Approximation schemes for viscosity solutions of fully nonlinear stochastic partial differential equations

Benjamin Seeger

Abstract We develop a method for constructing convergent approximation schemes for viscosity solutions of fully nonlinear stochastic partial differential equations. Our results apply to explicit finite difference schemes and Trotter-Kato splitting formulas, and error estimates are found for schemes approximating solutions of stochastic Hamilton-Jacobi equations.

1 Introduction

We develop a general program for constructing numerical schemes to approximate pathwise viscosity solutions of the initial value problem

$$\begin{cases} du = F(D^2u, Du)\,dt + \displaystyle\sum_{i=1}^{m} H^i(Du)\cdot dW^i \quad \text{in } \mathbb{R}^d \times (0,T] \quad \text{and} \\ u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^d, \end{cases} \tag{1}$$

where $T > 0$, $F \in C^{0,1}(\mathbb{S}^d \times \mathbb{R}^d)$[1] is degenerate elliptic, $H \in C^2(\mathbb{R}^d)$, $W = (W^1, W^2, \ldots, W^m) \in C([0,T], \mathbb{R}^m)$, and $u_0 \in BUC(\mathbb{R}^d)$[2].

When W is continuously differentiable or of bounded variation, (1) falls within the scope of the theory of viscosity solutions; see, for instance, [5]. However, a general continuous path W may be nowhere differentiable and have unbounded variation on every open interval, as is the case, for example, for Brownian paths with probability one. For such paths, the study of (1) requires the theory of pathwise (or stochastic) viscosity solutions put forth by Lions and Souganidis [10, 11, 12, 13].

Collège de France and Université Paris - Dauphine (CEREMADE), Place du Maréchal de Lattre de Tassigny, 75016 Paris, France, e-mail: seeger@ceremade.dauphine.fr

[1] $\mathbb{S}^d$ is the space of symmetric $d \times d$ matrices

[2] $BUC(\mathbb{R}^d)$ is the space of bounded and uniformly continuous functions on $\mathbb{R}^d$

In view of the robust stability properties of viscosity solutions, there is an extensive literature on the construction of approximation schemes for fully nonlinear equations, initiated by Crandall and Lions [6] and Souganidis [15, 16], who found error estimates for convergent approximations of Hamilton-Jacobi equations, and extended to second order equations by Barles and Souganidis [2] with a qualitative proof of convergence. Rates of convergence in the second order case have also been obtained in various cases, see for instance [1, 4, 8, 9, 17].

It turns out [14] that pathwise viscosity solutions are also quite amenable to various approximation schemes, although the methods are more involved due to the presence of the singular terms dW^i, as we describe in what follows.

2 A summary of the main results

We discuss first the general algorithm for the construction of schemes, and we present some specific examples to illustrate its use.

2.1 The scheme operator

The central object to be constructed is the scheme operator, which, for $h > 0$, $0 \le s \le t \le T$, and $\zeta \in C([0,T];\mathbb{R}^m)$, is a map $S_h(t,s;\zeta) : BUC(\mathbb{R}^d) \to BUC(\mathbb{R}^d)$. Then, given a partition $\mathcal{P} := \{0 = t_0 < t_1 < \cdots, t_N = T\}$ of $[0,T]$ with mesh-size $|\mathcal{P}| := \max_{n=0,1,\dots,N-1}(t_{n+1} - t_n)$ and a path $\zeta \in C([0,T];\mathbb{R}^m)$, we define the function $\tilde{u}_h(\cdot;\zeta,\mathcal{P})$ by

$$\begin{cases} \tilde{u}_h(\cdot,0;\zeta,\mathcal{P}) := u_0 \quad \text{and, for } n = 0,1,\dots,N-1 \text{ and } t \in (t_n,t_{n+1}], \\ \tilde{u}_h(\cdot,t;\zeta,\mathcal{P}) := S_h(t,t_n;\zeta)\tilde{u}_h(\cdot,t_n;\zeta,\mathcal{P}). \end{cases} \tag{2}$$

Piecewise smooth approximating paths $\{W_h\}_{h>0}$ and partitions $\{\mathcal{P}_h\}_{h>0}$ satisfying

$$\lim_{h\to 0^+} \|W_h - W\|_\infty = 0 = \lim_{h\to 0^+} |\mathcal{P}_h| \tag{3}$$

are then chosen in such a way that the function

$$u_h(x,t) := \tilde{u}_h(x,t;W_h,\mathcal{P}_h) \tag{4}$$

is an approximation of the solution of (5) for small $h > 0$.

2.2 The main examples

We focus here on finite difference schemes, while noting that the general convergence results apply to other approximations, for example, Trotter-Kato splitting formulas; see also [7].

To simplify the presentation, assume $d = m = 1$, $\|Du_0\|_\infty \le L$, F and H are both smooth with bounded derivatives, and F depends only on u_{xx}, so that (1) becomes

$$du = F(u_{xx})\, dt + H(u_x) \cdot dW \quad \text{in } \mathbb{R} \times (0,T] \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}, \tag{5}$$

or, in the first order case, when $F \equiv 0$,

$$du = H(u_x) \cdot dW \quad \text{in } \mathbb{R} \times (0,T] \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}. \tag{6}$$

Below, the various specifications for $\mathcal{P}_h$ and W_h, while technical, are all made in order to ensure that, for some fixed, sufficiently small $\lambda > 0$, the following generalized CFL condition holds:

$$\sup_{h>0} \sup_{n=0,1,2,\ldots,N-1} \frac{|W_h(t_{n+1}) - W_h(t_n)|}{h} \le \lambda. \tag{7}$$

The reason for this is discussed further in Section 4.

The first scheme is defined, for some $\varepsilon_h > 0$, by

$$\begin{aligned} S_h(t,s;\zeta)u(x) := u(x) &+ H\left(\frac{u(x+h) - u(x-h)}{2h}\right)(\zeta(t) - \zeta(s)) \\ &+ \left[F\left(\frac{u(x+h) + u(x-h) - 2u(x)}{h^2}\right)\right. \\ &+ \left.\varepsilon_h \left(\frac{u(x+h) + u(x-h) - 2u(x)}{h^2}\right)\right](t-s). \end{aligned} \tag{8}$$

Theorem 1 *Assume that, in addition to* (3)*,* W_h *and* $\mathcal{P}_h$ *satisfy*

$$|\mathcal{P}_h| < \frac{h^2}{\|F'\|_\infty} \quad \text{and} \quad \varepsilon_h := h\,\|H'\|_\infty \left\|\dot{W}_h\right\|_\infty \xrightarrow{h\to 0} 0.$$

Then, as $h \to 0$*, the function* u_h *defined by* (4) *using the scheme operator* (8) *converges locally uniformly to the pathwise viscosity solution* u *of* (5)*.*

For schemes approximating solutions of the pathwise Hamilton-Jacobi equation (6), we are able to obtain explicit error estimates. We focus here on the particular scheme defined, for some $\theta \in (0,1]$, by

$$\begin{aligned} S_h(t,s;\zeta)u(x) := u(x) &+ H\left(\frac{u(x+h) - u(x-h)}{2h}\right)(\zeta(t) - \zeta(s)) \\ &+ \frac{\theta}{2}\left(u(x+h) + u(x-h) - 2u(x)\right). \end{aligned} \tag{9}$$

Assume that $\omega : [0,\infty) \to [0,\infty)$ is the modulus of continuity of the fixed continuous path W on $[0,T]$. For $h > 0$, define ρ_h implicitly by

$$\lambda := \frac{(\rho_h)^{1/2}\omega((\rho_h)^{1/2})}{h} < \frac{\theta}{\|H'\|_\infty}, \tag{10}$$

and let the partition $\mathcal{P}_h$ and path W_h satisfy

$$\begin{cases} \mathcal{P}_h := \{n\rho_h \wedge T\}_{n\in\mathbb{N}_0}, \ M_h := \lfloor (\rho_h)^{-1/2} \rfloor, \\ \text{and, for } k \in \mathbb{N}_0 \text{ and } t \in [kM_h\rho_h, (k+1)M_h\rho_h), \\ W_h(t) := W(kM_h\rho_h) + \left(\dfrac{W((k+1)M_h\rho_h) - W(kM_h\rho_h)}{M_h\rho_h} \right)(t - kM_h\rho_h)\,. \end{cases} \tag{11}$$

Theorem 2 *There exists $C > 0$ depending only on L such that, if u_h is constructed using* (4) *and* (9) *with $\mathcal{P}_h$ and W_h as in* (10) *and* (11)*, and u is the pathwise viscosity solution of* (6)*, then*

$$\sup_{(x,t)\in\mathbb{R}^d\times[0,T]} |u_h(x,t) - u(x,t)| \le C(1+T)\omega((\rho_h)^{1/2}).$$

If $W \in C^{0,\alpha}([0,T])$, then the CFL condition (10) becomes $\rho_h = O(h^{2/(1+\alpha)})$, and the rate of convergence in Theorem 2 is $O(h^{\alpha/(1+\alpha)})$.

When W is a Brownian motion, then the CFL condition (10) can be chosen according to the Lévy modulus of continuity:

$$\lambda := \frac{(\rho_h)^{3/4}\left|\log \rho_h\right|^{1/2}}{h} < \frac{\theta}{\|H'\|_\infty}. \tag{12}$$

The proof of Theorem 2 can then be modified to show that, with probability one, for a deterministic constant $C > 0$ depending only on L and λ,

$$\limsup_{h\to 0} \sup_{(x,t)\in\mathbb{R}^d\times[0,T]} \frac{|u_h(x,t) - u(x,t)|}{h^{1/3}\left|\log h\right|^{1/3}} \le C(1+T).$$

The final example converges in distribution. Let λ, ρ_h, W_h, and $\mathcal{P}_h$ be given, for some probability space $(\mathcal{A}, \mathcal{G}, \mathbf{P})$, by

$$\begin{cases} \lambda := \dfrac{(\rho_h)^{3/4}}{h} \le \dfrac{\theta}{\|H'\|_\infty}, \quad M_h := \lfloor (\rho_h)^{-1/2} \rfloor, \\ \mathcal{P}_h := \{t_n\}_{n=0}^N = \{n\rho_h \wedge T\}_{n\in\mathbb{N}_0}, \\ \{\xi_n\}_{n=1}^\infty : \mathcal{A} \to \{-1,1\} \text{ are independent and Rademacher}, \\ W_h(0) = 0, \quad \text{and} \\ W_h(t) := W_h(kM_h\rho_h) + \dfrac{\xi_k}{\sqrt{M_h\rho_h}}(t - kM_h\rho_h) \\ \text{for } k \in \mathbb{N}_0,\ t \in [kM_h\rho_h, (k+1)M_h\rho_h). \end{cases} \tag{13}$$

Donsker's invariance principle (see [3]) implies that, as $h \to 0$, W_h converges in distribution to a Brownian motion W in the space $C([0,\infty),\mathbb{R})$.

Theorem 3 *If u_h is constructed using* (4) *and* (9) *with W_h and $\mathcal{P}_h$ as in* (13)*, and u is the solution of* (5)*, then, as $h \to 0$, u_h converges to u in distribution in the topology of local uniform convergence.*

3 The convergence proof: monotonicity and consistency

We next outline the proof of the general convergence result, which is based on a generalization of the method of half-relaxed limits from the theory of viscosity solutions, used by Barles and Souganidis [2] to prove the convergence of finite difference approximations of second order equations.

We always impose the following monotonicity condition on the scheme:

$$\begin{cases} \text{if } t_n \le t \le t_{n+1},\ t_n, t_{n+1} \in \mathcal{P}_h,\ \text{and } u, v \in BUC(\mathbb{R}^d),\ \text{then} \\ u \le v \quad \Rightarrow \quad S_h(t,t_n;W_h)u \le S_h(t,t_n;W_h)v, \end{cases} \tag{14}$$

and also that the scheme operator commutes with constants, that is, for all $u \in BUC(\mathbb{R}^d)$, $h > 0$, $0 \le s \le t < \infty$, $\zeta \in C([0,T],\mathbb{R}^m)$, and $k \in \mathbb{R}$,

$$S_h(t,s;\zeta)\,(u+k) = S_h(t,s;\zeta)u + k. \tag{15}$$

In addition, the scheme operator must satisfy a consistency requirement (see (19) below). We motivate such a condition by outlining the proof next, keeping the details light.

To that end, assume that, for some $u \in BUC(\mathbb{R}^d \times [0,T])$, $\lim_{h\to 0} u_h = u$ locally uniformly[3], and we attempt to show that u is a pathwise viscosity sub- and super-solution of (1). Recall (see [10]) that u is said to be a sub- (super)- solution of (1) if, whenever $I \subset [0,T]$ is an open interval; $\Phi \in C(I;C^2(\mathbb{R}^d))$ is a local-in-time, smooth-in-space solution of

$$d\Phi = \sum_{i=1}^m H^i(D\Phi)\cdot dW^i \quad \text{in } \mathbb{R}^d \times I; \tag{16}$$

$\psi \in C^1(I)$; and $u(x,t) - \Phi(x,t) - \psi(t)$ attains a strict maximum (minimum) at $(y,s) \in \mathbb{R}^d \times I$, then

$$\psi'(s) \le F(D^2\Phi(y,s), D\Phi(y,s)).$$

We show that u is a sub-solution, and the argument for super-solutions is similar.

[3] In general, the existence of such a limit is not guaranteed a priori, and one must work with so-called "half-relaxed" limits. To simplify the presentation, we avoid such details here.

Let I, Φ, ψ, and (y,s) be as above, and, for $h > 0$, let Φ_h be the local-in-time, smooth-in-space solution, constructed with the method of characteristics, of

$$\Phi_{h,t} = \sum_{i=1}^{m} H^i(D\Phi_h)\dot{W}_h^i \quad \text{in } \mathbb{R}^d \times I, \quad \Phi_h(\cdot, s) = \Phi(\cdot, t_0). \tag{17}$$

Since W_h converges to W uniformly as $h \to 0$, it follows that Φ_h converges in $C(I, C^2(\mathbb{R}^d))$ to Φ as $h \to 0$, with I made smaller if necessary, independently of h.

As a result, there exists $\{(y_h, s_h)\}_{h>0} \subset \mathbb{R}^d \times I$ such that $\lim_{h\to 0}(y_h, s_h) = (y, s)$ and

$$u_h(x,t) - \Phi_h(x,t) - \psi(t)$$

attains a local maximum at (y_h, s_h).

The mesh-size $|\mathcal{P}_h|$ converges to 0 as $h \to 0$, so, for sufficiently small h, there exists $n \in \mathbb{N}$ depending on h such that

$$t_n < s_h \le t_{n+1} \quad \text{and} \quad t_n, t_{n+1} \in I.$$

Then

$$u_h(\cdot, t_n) - \Phi_h(\cdot, t_n) - \psi(t_n) \le u_h(y_h, s_h) - \Phi_h(y_h, s_h) - \psi(s_h),$$

which leads to

$$u_h(\cdot, t_n) \le u_h(y_h, s_h) + \Phi_h(\cdot, t_n) - \Phi_h(y_h, s_h) + \psi(t_n) - \psi(s_h). \tag{18}$$

It is here that the monotonicity (14) of the scheme is used. Applying $S_h(s_h, t_n; W_h)$ to both sides of (18), using the fact that the scheme commutes with constants, and plugging in $x = y_h$, we arrive at

$$u_h(y_h, s_h) \le u_h(y_h, s_h) + S_h(s_h, t_n; W_h)\Phi_h(\cdot, t_n)(y_h) - \Phi_h(y_h, s_h) + \psi(t_n) - \psi(s_h),$$

whence

$$\frac{\psi(s_h) - \psi(t_n)}{s_h - t_n} \le \frac{S_h(s_h, t_n; W_h)\Phi_h(\cdot, t_n)(y_h) - \Phi_h(y_h, s_h)}{s_h - t_n}.$$

As $h \to 0$, the left-hand side converges to $\psi'(s)$. The right-hand side converges to $F(D^2\Phi(y,s), D\Phi(y,s))$ if we make the following consistency requirement: whenever Φ and Φ_h are as in respectively (16) and (17), we have

$$\lim_{s,t\in I,\ t-s\to 0} \frac{S_h(t,s;W_h)\Phi_h(\cdot,s) - \Phi_h(\cdot,s)}{t-s} = F(D^2\Phi, D\Phi). \tag{19}$$

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4 On the construction of the scheme operator

We discuss next the strategy for constructing scheme operators that satisfy the assumptions of the previous section, and, in particular, the need for regularizing the path W in general. We focus on the equation (6) and consider the scheme operator given by

$$\begin{aligned} S_h(t,s)u(x) &:= u(x) + H\left(\frac{u(x+h)-u(x-h)}{2h}\right)(W(t)-W(s)) \\ &\quad + \varepsilon_h\left(\frac{u(x+h)+u(x-h)-2u(x)}{h^2}\right)(t-s), \end{aligned} \tag{20}$$

which can be seen to be monotone for $0 \le t-s \le \rho_h$ as long as, for some $\theta \in (0,1]$,

$$\varepsilon_h := \frac{\theta h^2}{2(t-s)} \quad \text{and} \quad \lambda := \max_{|t-s|\le\rho_h} \frac{|W(t)-W(s)|}{h} \le \lambda_0 := \frac{\theta}{\|H'\|_\infty}. \tag{21}$$

For any $s,t \in [0,T]$ with $|s-t|$ sufficiently small, spatially smooth solutions Φ of (6) have the expansion

$$\begin{aligned} \Phi(x,t) &= \Phi(x,s) + H(\Phi_x(x,s))(W(t)-W(s)) \\ &\quad + H'(\Phi_x(x,s))^2\Phi_{xx}(x,s)(W(t)-W(s))^2 + O(|W(t)-W(s)|^3), \end{aligned} \tag{22}$$

so that, if $0 \le t-s \le \rho_h$, then, for some $C>0$ depending only on H,

$$\begin{aligned} \sup_{\mathbb{R}} |S_h(t,s)\Phi(\cdot,s) - \Phi(\cdot,t)| &\le C \sup_{r\in[s,t]} \left\|D^2\Phi(\cdot,r)\right\|_\infty \left(|W(t)-W(s)|^2 + h^2\right) \\ &\le C \sup_{r\in[s,t]} \left\|D^2\Phi(\cdot,r)\right\|_\infty (1+\lambda_0^2)h^2. \end{aligned}$$

Then (19) is satisfied if

$$\lim_{h\to 0} \frac{h^2}{\rho_h} = 0. \tag{23}$$

Both (21) and (23) can be achieved when W is continuously differentiable, or, more generally, if $W \in C^{0,\alpha}([0,T])$ with $\alpha > \frac{1}{2}$, by setting

$$\rho_h := \left(\frac{\lambda h}{[W]_{\alpha,T}}\right)^{1/\alpha}. \tag{24}$$

However, this approach fails as soon as the quadratic variation path

$$\langle W\rangle_T := \lim_{|\mathcal{P}|\to 0} \sum_{n=0}^{N-1} |W(t_{n+1}) - W(t_n)|^2$$

is non-zero, as (21) and (23) together imply that $\langle W \rangle_T = 0$. This rules out, for instance, the case where W is the sample path of a Brownian motion, or, more generally, any nontrivial semimartingale.

On the other hand, if $\{W_h\}_{h>0}$ is a family of piecewise smooth paths converging uniformly, as $h \to 0$, to W, then $\langle W_h \rangle_T = 0$ for each fixed $h > 0$, and therefore, W_h and ρ_h can be chosen so that (21) and (23) hold for W_h rather than W. As described in Section 2, such choices are related to the general CFL condition (7).

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