fig. 21. The spectral sequence $E^r_{p,q}$ illustrated for $n = 6$. (A) The term $E^r_{n-2,0}$. (B) The term $E^r_{n-3,1}$.

The Solomon–Tits theorem now implies that for $q \geq 2$ the homology groups $E^2_{p,q}$ can be nonzero only when $p+q$ is equal to $n-1$ or $n-2$, property (iii). We have verified our description of the E^2 page, as illustrated in Figure 20.

From the structure of the E^2 page, we can deduce the terms $E^r_{n-2,0}$ and $E^r_{n-3,1}$ are not the source or target of any nonzero differentials for any $r \geq 2$. See Figure 21. It follows that $E^2_{n-2,0} \cong E^\infty_{n-2,0}$ and $E^2_{n-3,1} \cong E^\infty_{n-3,1}$. Using formal properties of the spectral sequence, we see that there is a surjection

$$H_{n-2}(BA_n(\mathcal{O})') \longrightarrow E^2_{n-2,0} \cong H_{n-2}(\mathcal{T}_n(K))$$

and the term $E^2_{n-3,1}$ is a quotient of the kernel. But

$$E^2_{n-3,1} \cong \bigoplus_{\substack{V \subseteq K^n \\ \dim(V)=2}} \widetilde{H}_{n-4}(\mathcal{T}(K^n/V); H_1(BA(V)) \cong \bigoplus_{\substack{V \subseteq K^n \\ \dim(V)=2}} \mathrm{St}(K^n/V) \otimes H_1(BA(V)).$$

Proposition 3.26 imply that the group $H_1(BA(V))$ is nonzero, and the Steinberg module $\mathrm{St}(K^n/V)$ is nonzero for $n \geq 3$, so we conclude $E^2_{n-3,1} \neq 0$ in this range.

As in the proof of Theorem B, our map $\mathrm{Byk}_n(\mathcal{O}) \to \mathrm{St}_n(K)$ factors as (since $n \geq 3$, in degree $(n-2)$ we can conflate reduced and non-reduced homology)

$$\mathrm{Byk}_n(\mathcal{O}) \xrightarrow{\cong} H_{n-1}(BA_n(\mathcal{O}), BA_n(\mathcal{O})') \xrightarrow{\partial} \widetilde{H}_{n-2}(BA_n(\mathcal{O})') \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(K)) = \mathrm{St}_n(K).$$

Since $H_{n-2}(BA_n(\mathcal{O})) = 0$, the connecting homomorphism ∂ is surjective. We have proven that the map $H_{n-2}(BA_n(\mathcal{O})') \to H_{n-2}(\mathcal{T}_n(K))$ is not injective, so $\mathrm{Byk}_n(\mathcal{O}) \to \mathrm{St}_n(K)$ is not injective. ■

We now prove Theorem C, which says the generalized Bykovskiĭ presentation does not hold for Euclidean quadratic number rings that are not generated by units.

Proof of Theorem C. Let $\mathcal{O}_K$ be a quadratic number ring that is Euclidean but is not generated by units. By Proposition 3.28, $\mathcal{O}_K$ has detours. The claim now follows from Theorem 4.13. ■

Remark 4.14. See Section 1.2 for our notation for fundamental classes of apartments. In this notation, the proof of Theorem C in conjunction with Figure 18 shows that

$$\left[\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\sqrt{7}\\1\end{pmatrix}\right]\right]+\left[\left[\begin{pmatrix}\sqrt{7}\\1\end{pmatrix},\begin{pmatrix}8\\-3\end{pmatrix}\right]\right]+\left[\left[\begin{pmatrix}8\\-3\end{pmatrix},\begin{pmatrix}-3\\1\end{pmatrix}\right]\right]+\left[\left[\begin{pmatrix}-3\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right]\right]=0$$

is a relation in $\mathrm{St}_2(\mathbb{Q}(\sqrt{7}))$ that does not follow from the generalized Bykovskiĭ relations.

5 Open Questions

We end with some open questions. All examples of Euclidean domains for which the generalized Bykovskiĭ presentation is known to hold are generated by units. Conversely, all Euclidean domains for which the generalized Bykovskiĭ presentation is known to fail are not generated by units.

Question 5.1. For $\mathcal{O}_K$ a Euclidean domain, does the generalized Bykovskiĭ presentation hold if and only if $\mathcal{O}_K$ is generated by units?

The following question asks whether all relations in $\mathrm{St}_n(K)$ come from $\mathrm{St}_2(K)$.

Question 5.2. Let $\mathrm{Ker}_n(\mathcal{O}_K)$ denote the kernel of $\mathrm{Byk}_n(\mathcal{O}_K) \to \mathrm{St}_n(K)$. For $\mathcal{O}_K$ a Euclidean domain, is there a natural surjection

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathcal{O}_K)\times\mathrm{GL}_{n-2}(\mathcal{O}_K)}^{\mathrm{GL}_n(\mathcal{O}_K)}\mathrm{Ker}_2(\mathcal{O}_K)\boxtimes\mathrm{Byk}_{n-2}(\mathcal{O}_K)\longrightarrow\mathrm{Ker}_n(\mathcal{O}_K)?$$

The group $\mathrm{Ker}_n(\mathcal{O}_K)$ measures relations in the Steinberg module beyond those appearing in the Bykovskiĭ presentation. An affirmative answer would imply that

for $n \geq 4$ and all Euclidean domains $\mathcal{O}_K$, we have $H_1(\mathrm{GL}_n(\mathcal{O}_K); \mathrm{St}_n(K) \otimes \mathbb{Q}) = H_1(\mathrm{SL}_n(\mathcal{O}_K); \mathrm{St}_n(K) \otimes \mathbb{Q}) = 0$.

Vanishing results near the virtual cohomological dimension are the subject of several conjectures by Church–Farb–Putman [5]. In particular, they conjecture ([6, Conjecture 2]) that

$$H^{\nu_n - i}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0 \qquad \text{for } i < n-1.$$

This is supported by the available computations [9, Remark 5.3], and known for $i = 0$ by Lee–Szczarba [17, Theorem 1.3] and $i = 1$ by Church–Putman [8, Theorem A]. It is natural to ask the same question for other number rings; using [17, Theorem 1.3] and Theorem A it is also true for $i = 0, 1$ when we replace $\mathbb{Z}$ by the Gaussian integers or Eisenstein integers.

Conjecture 5.3. Let $\mathcal{O}_K$ denote the Gaussian integers or Eisenstein integers. Then

$$H^{\nu_n - i}(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q}) = H^{\nu_n - i}(\mathrm{GL}_n(\mathcal{O}_K); \mathbb{Q}) = 0 \qquad \text{for all } i < n-1.$$

This is supported by the available computations Dutour Sikirić–Gangl–Gunnells–Hanke–Schürmann–Yasaki [10, Tables 11, 12] or [11, Propositions 2.6, 2.10]:

$$H^{\nu_4 - i}(\mathrm{GL}_4(\mathcal{O}_K); \mathbb{Q}) = 0 \qquad \text{for } i \leq 2 \text{ and } \mathcal{O}_K \text{ the Gaussian integers or Eisenstein integers.}$$

For general Euclidean number rings, one might expect a similar vanishing result though possibly with a worse range.

One can also ask about integral versions of our vanishing result. In [21, Theorem 1.10] it was proven that for $n \geq 6$

$$H_1(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q})) = H_1(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q})) = 0.$$

Question 5.4. Is it true that for $n \geq 6$ we have

$$H_1(\mathrm{GL}_n(\mathcal{O}_K); \mathrm{St}_n(K)) = H_1(\mathrm{SL}_n(\mathcal{O}_K); \mathrm{St}_n(K)) = 0$$

when $\mathcal{O}_K$ is the Gaussian integers or Eisenstein integers? Can the range be improved?

The Bykovskiĭ presentation is also useful for computing the homology of congruence subgroups. For an ideal $J \subset \mathcal{O}_K$, let

$$\Gamma_n(J) \coloneqq \operatorname{Ker}\Big[\mathrm{GL}_n(\mathcal{O}_K) \to \mathrm{GL}_n(\mathcal{O}_K/J)\Big].$$

We say an ideal J has *the Lee–Szczarba property* (see [17, page 28]) if the natural map

$$H^{\nu_n}(\Gamma_n(J);\mathbb{Q}) \longrightarrow H_0(\Gamma_n(J);\mathrm{St}_n(K)\otimes\mathbb{Q}) \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(K)/\Gamma_n(J);\mathbb{Q})$$

is an isomorphism. For $\mathcal{O}_K = \mathbb{Z}$, the prime ideals with the Lee–Szczarba property are $(2), (3)$, and (5); see [17, Theorem 1.2] and [22, Theorem A]. The proof relies on the Bykovskiĭ presentation.

Question 5.5. Which prime ideals in the Gaussian integers or Eisenstein integers have the Lee–Szczarba property?

Often, $\widetilde{H}_{n-2}(\mathcal{T}_n(K)/\Gamma_n(J))$ is computable (see [22, Table 1]) so an answer to this question could yield concrete calculations.

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References

[1] Ash, A. and L. Rudolph. "The modular symbol and continued fractions in higher dimensions." *Invent. Math.* 55 (1979): 241–50. MR: 553998.

[2] Ashrafi, N. and P. Vámos. "On the unit sum number of some rings." *Q. J. Math.* 56 (2005): 1–12. MR: 2124574.

[3] Borel, A. and J.-P. Serre. "Corners and arithmetic groups." *Comment. Math. Helv.* 48 (1973): 436–91. Avec un appendice: arrondissement des variétés à coins, par A. Douady et L. Hérault. MR: 0387495.

[4] Bykovskiĭ, V. A. "Generating elements of the annihilating ideal for modular symbols." *Funktsional. Anal. i Prilozhen* 37 (2003): 27–38 95. MR: 2083229.

[5] Church, T., B. Farb, and A. Putman. "A Stability Conjecture for the Unstable Cohomology of SL_nZ Mapping Class Groups, and Aut (F_n)." *Algebraic Topology: Applications and New Directions*. 55–70. Contemp. Math., vol. 620. Providence, RI: Amer. Math. Soc, 2014. MR: 3290086.

[6] Church, T., B. Farb, and A. Putman. "Integrality in the Steinberg module and the top-dimensional cohomology of SL_nO_k." *Amer. J. Math.* 141 (2019): 1375–419. MR: 4011804.

[7] Charney, R. "A Generalization of a Theorem of Vogtmann." *Proceedings of the Northwestern Conference on Cohomology of Groups (Evanston, Ill., 1985)*, vol. 44 (1987): 107–25. MR: 885099.

[8] Church, T. and A. Putman. "The codimension-one cohomology of $SL_n\mathbb{Z}$." *Geom. Topol.* 21 (2017): 999–1032. MR: 3626596.

[9] Dutour Sikirić, M., P. Elbaz-Vincent, A. Kupers, and J. Martinet. "Voronoi complexes in higher dimensions, cohomology of $GL_N(\mathbb{Z})$ for $N \geq 8$ and the triviality of $K_8(\mathbb{Z})$." (2021): preprint, https://arxiv.org/abs/1910.11598.

[10] Dutour Sikirić, M., H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki. "On the cohomology of linear groups over imaginary quadratic fields." *J. Pure Appl. Algebra* 220 (2016): 2564–89. MR: 3457984.

[11] Dutour Sikirić, M., H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki. "On the topological computation of K_4 of the Gaussian and Eisenstein integers." *J. Homotopy Relat. Struct.* 14 (2019): 281–91. MR: 3913976.

[12] Garland, H. "p-adic curvature and the cohomology of discrete subgroups of p-adic groups." *Ann. Math.* 97 (1973): 375–423. MR: 0320180.

[13] Galatius, S., A. Kupers, and O. Randal-Williams. "E_∞-cells and general linear groups of finite fields." (2018).

[14] Kupers, A. and J. Miller. "Homological stability for topological chiral homology of completions." *Adv. Math.* 292 (2016): 755–827. MR: 3464033.

[15] Kupers, A., J. Miller, and P. Patzt. "Exceptional homological stability for general linear groups of Euclidean domains." (in preparation).

[16] Lee, R. and R. H. Szczarba. "The group $K_3(Z)$ is cyclic of order forty-eight." *Ann. Math. (2)* 104 (1976): 31–60. MR: 0442934.

[17] Lee, R. and R. H. Szczarba. "On the homology and cohomology of congruence subgroups." *Invent. Math.* 33 (1976): 15–53. MR: 0422498.

[18] Lee, R. and R. H. Szczarba. "On the torsion in $K_4(Z)$ and $K_5(Z)$." *Duke Math. J.* 45 (1978): 101–29. MR: 0491893.

[19] Maazen, H. *Homology Stability for the General Linear Group*. (1979).

[20] May, J.P. *A Concise Course in Algebraic Topology*. University of Chicago Press, 1999.

[21] Miller, J., R. Nagpal, and P. Patzt. "Stability in the high-dimensional cohomology of congruence subgroups." *Compos. Math.* 156 (2020): 822–61. MR: 4079629.

[22] Miller, J., P. Patzt, and A. Putman. "On the top dimensional cohomology groups of congruence subgroups of $SL_n(\mathbb{Z})$." *Geom. Topol.* (2021): preprint, https://arxiv.org/abs/1909.02661.

[23] Miller, J., *et al.* "Non-integrality of some Steinberg modules." *Journal of Topology 13.2* (2020): 441–59.

[24] Putman, A. and D. Studenmund. "The dualizing module and top-dimensional cohomology group of GL_n O." (2021): preprint, https://arxiv.org/abs/1909.01217.

[25] Quillen, D. "Finite Generation of the Groups K_i of Rings of Algebraic Integers." *Higher K-Theories*. 179–98. Springer, 1973.

[26] Quillen, D. "Homotopy properties of the poset of nontrivial p-subgroups of a group." *Adv. Math.* 28 (1978): 101–28.

[27] Rourke, C.P. and B.J. Sanderson. *Introduction to piecewise-linear topology*. Springer Study Edition. Berlin–New York: Springer, 1982. Reprint. MR: 665919

[28] Solomon, L. "The Steinberg Character of a Finite Group With BN Pair." *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*. 213–21. New York: Benjamin, 1969. MR: 0246951.

[29] Stark, H. M. *An Introduction to Number Theory*. Chicago, IL: Markham Publishing Co., 1970. MR: 0253973

[30] van der Kallen, W. "Homology stability for linear groups." *Invent. Math.* 60 (1980): 269–95. MR: 586429.

[31] van der Kallen, W. and E. Looijenga. "Spherical complexes attached to symplectic lattices." *Geom. Dedicata* 152 (2011): 197–211. MR: 2795243.