

Homogenization of a stochastically forced Hamilton-Jacobi equation[☆]

Benjamin Seeger

Université Paris-Dauphine & Collège de France, Place du Maréchal de Lattre de Tassigny, 75016 Paris, France

Received 18 December 2019; received in revised form 17 June 2020; accepted 1 November 2020

Available online 13 November 2020

Abstract

We study the homogenization of a Hamilton-Jacobi equation forced by rapidly oscillating noise that is colored in space and white in time. It is shown that the homogenized equation is deterministic, and, in general, the noise has an enhancement effect, for which we provide a quantitative estimate. As an application, we perform a noise sensitivity analysis for Hamilton-Jacobi equations forced by a noise term with small amplitude, and identify the scaling at which the macroscopic enhancement effect is felt. The results depend on new, probabilistic estimates for the large scale Hölder regularity of the solutions, which are of independent interest.

© 2020 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

MSC: 60H15; 35B27

Keywords: Stochastic homogenization; Stochastic Hamilton-Jacobi equation; Stochastic enhancement; Eikonal equation

1. Introduction

The purpose of this paper is to study the asymptotic behavior of stochastically forced Hamilton-Jacobi equations that take the form

$$u_t^\varepsilon + H(Du^\varepsilon) = F^\varepsilon(x, t, \omega) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \quad \text{and} \quad u^\varepsilon(x, 0, \omega) = u_0(x) \quad \text{in } \mathbb{R}^d \times \Omega, \quad (1.1)$$

where the initial datum u_0 belongs to $BUC(\mathbb{R}^d)$, the space of bounded, uniformly continuous functions, and $(\Omega, \mathbf{F}, \mathbf{P})$ is a given probability space.

We assume that

$$H : \mathbb{R}^d \rightarrow \mathbb{R} \text{ is convex with superlinear growth,} \quad (1.2)$$

and the noise term F^ε , which is scaled by a small parameter $\varepsilon > 0$, is white in time and smooth in space:

[☆] Partially supported by the National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship under Grant Number DMS-1902658.

E-mail address: seeger@ceremade.dauphine.fr.

$$\left\{ \begin{array}{l} F^\varepsilon(x, t, \omega) := F\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right), \quad \text{where} \\ F(x, t, \omega) := f(x, \omega) \cdot \dot{B}(t) = \sum_{i=1}^m f^i(x, \omega) \dot{B}^i(t, \omega), \\ f = (f^1, f^2, \dots, f^m) \text{ is a smooth, stationary-ergodic random field, and} \\ B = (B^1, B^2, \dots, B^m) : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m \text{ is a Brownian motion independent of } f. \end{array} \right. \quad (1.3)$$

More precise assumptions will be given in Section 2.

1.1. The homogenization result

Our main goal is to demonstrate that, as $\varepsilon \rightarrow 0$, the limiting behavior of (1.1) is governed by a deterministic, homogenized initial value problem

$$\bar{u}_t + \bar{H}(D\bar{u}) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \bar{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d. \quad (1.4)$$

Theorem 1.1. *Assume (1.2) and (1.3). Then there exists a deterministic, convex, super-linear Hamiltonian $\bar{H} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that, for all $u_0 \in BUC(\mathbb{R}^d)$, the solution u^ε of (1.1) converges locally uniformly with probability one to the viscosity solution $\bar{u}$ of (1.4).*

1.2. The enhancement effect

The scaling properties of Brownian motion imply that, in law,

$$F^\varepsilon(x, t, \omega) \stackrel{d}{=} \varepsilon^{1/2} f\left(\frac{x}{\varepsilon}, \omega\right) \cdot \dot{B}(t, \omega),$$

and so formally, as $\varepsilon \rightarrow 0$, the right-hand side of (1.1) converges to zero. Nevertheless, although singular terms no longer appear in (1.4), it turns out that the noise has a nontrivial effect on the limiting equation.

Theorem 1.2. *In addition to the hypotheses of Theorem 1.1, assume that f is not constant on $\mathbb{R}^d$. Then*

$$\bar{H}(p) > H(p) \quad \text{for all } p \in \mathbb{R}^d. \quad (1.5)$$

The particular case of the eikonal equation

$$u_t^\varepsilon + \frac{1}{2}|Du|^\varepsilon = F^\varepsilon(x, t, \omega) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \quad \text{and} \quad u^\varepsilon(\cdot, 0, \omega) = u_0 \quad \text{in } \mathbb{R}^d \times \Omega \quad (1.6)$$

is a simplified model for turbulent combustion, in which the evolving region

$$U_t^\varepsilon := \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R} : y > u^\varepsilon(x, t) \right\}$$

and its complement represent respectively “burnt” and “unburnt” regions in a rough, dynamic environment. The noise term F^ε corresponds to random turbulence, which, according to Theorem 1.2, gives rise to an average, large-scale enhancement effect on the velocity of the interface.

We will also investigate the effect that varying the strength of the noise has on the limiting problem. More precisely, for some $\theta \in \mathbb{R}$ and for f and B as in (1.3), we study the initial value problem

$$u_t^\varepsilon + H(Du^\varepsilon) = \varepsilon^\theta f\left(\frac{x}{\varepsilon}, \omega\right) \cdot \dot{B}(t, \omega) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \quad \text{and} \quad u^\varepsilon(x, 0, \omega) = u_0 \quad \text{in } \mathbb{R}^d \times \Omega. \quad (1.7)$$

The following result explains the relationship between the size of the noise (in terms of θ) and the enhancement property.

Theorem 1.3. *Under the assumptions of Theorem 1.2, let $\theta \in \mathbb{R}$ and let u^ε be the solution of (1.7).*

(a) If $\theta > 1/2$, then, as $\varepsilon \rightarrow 0$, u^ε converges locally uniformly in probability to the solution u of

$$u_t + H(Du) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.$$

(b) If $\theta < 1/2$, then, as $\varepsilon \rightarrow 0$, u^ε converges locally uniformly in $\mathbb{R}^d \times (0, \infty)$ in probability to $-\infty$.

(c) If $\theta = 1/2$, then there exists a deterministic, convex Hamiltonian $\bar{H} : \mathbb{R}^d \rightarrow \mathbb{R}$ with $\bar{H} > H$ such that, as $\varepsilon \rightarrow 0$, u^ε converges locally uniformly in probability to the solution $\bar{u}$ of

$$\bar{u}_t + \bar{H}(D\bar{u}) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \bar{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d \times \{0\}.$$

1.3. A regularity result

The convergence results are proved by applying the sub-additive ergodic theorem [1] to particular solutions of (1.1). A crucial tool in the analysis is a Hölder regularity estimate for solutions of

$$u_t + H(Du) = \sum_{i=1}^m f^i(x, \omega) \dot{B}^i(t, \omega) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \quad (1.8)$$

that is invariant under the scaling $(x, t) \rightarrow (x/\varepsilon, t/\varepsilon)$.

If the Brownian motion B is replaced with a continuously differentiable path, with $\dot{B}(t, \omega)$ bounded uniformly in $(t, \omega) \in [0, \infty) \times \Omega$, then (1.8) is a Hamilton-Jacobi equation of the form

$$u_t + \tilde{H}(Du, x, t) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad (1.9)$$

for some $\tilde{H} \in C(\mathbb{R}^d \times \mathbb{R}^d \times [0, \infty))$. The results of [17, 18, 20, 54] imply that the Hölder semi-norm of u can be locally controlled in terms of the growth of $\|f\|_\infty$, $\|\dot{B}\|_\infty$, $\|u\|_\infty$, and the growth of H in Du . However, none of these works apply to (1.8), where the right-hand side is not only unbounded, but nowhere point-wise defined.

The transformation

$$\tilde{u}(x, t, \cdot) := u(x, t, \cdot) - \sum_{i=1}^m f^i(x, \cdot) B^i(t, \cdot)$$

leads to the equation

$$\tilde{u}_t + H(D\tilde{u} + Df(x, \omega) \cdot B(t, \omega)) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega, \quad (1.10)$$

which, for each fixed $\omega \in \Omega$, is a classical Hamilton-Jacobi equation of the form (1.9). Applying known regularity results to (1.10) then yields Hölder estimates that depend on $\|Df\|_\infty$, which presents a major obstacle to finding estimates for (1.9) that are scale-invariant.

These issues are resolved by the following result, which is of independent interest.

Theorem 1.4. Fix $M, R > 0$, and assume that H satisfies (1.2), $f \in C_b^1(\mathbb{R}^d, \mathbb{R}^m)$, and B is a standard m -dimensional Brownian motion on the probability space $(\Omega, \mathbf{F}, \mathbf{P})$. Then there exist constants $C_1 = C_1(R, M) > 0$, $\alpha, \beta > 0$, and, for all $p \geq 1$, $C_2 = C_2(R, M, p) > 0$ such that, if

$$\|f\|_\infty \cdot \|Df\|_\infty + \|f\|_\infty + \|u(\cdot, 0)\|_\infty \leq M,$$

then, for all $\lambda \geq 1$,

$$\mathbf{P} \left(\sup_{(x,s), (y,t) \in B_R \times [1/R, R]} \frac{|u(x, s) - u(y, t)|}{|x - y|^\alpha + |s - t|^\beta} > C_1 + \lambda \right) \leq \frac{C_2 \|f\|_\infty^p}{\lambda^p}.$$

For fixed $\varepsilon > 0$, the equation (1.1) can be rewritten in the form

$$u_t^\varepsilon + H(Du^\varepsilon) = f^\varepsilon(x, \omega) \cdot \dot{B}^\varepsilon(t, \omega) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega,$$

where, for $x \in \mathbb{R}^d$ and $t \in [0, \infty)$,

$$f^\varepsilon(x, \cdot) = \varepsilon^{1/2} f(x/\varepsilon, \cdot) \quad \text{and} \quad B^\varepsilon(t, \cdot) = \varepsilon^{1/2} B(t/\varepsilon, \cdot).$$

Theorem 1.4 then immediately implies that, for all $\varepsilon > 0$ and $\lambda \geq 1$,

$$\mathbf{P} \left(\sup_{(x,s),(y,t) \in B_R \times [1/R, R]} \frac{|u^\varepsilon(x, s) - u^\varepsilon(y, t)|}{|x - y|^\alpha + |s - t|^\beta} > C_1 + \lambda \right) \leq \frac{C_2 \|f\|_\infty^p \varepsilon^{p/2}}{\lambda^p},$$

where C_1, C_2, α , and β are all independent of ε .

1.4. Background

In [55,56], the author studied general asymptotic problems for equations taking the form

$$du^\varepsilon + \sum_{i=0}^m H^i(Du^\varepsilon, x/\varepsilon) \cdot d\zeta^{i,\varepsilon}(t) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),$$

where each H^i satisfies a self-averaging property in the spatial variable (for example, periodic, almost-periodic, or stationary and ergodic dependence), and, for some path $\zeta \in C([0, \infty), \mathbb{R}^{m+1})$,

$$\zeta^\varepsilon = (\zeta^{0,\varepsilon}, \zeta^{1,\varepsilon}, \zeta^{2,\varepsilon}, \dots, \zeta^{m,\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} \zeta \quad \text{locally uniformly.}$$

The limiting equations take the form

$$d\bar{u} + \sum_{j=1}^M \bar{H}^j(D\bar{u}) \cdot d\tilde{\zeta}^j = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),$$

for some deterministic, spatially homogeneous Hamiltonians $\bar{H}^j$, $j = 1, 2, \dots, M$, a path $\tilde{\zeta} \in C([0, \infty), \mathbb{R}^M)$, and some $M \in \mathbb{N}$ possibly larger than $m + 1$. The results of the present paper can be placed within this framework by setting, for $(p, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty)$, $\varepsilon > 0$, and $i = 1, 2, \dots, m$,

$$H^0(p, y) = H(p), \quad \zeta^{0,\varepsilon}(t) = \zeta^0(t) = t, \quad H^i(p, y) = f^i(y), \quad \zeta^{i,\varepsilon}(t) = \varepsilon B^i\left(\frac{t}{\varepsilon}\right), \quad \text{and} \quad \zeta^i(t) = 0.$$

In this context, the fact that the limiting equation takes the form (1.4) with $\bar{H} \neq H$ can be translated as saying that each effective Hamiltonian is determined by the entire collection $(H^i)_{i=0}^m$, a phenomenon which was seen in [56] for a different class of problems. A form of Theorem 1.3(b) was proved in [55] in the case where $\theta = 0$, with B replaced with a sufficiently mild approximation B^ε converging to a Brownian motion as $\varepsilon \rightarrow 0$.

There is a vast literature on the stochastic homogenization of Hamilton-Jacobi equations like

$$u_t^\varepsilon + H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0 \quad \text{and} \quad u_t^\varepsilon + H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad (1.11)$$

set in a stationary-ergodic environment; for qualitative and quantitative results, and many variations and extensions, see [4,6,8,9,19,21,23,30,31,37,39,48,52–54,57,60]. The results of the present paper are unique in that the problem is a stochastic partial differential equation, and therefore, the time-dependent forcing term is not only unbounded, but not well-defined point-wise anywhere.

A specific example of the equations in (1.11), and another model for turbulent combustion, is the G -equation, for which the level sets of the solutions evolve according to the normal velocity

$$1 + V\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right) \cdot n,$$

where $V : \mathbb{R}^d \times [0, \infty) \times \Omega \rightarrow \mathbb{R}^d$ is a stationary-ergodic velocity field and $n \in S^{d-1}$ is the outward unit normal vector to the interface. Under the assumption that $|\mathbf{E}V| < 1$, where $\mathbf{E}$ denotes the expectation of the random field V , the evolving region will, on average, expand. In fact, the authors in [19,21,52] demonstrate that, over a long time and large range, the velocity is actually enhanced, that is, it is given by $\bar{a}(n)$ for some deterministic $\bar{a} \in C(S^{d-1}, \mathbb{R}_+)$ satisfying

$$\bar{a}(n) \geq 1 + \mathbf{E}V \cdot n \quad \text{for all } n \in S^{d-1}.$$

Moreover, under further assumptions on V , and for “most” directions $n \in S^{d-1}$, the inequality is strict. This should be compared with Theorem 1.2, in which an analogous strict enhancement property is observed for all $p \in \mathbb{R}^d$.

The homogenization of “viscous” Hamilton-Jacobi equations, or Hamilton-Jacobi-Bellman equations, like

$$\begin{aligned} u_t^\varepsilon - \varepsilon \operatorname{tr} \left[A \left(\frac{x}{\varepsilon}, \omega \right) D^2 u^\varepsilon \right] + H \left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega \right) &= 0 \quad \text{and} \\ u_t^\varepsilon - \varepsilon \operatorname{tr} \left[A \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega \right) D^2 u^\varepsilon \right] + H \left(Du^\varepsilon, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega \right) &= 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \end{aligned} \quad (1.12)$$

where A is a symmetric, nonnegative matrix with stationary and ergodic dependence, has also been an active area of study, see for instance [2,3,5,7,22,25,29,38,41,42,49,50]. A natural next step is to study the homogenization of equations like

$$u_t^\varepsilon - \varepsilon \operatorname{tr} \left[A \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega \right) D^2 u^\varepsilon \right] + H \left(Du^\varepsilon, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega \right) = F^\varepsilon(x, t) \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad (1.13)$$

where, as in (1.3), F^ε is a stationary-ergodic forcing that is white in time.

When $H(p, x, t) = \frac{1}{2}|p|^2$, A is the identity matrix, and F^ε is a space-time white noise, (1.13) is the KPZ equation, whose well-posedness was established in the seminal works of Hairer on regularity structures [35,36] (see also the work of Gubinelli et al. [34] on paracontrolled distributions). The KPZ equation can be related, through the Hopf-Cole transform, to the multiplicative stochastic heat equation. This connection is explored in various works [27, 32,33,51] that study homogenization problems in which the noise term formally vanishes as $\varepsilon \rightarrow 0$, but still has a nontrivial, enhancement-type effect on the deterministic limit. For example, in [33], the limiting heat equation exhibits an effective diffusivity, and the stochastic heat equation that describes the fluctuations has an effective variance.

A central part of understanding the macroscopic behavior of stochastically forced Hamilton-Jacobi equations concerns the existence of global solutions of (1.1) with stationary increments. In the homogenization literature, such solutions are known as correctors, and they play an important role in obtaining error estimates and determining fine properties of the effective Hamiltonian $\bar{H}$; see [22,26,48]. Additionally, they are related to the long-time behavior and invariant measures for the random dynamical system associated to the equation, as well as the asymptotic slopes of one-sided minimizers of the corresponding Lagrangian system. In this context, the so-called “shape functional” is exactly the effective Lagrangian $\bar{L}$, which is the convex conjugate of $\bar{H}$; see Lemma 4.2 below. The work of Bakhtin and Khanin [14], which deals with exactly such questions, puts forward a conjectured overarching theory for the global behavior of stochastically forced Hamilton-Jacobi equations, and the related forced Burgers equation. These conjectures are motivated by a body of existing results, including, but not limited to, [10–13,16,28,40]. Within this framework, the homogenization result, Theorem 1.1, provides insight into the average large scale behavior of the solutions. Moreover, the enhancement property in Theorem 1.2 (see Theorem 5.1 for a more precise statement) gives a better understanding of the effective Hamiltonian and the related shape functional. We believe that the regularity result, Theorem 1.4 (see also Proposition 3.1), and the methods used to prove it, can be an important tool for other parts of this ongoing program.

1.5. Organization

In Section 2, we list the main assumptions on the data and prove some preliminary results. The properties of the random Lagrangian fields, and especially their regularity, are discussed in Section 3. The main tool in this section is a decomposition method for bounding moments of stochastic integrals that are not martingales. The proof of the homogenization result and the identification of the effective Hamiltonian appear in Section 4, and the enhancement property is proved in Section 5. Finally, the results of Theorem 1.3 are proved in Section 6.

1.6. Notation

For a domain $U \subset \mathbb{R}^M \times \mathbb{R}^N$ and $\alpha, \beta \in (0, 1)$, $C_x^\alpha C_y^\beta(U)$ is the space of functions $f = f(x, y)$, where $x \in \mathbb{R}^M$ and $y \in \mathbb{R}^N$, that are α -Hölder continuous in $\mathbb{R}^M$ and β -Hölder continuous in $\mathbb{R}^N$, with the semi-norm

$$[f]_{C_x^\alpha C_y^\beta(U)} := \sup_{(x,y),(\tilde{x},\tilde{y}) \in U} \frac{|f(x,y) - f(\tilde{x},\tilde{y})|}{|x - \tilde{x}|^\alpha + |y - \tilde{y}|^\beta}.$$

For $k \in \mathbb{N}$, $C_b^k(U)$ is the space of functions with bounded and continuous derivatives through order k , and

$$\|f\|_{C^k} := \sum_{i=0}^k \|D^i f\|_\infty.$$

If X is a vector space with norm $\|\cdot\|_X$ and $f \in C(\mathbb{R}, X)$, then, for $-\infty < a < b < \infty$, we write

$$\|f\|_{[a,b],X} := \max_{t \in [a,b]} \|f(t)\|_X.$$

In the case where $X = \mathbb{R}$, if $\alpha \in (0, 1)$ and $f \in C^\alpha(\mathbb{R})$, we denote the α -Hölder semi-norm of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ on $[a, b] \subset \mathbb{R}$ by $[f]_{\alpha,[a,b]}$.

For a function f on $\mathbb{R}$ and $t \in \mathbb{R}$, we denote interchangeably by f_t and $f(t)$ the value of f at t , depending on notational convenience.

For $q \in (1, \infty)$, q' is the conjugate exponent $q/(q-1)$. Given a set A , $\mathbf{1}_A$ denotes the indicator function. The expectation with respect to the probability measure $\mathbf{P}$ is denoted by $\mathbf{E}$. When it does not cause confusion, we suppress the dependence of random variables on the parameter $\omega \in \Omega$.

2. Preliminaries

2.1. The Hamiltonian

We will always assume that

$$\begin{cases} H : \mathbb{R}^d \rightarrow \mathbb{R} \text{ is convex, and, for some } C > 1 \text{ and } q > 1, \\ \frac{1}{C}|p|^q - C \leq H(p) \leq C(|p|^q + 1) \quad \text{for all } p \in \mathbb{R}^d. \end{cases} \quad (2.1)$$

The Legendre transform of H

$$H^*(v) := \sup_{p \in \mathbb{R}^d} (p \cdot v - H(p))$$

has analogous bounds in terms of the conjugate exponent $q' := q(q-1)^{-1}$, that is, for a possibly different constant $C > 1$, we have

$$\frac{1}{C}|p|^{q'} - C \leq H^*(p) \leq C(|p|^{q'} + 1) \quad \text{for all } p \in \mathbb{R}^d. \quad (2.2)$$

We shall also impose that, for some $C > 0$,

$$|H^*(p_1) - H^*(p_2)| \leq C \left(1 + |p_1|^{q'-1} + |p_2|^{q'-1}\right) |p_1 - p_2| \quad \text{for all } p_1, p_2 \in \mathbb{R}^d. \quad (2.3)$$

2.2. The random field

For the random forcing term

$$F(x, t, \omega) = f(x, \omega) \cdot \dot{B}(t, \omega) = \sum_{i=1}^m f^i(x, \omega) \dot{B}^i(t, \omega) \quad (2.4)$$

and the probability measure $\mathbf{P}$, we assume the following:

$$B : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m \quad \text{is a standard Brownian motion,} \quad (2.5)$$

$$f(\cdot, \omega) \in C_b^1(\mathbb{R}^d, \mathbb{R}^m) \quad \text{with probability one,} \quad (2.6)$$

$$f \text{ and } B \text{ are independent,} \quad (2.7)$$

and there exists a group of transformations

$$(\tau_x)_{x \in \mathbb{R}^d} : \Omega \mapsto \Omega$$

satisfying

$$\tau_{x+y} = \tau_x \circ \tau_y \quad \text{for all } x, y \in \mathbb{R}^d, \quad (2.8)$$

$$f(x, \tau_y \omega) = f(x + y, \omega) \quad \text{and} \quad B(\cdot, \tau_y \omega) = B(\cdot, \omega) \quad \text{for all } x, y \in \mathbb{R}^d \text{ and } \omega \in \Omega, \quad (2.9)$$

$$\mathbf{P} \circ \tau_x = \mathbf{P} \quad \text{for all } x \in \mathbb{R}^d, \quad (2.10)$$

and

$$\text{if } A \in \sigma(f) \text{ and } \tau_x A = A \text{ for all } x \in \mathbb{R}^d, \text{ then } \mathbf{P}(A) \in \{0, 1\}. \quad (2.11)$$

Here, $\sigma(f) \subset \mathbf{F}$ is the σ -algebra generated by the random field f .

To avoid long lists of assumptions later on, we introduce the assumption that

$$\text{the random field } F \text{ satisfies (2.4) - (2.11)}. \quad (2.12)$$

Without loss of generality, it can be assumed that Ω and $\mathbf{P}$ have a product-like structure. That is, we may take $\Omega = X \times Y$ and $\mathbf{P} = \mu \otimes \nu$, where $X = C_b^1(\mathbb{R}^d \times \mathbb{R}^m)$, $Y = C([0, \infty), \mathbb{R}^m)$, μ is a measure on X that is stationary and ergodic with respect to translations in $\mathbb{R}^d$, and ν is the Wiener measure.

The stationarity and ergodicity of the shifts in space imply that, for some $M_0 > 0$,

$$\mathbf{P}(\|f\|_{C^1} \leq M_0) = 1. \quad (2.13)$$

2.3. Stability with respect to forcing terms

We next address the issue of well-posedness for viscosity solutions of the initial value problems

$$v_t + H(Dv) = \frac{\partial}{\partial t} \zeta(x, t) \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u(\cdot, 0) = u_0, \quad (2.14)$$

where $u_0 \in BUC(\mathbb{R}^d)$, H satisfies (2.1), and $\zeta \in C([0, \infty), C_b^1(\mathbb{R}^d))$. In particular, ζ is not sufficiently regular for (2.14) to be covered by the standard Crandall-Lions theory [24] of viscosity solutions of Hamilton-Jacobi equations. Instead, (2.14) is a special case of the “pathwise” equations studied by Lions and Souganidis [44–47, 58]. In the present situation, the well-posedness and stability of the equation is more straightforward than in these works, as a consequence of the additive structure of the noise and its smoothness in space.

For $u_0 \in BUC(\mathbb{R}^d)$ fixed, let

$$S_{u_0} : C^1([0, \infty), C_b^1(\mathbb{R}^d)) \mapsto C(\mathbb{R}^d \times [0, \infty)) \quad (2.15)$$

be the solution operator for (2.14), that is, $v = S_{u_0}(\zeta)$.

Lemma 2.1. *Assume H satisfies (2.1). Then S_{u_0} extends continuously to the space $C([0, \infty), C_b^1(\mathbb{R}^d))$.*

Proof. If v is the solution of (2.14), then the function $\tilde{v}$ defined by

$$\tilde{v}(x, t) := v(x, t) - \zeta(x, t) + \zeta(x, 0)$$

is a classical viscosity solution of the initial value problem

$$\tilde{v}_t + H(D\tilde{v} + D_x \zeta(x, t) - D_x \zeta(x, 0)) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \tilde{v}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.$$

The claim now follows from classical arguments from the theory of viscosity solutions. $\square$

The stability for the equation can also be directly seen on the level of the representation formula for the solution operator. For $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \times [0, \infty)$ with $s < t$, we define

$$\mathcal{A}(x, y, s, t) := \left\{ \gamma \in W^{1,\infty}([s, t], \mathbb{R}^d) : \gamma_s = x, \gamma_t = y \right\}$$

and

$$L(x, y, s, t; \zeta) := \inf \left\{ \int_s^t \left[H^*(\dot{\gamma}_r) + \frac{\partial \zeta}{\partial r}(\gamma_r, r) \right] dr : \gamma \in \mathcal{A}(x, y, s, t) \right\}, \quad (2.16)$$

and then, as shown in [43], for example,

$$S_{u_0}(\zeta)(x, t) = \inf_{y \in \mathbb{R}^d} (u_0(y) + L(y, x, s, t; \zeta)). \quad (2.17)$$

Lemma 2.1 then follows from Lemma 2.2 below, which establishes the stability of the Lagrangian with respect to the forcing term.

Lemma 2.2. Assume (2.1), let $L(\cdot, \zeta)$ be defined by (2.16), and fix $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \times [0, \infty)$ with $s < t$. Then the map

$$C^1([0, \infty), C_b^1(\mathbb{R}^d)) \ni \zeta \mapsto L(x, y, s, t, \zeta) \in \mathbb{R}$$

extends continuously to $\zeta \in C([0, \infty), C_b^1(\mathbb{R}^d))$.

Proof. Integrating (2.16) by parts yields

$$L(x, y, s, t; \zeta) = \zeta(y, t) - \zeta(x, s) + \inf \left\{ \int_s^t [H^*(\dot{\gamma}_r) - D_x \zeta(\gamma_r, r) \cdot \dot{\gamma}_r] dr : \gamma \in \mathcal{A}(x, y, s, t) \right\}. \quad (2.18)$$

The result now follows from classical arguments, in view of the super-linearity of H^* and the continuity of $D_x \zeta$ in both variables. $\square$

3. Properties of the random Lagrangian

We continue the discussion of the Lagrangians defined at the end of Section 2, and we investigate the case where the forcing term is random and given by

$$\zeta(x, t, \omega) := f(x) \cdot B(t, \omega) = \sum_{i=1}^m f^i(x) B^i(t, \omega) \quad \text{for } (x, t, \omega) \in \mathbb{R}^d \times [0, \infty) \times \Omega,$$

where $f \in C_b^1(\mathbb{R}^d)$ and B is a standard Brownian motion on the probability space $(\Omega, \mathbf{F}, \mathbf{P})$. Throughout this section, f is non-random, and the estimates will be uniform over certain bounded sets of $C_b^1(\mathbb{R}^d)$.

We obtain uniform growth bounds and regularity estimates for the random Lagrangian

$$L_f(x, y, s, t, \omega) := L(x, y, s, t; f \cdot B(\omega)) = \inf \left\{ \int_s^t H^*(\dot{\gamma}_r) dr + \int_s^t f(\gamma_r) \cdot dB_r(\omega) : \gamma \in \mathcal{A}(x, y, s, t) \right\}. \quad (3.1)$$

In view of Lemma 2.2, L_f is well-defined for any continuous sample path $B(\cdot, \omega)$.

3.1. Integrating non-adapted paths against Brownian motion

The methods used below resemble those of [18,42,54], which involve the manipulation of almost-minimizers of the Lagrangian action. The new difficulty is to find a way to control, for an arbitrary Lipschitz process γ , integrals of the form

$$\int_{r_1}^{r_2} f(\gamma_r) \cdot dB_r. \quad (3.2)$$

If γ is adapted with respect to the natural filtration of the Brownian motion, then standard Itô calculus implies that the moments of (3.2) can be bounded in terms of $\|f\|_\infty$, independently of the regularity of f or γ . Therefore, as a consequence of Kolmogorov's continuity criterion (see [59]), for any $\alpha \in (0, 1/2)$, (3.2) is α -Hölder continuous in r_1 and r_2 , and, for all $p \geq 1$, $T > 0$, $\lambda \geq 1$, and some $C = C(p, \alpha, T) > 0$,

$$\mathbf{P} \left(\sup_{r_1, r_2 \in [0, T]} \frac{1}{|r_1 - r_2|^\alpha} \left| \int_{r_1}^{r_2} f(\gamma_r) \cdot dB_r \right| > \lambda \right) \leq \frac{C \|f\|_\infty^p}{\lambda^p}. \quad (3.3)$$

However, if γ is not adapted, as is the case for the almost-minimizers in general, then Itô calculus does not apply, and the regularity of f and γ enter into the moment estimates.

In order to control (3.2) in such a way that allows us to obtain scale-invariant estimates for L_f , we decompose the integral into three parts: one that can be bounded by a deterministic constant, one which measures the regularity of γ , and a final random piece whose probability tails satisfy bounds resembling (3.3). Crucially, the various constants depend on $\|f\|_{C^1}$ only through an upper bound for the product $\|f\|_\infty \cdot \|Df\|_\infty$.

For $\sigma > 0$ and $\Lambda > 0$, define

$$F_{\sigma, \Lambda} := \left\{ f \in C_b^1(\mathbb{R}^d, \mathbb{R}^m) : \|f\|_\infty \leq \sigma \text{ and } \|Df\|_\infty \leq \Lambda \right\}. \quad (3.4)$$

Lemma 3.1. Fix $T > 0$, $\sigma_0 > 0$, $K > 0$, $q > 1$, and $\alpha \in (0, 1/2)$. Then there exists a constant $M = M(T, \sigma_0, K, q, \alpha) > 0$ and, for every $\sigma \in (0, \sigma_0)$ and $\Lambda > 0$ satisfying $\sigma \Lambda = K$, a random variable

$$\mathcal{D}_{\sigma, \Lambda} : \Omega \rightarrow \mathbb{R}_+$$

such that

(a) for any $p \geq 1$ and some constant $C = C(T, \sigma_0, K, p, q, \alpha) > 0$,

$$\mathbf{P}(\mathcal{D}_{\sigma, \Lambda} \geq \lambda) \leq \frac{C \sigma^p}{\lambda^p} \quad \text{for all } \lambda \geq 1,$$

and

(b) for all $\gamma \in W^{1, \infty}([0, T], \mathbb{R}^d)$, $f \in F_{\sigma, \Lambda}$, $\delta \in (0, 1)$, and $0 \leq s \leq r_1 \leq r_2 \leq t \leq T$,

$$\left| \int_{r_1}^{r_2} f(\gamma_r) \cdot dB_r \right| \leq \left(M \delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr + \frac{M + \mathcal{D}_{\sigma, \Lambda}}{\delta^q} \right) (r_2 - r_1)^\alpha.$$

We note that there is a different random variable $\mathcal{D}_{\sigma, \Lambda, T, \sigma_0, q, \alpha}$ corresponding to each choice of $(\sigma, \Lambda, T, \sigma_0, q, \alpha)$ satisfying the hypotheses of Lemma 3.1. However, we suppress the dependence on all but σ and Λ , as the dependence of $\mathcal{D}_{\sigma, \Lambda}$ on the other parameters will not play a role. The important aspect of the random variable $\mathcal{D}_{\sigma, \Lambda}$ is not its exact formula (see (3.11) below), but rather, the nature of the estimates it satisfies as claimed by the above lemma, and the fact that it does not depend on the Lipschitz path γ or $f \in F_{\sigma, \Lambda}$.

In order to prove Lemma 3.1, we will need a parameter-dependent generalization of the classical Kolmogorov continuity criterion. The proof follows a similar method, but we present it here for the sake of completeness.

Lemma 3.2. For some parameter set $\mathcal{M}$, let $(M_\mu)_{\mu \in \mathcal{M}} : \Omega \rightarrow \mathbb{R}_+$ and $(Z_\mu)_{\mu \in \mathcal{M}} : [0, T] \times \Omega \rightarrow \mathbb{R}$ be such that, for some positive constants $m > 0$, $\beta \in (0, 1)$, and $p \geq 1$,

$$\sup_{0 \leq s < t \leq T} \mathbf{E} \left[\sup_{\mu \in \mathcal{M}} \left(\frac{|Z_\mu(t) - Z_\mu(s)|}{(t-s)^{\beta+1/p}} - M_\mu \right)_+^p \right] \leq m.$$

Then, for all $\alpha \in (0, \beta)$, there exist constants $C_1 = C_1(\alpha, T)$ and $C_2 = C_2(p, \alpha, \beta, T) > 0$ such that, for all $\lambda \geq 1$,

$$\mathbf{P} \left(\sup_{\mu \in \mathcal{M}} ([Z_\mu]_{\alpha, [0, T]} - C_1 M_\mu) > \lambda \right) \leq \frac{C_2 m}{\lambda^p}.$$

Proof. Without loss of generality we take $T = 1$. For $n = 0, 1, 2, \dots$, define

$$D_n := \left\{ \frac{k}{2^n} : k = 0, 1, 2, \dots, 2^n \right\} \quad \text{and} \quad D := \bigcup_{n=0}^{\infty} D_n.$$

For some constant $A = A(\alpha) > 0$ to be determined and for each $n = 0, 1, 2, \dots$, define the event

$$\mathcal{A}_n := \left\{ |Z_\mu(s) - Z_\mu(t)| \leq (M_\mu + A\lambda)|s - t|^\alpha \text{ for all } \mu \in \mathcal{M} \text{ and } \{s, t\} \subset D_n \text{ such that } |s - t| = 2^{-n} \right\}.$$

Fix a pair $\{s, t\}$ in D_n satisfying $|s - t| = 2^{-n}$. Then

$$\begin{aligned} & \mathbf{P}(|Z_\mu(s) - Z_\mu(t)| > (M_\mu + A\lambda)|s - t|^\alpha \text{ for some } \mu \in \mathcal{M}) \\ & \leq \mathbf{P} \left(\sup_{\mu \in \mathcal{M}} \left(\frac{|Z_\mu(s) - Z_\mu(t)|}{|s - t|^{\beta+1/p}} - M_\mu \right)_+^p > (A\lambda)^p 2^{n(1+(\beta-\alpha)p)} \right) \\ & \leq m A^{-p} \lambda^{-p} 2^{-n(1+(\beta-\alpha)p)}, \end{aligned}$$

and therefore,

$$\mathbf{P}(\Omega \setminus \mathcal{A}_n) \leq m A^{-p} \lambda^{-p} 2^{-np(\beta-\alpha)}.$$

Now, fix $\omega \in \bigcap_{n=0}^{\infty} \mathcal{A}_n$ and $s, t \in D$, assume without loss of generality that $s < t$, and let $n \in \mathbb{N}$ be such that

$$2^{-n-1} < t - s \leq 2^{-n}.$$

Then, for some $M_1, M_2 \in \mathbb{N}$ and

$$\binom{s^i}{i=0}^{M_1}, \binom{t^j}{j=0}^{M_2} \subset D,$$

we can write

$$s = s^{M_1} > s^{M_1-1} > s^{M_1-2} > \dots > s^1 > s^0 \quad \text{and} \quad t = t^{M_2} < t^{M_2-1} < t^{M_2-2} < \dots < t^1 < t^0,$$

where

$$\begin{cases} s^i \in D_{n+i} \text{ for } i = 0, 1, 2, \dots, M_1, & t^j \in D_{n+j} \text{ for } j = 0, 1, 2, \dots, M_2, \\ s^i - s^{i-1} = 2^{-n-i}, & t^{j-1} - t^j = 2^{-n-j}, \text{ and} \\ t_0 - s_0 = 2^{-n}. \end{cases}$$

It follows that, for all $\mu \in \mathcal{M}$,

$$\begin{aligned} |Z_\mu(s) - Z_\mu(t)| & \leq |Z_\mu(s^0) - Z_\mu(t^0)| + \sum_{i=1}^{M_1} |Z_\mu(s^i) - Z_\mu(s^{i-1})| + \sum_{j=1}^{M_2} |Z_\mu(t^j) - Z_\mu(t^{j-1})| \\ & \leq (M_\mu + A\lambda) \left(2^{-n\alpha} + \sum_{i=1}^{M_1} 2^{-(n+i)\alpha} + \sum_{j=1}^{M_2} 2^{-(n+j)\alpha} \right) \end{aligned}$$

$$\begin{aligned} &\leq (M_\mu + A\lambda)2^\alpha \left(1 + \sum_{i=1}^{M_1} 2^{-i\alpha} + \sum_{j=1}^{M_2} 2^{-j\alpha} \right) |t-s|^\alpha \\ &\leq 3 \cdot 2^\alpha (M_\mu + A\lambda) |t-s|^\alpha. \end{aligned}$$

Choosing now $A(\alpha) = 3^{-1}2^{-\alpha}$, we conclude that, if $\omega \in \bigcap_{n=0}^{\infty} \mathcal{A}_n$, then, for some constant C_1 as in the statement of the lemma,

$$\sup_{\mu \in \mathcal{M}} \left(\sup_{s, t \in [0, 1]} \frac{|Z_\mu(s) - Z_\mu(t)|}{|t-s|^\alpha} - C_1 M_\mu \right) \leq \lambda.$$

Therefore, for some $C_2 = C_2(p, \alpha, \beta, T) > 0$,

$$\mathbf{P} \left(\sup_{\mu \in \mathcal{M}} ([Z_\mu]_{\alpha, [0, 1]} - C_1 M_\mu) > \lambda \right) \leq \sum_{n=0}^{\infty} \mathbf{P}(\Omega \setminus \mathcal{A}_n) \leq m A(\alpha)^{-p} \lambda^{-p} \sum_{n=0}^{\infty} 2^{-np(\beta-\alpha)} \leq C_2 m \lambda^{-p}. \quad \square$$

Proof of Lemma 3.1. We define the parameter set

$$\mathcal{M} := (0, 1) \times \{(s, t) : 0 \leq s \leq t \leq T\} \times W^{1, \infty}([0, T], \mathbb{R}^d) \times F_{\sigma, \Lambda},$$

where $F_{\sigma, \Lambda}$ is the subspace of $C^1(\mathbb{R}^d, \mathbb{R}^m)$ defined in (3.4), and, for each $\mu = (\delta, (s, t), \gamma, f) \in \mathcal{M}$ and $u \in [0, T]$, the stochastic process

$$Z_\mu(u) := \delta^q \int_0^u f(\gamma_r) \mathbf{1}_{[s, t]}(r) \cdot dB_r. \quad (3.5)$$

We first show that there exists a constant $M = M(T, \sigma_0, K, q) > 0$ and, for all $p \geq 1$, a constant $C = C(T, \sigma_0, K, p, q) > 0$ such that

$$\sup_{0 \leq r_1 \leq r_2 \leq T} \mathbf{E} \left[\sup_{\mu \in \mathcal{M}} \left(\frac{|Z_\mu(r_2) - Z_\mu(r_1)|}{(r_2 - r_1)^{1/2}} - M \delta^q \left(\delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr + \frac{1}{\delta^q} \right) \right)_+^p \right] \leq C \sigma^p. \quad (3.6)$$

Fix $r_1, r_2 \in [0, T]$ with $r_1 \leq r_2$. Then

$$Z_\mu(r_1) - Z_\mu(r_2) = \delta^q \int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s, t]}(r) \cdot dB_r = \delta^q \int_{r_1 \vee s}^{r_2 \wedge t} f(\gamma_r) \cdot dB_r.$$

We now split into several cases, depending on the relative sizes and positions of the intervals $[r_1, r_2]$ and $[s, t]$.

Case 1. Assume first that

$$r_2 - r_1 \leq \frac{\sigma^q}{\Lambda^q}. \quad (3.7)$$

Integrating by parts, we have

$$\int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s, t]}(r) \cdot dB_r = f(\gamma_{r_2 \wedge t}) \cdot (B_{r_2 \wedge t} - B_{r_1 \vee s}) + \int_{r_1 \vee s}^{r_2 \wedge t} Df(\gamma_r) \dot{\gamma}_r \cdot (B_r - B_{r_1 \vee s}) dr.$$

Set

$$X_0 := \max_{u, v \in [r_1, r_2]} |B_u - B_v|.$$

Young's inequality and (3.7) then give, for some constant $C = C(\sigma_0, q) > 0$,

$$\begin{aligned}
\left| \int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s,t]}(r) \cdot dB_r \right| &\leq \sigma X_0 + \Lambda X_0 \int_{r_1 \vee s}^{r_2 \wedge t} |\dot{\gamma}_r| dr \\
&\leq \sigma \left(X_0 + X_0 \left(\int_s^t |\dot{\gamma}_r|^{q'} dr \right)^{1/q'} \right) \\
&\leq \sigma \left(X_0 + C \frac{X_0^q}{\delta^q (r_2 - r_1)^{(q-1)/2}} \right) + (r_2 - r_1)^{1/2} \delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr.
\end{aligned}$$

Therefore,

$$\sup_{\mu \in \mathcal{M}} \left(|Z_\mu(r_2) - Z_\mu(r_1)| - \delta^{q'+q} \int_s^t |\dot{\gamma}_r|^{q'} dr (r_2 - r_1)^{1/2} \right) \leq \sigma \left(X_0 + \frac{C X_0^q}{(r_2 - r_1)^{(q-1)/2}} \right),$$

and (3.6) holds in this case for any $M > 0$, in view of the fact that, for any $m > 0$, there exists a constant $c(m) > 0$ such that

$$\mathbf{E} X_0^m \leq c(r_2 - r_1)^{m/2}.$$

Case 2. Assume now that

$$r_2 - r_1 > \frac{\sigma^q}{\Lambda^q}. \quad (3.8)$$

Set

$$h := \frac{\sigma}{\Lambda} (r_2 - r_1)^{1/q'}, \quad (3.9)$$

and let $N \in \mathbb{N}$ be such that

$$\frac{r_2 - r_1}{h} \leq N < \frac{r_2 - r_1}{h} + 1.$$

Note that (3.8) implies that Nh is proportional to the size of the interval $[r_1, r_2]$, and, in particular,

$$r_2 - r_1 \leq Nh < 2(r_2 - r_1).$$

For $k = 0, 1, 2, \dots, N-1$, set $\tau_k := r_1 + kh$ and $\tau_N = r_2$, and, for $k = 1, 2, \dots, N$, define

$$X_k = \max_{u, v \in [\tau_{k-1}, \tau_k]} |B_u - B_v|.$$

We claim that, for all $\varepsilon > 0$, $0 \leq s \leq t \leq T$, $\gamma \in \mathcal{A}$, and $f \in F_{\sigma, \Lambda}$,

$$\left| \int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s,t]}(r) \cdot dB_r \right| \leq \sigma \sum_{k=1}^N X_k + \Lambda h^{1/q} \left(\frac{1}{q \varepsilon^q} \sum_{k=1}^N X_k^q + \frac{\varepsilon^{q'}}{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr \right). \quad (3.10)$$

The proof of (3.10) will depend on whether $t - s$ is small or large compared to $r_2 - r_1$.

Choose $m, n \in \mathbb{N}$ such that

$$\tau_{m-1} < s \leq \tau_m \quad \text{and} \quad \tau_n \leq t < \tau_{n+1},$$

and, in what follows, define $\tau_{-1} := -\infty$, $\tau_{N+1} := +\infty$, and $X_0 = X_{N+1} = 0$ for consistency. Observe that $n \geq m-1$.

Case 2a. If $n = m-1$, that is,

$$\tau_{m-1} < s \leq t < \tau_m,$$

then

$$\int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s,t]}(r) \cdot dB_r = f(\gamma_t) \cdot (B_t - B_s) - \int_s^t Df(\gamma_r) \dot{\gamma}_r \cdot (B_r - B_s) dr$$

and so Young's inequality and the fact that $t - s \leq h$ yield

$$\begin{aligned} \left| \int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s,t]}(r) \cdot dB_r \right| &\leq \sigma X_m + \Lambda X_m \int_s^t |\dot{\gamma}_r| dr \\ &\leq \sigma X_m + \Lambda h^{1/q} X_m \left(\int_s^t |\dot{\gamma}_r|^{q'} dr \right)^{1/q'} \\ &\leq \sigma X_m + \Lambda h^{1/q} \left(\frac{1}{q \varepsilon^q} X_m^q + \frac{\varepsilon^{q'}}{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr \right). \end{aligned}$$

Therefore, (3.10) holds.

Case 2b. Assume now that $n > m - 1$. Then

$$\begin{aligned} \int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s,t]}(r) \cdot dB_r &= \sum_{k=m_{\tau_{k-1}}}^{n+1} \int_{\tau_{k-1}}^{\tau_k} f(\gamma_r) \mathbf{1}_{[r_1 \vee s, r_2 \wedge t]}(r) \cdot dB_r \\ &= \text{I} - \text{II}, \end{aligned}$$

where

$$\text{I} := f(\gamma_{\tau_m}) \cdot (B_{\tau_m} - B_{s \vee r_1}) + f(\gamma_{t \wedge r_2}) \cdot (B_{t \wedge r_2} - B_{\tau_n}) + \sum_{k=m+1}^n f(\gamma_{\tau_k}) \cdot (B_{\tau_k} - B_{\tau_{k-1}})$$

and

$$\begin{aligned} \text{II} &:= \int_{s \vee r_1}^{\tau_m} Df(\gamma_r) \dot{\gamma}_r \cdot (B_r - B_{s \vee r_1}) dr + \int_{\tau_n}^{t \wedge r_2} Df(\gamma_r) \dot{\gamma}_r \cdot (B_r - B_{\tau_n}) dr \\ &\quad + \sum_{k=m+1}^n \int_{\tau_{k-1}}^{\tau_k} Df(\gamma_r) \dot{\gamma}_r \cdot (B_r - B_{\tau_{k-1}}) dr. \end{aligned}$$

The inequality (3.10) is then a consequence of the estimates

$$|\text{I}| \leq \sigma \sum_{k=m}^{n+1} X_k \leq \sigma \sum_{k=1}^N X_k$$

and

$$\begin{aligned} |\text{II}| &\leq \Lambda \left(X_m \int_{s \vee r_1}^{\tau_m} |\dot{\gamma}_r| dr + X_{n+1} \int_{\tau_n}^{t \wedge r_2} |\dot{\gamma}_r| dr + \sum_{k=m+1}^n X_k \int_{\tau_{k-1}}^{\tau_k} |\dot{\gamma}_r| dr \right) \\ &\leq \Lambda h^{1/q} \left(X_m \left(\int_s^{\tau_m} |\dot{\gamma}_r|^{q'} dr \right)^{1/q'} + X_{n+1} \left(\int_{\tau_n}^t |\dot{\gamma}_r|^{q'} dr \right)^{1/q'} + \sum_{k=m+1}^n X_k \left(\int_{\tau_{k-1}}^{\tau_k} |\dot{\gamma}_r|^{q'} dr \right)^{1/q'} \right) \end{aligned}$$

$$\begin{aligned} &\leq \Lambda h^{1/q} \left(\frac{1}{q\varepsilon^q} \sum_{k=m}^{n+1} X_k^q + \frac{\varepsilon^{q'}}{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr \right) \\ &\leq \Lambda h^{1/q} \left(\frac{1}{q\varepsilon^q} \sum_{k=1}^N X_k^q + \frac{\varepsilon^{q'}}{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr \right). \end{aligned}$$

We now set

$$\varepsilon := \delta \left(N^{1/q} h^{1/2} \right)^{1/q'},$$

so that (3.10) becomes

$$\left| \int_{r_1}^{r_2} f(\gamma_r) \mathbf{1}_{[s,t]}(r) \cdot dB_r \right| \leq \sigma \sum_{k=1}^N X_k + \Lambda h^{1/q} \left(\frac{(N^{1/q} h^{1/2})^{1-q}}{q\delta^q} \sum_{k=1}^N X_k^q + \frac{\delta^{q'} N^{1/q} h^{1/2}}{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr \right).$$

For $k = 1, 2, \dots, N$, the constants

$$a_k := \mathbf{E} X_k \quad \text{and} \quad b_k := \mathbf{E} X_k^q,$$

satisfy, for some $a > 0$ and $b = b(q) > 0$,

$$a_k \leq ah^{1/2} \quad \text{and} \quad b_k \leq bh^{q/2},$$

and so, using the definition of h in (3.9),

$$\sigma \sum_{k=1}^N a_k \leq \sigma N h^{1/2} \leq 2a\sigma(r_2 - r_1)h^{-1/2} = 2aK^{1/2}(r_2 - r_1)^{\frac{q+1}{2q}}$$

and

$$\begin{aligned} \Lambda h^{1/q} \cdot (N^{1/q} h^{1/2})^{1-q} \sum_{k=1}^N b_k &\leq b\Lambda(Nh)^{1/q} h^{1/2} \\ &\leq 2^{1/q} b\Lambda(r_2 - r_1)^{1/2} h^{1/2} \\ &= 2^{1/q} bK^{1/2}(r_2 - r_1)^{\frac{q+1}{2q}}. \end{aligned}$$

Similarly,

$$\Lambda h^{1/q} N^{1/q} h^{1/2} \leq 2^{1/q} \Lambda(r_2 - r_1)^{1/q} h^{1/2} = 2^{1/q} K^{1/2}(r_2 - r_1)^{\frac{q+1}{2q}},$$

and therefore, for some constant $M = M(T, K, q) > 0$,

$$\begin{aligned} &\sup_{\mu \in \mathcal{M}} \left(|Z_\mu(r_2) - Z_\mu(r_1)| - M\delta^q \left(\delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr + \frac{1}{\delta^q} \right) (r_2 - r_1)^{1/2} \right)_+ \\ &\leq M \left(\sigma \left| \sum_{k=1}^N (X_k - a_k) \right| + \Lambda(r_2 - r_1)^{1/q} h^{1/2} \cdot \frac{1}{Nh^{q/2}} \left| \sum_{k=1}^N (X_k^q - b_k) \right| \right). \end{aligned}$$

The collections $(X_k - a_k)_{k=1}^N$ and $(X_k^q - b_k)_{k=1}^N$ consist of independent, mean-zero random variables that satisfy, for any $k = 1, 2, \dots, N$, $m \geq 1$, and some constants $a' = a'(m) > 0$ and $b' = b'(q, m)$,

$$\mathbf{E} |X_k - a_k|^m \leq a' h^{m/2} \quad \text{and} \quad \mathbf{E} |X_k^q - b_k|^m \leq b' h^{mq/2}.$$

Therefore, for each $p \geq 1$, there exist constants $A = A(p) > 0$ and $B = B(p, q) > 0$ such that

$$\mathbf{E} \left| \sum_{k=1}^N (X_k - a_k) \right|^p \leq AN^{p/2} h^{p/2}$$

and

$$\mathbf{E} \left| \sum_{k=1}^N (X_k^q - b_k) \right|^p \leq BN^{p/2} h^{pq/2}.$$

Combining all terms in the inequality leads to (3.6).

We now take p large enough that

$$\alpha < \frac{1}{2} - \frac{1}{p}.$$

Then (3.6) and Lemma 3.2 imply that, for some $C = C(T, \sigma_0, K, p, q, \alpha) > 0$ and $M = M(T, \sigma_0, K, q, \alpha) > 0$, possibly different from above, and for all $\lambda \geq 1$ and

$$p > \frac{2}{1 - 2\alpha},$$

we have

$$\mathbf{P} \left(\sup_{\mu \in \mathcal{M}} \left([Z_\mu]_{\alpha, [0, T]} - M\delta^q \left(\delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr + \frac{1}{\delta^q} \right) \right) > \lambda \right) \leq \frac{C\sigma^p}{\lambda^p}.$$

By changing the value of C , if necessary, depending only on σ_0 , the same can be accomplished for any $p \geq 1$. The proof of Lemma 3.1 is then finished upon defining

$$\mathcal{D}_{\sigma, \Lambda} := \sup_{\mu \in \mathcal{M}} \left([Z_\mu]_{\alpha, [0, T]} - M\delta^q \left(\delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr + \frac{1}{\delta^q} \right) \right). \quad \square \quad (3.11)$$

We now use Lemma 3.1 to estimate the growth of the Lagrangian, and the $W^{1, q'}$ -norms for almost-minimizers of L_f , in terms of $\mathcal{D}_{\sigma, \Lambda}$. Note that the exponent q' in Lemma 3.1 is chosen so as to match the growth of H^* in the definition of L_f .

Lemma 3.3. Assume that H satisfies (2.1). Then there exists $C = C(T, \sigma_0, K, q, \alpha) > 0$ such that, if $\sigma \in (0, \sigma_0)$, $\sigma\Lambda = K$, $f \in F_{\sigma, \Lambda}$, and $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \times [0, T]$ with $s < t$, then

$$-C(1 + \mathcal{D}_{\sigma, \Lambda})(t - s)^\alpha + \frac{1}{C} \frac{|y - x|^{q'}}{(t - s)^{q'-1}} \leq L_f(x, y, s, t) \leq C(1 + \mathcal{D}_{\sigma, \Lambda})(t - s)^\alpha + C \frac{|y - x|^{q'}}{(t - s)^{q'-1}}, \quad (3.12)$$

and, if $\gamma : [s, t] \times \Omega \rightarrow \mathbb{R}^d$ is such that $\gamma \in \mathcal{A}(x, y, s, t)$ and

$$L_f(x, y, s, t) + 1 \geq \int_s^t H^*(\dot{\gamma}_r) dr + \int_s^t f(\gamma_r) \cdot dB_r \quad \text{with probability one,} \quad (3.13)$$

then

$$\int_s^t |\dot{\gamma}_r|^{q'} dr \leq C \left(\frac{|y - x|^{q'}}{(t - s)^{q'-1}} + \mathcal{D}_{\sigma, \Lambda} + 1 \right). \quad (3.14)$$

Proof. We apply the Lagrangian action to the linear path

$$\ell := \left(x + \frac{y - x}{t - s}(r - s) \right)_{r \in [s, t]} \in \mathcal{A}(x, y, s, t),$$

and, appealing to Lemma 3.1 with $\delta = 1$, we find

$$L_f(x, y, s, t) \leq \int_s^t H^*(\dot{\ell}_r) dr + \int_s^t f(\ell_r) \cdot dB_r \leq C \left(\frac{|y-x|^{q'}}{(t-s)^{q'-1}} + (1 + \mathcal{D}_{\sigma, \Lambda})(t-s)^\alpha \right).$$

This establishes the upper bound of (3.12).

Next, (2.2) and Lemma 3.1 yield, for any $\delta \in (0, 1)$ and $\gamma \in \mathcal{A}(x, y, s, t)$,

$$\int_s^t H^*(\dot{\gamma}_r) dr + \int_s^t f(\gamma_r) \cdot dB_r \geq \left(\frac{1}{C} - M\delta^{q'} \right) \int_s^t |\dot{\gamma}_r|^{q'} dr - \frac{1}{\delta^q} (M + \mathcal{D}_{\sigma, \Lambda})(t-s)^\alpha. \quad (3.15)$$

Taking δ sufficiently small, employing Jensen's inequality, and taking the infimum over $\gamma \in \mathcal{A}(x, y, s, t)$ gives the lower bound in (3.12). Finally, (3.14) is a consequence of (3.13), (3.15), and the upper bound in (3.12). $\square$

3.2. The regularity estimate

For $R > 0$ and $0 < \tau < T$, define the domain

$$U_{\tau, R, T} := \left\{ (x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \times [0, T] : |x - y| \leq R \text{ and } \tau < t - s < T \right\}.$$

Proposition 3.1. Assume H satisfies (2.1), and let $\sigma_0 > 0$, $K > 0$, and $0 < \theta < \frac{q}{2+q}$. Then there exist $M = M(R, T, \tau, \sigma_0, K, \theta, q) > 0$ and, for any $p \geq 1$, $C = C(T, \sigma_0, K, \theta, p, q) > 0$ such that, if $\sigma \in (0, \sigma_0)$ and $\sigma \Lambda = K$, then

$$\mathbf{P} \left(\sup_{f \in F_{\sigma, \Lambda}} [L_f]_{C_{x, y}^{\theta} C_{s, t}^{\theta/q}(U_{\tau, R, T})} > M + \lambda \right) \leq \frac{CM^p \sigma^p}{\lambda^p} \quad \text{for all } \lambda \geq 1.$$

Proof. We choose $\alpha \in (0, 1/2)$ from Lemma 3.1 sufficiently close to $1/2$ that

$$\theta = \frac{\alpha q}{1 + \alpha q}. \quad (3.16)$$

In what follows, the constants $C > 0$ and $M > 0$ depend on $R, T, \tau, \sigma_0, K, q$, and θ (and therefore α) unless otherwise specified, and may change from line to line.

Step 1: Spatial increments. Fix $(x, y, s, t), (x, \tilde{y}, s, t) \in U_{R, T, \tau}$ and $v \in (0, 1)$, and let $\gamma \in \mathcal{A}(x, y, s, t)$ satisfy

$$L_f(x, y, s, t) + v \geq \int_s^t H^*(\dot{\gamma}_r) dr + \int_s^t f(\gamma_r) \cdot dB_r. \quad (3.17)$$

Set

$$h := c(t-s)|\tilde{y} - y|^{q/(1+\alpha q)},$$

where $c = c(q, \alpha, R)$ is chosen so that $s < t - h < t$, and define $\tilde{\gamma} \in \mathcal{A}(x, \tilde{y}, s, t)$ by

$$\tilde{\gamma}_r := \begin{cases} \gamma_r & \text{for } r \in [s, t-h], \text{ and} \\ \gamma_r + \frac{\tilde{y} - y}{h}(r - t + h) & \text{for } r \in [t-h, t]. \end{cases}$$

Observe that

$$\|\dot{\tilde{\gamma}}\|_{q'} \leq \|\dot{\gamma}\|_{q'} + \left(\int_{t-h}^t \left| \frac{\tilde{y} - y}{h} \right|^{q'} dr \right)^{1/q'} = \|\dot{\gamma}\|_{q'} + \frac{|\tilde{y} - y|}{h^{1/q}} \leq \|\dot{\gamma}\|_{q'} + C|\tilde{y} - y|^{\alpha q/(1+\alpha q)} \leq \|\dot{\gamma}\|_{q'} + C, \quad (3.18)$$

and so, by Lemma 3.3,

$$\int_s^t \left(|\dot{\gamma}_r|^{q'} + |\dot{\tilde{\gamma}}_r|^{q'} \right) dr \leq C(1 + \mathcal{D}_{\sigma, \Lambda}).$$

The definition of L_f then yields

$$L_f(x, \tilde{y}, s, t) - L_f(x, y, s, t) - v \leq \int_{t-h}^t \left[H^* \left(\dot{\gamma}_r + \frac{\tilde{y} - y}{h} \right) - H^*(\dot{\gamma}_r) \right] dr + \int_{t-h}^t (f(\tilde{\gamma}_r) - f(\gamma_r)) \cdot dB_r.$$

To bound the first integral, we use (2.3) and (3.18) to obtain

$$\begin{aligned} \int_{t-h}^t \left[H^* \left(\dot{\gamma}_r + \frac{\tilde{y} - y}{h} \right) - H^*(\dot{\gamma}_r) \right] dr &\leq C \int_{t-h}^t \left(1 + |\dot{\gamma}_r|^{q'-1} + |\dot{\tilde{\gamma}}_r|^{q'-1} \right) dr \frac{|\tilde{y} - y|}{h} \\ &\leq C \left(|\tilde{y} - y| + \frac{|\tilde{y} - y|}{h} \int_{t-h}^t \left(|\dot{\gamma}_r|^{q'-1} + |\dot{\tilde{\gamma}}_r|^{q'-1} \right) dr \right) \\ &\leq C \left(|\tilde{y} - y| + |\tilde{y} - y|^{\alpha q / (1 + \alpha q)} (1 + \mathcal{D}_{\sigma, \Lambda})^{1/q} \right). \end{aligned}$$

Applying Lemma 3.1 in conjunction with Lemma 3.3 gives

$$\left| \int_{t-h}^t f(\gamma_r) \cdot dB_r \right| \leq C \left(\int_s^t |\dot{\gamma}_r|^{q'} dr + 1 + \mathcal{D}_{\sigma, \Lambda} \right) h^\alpha \leq C(1 + \mathcal{D}_{\sigma, \Lambda}) h^\alpha,$$

and, because of (3.18), the same estimate holds with $\tilde{\gamma}$ in place of γ .

Note that $h^\alpha \leq C|\tilde{y} - y|^{\alpha q / (1 + \alpha q)}$. Then, by sending $v \rightarrow 0$ and exchanging the roles of $\tilde{y}$ and y , we conclude that, for some $M > 0$,

$$|L_f(x, \tilde{y}, s, t) - L_f(x, y, s, t)| \leq M(1 + \mathcal{D}_{\sigma, \Lambda})|\tilde{y} - y|^{\alpha q / (1 + \alpha q)}.$$

Similar arguments give, for all $(x, y, s, t), (\tilde{x}, y, s, t) \in U_{R, T, \tau}$,

$$|L_f(\tilde{x}, y, s, t) - L_f(x, y, s, t)| \leq M(1 + \mathcal{D}_{\sigma, \Lambda})|\tilde{x} - x|^{\alpha q / (1 + \alpha q)}.$$

Step 2: time increments. Fix $(x, y, s, t), (x, y, s, \tilde{t}) \in U_{R, T, \tau}$ with $\tilde{t} < t$. Then the sub-additivity of L_f (see (4.9)) yields

$$L_f(x, y, s, t) - L_f(x, y, s, \tilde{t}) \leq L_f(y, y, \tilde{t}, t).$$

Bounding $L_f(y, y, \tilde{t}, t)$ from above by applying the Lagrangian action to the constant path $\gamma \equiv y$ gives, for some $C = C(R, q) > 0$,

$$L_f(x, y, s, t) - L_f(x, y, s, \tilde{t}) \leq C(t - \tilde{t}) + f(y) \cdot (B(t) - B(\tilde{t})) \leq C(1 + \sigma[B]_{\alpha, [0, T]}) (t - \tilde{t})^\alpha. \quad (3.19)$$

We next find a lower bound for $L_f(x, y, s, t) - L_f(x, y, s, \tilde{t})$. For $v \in (0, 1)$, let $\gamma \in \mathcal{A}(x, y, s, t)$ once more satisfy (3.17), set

$$h := c(t - s)(t - \tilde{t})^{1/(1 + \alpha q)},$$

where $c = c(\tau) > 0$ is chosen so that $s < t - h < t$, and define $\tilde{\gamma} \in \mathcal{A}(x, y, s, \tilde{t})$ by

$$\tilde{\gamma}_r := \begin{cases} \gamma_r & \text{for } r \in [s, \tilde{t} - h), \text{ and} \\ \gamma_r + \frac{y - \gamma_{\tilde{t}}}{h}(r - \tilde{t} + h) & \text{for } r \in [\tilde{t} - h, \tilde{t}]. \end{cases}$$

Then

$$\begin{aligned}
L_f(x, y, s, \tilde{t}) - L_f(x, y, s, t) &\leq \int_{\tilde{t}-h}^{\tilde{t}} \left[H^* \left(\dot{\gamma}_r + \frac{y - \gamma_{\tilde{t}}}{h} \right) - H^* (\dot{\gamma}_r) \right] dr - \int_{\tilde{t}}^t H^* (\dot{\gamma}_r) dr \\
&\quad + \int_{\tilde{t}-h}^{\tilde{t}} (f(\tilde{\gamma}_r) - f(\gamma_r)) \cdot dB_r - \int_{\tilde{t}}^t f(\gamma_r) \cdot dB_r.
\end{aligned} \tag{3.20}$$

Observe that

$$|y - \gamma_{\tilde{t}}| \leq \int_{\tilde{t}}^t |\dot{\gamma}_r| dr \leq \left(\int_s^t |\dot{\gamma}_r|^{q'} dr \right)^{1/q'} (t - \tilde{t})^{1/q},$$

which implies that

$$\begin{aligned}
\|\dot{\gamma}\|_{L^{q'}[s, \tilde{t}]} &\leq \|\dot{\gamma}\|_{L^{q'}[s, t]} + \left(\int_{\tilde{t}-h}^{\tilde{t}} \left| \frac{y - \gamma_{\tilde{t}}}{h} \right|^{q'} dr \right)^{1/q'} \\
&= \|\dot{\gamma}\|_{L^{q'}[s, t]} + \frac{|y - \gamma_{\tilde{t}}|}{h^{1/q}} \\
&\leq \|\dot{\gamma}\|_{L^{q'}[s, t]} + C \|\dot{\gamma}\|_{L^{q'}[s, t]} (t - \tilde{t})^{\alpha/(1+\alpha q)} \\
&\leq C \|\dot{\gamma}\|_{L^{q'}[s, t]},
\end{aligned}$$

and, therefore, in view of Lemma 3.3,

$$\int_s^{\tilde{t}} \left(|\dot{\gamma}_r|^{q'} + |\dot{\tilde{\gamma}}_r|^{q'} \right) dr \leq C(1 + \mathcal{D}_{\sigma, \Lambda}).$$

The first two integrals on the right-hand side of (3.20) can then be bounded using (2.2) and (2.3) to obtain

$$\begin{aligned}
&\int_{\tilde{t}-h}^{\tilde{t}} \left[H^* \left(\dot{\gamma}_r + \frac{y - \gamma_{\tilde{t}}}{h} \right) - H^* (\dot{\gamma}_r) \right] dr - \int_{\tilde{t}}^t H^* (\dot{\gamma}_r) dr \\
&\leq C(t - \tilde{t}) + C \int_{\tilde{t}-h}^{\tilde{t}} \left(1 + |\dot{\gamma}_r|^{q'-1} + |\dot{\tilde{\gamma}}_r|^{q'-1} \right) dr \frac{|y - \gamma_{\tilde{t}}|}{h} \\
&\leq C(t - \tilde{t}) + Ch + C \int_{\tilde{t}-h}^{\tilde{t}} \left(|\dot{\tilde{\gamma}}_r|^{q'-1} + |\dot{\gamma}_r|^{q'-1} \right) dr \left(\int_s^t |\dot{\gamma}_r|^{q'} dr \right)^{1/q'} \frac{(t - \tilde{t})^{1/q}}{h} \\
&\leq C(t - \tilde{t}) + C(t - \tilde{t})^{\alpha q/(1+\alpha q)} + C(1 + \mathcal{D}_{\sigma, \Lambda})(t - \tilde{t})^{\alpha/(1+\alpha q)} \\
&\leq C(1 + \mathcal{D}_{\sigma, \Lambda})(t - \tilde{t})^{\alpha/(1+\alpha q)}.
\end{aligned}$$

Additionally, Lemma 3.1 gives

$$\left| \int_{\tilde{t}-h}^{\tilde{t}} f(\gamma_r) \cdot dB_r \right| + \left| \int_{\tilde{t}-h}^{\tilde{t}} f(\tilde{\gamma}_r) \cdot dB_r \right| \leq C(1 + \mathcal{D}_{\sigma, \Lambda})h^\alpha$$

and

$$\left| \int_{\tilde{t}}^t f(\gamma_r) \cdot dB_r \right| \leq C(1 + \mathcal{D}_{\sigma, \Lambda})(t - \tilde{t})^\alpha.$$

Sending $v \rightarrow 0$, combining all of the lower bounds with the upper bound (3.19), and applying a similar argument in the s variable, we conclude that, for some $M > 0$,

$$|L(x, y, \tilde{s}, \tilde{t}) - L(x, y, s, t)| \leq M(1 + \mathcal{D}_{\sigma, \Lambda} + \sigma[B]_{\alpha, [0, T]}) \left(|s - \tilde{s}|^{\alpha/(1+\alpha q)} + |t - \tilde{t}|^{\alpha/(1+\alpha q)} \right).$$

Step 3. Combining all of the estimates obtained in Steps 1 and 2 gives

$$\sup_{f \in F_{\sigma, \Lambda}} [L_f]_{C_{x,y}^{\theta} C_{s,t}^{\theta/q}(U_{R,T,\tau})} - M \leq M(\mathcal{D}_{\sigma, \Lambda} + \sigma[B]_{\alpha, [0, T]}).$$

The result now follows as a consequence of Lemma 3.1 and the fact that $[B]_{\alpha, [0, T]} \in L^p(\Omega, \mathbf{P})$ for all $p \geq 1$. $\square$

Proposition 3.1 can be used to obtain regularity estimates for solutions u of

$$u_t + H(Du) = \sum_{i=1}^m f^i(x) \dot{B}^i(t, \omega) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \quad \text{and} \quad u(\cdot, 0, \cdot) = u_0 \quad \text{in } \mathbb{R}^d \times \Omega. \quad (3.21)$$

Although we do not directly use the following result in the later parts of the paper, its statement is of independent interest.

Theorem 3.1. Assume that H satisfies (2.1), $0 < \theta < \frac{q}{2+q}$, and $\mathcal{R} > 1$. Then there exists a constant $C_1 = C_1(\mathcal{R}, q, \theta) > 0$, and for $p \geq 1$, a constant $C_2 = C_2(\mathcal{R}, p, q, \theta) > 0$ such that, whenever $B : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m$ is a standard Brownian motion; $f \in C_b^1(\mathbb{R}^d; \mathbb{R}^m)$ and $u_0 \in BUC(\mathbb{R}^d)$ satisfy

$$\|f\|_\infty \cdot \|Df\|_\infty + \|f\|_\infty + \|u_0\|_\infty \leq \mathcal{R};$$

and u is the solution of (3.21), then, for all $\lambda \geq 1$,

$$\mathbf{P} \left(\sup_{(x,s), (y,t) \in B_{\mathcal{R}} \times [1/\mathcal{R}, \mathcal{R}]} \frac{|u(x, s) - u(y, t)|}{|x - y|^\theta + |s - t|^{\theta/q}} > C_1 + \lambda \right) \leq \frac{C_2 \|f\|_\infty^p}{\lambda^p}.$$

Proof. The function u is given by

$$u(x, t) = \inf_{y \in \mathbb{R}^d} (u_0(y) + L_f(y, x, 0, t)).$$

Assume that $t > 1/\mathcal{R}$ and $|x| \leq \mathcal{R}$. Then Lemma 3.3 gives the upper bound, for some $C = C(\mathcal{R}, q, \theta) > 0$,

$$u(x, t) \leq u_0(x) + L_f(x, x, 0, t) \leq C(1 + \mathcal{D}_{\sigma, \Lambda}).$$

It then follows that the infimum in the definition of u can be taken over $|y - x| \leq R'$, where, for another constant $C = C(\mathcal{R}, q, \theta) > 0$ and for all $\omega \in \Omega$,

$$R'(\omega) := C(1 + \mathcal{D}_{\sigma, \Lambda}(\omega)).$$

It can be verified that the constant M from Proposition 3.1 has polynomial growth in R , that is, for some $M' = M'(T, \tau, \sigma_0, K, \theta, q) > 0$ and $a = a(\theta, q) > 0$,

$$M(R, T, \tau, \sigma_0, K, \theta, q) \leq M'(1 + R^a).$$

We also have, for all $x, y \in B_{\mathcal{R}}$ and $s, t \in [1/\mathcal{R}, \mathcal{R}]$,

$$|u(x, s) - u(y, t)| \leq \sup_{|x-z| \leq R', |y-z| \leq R'} |L_f(z, x, 0, s) - L_f(z, y, 0, t)|.$$

From this, we conclude that, for some $C_1 = C_1(\mathcal{R}, q, \theta) > 0$ and $C_2 = C_2(\mathcal{R}, p, q, \theta) > 0$ and for any $\lambda, \lambda' \geq 1$,

$$\begin{aligned} \mathbf{P}\left([u]_{C_x^\theta C_t^{\theta/q}(B_{1/\mathcal{R}} \times [1/\mathcal{R}, \mathcal{R}])} > C_1 + \lambda\right) &\leq \mathbf{P}(R' > \lambda') + \mathbf{P}\left([Lf]_{C_{x,y}^\theta C_{s,t}^{\theta/q}(B_{\lambda'}^2 \times [1/\mathcal{R}, \mathcal{R}]^2)} > C_1 + \lambda\right) \\ &\leq C_2 \left(\frac{1}{(\lambda')^{p(1+a)}} + \frac{1 + (\lambda')^{pa(1+a)}}{\lambda^{p(1+a)}} \right) \|f\|_\infty^{p(1+a)}. \end{aligned}$$

The proof is finished upon choosing $\lambda' = \lambda^{1/(1+a)}$ and using the fact that $\|f\|_\infty^{1+a} \leq \mathcal{R}^a \|f\|_\infty$. $\square$

4. Homogenization

We now turn to the proof of Theorem 1.1, which concerns the convergence, as $\varepsilon \rightarrow 0$, of solutions of the stochastically perturbed initial value problem

$$u_t^\varepsilon + H(Du^\varepsilon) = F\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \quad \text{and} \quad u^\varepsilon(x, 0, \omega) = u_0(x) \quad \text{in } \mathbb{R}^d \times \Omega \quad (4.1)$$

to the solution of the effective equation

$$\bar{u}_t + \bar{H}(D\bar{u}) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \bar{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d. \quad (4.2)$$

We restate the theorem here with more precise hypotheses and conclusions. Recall that F is given by

$$F(x, t, \omega) = \sum_{i=1}^m f^i(x, \omega) \dot{B}^i(t, \omega) \quad \text{for } (x, t) \in \mathbb{R}^d \times [0, \infty),$$

where $f(\cdot, \omega) \in C_b^1(\mathbb{R}^d)$ is a stationary-ergodic random field and B is a standard m -dimensional Brownian motion that is independent of f .

Below, the constant $M_0 > 0$ is such that

$$\mathbf{P}(\|f\|_{C^1} \leq M_0) = 1.$$

Theorem 4.1. Assume that H satisfies (2.1) and the random field F satisfies (2.12). Then there exists a deterministic, convex $\bar{H} : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the bounds

$$\frac{1}{C}|p|^q - C \leq \bar{H}(p) \leq C(|p|^q + 1) \quad \text{for some } C = C(M_0) > 1 \text{ and all } p \in \mathbb{R}^d \quad (4.3)$$

such that, if $u_0 \in BUC(\mathbb{R}^d)$ and u^ε solves (4.1), then, as $\varepsilon \rightarrow 0$ and with probability one, u^ε converges locally uniformly to the viscosity solution $\bar{u}$ of (4.2).

Much of the proofs that follow proceed similarly to those in [38,49,53,54,57], with some new difficulties arising because of the singular nature of the forcing term.

The solution u^ε has the control-theory representation

$$u^\varepsilon(x, t) := \inf_{y \in \mathbb{R}^d} \left\{ u_0(y) + L^\varepsilon(y, x, s, t, \omega) \right\}, \quad (4.4)$$

where

$$L^\varepsilon(x, y, s, t, \omega) := \inf \left\{ \int_s^t \left[H^*(\dot{\gamma}_r) + F\left(\frac{\gamma_r}{\varepsilon}, \frac{r}{\varepsilon}, \omega\right) \right] dr : \gamma \in \mathcal{A}(x, y, s, t) \right\}. \quad (4.5)$$

Indeed, (4.4) holds if B is replaced with a continuously differentiable path (see [43]), and, as a consequence of Lemmas 2.1 and 2.2, the equality continues to hold for arbitrary continuous B , and, in particular, sample paths of Brownian motion.

It will be useful to rewrite L^ε in two different ways. First, if we set $B^\varepsilon(r, \omega) := \varepsilon^{1/2} B(r/\varepsilon, \omega)$, then

$$L^\varepsilon(x, y, s, t, \omega) = \inf \left\{ \int_s^t H^*(\dot{\gamma}_r) dr + \varepsilon^{1/2} \int_s^t f(\varepsilon^{-1} \gamma_r, \omega) \cdot dB_r^\varepsilon(\omega) : \gamma \in \mathcal{A}(x, y, s, t) \right\}. \quad (4.6)$$

Also, by rescaling the paths $\gamma \in \mathcal{A}(x, y, s, t)$, we see that

$$L^\varepsilon(x, y, s, t, \omega) = \varepsilon L\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{s}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right),$$

where $L := L^1$ is given by

$$L(x, y, s, t, \omega) := \inf \left\{ \int_s^t H^*(\dot{\gamma}_r) dr + \int_s^t f(\gamma_r, \omega) \cdot dB_r(\omega) : \gamma \in \mathcal{A}(x, y, s, t) \right\}. \quad (4.7)$$

We note that, for $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \times [0, T]$ and $\omega \in \Omega$,

$$L(x, y, s, t, \omega) = L_{f(\cdot, \omega)}(x, y, s, t, \omega),$$

where, for fixed $f \in C_b^1(\mathbb{R}^d, \mathbb{R}^m)$, L_f is defined as in (3.1).

4.1. Uniform regularity of the Lagrangians

We use Proposition 3.1 and the Borel-Cantelli lemma to show that the family $(L^\varepsilon)_{\varepsilon > 0}$ is locally uniformly equicontinuous for ε smaller than some random threshold.

Recall that we define, for $R > 0$ and $0 < \tau < T$, the domain

$$U_{\tau, R, T} := \left\{ (x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \times [0, T] : |x - y| \leq R \text{ and } \tau < t - s < T \right\},$$

and the parameters $\alpha \in (0, 1/2)$ and $\theta \in (0, q/(2+q))$ are related by

$$\theta = \frac{\alpha q}{1 + \alpha q}.$$

Lemma 4.1. *Let L^ε be given by (4.5), where H and F are as in (2.1) and (2.12). Then, for all $0 < \theta < \frac{q}{2+q}$, $R > 0$, and $0 < \tau < T$, there exists a constant $C = C(R, T, \tau, M_0, q, \theta) > 0$ and a random variable $\varepsilon_0 : \Omega \rightarrow \mathbb{R}_+$ independent of f such that $\mathbf{P}(\varepsilon_0 > 0) = 1$ and*

$$\mathbf{P}\left([L^\varepsilon]_{C_{x,y}^\theta C_{s,t}^{\theta/q}(U_{R,T,\tau})} \leq C \text{ for all } 0 < \varepsilon < \varepsilon_0\right) = 1.$$

Proof. For each fixed $\varepsilon \in (0, 1)$, we apply Proposition 3.1 using the formula (4.6) for L^ε , with

$$f^\varepsilon := \varepsilon^{1/2} f(\cdot/\varepsilon) \quad \text{and} \quad B^\varepsilon = \varepsilon^{1/2} B(\cdot/\varepsilon).$$

Then $f^\varepsilon \in F_{\sigma^\varepsilon, \Lambda^\varepsilon}$, where

$$\sigma^\varepsilon := M_0 \varepsilon^{1/2}, \quad \Lambda^\varepsilon := \frac{M_0}{\varepsilon^{1/2}},$$

and $F_{\sigma^\varepsilon, \Lambda^\varepsilon}$ is defined as in (3.4). Note that, $K := \sigma^\varepsilon \Lambda^\varepsilon = M_0^2$ is independent of $\varepsilon > 0$. Then there exists a constant $C_1 = C_1(R, T, \tau, M_0, q, \theta) > 0$ and, for all $p \geq 1$, a constant $C_2 = C_2(R, T, \tau, M_0, p, q, \theta) > 0$ such that, for all $\lambda \geq 1$ and $\varepsilon \in (0, 1)$,

$$\mathbf{P}\left(\sup_{f \in F_{\sigma^\varepsilon, \Lambda^\varepsilon}} [L_f]_{C_{x,y}^\theta C_{s,t}^{\theta/q}(U_{R,T,\tau})} > C_1 + \lambda\right) \leq C_2 \varepsilon^{p/2} \lambda^{-p}.$$

Define

$$A_\varepsilon := \left\{ \sup_{f \in F_{\sigma^\varepsilon, \Lambda^\varepsilon}} [L_f]_{C_{x,y}^\theta C_{s,t}^{\theta/q}(U_{R,T,\tau})} > C_1 + 1 \right\},$$

and observe that A_ε is independent of the random field $\{\omega \mapsto f(\cdot, \omega)\}$. Because

$$\mathbf{P}(A_{2^{-k}}) \leq C_2 2^{-kp/2},$$

the Borel-Cantelli lemma yields that, for some $k_0 : \Omega \rightarrow \mathbb{N}$ that is independent of the random field f , for $\mathbf{P}$ -almost every $\omega \in \Omega$, and for all $k \geq k_0(\omega)$,

$$\left[L^{2^{-k}}(\cdot, \omega) \right]_{C_{x,y}^{\theta} C_{s,t}^{\theta/q}(U_{R,T,\tau})} \leq C_1 + 1.$$

For $\omega \in \Omega$ and $0 < \varepsilon < \varepsilon_0(\omega) := 2^{-k_0(\omega)}$, we choose $k > k_0(\omega)$ such that

$$2^{-k-1} < \varepsilon \leq 2^{-k}.$$

Then

$$L^\varepsilon(x, y, s, t, \cdot) = \varepsilon L\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{s}{\varepsilon}, \frac{t}{\varepsilon}, \cdot\right) = 2^{k+1} \varepsilon L^{2^{-k-1}}\left(2^{-k-1} \varepsilon^{-1} x, 2^{-k-1} \varepsilon^{-1} y, 2^{-k-1} \varepsilon^{-1} s, 2^{-k-1} \varepsilon^{-1} t, \cdot\right),$$

and therefore, for $\mathbf{P}$ -almost every $\omega \in \Omega$ and all $\varepsilon \in (0, \varepsilon_0(\omega))$,

$$[L^\varepsilon(\cdot, \omega)]_{C_{x,y}^{\theta} C_{s,t}^{\theta/q}(U_{R,T,\tau})} \leq 2(C_1 + 1). \quad \square$$

4.2. The stationary-ergodic, spatio-temporal environment

The temporal white noise term $\dot{B}$ is stationary, uncorrelated, and independent from f , and, as a consequence, the spatio-temporal environment generated by the random field F is stationary-ergodic. More precisely, we may assume without loss of generality that the probability measure $\mathbf{P}$ is such that there exists a collection of transformations

$$(\tau'_t)_{t \geq 0} : \Omega \rightarrow \Omega$$

such that, for all $s, t \geq 0$ and $\omega \in \Omega$,

$$\tau'_{s+t} = \tau'_s \circ \tau'_t, \quad \mathbf{P} \circ \tau'_t = \mathbf{P}, \quad B(s, \tau'_t \omega) = B(t + s, \omega) - B(t, \omega), \quad \text{and} \quad f(\cdot, \tau'_t \omega) = f(\cdot, \omega).$$

For $(x, t) \in \mathbb{R}^d \times [0, \infty)$, we now set

$$\tau_{x,t} := \tau_x \circ \tau'_t : \Omega \rightarrow \Omega.$$

It is clear that $\tau_{x,t}$ preserves $\mathbf{P}$ for any $(x, t) \in \mathbb{R}^d \times [0, \infty)$. Moreover the collection is ergodic with respect to $\mathbf{P}$:

$$\text{if } A \in \mathbf{F} \text{ and } \tau_{x,t} A = A \text{ for all } (x, t) \in \mathbb{R}^d \times [0, \infty), \text{ then } \mathbf{P}(A) \in \{0, 1\}.$$

As noted in Section 2, we may assume that $\Omega = X \times Y$ and $\mathbf{P} = \mu \otimes \nu$, where $X = C_b^1(\mathbb{R}^d, \mathbb{R}^m)$, $Y \in C([0, \infty), \mathbb{R}^m)$, μ is a probability measure that is stationary and ergodic with respect to translations in space, and ν is the Wiener measure. The transformation τ'_t can then be realized as

$$\tau'_t B := B(t + \cdot) - B(t) \quad \text{for all } B \in Y,$$

and the stationarity of ν with respect to τ'_t is a consequence of the Markov property of Brownian motion.

4.3. Identification of the effective Lagrangian

We next use the sub-additive ergodic theorem to establish the almost-sure, local uniform convergence of L^ε to a deterministic, effective quantity. This relies on the sub-additivity and stationarity of L defined by (4.7), namely, for all $x, y, z \in \mathbb{R}^d$, $s < r < t$, $q \in [0, \infty)$, and $\omega \in \Omega$,

$$L(x, y, s, t, \tau_{z,q} \omega) = L(x + z, y + z, s + q, t + q, \omega) \quad (4.8)$$

and

$$L(x, y, s, t, \omega) \leq L(x, z, s, r, \omega) + L(z, y, r, t, \omega), \quad (4.9)$$

both of which can be proved using appropriate manipulations of the minimizing paths in the definition of L , invoking the stationarity of F to prove the former.

We first identify the effective Lagrangian as the long-time average of L .

Lemma 4.2. Assume (2.1), (2.12), and that L is given by (4.7). Then there exists a deterministic, convex function $\bar{L} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that, for some $C = C(M_0) > 1$,

$$\frac{1}{C} |p|^{q'} - C \leq \bar{L}(p) \leq C(|p|^{q'} + 1) \quad \text{for all } p \in \mathbb{R}^d, \quad (4.10)$$

and, with probability one and for all $R > 0$,

$$\lim_{T \rightarrow +\infty} \sup_{|p| \leq R} \left| \frac{1}{T} L(0, Tp, 0, T, \cdot) - \bar{L}(p) \right| = 0.$$

Proof. *Step 1: identifying the limit.* Fix $p \in \mathbb{Q}^d$, define the process ϕ by

$$\phi([a, b], \omega) := L(ap, bp, a, b, \omega) \quad \text{for } 0 \leq a < b,$$

and, for $t \geq 0$, define the measure-preserving transformation $\sigma_t : \Omega \rightarrow \Omega$ by $\sigma_t := \tau_{tp, t}$.

In order to apply the sub-additive ergodic theorem of Akcoglu and Krengel [1], we need to verify that ϕ is a stationary, sub-additive process with respect to $(\sigma_t)_{t \geq 0}$, that is, for all $\omega \in \Omega$,

$$\begin{cases} \phi([a, b], \sigma_t \omega) = \phi([a, b] + t, \omega) & \text{for all } a < b \text{ and } t \geq 0, \\ \phi([a, c], \omega) \leq \phi([a, b], \omega) + \phi([b, c], \omega) & \text{for all } a < b < c, \text{ and} \\ \inf_{T > 0} \frac{1}{T} \mathbf{E} \phi([0, T], \cdot) > -\infty. \end{cases} \quad (4.11)$$

The stationarity and sub-additivity of ϕ follow from (4.8) and (4.9), which give

$$\phi([a, b], \sigma_t \omega) = L(ap, bp, a, b, \tau_{tp, t} \omega) = L((a+t)p, (b+t)p, a+t, b+t, \omega) = \phi([a, b] + t, \omega)$$

and

$$\phi([a, c], \omega) = L(ap, cp, a, c, \omega) \leq L(ap, bp, a, b, \omega) + L(bp, cp, b, c, \omega) = \phi([a, b], \omega) + \phi([b, c], \omega).$$

It remains to prove the third item of (4.11). This requires bounds for the long-time averages

$$\frac{1}{T} \int_0^T f(\gamma_r) \cdot dB_r.$$

The difficulty is that, as $T \rightarrow \infty$, this quantity need not converge to 0, in view of the fact that the almost-minimizing path γ need not be adapted to the Brownian motion B , as was the case in the proof of Lemma 3.1.

Fix $\gamma \in \mathcal{A}(0, Tp, 0, T)$ and $h > 0$ to be determined, and let $N \in \mathbb{N}$ be such that $T/h \leq N < T/h + 1$. Set $\tau_k := kh$ for $k = 0, 1, 2, \dots, N-1$, $\tau_N = T$, and

$$X_k := \max_{r \in [\tau_{k-1}, \tau_k]} |B_r - B_{\tau_{k-1}}|.$$

Then Young's inequality gives, for any $\delta > 0$ and some $C = C(q) > 0$,

$$\begin{aligned} \frac{1}{T} \int_0^T f(\gamma_r) \cdot dB_r &= \frac{1}{T} \sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} f(\gamma_r) \cdot dB_r \\ &= \frac{1}{T} \sum_{k=1}^N \left(f(\gamma_{\tau_k}) \cdot (B_{\tau_k} - B_{\tau_{k-1}}) - \int_{\tau_{k-1}}^{\tau_k} Df(\gamma_r) \cdot \dot{\gamma}_r \cdot (B_r - B_{\tau_{(k-1)h}}) dr \right) \\ &\geq -\frac{\delta^{q'}}{T} \int_0^T |\dot{\gamma}_r|^{q'} dr - \frac{\|f\|_\infty}{T} \sum_{k=1}^N X_k - \frac{Ch \|Df\|_\infty^q}{\delta^q T} \sum_{k=1}^N X_k^q. \end{aligned}$$

Choosing the constant δ small enough relative to the lower bound for H^* in (2.2) gives, for some $C > 1$,

$$\begin{aligned} \frac{1}{T} \int_0^T H^*(\dot{\gamma}_r) dr + \frac{1}{T} \int_0^T f(\gamma_r) \cdot dB_r &\geq \frac{1}{CT} \int_0^T |\dot{\gamma}_r|^{q'} dr - C \left(1 + \frac{\|f\|_\infty}{T} \sum_{k=1}^N X_k + \frac{h \|Df\|_\infty^q}{T} \sum_{k=1}^N X_k^q \right) \\ &\geq -C \left(1 + \frac{\|f\|_\infty}{T} \sum_{k=1}^N X_k + \frac{h \|Df\|_\infty^q}{T} \sum_{k=1}^N X_k^q \right). \end{aligned}$$

Taking expectations, we find that, for some constant $C > 0$ independent of T ,

$$\begin{aligned} \frac{1}{T} \mathbf{E} L(0, Tp, 0, T, \omega) &\geq -C \left(1 + \frac{\|f\|_\infty N h^{1/2}}{T} + \frac{\|Df\|_\infty^q N h^{1+q/2}}{T} \right) \\ &\geq -C \left(1 + \frac{\|f\|_\infty}{h^{1/2}} + \|Df\|_\infty^q h^{q/2} \right). \end{aligned}$$

We now choose

$$h := \frac{\|f\|_\infty^{2/(1+q)}}{\|Df\|_\infty^{2q/(1+q)}},$$

which leads to the lower bound

$$\frac{1}{T} \mathbf{E} L(0, Tp, 0, T, \omega) \geq -C(1 + K^{q/(q+1)}),$$

and so (4.11) is proved.

The sub-additive ergodic theorem of [1] then yields the existence of $\Omega_0 \in \mathbf{F}$ with $\mathbf{P}(\Omega_0) = 1$ and a random field $\bar{L} : \mathbb{Q}^d \times \Omega \rightarrow \mathbb{R}$ such that

$$\bar{L}(p, \omega) = \lim_{T \rightarrow +\infty} \frac{1}{T} \phi([0, T], \omega) = \lim_{T \rightarrow +\infty} \frac{1}{T} L(0, Tp, 0, T, \omega) \quad \text{for all } \omega \in \Omega_0 \text{ and } p \in \mathbb{Q}^d.$$

For $T > 0$, $p \in \mathbb{R}^d$, and $\omega \in \Omega_0$, set

$$\ell_T(p, \omega) := \frac{1}{T} L(0, Tp, 0, T, \omega) = L^{1/T}(0, p, 0, 1, \omega).$$

Fix $N \in \mathbb{N}$. Then, by Lemma 4.1, there exists $\Omega^N \in \mathbf{F}$ such that $\Omega^N \subset \Omega_0$ and $\mathbf{P}(\Omega^N) = 1$, and, for some $\varepsilon_N : \Omega^N \rightarrow \mathbb{R}_+$ and for all $\omega \in \Omega^N$, the collection

$$(\ell_T(\cdot, \omega))_{T > \varepsilon_N^{-1}(\omega)}$$

is equicontinuous on B_N . Set

$$\Omega_1 := \bigcap_{N \in \mathbb{N}} \Omega^N.$$

Then $\mathbf{P}(\Omega_1) = 1$, and, for any $\omega \in \Omega_1$, $\bar{L}(\cdot, \omega)$ can be extended to a continuous function on all of $\mathbb{R}^d$, and moreover, as $T \rightarrow \infty$, $\ell_T(\cdot, \omega)$ converges locally uniformly to $\bar{L}(\cdot, \omega)$.

Step 2: the limit is deterministic. Fix $p \in \mathbb{R}^d$ and $(y, s) \in \mathbb{R}^d \times [0, \infty)$. Then Lemma 4.1 implies that there exists a modulus $\rho : [0, \infty) \rightarrow [0, \infty)$ such that, for all $\omega \in \Omega_1$ and sufficiently large T ,

$$\begin{aligned} \left| \bar{L}(p, \omega) - \frac{1}{T} L(0, Tp, 0, T, \tau_{y,s}\omega) \right| &= \left| \bar{L}(p, \omega) - L^{1/T} \left(\frac{y}{T}, p + \frac{y}{T}, \frac{s}{T}, 1 + \frac{s}{T}, \omega \right) \right| \\ &\leq \left| \bar{L}(p, \omega) - L^{1/T}(0, p, 0, 1, \omega) \right| + \rho \left(\frac{|y|}{T} + \frac{s}{T} \right). \end{aligned}$$

Sending $T \rightarrow \infty$ yields

$$\bar{L}(p, \omega) = \bar{L}(p, \tau_{y,s}\omega)$$

for all $p \in \mathbb{R}^d$, $(y, s) \in \mathbb{R}^d \times [0, \infty)$, and $\omega \in \Omega_1$. The ergodicity of the group $(\tau_\cdot)$ then implies that $\omega \mapsto \bar{L}(\cdot, \omega)$ is constant, and, in fact,

$$\bar{L}(p) := \lim_{T \rightarrow \infty} \frac{1}{T} \mathbf{E} L(0, Tp, 0, T, \cdot).$$

Step 3: convexity and estimates. The convexity can be seen in a standard way from the sub-additivity of L (see, for instance, [54, 57]). To prove (4.10), we appeal to Lemma 3.3, which yields a constant $C = C(M_0, q) > 1$ such that, for all $T > 0$,

$$\frac{1}{C} |p|^{q'} - C \leq \mathbf{E} L^{1/T}(0, p, 0, 1, \omega) \leq C (|p|^{q'} + 1).$$

Letting $T \rightarrow \infty$ finishes the proof. $\square$

Before we continue, we give an example to show that, in general, the assumption of almost-sure boundedness for f cannot be dropped in the above argument. Consider the stationary sub-additive process

$$\psi([a, b], \omega) := \inf \left\{ \frac{1}{2} \int_a^b |\dot{\gamma}_t|^2 dt + \int_a^b \gamma_t \cdot dB_t : \gamma \in \mathcal{A}(0, 0, a, b) \right\};$$

here $d = m = 1$ and B is a one-dimensional Brownian motion. A straightforward computation reveals that the action is minimized by the path

$$[a, b] \ni t \mapsto \gamma_t^* := \int_a^t \left(B_s - \frac{1}{b-a} \int_a^b B_r dr \right) ds,$$

which leads to the identity

$$\psi([a, b], \omega) = \frac{1}{2} \left[\frac{1}{b-a} \left(\int_a^b B_r dr \right)^2 - \int_a^b B_r^2 dr \right].$$

Then

$$\mathbf{E} \frac{1}{T} \psi([0, T], \cdot) = -\frac{T}{12},$$

which is unbounded in T .

4.4. The local uniform convergence of L^ε

Fix $\varepsilon > 0$ and $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \times [0, \infty)$ with $s < t$. Then Lemma 4.2 and the stationarity of L^ε yield

$$\mathbf{E} L^\varepsilon(x, y, s, t, \cdot) = \mathbf{E} \varepsilon L \left(0, \frac{y-x}{\varepsilon}, 0, \frac{t-s}{\varepsilon}, \cdot \right) \xrightarrow{\varepsilon \rightarrow 0} (t-s) \bar{L} \left(\frac{y-x}{t-s} \right).$$

The next lemma establishes the local-uniform convergence of L^ε with probability one, using a standard argument that combines the multi-parameter ergodic theorem and Egoroff's theorem. Such an argument has been used several times throughout the literature on the stochastic homogenization of Hamilton-Jacobi equations; see, for example, [5–7, 23, 38, 39, 41, 42, 54].

Lemma 4.3. *Assume (2.1) and (2.12). Then*

$$\mathbf{P} \left(\lim_{\varepsilon \rightarrow 0} \max_{|x| \leq R} \max_{|y| \leq R} \max_{s, t \in [0, T], t-s > \tau} \left| L^\varepsilon(x, y, s, t, \cdot) - (t-s) \bar{L} \left(\frac{y-x}{t-s} \right) \right| = 0 \text{ for all } R > 0, 0 < \tau < T \right) = 1.$$

Proof. Let $\Omega_0 \in \mathbf{F}$ be the event of full probability for which the conclusion of Lemma 4.2 holds. Then Egoroff's Theorem implies that, for all $\eta > 0$, there exists an event $G_\eta \subset \Omega_0$ such that $\mathbf{P}(G_\eta) \geq 1 - \eta$ and, for any $M > 0$,

$$\lim_{T \rightarrow \infty} \sup_{|p| \leq M} \sup_{\omega \in G_\eta} \left| \frac{1}{T} L(0, Tp, 0, T, \omega) - \bar{L}(p) \right| = 0.$$

Fix $R > 0$ and $T > 0$, and, for $\omega \in \Omega_0$, define

$$A_\eta^\varepsilon(\omega) := \{(x, s) \in B_R \times [0, T] : \tau_{x/\varepsilon, s/\varepsilon} \omega \in G_\eta\}.$$

The multiparameter ergodic theorem (see Becker [15]) then yields that, for some $\Omega_\eta \in \mathbf{F}$ satisfying $\Omega_\eta \subset \Omega_0$ and $\mathbf{P}(\Omega_\eta) = 1$, and for all $\omega \in \Omega_\eta$,

$$\frac{|A_\eta^\varepsilon(\omega)|}{|B_R \times [0, T]|} = \frac{1}{|B_R \times [0, T]|} \int_{B_R \times [0, T]} \mathbf{1}_{G_\eta}(\tau_{(x/\varepsilon, s/\varepsilon)} \omega) dx ds \xrightarrow{\varepsilon \rightarrow 0} \mathbf{P}(G_\eta),$$

and so, for some $\varepsilon_\eta : \Omega \rightarrow \mathbb{R}_+$ and for all $\omega \in \Omega_\eta$, $\varepsilon_\eta(\omega) > 0$ and

$$|A_\eta^\varepsilon(\omega)| \geq (1 - 2\eta) |B_R \times [0, T]| \quad \text{for all } 0 < \varepsilon < \varepsilon_1(\omega).$$

Now, fix $\omega \in \Omega_\eta$, $0 < \varepsilon < \varepsilon_\eta(\omega) \wedge \varepsilon_0(\omega)$, $(x, y) \in B_R^2$, and $s, t \in [0, T]$ satisfying $t - s \geq \tau$. Then there exists $(x_\varepsilon, s_\varepsilon) \in A_\eta^\varepsilon(\omega)$ such that, for some constant $c = c(d, R, T) > 0$,

$$|x - x_\varepsilon| + |s - s_\varepsilon| \leq c\eta^{1/(d+1)}.$$

Choosing η sufficiently small, depending on τ , we have $t - s_\varepsilon > \tau/2$.

Set

$$p_\varepsilon := \frac{y - x_\varepsilon}{t - s_\varepsilon} \quad \text{and} \quad T_\varepsilon := \frac{t - s_\varepsilon}{\varepsilon}.$$

Note that, for some constant $M = M(R, \tau) > 0$, $|p_\varepsilon| \leq M$.

We now invoke Lemma 4.1, which gives a deterministic modulus $\rho : [0, \infty) \rightarrow [0, \infty)$, depending only on R, T , and τ , such that, for all $\omega \in \Omega_\eta$ and $\varepsilon \in (0, \varepsilon_0(\omega) \wedge \varepsilon_\eta(\omega))$,

$$\begin{aligned} & \left| L^\varepsilon(x, y, s, t, \omega) - (t - s) \bar{L}\left(\frac{y - x}{t - s}\right) \right| \leq \left| L^\varepsilon(x_\varepsilon, y, s_\varepsilon, t, \omega) - (t - s_\varepsilon) \bar{L}\left(\frac{y - x_\varepsilon}{t - s_\varepsilon}\right) \right| \\ & + \left| L^\varepsilon(x, y, s, t, \omega) - L^\varepsilon(x_\varepsilon, y, s_\varepsilon, t, \omega) \right| + \left| (t - s) \bar{L}\left(\frac{y - x}{t - s}\right) - (t - s_\varepsilon) \bar{L}\left(\frac{y - x_\varepsilon}{t - s_\varepsilon}\right) \right| \\ & \leq \rho(\eta) + T \left| \frac{1}{T_\varepsilon} L(0, T_\varepsilon p_\varepsilon, 0, T_\varepsilon, \tau_{(x_\varepsilon/\varepsilon, s_\varepsilon/\varepsilon)} \omega) - \bar{L}(p_\varepsilon) \right| \\ & \leq \rho(\eta) + T \sup_{|p| \leq M} \sup_{\tilde{\omega} \in G_\eta} \left| \frac{1}{T_\varepsilon} L(0, T_\varepsilon p, 0, T_\varepsilon, \tilde{\omega}) - \bar{L}(p) \right|. \end{aligned}$$

Sending $\varepsilon \rightarrow 0$ gives

$$\limsup_{\varepsilon \rightarrow 0} \max_{|x| \leq R} \max_{|y| \leq R} \max_{s, t \in [0, T], t - s > \tau} \left| L^\varepsilon(x, y, s, t, \omega) - (t - s) \bar{L}\left(\frac{y - x}{t - s}\right) \right| \leq \rho(\eta).$$

For $n \in \mathbb{N}$, set $\eta_n := \frac{1}{n}$. If

$$\tilde{\Omega} := \bigcup_{n \in \mathbb{N}} \Omega_{\eta_n},$$

we then have $\mathbf{P}(\tilde{\Omega}) = 1$ and, for all $\omega \in \tilde{\Omega}$,

$$\lim_{\varepsilon \rightarrow 0} \max_{|x| \leq R} \max_{|y| \leq R} \max_{s, t \in [0, T], t - s > \tau} \left| L^\varepsilon(x, y, s, t, \omega) - (t - s) \bar{L}\left(\frac{y - x}{t - s}\right) \right| = 0.$$

The result is finished in a standard way upon repeating the argument for a countable collection of R, T , and τ , and taking the countable intersection of the resulting events of full probability. $\square$

4.5. The effective Hamiltonian and the homogenization of the equation

Define the convex function

$$\overline{H}(p) := \overline{L}^*(p) := \sup_{q \in \mathbb{R}^d} (p \cdot q - \overline{L}(q)),$$

which, in view of (4.10), immediately satisfies (4.3) (although in the following section, we will provide a sharper lower bound).

The proof of Theorem 4.1 will follow from the local uniform convergence of the Lagrangians, as well as the fact that, because $\overline{H}$ is convex and $u_0 \in BUC(\mathbb{R}^d)$, $\bar{u}$ is given by the Hopf-Lax formula

$$\bar{u}(x, t) = \inf_{y \in \mathbb{R}^d} \left(u_0(y) + t \overline{L} \left(\frac{x - y}{t} \right) \right). \quad (4.12)$$

Proof of Theorem 4.1. Let $\Omega_1 \in \mathbf{F}$ be the event of full probability for which the conclusion of Lemma 4.3 holds.

Step 1: coercivity bounds. We first demonstrate that there exists $\Omega_2 \subset \Omega_1$ such that $\mathbf{P}(\Omega_2) = 1$, as well as $\varepsilon_2 : \Omega \rightarrow \mathbb{R}_+$ and a constant $C = C(T, M_0, q, \alpha) > 1$ such that, for all $x, y \in \mathbb{R}^d$, $0 \leq s < t \leq T$, $\omega \in \Omega_2$, and $0 < \varepsilon < \varepsilon_2(\omega)$,

$$-C(t-s)^\alpha + \frac{1}{C} \frac{|y-x|^{q'}}{(t-s)^{q'-1}} \leq L^\varepsilon(y, x, s, t, \omega) \leq C \frac{|y-x|^{q'}}{(t-s)^{q'-1}} + C(t-s)^\alpha. \quad (4.13)$$

In view of Lemma 3.3, there exist random variables $(\mathcal{D}_\varepsilon)_{\varepsilon>0} : \Omega \rightarrow \mathbb{R}_+$ such that, for some constant $C = C(T, M_0, q, \alpha) > 1$ and for all $\varepsilon > 0$, $x, y \in \mathbb{R}^d$, and $0 \leq s < t \leq T$,

$$-C(\mathcal{D}_\varepsilon + 1)(t-s)^\alpha + \frac{1}{C} \frac{|y-x|^{q'}}{(t-s)^{q'-1}} \leq L^\varepsilon(y, x, s, t, \omega) \leq C \frac{|y-x|^{q'}}{(t-s)^{q'-1}} + C(\mathcal{D}_\varepsilon + 1)(t-s)^\alpha,$$

and, for all $p \geq 1$, there exists a constant $C' = C'(T, M_0, p, q, \alpha) > 0$ such that, for all $\varepsilon > 0$ and $\lambda \geq 1$,

$$\mathbf{P}(\mathcal{D}_\varepsilon > \lambda) \leq \frac{C' \varepsilon^{p/2}}{\lambda^p}.$$

The Borel-Cantelli lemma yields the existence of $k_0 : \Omega \rightarrow \mathbb{N}$ and $\Omega_2 \subset \Omega_1$ with $\mathbf{P}(\Omega_2) = 1$ such that, for all $\omega \in \Omega_2$ and $k \geq k_0(\omega)$,

$$\mathcal{D}_{2^{-k}}(\omega) \leq 1,$$

and so, for all $\omega \in \Omega_2$, $k \geq k_0(\omega)$, $x, y \in \mathbb{R}^d$, and $s, t \in [0, T]$ with $s < t$,

$$-C(t-s)^\alpha + \frac{1}{C} \frac{|y-x|^{q'}}{(t-s)^{q'-1}} \leq L^{2^{-k}}(y, x, s, t, \omega) \leq C \frac{|y-x|^{q'}}{(t-s)^{q'-1}} + C(t-s)^\alpha.$$

Now choose $\varepsilon < \varepsilon_2(\omega) := 2^{-k_0(\omega)}$ and let $k > k_0(\omega)$ be such that $2^{-k-1} < \varepsilon \leq 2^{-k}$. Then, for all $x, y \in \mathbb{R}^d$ and $0 \leq s < t \leq T$,

$$L^\varepsilon(x, y, s, t, \omega) = \varepsilon L \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{s}{\varepsilon}, \frac{t}{\varepsilon}, \omega \right) = (2^k \varepsilon) L^{2^{-k}} \left(\frac{x}{2^k \varepsilon}, \frac{y}{2^k \varepsilon}, \frac{s}{2^k \varepsilon}, \frac{t}{2^k \varepsilon}, \omega \right).$$

It follows that (4.13) holds upon replacing C with $2C$, because $1 \leq 2^k \varepsilon < 2$.

Step 2: localization. Let $\tau > 0$ be fixed. We claim that there exists a deterministic M depending only on T , τ , and $\|u_0\|_\infty$ such that, for all $(x, t) \in \mathbb{R}^d \times [0, T]$, $\omega \in \Omega_2$, and $\varepsilon \in (0, \varepsilon_2(\omega))$,

$$u^\varepsilon(x, t, \omega) = \inf_{y \in B_M(x)} (u_0(y) + L^\varepsilon(y, x, 0, t, \omega)). \quad (4.14)$$

Setting $y = x$ in the definition of u^ε and using (4.13), we find that there exists a constant $C = C(T) > 0$ such that, for all $\omega \in \Omega_2$, $\varepsilon \in (0, \varepsilon_2(\omega))$, and $(x, t) \in \mathbb{R}^d \times [0, T]$,

$$u^\varepsilon(x, t, \omega) \leq C.$$

The lower bound in (4.13) then yields that, if $|y - x| > M$ and M is chosen large enough depending only on T and τ , then, for all $\varepsilon \in (0, \varepsilon_2(\omega))$,

$$u_0(y) + L^\varepsilon(y, x, 0, t, \omega) \geq -\|u_0\|_\infty + \frac{1}{C} \frac{|y - x|^{q'}}{\tau^{q'}} - C \geq -\|u_0\|_\infty + \frac{M^{q'}}{C\tau} - C > u^\varepsilon(x, t, \omega).$$

This establishes (4.14).

Step 3. By (4.14) and Lemma 4.3, we have, for all $\omega \in \Omega_2$,

$$\sup_{(x,t) \in B_R \times [\tau, T]} |u^\varepsilon(x, t, \omega) - \bar{u}(x, t)| \leq \sup_{x \in B_R} \sup_{y \in B_{R+M}} \sup_{\tau < t < T} \left| L^\varepsilon(y, x, 0, t, \omega) - t\bar{L}\left(\frac{x-y}{t}\right) \right| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The bounds in (4.13) give, for $(x, t) \in B_R \times [0, \tau]$ and $\omega \in \Omega_2$,

$$u^\varepsilon(x, t, \omega) \leq u_0(x) + C\tau^\alpha,$$

and, if $\rho : [0, \infty) \rightarrow [0, \infty)$ is the modulus of continuity for u_0 ,

$$u^\varepsilon(x, t, \omega) \geq u_0(x) - C\tau^\alpha + \inf_{r \geq 0} \left(-\rho(r) + \frac{1}{C} \frac{r^{q'}}{\tau^{q'-1}} \right).$$

Define

$$\tilde{\rho}(\tau) := \sup_{r \geq 0} \left(\rho(r) - \frac{1}{C} \frac{r^{q'}}{\tau^{q'-1}} \right),$$

and note that $\tilde{\rho} : [0, \infty) \rightarrow [0, \infty)$ satisfies $\lim_{\tau \rightarrow 0^+} \tilde{\rho}(\tau) = 0$. As a result, for all $\omega \in \Omega_2$ and $\varepsilon \in (0, \varepsilon_2(\omega))$,

$$\sup_{(x,t) \in B_R \times [0, \tau]} |u^\varepsilon(x, t, \omega) - u_0(x)| \leq C\tau^\alpha + C\tilde{\rho}(\tau).$$

A similar argument gives

$$\sup_{(x,t) \in B_R \times [0, \tau]} |\bar{u}(x, t, \omega) - u_0(x)| \leq C\tau + \tilde{\rho}(\tau),$$

and so, for all $\tau > 0$ and $\omega \in \Omega_2$,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{(x,t) \in B_R \times [0, T]} |u^\varepsilon(x, t, \omega) - \bar{u}(x, t)| \leq C(\tau + \tau^\alpha) + \tilde{\rho}(\tau).$$

The proof is finished upon letting $\tau \rightarrow 0$. $\square$

5. Enhancement

If L is defined as in (4.7), then, for all $T > 0$ and $v \in \mathbb{R}^d$,

$$\mathbf{E} \frac{1}{T} L(0, Tv, 0, T, \cdot) \leq H^*(v) + \frac{1}{T} \mathbf{E} \int_0^T f(tv, \cdot) \cdot dB_t = H^*(v),$$

so that, in general, $\bar{L} \leq H^*$ and

$$\bar{H} \geq H.$$

This is actually an equality if f is equal to a fixed, deterministic constant $\bar{f} \in \mathbb{R}^d$, since then, for each $v \in \mathbb{R}^d$ and for $\mathbf{P}$ -almost every $\omega \in \Omega$,

$$\frac{1}{T} L(0, Tv, 0, Tv, \omega) = H^*(v) + \bar{f} \cdot \frac{B(T, \omega)}{T} \xrightarrow{T \rightarrow \infty} H^*(v).$$

It turns out that this is the only situation in which the two Hamiltonians are equal. In fact, a consequence of the isotropic nature of the temporal noise is that, if f is nonconstant, then $\overline{H}$ is actually greater than H everywhere.

In order to facilitate the following arguments, we will assume additionally that

$$\begin{cases} \text{for some } \kappa \in (0, 1) \text{ and } M_0 > 0, \\ f(\cdot, \omega) \in C^{1,\kappa}(\mathbb{R}^d, \mathbb{R}^m) \text{ with probability one, and} \\ \|f\|_\infty + \|Df\|_\infty + [Df]_\kappa \leq M_0. \end{cases} \quad (5.1)$$

Let $\mathbf{v} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathbf{p} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfy

$$\mathbf{p}(v) \in \arg \max_{p \in \mathbb{R}^d} \{p \cdot v - H(p)\} \quad \text{and} \quad \mathbf{v}(p) \in \arg \max_{v \in \mathbb{R}^d} \{p \cdot v - H^*(v)\},$$

and, for $\rho > 0$ and $v \in \mathbb{R}^d$, set

$$G(v) := \sup_{\rho \in (0, 1]} \frac{1}{\rho} \max_{|z| \leq \rho} \{H^*(v+z) - H^*(v) - \mathbf{p}(v) \cdot z\}.$$

Note that (2.3) and the convexity of H^* imply that

$$0 \leq G(v) \leq C(1 + |v|^{q'-1}) \quad \text{for all } v \in \mathbb{R}^d.$$

Theorem 5.1. Assume that H satisfies (2.1) and the random field F satisfies (2.12) and (5.1). Let $\overline{H}$ be the effective Hamiltonian from Theorem 4.1. Then there exists $c = c(M_0, \kappa) > 0$ such that, for all $\lambda \in (0, \kappa)$,

$$\overline{H}(p) \geq H(p) + \frac{c(\mathbf{E}|Df(0)|^2)^{2+1/\lambda}}{1 + G(\mathbf{v}(p))} \quad \text{for all } p \in \mathbb{R}^d. \quad (5.2)$$

As an example, consider the Hamiltonian

$$H(p) := \frac{1}{q}|p|^q,$$

whose Legendre transform is given by

$$H^*(v) = \frac{1}{q'}|v|^{q'}.$$

We then have

$$\mathbf{p}(v) = DH^*(v) = |v|^{q'-2}v, \quad \mathbf{v}(p) = DH(p) = |p|^{q-2}p,$$

and, for some constant $C = C(q) > 0$,

$$G(v) \leq C(1 + |v|)^{q'-2},$$

so that (5.2) becomes, for some $c > 0$,

$$\overline{H}(p) \geq \frac{1}{q}|p|^q + c(1 + |p|)^{\frac{q-2}{q-1}}.$$

Proof of Theorem 5.1. Fix $v \in \mathbb{R}^d$, $M > 0$, and $N \in \mathbb{N}$. We define a path $\gamma \in \mathcal{A}(0, NMv, 0, NM)$ as follows: set

$$\eta_r := 1 - |2r - 1| \quad \text{for } r \in [0, 1],$$

and, for a sequence $(u_k)_{k \in \mathbb{N}} \subset B_1 \subset \mathbb{R}^d$ and

$$0 < \delta \leq \frac{M}{2}, \quad (5.3)$$

define

$$\gamma_r := vr + \delta u_k \eta \left(\frac{r - kM}{M} \right) \quad \text{for } r \in [kM, (k+1)M] \text{ and } k = 0, 1, 2, \dots$$

Then

$$\frac{1}{NM} L(0, NMv, 0, NM, \omega) \leq \frac{1}{NM} \int_0^{NM} H^*(\dot{\gamma}_r) dr + \frac{1}{NM} \int_0^{NM} f(\gamma_r, \omega) \cdot dB_r(\omega).$$

Since $|\dot{\gamma} - v| \leq \frac{2\delta}{M} \leq 1$ and $\frac{1}{NM} \int_0^{NM} \dot{\gamma}_r dr = v$, we find that

$$\begin{aligned} \frac{1}{NM} \int_0^{NM} H^*(\dot{\gamma}_r) dr &= H^*(v) + \mathbf{p}(v) \cdot \frac{1}{NM} \int_0^{NM} (\dot{\gamma}_r - v) dr \\ &\quad + \frac{1}{NM} \int_0^{NM} (H^*(\dot{\gamma}_r) - H^*(v) - \mathbf{p}(v) \cdot (\dot{\gamma}_r - v)) dr \\ &\leq H^*(v) + \frac{2\delta}{M} G(v). \end{aligned}$$

For $r \geq 0$, set $B^M(r) := M^{-1/2} B(Mr)$. Then

$$\begin{aligned} \frac{1}{NM} \int_0^{NM} f(\gamma_r, \omega) \cdot dB_r(\omega) &- \frac{1}{NM} \int_0^{NM} f(vr, \omega) \cdot dB_r(\omega) \\ &= \frac{\delta}{NM} \sum_{k=0}^{N-1} u_k \cdot \int_{kM}^{(k+1)M} \int_0^1 Df \left(vr + s\delta\eta \left(\frac{r - kM}{M} \right) u_k, \omega \right) ds \eta \left(\frac{r - kM}{M} \right) dB_r(\omega) \\ &= \frac{\delta}{NM^{1/2}} \sum_{k=0}^{N-1} u_k \cdot \int_0^1 \int_0^1 Df (Mv(r+k) + s\delta\eta(r) u_k, \omega) ds \eta(r) dB_{r+k}^M(\omega). \end{aligned}$$

The choice of the sequence $(u_k)_{k \in \mathbb{N}}$ was arbitrary, and so

$$\begin{aligned} \frac{1}{NM} L(0, NMv, 0, NM, \omega) &\leq H^*(v) + \frac{2\delta}{M} G(v) \\ &\quad + \frac{1}{NM} \int_0^{NM} f(vr, \omega) \cdot dB_r + \frac{\delta}{NM^{1/2}} \sum_{k=0}^{N-1} Z_\delta(v, \sigma_k \omega), \end{aligned} \tag{5.4}$$

where $\sigma_k := \tau_{Mvk, Mk}$ and

$$Z_\delta(v, \omega) := \min_{u \in B_1} u \cdot \int_0^1 \int_0^1 Df (Mvr + s\delta\eta(r)u, \omega) ds \eta(r) dB_r^M(\omega).$$

The random field Z_δ takes the form

$$Z_\delta(v, \omega) = \min_{u \in B_1} u \cdot Y(v, \delta u, \omega),$$

where

$$Y(v, y, \omega) := \int_0^1 \int_0^1 Df (Mvr + sy\eta(r), \omega) ds \eta(r) dB_r^M(\omega).$$

In view of (5.1), for all $m \geq 1$, there exists a constant $C = C(m) > 0$ such that, for all $y_1, y_2 \in B_1$,

$$\mathbf{E} |Y(v, y_1, \cdot) - Y(v, y_2, \cdot)|^m \leq C M_0^m |y_1 - y_2|^{\kappa m}.$$

The Kolmogorov continuity criterion implies that, for any $\lambda \in (0, \kappa)$ and $\mathbf{P}$ -almost every $\omega \in \Omega$, $Y(v, \cdot, \omega) \in C^\lambda(B_1)$, and, moreover, for some constant $C = C(M_0, \lambda)$,

$$\sup_{v \in \mathbb{R}^d} \mathbf{E}[Y(v, \cdot, \cdot)]_{C^\lambda(B_1)} \leq C.$$

We then find that $Z_\delta(v, \cdot) \in L^1(\Omega, \mathbf{P})$, since

$$|Z_\delta(v, \omega)| \leq \sup_{|u| \leq 1} |Y(v, \delta u, \omega)| \leq |Y(v, 0, \omega)| + [Y(v, \cdot, \omega)]_{C^\lambda(B_1)} \delta^\lambda.$$

Moreover,

$$\mathbf{E} Z_\delta(v, \cdot) \leq \mathbf{E} Z_0(v, \cdot) + C \delta^\lambda = -\mathbf{E} |Y(v, 0, \cdot)| + C \delta^\lambda.$$

Recall that we may assume, in view of the independence of f and B , that $\mathbf{P}$ takes the form $\mu \otimes \nu$ on the probability space $C_b^1(\mathbb{R}^d, \mathbb{R}^m) \times C([0, \infty), \mathbb{R}^m)$, where μ is a probability measure that is stationary and ergodic with respect to spatial translations, and ν is the Wiener measure (see subsection 4.2). The rotational invariance of ν then yields

$$\begin{aligned} \mathbf{E} |Y(v, 0, \cdot)| &= \int_{C_b^1(\mathbb{R}^d, \mathbb{R}^m)} \int_{C([0, \infty), \mathbb{R}^m)} \left| \int_0^1 Df(Mvr) \eta(r) \cdot dB_r \right| d\nu(B) d\mu(f) \\ &= \int_{C_b^1(\mathbb{R}^d, \mathbb{R}^m)} \int_{C([0, \infty), \mathbb{R}^m)} \left| \int_0^1 |Df(Mvr)| \eta(r) dB_r^1 \right| d\nu(B) d\mu(f). \end{aligned}$$

For fixed $f \in C_b^1(\mathbb{R}^d, \mathbb{R}^m)$, the random variable

$$C([0, \infty), \mathbb{R}^m) \ni B \mapsto \int_0^1 |Df(Mvr)| \eta(r) dB_r^1$$

is a stochastic integral with deterministic integrand, and is therefore a Gaussian random variable with respect to the probability measure ν . Its standard deviation can therefore be computed, according to Itô's formula, as

$$\int_{C([0, \infty), \mathbb{R}^m)} \left| \int_0^1 |Df(Mvr)| \eta(r) dB_r^1(\cdot) \right| d\nu(B) = \sqrt{\frac{2}{\pi}} \left(\int_0^1 |Df(Mvr)|^2 \eta(r)^2 dr \right)^{1/2}.$$

Since $0 \leq \eta \leq 1$, we have

$$\int_0^1 |Df(Mvr)|^2 \eta(r)^2 dr \leq \|Df\|_\infty \left(\int_0^1 |Df(Mvr)|^2 \eta(r)^2 dr \right)^{1/2}.$$

It now follows from Fubini's theorem and the stationarity of f that

$$\begin{aligned} \mathbf{E} |Y(v, 0, \cdot)| &= \sqrt{\frac{2}{\pi}} \int_{C_b^1(\mathbb{R}^d, \mathbb{R}^m)} \left(\int_0^1 |Df(Mvr)|^2 \eta(r)^2 dr \right)^{1/2} d\mu(f) \\ &\geq \frac{1}{\|Df\|_\infty} \sqrt{\frac{2}{\pi}} \int_{C_b^1(\mathbb{R}^d, \mathbb{R}^m)} \int_0^1 |Df(Mvr)|^2 \eta(r)^2 dr d\mu(f) \end{aligned}$$

$$= \frac{1}{3 \|Df\|_\infty} \sqrt{\frac{2}{\pi}} \mathbf{E} |Df(0)|^2.$$

We then set

$$\delta := \bar{c} \left(\mathbf{E} |Df(0)|^2 \right)^{1/\lambda} \quad (5.5)$$

for some sufficiently small constant $\bar{c} = \bar{c}(M_0, \lambda) > 0$, and we conclude that, for a further constant $c = c(M_0, \kappa) > 0$,

$$\sup_{v \in \mathbb{R}^d} \mathbf{E} Z_\delta(v, \cdot) \leq -c \mathbf{E} |Df(0)|^2.$$

Taking the expectation of both sides of (5.4), we obtain, for some $c = c(M_0, \kappa) > 0$,

$$\frac{1}{NM} \mathbf{E} L(0, NMv, 0, NM, \cdot) \leq H^*(v) + \frac{2\delta}{M} G(v) + \frac{\delta}{M^{1/2}} \mathbf{E} Z_\delta(v, \cdot) \leq H^*(v) + c\delta \left(\frac{1}{M} G(v) - \frac{\mathbf{E} |Df(0)|^2}{M^{1/2}} \right),$$

and so, sending $N \rightarrow \infty$,

$$\bar{L}(v) \leq H^*(v) + c\delta \left(\frac{1}{M} G(v) - \frac{\mathbf{E} |Df(0)|^2}{M^{1/2}} \right).$$

We now choose

$$M := 4 \frac{(1 + G(v))^2}{(\mathbf{E} |Df(0)|^2)^2}.$$

Observe that, if $\bar{c}$ in (5.5) is chosen so that $\bar{c} M_0^{4+2/\lambda} \leq 2$, then

$$\delta = \bar{c} \left(\mathbf{E} |Df(0)|^2 \right)^{1/\lambda} \leq \frac{\bar{c} M_0^{4+2/\lambda}}{(\mathbf{E} |Df(0)|^2)^2} \leq \frac{M}{2},$$

so that (5.3) is satisfied. For this choice of M , we obtain, for some constant $c = c(M_0, \kappa, \lambda) > 0$,

$$\bar{L}(v) \leq H^*(v) - c\delta \frac{(\mathbf{E} |Df(0)|^2)^2}{1 + G(v)} = H^*(v) - c\bar{c} \frac{(\mathbf{E} |Df(0)|^2)^{2+1/\lambda}}{1 + G(v)},$$

and therefore, for all $p \in \mathbb{R}^d$,

$$\bar{H}(p) \geq \sup_{v \in \mathbb{R}^d} \left\{ p \cdot v - H^*(v) + c\bar{c} \frac{(\mathbf{E} |Df(0)|^2)^{2+1/\lambda}}{1 + G(v)} \right\} \geq H(p) + c\bar{c} \frac{(\mathbf{E} |Df(0)|^2)^{2+1/\lambda}}{1 + G(v(p))}. \quad \square$$

6. Noise of varying strength

We conclude by studying, for $\theta \in \mathbb{R}$, $\varepsilon > 0$, and a stationary-ergodic random field f and Brownian motion B satisfying (2.5) - (2.11), the initial value problem

$$u_t^\varepsilon + H(Du^\varepsilon) = \varepsilon^\theta f\left(\frac{x}{\varepsilon}, \omega\right) \cdot \dot{B}(t, \omega) \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \quad \text{and} \quad u^\varepsilon(x, 0, \omega) = u_0(x) \quad \text{in } \mathbb{R}^d \times \Omega. \quad (6.1)$$

The strength of the noise determines the nature of the enhancement effect for vanishing ε . Namely, when θ is equal to the scaling critical exponent $1/2$, the enhancement property can be exactly characterized using the results from the previous section. When $\theta > 1/2$, the noise is macroscopically insignificant, while taking $\theta < 1/2$ gives rise to infinite velocity.

Theorem 6.1. Assume $u_0 \in BUC(\mathbb{R}^d)$, (2.1), and f and B satisfy (2.5) - (2.11) and (5.1).

(a) If $\theta > 1/2$, then, as $\varepsilon \rightarrow 0$, u^ε converges locally uniformly in probability to the solution u of

$$u_t + H(Du) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.$$

- (b) If $\theta < 1/2$ and f is nonconstant, then, as $\varepsilon \rightarrow 0$, u^ε converges locally uniformly in $\mathbb{R}^d \times (0, \infty)$ in probability to $-\infty$.
- (c) If $\theta = 1/2$, then there exists a deterministic, convex Hamiltonian $\overline{H} : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying (4.3) and (5.2) such that, as $\varepsilon \rightarrow 0$, u^ε converges locally uniformly in probability to the solution $\overline{u}$ of

$$\overline{u}_t + \overline{H}(D\overline{u}) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d \times \{0\}.$$

Proof. Replacing the Brownian motion B with $t \mapsto \varepsilon^{1/2} B(t/\varepsilon)$ and invoking Lemma 2.1, it follows that it suffices to prove the appropriate local uniform limits, with probability one, for the function

$$\tilde{u}^\varepsilon(x, t, \omega) := \inf_{y \in \mathbb{R}^d} (u_0(y) + L_{\varepsilon, f}(y, x, 0, t, \omega)), \quad (6.2)$$

where, for $\varepsilon > 0$, $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \times [0, \infty)$ with $s < t$, and $\omega \in \Omega$,

$$L_{\varepsilon, f}(x, y, s, t, \omega) := \inf \left\{ \int_s^t H^*(\dot{\gamma}_r) dr + \varepsilon^{\theta-1/2} \int_s^t f(\varepsilon^{-1} \gamma_r, \omega) \cdot dB_{r/\varepsilon}(\omega) : \gamma \in \mathcal{A}(x, y, s, t) \right\}.$$

(a) Similarly to the proof of Theorem 4.1, the result will follow from two facts:

$$\left\{ \begin{array}{l} \text{there exists } C > 1 \text{ and } \varepsilon_0 : \Omega \rightarrow \mathbb{R}_+ \text{ such that, with probability one,} \\ \text{for all } \varepsilon \in (0, \varepsilon_0), x, y \in \mathbb{R}^d, \text{ and } s, t \in [0, T] \text{ with } s < t, \\ -C(t-s)^\alpha + \frac{1}{C} \frac{|y-x|^{q'}}{(t-s)^{q'-1}} \leq L_{\varepsilon, f}(x, y, s, t) \leq C \left(\frac{|y-x|^{q'}}{(t-s)^{q'-1}} + (t-s)^\alpha \right), \end{array} \right. \quad (6.3)$$

and

$$\left\{ \begin{array}{l} \text{for all } R > 0 \text{ and } 0 < \tau < T, \\ \lim_{\varepsilon \rightarrow 0} \sup_{x, y \in B_R} \sup_{s, t \in [0, T], \tau \leq t-s \leq T} \left| L_{\varepsilon, f}(x, y, s, t) - (t-s) H^* \left(\frac{y-x}{t-s} \right) \right| = 0. \end{array} \right. \quad (6.4)$$

Setting

$$\sigma^\varepsilon := M_0 \varepsilon^{1/2}, \quad \Lambda^\varepsilon := M_0 \varepsilon^{-1/2}, \quad F_\varepsilon := F_{\sigma^\varepsilon, \Lambda^\varepsilon}, \quad \text{and} \quad \mathcal{D}_\varepsilon := \mathcal{D}_{\sigma^\varepsilon, \Lambda^\varepsilon},$$

we see that, by Lemma 3.1, there exists a constant $C = C(T, M_0) > 0$ such that, for all $\delta \in (0, 1)$, $0 \leq s \leq r_1 \leq r_2 \leq t \leq T$, $f \in F_\varepsilon$, and Lipschitz γ ,

$$\left| \int_{r_1}^{r_2} f(\varepsilon^{-1} \gamma_r) \cdot dB_{r/\varepsilon} \right| \leq \left(C \delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr + \frac{C + \mathcal{D}_\varepsilon}{\delta^q} \right) (r_2 - r_1)^\alpha, \quad (6.5)$$

and, for some $C = C(T, M_0, p) > 0$ and all $\lambda \geq 1$,

$$\mathbf{P}(\mathcal{D}_\varepsilon > \lambda) \leq \frac{C \varepsilon^{p/2}}{\lambda^{p/2}}.$$

We first prove (6.3). Since $\theta > 1/2$, similar arguments as in the proof of Lemma 3.3 give, for some $C > 1$ and all $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \times [0, \infty)$ with $s < t$ and $\varepsilon \in (0, 1)$,

$$\frac{1}{C} \frac{|y-x|^{q'}}{(t-s)^{q'-1}} - C(1 + \mathcal{D}_\varepsilon)(t-s)^\alpha \leq L_{\varepsilon, f}(x, y, s, t) \leq C \left(\frac{|y-x|^{q'}}{(t-s)^{q'-1}} + (1 + \mathcal{D}_\varepsilon)(t-s)^\alpha \right).$$

It follows from the Borel-Cantelli lemma that there exists $k_0 : \Omega \rightarrow \mathbb{N}$ such that, for a possibly different constant $C > 1$, with probability one, for all $k \geq k_0$ and all $(x, y, s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \times [0, T]$ with $s < t$,

$$-C(t-s)^\alpha + \frac{1}{C} \frac{|y-x|^{q'}}{(t-s)^{q'-1}} \leq L_{2^{-k}, f}(x, y, s, t) \leq C \left(\frac{|y-x|^{q'}}{(t-s)^{q'-1}} + (t-s)^\alpha \right).$$

Now let $0 < \varepsilon < \varepsilon_0 := 2^{-k_0}$, and choose $k > k_0$ so that

$$2^{-k-1} < \varepsilon \leq 2^{-k}.$$

Set $\tau = 2^k \varepsilon$, which satisfies $\tau \in (1/2, 1]$. A straightforward scaling argument yields

$$L_{\varepsilon, f}(x, y, s, t) = \tau L_{2^{-k}, \tau^{\theta-1/2} f} \left(\frac{x}{\tau}, \frac{y}{\tau}, \frac{s}{\tau}, \frac{t}{\tau} \right), \quad (6.6)$$

and therefore, for yet another $C > 1$, we find that, with probability one, (6.3) holds for all (x, y, s, t) and $\varepsilon \in (0, \varepsilon_0)$.

We now establish (6.4). Fix $R > 0$ and $0 < \tau < T$. Then (6.5) implies that there exists $C = C(R, T, \tau, q, M_0) > 0$ such that, with probability one, for all $x, y \in B_R$ and $s, t \in [0, T]$ with $t - s \geq \tau$,

$$\begin{aligned} L_{\varepsilon, f}(x, y, s, t) &\leq (t-s)H^* \left(\frac{y-x}{t-s} \right) + \varepsilon^{\theta-1/2} \int_s^t f \left(\frac{1}{\varepsilon} \left(x + \frac{y-x}{t-s} r \right) \right) dB_{r/\varepsilon} \\ &\leq (t-s)H^* \left(\frac{y-x}{t-s} \right) + C\varepsilon^{\theta-1/2} (1 + \mathcal{D}_\varepsilon). \end{aligned}$$

For the lower bound, let $v \in (0, 1)$ and $\gamma \in \mathcal{A}(x, y, s, t)$ satisfy

$$L_{\varepsilon, f}(x, y, s, t) + v \geq \int_s^t H^*(\dot{\gamma}_r) dr + \varepsilon^{\theta-1/2} \int_s^t f(\varepsilon^{-1} \gamma_r) \cdot dB_{r/\varepsilon}.$$

In view of (6.3), we then have

$$\int_s^t |\dot{\gamma}_r|^{q'} dr \leq C \left(1 + \frac{1 + \mathcal{D}_\varepsilon}{\delta^q} + \delta^{q'} \int_s^t |\dot{\gamma}_r|^{q'} dr \right),$$

and then rearranging terms and choosing δ sufficiently small yields

$$\int_s^t |\dot{\gamma}_r|^{q'} dr \leq C(1 + \mathcal{D}_\varepsilon).$$

It follows from Jensen's inequality that

$$L_{\varepsilon, f}(x, y, s, t) + v \geq (t-s)H^* \left(\frac{y-x}{t-s} \right) - \varepsilon^{\theta-1/2} (1 + \mathcal{D}_\varepsilon).$$

As v was arbitrary, we conclude, combining the upper and lower bounds, that

$$\sup_{f \in F_\varepsilon} \sup_{x, y \in B_R} \sup_{s, t \in [0, T], t-s \geq \tau} \left| L_{\varepsilon, f}(x, y, s, t) - (t-s)H^* \left(\frac{y-x}{t-s} \right) \right| \leq C\varepsilon^{\theta-1/2} (1 + \mathcal{D}_\varepsilon).$$

As before, using the Borel-Cantelli lemma, the above expression converges with probability one, as $k \rightarrow \infty$ along the subsequence $\varepsilon_k = 2^{-k}$, to 0. The convergence over all $\varepsilon \rightarrow 0$ can be seen by once again appealing to the scaling relationship (6.6).

(b) Let $v_0 \in \mathbb{R}^d$ be such that

$$H^*(v_0) = \min_{v \in \mathbb{R}^d} H^*(v).$$

Then, since $\theta < 1/2$, we have, for all $\varepsilon \in (0, 1)$,

$$L_{\varepsilon, f}(x, y, s, t) \leq (t-s)H^*(v_0) + \varepsilon^{\theta-1/2} \left(-(t-s)H^*(v_0) + L^\varepsilon(x, y, s, t) \right),$$

where L^ε is given by (4.5). It follows that

$$\begin{aligned}\tilde{u}^\varepsilon(x, t) &\leq \inf_{y \in \mathbb{R}^d} \left(u_0(y) + tH^*(v_0) + \varepsilon^{\theta-1/2} \left(-tH^*(v_0) + L^\varepsilon(y, x, 0, t) \right) \right) \\ &\leq u_0(x - tv_0) + tH^*(v_0) + \varepsilon^{\theta-1/2} \left(-tH^*(v_0) + L^\varepsilon(x - tv_0, x, 0, t) \right).\end{aligned}$$

Lemma 4.3 yields that, with probability one, locally uniformly in $\mathbb{R}^d \times (0, \infty)$,

$$L^\varepsilon(x - tv_0, x, 0, t) \xrightarrow{\varepsilon \rightarrow 0} t\bar{L}(v_0) < tH^*(v_0),$$

where the strict inequality is due to Theorem 5.1. The result follows.

(c) This is a consequence of Theorems 4.1 and 5.1. $\square$

Declaration of competing interest

There is no competing interest.

Acknowledgements

The author would like to thank Panagiotis Souganidis and Pierre Cardaliaguet for many helpful comments and suggestions in the preparation of this work.

References

- [1] M.A. Akcoglu, U. Krengel, Ergodic theorems for superadditive processes, *J. Reine Angew. Math.* 323 (1981) 53–67.
- [2] S. Armstrong, P. Cardaliaguet, Stochastic homogenization of quasilinear Hamilton-Jacobi equations and geometric motions, *J. Eur. Math. Soc.* 20 (4) (2018) 797–864.
- [3] S.N. Armstrong, P. Cardaliaguet, Quantitative stochastic homogenization of viscous Hamilton-Jacobi equations, *Commun. Partial Differ. Equ.* 40 (3) (2015) 540–600.
- [4] S.N. Armstrong, P. Cardaliaguet, P.E. Souganidis, Error estimates and convergence rates for the stochastic homogenization of Hamilton-Jacobi equations, *J. Am. Math. Soc.* 27 (2) (2014) 479–540.
- [5] S.N. Armstrong, P.E. Souganidis, Stochastic homogenization of Hamilton-Jacobi and degenerate Bellman equations in unbounded environments, *J. Math. Pures Appl.* (9) 97 (5) (2012) 460–504.
- [6] S.N. Armstrong, P.E. Souganidis, Stochastic homogenization of level-set convex Hamilton-Jacobi equations, *Int. Math. Res. Not.* 15 (2013) 3420–3449.
- [7] S.N. Armstrong, H.V. Tran, Stochastic homogenization of viscous Hamilton-Jacobi equations and applications, *Anal. PDE* 7 (8) (2014) 1969–2007.
- [8] S.N. Armstrong, H.V. Tran, Y. Yu, Stochastic homogenization of a nonconvex Hamilton-Jacobi equation, *Calc. Var. Partial Differ. Equ.* 54 (2) (2015) 1507–1524.
- [9] S.N. Armstrong, H.V. Tran, Y. Yu, Stochastic homogenization of nonconvex Hamilton-Jacobi equations in one space dimension, *J. Differ. Equ.* 261 (5) (2016) 2702–2737.
- [10] Y. Bakhtin, The Burgers equation with Poisson random forcing, *Ann. Probab.* 41 (4) (2013) 2961–2989.
- [11] Y. Bakhtin, Ergodic theory of the Burgers equation, in: *Probability and Statistical Physics in St. Petersburg*, in: *Proc. Sympos. Pure Math.*, vol. 91, Amer. Math. Soc., Providence, RI, 2016, pp. 1–49.
- [12] Y. Bakhtin, Inviscid Burgers equation with random kick forcing in noncompact setting, *Electron. J. Probab.* 21 (2016) 37.
- [13] Y. Bakhtin, E. Cator, K. Khanin, Space-time stationary solutions for the Burgers equation, *J. Am. Math. Soc.* 27 (1) (2014) 193–238.
- [14] Y. Bakhtin, K. Khanin, On global solutions of the random Hamilton-Jacobi equations and the KPZ problem, *Nonlinearity* 31 (4) (2018) R93–R121.
- [15] M.E. Becker, Multiparameter groups of measure-preserving transformations: a simple proof of Wiener's ergodic theorem, *Ann. Probab.* 9 (3) (1981) 504–509.
- [16] A. Boritchev, K. Khanin, On the hyperbolicity of minimizers for 1D random Lagrangian systems, *Nonlinearity* 26 (1) (2013) 65–80.
- [17] P. Cannarsa, P. Cardaliaguet, Regularity results for eikonal-type equations with nonsmooth coefficients, *NoDEA Nonlinear Differ. Equ. Appl.* 19 (6) (2012) 751–769.
- [18] P. Cardaliaguet, A note on the regularity of solutions of Hamilton-Jacobi equations with superlinear growth in the gradient variable, *ESAIM Control Optim. Calc. Var.* 15 (2) (2009) 367–376.
- [19] P. Cardaliaguet, J. Nolen, P.E. Souganidis, Homogenization and enhancement for the G -equation, *Arch. Ration. Mech. Anal.* 199 (2) (2011) 527–561.
- [20] P. Cardaliaguet, L. Silvestre, Hölder continuity to Hamilton-Jacobi equations with superquadratic growth in the gradient and unbounded right-hand side, *Commun. Partial Differ. Equ.* 37 (9) (2012) 1668–1688.

- [21] P. Cardaliaguet, P.E. Souganidis, Homogenization and enhancement of the G -equation in random environments, *Commun. Pure Appl. Math.* 66 (10) (2013) 1582–1628.
- [22] P. Cardaliaguet, P.E. Souganidis, On the existence of correctors for the stochastic homogenization of viscous Hamilton-Jacobi equations, *C. R. Math. Acad. Sci. Paris* 355 (7) (2017) 786–794.
- [23] A. Ciomaga, P.E. Souganidis, H.V. Tran, Stochastic homogenization of interfaces moving with changing sign velocity, *J. Differ. Equ.* 258 (4) (2015) 1025–1057.
- [24] M.G. Crandall, P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Am. Math. Soc.* 277 (1) (1983) 1–42.
- [25] A. Davini, E. Kosygina, Homogenization of viscous and non-viscous HJ equations: a remark and an application, *Calc. Var. Partial Differ. Equ.* 56 (4) (2017) 95.
- [26] A. Davini, A. Siconolfi, Exact and approximate correctors for stochastic Hamiltonians: the 1-dimensional case, *Math. Ann.* 345 (4) (2009) 749–782.
- [27] A. Dunlap, Y. Gu, L. Ryzhik, O. Zeitouni, The random heat equation in dimensions three and higher: the homogenization viewpoint, preprint arXiv:1808.07557 [math.PR].
- [28] W. E, K. Khanin, A. Mazel, Y. Sinai, Invariant measures for Burgers equation with stochastic forcing, *Ann. Math. (2)* 151 (3) (2000) 877–960.
- [29] W.M. Feldman, J.-B. Fermanian, B. Ziliotto, An example of failure of stochastic homogenization for viscous Hamilton-Jacobi equations without convexity, arXiv:1905.07295 [math.AP].
- [30] W.M. Feldman, P.E. Souganidis, Homogenization and non-homogenization of certain non-convex Hamilton-Jacobi equations, *J. Math. Pures Appl. (9)* 108 (5) (2017) 751–782.
- [31] H. Gao, Random homogenization of coercive Hamilton-Jacobi equations in 1d, *Calc. Var. Partial Differ. Equ.* 55 (2) (2016) 30.
- [32] Y. Gu, G. Bal, Homogenization of parabolic equations with large time-dependent random potential, *Stoch. Process. Appl.* 125 (1) (2015) 91–115.
- [33] Y. Gu, L. Ryzhik, O. Zeitouni, The Edwards-Wilkinson limit of the random heat equation in dimensions three and higher, *Commun. Math. Phys.* 363 (2) (2018) 351–388.
- [34] M. Gubinelli, P. Imkeller, N. Perkowski, Paracontrolled distributions and singular PDEs, *Forum Math. Pi* 3 (2015) e6.
- [35] M. Hairer, Solving the KPZ equation, *Ann. Math. (2)* 178 (2) (2013) 559–664.
- [36] M. Hairer, A theory of regularity structures, *Invent. Math.* 198 (2) (2014) 269–504.
- [37] A. Hajej, Stochastic homogenization of a front propagation problem with unbounded velocity, *J. Differ. Equ.* 262 (7) (2017) 3805–3836.
- [38] W. Jing, P.E. Souganidis, H.V. Tran, Stochastic homogenization of viscous superquadratic Hamilton-Jacobi equations in dynamic random environment, *Res. Math. Sci.* 4 (2017) 6.
- [39] W. Jing, P.E. Souganidis, H.V. Tran, Large time average of reachable sets and applications to homogenization of interfaces moving with oscillatory spatio-temporal velocity, *Discrete Contin. Dyn. Syst., Ser. S* 11 (5) (2018) 915–939.
- [40] K. Khanin, K. Zhang, Hyperbolicity of minimizers and regularity of viscosity solutions for a random Hamilton-Jacobi equation, *Commun. Math. Phys.* 355 (2) (2017) 803–837.
- [41] E. Kosygina, F. Rezakhanlou, S.R.S. Varadhan, Stochastic homogenization of Hamilton-Jacobi-Bellman equations, *Commun. Pure Appl. Math.* 59 (10) (2006) 1489–1521.
- [42] E. Kosygina, S.R.S. Varadhan, Homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium, *Commun. Pure Appl. Math.* 61 (6) (2008) 816–847.
- [43] P.-L. Lions, Generalized Solutions of Hamilton-Jacobi Equations, *Research Notes in Mathematics*, vol. 69, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.
- [44] P.-L. Lions, P.E. Souganidis, Fully nonlinear stochastic partial differential equations, *C. R. Acad. Sci. Paris, Sér. I Math.* 326 (9) (1998) 1085–1092.
- [45] P.-L. Lions, P.E. Souganidis, Fully nonlinear stochastic partial differential equations: non-smooth equations and applications, *C. R. Acad. Sci. Paris, Sér. I Math.* 327 (8) (1998) 735–741.
- [46] P.-L. Lions, P.E. Souganidis, Fully nonlinear stochastic pde with semilinear stochastic dependence, *C. R. Acad. Sci. Paris, Sér. I Math.* 331 (8) (2000) 617–624.
- [47] P.-L. Lions, P.E. Souganidis, Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations, *C. R. Acad. Sci. Paris, Sér. I Math.* 331 (10) (2000) 783–790.
- [48] P.-L. Lions, P.E. Souganidis, Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting, *Commun. Pure Appl. Math.* 56 (10) (2003) 1501–1524.
- [49] P.-L. Lions, P.E. Souganidis, Homogenization of “viscous” Hamilton-Jacobi equations in stationary ergodic media, *Commun. Partial Differ. Equ.* 30 (1–3) (2005) 335–375.
- [50] P.-L. Lions, P.E. Souganidis, Stochastic homogenization of Hamilton-Jacobi and “viscous”-Hamilton-Jacobi equations with convex nonlinearities—revisited, *Commun. Math. Sci.* 8 (2) (2010) 627–637.
- [51] C. Mukherjee, A. Shamov, O. Zeitouni, Weak and strong disorder for the stochastic heat equation and continuous directed polymers in $d \geq 3$, *Electron. Commun. Probab.* 21 (2016) 61.
- [52] J. Nolen, A. Novikov, Homogenization of the G -equation with incompressible random drift in two dimensions, *Commun. Math. Sci.* 9 (2) (2011) 561–582.
- [53] F. Rezakhanlou, J.E. Tarver, Homogenization for stochastic Hamilton-Jacobi equations, *Arch. Ration. Mech. Anal.* 151 (4) (2000) 277–309.
- [54] R.W. Schwab, Stochastic homogenization of Hamilton-Jacobi equations in stationary ergodic spatio-temporal media, *Indiana Univ. Math. J.* 58 (2) (2009) 537–581.
- [55] B. Seeger, Homogenization of pathwise Hamilton-Jacobi equations, *J. Math. Pures Appl. (9)* 110 (2018) 1–31.
- [56] B. Seeger, Scaling limits and homogenization of mixing Hamilton-Jacobi equations, *Commun. Partial Differ. Equ.* (2020), <https://doi.org/10.1080/03605302.2020.1831020>, in press.

- [57] P.E. Souganidis, Stochastic homogenization of Hamilton-Jacobi equations and some applications, *Asymptot. Anal.* 20 (1) (1999) 1–11.
- [58] P.E. Souganidis, Pathwise solutions for fully nonlinear first- and second-order partial differential equations with multiplicative rough time dependence, in: *Singular Random Dynamics*, in: *Lecture Notes in Math.*, vol. 2253, Springer, Cham, 2019, pp. 75–220.
- [59] D.W. Stroock, S.R.S. Varadhan, *Multidimensional Diffusion Processes*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 233, Springer-Verlag, Berlin-New York, 1979.
- [60] B. Ziliotto, Stochastic homogenization of nonconvex Hamilton-Jacobi equations: a counterexample, *Commun. Pure Appl. Math.* 70 (9) (2017) 1798–1809.