# A lower bound on the quantitative version of the transversality theorem

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December 29, 2022

**Abstract**

The present paper studies a quantitative version of the transversality theorem. More precisely, given a continuous function $f \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$ and a manifold $W \subset \mathbb{R}^m$ of dimension $p$, a sharpness result on the upper quantitative estimate of the $(d+p-m)$-dimensional Hausdorff measure of the set $\mathcal{Z}_W^f = \{x \in [0,1]^d : f(x) \in W\}$, which was achieved in [8], will be proved in terms of power functions.

**Keyword.** Transversality lemma, quantitative estimates

**AMS Mathematics Subject Classification.** 46T20

## 1 Introduction

Let $g : X \to Y$ be a $\mathcal{C}^1$ map between two smooth manifolds $X$ of dimension $d$ and $Y$ of dimension $m$ . For any smooth submanifold $W \subseteq Y$ of dimension $p$, we say that the function $g$ is transverse to $W$ and write $g \pitchfork W$ if

$$(dg)_p(T_pX) + T_{g(p)}(W) \ = \ T_{g(p)}(Y) \quad \text{for all } p \in g^{-1}(W).$$

The transversality lemma, which is the key to studying Thom's transversality theorem [10, 11, 12], shows that the set of transverse maps is dense [9]. In particular, for any continuous function $f : [0,1]^d \to \mathbb{R}^m$ and any $\varepsilon > 0$, there exists a $\mathcal{C}^1$ function $f_\varepsilon : [0,1]^d \to \mathbb{R}^m$ such that

$$\|f_\varepsilon - f\|_{\mathcal{C}^1} \ \leq \ \varepsilon \qquad \text{and} \qquad f_\varepsilon \pitchfork W.$$

For every $h \in \mathcal{C}\big([0,1]^d, \mathbb{R}^m\big)$, consider the set

$$\mathcal{Z}_W^h \ := \ \Big\{x \in [0,1]^d : h(x) \in W\Big\}. \tag{1.1}$$

If $h$ is smooth and transverse to $W$, then $\mathcal{Z}_W^h$ is a $(d+p-m)$-dimensional smooth manifold. Hence, its $(d+p-m)$-dimensional Hausdorff measure is finite. In the spirit of metric entropy, which was used in the study of compactness estimates for solution sets of hyperbolic conservation laws [1, 2, 3, 7] and Hamilton-Jacobi equations [4, 5, 6], a natural question is to perform a quantitative analysis of the measure of $\mathcal{Z}_W^f$. Namely, how small can one make this measure, by an $\varepsilon$-perturbation of $f$? To formulate more precisely the result, given $f \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$, one defines

$$\mathcal{N}_W^f(\varepsilon) \;:=\; \inf_{\|h-f\|_{\mathcal{C}^0}\le\varepsilon} \mathcal{H}^{d+p-m}\left(\mathcal{Z}_W^h\right) \tag{1.2}$$

to be the smallest $(d+p-m)$-Hausdorff measure of $\mathcal{Z}_W^h$ among all functions $h \in \mathcal{C}\left([0,1]^d, \mathbb{R}^m\right)$ with $\|h-f\|_{\mathcal{C}^0} \le \varepsilon$. In [8], an upper bound on the number $\mathcal{N}_W^f(\varepsilon)$ was recently established and applied to provide quantitative estimates on the number of shock curves in entropy weak solutions of scalar conservation laws with strictly convex fluxes. Specifically, for $f \in \mathcal{C}^\alpha([0,1]^d, \mathbb{R}^m)$ with Hölder norm $\|f\|_{\mathcal{C}^{0,\alpha}}$ and $\varepsilon > 0$ sufficiently small, there exists a constant $C_W > 0$ that depends only on $W$ such that

$$\mathcal{N}_W^f(\varepsilon) \;\le\; C_W \cdot \left(\frac{\|f\|_{\mathcal{C}^{0,\alpha}}}{\varepsilon}\right)^{\frac{m-p}{\alpha}}. \tag{1.3}$$

The blow up rate $\left(\frac{1}{\varepsilon}\right)^{\frac{m-p}{\alpha}}$ with respect to $\varepsilon$ is shown to be the best bound in terms of power function in [8, Example 3.1] for a class of Lipscthiz functions ($\alpha = 1$) in the scalar case ($d = m = 1$). However, this still remains open for the multi-dimensional cases. Hence, the present paper aims to address the sharpness of (1.3) for general continuous function $f \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$ with $d, m \ge 1$. In particular, we achieve the following lower quantitative estimate for the class of Hölder continuous functions.

**Theorem 1.1** *Assume that $p < m \le p + d$ and $W \subset \mathbb{R}^m$ is a $\mathcal{C}^1$-manifold of dimension $p$. For every $0 < \alpha \le 1$ and $\lambda > 0$, there exists a Hölder continuous function $f : [0,1]^d \to \mathbb{R}^m$ with exponent $\alpha$ and the Hölder norm $\lambda$ such that*

$$\mathcal{N}_W^f(\varepsilon) \;\ge\; C_{[W,\alpha,\lambda]} \cdot \left(\frac{1}{\varepsilon \cdot 2^{4\cdot\sqrt{\alpha|\log_2 \varepsilon|}}}\right)^{\frac{m-p}{\alpha}}$$

*for some constant $C_{[W,\alpha,\lambda]} > 0$ that depends only on $W$, $\alpha$, and $\lambda$.*

Here the constant $C_{[W,\alpha,\lambda]}$ is explicitly computed in Remark 2.4. Moreover, by using the concept of modulus of continuity and its inverse in Definition 2.1, a general result for continuous functions will be proved in Theorem 2.3 of Section 2. This can be easily extended to the case of continuous functions $f : X \to Y$ where $X, Y$ are smooth manifolds and $W \subseteq Y$ is a smooth submanifold of $Y$. Finally, we remark that the factor $2^{4\cdot\sqrt{\alpha|\log_2 \varepsilon|}}$ in Theorem 1.1 is necessary. Indeed, we shall prove in the Proposition 2.1 that the estimate on $\mathcal{N}_W^f(\varepsilon)$ in (1.3) is not actually sharp for the case $\alpha = d = m = 1$, $p = 0$, and $W = \{0\}$. This leads to an open question on the sharp estimate for $\mathcal{N}_W^f(\varepsilon)$.

## 2 A lower bound on $\mathcal{N}_W^f(\varepsilon)$

In this section, we will establish a lower quantitative estimate on the Hausdorff measure of $\mathcal{Z}_W^f$ for a constructed continuous $f \in \mathcal{C}\left([0,1]^d, \mathbb{R}^m\right)$ which admits a given modulus of continuity and the set $W \subseteq \mathbb{R}^m$ being a $\mathcal{C}^1$ manifold with $\dim(W) = p$. For the sake of simplicity, we shall assume that $W$ consists of only one chart $\mathbb{R}^m$, i.e.,

**(A1).** *There exists a $\mathcal{C}^1$ diffeomorphism $\phi$ between open subsets $U, V \subset \mathbb{R}^m$ such that $W \subset U$ and $\phi(W) = \mathbb{R}^p \times \{0\} \cap V$ and*

$$0 \ < \ \gamma_W \ \doteq \ 2\sqrt{m-p} \cdot \left( \frac{\sup_{x \in U} |\nabla \phi(x)|}{\inf_{x \in U} |\nabla \phi(x)|} \right) \ < \ \infty. \tag{2.4}$$

For a general $\mathcal{C}^1$ manifold $W$ consists of multiple charts, one can just restrict the construction of $f$ in a single chart of $W$ which has a smallest constant $\gamma_W$ among other charts. Toward to the main result, let us now recall some basic concepts on the modulus of continuity and its inverse.

**Definition 2.1** *Given subsets $U \subseteq \mathbb{R}^d$ and $V \subseteq \mathbb{R}^m$, let $h : U \to V$ be continuous. The minimal modulus of continuity of $h$ is given by*

$$\omega_h(\delta) \ = \sup_{x,y \in U, |x-y| \le \delta} |h(y) - h(x)| \qquad \text{for all } \delta \in [0, \mathrm{diam}(U)]. \tag{2.5}$$

*The inverse of the minimal modulus of continuity of $h$ is the map $s \to \Psi_h(s)$ is defined by*

$$\Psi_h(s) \ := \ \sup \{\delta \ge 0 : |h(x) - h(y)| \le s \ \ \text{for all } |x-y| \le \delta, x, y \in U\} \tag{2.6}$$

*for all $s \ge 0$.*

It is clear that $\Psi_h(s) = \infty$ for all $s \in [M_h, \infty[$ with $M_h := \sup_{x,y \in U} |h(x) - h(y)|$. In particular, if $h$ is a constant function then $\Psi_h(s) = \infty$ for all $s \ge 0$. Otherwise, by the continuity of $h$, it holds

$$\Psi_h(0) \ = \ 0 \qquad \text{and} \qquad 0 \ < \ \Psi_h(s) \ \le \ \mathrm{diam}(U) \qquad \text{for all } s \in ]0, M_h[.$$

Moreover, $\Psi_h(\cdot) : [0, \infty[ \to [0, \infty[$ is increasing and superadditive

$$\Psi_h(s_1 + s_2) \ \ge \ \Psi_h(s_1) + \Psi_h(s_2) \qquad \text{for all } s_1, s_2 \ge 0.$$

If the map $\delta \to \omega_h(\delta)$ is strictly increasing in $[0, \mathrm{diam}(U)[$ then $\Psi_h$ is the inverse of $\omega_h$, i.e.,

$$\Psi_h(s) \ = \ \omega_h^{-1}(s) \qquad \text{for all } s \in [0, M_h[.$$

From the above observations, we define a modulus of continuity as follows:

**Definition 2.2** *A function $\beta : [0, \infty] \to [0, \infty]$ is called a modulus of continuity if it is increasing, subadditive, and satisfies*

$$\lim_{\delta \to 0+} \beta(\delta) \ = \ \beta(0) \ = \ 0.$$

*We say that a continuous function $f : U \subset \mathbb{R}^d \to \mathbb{R}^m$ admits $\beta$ as a modulus of continuity if*

$$\sup_{x,y \in U, |x-y| \le s} |f(x) - f(y)| \le \beta(s) \qquad \text{for all } s \ge 0. \tag{2.7}$$

The main result in this paper is stated as follows:

**Theorem 2.3** *In addition to* **(A1)**, *assume that* $p < m \le p + d$. *For every modulus of continuity* $\beta$, *there exists a continuous function* $f : [0,1]^d \to \mathbb{R}^m$ *that admits* $\beta$ *as a modulus continuity and for* $\varepsilon > 0$ *sufficiently small*

$$\mathcal{N}_W^f(\varepsilon) \ \ge \ \left(\frac{16}{\Psi_\beta(\gamma_W\varepsilon)}\right)^{m-p} \cdot 2^{-4(m-p)\cdot\sqrt{\left|\log_2(\Psi_\beta(\gamma_W\varepsilon))\right|}}. \tag{2.8}$$

**Proof.** The proof is divided into three main steps:

<u>*Step 1.*</u> Consider the case $W = \{0\}$ and $p = 0$. We claim that

**(G).** *There exists a continuous function* $\tilde{f} : [0,1]^d \to \mathbb{R}^m$ *that admits* $\beta$ *as a modulus of continuity and for every* $0 < \varepsilon < \frac{1}{2\sqrt{m}} \cdot \beta(2^{-5})$ *it holds*

$$\mathcal{N}_{\{0\}}^{\tilde{f}}(\varepsilon) \ \ge \ \left(\frac{16}{\Psi_\beta(2\sqrt{m}\varepsilon)}\right)^m \cdot 2^{-4m\cdot\sqrt{\left|\log_2\left(\Psi_\beta\left(2\sqrt{m}\varepsilon\right)\right)\right|}} \tag{2.9}$$

*with* $\Psi_\beta$ *being the inverse of the minimal modulus of continuity of* $\beta$.

The construction of a desired function $\tilde{f} \in \mathcal{C}([0,1],\mathbb{R}^m)$ in **(G).** will be done as follows:

**1.** Let's first divide $[0,1]$ into countably infinite subintervals $[s_n, s_{n+1}]$ with

$$s_1 \ = \ 0, \qquad s_n \ = \ \sum_{\ell=1}^{n} 2^{-\ell} \qquad \text{for all } n \ge 2.$$

For every $n \ge 1$, we define $u_n : [0,1] \to \mathbb{R}$ by

$$u_n(s) \ = \ \sum_{k=0}^{2^{n^2}-1} c_n(s - s_n - 4k\ell_n), \qquad \ell_n \ = \ 2^{-n^2-n-2},$$

where $c_n : [0,1] \to \mathbb{R}$ is a sample function with $\mathrm{supp}(c_n) \subseteq [0, 4\ell_n]$ such that for all $s \in [0, 2\ell_n]$

$$c_n(s) \ = \ -\,c_n(4\ell_n - s) \ = \ \frac{\beta(s)}{2} \cdot \chi_{[0,\ell_n[}(s) + \frac{\beta(2\ell_n - s)}{2} \cdot \chi_{[\ell_n,2\ell_n]}(s)\,. \tag{2.10}$$

The function $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m) \in \mathcal{C}([0,1],\mathbb{R}^m)$ is defined by

$$\tilde{f}(x) \ = \ \frac{1}{\sqrt{m}} \cdot (r(x_1), \ldots, r(x_m)) \quad \text{for all } x = (x_1, \ldots, x_d) \in [0,1]^d \tag{2.11}$$

with

$$r(s) \ \doteq \ \sum_{n=1}^{\infty} u_n(s) \qquad \text{for all } s \in [0,1].$$

Since the modulus of continuity of $r$ is bounded by $\beta$, the modulus of continuity of $\tilde{f}$ is also bounded by $\beta$. Indeed, for every $s \ge 0$, one estimates

$$\begin{aligned}\omega_{\tilde{f}}(s) \ &= \ \sup_{x,y\in[0,1]^d, |x-y|\le s} |\tilde{f}(x) - \tilde{f}(y)| \\ &= \ \sup_{x,y\in[0,1]^d, |x-y|\le s} \frac{1}{\sqrt{m}} \cdot \left(\sum_{i=1}^{m} |r(x_i) - r(y_i)|^2\right)^{\frac{1}{2}} \ \le \ \beta(s).\end{aligned}$$

Assume that for every $\varepsilon > 0$ satisfying

$$\frac{1}{2\sqrt{m}} \cdot \beta\left(\frac{\ell_{n_0+1}}{2}\right) \;\leq\; \varepsilon \;\leq\; \frac{1}{2\sqrt{m}} \cdot \beta\left(\frac{\ell_{n_0}}{2}\right), \tag{2.12}$$

it holds

$$\mathcal{N}^{\tilde{f}}_{\{0\}}(\varepsilon) \;=\; \inf_{\|g-f\|_{\mathcal{C}^0}\leq\varepsilon} \mathcal{H}^{d-m}\left(\mathcal{Z}^{g}_{\{0\}}\right) \;\geq\; 2^{mn_0^2}. \tag{2.13}$$

In this case, by the properties of an inverse of the minimal modulus of continuity in (2.6), we have that

$$\Psi_\beta(2\sqrt{m}\varepsilon) \;\geq\; \Psi_\beta\left(\beta\left(\frac{\ell_{n_0+1}}{2}\right)\right) \;\geq\; \frac{\ell_{n_0+1}}{2} \;=\; 2^{-(n_0+1)^2-(n_0+1)-3} \;\geq\; 2^{-(n_0+2)^2}.$$

Thus, one has

$$n_0 \;\geq\; -2+\sqrt{-\log_2 \Psi_\beta\big(2\sqrt{m}\varepsilon\big)}$$

and (2.9) follows from (2.13).

**2.** In the next two steps, we shall prove (2.13). For every $n \geq 1$ and $k \in \{0, 1, \dots, 2^{n^2}-1\}$, set

$$a_{n,k} \;=\; s_n + (4k+1)\ell_n, \qquad b_{n,k} \;=\; s_n + (4k+3)\ell_n,$$

we shall denote by

$$\square_{n,\iota} \;=\; [a_{n,\iota_1}, b_{n,\iota_1}] \times \cdots \times [a_{n,\iota_m}, b_{n,\iota_m}] \quad \text{for all } \iota \in \{0,1,\dots,2^{n^2}-1\}^m. \tag{2.14}$$

Fix $g \in \mathcal{C}\big([0,1]^d, \mathbb{R}^m\big)$ with $\|\tilde{f} - g\|_{\mathcal{C}^0} \leq \varepsilon$. By the definition of $\mathcal{Z}^g_{\{0\}}$, we have

$$\mathcal{Z}^g_{\{0\}} \;\supseteq\; \bigcup_{n\geq 1, \iota\in\{0,1,\dots,2^{n^2}-1\}^m} \left(\bigcup_{z\in[0,1]^{d-m}} \mathcal{Z}_{n,\iota}(z)\times\{z\}\right)$$

with

$$\mathcal{Z}_{n,\iota}(z) \;=\; \{y \in \square_{n,\iota} : g(y_1, ..., y_m, z_1, ..., z_{d-m}) = 0\}.$$

Assume that for every $1 \leq n \leq n_0$ and $\iota \in \{0, 1, \dots, 2^{n^2}-1\}^m$, the set

$$\mathcal{Z}_{n,\iota}(z) \;\neq\; \varnothing \qquad \text{for all } z \in [0,1]^{d-m}. \tag{2.15}$$

In this case, we can bound the $(d-m)$-Hausdorff measure of $\mathcal{Z}^g_{\{0\}}$ by

$$\begin{aligned}
\mathcal{H}^{d-m}\left(\mathcal{Z}^g_{\{0\}}\right) \;&\geq\; \sum_{n=1}^{\infty} \sum_{\iota\in\{0,1,\dots,2^{n^2}-1\}^m} \mathcal{H}^{d-m}\left(\bigcup_{z\in[0,1]^{d-m}} \mathcal{Z}_{n,\iota}(z)\times\{z\}\right) \\
&\geq\; \sum_{n=1}^{n_0} \sum_{\iota\in\{0,1,\dots,2^{n^2}-1\}^m} \mathcal{H}^{d-m}\left([0,1]^{d-m}\right) \;=\; \sum_{n=1}^{n_0} 2^{mn^2} \;\geq\; 2^{mn_0^2},
\end{aligned}$$

and this yields (2.13).

**3.** To complete the proof, we need to verify (2.15). Fix $n \in \{1, \ldots, n_0\}$, $\iota \in \{0, 1, \ldots, 2^{n^2}-1\}^m$, and $z \in [0,1]^{d-m}$, we consider the continuous map $h^z : \square_{n,\iota} \to \mathbb{R}^m$ such that

$$h^z(y) \;=\; y + \frac{\sqrt{m}\cdot \ell_n}{\beta(\ell_n/2)}\cdot g(y,z) \qquad \text{for all } y \in \square_{n,\iota}. \tag{2.16}$$

Notice that $\square_{n,\iota} \subseteq [0,1]^m$ is a cube of size $2\ell_n$ centered at $c^{\iota,n}$ with

$$c_i^{\iota,n} \;=\; s_n + (4\iota_i + 2)\ell_n \qquad \text{for all } i \in \{1,2,\ldots,m\}.$$

Recall (2.11), (2.12), and $\|\tilde{f} - g\|_{\mathcal{C}^0} \le \varepsilon$, for every $y \in \square_{n,\iota}$ and $i \in \{1,2,\ldots,m\}$, set $s := y_i - s_n - 4\iota_i\ell_n \in [\ell_n, 3\ell_n]$, we estimate

$$\begin{aligned}
\left|h_i^z(y) - c_i^{\iota,n}\right| \;&=\; \left|y_i + \frac{\sqrt{m}\cdot\ell_n}{\beta(\ell_n/2)}\cdot g_i(y,z) - s_n - (4\iota_i+2)\ell_n\right| \\
&\le\; \frac{\sqrt{m}\cdot\ell_n}{\beta(\ell_n/2)}\cdot\varepsilon + \left|y_i + \frac{\sqrt{m}\cdot\ell_n}{\beta(\ell_n/2)}\cdot f_i(y,z) - s_n - (4\iota_i+2)\ell_n\right| \\
&\le\; \frac{\ell_n}{2} + \left|y_i + \frac{\ell_n}{\beta(\ell_n/2)} r(y_i) - s_n - (4\iota_i+2)\ell_n\right| \\
&=\; \frac{\ell_n}{2} + \left|s - 2\ell_n + \ell_n\cdot\frac{c_n(s)}{\beta(\ell_n/2)}\right|.
\end{aligned} \tag{2.17}$$

By the definition of $c_n$ in (2.10), both cases $s \in [\ell_n, 2\ell_n]$ and $s \in [2\ell_n, 3\ell_n]$ are similar, we shall bound $\left|h_i^z(y) - c_i^{\iota,n}\right|$ for $s \in [\ell_n, 2\ell_n]$. In this case, we have that

$$\left|h_i^z(y) - c_i^{\iota,n}\right| \;=\; \frac{\ell_n}{2} + \left|s - 2\ell_n + \ell_n\cdot\frac{\beta(2\ell_n - s)}{2\beta(\ell_n/2)}\right|$$

If $s \ge \dfrac{3\ell_n}{2}$ then $\left|h_i^z(y) - c_i^{\iota,n}\right| \le \dfrac{\ell_n}{2} + \max\left\{2\ell_n - s, \ell_n\cdot\dfrac{\beta(2\ell_n-s)}{2\beta(\ell_n/2)}\right\} \le \ell_n$. Otherwise, if $\ell_n \le s < \dfrac{3\ell_n}{2}$ then by the subadditivity of $\beta$, we have

$$-\ell_n \;=\; -\frac{\ell_n}{2} - \left(\ell_n - \ell_n\cdot\frac{\beta(\ell_n/2)}{2\beta(\ell_n/2)}\right) \;\le\; h_i^z(y) - c_i^{\iota,n} \;\le\; \frac{\ell_n}{2} - \frac{\ell_n}{2} + \ell_n\cdot\frac{\beta(\ell_n)}{2\beta(\ell_n/2)} \;\le\; \ell_n.$$

Thus, the map $y \mapsto h^z(y)$ is invariant in $\square_{n,\iota}$. Finally, by Brouwer's fixed point theorem, $h^z$ has a fixed point $y_z \in \square_{n,\iota}$, and (2.16) implies that $y_z$ belongs to the set $\mathcal{Z}_{n,\iota}(z)$ in (2.15). The proof of **(G)**is complete.

*Step 2.* For every given $r_0 > 0$, we shall prove our result for the case $W = [-r_0, r_0]^p \times \{0\}^{m-p}$. From **(G)**, there exists a function $\tilde{g} \in \mathcal{C}([0,1]^d, \mathbb{R}^{m-p})$ such that

- $\tilde{g}$ admits $\beta$ as a modulus of continuity;
- For every $0 < \varepsilon < \frac{1}{2\sqrt{m-p}}\cdot\beta(2^{-5})$, it holds

$$\mathcal{N}^g_{\{0\}}(\varepsilon) \;\ge\; \left(\frac{16}{\Psi_\beta(2\sqrt{m-p}\cdot\varepsilon)}\right)^{m-p}\cdot 2^{-4(m-p)\cdot\sqrt{\left|\log_2\left(\Psi_\beta\left(2\sqrt{m-p}\cdot\varepsilon\right)\right)\right|}}. \tag{2.18}$$

The continuous function $g : [0,1]^d \to \mathbb{R}^m$ defined by

$$g(x) \;=\; (0, \tilde{g}(x)) \qquad \text{for all } x \in [0,1]^d,$$

admits $\beta$ as a modulus of continuity. Moreover, if $0 < \varepsilon \leq \min\left\{ \frac{1}{2\sqrt{m-p}} \cdot \beta(2^{-5}), r_0 \right\}$ then for every function $h = (h_1, ..., h_m) \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$ with $\|h - g\|_{\mathcal{C}^0} \leq \varepsilon$, it holds

$$h_i(x) \;\in\; [-r_0, r_0] \qquad \text{for all } i \in \{1, \ldots, p\}, x \in [0,1]^d.$$

Thus, we can bound the $(d - m + p)$ Hausdorff measure of $\mathcal{Z}^h_W$ by

$$\begin{aligned} \mathcal{H}^{d-m+p}\left(\mathcal{Z}^h_W\right) \;&=\; \mathcal{H}^{d+m-p}\left(\left\{x \in [0,1]^d : h(x) \in [-r_0, r_0]^p \times \{0\}^{m-p}\right\}\right) \\ &=\; \mathcal{H}^{d-m+p}\left(\{x \in [0,1]^d : (h_{p+1}(x), ..., h_m(x)) \in \{0\}^{m-p}\}\right) \\ &\geq\; \inf_{\|b - \tilde{g}\|_{\mathcal{C}^0} \leq \varepsilon} \mathcal{H}^{d-m+p}\left(\mathcal{Z}^b_{\{0\}}\right). \end{aligned} \tag{2.19}$$

Substituting (2.18) into (2.19), we obtain that

$$\begin{aligned} \mathcal{N}^g_W(\varepsilon) \;&=\; \inf_{\|h-g\|_{\mathcal{C}^0} \leq \varepsilon} \mathcal{H}^{d-m+p}\left(\mathcal{Z}^h_W\right) \;\geq\; \inf_{\|b-\tilde{g}\|_{\mathcal{C}^0} \leq \varepsilon} \mathcal{H}^{d-m+p}\left(\mathcal{Z}^b_{\{0\}}\right) \\ &\geq\; \left(\frac{16}{\Psi_\beta(2\sqrt{m-p} \cdot \varepsilon)}\right)^{m-p} \cdot 2^{-4(m-p)\cdot\sqrt{\left|\log_2\left(\Psi_\beta\left(2\sqrt{m-p}\cdot\varepsilon\right)\right)\right|}}. \end{aligned} \tag{2.20}$$

<u>*Step 3.*</u> To complete the proof, we shall establish (2.8) for a $\mathcal{C}^1$-smooth manifold $W \subset \mathbb{R}^m$ satisfying **(A1)**. Without loss of generality, assume that for some $r_0 > 0$

$$W_{r_0} \;\doteq\; [-\tilde{r}_0, \tilde{r}_0]^p \times \{0\}^{m-p} \;\subseteq\; \phi(W),$$

we consider $g$ for $r_0 = \tilde{r}_0/\lambda_2$ in Step 2 with

$$\lambda_1 \;\doteq\; \inf_{x \in U} |\nabla\phi(x)| \qquad \text{and} \qquad \lambda_2 \;\doteq\; \sup_{x \in U} |\nabla\phi(x)|. \tag{2.21}$$

The desired function $f : [0,1]^d \to \mathbb{R}^m$ is defined by

$$f(x) \;=\; \phi^{-1} \circ [\lambda_1 \cdot g(x)] \qquad \text{for all } x \in [0,1]. \tag{2.22}$$

Indeed, $f$ admits $\beta$ as a modulus of continuity since for every $x, y \in [0,1]^d$, it holds

$$|f(y) - f(x)| \;\leq\; \frac{\lambda_1 \cdot |g(y) - g(x)|}{\inf_{z \in U} |\nabla\phi(z)|} \;=\; |g(y) - g(x)|.$$

To verify (2.8), let $h \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$ be such that $\|h - f\|_{\mathcal{C}^0} \leq \varepsilon$. From (2.21) and (2.22), one has that

$$\left\|\frac{\phi \circ h}{\lambda_1} - g\right\|_{\mathcal{C}^0} \;=\; \frac{1}{\lambda_1} \cdot \|\phi \circ h - \phi \circ f\|_{\mathcal{C}^0} \;\leq\; \frac{\lambda_2}{\lambda_1} \cdot \|h - f\|_{\mathcal{C}^0} \;\leq\; \frac{\lambda_2 \varepsilon}{\lambda_1},$$

and this implies

$$\begin{aligned} \mathcal{H}^{d-m+p}\left(\mathcal{Z}^h_W\right) \;&=\; \mathcal{H}^{d-m+p}\left(\left\{x \in [0,1]^d : (\phi \circ h)(x) \in \phi(W)\right\}\right) \\ &\geq\; \mathcal{H}^{d-m+p}\left(\left\{x \in [0,1]^d : (\phi \circ h)(x) \in [-\tilde{r}_0, \tilde{r}_0]^p \times \{0\}^{m-p}\right\}\right) \\ &=\; \mathcal{H}^{d-m+p}\left(\mathcal{Z}^{\phi\circ h}_{W_{r_0}}\right) \;\geq\; \mathcal{N}^g_{W_{r_0}}\left(\frac{\lambda_2 \varepsilon}{\lambda_1}\right). \end{aligned} \tag{2.23}$$

Finally, recalling (2.20), we get for every $h \in \mathcal{C}([0,1]^d, \mathbb{R}^m)$ with $\|h-f\|_{\mathcal{C}^0} \leq \varepsilon$ that

$$\mathcal{H}^{d-m+p}\left(\mathcal{Z}_W^h\right) \;\geq\; \left(\frac{16}{\Psi_\beta(2\sqrt{m-p}\cdot\lambda_2\varepsilon/\lambda_1)}\right)^{m-p} \cdot 2^{-4(m-p)\cdot\sqrt{\left|\log_2\left(\Psi_\beta\left(2\sqrt{m-p}\cdot\lambda_2\varepsilon/\lambda_1\right)\right)\right|}},$$

and (2.4) yields (2.8). □

Notice that if $\beta(s) = \lambda s^\alpha$ for some $\lambda > 0$ and $\alpha \in ]0,1]$ then from (2.6), it holds

$$\Psi_\beta(s) \;=\; \left(\frac{s}{\lambda}\right)^{\frac{1}{\alpha}} \qquad \text{for all } s \in [0,\infty[.$$

In this case, we achieve an explicit estimate in (2.8) by a direct computation. More precisely, we have the following remark.

**Remark 2.4** *Under the same setting in Theorem 2.3, if $\beta(s) = \lambda s^\alpha$ for some $\lambda > 0$, $\alpha \in (0,1]$ then there exists a Hölder continuous function $f : [0,1]^d \to \mathbb{R}^m$ with exponent $\alpha$ and Hölder norm $\lambda$ such that*

$$\mathcal{N}_W^f(\varepsilon) \;\geq\; C_{[W,\alpha,\lambda]} \cdot \left(\frac{1}{\varepsilon}\right)^{\frac{m-p}{\alpha}} \cdot 2^{-\frac{4(m-p)}{\alpha^{1/2}}\cdot\sqrt{\left|\log_2(\gamma_W\varepsilon/\lambda)\right|}}.$$

*This particularly yields Theorem 1.1.*

Finally, we remark that the factor $2^{-4m\cdot\sqrt{\left|\log_2\left(\Psi_\beta\left(2\sqrt{m}\varepsilon\right)\right)\right|}}$ in Theorem 2.3 is necessary . In other words, the estimate on $\mathcal{N}_W^f(\varepsilon)$ in (1.3) is not actually sharp for the case $\alpha = d = m = 1$, $p = 0$, and $W = \{0\}$.

**Proposition 2.1** *Assume that $d = m = 1$, $p = 0$, $W = \{0\}$ and $\beta(s) = s$ for all $s \geq 0$. Then Theorem 2.3 does not hold if the factor $2^{-4(m-p)\cdot\sqrt{\left|\log_2(\Psi_\beta(\gamma_W\varepsilon))\right|}}$ in 2.8 is replaced by any positive constant.*

**Proof.** Arguing by contradiction, suppose that there exists a function $f \in \mathcal{C}([0,1], \mathbb{R})$ and a constant $C_f \in (0,1]$ such that $f$ admits $\beta$ as a modulus of continuity and

$$\mathcal{N}_{\{0\}}^f \;\geq\; \frac{C_f}{\varepsilon} \qquad \text{for all } \varepsilon > 0 \text{ small.} \tag{2.24}$$

**1.** We first claim that for every $0 < \varepsilon \leq \dfrac{C_f^2}{12}$ there exists $(y_i)_{i=1}^N \in [0,1]$ such that

$$N \;\geq\; \frac{C_f^2}{36\varepsilon}, \qquad |f(y_i)| \;\geq\; \frac{\varepsilon}{2}, \qquad |y_i - y_j| \;\geq\; \frac{2\varepsilon}{C_f} \quad \text{for all } i \neq j. \tag{2.25}$$

Indeed, dividing $[0,1]$ into $K_0 = \lfloor \frac{C_f}{3\varepsilon} \rfloor$ subintervals $[a_i, a_{i+1}]$ of length

$$\frac{2\varepsilon}{C_f} \;\leq\; \ell_i \;\leq\; \frac{3\varepsilon}{C_f} \qquad \text{for all } i \in \{0,\dots,K_0-1\}, \tag{2.26}$$

we consider a function $h_\varepsilon \in \mathcal{C}([0,1], \mathbb{R})$ which is defined in $[a_i, a_{i+1}]$ for every $i \in \{0,\dots,K_0-1\}$ as follows:

- If $\max_{x\in[a_i,a_{i+1}]} |f(x)| \le \frac{\varepsilon}{2}$ then we set

$$h_\varepsilon(x) \;=\; \begin{cases} f(a_i) + x - a_i, & a_i \le x \le a_i - f(a_i) + \frac{\varepsilon}{2}, \\ \frac{\varepsilon}{2}, & a_i - f(a_i) + \frac{\varepsilon}{2} \le x \le a_{i+1} + f(a_{i+1}) - \frac{\varepsilon}{2}, \\ f(a_{i+1}) - x + a_{i+1}, & a_{i+1} + f(a_{i+1}) - \frac{\varepsilon}{2} \le x \le a_{i+1}. \end{cases} \tag{2.27}$$

  It is clear that $h_\varepsilon$ has at most 2 zeros on $[a_i, a_{i+1}]$, and $\|h_\varepsilon - f\|_{\mathcal{C}^0} \le \|h_\varepsilon\|_{\mathcal{C}^0} + \|f\|_{\mathcal{C}^0} \le \varepsilon_0$.

- Otherwise, if $\max_{x\in[a_i,a_{i+1}]} |f(x)| > \frac{\varepsilon}{2}$ then we divide $[a_i, a_{i+1}]$ into $K_1 = \lceil \frac{3}{C_f} \rceil$ subintervals $\big[a_i^j, a_i^{j+1}\big]$ of length at most $\varepsilon$. For every $j \in \{0, \dots, K_1 - 1\}$, we set

$$h_\varepsilon\big(\theta \cdot a_i^j + (1-\theta) \cdot a_i^{j+1}\big) \;=\; \theta \cdot f\big(a_i^j\big) + \big(1-\theta\big) \cdot f\big(a_i^{j+1}\big), \qquad \theta \in [0,1]. \tag{2.28}$$

  In this case, $h_\varepsilon$ has at most $\frac{3}{C} + 1$ zeros on $[a_i, a_{i+1}]$ and

$$\|h_\varepsilon - f\|_{\mathcal{C}^0([a_i,a_{i+1}])} \;\le\; \max_{0\le j\le K_1-1} \sup_{|x-y|\le a_i^{j+1} - a_i^j} |f(x) - f(y)| \;\le\; \beta(\varepsilon) \;=\; \varepsilon.$$

Thus, set $\mathcal{I} = \big\{ i \in \{0, \dots, K_0 - 1\} : \max_{x\in[a_i,a_{i+1}]} |f(x)| > \varepsilon/2 \big\}$ and $\eta = \#\mathcal{I}$. By (1.2) and (2.24), we have

$$\frac{C_f}{\varepsilon} \;\le\; \mathcal{H}^0\left(\mathcal{Z}^{h_\varepsilon}_{\{0\}}\right) \;\le\; \eta \cdot \left(\frac{3}{C_f} + 1\right) + (K_0 - \eta) \cdot 2 \;\le\; \eta \cdot \left(\frac{3}{C_f} + 1\right) + \left(\frac{3}{C_f} - \eta\right) \cdot 2,$$

and (2.24) yields

$$\eta \;\ge\; \frac{C_f^2 - 6\varepsilon}{3(3 - C_f)\varepsilon} \;\ge\; \frac{C_f^2}{18\varepsilon}.$$

For every $i \in \mathcal{I}$, let $z_i \in \operatorname{argmax}_{x\in[a_i,a_{i+1}]} |f(x)|$ be such that $|f(z_i)| \ge \varepsilon/2$. From the first inequality of (2.26), one can pick a desired set of at least $N \ge \frac{C_f^2}{36\varepsilon}$ points $y_i$ from the set $\{z_i : i \in \mathcal{I}\}$ which satisfies (2.25).

**2.** Using (2.25), we show that

$$\mathcal{N}^f_{\{0\}}(\varepsilon/4) \;\le\; \left(1 - \frac{C_f^2}{12}\right) \cdot \frac{4}{\varepsilon} \qquad \text{for all } 0 \le \varepsilon \le \frac{2\varepsilon}{C_f} \tag{2.29}$$

Divide $[0,1]$ into $K_\varepsilon = \lceil \frac{4}{\varepsilon} \rceil + 1$ subintervals $[b_k, b_{k+1}]$ with length smaller than $\varepsilon/4$, let $g_\varepsilon : [0,1] \to \mathbb{R}$ be a continuous such that for all $k \in \{0, \dots, K_\varepsilon - 1\}$, it holds

$$g_\varepsilon\big(\theta \cdot b_k + (1-\theta) \cdot b_{k+1}\big) \;=\; \theta \cdot f\big(b_k\big) + \big(1-\theta\big) \cdot f\big(b_{k+1}\big), \qquad \theta \in [0,1].$$

Up to a small variation, we can assume that $f(b_k) \ne 0$ for every $k \in \{1, \dots, K_\varepsilon\}$ so that $g_\varepsilon$ has at most one each of the intervals $[b_k, b_{k+1}]$. By the construction, one has

$$\|g_\varepsilon - f\|_{C^0} \;\le\; \max_{0\le k\le K_\varepsilon-1} |f(b_{k+1}) - f(b_k)| \;\le\; \max_{0\le k\le K_\varepsilon-1} |b_{k+1} - b_k| \;\le\; \frac{\varepsilon}{4}.$$

For every $k \in \{0,\dots,K_\varepsilon - 1\}$ such that $y_i \in [b_k, b_{k+1}]$ for some $i \in \{0,\dots,N\}$, it holds for all $x \in [b_k, b_{k+1}]$ that

$$|g_\varepsilon(x)| \ \geq\ |f(x)| - \frac{\varepsilon}{4} \ \geq\ |f(y_i)| - |\beta(|x-y_i|)| - \frac{\varepsilon}{4} \ >\ \frac{\varepsilon}{4} - |b_{k+1} - b_k| \ >\ 0.$$

In this case, $g_\varepsilon$ has non-zero on the at least $N$ intervals $[b_k], b_{k+1}$. Thus, we have

$$\mathcal{N}^f_{\{0\}}(\varepsilon/4) \ \leq\ \mathcal{H}^0\left(\mathcal{Z}^{g_\varepsilon}_{\{0\}}\right) \ \leq\ K_\varepsilon - N \ \leq\ \frac{4}{\varepsilon} + 1 - \frac{C_f^2}{36\varepsilon}$$

and this yields (2.29).

**3.** Finally, applying (2.25) $n$ times, we find that

$$\mathcal{N}^f_{\{0\}}\left(\frac{\varepsilon}{4^n}\right) \ \leq\ \left(1 - \frac{C_f^2}{12}\right)^n \cdot \frac{4^n}{\varepsilon}.$$

Thus, (2.24) does not holds for $\varepsilon$ replaced by $\frac{\varepsilon}{4^n}$ with $n \geq 1$ sufficiently large so that $\left(1 - \frac{C_f^2}{12}\right)^n < C_f$. This concludes the proof. $\square$

**Acknowledgments.** This research by K. T. Nguyen was partially supported by National Science Foundation grant DMS-2154201.

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