Mathematische Zeitschrift

Hardy spaces for a class of singular domains

A.-K. Gallagher[1] · P. Gupta[2] · L. Lanzani[3] · L. Vivas[4]

Abstract

We set a framework for the study of Hardy spaces inherited by complements of analytic hypersurfaces in domains with a prior Hardy space structure. The inherited structure is a filtration, various aspects of which are studied in specific settings. For punctured planar domains, we prove a generalization of a famous rigidity lemma of Kerzman and Stein. A stabilization phenomenon is observed for egg domains. Finally, using proper holomorphic maps, we derive a filtration of Hardy spaces for certain power-generalized Hartogs triangles, although these domains fall outside the scope of the original framework.

1 Introduction

In this paper, we construct Hardy spaces for a class of domains, which includes the punctured unit disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ and the product domain $\mathbb{D} \times \mathbb{D}^*$ as particularly simple, but enlightening, examples. Although our class of domains is not biholomorphically invariant, it is possible to

✉ L. Vivas
   vivas@math.osu.edu

   A.-K. Gallagher
   anne.g@gallagherti.com

   P. Gupta
   purvigupta@iisc.ac.in

   L. Lanzani
   llanzani@syr.edu

[1] Gallagher Tool & Instrument LLC, Redmond, WA, USA

[2] Department of Mathematics, Indian Institute of Science, Bangalore, India

[3] Department of Mathematics, Syracuse University, Syracuse, NY, USA

[4] Department of Mathematics, The Ohio State University, Columbus, OH, USA

push the construction forward under certain biholomorphisms. This allows us to construct Hardy spaces for the Hartogs triangle, $\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$, and compute the relevant Szegő kernels. In fact, this was the original motivation for this work. The Hartogs triangle is of classical importance in several complex variables, see [29], and serves as an important example of a singular domain since its boundary fails to be even locally graph-like at one point. While $\mathbb{H}$ and its generalizations have received a lot of attention from the point of view of the $\bar{\partial}$-problem, e.g., [6,8,9,20,21], and Bergman spaces, e.g., [5,7,13–15,17,23], Hardy spaces for $\mathbb{H}$ were considered for the first time only recently by Monguzzi in [22]. Independently of Monguzzi, we had constructed a different Hardy space for the Hartogs triangle, and this discrepancy led us to recognize the central phenomenon of this paper. Before we describe this phenomenon, we clarify the main terminology used in this work.

Since there is no unified notion of a Hardy space in the literature, we state here our minimum criteria for using this term. A Hilbert space of functions $\mathfrak{H}$ on the boundary of a domain is deemed a Hardy space *only if* there is a reproducing kernel Hilbert space (in the sense of Aronszajn in [1]) $\mathfrak{X}$ of holomorphic functions on the domain such that

(a) functions in a dense subspace $\mathcal{A} \subset \mathfrak{X}$ admit boundary values in $\mathfrak{H}$, and
(b) this identification of $\mathcal{A}$ with a subspace of $\mathfrak{H}$ is an isometry that extends to an isometric isomorphism between $\mathfrak{X}$ and $\mathfrak{H}$.

We note that in all the explicit examples in this paper, $\mathfrak{X}$ is directly defined in terms of an exhaustion procedure on the domain, see Sects. 2, 5, and 6. However, our general setting is not conducive to this process, and $\mathfrak{X}$ is only abstractly defined, for more details, see (3.1) and subsequent paragraphs.

To describe the class of domains under consideration, we start with a domain $\Omega \Subset \mathbb{C}^n$ and a Borel measure ν supported on its boundary, $b\Omega$, that admits a Hardy space structure. This structure is then inherited by domains that are obtained from Ω by removing analytic hypersurfaces that are component-wise minimally defined, see Definition 3.10. We refer to any such domain as a 'hypersurface-deleted domain', and denote it by Ω^*. We call this process the 'inheritance scheme', and the pair (Ω, ν) the 'parent space'.

In a notable departure from the classical theory, it turns out that under appropriate assumptions on the parent space, any hypersurface-deleted domain is associated to a filtration of Hardy spaces, as opposed to a single such space. This is due to the fact that functions holomorphic on Ω^* can be singular along the deleted hypersurface, but all orders of singularities cannot be captured in a single Hardy space, see the discussion at the beginning of Sect. 2.2. We demonstrate via explicit examples that this filtration may or may not stabilize, depending on the choice of ν and the deleted hypersurface.

1.1 Function-theoretic context

In [26], Poletsky and Stessin give a construction of Hardy spaces for hyperconvex domains in $\mathbb{C}^n$. We note that, while Ω^* is pseudoconvex whenever Ω is, it is never hyperconvex. Our construction therefore covers a new class of domains.

Note that this class of domains is however uninteresting from the point of view of Bergman space theory, since the Bergman space for Ω^* equals the Bergman space for Ω, see [24, Proposition 1.14]. Additionally, our approach does not lead to meaningful Hardy spaces of *harmonic* functions because $b\Omega$ is not, in general, a uniqueness set for harmonic functions on Ω^*. For instance, if $\Omega^* = \mathbb{D} \setminus \{0\}$, then $\operatorname{Re} z$ and $\operatorname{Re} \frac{1}{z}$ are both harmonic on $\mathbb{D}^*$ but coincide on $b\mathbb{D}$.

1.2 Boundary-based approach to Hardy spaces

The lack of a general exhaustion procedure to construct $\mathfrak{X}$ shifts the burden of the construction to the dense subspace $\mathcal{A}$. In the classical setting of the unit disk, $\mathcal{A}$ is the disk algebra, i.e., the space of holomorphic functions on $\mathbb{D}$ that are continuous up to the boundary. If we extend this definition verbatim to the punctured disk, since $\overline{\mathbb{D}^*} = \overline{\mathbb{D}}$, it would lead to the same Hardy space, which does not capture a significant class of holomorphic functions on $\mathbb{D}^*$. Our construction overcomes this issue. When (Ω, ν) is the parent space, we consider $\mathcal{A}$ to be

$$\mathcal{A}(\Omega, \nu) := \mathcal{O}(\Omega) \cap \mathcal{C}(\Omega \cup \text{supp}\, \nu).$$

Moreover, for (Ω^*, ν), we work with *subspaces* of $\mathcal{O}(\Omega^*) \cap \mathcal{C}(\Omega^* \cup \text{supp}\, \nu)$ which have prescribed singularity along the deleted hypersurface. Under appropriate assumptions (see Definition 3.5 for details), the $L^2(\nu)$-completion of $\mathcal{A}|_{\text{supp}\, \nu}$ is a reproducing kernel Hilbert space on the domain in consideration. Hence, we call it a Hardy space and refer to its reproducing kernels as a Szegő kernels.

We point out that there may be kernel functions $c(z, \cdot)$ that have the *reproducing property for $\mathcal{A}$*, namely, for all z in the domain

$$F(z) = \int_{\text{supp}\, \nu} F(w) \cdot c(z, w)\, d\nu(w) \quad \forall F \in \mathcal{A}, \tag{1.1}$$

but are not the Szegő kernel for the associated Hardy space. For instance, this is the case for the Cauchy kernel of any smoothly bounded planar domain $\Omega \neq \mathbb{D}$. Our boundary-based approach is particularly suited to obtaining such boundary integral representation formulas.

1.3 Description of results

We first state conditions on the parent space (Ω, ν) that lead to a Hardy space for Ω, see Definitions 3.1 and 3.4. Then we provide the inheritance scheme that gives a filtration of Hardy spaces for Ω^*, see Theorem 3.12. For each level of the filtration, we produce new kernels that have the reproducing property (1.1). Moreover, we give a sufficient condition for these kernels to agree with the Szegő kernels, see Proposition 3.14. We then proceed to analyse the framework via some examples.

In Theorem 4.2, we consider simply connected planar domains with finitely many points removed. For this class of domains, we formulate and extend a famous rigidity lemma of Kerzman and Stein [18], i.e., if $\Omega \Subset \mathbb{C}$ is simply connected then the Cauchy kernel on Ω coincides with the Szegő kernel for Ω if and only if Ω is a disc. We next identify a family of domains for which the filtration of Hardy spaces stabilizes. These are egg domains, sometimes known as complex ellipsoids, in $\mathbb{C}^2$ from which a single hyperplane has been deleted, and we observe that the stabilization occurs at different levels depending on the choice of boundary measure, see Theorem 5.1. Finally, we use proper holomorphic maps to transfer the filtered Hardy space structure on $\mathbb{D} \times \mathbb{D}^*$ to a class of non hypersurface-deleted domains, i.e., the Hartogs triangle and its rational power generalizations that were first introduced in [13,14]. We also produce explicitly the Szegő kernels for these domains in Theorems 6.1 and 6.2.

1.4 Structure of this paper

In Sect. 2, we consider the punctured disk as this exemplifies the general construction of the filtration of Hardy spaces. In Sect. 3, we provide the general framework and prove

the main inheritance results. Section 4 is specialized to the setting of planar domains, for which more explicit formulas can be proved by means of conformal mapping, along with the aforementioned rigidity result. The egg domains are dealt with in Sect. 5, and $\mathbb{D} \times \mathbb{D}^*$, the Hartogs triangle and its rational power generalizations are treated in Sect. 6.

2 Motivating example

We consider the open unit disk $\mathbb{D}$ and the arc-length measure σ_{S^1} on $b\mathbb{D}$ as the parent space, and the punctured disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ as the hypersurface-deleted domain. Using the basic descriptions for the L^2-Hardy space for the disk detailed in Sect. 2.1, we derive a filtration of Hardy spaces for $(\mathbb{D}^*, \sigma_{S^1})$ in Sect. 2.2. Throughout this section, we omit σ_{S^1} from the notation for the relevant function spaces.

2.1 Hardy Space for the unit disk

The classical L^2-Hardy space $\mathcal{H}^2(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$ that are finite in the norm given by

$$\|F\|_{\mathcal{H}^2(\mathbb{D})} := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 \, d\theta \right)^{\frac{1}{2}}.$$

Note that for any $F \in \mathcal{H}^2(\mathbb{D})$ with $F(z) = \sum_{j=0}^{\infty} a_j z^j$, it follows that

$$\|F\|_{\mathcal{H}^2(\mathbb{D})} = \left(\sum_{j=0}^{\infty} |a_j|^2 \right)^{\frac{1}{2}} < \infty.$$

This characterization facilitates the identification of $\mathcal{H}^2(\mathbb{D})$ as a reproducing kernel Hilbert space, by way of considering the inner product

$$\langle F, G \rangle := \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) \overline{G(re^{i\theta})} \, d\theta \qquad \text{for } F, G \in \mathcal{H}^2(\mathbb{D}),$$

and the evaluation operators $F \mapsto F(z)$ for $z \in \mathbb{D}$ and $F \in \mathcal{H}^2(\mathbb{D})$. Moreover, a truncation of power series argument gives that the disk algebra $\mathcal{A}(\mathbb{D}) = \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ is a dense subspace of $\mathcal{H}^2(\mathbb{D})$. Next, the restriction to the boundary map from $\mathcal{A}(\mathbb{D}) \subset \mathcal{H}^2(\mathbb{D})$ to $\mathcal{A}(\mathbb{D})|_{b\mathbb{D}} \subset L^2(b\mathbb{D})$ extends to an isometric isomorphism, up to a multiplicative constant,

$$\Phi : \mathcal{H}^2(\mathbb{D}) \longrightarrow \overline{\mathcal{A}(\mathbb{D})|_{b\mathbb{D}}}^{L^2(b\mathbb{D})}$$

$$F(z) = \sum_{j=0}^{\infty} a_j z^j \mapsto \Phi(F)(e^{i\theta}) = \sum_{j=0}^{\infty} a_j e^{ij\theta},$$

where $\sum_{j=0}^{\infty} a_j e^{ij\theta}$ is the representation of $\Phi(F)$ as its Fourier series. We call the closure of $\mathcal{A}(\mathbb{D})|_{b\mathbb{D}}$ in $L^2(b\mathbb{D})$ the Hardy space $\mathfrak{H}^2(\mathbb{D})$ for $(\mathbb{D}, \sigma_{S^1})$. Note that if we set $\mathfrak{X}$ as $(\mathcal{H}^2(\mathbb{D}), \frac{1}{\sqrt{2\pi}}\|.\|_{\mathcal{H}^2(\mathbb{D})})$ and $\mathcal{A}$ as $\mathcal{A}(\mathbb{D})$, then $\mathfrak{H} = \mathfrak{H}^2(\mathbb{D})$ satisfies the minimum criterion of a Hardy space stated in the introduction.

The Szegő kernel s for $\mathfrak{H}^2(\mathbb{D})$ may now be derived from the Cauchy integral formula for $F \in \mathcal{A}(\mathbb{D})$, which says that

$$F(z) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{F(w)}{w - z}\, dw = \frac{1}{2\pi} \int_{b\mathbb{D}} \frac{F(w)}{1 - z\overline{w}}\, d\sigma_{S^1}(w).$$

Since s is uniquely determined by such a reproducing property and the fact that $\overline{s(z, .)} \in \mathfrak{H}^2(\mathbb{D})$ for $z \in \mathbb{D}$, see Proposition 3.3, we have that

$$s(z, w) = \frac{1}{2\pi} \frac{1}{1 - z\overline{w}} \quad \text{for } z \in \mathbb{D}, w \in b\mathbb{D}.$$

2.2 Hardy spaces on the punctured disk

In an attempt to develop a Hardy space theory for the punctured disk, one might first consider $\mathcal{O}(\mathbb{D}^*) \cap \mathcal{C}(\overline{\mathbb{D}^*})$. However, $\overline{\mathbb{D}^*} = \overline{\mathbb{D}}$, so this approach would only lead to the rediscovery of the Hardy space on the unit disk. One might also try to construct a Hardy space for $\mathbb{D}^*$ by considering the closure of $\left(\mathcal{O}(\mathbb{D}^*) \cap \mathcal{C}(\mathbb{D}^* \cup b\mathbb{D})\right)_{|b\mathbb{D}}$ with respect to $L^2(b\mathbb{D})$. This fails, too, as pointwise evaluation on this class of $L^2(b\mathbb{D})$-functions is not bounded for any point in $\mathbb{D}^*$. To wit, consider the functions

$$F_k(z) := \sum_{j=1}^{k} \frac{1}{j z^j}, \quad k \in \mathbb{N}.$$

Clearly, $F_k \in \mathcal{O}(\mathbb{D}^*) \cap \mathcal{C}(\mathbb{D}^* \cup b\mathbb{D})$, while

$$\|(F_k)_{|b\mathbb{D}}\|_{L^2(b\mathbb{D})} \leq \sqrt{2\pi} \left(\sum_{j=1}^{\infty} j^{-2}\right)^{\frac{1}{2}} < \infty \ \ \forall k \in \mathbb{N}.$$

Since $F_k(z)$ diverges as $k \to \infty$ for any $z \in \mathbb{D}^*$, it follows that the pointwise evaluation operator is not a bounded operator on $\left(\left(\mathcal{O}(\mathbb{D}^*) \cap \mathcal{C}(\mathbb{D}^* \cup b\mathbb{D})\right)_{|b\mathbb{D}}, \|.\|_{L^2(b\mathbb{D})}\right)$ for any point in $\mathbb{D}^*$. This failure stems from allowing holomorphic functions on $\mathbb{D}^*$ with essential singularities at the origin. Thus, we allow poles of prescribed order at the origin, that is, for $k \in \mathbb{N}_0$, consider the following subset of $\mathcal{O}(\mathbb{D}^*) \cap \mathcal{C}(\mathbb{D}^* \cup b\mathbb{D})$

$$\mathcal{A}_k(\mathbb{D}^*) = \left\{ F : \mathbb{D}^* \cup b\mathbb{D} \longrightarrow \mathbb{C} : F(z) = \left(z^{-k} G(z)\right)|_{\mathbb{D}^*} \text{ for some } G \in \mathcal{A}(\mathbb{D}) \right\}. \quad (2.1)$$

For each $k \in \mathbb{N}_0$ define $\mathfrak{H}_k^2(\mathbb{D}^*)$ to be the closure of $\mathcal{A}_k(\mathbb{D}^*)|_{b\mathbb{D}}$ with respect to $L^2(b\mathbb{D})$. It immediately follows from $z|_{b\mathbb{D}} \neq 0$ that

$$\mathfrak{H}_k^2(\mathbb{D}^*) = \left\{ f \in L^2(b\mathbb{D}) : f(z) = z^{-k} g(z) \text{ for some } g \in \mathfrak{H}^2(\mathbb{D}) \right\}.$$

In particular, any function $f \in \mathfrak{H}_k^2(\mathbb{D}^*)$ is represented by its Fourier series $\sum_{j=-k}^{\infty} \hat{f}_j e^{ij\theta}$ where $\sum_{j=-k}^{\infty} |\hat{f}_j|^2 < \infty$. Note that $\mathfrak{H}_0^2(\mathbb{D}^*) = \mathfrak{H}^2(\mathbb{D})$, $\mathfrak{H}_k^2(\mathbb{D}^*) \subsetneq \mathfrak{H}_{k+1}^2(\mathbb{D}^*)$ for any $k \in \mathbb{N}_0$, and $\bigcup_{k=0}^{\infty} \mathfrak{H}_k^2(\mathbb{D}^*)$ is dense in $L^2(b\mathbb{D})$.

We can also derive the Szegő kernel s_k for $\mathfrak{H}_k^2(\mathbb{D}^*)$ directly from the Szegő kernel s for $\mathfrak{H}^2(\mathbb{D})$. That is, for $F \in \mathcal{A}_k(\mathbb{D}^*)$ given, let $G \in \mathcal{A}(\mathbb{D})$ such that $F(z) = z^{-k} G(z)$ for $z \in \mathbb{D}^*$. Then for $z \in \mathbb{D}^*$, we get

$$z^k F(z) = G(z) = \int_{b\mathbb{D}} G(w) s(z, w)\, d\sigma_{S^1}(w) = \int_{b\mathbb{D}} w^k F(w) s(z, w)\, d\sigma_{S^1}(w).$$

Thus, the kernel given by

$$s_k(z, w) = \frac{w^k}{z^k} s(z, w) = \frac{1}{2\pi} \frac{w^k}{z^k (1 - z\overline{w})} = \frac{1}{2\pi} \frac{1}{(z\overline{w})^k (1 - z\overline{w})}$$

exhibits the reproducing property for $\mathfrak{H}_k^2(\mathbb{D}^*)$, and $\overline{s_k(z, .)} \in \mathfrak{H}_k^2(\mathbb{D}^*)$ for all $z \in \mathbb{D}^*$. Hence s_k is the Szegő kernel for $\mathfrak{H}_k^2(\mathbb{D}^*)$.

Lastly, we remark that $\mathfrak{H}_k^2(\mathbb{D}^*)$ satisfies the minimum criteria, laid out in Sect. 1, for a space $\mathfrak{H}$ to be called a reproducing kernel Hilbert space. Here $\mathcal{A}$ corresponds to $\mathcal{A}_k(\mathbb{D}^*)$, while $\mathfrak{X}$ is the space $\mathcal{H}_k^2(\mathbb{D}^*)$ consisting of $F \in \mathcal{O}(\mathbb{D}^*)$ which satisfy

$$\|F\|_{\mathcal{H}_k^2(\mathbb{D}^*)} := \sup_{0 < r < 1} \left(\frac{r^{2k}}{2\pi} \int_0^{2\pi} \left| F(re^{i\theta}) \right|^2 d\theta \right)^{\frac{1}{2}} < \infty.$$

It follows that

$$\mathcal{H}_k^2(\mathbb{D}^*) = \left\{ F \in \mathcal{O}(\mathbb{D}^*) : F(z) = (z^{-k} G(z))|_{\mathbb{D}^*} \text{ for some } G \in \mathcal{H}^2(\mathbb{D}) \right\}. \tag{2.2}$$

Moreover, the Laurent series for any function in $\mathcal{H}_k^2(\mathbb{D}^*)$ is of the form $\sum_{j=-k}^\infty a_j z^j$ with $\sum_{j=-k}^\infty |a_j|^2 < \infty$. This implies that $\mathcal{H}_k^2(\mathbb{D}^*)$ is a Hilbert space. Furthermore, pointwise evaluation is bounded on $\mathcal{H}_k^2(\mathbb{D}^*)$. This follows from pointwise evaluation being bounded on $\mathcal{H}^2(\mathbb{D})$, characterization (2.2), and the fact that $z|_{\mathbb{D}^*} \neq 0$. Thus, $\mathcal{H}_k^2(\mathbb{D}^*)$ is a reproducing kernel Hilbert space. Finally, $\mathcal{H}_k^2(\mathbb{D}^*)$ and $\mathfrak{H}_k^2(\mathbb{D}^*)$ can be seen to be isometrically isomorphic, up to a constant factor, by mapping the j-th Laurent series coeffient of $F \in \mathcal{H}_k^2(\mathbb{D}^*)$ to the j-th Fourier coefficient of $F|_{b\mathbb{D}}$ for all $j \geq k$.

3 Hardy spaces on hypersurface-deleted domains

The construction of the Hardy spaces for $\mathbb{D}^*$ suggests a general inheritance scheme for the construction of Hardy spaces for domains that are obtained by removing certain complex hypersurfaces from a given domain. As in the case of $\mathbb{D}^*$ in Sect. 2, one starts with a domain Ω and a boundary measure ν that together carry their own Hardy space structure. We henceforth refer to such a pair (Ω, ν) as a *parent space*.

We detail requirements on the parent space (Ω, ν) in Sect. 3.1. In Sect. 3.2, we describe the class of complex hypersurfaces that will be removed from Ω to produce the so-called hypersurface-deleted domain Ω^*. The inheritance scheme is described in Sect. 3.3.

3.1 Requirements on the parent space

We consider a domain $\Omega \Subset \mathbb{C}^n$ equipped with a finite Borel measure ν on its topological boundary $b\Omega$. We denote the support of ν by T, and set

$$\Omega_T := \Omega \cup T.$$

We discuss some conditions that allow us to identify reproducing kernel Hilbert spaces of holomorphic functions on Ω that admit boundary values on T for, at least, a dense subspace.

Definition 3.1 Let (Ω, ν) be as above, and $\mathcal{F}$ be a family of complex-valued functions on Ω_T. Then $\mathcal{F}$ is said to be *weakly admissible* if and only if

(i) $F|_T \in L^2(\nu)$ for any $F \in \mathcal{F}$, and
(ii) for any compact set $K \subset \Omega$, there exists a $C_K > 0$ such that

$$\sup \big\{ |F(z)| : z \in K \big\} \leq C_K \|F|_T\|_{L^2(\nu)} \quad \text{for all} \quad F \in \mathcal{F}.$$

If we further assume that $\mathcal{F}$ is closed under subtraction, then each element of $\mathcal{F}$ is uniquely determined by its values along T.

We focus on the family of holomorphic functions given by

$$\mathcal{A}(\Omega, \nu) := \mathcal{O}(\Omega) \cap \mathcal{C}(\Omega_T).$$

Note that $\mathcal{A}(\Omega, \nu)$ is an algebra over $\mathbb{C}$. It satisfies condition (i) in Definition 3.1 because $\mathcal{C}(T) \subset L^2(\nu)$ whenever ν is a finite Borel measure.

Definition 3.2 Let (Ω, ν) be such that $\mathcal{A}(\Omega, \nu)$ is weakly admissible. We define the *pre-Hardy space* associated to (Ω, ν) as

$$\mathfrak{H}^2(\Omega, \nu) := \overline{\mathcal{A}(\Omega, \nu)|_T}^{L^2(\nu)},$$

where

$$\mathcal{A}(\Omega, \nu)|_T := \big\{ f : T \to \mathbb{C}, \ f = F|_T \ \text{for some} \ F \in \mathcal{A}(\Omega, \nu) \big\}.$$

Proposition 3.3, and the subsequent discussion, justifies the nomenclature introduced in Definition 3.1. Note that despite the nonstandard terminology, the following proposition is standard in functional analysis.

Proposition 3.3 *Suppose that $\mathcal{A}(\Omega, \nu)$ is weakly admissible. Then for any $z \in \Omega$, there exists a unique bounded linear functional*

$$\mathrm{Ev}_z : \ \mathfrak{H}^2(\Omega, \nu) \to \mathbb{C}$$

such that $\mathrm{Ev}_z(F|_T) = F(z)$ *for any* $F \in \mathcal{A}(\Omega, \nu)$. *Furthermore, there exists a unique function* $s : \Omega \times T \to \mathbb{C}$ *such that*

(1) $\overline{s(z, .)} \in \mathfrak{H}^2(\Omega, \nu)$ for all $z \in \Omega$, and
(2) Ev_z and $s(z, .)$ are related through the integral representation given by

$$\mathrm{Ev}_z(f) = \langle f(.), \overline{s(z, .)} \rangle_{L^2(\nu)} = \int_T f(w) s(z, w) \, d\nu(w) \ \text{for any} \ f \in \mathfrak{H}^2(\Omega, \nu).$$

We refer to the function s as the *Szegő kernel for* $\mathfrak{H}^2(\Omega, \nu)$.

Proof Note that $\mathcal{A}(\Omega, \nu)|_T$ is a normed vector space when endowed with the norm for $L^2(\nu)$. The existence of $\mathrm{Ev}_z(f)$ follows from the Bounded Linear Extension Theorem applied to the evaluation $F|_T \mapsto F(z)$ for $F \in \mathcal{A}(\Omega, \nu)|_T$. An application of the Riesz Representation Theorem then yields the existence and uniqueness of $s(z, .)$.

In the literature, Hardy spaces are considered as examples of reproducing kernel Hilbert spaces on Ω. Note that $\mathfrak{H}^2(\Omega, \nu)$ contains functions that *a priori* are defined only on $T \subseteq b\Omega$. With an additional assumption on $\mathcal{A}(\Omega, \nu)$, $\mathfrak{H}^2(\Omega, \nu)$ may be identified with a function space on Ω, and hence may be considered as a reproducing kernel Hilbert space on Ω.

To identify the appropriate function space on Ω for a given weakly admissible $\mathcal{A}(\Omega, \nu)$, we note first that $\mathrm{Ev}_{(.)}(f)$ is holomorphic on Ω for all $f \in \mathfrak{H}^2(\Omega, \nu)$. This is obvious if there exists an $F \in \mathcal{A}(\Omega, \nu)$ such that $F|_T = f$. It is also true for general $f \in \mathfrak{H}^2(\Omega, \nu)$ because

the uniform boundedness of the evaluation operators on compacta, see (ii) in Definition 3.1, says that $\mathrm{Ev}_{(.)}(f)$ is the normal limit of holomorphic functions. Thus, the map

$$\mathcal{I} : \mathfrak{H}^2(\Omega, \nu) \longrightarrow \mathcal{O}(\Omega) \tag{3.1}$$
$$f \;\mapsto\; F, \text{ where } F(z) := \mathrm{Ev}_z(f)$$

is well-defined. Denote by $\mathfrak{X}(\Omega, \nu) := \mathcal{I}\big(\mathfrak{H}^2(\Omega, \nu)\big) \subset \mathcal{O}(\Omega)$. The injectivity of $\mathcal{I}$ can be stated through a condition on certain Cauchy sequences in $\mathcal{A}(\Omega, \nu)$. We formulate this condition for general function spaces as follows.

Definition 3.4 Let (Ω, ν) be as above, and $\mathcal{F}$ be a weakly admissible family of complex-valued functions on Ω_T. Then $\mathcal{F}$ is said to be *strongly admissible* if for any sequence $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ for which $\{(F_n)|_T\}_{n \in \mathbb{N}}$ is Cauchy in $L^2(\nu)$ and $F_n \to 0$ uniformly on compacta in Ω as $n \to \infty$, the sequence $\{(F_n)|_T\}_{n \in \mathbb{N}}$ converges to 0 in $L^2(\nu)$ as $n \to \infty$.

Now suppose (Ω, ν) is such that $\mathcal{A}(\Omega, \nu)$ is strongly admissible. Then we may equip $\mathfrak{X}(\Omega, \nu)$ with a reproducing kernel Hilbert space structure via $\mathcal{I}$. This allows us to identify $\mathfrak{H}^2(\Omega, \nu)$ with a reproducing kernel Hilbert space on Ω, and hence we can make the following definition.

Definition 3.5 Let (Ω, ν) be such that $\mathcal{A}(\Omega, \nu)$ is strongly admissible. The *Hardy space* of (Ω, ν) is $\mathfrak{H}^2(\Omega, \nu)$.

We note that we do not have an independent description of $\mathfrak{X}(\Omega, \nu)$ in this general setting of strongly admissible function spaces. However, in all the examples considered in this paper, $\mathfrak{X}(\Omega, \nu)$ is independently described using an exhaustion-based approach, see the spaces denoted by $\mathcal{H}^2(.)$ in Sects. 2, 5 and 6.

Examples of (Ω, ν) for which $\mathcal{A}(\Omega, \nu)$ is strongly admissible include

(1) (Ω, σ), where $\Omega \subset \mathbb{C}$ is a $\mathcal{C}^{1,\alpha}$-smooth bounded domain, and σ is the arc-length measure on $b\Omega$, see the discussion at the beginning of Sect. 4.
(2) $(\mathbb{D}^n, \sigma_{S^1} \times \ldots \times \sigma_{S^1})$, where σ_{S^1} is the arc-length measures of the unit circle in the j-th coordinate, and $T = (b\mathbb{D})^n$, and
(3) (Ω, σ), where $\Omega \subset \mathbb{C}^n$ is a $\mathcal{C}^2$-smooth bounded domain, σ is the surface measure of $b\Omega$, and $T = b\Omega$, see [30].

On the other hand, recall from Sect. 2.2 that $\mathcal{A}(\mathbb{D}^*, \sigma_{S^1})$ is not even a weakly admissible subspace of $L^2(b\mathbb{D}, \sigma_{S^1})$. Conditions analogous to weak and strong admissibility, albeit in a broader context, were identified in [1, Theorem p. 347]. An example is also given therein to demonstrate the inequivalence of the two conditions, see [1, p. 349].

3.2 Requirements on the hypersurface

We first recall some standard notions from analytic geometry. Let $K \Subset \mathbb{C}^n$ be a bounded set.

Definition 3.6 Denote by $\mathcal{O}(K)$ the set of equivalence classes of

$$\{(f, \omega) : \omega \text{ is an open neighborhood of } K \text{ and } f : \omega \to \mathbb{C} \text{ is holomorphic}\}$$

modulo the equivalence relation $(f_1, \omega_1) \sim (f_2, \omega_2)$ if and only if there is an open neighborhood $\omega \subset \omega_1 \cap \omega_2$ of K such that $f_1|_\omega = f_2|_\omega$. The equivalence class of (f, ω) will be denoted simply by f, which we call the *germ of an analytic function on K*. Note that $\mathcal{O}(K)$ forms a ring under multiplication and addition.

Definition 3.7 Let $\omega \subset \mathbb{C}^n$ be an open set. A closed subset V of ω is an *analytic variety in* ω if for any $z \in \omega$ there exists a neighborhood $U(z) \subset \omega$ such that $U(z) \cap V$ is the common zero set of some nontrivial $f_1, \ldots, f_k \in \mathcal{O}(U(z))$ for some $k \in \mathbb{N}$. We say that V is a *locally principal variety in* ω if k may be chosen equal to 1 for any $z \in \omega$.

Definition 3.8 Define $\mathscr{V}(K)$ to be the set of equivalence classes of

$$\{(V, \omega) : \omega \text{ is an open neighborhood of } K, V \subsetneq \omega \text{ is a locally principal variety in } \omega\}$$

modulo the equivalence relation $(V_1, \omega_1) \sim (V_2, \omega_2)$ if and only if there is an open neighborhood $\omega \subset \omega_1 \cap \omega_2$ of K such that $V_1|_\omega = V_2|_\omega$. The equivalence class of (V, ω) will be denoted simply by V, which we call the *germ of an analytic hypersurface in* K.

We next focus on the situation when $K = \overline{\Omega}$ for $\Omega \Subset \mathbb{C}^n$ is a domain. Note that the zero set of any nontrivial $f \in \mathcal{O}(\overline{\Omega})$ gives rise to an element $V \in \mathscr{V}(\overline{\Omega})$, but not every element in $\mathscr{V}(\overline{\Omega})$ arises this way. If $V \in \mathscr{V}(\overline{\Omega})$ is indeed the zero set of a single $f \in \mathcal{O}(\overline{\Omega})$, then V is called *principal* and such an f *a defining function for* V. A principal germ V is called *minimally defined* if it admits a defining function $f \in \mathcal{O}(\overline{\Omega})$ such that, whenever $U \subset \overline{\Omega}$ is an open set (in the relative topology) and $g \in \mathcal{O}(U)$ vanishes on $U \cap V$, then $f|_U$ divides g in $\mathcal{O}(U)$. We call such an f a *minimal defining function of V in* $\mathcal{O}(\overline{\Omega})$. It follows from a standard argument that minimal defining functions are unique up to non-vanishing holomorphic factors. We state this as a lemma for easy reference.

Lemma 3.9 *Let V be a minimally defined germ of an analytic hypersurface in $\overline{\Omega}$. Suppose $f, g \in \mathcal{O}(\overline{\Omega})$ are two minimal defining functions of V. Then there is an $h \in \mathcal{O}(\overline{\Omega})$ such that $f = hg$, and h does not vanish on $\overline{\Omega}$.*

Finally, $V \in \mathscr{V}(\overline{\Omega})$ is said to be *irreducible* if it cannot be expressed as $V_1 \cup V_2$ for elements $V_1, V_2 \in \mathscr{V}(\overline{\Omega})$ distinct from V. Note that for any $V \in \mathscr{V}(\overline{\Omega})$, there is an $m \in \mathbb{N}$ such that $V \cap \Omega = \cup_{j=1}^m (V_j \cap \Omega)$, where each V_j is an irreducible germ of an analytic hypersurface in $\overline{\Omega}$, see [10, § 5.4].

Subsequently, we consider domains as follows.

Definition 3.10 Let $\Omega \Subset \mathbb{C}^n$ be a domain. Let $V \in \mathscr{V}(\overline{\Omega})$ be a finite union of irreducible, minimally defined germs of analytic hypersurfaces on $\overline{\Omega}$. Then

$$\Omega^* = \Omega \setminus V$$

is called a *hypersurface-deleted domain*.

We now discuss some examples of hypersurface-deleted domains. In the planar case, if $\Omega \Subset \mathbb{C}$ is a domain and $V \subset \mathscr{V}(\overline{\Omega})$, then $\Omega \cap V = \{a_1, \ldots, a_m\}$ for some $a_1, \ldots, a_m \in \Omega$ and $m \in \mathbb{N}$. It is immediate to see that $f_j(z) = z - a_j$ is a minimal defining function of $\{a_j\}$ in $\mathcal{O}(\overline{\Omega})$. Thus, $\Omega \setminus V = \Omega \setminus \{a_1, \ldots, a_m\}$ is a hypersurface-deleted domain.

A further class of examples, which includes bounded convex domains in $\mathbb{C}^n$, is provided by the following result. Note that the result implies that for such Ω, $\Omega \setminus V$ is a hypersurface-deleted domain for any $V \in \mathscr{V}(\overline{\Omega})$.

Proposition 3.11 *Let $n > 1$. Suppose $\Omega \Subset \mathbb{C}^n$ is a domain such that $\overline{\Omega}$ admits a Stein neighborhood basis and $H^2(\overline{\Omega}; \mathbb{Z}) = 0$. Then any irreducible $V \in \mathscr{V}(\overline{\Omega})$ is minimally defined.*

Proof The proof is well-known. For the reader's convenience, we highlight the main steps of the argument. Recall that a Cousin II distribution on the compact set $\overline{\Omega}$ is a collection $\{(U_\iota, f_\iota)\}_{\iota \in I}$, where $\{U_\iota\}_{\iota \in I}$ is a (relatively) open cover of $\overline{\Omega}$, and $f_\iota \in \mathcal{O}(U_\iota)$ with $f_\iota|_{U_\iota \cap U_J} = h_{\iota J} \cdot f_J|_{U_\iota \cap U_J}$ for some nonvanishing $h_{\iota J} \in \mathcal{O}(U_\iota \cap U_J)$. The hypothesis on $\overline{\Omega}$ implies that, given such a Cousin II distribution, there is an $f \in \mathcal{O}(\overline{\Omega})$ such that $f_\iota = h_\iota \cdot f|_{U_\iota}$ for some nonvanishing $h_\iota \in \mathcal{O}(U_\iota)$, for all $\iota \in I$, i.e., $\overline{\Omega}$ is a Cousin II set, see [11].

Let $V \in \mathcal{V}(\overline{\Omega})$ be irreducible. Then V admits a local minimal defining function at each point of $V \cap \overline{\Omega}$, see [10, §2.8.]. By compactness and Lemma 3.9, there is a finite Cousin II distribution, $\{U_i, f_i\}_{i \in \{1,...,m\}}$, such that f_i is a minimal defining function of $V \cap U_i$ for $i \in \{1, ..., m\}$. We claim that the Cousin II solution, $f \in \mathcal{O}(\overline{\Omega})$, for this distribution is a minimal defining function of V in $\mathcal{O}(\overline{\Omega})$. First observe that $f|_{U_i \cap V} = (h_i^{-1} \cdot f_i)|_{U_i \cap V} = 0$ for $i \in \{1, ..., m\}$. Thus, f vanishes on V. Next, let $U \subset \overline{\Omega}$ be a (relatively) open subset and $g \in \mathcal{O}(U)$ be such that g vanishes on $U \cap V$. Since each f_i is minimal, it follows that each f_i divides g in $\mathcal{O}(U \cap U_i)$. Furthermore, $f|_{U_i}$ divides f_i in $\mathcal{O}(U_i)$, in particular $f|_{U \cap U_i}$ divides f_i in $\mathcal{O}(U \cap U_i)$ for each i. Therefore $f|_{U \cap U_i}$ divides g in $\mathcal{O}(U \cap U_i)$. That is, $f|_U$ divides g locally and hence in $\mathcal{O}(U)$ since f and g are globally defined in U.

In general, if $V \in \mathcal{V}(\overline{\Omega})$ is principal, then any defining function $f \in \mathcal{O}(\overline{\Omega})$ of V is minimal if and only if $\{z \in \omega : \det Df(z) = 0\}$ is nowhere dense in $V \cap \omega$ for some open neighborhood ω of $\overline{\Omega}$, see [10, § 2.9]. Thus, by this criterion, $\Omega \setminus V$, where V is an affine hyperplane, is always a hypersurface-deleted domain.

3.3 The inheritance scheme

We first construct Hardy spaces for triples of the form (Ω, ν, V), such that

(i) $\Omega \Subset \mathbb{C}^n$ is a domain, ν is a finite Borel measure on $b\Omega$,
(ii) V is an irreducible, minimally defined germ of an analytic hypersurface in $\overline{\Omega}$, and
(iii) $\Omega \cap V \neq \emptyset$ and $\nu(T \cap V) = 0$, where $T = \operatorname{supp}(\nu)$.

The case of general hypersurface-deleted domains is discussed at the end of this subsection. As before, $\Omega_T = \Omega \cup T$, $\Omega^* = \Omega \setminus V$, and $\mathcal{A}(\Omega, \nu)$ is as in Definition 3.1. We also set $T^* := T \setminus V$. Let $\psi \in \mathcal{O}(\overline{\Omega})$ be a minimal defining function of V. Then for any non-negative integer k, we consider the following subset of $\mathcal{O}(\Omega^*) \cap \mathcal{C}(\Omega^* \cup T^*)$

$$\mathcal{A}_k(\Omega^*, \nu) := \Big\{ F : \Omega^* \cup T^* \to \mathbb{C} : F = (\psi^{-k} G)|_{\Omega^* \cup T^*} \text{ for some } G \in \mathcal{A}(\Omega, \nu)$$

$$\text{and } F|_{T^*} \in L^2(\nu) \Big\}. \quad (3.2)$$

Note that it follows from Lemma 3.9, that $\mathcal{A}_k(\Omega^*, \nu)$ does not depend on the choice of minimal defining function of V. Hence, we make no reference to ψ in our notation and work with a fixed choice of ψ for the purpose of our proofs.

We identify $\mathcal{A}_k(\Omega^*, \nu)$ with a function space on $\Omega^* \cup T$ by extending its members trivially, by zero, to $T \cap V$, which is a measure-zero set. Then the space of boundary values of $\mathcal{A}_k(\Omega^*, \nu)$, i.e.,

$$\mathcal{A}_k(\Omega^*, \nu)|_T = \big\{ F|_T : F \in \mathcal{A}_k(\Omega^*, \nu) \big\}$$

is a subspace of $L^2(\nu)$. Note that as subspaces of $L^2(\nu)$, $\mathcal{A}_k(\Omega^*, \nu)|_T = \mathcal{A}_k(\Omega^*, \nu)|_{T^*}$. This allows us to speak of the notion of weak and strong admissibility for $\mathcal{A}_k(\Omega^*, \nu)$. The spaces $\mathcal{A}_k(\Omega^*, \nu)$ always inherit the properties of weak and strong admissibility from $\mathcal{A}(\Omega, \nu)$.

Theorem 3.12 *For (Ω, v, V) satisfying (i), (ii) and (iii) above, the following holds.*

(1) If $\mathcal{A}(\Omega, v)$ is weakly admissible, then so is $\mathcal{A}_k(\Omega^*, v)$ for any $k \in \mathbb{N}_0$.
(2) If $\mathcal{A}(\Omega, v)$ is strongly admissible, then so is $\mathcal{A}_k(\Omega^*, v)$ for any $k \in \mathbb{N}_0$.

Proof For the proof of part (1), fix a $k \in \mathbb{N}_0$ and suppose that $\mathcal{A}(\Omega, v)$ is weakly admissible. We need to show that for any compact set $K \subset \Omega^*$, there exists a constant $c_K > 0$ such that the evaluation operators

$$\mathrm{Ev}_z : \mathcal{A}_k(\Omega^*, v) \longrightarrow \mathbb{C}$$
$$F \mapsto \mathrm{Ev}_z(F) := F(z), \quad z \in K,$$

are uniformly bounded on K. For that, let $F \in \mathcal{A}_k(\Omega^*, v)$. Then $F = (\psi^{-k}G)|_{\Omega^* \cup T^*}$ for some $G \in \mathcal{A}(\Omega, v)$ and $F|_T \in L^2(v)$. Since $\mathcal{A}(\Omega, v)$ is weakly admissible and K is compact in Ω^*, hence in Ω, it follows that there exists a constant $C_K > 0$ such that

$$|\mathrm{Ev}_z(G)| \leq C_K \|G|_T\|_{L^2(v)} \quad \forall z \in K.$$

Therefore,

$$|\mathrm{Ev}_z(F)| = \left|\psi^{-k}(z)\right| \cdot |\mathrm{Ev}_z(G)| \leq C_K \left|\psi^{-k}(z)\right| \|G|_T\|_{L^2(v)} \quad \forall z \in K.$$

Since $K \subset \Omega^*$, ψ is continuous and nonvanishing on K, and $v(V \cap T) = 0$, there exists a constant $\widetilde{C_K} > 0$ such that

$$|\mathrm{Ev}_z(F)| \leq \widetilde{C_K} \|G|_T\|_{L^2(v)} = \widetilde{C_k} \left\|(\psi^k \cdot F)|_T\right\|_{L^2(v)} \quad \forall z \in K.$$

As $\psi|_T$ is bounded and $F|_T \in L^2(v)$, there is a constant c_K such that

$$|\mathrm{Ev}_z(F)| \leq c_K \|F|_T\|_{L^2(v)}.$$

This concludes the proof of part (1).

To prove part (2), let $k \in \mathbb{N}_0$ and suppose that $\mathcal{A}(\Omega, v)$ is strongly admissible. Let $\{(F_n)\}_{n \in \mathbb{N}} \subset \mathcal{A}_k(\Omega^*, v)$ be a sequence such $\{F_n|_T\}_{n \in \mathbb{N}}$ is Cauchy in $L^2(v)$ and $F_n \longrightarrow 0$ uniformly on compacta in Ω^*. Then for any $n \in \mathbb{N}$, $F_n = (\psi^{-k}G_n)|_{\Omega^* \cup T^*}$ for some $G_n \in \mathcal{A}(\Omega, v)$. Therefore,

$$\|(G_n - G_m)|_T\|_{L^2(v)} = \left\|(\psi^k \cdot F_n - \psi^k \cdot F_m)|_T\right\|_{L^2(v)}.$$

Since ψ is bounded on T, it follows that $\{(G_n)|_T\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(v)$. Furthermore, $\mathcal{A}(\Omega, v)$ is weakly admissible, and so for any compact set $K \subset \Omega$, there exists a constant $C_K > 0$ such that

$$|G_n(z) - G_m(z)| \leq C_K \|(G_n - G_m)|_T\|_{L^2(v)} \quad \forall z \in K,$$

i.e., $\{G_n\}_{n \in \mathbb{N}}$ converges uniformly on compacta in Ω. Thus, there exists a $G \in \mathcal{O}(\Omega)$ such that $G_n(z) \longrightarrow G(z)$ for all $z \in \Omega$ as $n \to \infty$. However, for $z \in \Omega^*$, $G_n(z) = \psi^k(z)F_n(z) \longrightarrow 0$ as $n \to \infty$. Therefore, $G(z) = 0$ for all $z \in \Omega^*$. This implies that $G \equiv 0$ on Ω, and $G_n \longrightarrow 0$ uniformly on compacta in Ω. Since $\mathcal{A}(\Omega, v)$ is strongly admissible, it follows that $(G_n)|_T \longrightarrow 0$ in $L^2(v)$ as $n \to \infty$. This in turn implies that $(F_n)|_T \longrightarrow 0$ in $L^2(v)$ as $n \to \infty$. Thus, $\mathcal{A}_k(\Omega^*, v)$ is strongly admissible. $\qquad\blacksquare$

We are now set to define the central objects of this discussion.

Definition 3.13 Let (Ω, ν) be such that $\mathcal{A}(\Omega, \nu)$ is weakly admissible and $k \in \mathbb{N}_0$. *The k-th pre-Hardy space* $\mathfrak{H}_k^2(\Omega^*, \nu)$ *is the closure of* $\mathcal{A}_k(\Omega^*, \nu)|_T$ *in* $L^2(\nu)$. *If* $\mathcal{A}(\Omega, \nu)$ *is strongly admissible, we call* $\mathfrak{H}_k^2(\Omega^*, \nu)$ *the k-th Hardy space of* (Ω, ν, V).

Note that
$$\mathcal{A}_0(\Omega^*, \nu) = \mathcal{A}(\Omega, \nu)|_{\Omega^* \cup T^*},$$

i.e., $\mathcal{A}_0(\Omega^*, \nu)$ does not lead to a new space. Furthermore,
$$\mathcal{A}_0(\Omega^*, \nu) \subseteq \mathcal{A}_1(\Omega^*, \nu) \subseteq \cdots \subseteq \mathcal{A}_k(\Omega^*, \nu) \subseteq \ldots, \tag{3.3}$$

and, for any $\ell \in \mathbb{N}_0$, the spaces $\psi^\ell \mathcal{A}_k(\Omega^*, \nu) := \{\psi^\ell \cdot F : F \in \mathcal{A}_k(\Omega^*, \nu)\}$ satisfy the inclusions
$$\psi^\ell \mathcal{A}_k(\Omega^*, \nu) \subseteq \mathcal{A}_{k-\ell}(\Omega^*, \nu) \quad \text{whenever} \quad \ell \leq k. \tag{3.4}$$

The collection $\{\mathfrak{H}_k^2(\Omega^*, \nu)\}_k$ inherits these properties. That is, $\mathfrak{H}_0^2(\Omega^*, \nu) = \mathfrak{H}^2(\Omega, \nu)$. Furthermore,
$$\mathfrak{H}_0^2(\Omega^*, \nu) \subseteq \mathfrak{H}_1^2(\Omega^*, \nu) \subseteq \cdots \subseteq \mathfrak{H}_k^2(\Omega^*, \nu) \ldots, \tag{3.5}$$

as well as
$$\psi^\ell \mathfrak{H}_k^2(\Omega^*, \nu) \subseteq \mathfrak{H}_{k-\ell}^2(\Omega^*, \nu) \quad \text{whenever} \quad \ell \leq k. \tag{3.6}$$

Applying Proposition 3.3 to $\mathcal{A}_k(\Omega^*, \nu)$, we see that $\mathfrak{H}_k^2(\Omega^*, \nu)$ possesses a Szegő kernel s_k for any $k \in \mathbb{N}_0$. Moreover, the Szegő kernel s for $\mathfrak{H}^2(\Omega, \nu)$ generates a family of kernels with the reproducing property for $\mathfrak{H}_k^2(\Omega^*, \nu)$.

Proposition 3.14 *Let* (Ω, ν) *be such that* $\mathcal{A}(\Omega, \nu)$ *is weakly admissible. Let* $\varphi \in \mathcal{A}(\Omega, \nu)$ *be such that* $\varphi = h\psi$ *where* ψ *is a minimal defining function of V and* $h \in \mathcal{A}(\Omega, \nu)$ *is nonvanishing on* $\Omega_T \setminus V$. *Then*
$$c_{k,\varphi}(z, w) := \frac{\varphi^k(w)}{\varphi^k(z)} s(z, w), \quad z \in \Omega^*, w \in T \tag{3.7}$$

has the reproducing property for $\mathfrak{H}_k^2(\Omega^*, \nu)$. *Moreover, if h is nowhere vanishing on* Ω_T *and* $|\varphi|$ *is constant on T, then* $c_{k,\varphi}$ *is the Szegő kernel for* $\mathfrak{H}_k^2(\Omega^*, \nu)$ *for all* $k \in \mathbb{N}_0$.

Proof Let $k \in \mathbb{N}_0$ and $F \in \mathcal{A}_k(\Omega^*, \nu)$. Then there is a $G \in \mathcal{A}(\Omega, \nu)$ such that $F = (\psi^{-k}G)|_{\Omega^* \cup T^*}$ and $F|_T \in L^2(\nu)$. Since $h^k G \in \mathcal{A}(\Omega, \nu)$, it follows that for any $z \in \Omega^*$
$$F(z) = \varphi^{-k}(z)\varphi^k(z)F(z) = \varphi^{-k}(z)h^k(z)G(z) = \varphi^{-k}(z) \int_T h^k(w)G(w)\, s(z, w)\, d\nu(w).$$

Thus,
$$F(z) = \int_T F(w)\, c_{k,\varphi}(z, w)\, d\nu(w) \quad \text{for any } z \in \Omega^*.$$

The reproducing property of $c_{k,\varphi}$ for $\mathfrak{H}_k^2(\Omega^*, \nu)$ then follows from the density of $\mathcal{A}_k(\Omega^*, \nu)|_T$ in $\mathfrak{H}_k^2(\Omega^*, \nu)$ with respect to $L^2(\nu)$.

It remains to show that if h is nonvanishing on Ω_T and $|\varphi|$ equals some constant $c \geq 0$ on T, then $\overline{c_{k,\varphi}(z, .)} \in \mathfrak{H}_k^2(\Omega^*, \nu)$ for any $z \in \Omega^*$. Note first that $c \neq 0$ since neither h nor ψ vanish on T^*. Thus, as an aside, observe that φ does not vanish on T and, in particular,

$V \cap T = \emptyset$. Since $\overline{s(z,.)} \in \mathfrak{H}_{\bar{k}}^2(\Omega, \nu)$ for any $z \in \Omega$, it follows that there exists a sequence $\{S_n(z,.)\}_{n \in \mathbb{N}}$ such that $\overline{S_n(z,.)} \in \mathcal{A}(\Omega, \nu)$ for all $n \in \mathbb{N}$ and

$$\left\| s(z,.) - S_n(z,.)|_T \right\|_{L^2(\nu)} \longrightarrow 0 \text{ as } n \to \infty, \ \forall z \in \Omega.$$

This, and the fact that $\varphi^k(.)\varphi^{-k}(z)$ is bounded on T for any fixed $z \in \Omega^*$, implies that

$$\left\| c_{k,\varphi}(z,.) - \left(\varphi^k(.)\varphi^{-k}(z)S_n(z,.) \right)|_T \right\|_{L^2(\nu)} \longrightarrow 0 \text{ as } n \to \infty, \ \forall z \in \Omega^*.$$

To see that $\overline{\varphi^k(.)\varphi^{-k}(z)S_n(z,.)}$ is in $\mathcal{A}_k(\Omega^*, \nu)$ for any $z \in \Omega^*$, we first note

$$\overline{\varphi^k(w)} = c^{2k}\varphi^{-k}(w) \ \forall \ w \in T.$$

It then suffices to show that $\varphi^{-k}(.)\overline{S_n(z,.)}$ is in $\mathcal{A}_k(\Omega^*, \nu)$ for any $z \in \Omega^*$. Since $h \in \mathcal{A}(\Omega, \nu)$ is nonvanishing on Ω_T, it follows that $h^{-k}(.)\overline{S_n(z,.)} \in \mathcal{A}(\Omega, \nu)$. Thus, by the definition of $\mathcal{A}_k(\Omega^*, \nu)$, it remains to show that $\left(\psi^{-k}(.)h^{-k}(.)\overline{S_n(z,.)} \right)|_T$ is in $L^2(\nu)$. This membership holds because $\psi \cdot h = \varphi$ is a nonvanishing continuous function on T. This concludes the proof of $c_{k,\varphi}$ being the Szegő kernel for $\mathfrak{H}_{\bar{k}}^2(\Omega^*, \nu)$.

Remark 3.15 Note that replacing the Szegő kernel for $\mathfrak{H}^2(\Omega, \nu)$ in (3.7) with *any* other kernel with the reproducing property for $\mathfrak{H}^2(\Omega, \nu)$, yields yet another family of kernels with the reproducing property for $\mathfrak{H}_{\bar{k}}^2(\Omega^*, \nu)$.

We briefly discuss the especially favorable situation when $V \cap T = \emptyset$. In this case the requirement that $F|_{T^*} \in L^2(\nu)$ in the definition of $\mathcal{A}_k(\Omega^*, \nu)$ is redundant. Moreover, the containment relations in (3.3) and (3.5) are strict, i.e.,

$$\mathcal{A}_\ell(\Omega^*, \nu) \subsetneq \mathcal{A}_k(\Omega^*, \nu) \quad \text{and} \quad \mathfrak{H}_{\bar{\ell}}^2(\Omega^*, \nu) \subsetneq \mathfrak{H}_{\bar{k}}^2(\Omega^*, \nu), \quad \text{when } \ell \le k, \ \ell \in \mathbb{N}_0,$$

and those in (3.4) and (3.6) are equalities, i.e.,

$$\psi^\ell \mathcal{A}_k(\Omega^*, \nu) = \mathcal{A}_{k-\ell}(\Omega^*, \nu) \quad \text{and} \quad \psi^\ell \mathfrak{H}_{\bar{k}}^2(\Omega^*, \nu) = \mathfrak{H}_{\overline{k-\ell}}^2(\Omega^*, \nu), \quad \text{when } \ell \le k, \ \ell \in \mathbb{N}_0.$$

Theorem 5.1 provides examples of (Ω, ν, V) that exhibit the dual phenomenon, i.e., the containments (3.5) stabilize to equalities, while the containments in (3.6) are strict.

In the classical construction, the Hardy space $\mathfrak{H}^2(\Omega, \nu)$ is a module over the algebra $\mathcal{A}(\Omega, \nu)$. This phenomenon cannot percolate to $\mathfrak{H}_{\bar{k}}^2(\Omega^*, \nu)$ as, in general, $\mathcal{A}_k(\Omega^*, \nu)$ is not even an algebra. However, when $V \cap T = \emptyset$, the union $\bigcup_{k=0}^{\infty} \mathcal{A}_k(\Omega^*, \nu)$ is a filtered algebra over $\mathbb{C}$ since

$$\mathcal{A}_k(\Omega^*, \nu) \cdot \mathcal{A}_j(\Omega^*, \nu) \subseteq \mathcal{A}_{k+j}(\Omega^*, \nu), \quad j, k \in \mathbb{N}_0.$$

The space $\bigcup_{k=0}^{\infty} \mathfrak{H}_{\bar{k}}^2(\Omega^*, \nu)$ is then a filtered module over this filtered algebra since

$$\mathcal{A}_k(\Omega^*, \nu) \cdot \mathfrak{H}_{\bar{j}}^2(\Omega^*, \nu) \subseteq \mathfrak{H}_{\overline{k+j}}^2(\Omega^*, \nu), \quad j, k \in \mathbb{N}_0.$$

We now consider the general case, i.e., $V = V_1 \cup \cdots \cup V_m$, where each V_j is an irreducible, minimally defined germ of an analytic hypersurface in $\overline{\Omega}$. Let $\psi_j \in \mathcal{O}(\overline{\Omega})$ be a minimal defining function of V_j, $j \in \{1, \ldots, m\}$. Then $\psi = \psi_1 \cdot \cdots \cdot \psi_m \in \mathcal{O}(\overline{\Omega})$ is a minimal defining function of V. One could proceed as in Definition 3.2 using ψ. However, this approach leads to an incomplete picture of the relevant spaces as each irreducible germ can independently yield a one-parameter family of spaces. For instance, consider the example Ω_P^* at the beginning of Sect. 4, and compare the spaces in (4.1) to the above definition where all the factors of ψ would appear with the same exponent.

To remedy this issue we proceed inductively. We write

$$\Omega_\ell^* = \Omega \setminus (V_1 \cup \cdots \cup V_\ell), \quad \ell \in \{1, \ldots, m\},$$

and define $\mathcal{A}_{\mathbf{k}}(\Omega_1^*, \nu)$ as in Definition 3.2 for $\mathbf{k} \in \mathbb{N}_0$. For $\ell \geq 2$, consider multi-indices $\mathbf{k} = \{k_1, \ldots, k_\ell\}$ and $\mathbf{k}' = \{k_1, \ldots, k_{\ell-1}\}$ with $k_j \in \mathbb{N}_0$, and define

$$\mathcal{A}_{\mathbf{k}}(\Omega_\ell^*, \nu) := \Big\{ F : \Omega^* \cup T^* \to \mathbb{C} : F = (\psi_\ell^{-k_\ell} G)|_{\Omega^* \cup T^*} \text{ for some } G \in \mathcal{A}_{\mathbf{k}'}(\Omega_{\ell-1}^*, \nu)$$

$$\text{and } F|_{T^*} \in L^2(\nu) \Big\}.$$

The inductive nature of this definition allows for the iterated application of Theorem 3.12 and Proposition 3.14. In particular, if (Ω, ν) is such that $\mathcal{A}(\Omega, \nu)$ is strongly admissible, then

$$\mathfrak{H}_{\mathbf{k}}^2(\Omega^*, \nu) := \overline{\mathcal{A}_{\mathbf{k}}(\Omega_m^*, \nu)|_T}^{L^2(\nu)} \tag{3.8}$$

is a reproducing kernel Hilbert space on Ω for any $\mathbf{k} = \mathbb{N}_0^m$, and we call it the $\mathbf{k}$-*th Hardy space* of (Ω, ν, V).

4 Planar domains

In this section, we apply the scheme described in Sect. 3.3 to hypersurface-deleted planar domains. Note that the case of the punctured disk is covered in Sect. 2.2.

Recall that any hypersurface-deleted planar domain may be written as

$$\Omega_P^* = \Omega \setminus P, \quad P = \{p_1, ..., p_m\} \subset \Omega,$$

see Definition 3.10 and the subsequent discussion. We henceforth refer to Ω_P^* as an *m-punctured domain*. Here, we consider $\Omega \Subset \mathbb{C}$ of class $\mathcal{C}^{1,\alpha}$ for $\alpha \in (0, 1)$, and the arc-length measure σ on $b\Omega$ so that $V \cap \text{supp}(\sigma) = \emptyset$. Under these assumptions, $\mathcal{A}(\Omega, \sigma)$ is strongly admissible. This is because $\mathcal{A}(\Omega, \sigma) \subset E^2(\Omega)$, the classical Smirnov–Hardy space of Ω, which is strongly admissible due to the existence of nontangential limits in $L^2(\sigma)$, see [12, Theorem 10.3 & Section 10.5]. In fact, $\mathcal{A}(\Omega, \sigma)|_{b\Omega}$ is dense in $E^2(\Omega)|_{b\Omega}$, see [12, Theorem 10.6 & Section 10.5]. Thus, the Hardy space $\mathfrak{H}^2(\Omega, \sigma)$ coincides with the classical Hardy space on Ω. Now we can either apply the inductive scheme of Sect. 3.3 or, equivalently, consider the closure in $L^2(\sigma)$ of the strongly admissible space of boundary values of

$$\mathcal{A}_{\mathbf{k}}(\Omega_P^*, \sigma) = \{F \in \mathcal{O}(\Omega_P^*) : (z - p_1)^{k_1} \cdot ... \cdot (z - p_m)^{k_m} F(z) \in \mathcal{A}(\Omega, \sigma)\} \tag{4.1}$$

for $\mathbf{k} = (k_1, ..., k_m) \in \mathbb{N}_0^m$. Either construction gives a family of Hardy spaces $\left\{\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)\right\}_{\mathbf{k} \in \mathbb{N}_0^m}$ such that

$$\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma) \subsetneqq \mathfrak{H}_{\mathbf{k}'}^2(\Omega_P^*, \sigma) \quad \text{whenever} \quad k_j \leq k_j', \ j \in \{1, ..., m\}.$$

Note that each $\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)$ is the space of L^2-boundary values of holomorphic functions on Ω that have poles of orders at most $k_1, ..., k_m$ at $p_1, ..., p_m$, respectively. Applying Proposition 3.14 and Remark 3.15 iteratively, we obtain the following result.

Proposition 4.1 *Let* $\boldsymbol{\varphi} = (\phi_1, ..., \phi_m) \in \mathcal{A}(\Omega, \sigma)^m$, $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{N}_0^m$ *and* $\boldsymbol{\varphi}^{\pm \mathbf{k}} = \phi_1^{\pm k_1} \cdot ... \cdot \phi_m^{\pm k_m}$. *Suppose each* ϕ_j *vanishes only at* p_j, $j = 1, ..., m$. *Suppose* $c(z, w)$ *is a kernel with the reproducing property for* $\mathfrak{H}^2(\Omega, \sigma)$. *Then*

$$\boldsymbol{\varphi}(z)^{-\mathbf{k}} c(z, w) \boldsymbol{\varphi}(w)^{\mathbf{k}}, \quad z \in \Omega_P^*, w \in b\Omega,$$

has the reproducing property for $\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)$. Further, if each ϕ_j has a simple zero at p_j and $|\phi_j|$ is constant on $b\Omega$, then

$$c_{\mathbf{k},\boldsymbol{\varphi}}(z, w) := \frac{\boldsymbol{\varphi}(w)^{\mathbf{k}}}{\boldsymbol{\varphi}(z)^{\mathbf{k}}} s(z, w), \quad z \in \Omega_P^*, w \in b\Omega,$$

is the Szegő kernel for $\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)$ for all $\mathbf{k} \in \mathbb{N}_0^m$.

In addition to the Szegő kernel, we discuss a generalization of the Cauchy kernel for $\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)$, $\mathbf{k} \in \mathbb{N}_0^m$. Recall that the classical Cauchy kernel

$$\mathcal{C}(z, w) = \frac{1}{2\pi i} \frac{1}{w - z}$$

is a holomorphic function on $\mathbb{C} \times \mathbb{C} \setminus \{z = w\}$ such that

$$\frac{j^*\big(\mathcal{C}(z, w)dw\big)}{d\sigma(w)}$$

has the reproducing property for $\mathfrak{H}^2(\Omega, \sigma)$, where $j : b\Omega \rightarrow \mathbb{C}$ is the inclusion map. Applying Proposition 4.1 to this kernel, we obtain the following analog of the Cauchy integral formula for m-times punctured domains

$$F(z) = \frac{1}{2\pi i} \int_{b\Omega} \underbrace{\frac{(w - p_1)^{k_1} \cdots (w - p_m)^{k_m}}{(z - p_1)^{k_1} \cdots (z - p_m)^{k_m} (w - z)}}_{=: 2\pi i\, \mathcal{C}_{\mathbf{k}}(z, w)} F(w)\, dw$$

for $F \in \mathcal{A}_{\mathbf{k}}(\Omega_P^*, \sigma)$ and $z \in \Omega_P^*$. We call $\mathcal{C}_{\mathbf{k}}(z, w)$ the *Cauchy $\mathbf{k}$-kernel for m punctures*. Note that it is a meromorphic function on $\mathbb{C} \times \mathbb{C} \setminus \{z = w\}$ whose poles depend solely on the location of the punctures. When written with respect to σ, the integral kernel in the above formula is, in fact,

$$\mathcal{C}_{\mathbf{k}}^{\Omega_P^*}(z, w) := \mathcal{C}_{\mathbf{k}}(z, w)\dot{\gamma}(w),$$

where $w = \gamma(t)$ is the arc-length parametrization of $b\Omega$. It follows that $\mathcal{C}_{\mathbf{k}}^{\Omega_P^*}(z, w)$ has the reproducing property for $\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)$. In contrast to $\mathcal{C}_{\mathbf{k}}^{\Omega_P^*}$, the Szegő kernel, $s_{\mathbf{k}}$, of $\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)$ is, in general, not known explicitly. However, for simply connected Ω, Theorem 4.2 below gives a formula for $s_{\mathbf{k}}$ in terms of the Szegő kernel for $\mathfrak{H}^2(\Omega, \sigma)$. It also shows that the two kernels, $s_{\mathbf{k}}$ and $\mathcal{C}_{\mathbf{k}}^{\Omega_P^*}$, coincide if and only if Ω_P^* is a disk punctured at its center. This rigidity result extends the Kerzman–Stein Lemma ([18, Lemma 7.1]) to the case of m-punctured domains.

Theorem 4.2 *Let $\Omega \Subset \mathbb{C}$ be a $\mathcal{C}^{1,\alpha}$-smooth simply connected domain, and $P = \{p_1, ..., p_m\} \subset \Omega$. Let $\mu : \Omega \rightarrow \mathbb{D}$ be a biholomorphism with $q_j = \mu(p_j)$, $j = 1, ..., m$.*

(1) The Szegő kernel for $\mathfrak{H}_{\mathbf{k}}^2(\Omega_P^*, \sigma)$ is given by

$$s_{\mathbf{k}}(z, w) = \boldsymbol{\varphi}_0^{-\mathbf{k}}(z)\, s(z, w)\, \overline{\boldsymbol{\varphi}_0^{-\mathbf{k}}(w)}, \quad z \in \Omega_P^*, w \in b\Omega,$$

where, $\boldsymbol{\varphi}_0 = \big(M_{q_1} \circ \mu, ..., M_{q_m} \circ \mu\big)$ for $M_q(\zeta) = \dfrac{\zeta - q}{1 - \overline{q}\zeta}$, $(q, \zeta) \in \mathbb{D} \times \overline{\mathbb{D}}$.

(2) $\mathcal{C}_{\mathbf{k}}^{\Omega_P^*}(z, w) = s_{\mathbf{k}}(z, w)$ for some $\mathbf{k} \in \mathbb{N}_0^m$ if and only if Ω_P^* is a disk punctured at its center.

To prove Theorem 4.2, we use the fact that the Szegő kernel s of $\mathfrak{H}^2(\Omega, \sigma)$ is $S|_{\Omega \times b\Omega}$, where S is the continuous extension of the Szegő kernel for $E^2(\Omega)$ to $\overline{\Omega} \times \overline{\Omega} \setminus \{(z, z) : z \in b\Omega\}$. Note that $S(z, w) = \overline{S(w, z)}$ for $z, w \in \overline{\Omega} \times \overline{\Omega} \setminus \{(z, z) : z \in b\Omega\}$. The continuous extension of the Szegő kernel for $E^2(\Omega)$ follows from three facts. Firstly, this is true for the classical Szegő kernel $S^{\mathbb{D}}$ of the disk. Secondly, the derivative of any biholomorphism β from Ω onto $\mathbb{D}$ admits a continuous nonvanishing square root on $\overline{\Omega}$, see [26, Theorem 3.5]. And lastly, the Szegő kernel for $E^2(\Omega)$ can be expressed in terms of $S^{\mathbb{D}}$ and $\sqrt{\beta'}$, see the transformation law in [19, Lemma 5.3].

Proof of Theorem 4.2 Since μ extends continuously to $\overline{\Omega}$, we have that $\boldsymbol{\varphi}_0 \in \mathcal{A}(\Omega, \sigma)^m$. Moreover, since $|M_q| \equiv 1$ on $b\mathbb{D}$, and M_q only has a simple zero at q, the same is true of each ϕ_j on $b\Omega$ and at p, respectively. Thus, by Proposition 4.1, $c_{\mathbf{k}, \boldsymbol{\varphi}_0}$ is the Szegő kernel for $\mathfrak{H}^2_{\mathbf{k}}(\Omega_P^*, \sigma)$. Since $M_q \circ \mu = \overline{(M_q \circ \mu)^{-1}}$ on $b\Omega$, the first claim follows.

Next, observe that $S_{\mathbf{k}}(z, w) = \boldsymbol{\varphi}_0(z)^{-\mathbf{k}} S(z, w) \overline{\boldsymbol{\varphi}_0(w)^{-\mathbf{k}}}$ extends $s_{\mathbf{k}}$ continuously to $(\overline{\Omega} \setminus P)^2 \setminus \{(z, z) : z \in b\Omega\}$, and $S_{\mathbf{k}}(z, w) = \overline{S_{\mathbf{k}}(w, z)}$. Thus, if $\mathcal{C}_{\mathbf{k}}^{\Omega_P^*} = s_{\mathbf{k}}$, it must be that for $z, w \in b\Omega, z \neq w$,

$$\mathcal{C}_{\mathbf{k}}^{\Omega_P^*}(S(z, w)) - \overline{\mathcal{C}_{\mathbf{k}}^{\Omega_P^*}(S(z, w))} = \frac{1}{2\pi i} \frac{e(S(z, w))}{w - z} \left(\dot{\gamma}(w) - \frac{1}{|e(S(z, w))|^2} \widetilde{\dot{\gamma}(z)} \right) = 0, \tag{4.2}$$

where $e(S(z, w)) = \dfrac{(w - p_1)^{k_1} \cdots (w - p_m)^{k_m}}{(z - p_1)^{k_1} \cdots (z - p_m)^{k_m}}$, and $\widetilde{\dot{\gamma}(z)} = \overline{\dot{\gamma}(z)} \dfrac{w - z}{\overline{w} - \overline{z}}$ is the vector obtained from reflecting $\dot{\gamma}(z)$ in the chord determined by w and z. Thus, as in the proof of the classical Kerzman–Stein Lemma, (4.2) implies that for any two distinct points $z, w \in b\Omega$, the chord connecting w and z meets the boundary curve with the same angle at both points. But this can only happen if $b\Omega$ is a circle [27], i.e., $\Omega = \mathbb{D}_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$ for some $a \in \mathbb{C}$ and $r > 0$. In this case $|\dot{\gamma}(w)| = |\widetilde{\dot{\gamma}(z)}|$ for all $z, w \in b\mathbb{D}_r(a)$, and so $|e(S(z, w))| \equiv 1$ for $z, w \in b\mathbb{D}_r(a)$. If $\mathbf{k} \in \mathbb{N}_0^m$, this yields that $|(w - p_1) \cdots (w - p_m)|$ is constant on $b\mathbb{D}_r(a)$, which is only possible if $P = \{a\}$. $\square$

Theorem 4.2 is stated only for simply connected domains because of the limited applicability of Proposition 4.1. In particular, if Ω is multiply connected, then the conditions on ϕ_j, assumed in Proposition 4.1, may not be attainable. For example, if $\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}$ and $V = \{a\}$ for some $a \in \Omega$, then there is no $\phi \in \mathcal{A}(\Omega, \sigma)$ that has a simple zero at a and is such that $|\phi| \equiv C$ on $b\Omega$. This is because, owing to the argument and maximum principles, $N(\xi) := \frac{1}{2\pi i} \int_{b\Omega} \frac{\phi'(w)}{\phi(w) - \xi} dw$ is a continuous, integer-valued function on $\mathbb{D}_C(0)$ and hence a constant. If ϕ had a simple zero, then $N \equiv 1$ on $\mathbb{D}_C(0)$, forcing ϕ to be a homeomorphism between Ω and $\mathbb{D}_C(0)$, which is impossible.

However, in the case when Ω is finitely connected, the Szegő kernel for $\mathfrak{H}^2_{\mathbf{k}}(\Omega_P^*, \sigma)$ enjoys a transformation law under biholomorphisms. The proof goes along classical arguments in [4, Ch. 12] and [19, Lemma 5.3], after taking into account the boundary regularity of conformal maps between $\mathcal{C}^{1,\alpha}$-smooth domains, see [2, App. A].

Theorem 4.3 *Suppose $\Omega, D \Subset \mathbb{C}$ are $\mathcal{C}^{1,\alpha}$-smooth domains, and $\mu : \Omega \to D$ is a biholomorphism. Then, for $\mathbf{k} \in \mathbb{N}_0^m$,*

$$s_{\mathbf{k}}^{\Omega_P^*}(z, w) = \sqrt{\mu'(z)} \left(s_{\mathbf{k}}^{D_{\mu(P)}^*}(\mu(z), \mu(w)) \right) \overline{\sqrt{\mu'(w)}}, \quad z \in \Omega_P^*, w \in b\Omega,$$

where $s_{\mathbf{k}}^{\Omega_P^*}$ and $s_{\mathbf{k}}^{D_{\mu(P)}^*}$ denote the Szegő kernels for $\mathfrak{H}^2_{\mathbf{k}}(\Omega_P^*, \sigma)$ and $\mathfrak{H}^2_{\mathbf{k}}(D_{\mu(P)}^*, \sigma)$, respectively.

5 Hypersurface-deleted egg domains as examples of finite stabilization

In this section, we consider triples of the form $\left(\mathcal{E}_p, \nu, \{z_2 = 0\}\right)$, $p \in \mathbb{N}$, where

$$\mathcal{E}_p = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2p} + |z_2|^{2p} < 1\right\}, \tag{5.1}$$

and the measure ν on $b\mathcal{E}_p$ is either

(a) σ, the Euclidean surface area measure, or
(b) ω_p, the Monge–Ampère boundary measure associated to the exhaustion function

$$\varphi_p(z_1, z_2) = \frac{1}{2p} \log\left(|z_1|^{2p} + |z_2|^{2p}\right).$$

Note that ω_p is also the Leray–Levi measure associated to the defining function

$$\rho_p(z_1, z_2) = \frac{2\pi}{p} \left(|z_1|^{2p} + |z_2|^{2p} - 1\right).$$

In the case of the ball, or $p = 1$, the two measures coincide and $\mathfrak{H}^2(\mathcal{E}_1, \sigma) = \mathfrak{H}^2(\mathcal{E}_1, \omega_1)$. In all other cases, $\mathfrak{H}^2(\mathcal{E}_p, \sigma) \subsetneq \mathfrak{H}^2(\mathcal{E}_p, \omega_p)$. We show that this discrepancy, owing to different choices of measure, is amplified in the case of $\mathcal{E}_p^* = \mathcal{E}_p \setminus \{z_2 = 0\}$. Moreover, this setting yields examples of nontrivially stabilizing filtrations of Hardy spaces.

For some context, note that $\mathfrak{H}^2(\mathcal{E}_p, \sigma)$ is the space of boundary values of the classical Hardy space on $\mathcal{E}_p$ as defined by Stein in [30], while $\mathfrak{H}^2(\mathcal{E}_p, \omega_p)$ is the space of boundary values of the Poletsky–Stessin Hardy space associated to φ_p on $\mathcal{E}_p$, see [25]. The latter spaces have been studied by Hansson in [16], Şahin in [28], and Barrett–Lanzani in [3]. Later, we encounter the limiting case of $\left(\mathcal{E}_p, \omega_p, \{z_2 = 0\}\right)$ as $p \to \infty$. To wit, if $\mathcal{E}_\infty = \lim\limits_{p \to \infty} \mathcal{E}_p$ in the Hausdorff metric, and ω_∞ is the Monge–Ampère measure corresponding to the function

$$\varphi_\infty(z_1, z_2) = \lim_{p \to \infty} \varphi_p(z_1, z_2) = \log \max\{|z_1|, |z_2|\},$$

then $\mathcal{E}_\infty^* = \mathcal{E}_\infty \setminus \{z_2 = 0\}$ is $\mathbb{D} \times \mathbb{D}^*$ and $\omega_\infty = \sigma_{S^1} \times \sigma_{S^1}$, see [25, § 4]. Since $\{z_2 = 0\}$ does not intersect $\text{supp}(\omega_\infty) = b\mathbb{D} \times b\mathbb{D}$, $\{\mathfrak{H}_k^2(\mathcal{E}_\infty^*, \omega_\infty)\}_{k \in \mathbb{N}_0}$ does not stabilize, and $z_2^\ell \mathfrak{H}_k^2(\mathcal{E}_\infty^*, \omega_\infty) = \mathfrak{H}_{k-\ell}^2(\mathcal{E}_\infty^*, \omega_\infty)$ for $\ell \leq k$. The behavior of this filtration is quite different when $p < \infty$.

Theorem 5.1 *Let $p \in \mathbb{N}$. Then $\{\mathfrak{H}_k^2(\mathcal{E}_P^*, \sigma)\}_{k \in \mathbb{N}_0}$ stabilizes at $k = 0$, i.e.,*

$$\mathfrak{H}_k^2(\mathcal{E}_p^*, \sigma) = \mathfrak{H}_0^2(\mathcal{E}_p^*, \sigma), \quad \forall k \in \mathbb{N}_0.$$

On the other hand, $\{\mathfrak{H}_k^2(\mathcal{E}_P^*, \omega_p)\}_{k \in \mathbb{N}_0}$ stabilizes at $k = p - 1$, i.e.,

$$\mathfrak{H}_0^2(\mathcal{E}_p^*, \omega_p) \subsetneq \mathfrak{H}_1^2(\mathcal{E}_p^*, \omega_p) \subsetneq \cdots \subsetneq \mathfrak{H}_{p-1}^2(\mathcal{E}_p^*, \omega_p) = \mathfrak{H}_k^2(\mathcal{E}_p^*, \omega_p), \quad \forall k \geq p.$$

Moreover, $\mathfrak{H}_0^2(\mathcal{E}_p^*, \omega_p) \supsetneq z_2 \mathfrak{H}_1^2(\mathcal{E}_p^*, \omega_p) \supsetneq \cdots \supsetneq z_2^k \mathfrak{H}_k^2(\mathcal{E}_p^*, \omega_p) \supsetneq \cdots$.

In order to prove Theorem 5.1, we describe the relevant Hardy spaces. Here, $j : b\mathcal{E}_p \to \mathbb{C}^2$ denotes the inclusion map, and d^c is the real operator $i(\bar{\partial} - \partial)$. Dropping the subscripts of φ_p and ρ_p, we have that

$$\omega_p = j^*(d^c\varphi \wedge dd^c\varphi) = \frac{j^*(\partial\rho \wedge \bar{\partial}\partial\rho)}{(2\pi i)^2} = \frac{-\det\begin{pmatrix} 0 & \rho_{\bar{z}_1} & \rho_{\bar{z}_2} \\ \rho_{z_1} & \rho_{z_1\bar{z}_1} & \rho_{z_1\bar{z}_2} \\ \rho_{z_2} & \rho_{z_2\bar{z}_1} & \rho_{z_2\bar{z}_2} \end{pmatrix}}{\pi^2 |\nabla\rho|} \sigma, \quad \text{on } b\mathcal{E}_p,$$

where ρ_{z_j} is the first order partial derivative of ρ with respect to z_j, $j \in \{1, 2\}$, and $\rho_{z_j \bar{z}_k}$ is the second order partial derivative of ρ with respect to z_j and $\bar{z}_k$, $j, k \in \{1, 2\}$. For ease of computation, we parametrize $(b\mathcal{E}_p)_* := b\mathcal{E}_p \setminus \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}$ as

$$\vartheta : (s, \theta_1, \theta_2) \mapsto \left(s^{\frac{1}{2p}} e^{i\theta_1}, (1-s)^{\frac{1}{2p}} e^{i\theta_2} \right), \quad (s, \theta_1, \theta_2) \in (0, 1) \times [0, 2\pi)^2. \quad (5.2)$$

Since $b\mathcal{E}_p \cap \{z_1 z_2 = 0\}$ is a set of measure zero for both σ and ω_p, we have that

$$L^2(b\mathcal{E}_p, \nu) \cong L^2\left((0, 1) \times [0, 2\pi)^2; \vartheta^* \nu\right), \quad \nu = \sigma, \omega_p, \quad (5.3)$$

via the map $f \mapsto f|_{(b\mathcal{E}_p)_*} \circ \vartheta$. It is easy to check that

$$\vartheta^* d\sigma = \frac{1}{2p} \frac{\sqrt{(s)^{2-\frac{1}{p}} + (1-s)^{2-\frac{1}{p}}}}{s^{1-\frac{1}{p}}(1-s)^{1-\frac{1}{p}}} \, ds \, d\theta_1 \, d\theta_2 \approx \frac{ds \, d\theta_1 \, d\theta_2}{s^{1-\frac{1}{p}}(1-s)^{1-\frac{1}{p}}}, \quad \text{and}$$

$$\vartheta^* \omega_p = ds \, d\theta_1 \, d\theta_2.$$

Here, $a(r) \approx b(r)$ means that there are constants $c, C > 0$ such that $cb(r) \leq a(r) \leq Cb(r)$ for all r. For the sake of brevity, we drop all references to ϑ, use (s, θ_1, θ_2) as coordinates on $b\mathcal{E}_p$, and abbreviate $||f||_{L^2(b\mathcal{E}_p, \nu)}$ to $||f||_\nu$. We now provide descriptions of the spaces $\mathfrak{H}^2(\mathcal{E}_p, \sigma)$ and $\mathfrak{H}^2(\mathcal{E}_p, \omega_p)$ in terms of L^2-convergent series expansions.

Proposition 5.2 *Let $p \in \mathbb{N}$. Then*

$$\mathfrak{H}^2(\mathcal{E}_p, \sigma) = \left\{ \sum_{j,\ell \geq 0} a_{j,\ell} s^{\frac{j}{2p}} (1-s)^{\frac{\ell}{2p}} e^{i(j\theta_1 + \ell \theta_2)} : \sum_{j,\ell \geq 0} |a_{j,\ell}|^2 \beta\left(\frac{j+1}{p}, \frac{\ell+1}{p}\right) < \infty \right\}, \quad (5.4)$$

$$\mathfrak{H}^2(\mathcal{E}_p, \omega_p) = \left\{ \sum_{j,\ell \geq 0} a_{j,\ell} s^{\frac{j}{2p}} (1-s)^{\frac{\ell}{2p}} e^{i(j\theta_1 + \ell \theta_2)} : \sum_{j,\ell \geq 0} |a_{j,\ell}|^2 \beta\left(\frac{j}{p}+1, \frac{\ell}{p}+1\right) < \infty \right\}, \quad (5.5)$$

where $\beta(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds$ is the Euler beta function. In particular, $\mathfrak{H}^2(\mathcal{E}_1, \sigma) = \mathfrak{H}^2(\mathcal{E}_1, \omega_1)$, and $\mathfrak{H}^2(\mathcal{E}_p, \sigma) \subsetneq \mathfrak{H}^2(\mathcal{E}_p, \omega_p)$ when $p > 1$.

Proof We first prove (5.5). In view of (5.3), any $f \in L^2(b\mathcal{E}_p, \omega_p)$ may be written as

$$f(s, \theta_1, \theta_2) = \sum_{(j,\ell) \in \mathbb{Z}^2} \hat{f}_{j,\ell}(s) e^{i(j\theta_1 + \ell \theta_2)}, \quad (5.6)$$

where $\{\hat{f}_{j,\ell}(s)\}_{(j,\ell) \in \mathbb{Z}^2}$ are the Fourier coefficients of $f(s, ., .)$, and $\sum_{(j,\ell) \in \mathbb{Z}^2} ||\hat{f}_{j,\ell}||^2_{L^2(0,1)} < \infty$. Now, for $F \in \mathcal{A}(\mathcal{E}_p, \omega_p)$, we may write

$$F(z_1, z_2) = \begin{cases} \sum_{j,\ell \geq 0} a_{j,\ell} z_1^j z_2^\ell, & \text{if } (z_1, z_2) \in \mathcal{E}_p, \\ \sum_{j,\ell \in \mathbb{Z}} \hat{F}_{j,\ell}(s) e^{i(j\theta_1 + \ell \theta_2)}, & \text{if } (z_1, z_2) = \left(s^{\frac{1}{2p}} e^{i\theta_1}, (1-s)^{\frac{1}{2p}} e^{i\theta_2} \right) \in b\mathcal{E}_p, \end{cases}$$

where in $\mathcal{E}_p$, the power series converges uniformly on compact subsets, and on $b\mathcal{E}_p$, the series converges in $L^2(\omega_p)$. Next, for $s \in (0, 1)$, F is continuous on the closed polydisk $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq s^{1/2p}, |z_2| \leq (1-s)^{1/2p}\}$. Hence,

$$\hat{F}_{j,\ell}(s) = \frac{s^{\frac{j}{2p}}(1-s)^{\frac{\ell}{2p}}}{(2\pi i)^2} \lim_{r \to 1^-} \iint_{\substack{|w_1|^{2p}=r(1-s) \\ |w_2|^{2p}=rs}} \frac{F(w_1, w_2)}{w_1^{j+1} w_2^{\ell+1}} dw_1 dw_2 = \begin{cases} a_{j,\ell} s^{\frac{j}{2p}}(1-s)^{\frac{\ell}{2p}}, & j, \ell \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$\sum_{j,\ell\in\mathbb{Z}}||\widehat{F}_{j,\ell}||^2_{L^2(0,1)} = \sum_{j,\ell\geq 0}\int_0^1 |a_{j,\ell}|^2 s^{\frac{j}{p}}(1-s)^{\frac{\ell}{p}}\,ds = \sum_{j,\ell\geq 0}|a_{j,\ell}|^2\beta\left(\frac{j}{p}+1,\frac{\ell}{p}+1\right) < \infty.$$

Thus, we obtain the characterization in (5.5) for a dense subspace. By taking $L^2(\omega_p)$-limits of sequences in $\mathcal{A}(\mathcal{E}_p,\omega_p)$, the expansion for any $f\in\mathfrak{H}^2(\mathcal{E}_p,\omega_p)$ can be established. The argument for (5.4) runs along similar lines.

Now, since $\beta\left(\frac{j}{p}+1,\frac{\ell}{p}+1\right)\leq\beta\left(\frac{j+1}{p},\frac{\ell+1}{p}\right)$ for all $j,\ell\geq 0$, we have that $\mathfrak{H}^2(\mathcal{E}_p,\sigma)\subseteq\mathfrak{H}^2(\mathcal{E}_p,\omega_p)$, with equality when $p=1$. To show strict containment for any $p>1$, we consider the series $f(s,\theta_1,\theta_2)=\sum_{j,\ell\geq 0}a_{j,\ell}s^{\frac{j}{2p}}(1-s)^{\frac{\ell}{2p}}e^{i(j\theta_1+\ell\theta_2)}$, with

$$a_{j,\ell} = \begin{cases} \frac{\beta(m+1,n+1)^{-1/2}}{mn}, & \text{when } \frac{j}{p}=m\in\mathbb{N},\ \frac{\ell}{p}=n\in\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $||f||^2_{\omega_p} = 4\pi^2\sum_{j,\ell\geq 0}|a_{j,\ell}|^2\beta\left(\frac{j}{p}+1,\frac{\ell}{p}+1\right) = 4\pi^2\sum_{m,n\geq 0}(mn)^{-2} < \infty$, but since

$$||f||^2_\sigma \approx \sum_{j,\ell\geq 0}|a_{j,\ell}|^2\beta\left(\frac{j+1}{p},\frac{\ell+1}{p}\right)\geq c\sum_{m,n\geq 1}\frac{1}{m^2n^2}\frac{m^{1-\frac{1}{p}}n^{1-\frac{1}{p}}}{(m+n)^{\frac{2}{p}-2}},$$

f does not converge in $L^2(\sigma)$.

Proof of Theorem 5.1 Fix a $p\in\mathbb{N}$. First, we consider $\sigma\approx s^{\frac{1}{p}-1}(1-s)^{\frac{1}{p}-1}\,ds\,d\theta_1\,d\theta_2$. It is clear that

$$z_2^{-k}\big|_{b\mathcal{E}_p} = (1-s)^{-k/2p}e^{-ik\theta_2}\in L^2(b\mathcal{E}_p,\sigma) \iff k<1.$$

Now suppose there is a $g\in\mathfrak{H}^2_1(\mathcal{E}_p^*,\sigma)\setminus\mathfrak{H}^2_0(\mathcal{E}_p^*,\sigma)$. Then

(i) $g\in L^2(\mathcal{E}_p,\sigma)\setminus\mathfrak{H}^2_0(\mathcal{E}_p^*,\sigma)$;

(ii) $(z_2 g)|_{b\mathcal{E}_p} = \sum_{j,\ell\geq 0}a_{j,\ell}s^{\frac{j}{2p}}(1-s)^{\frac{\ell}{2p}}e^{i(j\theta+\ell\phi)}$ with $\sum_{j,\ell\geq 0}|a_{j,\ell}|^2\beta\left(\frac{j+1}{p},\frac{\ell+1}{p}\right) < \infty.$

Writing $g=\sum_{j,\ell\in\mathbb{Z}}\hat{g}_{j,\ell}(s)\,e^{i(j\theta_1+\ell\theta_2)}$, we obtain from (ii) that

$$\hat{g}_{j,\ell} = \begin{cases} a_{j,\ell+1}, & \text{if } j\geq 0,\ \ell\geq -1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$||g||^2_\sigma \approx \sum_{j\geq 0}|a_{j,0}|^2\int_0^1\frac{s^{\frac{j+1}{p}-1}\,ds}{1-s} + \sum_{j,\ell\geq 0}|a_{j,\ell+1}|^2\beta\left(\frac{j+1}{p},\frac{\ell+1}{p}\right),$$

which is finite only if $a_{j,0}=0$ for all $j\geq 0$, and $\sum_{j,\ell\geq 0}|a_{j,\ell+1}|^2\beta\left(\frac{j+1}{p},\frac{\ell+1}{p}\right) < \infty$. In that case, $g\in\mathfrak{H}^2_0(\mathcal{E}_p^*,\sigma)$, which contradicts (i). Thus, $\mathfrak{H}^2_1(\mathcal{E}_p^*,\sigma) = \mathfrak{H}^2_0(\mathcal{E}_p^*,\sigma)$. A similar argument shows that $\mathfrak{H}^2_k(\mathcal{E}_p^*,\sigma) = \mathfrak{H}^2_0(\mathcal{E}_p^*,\sigma)$ for all $k\in\mathbb{N}_0$.

In the case of $\omega_p = ds\,d\theta_1\,d\theta_2$, we have that

$$z_2^{-k}|_{b\mathcal{E}_p} = (1-s)^{-k/2p}e^{-ik\theta_2} \in L^2(b\mathcal{E}_p,\omega_p) \iff k < p.$$

Thus, $z_2^{-k} \in \mathfrak{H}_k^2(\mathcal{E}_p^*,\omega_p) \setminus \mathfrak{H}_{k-1}^2(\mathcal{E}_p^*,\omega_p)$ as long as $k \le p-1$. For $k \ge p$, we may argue, as in the case of σ above, that $\mathfrak{H}_k^2(\mathcal{E}_p^*,\omega_p) = \mathfrak{H}_0^2(\mathcal{E}_p^*,\omega_p)$.

Finally, we show that $\mathfrak{H}_{k-1}^2(\mathcal{E}_p^*,\omega_p) \supsetneq z_2\mathfrak{H}_k^2(\mathcal{E}_p^*,\omega_p)$ for any $k \in \mathbb{N}_0$. In view of the stabilization, when $k \ge p$, it suffices to show that $\mathfrak{H}_{p-1}^2(\mathcal{E}_p^*,\omega_p) \supsetneq z_2\mathfrak{H}_{p-1}^2(\mathcal{E}_p^*,\omega_p)$. This is clear since $z_2^{-(p-1)} \in \mathfrak{H}_{p-1}^2(\mathcal{E}_p^*,\omega_p)$, but $z_2^{-p} \notin L^2(b\mathcal{E}_p,\omega_p)$. For $k < p$, let

$$f(s,\theta_1,\theta_2) = \sum_{m\ge 0}(m+1)^{-\frac{k}{2p}}\left(s^{\frac{1}{2p}}e^{i\theta_1}\right)^{mp}, \text{ and}$$

$$h(s,\theta_1,\theta_2) = \left((1-s)^{\frac{1}{2p}}e^{i\theta_2}\right)^{-(k-1)}f(s,\theta_1,\theta_2).$$

Since, for any fixed $r > 0$, $\beta(m+1,r) \sim (m+1)^{-r}$ as $m \to \infty$, we have that

$$\|f\|_{\omega_p}^2 = \sum_{m\ge 0}(m+1)^{-\frac{k}{p}}\beta(m+1,1) \lesssim \sum_{m\ge 0}m^{-1-\frac{k}{p}} < \infty.$$

Thus, $z_2^{k-1}h = f \in \mathfrak{H}_0^2(\mathcal{E}_p^*,\omega_p)$. Moreover,

$$\|h\|_{\omega_p}^2 = \sum_{m\ge 0}(m+1)^{-\frac{k}{p}}\beta\left(m+1,1+\frac{1}{p}-\frac{k}{p}\right) \lesssim \sum_{m\ge 0}m^{-1-\frac{1}{p}} < \infty.$$

Thus, $h \in \mathfrak{H}_{k-1}^2(\mathcal{E}_p^*,\omega_p)$. But $\|z_2^{-1}h\|_{\omega_p}^2 \gtrsim \sum_{m\ge 0}m^{-1}$ is not finite. Thus, there is no $g \in \mathfrak{H}_k^2(\mathcal{E}_p^*,\omega_p)$ such that $z_2 g = h$. That is, $h \in \mathfrak{H}_{k-1}^2(\mathcal{E}_p^*,\omega_p) \setminus z_2\mathfrak{H}_k^2(\mathcal{E}_p^*,\omega_p)$. $\square$

Remark 5.3 The egg domains $\mathcal{E}_p$ may be endowed with other natural boundary measures. For example, in [3, Def. 43], the authors consider the family of measures $\{\nu_\tau = f|\mathscr{L}|^{1-\tau}\sigma\}_{\tau\in[0,1]}$ on $b\mathcal{E}_p$, where f is any positive continuous function on $b\mathcal{E}_p$, and

$$|\mathscr{L}| = -4|\nabla\rho|^{-3}\det\begin{pmatrix} 0 & \rho_{\bar{z}_k} \\ \rho_{z_j} & \rho_{z_j\bar{z}_k}\end{pmatrix}_{1\le j,k\le 2}$$

for any defining function ρ of $\mathcal{E}_p$. The measures σ and ω_p correspond to ν_1 ($f \equiv 1$) and ν_0 ($f = |\nabla\rho_p|^2/4\pi^2$), respectively. It is also worth noting that the Fefferman hypersurface measure on $b\mathcal{E}_p$ is precisely $\nu_{2/3}$ ($f \equiv 1$). Analogous computations can be carried out to obtain explicit descriptions of the spaces $\mathfrak{H}_k^2(\mathcal{E}_p^*,\nu_\tau)$. Note, in particular, that the filtration corresponding to the measure ν_τ stabilizes at $k = \lceil p(1-\tau)+\tau\rceil - 1$, where $\lceil\cdot\rceil$ is the ceiling function.

6 Hartogs triangles: an application

We construct filtered modules of Hardy spaces for certain power-generalized Hartogs triangles. This family of domains was first introduced in [13,14]. Specifically, we consider domains of the form

$$\mathbb{H}_{m/n} := \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^m < |z_2|^n < 1\}, \quad m,n \in \mathbb{N}, \gcd(m,n) = 1,$$

where $\mathbb{T} = b\mathbb{D} \times b\mathbb{D}$ is endowed with the product measure $\sigma_{\mathbb{T}} := \sigma_{S^1} \times \sigma_{S^1}$. Although $\mathbb{H}_{m/n}$ is not a hypersurface-deleted domain, it is a proper holomorphic image of the hypersurface-deleted domain $\mathbb{D} \times \mathbb{D}^*$ via

$$\Theta_{m/n} : (z_1, z_2) \mapsto (z_1^n z_2^n, z_2^m).$$

Note that $\Theta_{m/n}$ maps $\mathbb{T}$ to $\mathbb{T}$, and $\Theta_{m/n}^* : f \mapsto f \circ \Theta_{m/n}$ induces an isometric isomorphism from $L^2(\mathbb{T}, \sigma_{\mathbb{T}})$ onto a closed subspace of $L^2(\mathbb{T}, \sigma_{\mathbb{T}})$. Thus, we can deduce the Szegő kernels for $\mathbb{H}_{m/n}$ from those for $\mathbb{D} \times \mathbb{D}^*$. To do this, we first treat the case of $\mathbb{D} \times \mathbb{D}^*$ in Sect. 6.1. In Sects. 6.2 and 6.3, we treat the case of the standard and the power-generalized Hartogs triangles, respectively. As done in Sect. 2, we omit the measure $\sigma_{\mathbb{T}}$ from the notation for the relevant functions spaces. Moreover, we use polar coordinates (θ_1, θ_2) on $\mathbb{T}$.

6.1 Hardy spaces on $\mathbb{D} \times \mathbb{D}^*$

We construct the Hardy spaces for $\mathbb{D} \times \mathbb{D}^*$ by executing the inheritance scheme in Sect. 3.3 for the triple $(\mathbb{D}^2, \sigma_{\mathbb{T}}, \{z_2 = 0\})$. To implement the scheme, consider, for $k \in \mathbb{N}_0$, the following subset of $\mathcal{O}(\mathbb{D} \times \mathbb{D}^*) \cap \mathcal{C}\big((\mathbb{D} \times \mathbb{D}^*) \cup \mathbb{T}\big)$

$$\mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*) = \left\{ F : (\mathbb{D} \times \mathbb{D}^*) \cup \mathbb{T} \to \mathbb{C} : F(z_1, z_2) = \left(z_2^{-k} G(z_1, z_2) \right) |_{(\mathbb{D} \times \mathbb{D}^*) \cup \mathbb{T}} \right.$$
$$\left. \text{for some } G \in \mathcal{A}(\mathbb{D}^2) = \mathcal{O}(\mathbb{D}^2) \cap \mathcal{C}(\mathbb{D}_{\mathbb{T}}^2) \right\}.$$

For each $k \in \mathbb{N}_0$, set $\mathfrak{H}_k^2(\mathbb{D} \times \mathbb{D}^*)$ to be the closure of $\mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*)|_{\mathbb{T}}$ in $L^2(\mathbb{T})$.

As in the case of $\mathbb{D}$ and $\mathbb{D}^*$, a precise description of these spaces in terms of Fourier series expansions can be given as follows. For $k \in \mathbb{N}_0$,

$$\mathfrak{H}_k^2(\mathbb{D} \times \mathbb{D}^*) = \left\{ \sum_{(j,\ell) \in \mathbb{Z}^2} \hat{f}_{j,\ell} \, e^{i(j\theta_1 + \ell\theta_2)} \in L^2(\mathbb{T}) : \hat{f}_{j,\ell} = 0, \text{ if } \max\{j, \ell + k\} < 0 \right\}.$$
$$(6.1)$$

Moreover, the Szegő kernel s_k for $\mathfrak{H}_k^2(\mathbb{D} \times \mathbb{D}^*)$ can be obtained by applying the Cauchy integral formula for $\mathbb{D}^2$ to $z_2^k F(z_1, z_2)$ for $F \in \mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*)$. This yields

$$s_k(z, w) = \frac{1}{(2\pi)^2} \frac{1}{(z_2 \overline{w}_2)^k (1 - z_2 \overline{w}_2)(1 - z_1 \overline{w}_1)}, \quad z \in \mathbb{D} \times \mathbb{D}^*, w \in \mathbb{T}. \quad (6.2)$$

We briefly note that in order to verify that $\mathfrak{H}_k^2(\mathbb{D} \times \mathbb{D}^*)$ indeed satisfies the minimum criterion for being a Hardy space, we may take $\mathfrak{X}$ to be

$$\mathcal{H}_k^2(\mathbb{D} \times \mathbb{D}^*) := \left\{ F \in \mathcal{O}(\mathbb{D} \times \mathbb{D}^*) : ||F||_{\mathcal{H}_k^2(\mathbb{D} \times \mathbb{D}^*)} < \infty \right\},$$

where

$$||F||_{\mathcal{H}_k^2(\mathbb{D} \times \mathbb{D}^*)} := \sup_{0 < s, r < 1} \left(\frac{r^{2k}}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |F(se^{i\theta_1}, re^{i\theta_2})|^2 \, d\theta_1 d\theta_2 \right)^{\frac{1}{2}},$$

with norm, a constant multiple of, $||.||_{\mathcal{H}_k^2(\mathbb{D} \times \mathbb{D}^*)}$.

6.2 The standard Hartogs triangle

For the sake of exposition, we first consider,

$$\mathbb{H} = \mathbb{H}_{1/1} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\},$$

for which $\Theta = \Theta_{1/1}$ is, in fact, a biholomorphism. This map allows us to describe both a boundary-based construction and an exhaustion-based construction of Hardy spaces for $\mathbb{H}$. We are primarily interested in the former approach.

For $k \in \mathbb{N}_0$, let

$$\mathcal{A}_k(\mathbb{H}) = \left\{ F \in \mathcal{O}(\mathbb{H}) \cap \mathcal{C}(\mathbb{H} \cup \mathbb{T}) : z_2^k F(z_1, z_2) \text{ is bounded at } (0, 0) \right\},$$

and $\mathfrak{H}_k^2(\mathbb{H})$ be the closure of $\mathcal{A}_k(\mathbb{H})|_{\mathbb{T}}$ in $L^2(\mathbb{T})$. As in the case of $\mathbb{D}^*$ and $\mathbb{D} \times \mathbb{D}^*$, we can describe these spaces and their Szegő kernels explicitly.

Theorem 6.1 *Let* $k \in \mathbb{N}_0$. *Then*

$$\mathfrak{H}_k^2(\mathbb{H}) = \left\{ \sum_{(j,\ell) \in \mathbb{Z}^2} \hat{f}_{j,\ell} \, e^{i(j\theta_1 + \ell\theta_2)} \in L^2(\mathbb{T}) : \hat{f}_{j,\ell} = 0, \text{ if } \max\{j, \ell + j + k\} < 0 \right\}. \tag{6.3}$$

In particular, the filtration $\{\mathfrak{H}_k^2(\mathbb{H})\}_{k \in \mathbb{N}_0}$ *does not stabilize. Moreover,*

$$s_k(z, w) = \frac{1}{4\pi^2} \frac{(z_2 \overline{w}_2)^{-(k-1)}}{(z_2 \overline{w}_2 - z_1 \overline{w}_1)(1 - z_2 \overline{w}_2)}, \quad z \in \mathbb{H}, \ w \in \mathbb{T}, \tag{6.4}$$

is the Szegő kernel for $\mathfrak{H}_k^2(\mathbb{H})$.

Proof Fix a $k \in \mathbb{N}_0$. Our proof relies on the fact that $\Theta^* : F|_{\mathbb{T}} \mapsto (F \circ \Theta)|_{\mathbb{T}}$ is an isometric isomorphism between $\mathcal{A}_k(\mathbb{H})|_{\mathbb{T}}$ and $\mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*)|_{\mathbb{T}}$ in the $L^2(\mathbb{T})$-norm. The isometry follows from an integration by substitution argument. For the isomorphism, note that $F \in \mathcal{A}_k(\mathbb{H})$ if and only if the function $(z_1, z_2) \mapsto z_2^k F(z_1 z_2, z_2)$ is holomorphic on $\mathbb{D} \times \mathbb{D}^*$, bounded on a neighborhood of $\{z_2 = 0\}$, and continuous up to $\mathbb{T}$. This is true if and only if $z_2^k F(z_1 z_2, z_2) = G(z_1, z_2)|_{\mathbb{D} \times \mathbb{D}^*}$ for some $G \in \mathcal{A}(\mathbb{D}^2)$. In other words, $F|_{\mathbb{T}} \in \mathcal{A}_k(\mathbb{H})_{\mathbb{T}}$ if and only if $(\Theta^* F)|_{\mathbb{T}} \in \mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*)|_{\mathbb{T}}$. Now, Θ^* extends to an isometry between $\mathfrak{H}_k^2(\mathbb{H})$ and $\mathfrak{H}_k^2(\mathbb{D} \times \mathbb{D}^*)$ which, in terms of Fourier expansions, is given by

$$\Theta^* : \sum_{(j,\ell) \in \mathbb{Z}^2} \hat{f}_{j,\ell} \, e^{i(j\theta_1 + \ell\theta_2)} \mapsto \sum_{(j,\ell) \in \mathbb{Z}^2} \hat{f}_{j,\ell} \, e^{i(j\theta_1 + (j+\ell)\theta_2)}.$$

The characterization in (6.3) now follows from that of $\mathfrak{H}_k^2(\mathbb{D} \times \mathbb{D}^*)$ in (6.1).

Finally, for any $F \in \mathcal{A}_k(\mathbb{H})$, the reproducing property of the Szegő kernel $s_k^{\mathbb{D} \times \mathbb{D}^*}$ for $\mathfrak{H}_k^2(\mathbb{D} \times \mathbb{D}^*)$ applies to $\Theta^* F \in \mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*)$. We obtain that

$$F(z) = \int_{\mathbb{T}} (\Theta^* F)(w) \, s_k^{\mathbb{D} \times \mathbb{D}^*} \left(\Theta^{-1}(z), \Theta^{-1}(w) \right) d\sigma_{\mathbb{T}}(w), \quad z \in \mathbb{H}, w \in \mathbb{T}.$$

Now, a straightforward computation yields the reproducing property of s_k as defined in (6.4). It is also clear that $\overline{s_k(z, \cdot)} \in \mathcal{A}_k(\mathbb{H})$ for any $z \in \mathbb{H}$.

We briefly discuss an exhaustion-based construction of Hardy spaces $\mathcal{H}_k^2(\mathbb{H}), k \in \mathbb{N}_0$, for $\mathbb{H}$, which in the case of $k = 1$ is the space constructed by Monguzzi in [22]. For $k \in \mathbb{N}_0$, let

$$\mathcal{H}_k^2(\mathbb{H}) = \left\{ F \in \mathcal{O}(\mathbb{H}) : ||F||_{\mathcal{H}_k^2(\mathbb{H})} < \infty \right\},$$

where

$$\|F\|_{\mathcal{H}_k^2(\mathbb{H})} := \sup_{0 < s, r < 1} \left(\frac{r^{2k}}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left| F(rse^{i\theta_1}, re^{i\theta_2}) \right|^2 d\theta_1 \, d\theta_2 \right)^{\frac{1}{2}}.$$

Rather than establish a direct isometric isomorphism, up to a factor, between $\mathcal{H}_k^2(\mathbb{H})$ and $\mathfrak{H}_k^2(\mathbb{H})$, we argue that $\Theta^* : F \mapsto F \circ \Theta$ is an isometric isomorphism between $\mathcal{H}_k^2(\mathbb{H})$ and $\mathcal{H}_k^2(\mathbb{D} \times \mathbb{D}^*)$. From the proof of Theorem 6.1, we know that Θ^* is an isomorphism between $\mathcal{A}_k(\mathbb{H})$ and $\mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*)$. Since these spaces are dense in the respective $\mathcal{H}^2$-spaces, it suffices to show that Θ^* is an isometry from $(\mathcal{A}_k(\mathbb{H}), \|.\|_{\mathcal{H}_k^2(\mathbb{H})})$ to $(\mathcal{A}_k(\mathbb{D} \times \mathbb{D}^*), \|.\|_{\mathcal{H}_k^2(\mathbb{D} \times \mathbb{D}^*)})$. This is a standard computation, by way of integration by substitution.

6.3 The (rational) power-generalized hartogs triangles

We now consider the general case of $\mathbb{H}_{m/n}$. For $k \in \mathbb{N}_0$, define

$$\mathcal{A}_k(\mathbb{H}_{m/n}) = \left\{ F \in \mathcal{O}(\mathbb{H}_{m/n}) \cap \mathcal{C}(\mathbb{H}_{m/n} \cup \mathbb{T}) : z_2^k F(z_1, z_2) \text{ is bounded at } (0, 0) \right\}.$$

Let $\mathfrak{H}_k^2(\mathbb{H}_{m/n})$ be the closure of $\mathcal{A}_k(\mathbb{H}_{m/n})|_{\mathbb{T}}$ in $L^2(\mathbb{T})$. As in the case $m = n = 1$, using $\Theta_{m/n}^*$, we see that

$$\mathfrak{H}_k^2(\mathbb{H}_{m/n}) = \left\{ \sum_{(j,\ell) \in \mathbb{Z}^2} \hat{f}_{j,\ell} \, e^{i(j\theta_1 + \ell\theta_2)} \in L^2(\mathbb{T}) : \hat{f}_{j,\ell} = 0, \text{ if } \max\{j, nj + ml + mk\} < 0 \right\}.$$

Next, we use the map $\Theta_{m/n}$ to compute the Szegő kernel for $\mathfrak{H}_k^2(\mathbb{H}_{m/n})$.

Theorem 6.2 *Let $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Set*

$$\mathcal{P}_{m,n}(a, b) = \sum_{r=0}^{m-1} (a)^r (b)^{n - \lfloor \frac{nr}{m} \rfloor}, \qquad (a, b) \in \mathbb{C}^2.$$

Then, for $k \in \mathbb{N}_0$,

$$s_k(z, w) = \frac{1}{4\pi^2} \frac{(z_2 \overline{w}_2)^{-k} \, \mathcal{P}_{m,n}(z_1 \overline{w}_1, z_2 \overline{w}_2)}{((z_2 \overline{w}_2)^n - (z_1 \overline{w}_1)^m)(1 - z_2 \overline{w}_2)}, \quad z \in \mathbb{H}_{m/n}, \ w \in \mathbb{T}, \tag{6.5}$$

is the Szegő kernel for $\mathfrak{H}_k^2(\mathbb{H}_{m/n})$.

In order to prove Theorem 6.2, we need the following two lemmas. The proofs are straightforward applications of integration by substitution and partial fraction decompositions, so they are omitted.

Lemma 6.3 *Suppose $f \in \mathcal{C}(b\mathbb{D})$. Then*

$$\int_{|\zeta|=1} \zeta^{n-1} f(\zeta^n) \, d\zeta = \int_{|z|=1} f(z) \, dz. \tag{6.6}$$

More generally, if $n \in \mathbb{N}$, $a \in \mathbb{C} \setminus S^1$, and $a_1, ..., a_n$ denote the nth-roots of a (counting multiplicity). Then

$$\sum_{j=1}^n \left(\int_{|\zeta|=1} \frac{f(\zeta^n)}{\zeta - a_j} \, d\zeta \right) = n \int_{|w|=1} \frac{f(w)}{w - a} \, dw. \tag{6.7}$$

Lemma 6.4 *Let $b \in \mathbb{C} \setminus \{0\}$ and $b_1, ..., b_m$ denote the mth-roots of b (counting multiplicty). Then*

$$\sum_{\ell=1}^{m} \frac{b_\ell^n}{(x - b_\ell^n)(y - b_\ell)} = \frac{mb^{n+1} \sum_{p,q=0}^{m-1} c_{p,q}\, x^p y^q}{(x^m - b^n)(y^m - b)},$$

where

$$(6.8) \qquad\qquad c_{p,q} = \begin{cases} 0, & if \quad np + 1 + q \not\equiv 0 \ (mod\ m), \\ b^{-\frac{np+1+q}{m}}, & if \quad np + 1 + q \equiv 0 \ (mod\ m). \end{cases}$$

Proof of Theorem 6.2 For any $k \in \mathbb{N}_0$, s_k, as defined in (6.5), satisfies $\overline{s_k(z, \cdot)} \in \mathcal{A}_k(\mathbb{H}_{m/n})$ for all $z \in \mathbb{H}_{m/n}$. Thus, it suffices to show that s_k has the reproducing property for $\mathfrak{H}_k^2(\mathbb{H}_{m/n})$. Since $|z_2|\big|_{\mathbb{T}} \equiv 1$, by Proposition 3.14, we only need to prove this for $k = 0$.

Recall that $\Theta_{m/n}(\zeta_1, \zeta_2) = (\zeta_1^n \zeta_2^n, \zeta_2^m)$ maps $\mathbb{D} \times \mathbb{D}^*$ onto $\mathbb{H}_{m/n}$. Given $F \in \mathcal{A}_0(\mathbb{H}_{m/n})$ and $z = (z_1, z_2) \in \mathbb{H}_{m/n}$, let $z_{11}, ..., z_{1n}$ and $z_{21}, ..., z_{2m}$ denote the nth-roots and m^{th}-roots of z_1 and z_2, respectively, so that $F(z_1, z_2) = F(z_{1j}^n, z_{2\ell}^m)$ for any $1 \le j \le n$ and $1 \le \ell \le m$. Thus,

$$F(z_1, z_2) = \frac{1}{mn} \sum_{\ell=1}^{m} \sum_{j=1}^{n} (F \circ \Theta_{m/n}) \left(\frac{z_{1j}}{z_{2\ell}}, z_{2\ell} \right).$$

We apply the Cauchy integral formula for $\mathbb{D}^2$ to $(F \circ \Theta_{m/n}) \in \mathcal{A}_0(\mathbb{D} \times \mathbb{D}^*) = \mathcal{A}(\mathbb{D}^2)$, and obtain the following sequence of arguments.

$$\begin{aligned} (2\pi i)^2 mn F(z_1, z_2) &= \sum_{\ell=1}^{m} \sum_{j=1}^{n} \iint_{\mathbb{T}} \frac{(F \circ \Theta_{m/n})(\zeta_1, \zeta_2)}{\left(\zeta_1 - \frac{z_{1j}}{z_{2\ell}} \right)(\zeta_2 - z_{2\ell})} \, d\zeta_1 \, d\zeta_2 \\ &= \sum_{\ell=1}^{m} \int_{|\zeta_2|=1} \left(\sum_{j=1}^{n} \int_{|\zeta_1|=1} \frac{F(\zeta_1^n \zeta_2^n, \zeta_2^m)}{\zeta_1 - \frac{z_{1j}}{z_{2\ell}}} \, d\zeta_1 \right) \frac{d\zeta_2}{(\zeta_2 - z_{2\ell})} \\ &\overset{\zeta_1 \zeta_2 \mapsto \xi}{=} \sum_{\ell=1}^{m} \int_{|\zeta_2|=1} \left(\sum_{j=1}^{n} \int_{|\xi|=1} \frac{F(\xi^n, \zeta_2^m)}{\left(\xi - \zeta_2 \frac{z_{1j}}{z_{2\ell}} \right)} \, d\xi \right) \frac{d\zeta_2}{(\zeta_2 - z_{2\ell})} \\ &\overset{(6.7)}{=} n \sum_{\ell=1}^{m} \int_{|\zeta_2|=1} \left(\int_{|w_1|=1} \frac{F(w_1, \zeta_2^m)}{\left(w_1 - \zeta_2^n \frac{z_1}{z_{2\ell}^n} \right)} \, dw_1 \right) \frac{d\zeta_2}{(\zeta_2 - z_{2\ell})} \\ &= n \iint_{\mathbb{T}} \frac{F(w_1, \zeta_2^m)}{-w_1} \left(\sum_{\ell=1}^{m} \frac{z_{2\ell}^n}{(\zeta_2^n \frac{z_1}{w_1} - z_{2\ell}^n)(\zeta_2 - z_{2\ell})} \right) \, dw_1 \, d\zeta_2. \end{aligned}$$

Now, by Lemma 6.4 (with $x = \frac{\zeta_2^n z_1}{w_1}$, $y = \zeta_2$ and $b = z_2$), we have that

$$(2\pi i)^2 mn F(z_1, z_2) = mn \iint_{\mathbb{T}} \frac{F(w_1, \zeta_2^m)}{-w_1} \left(\frac{z_2^{n+1} \sum_{p,q=0}^{m-1} c_{p,q} \left(\frac{\zeta_2^n z_1}{w_1} \right)^p \zeta_2^q}{\zeta_2^{mn} (\frac{\overline{z}_1^m}{w_1^m} - z_2^n)(\zeta_2^m - z_2)} \right) dw_1 \, d\zeta_2$$

$$= mn \iint_{\mathbb{T}} F(w_1, \zeta_2^m) \left(\frac{\zeta_2^{m-1} \sum_{p,q=0}^{m-1} \left(z_2^{-\frac{np+1+q}{m}} c_{p,q} \right) (z_1 \overline{w}_1)^p (z_2 \overline{\zeta}_2^m)^{(n+1) - \frac{np+q+1}{m}}}{\left((z_2 \overline{\zeta}_2^m)^n - z_1^m \overline{w}_1^m \right) (\zeta_2^m - z_2)} \right) \frac{dw_1}{w_1} \, d\zeta_2,$$

where $c_{p,q}$ are as in (6.8) (with $b = z_2$). Applying (6.6) in the ζ_2 variable, we get

$$(2\pi i)^2 F(z_1, z_2) = \iint_{\mathbb{T}} F(w_1, w_2) \frac{\sum_{p,q=0}^{m-1} \widetilde{c}_{p,q} (z_1 \overline{w}_1)^p (z_2 \overline{w}_2)^{n+1-\frac{np+1+q}{m}}}{(z_2^n \overline{w}_2^n - z_1^m \overline{w}_1^m)(1 - z_2 \overline{w}_2)} \frac{dw_1}{w_1} \frac{dw_2}{w_2},$$

where

$$\sum_{p,q=0}^{m-1} \widetilde{c}_{p,q} (a)^p (b)^{n+1-\frac{np+1+q}{m}} = \sum_{r=0}^{m-1} (a)^r (b)^{n-\lfloor \frac{nr}{m} \rfloor} = \mathcal{P}_{m,n}(a, b).$$

This settles our claim, once we observe that

$$\sum_{p,q=0}^{m-1} \widetilde{c}_{p,q} (a)^p (b)^{n+1-\frac{np+1+q}{m}} = \sum_{r=0}^{m-1} (a)^r (b)^{n-\lfloor \frac{nr}{m} \rfloor} = \mathcal{P}_{m,n}(a, b).$$

□

In view of our minimum criterion for a Hardy space, we end this subsection with an exhaustion-based definition of Hardy spaces for $\mathbb{H}_{m/n}$. For $k \in \mathbb{N}_0$, let

$$\mathcal{H}_k^2(\mathbb{H}_{m/n}) = \left\{ F \in \mathcal{O}(\mathbb{H}_{m/n}) : ||F||_{\mathcal{H}_k^2(\mathbb{H}_{m/n})} < \infty \right\},$$

where

$$||F||_{\mathcal{H}_k^2(\mathbb{H}_{m/n})} := \sup_{0 < s, r < 1} \left(\frac{r^{2k}}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left| F(r^{\frac{n}{m}} s^{\frac{1}{m}} e^{i\theta_1}, r e^{i\theta_2}) \right|^2 d\theta_1 d\theta_2 \right)^{\frac{1}{2}}.$$

6.4 L^p-regularity of the Szegő projection

We briefly remark on the L^p-mapping properties of the projection operator induced by the Szegő kernel s_k for $\mathfrak{H}_k^2(\mathbb{H}_{m/n})$, $k \in \mathbb{N}_0$, $m, n \in \mathbb{N}$, $\gcd(m, n) = 1$. In [22], Monguzzi shows that when $k = m = n = 1$, the densely defined operator

$$\mathbb{S}_k : L^2(\mathbb{T}) \cap L^p(\mathbb{T}) \to \mathfrak{H}_k^2(\mathbb{H}_{m/n}) \tag{6.8}$$

$$f \mapsto \mathbb{S}_k f := \left(z \mapsto \int_{\mathbb{T}} f(w) \cdot s_k(z, w) \, d\sigma_{\mathbb{T}}(w) \right) \Big|_{\mathbb{T}}$$

extends to a bounded operator from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ for any $p \in (1, \infty)$. This is done by realizing $\mathbb{S}_1$ as a Fourier multiplier operator on the 2-dimensional torus $\mathbb{T}$. As a generalization of this fact, we note that $\mathbb{S}_k$ is the Fourier multiplier operator

$$f(e^{i\theta_1}, e^{i\theta_2}) = \sum_{(j,\ell)\in\mathbb{Z}^2} \hat{f}_{j,\ell}\, e^{ij\theta_1} e^{i\ell\theta_2} \mapsto \sum_{(j,\ell)\in\mathbb{Z}^2} c(j,\ell)\hat{f}_{j,\ell}\, e^{ij\theta_1} e^{i\ell\theta_2}, \quad f \in L^2(\mathbb{T}),$$

where, using the convention that $\mathrm{sgn}(0) = 0$,

$$c(j,\ell) = \frac{1 + \mathrm{sgn}(j+1)}{2} \cdot \frac{1 + \mathrm{sgn}(nj + ml + mk + 1)}{2}, \quad (j,\ell) \in \mathbb{Z}^2.$$

Then, by the same argument, (6.8) extends to a bounded operator from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$, $1 < p < \infty$.

The Szegő projections considered above do not exhibit the irregularity properties of the Bergman projection, see [7,14], because the underlying Hardy spaces are supported only on the distinguished boundary of the domain. It is possible that if one considers Hardy spaces supported on the full boundary of the domain, then a stronger connection with the Bergman projection will emerge.

References

1. Aronszajn, N.: Theory of reproducing kernels. Trans. AMS **68**(3), 337–404 (1950)
2. Baratchart, L., Fischer, Y., Leblond, J.: Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation. Complex Var. Elliptic Equ. **59**(4), 504–538 (2014)
3. Barrett, D.E., Lanzani, L.: The spectrum of the Leray transform for convex Reinhardt domains in $\mathbb{C}^2$. J. Funct. Anal. **257**(9), 2780–2819 (2009)
4. Bell, S.R.: The Cauchy transform, potential theory and conformal mapping. CRC Press, Boca Raton (1992)
5. Chakrabarti, D., Shaw, M.-C.: Sobolev regularity of the $\bar{\partial}$-equation on the Hartogs triangle. Math. Ann. **356**(1), 241–258 (2013)
6. Chakrabarti, D., Zeytuncu, Y.: L^p mapping properties of the Bergman projection on the Hartogs triangle. Proc. Am. Math. Soc. **144**(4), 1643–1653 (2016)
7. Chakrabarti, D., Edholm, L., McNeal, J.: Duality and approximation of Bergman spaces. Adv. Math. **341**, 616–656 (2019)
8. Chaumat, J., Chollet, A.-M.: Régularité höldérienne de l'opérateur $\bar{\partial}$ sur le triangle de Hartogs. Ann. Inst. Fourier (Grenoble) **41**(4), 867–882 (1991)
9. Chen, L., McNeal, J.D.: A solution operator for $\bar{\partial}$ on the Hartogs triangle and L^p estimates. Math. Ann. (to appear)
10. Chirka, E.M.: Complex analytic sets, vol. 46. Springer Science & Business Media, Berlin (2012)
11. Dales, H.G.: The ring of holomorphic functions on a Stein compact set as a unique factorization domain. Proc. Am. Math. Soc. **44**(1), 88–92 (1974)
12. Duren, P.L.: Theory of H^p Spaces. Academic press, Cambridge (1970)
13. Edholm, L.: Bergman theory of certain generalized Hartogs triangles. Pac. J. Math. **284**(2), 327–342 (2016)
14. Edholm, L., McNeal, J.: The Bergman projection on fat Hartogs triangles: L^p boundedness. Proc. Am. Math. Soc. **144**(5), 2185–2196 (2016)
15. Edholm, L., McNeal, J.: Bergman subspaces and subkernels: Degenerate L^p mapping and zeroes. J. Geom. Anal. **27**(4), 2658–2683 (2017)
16. Hansson, T.: On Hardy spaces in complex ellipsoids. Ann. Inst. Fourier (Grenoble) **49**(5), 1477–1501 (1999)
17. Huo, Z., Wick, B.D.: Weighted estimates for the Bergman projection on the Hartogs triangle. J. Funct. Anal. **279**(9), 108727 (2020)
18. Kerzman, N., Stein, E.M.: The Cauchy kernel, the Szegő kernel, and the Riemann mapping function. Math. Ann. **236**(1), 85–93 (1978)
19. Lanzani, L.: Szegő projection vs. potential theory for non-smooth planar domains. Indiana U. Math. J. **48**(2), 537–555 (1998)
20. Laurent-Thiébaut, C., Shaw, M.-C.: Solving $\bar{\partial}$ with prescribed support on Hartogs triangles in $\mathbb{C}^2$ and $\mathbb{CP}^2$. Trans. Am. Math. Soc. **371**(9), 6531–6546 (2019)
21. Ma, L., Michel, J.: $\mathcal{C}^{k,\alpha}$-estimates for the $\bar{\partial}$-equation on the Hartogs triangle. Math. Ann. **294**(1), 661–675 (1992)

22. Monguzzi, A.: Holomorphic function spaces on the Hartogs triangle. Math. Nachr. (to appear)
23. Nagel, A., Pramanik, M.: Bergman spaces under maps of monomial type. arXiv preprint arXiv:2002.02915, 2020
24. Ohsawa, T.: Analysis of several complex variables, 211th edn. American Mathematical Society, Providence (2002)
25. Poletsky, E.A., Stessin, M.I.: Hardy and Bergman spaces on hyperconvex domains and their composition operators. Indiana Univ. Math. J. **57**(5), 2153–2201 (2008)
26. Pommerenke, C.: Boundary behaviour of conformal maps, vol. 299. Springer Science & Business Media, Berlin (2013)
27. Rademacher, H., Toeplitz, O.: The enjoyment of mathematics. Princeton U. press, Princeton (1957)
28. Sahin, S.: Poletsky-Stessin Hardy spaces on Complex Ellipsoids in $\mathbb{C}^n$. Complex Anal. Oper. Theory **10**(2), 295–309 (2016)
29. Shaw, M.-C.: The Hartogs triangle in complex analysis. Contemp. Math. **646**, (2015)
30. Stein, E.M.: Boundary behavior of holomorphic functions of several complex variables.(MN-11). Princeton University Press, Princeton (2015)