PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, Number 6, June 2020, Pages 2525–2537
https://doi.org/10.1090/proc/14921
Article electronically published on February 4, 2020

NON-AUTONOMOUS PARABOLIC BIFURCATION

LIZ VIVAS

(Communicated by Filippo Bracci)

ABSTRACT. Let $f(z) = z+z^2+O(z^3)$ and $f_\epsilon(z) = f(z)+\epsilon^2$. A classical result in parabolic bifurcation in one complex variable is the following: if $N - \frac{\pi}{\epsilon} \to 0$ we obtain $(f_\epsilon)^N \to \mathcal{L}_f$, where $\mathcal{L}_f$ is the Lavaurs map of f. In this paper we study a *non-autonomous* parabolic bifurcation. We focus on the case of $f_0(z) = \frac{z}{1-z}$. Given a sequence $\{\epsilon_i\}_{1\leq i\leq N}$, we denote $f_n(z) = f_0(z) + \epsilon_n^2$. We give sufficient and necessary conditions on the sequence $\{\epsilon_i\}$ that imply that $f_N \circ \dots f_1 \to \mathrm{Id}$ (the Lavaurs map of f_0). We apply our results to prove parabolic bifurcation phenomenon in two dimensions for some class of maps.

1. Introduction

The theory of parabolic bifurcation has been extensively studied in one dimension starting with the pioneering work of Lavaurs and Douady, as well as Shishikura [4, 6, 9]. In recent years, parabolic bifurcation has been explored in several dimensions; Bedford, Smillie, and Ueda studied semiparabolic bifurcations [3] and Bianchi [2] studied parabolic bifurcations for a class of maps in $\mathbb{C}^2$. Also the recent works of Dujardin and Lyubich [5] and Astorg et al. [1] have shown applications to new phenomena in several dimensions using higher dimension parabolic bifurcations.

In this article we propose to study parabolic bifurcation in two dimensions by considering *non-autonomous* sequences of one dimensional Möbius maps. Let us recall the result in one dimension and explain our result.

The following is a classical result by Lavaurs [6].

Theorem 1 (Lavaurs)**.** *Let f be defined in a neighborhood V of the origin and be of the form $f(z) = z + z^2 + O(z^3)$. Consider the perturbation of f as follows: given $\epsilon > 0$ let $f_\epsilon(z) := f(z) + \epsilon^2$. If we take a sequence of number N_ϵ such that $N_\epsilon - \frac{\pi}{\epsilon} \to 0$, then we obtain the following:*

$$(f_\epsilon)^{N_\epsilon} \to \mathcal{L}_f,$$

where $\mathcal{L}_f$ is the Lavaurs map of f.

In this paper we study the following question.

Question. Let f be defined in a neighborhood V of the origin and be of the form $f(z) = z+z^2+O(z^3)$ as above. Consider different perturbations of f as follows: fix

---

Received by the editors May 10, 2019, and, in revised form, August 28, 2019, and October 21, 2019.

2010 *Mathematics Subject Classification.* Primary 32H50, 37F45.

The author was partially supported by the NSF - National Science Foundation Grant DMS-1800777.

$N > 0$ then given $\epsilon_k = \epsilon_k(N) > 0$ let $f_k(z) := f(z) + \epsilon_k^2$. Under which conditions on $\epsilon_1, \epsilon_2, \ldots, \epsilon_N$ do we obtain:

$$f_N \circ f_{N-1} \circ \ldots f_2 \circ f_1 \to \mathcal{L}_f, \tag{1}$$

where $\mathcal{L}_f$ is the Lavaurs map of f?

When (1) holds we say that "non-autonomous bifurcation" holds for f.

In this paper we focus in the case of $f(z) = \frac{z}{1-z}$ and prove a sufficient and necessary condition on the sequence $\{\epsilon_k\}_{1 \le k \le N}$ as follows.

Theorem 2. *Fix N large. Consider $\{\epsilon_k\}, 1 \le k \le N$ a sequence such that:*

$$\epsilon_k = \frac{\pi}{N} + \frac{\alpha(k)}{N^2} \tag{2}$$

(each $\alpha(k)$ might depend on N) such that $\alpha(k)$ are bounded and $\alpha(k) + \alpha(N-k) = O(1/N)$. Let $f_k(z) := \frac{z}{1-z} + \epsilon_k^2$. Then we have:

$$f_N \circ f_{N-1} \circ \ldots f_2 \circ f_1 \to \mathrm{Id} \tag{3}$$

for all $z \in K$, where K is a compact subset of $\mathbb{C}$.

We call this result "non-autonomous bifurcation" for a Möbius transformation, since in this case the Lavaurs map of f is simply the identity. We prove also that the condition is necessary (see Section 2.4).

Remark 1. In order to prove our main result we use the theory of orthogonal functions. This is to our knowledge the first time that we have a connection to this field. More importantly, we only use a particular version of a general estimate for orthogonal polynomials. We believe that the general version of this theorem must have its corresponding bifurcation version. See Section 5 for more details.

The structure of the paper is as follows. In the next section we set the notation as well as our theorems and the proofs. In Section 3, we give examples of sequences that satisfy the conditions. In Section 4 we apply our results to prove parabolic bifurcation for specific families of maps in two dimensions. In the last section, we formulate some questions and remarks.

2. Non-parabolic bifurcation

We give a sequence of positive real numbers $\epsilon_1, \epsilon_2, \ldots$. Consider the following functions:

$$f_k(z) = f_{\epsilon_k}(z) = \frac{z}{1-z} + \epsilon_k^2$$

for $k \ge 1$. Set

$$F_n = f_n \circ f_{n-1} \circ \ldots f_2 \circ f_1.$$

We prove the following technical version of our theorem. We see at the end of this section how Theorem 3 implies Theorem 2.

Theorem 3. *Fix N large. Consider $\{\epsilon_k\}, 1 \le k \le N$ a sequence such that:*

$$\left| \sum_{k=1}^{N} \left(\frac{\pi^2}{N^2} - \epsilon_k^2 \right) \frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2} \right| < \frac{A}{N} \tag{4}$$

and

$$\left|\frac{\pi^2}{N^2} - \epsilon_k^2\right| < \frac{A}{N^3}$$

for all $1 \leq k \leq N$ *and a fixed constant* A *independent of* N. *Then:*

$$|F_N(z) - z| < C/N$$

for all $z \in K$, *where* K *is a compact subset of* $\mathbb{C}$ *and some* C *independent of* N.

(Note that when $\epsilon_k = \epsilon = \pi/N$ then the conditions on the theorem are satisfied trivially. The conclusion that $\lim_{N\to\infty}(f_\epsilon)^N(z) = z$ is a particular case of the classical bifurcation theorem in one dimension.)

Fix $N \geq 1$. Since each f_n is a Möbius transformation, then we can compute the specific formula for F_k by computing the product of the matrices related to each. Then

$$F_k(z) = \frac{A_k z + C_k}{B_k z + D_k}; \tag{5}$$

then:

$$\begin{pmatrix} A_k & C_k \\ B_k & D_k \end{pmatrix} = \begin{pmatrix} 1-\epsilon_k^2 & \epsilon_k^2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_{k-1} & C_{k-1} \\ B_{k-1} & D_{k-1} \end{pmatrix}.$$

Lemma 1. *Set* $t_k = 2 - \epsilon_k^2$ *the trace of each matrix above. Consider the sequence* $p_0 = 0, p_1 = 1$ *and* $q_0 = 1, q_1 = 1$ *and for* $k \geq 1$*:*

$$\begin{aligned} p_{k+1} &= t_k p_k - p_{k-1}, \\ q_{k+1} &= t_k q_k - q_{k-1}. \end{aligned} \tag{6}$$

Then for any $n \geq 1$ *we have:*

$$\begin{aligned} A_n &= p_{n+1} - p_n, C_n = q_n - q_{n+1}, \\ B_n &= -p_n, \qquad D_n = q_n. \end{aligned}$$

Proof of Lemma 1. It follows directly using induction. □

Although the following statement is, as mentioned above a particular case of the general parabolic bifurcation in one variable, we redo the proof here as a preparation step for the proof of Theorem 3.

Lemma 2. *Fix* N. *Suppose that all* $\epsilon_i = \epsilon$ *and the condition:*

$$N - \pi/\epsilon \to \sigma.$$

Then $F_N(z) \to \frac{z}{1-\sigma z}$. *When* $\sigma = 0$ *we obtain* $F_N(z) \to \mathrm{Id}$.

Proof. The equation (6) is a generalization of the classical Chebyshev polynomials. Note that the classical Chebyshev polynomial corresponds to the case of the same ϵ_i, that is, the classical parabolic bifurcation on one variable. Indeed, if we have all $\epsilon_i = \epsilon$ and $t_i = x = 2 - \epsilon^2$, then it's well known that:

$$p_k = \frac{\sin(k\theta)}{\sin(\theta)} \qquad \text{and} \qquad q_k = \frac{\sin(k\theta) - \sin((k-1)\theta)}{\sin(\theta)},$$

where $x = 2\cos(\theta)$. When $x = 2 - \epsilon^2$, then $\theta = \epsilon + O(\epsilon^3)$. Suppose $N - \pi/\epsilon \to \sigma$. In that case $N - \pi/\epsilon - \sigma = o_N(1)$ and we can write $\theta = \frac{\pi}{N} + \frac{\pi\sigma}{N^2} + o\left(\frac{1}{N^2}\right)$. Then

$$p_N = \frac{\sin(N\theta)}{\sin(\theta)} = \frac{\sin\left(\pi + \frac{\pi\sigma}{N} + o\left(\frac{1}{N}\right)\right)}{\sin\left(\frac{\pi}{N} + \frac{\pi\sigma}{N^2} + o\left(\frac{1}{N^2}\right)\right)} = -\sigma + o_N(1),$$

$$p_{N+1} = \frac{\sin((N+1)\theta)}{\sin(\theta)} = \frac{\sin\left(\pi + \frac{\pi+\sigma\pi}{N} + O\left(\frac{\pi 0_N(1)}{N}\right)\right)}{\sin\left(\frac{\pi}{N} + \frac{\pi o_N(1)}{N^2} + O\left(\frac{1}{N^3}\right)\right)} = -1 - \sigma + o_N(1),$$

and similarly $p_{N-1} = 1 - \sigma + o_N(1)$, which translated to the element of our matrix:

$$A_N = D_N = -1 + o_N(1), B_N = \sigma + o_N(1), C_N = o_N(1).$$

Therefore, when $N \to \infty$, $A_N = D_N \to -1, B_N \to \sigma, C_N \to 0$ so $F_N(z) \to \frac{z}{1-\sigma z}$. □

As it is clear from the proof of the lemma above, if we have estimates on p_N and q_N, then we immediately have the estimates for A_N, B_N, C_N, D_N.

All our theorems on the non-autonomous case will deal with the case $\sigma = 0$ (also known as the phase 0 case) for ease of notation. Similar conditions as equation (4) can be displayed for the case $\sigma \neq 0$, but they are cumbersome. We include a note about this in the last Section 2.4.

2.1. **Orthogonal polynomials.** We review here some facts about orthogonal polynomials. We use the following lemma from [8].

Lemma 3. *Consider the sequence $p_0 = 0, p_1 = 1$ and for $k \geq 1$:*

$$p_{k+1} = (x + a_k)p_k - p_{k-1}. \tag{7}$$

Let $x = 2\cos(\theta)$; then we have the following equality:

$$\sin(\theta)p_n(x) = |\phi_n| \sin(n\theta - \arg(\phi_n)), \tag{8}$$

where $\phi_n = 1 + \delta_n = 1 + \sum_{j=1}^{n-1} a_j p_j e^{ij\theta}$ for $n \geq 2$ and $\phi_1 = 1$.

For simplification we will use the following terminology for the classical Chebyshev polynomials $U_0 = 0, U_1 = 1$ and for $k \geq 1, U_{k+1} = xU_k - U_{k-1}$. In that case $U_k = \sin(k\theta)/\sin(\theta)$ for $x = 2\cos(\theta)$.

Lemma 4. *Consider the following two sequences:*

$$p_0 = 0, p_1 = 1, p_{k+1} = (x + a_k)p_k - p_{k-1}, k \geq 1,$$
$$U_0 = 0, U_1 = 1, U_{k+1} = xU_k - U_{k-1}, k \geq 1.$$

Let $x = 2\cos(\theta)$. Suppose there exists $\epsilon > 0$ and $m \in \mathbb{N}$ such that the sequence $\{a_i\}$ satisfies:

$$\sum_{j=1}^{m-1} |a_j p_j| \leq \epsilon \sin(\theta); \tag{9}$$

then

$$|p_n - U_n| \leq \epsilon$$

for all $1 \leq n \leq m$.

Proof. We use equation (8):

$$\sin(\theta)p_n(x) = \sin(n\theta)(1 + \mathrm{Re}(\delta_n)) - \cos(n\theta)\mathrm{Im}(\delta_n),$$
$$\sin(\theta)p_n(x) = \sin(n\theta) + \sin(n\theta)\mathrm{Re}(\delta_n) - \cos(n\theta)\mathrm{Im}(\delta_n),$$
$$\sin(\theta)(p_n - U_n) = -\mathrm{Im}(\delta_n e^{-in\theta}).$$

Recall that $\delta_n = \sum_{j=1}^{n-1} a_j p_j e^{ij\theta}$; then $|\delta_n| \leq \sum_{j=1}^{n-1} |a_j p_j e^{ij\theta}| = \sum_{j=1}^{n-1} |a_j p_j| < \epsilon \sin(\theta)$ and we immediately obtain the desired result. □

2.2. **Proof of Theorem 3.** We are ready now to prove Theorem 3. Fix $N > 0$ large. We use the lemmas referred to above with the following choices: $x = 2\cos(\theta)$ where $\theta = \frac{\pi}{N}$; then we have explicit values and estimates for U_i for all i in terms of N. In particular $|U_i| \leq \frac{1}{\sin(\theta)} \leq \frac{2N}{\pi} \leq N$. Our goal is to prove that under certain conditions on a_k, then p_n and U_n are very close to each other.

Lemma 5. *Fix $N > 0$. Let $x = 2\cos(\theta)$ where $\theta = \frac{\pi}{N}$. Given a sequence $\{a_i\}$ for $1 \leq i \leq N$, suppose there exists a fixed $C > 0$ constant such that*

$$|a_i| \leq \frac{C}{N^3} \leq \frac{1}{2N^2}; \tag{10}$$

then we have $|p_i - U_i| < 2C$ for $1 \leq i \leq N + 1$.

Proof. From the proof of the last lemma we have

$$\frac{1}{N}|p_n - U_n| \leq |\sin(\theta)||p_n - U_n| \leq |\delta_n| \leq \sum_{j=1}^{n-1} |a_j p_j|.$$

We use induction: the property is obvious for $i = 1$. Assume the bound holds for $i \in [1, n-1]$; then for $i = n \leq N + 1$ we have

$$\begin{aligned}\sum_{j=1}^{n-1} |a_j p_j| &\leq \sum_{j=1}^{n-1} |a_j U_j| + 2C \sum_{j=1}^{n-1} |a_j| \\ &\leq N \sum_{j=1}^{n-1} |a_j| + 2C \sum_{j=1}^{n-1} |a_j| \\ &\leq (N + 2C)(n-1)\frac{C}{N^3} \\ &\leq (N + 2C)\frac{C}{N^2}.\end{aligned}$$

Then

$$\frac{1}{N}|p_n - U_n| \leq (N + 2C)\frac{C}{N^2},$$
$$|p_n - U_n| \leq (N + 2C)\frac{C}{N} = C + \frac{2C^2}{N} \leq C + C = 2C,$$

which concludes the proof. □

Proposition 1. *Fix N. Let $x = 2\cos\frac{\pi}{N}$. Suppose that*

$$\left|\sum_{k=1}^{N} a_k U_k^2\right| \leq \frac{C}{N} \quad \text{and} \quad |a_k| \leq \frac{C}{N^3} \tag{11}$$

for all $1 \le k \le N$ *and a fixed constant* C *independent of* N*. Then:*

$$|p_N| \le C'/N, |p_{N+1}+1| \le C'/N$$

for some C' *independent of* N*.*

Proof. We use (8) for $\theta = \pi/N$:

$$\begin{aligned}\sin(\theta)p_N &= |\phi_N|\sin(N\theta - \arg(\phi_N))\\ &= |\phi_N|\sin(\pi - \arg(\phi_N))\\ &= |\phi_N|\sin(\arg(\phi_N)) = \mathrm{Im}(\phi_N) = \mathrm{Im}(\delta_N)\\ &= a_1p_1\sin(\theta) + a_2p_2\sin(2\theta) + \ldots + a_{N-1}p_{N-1}\sin((N-1)\theta).\end{aligned}$$

Then

$$\begin{aligned}|p_N| &= \frac{1}{\sin(\theta)}|a_1p_1\sin(\theta) + a_2p_2\sin(2\theta) + \ldots + a_{N-1}p_{N-1}\sin((N-1)\theta)|\\ &= |a_1p_1U_1 + a_2p_2U_2 + \ldots + a_{N-1}p_{N-1}U_{N-1}|\\ &\le |a_1U_1^2 + a_2U_2^2 + \ldots + a_{N-1}U_{N-1}^2| + 2C\sum_{i=1}^{N-1}|a_iU_i|\\ &\le \frac{C}{N} + 2C\frac{C}{N^3}N^2 = \frac{C'}{N}.\end{aligned}$$

Similarly for p_{N+1} we have:

$$\sin(\theta)(p_{N+1} - U_{N+1}) = -\mathrm{Im}(\delta_{N+1}e^{-i\pi - i\theta}) = \mathrm{Im}(\delta_{N+1}e^{-i\theta}),$$

where

$$\delta_{N+1} = \sum_{k=1}^{N} a_kp_ke^{ik\theta};$$

then

$$e^{-i\theta}\delta_{N+1} = \sum_{k=1}^{N} a_kp_ke^{i(k-1)\theta}$$

so

$$\mathrm{Im}(\delta_{N+1}e^{-i\theta}) = \sum_{k=1}^{N} a_kp_k\sin((k-1)\theta)$$

and we obtain:

$$p_{N+1} - U_{N+1} = \sum_{k=2}^{N} a_kp_kU_{k-1}.$$

Using the fact that $\left|\sum_{k=1}^N a_kU_k^2\right| < \frac{C}{N}$ implies that $\left|\sum_{k=1}^N a_kU_kU_{k-1}\right| < \frac{C''}{N}$ and with the same idea that for p_N we obtain

$$|p_{N+1} - U_{N+1}| = |p_{N+1} + 1| < \frac{C'}{N}.$$

□

We are almost done proving Theorem 3, however, we still need analogue bounds for q_n.

Lemma 6. *Consider the sequences* $p_0 = 0, p_1 = 1$ *and* $q_0 = 1, q_1 = 1$ *and for* $k \geq 1$*:*

$$p_{k+1} = t_k p_k - p_{k-1},$$
$$q_{k+1} = t_k q_k - q_{k-1}.$$

Then

$$q_k = p_k - \widetilde{p}_{k-1},$$

where the sequence $\widetilde{p}_k$ *is given by the conditions* $\widetilde{p}_0 = 0, \widetilde{p}_1 = 1$ *and for* $k \geq 1$ *we have*

$$\widetilde{p}_{k+1} = t_{k+1}\widetilde{p}_k - \widetilde{p}_{k-1}.$$

Proof. The proof follows immediately by writing down $q_k - p_k$ and checking the corresponding initial conditions. □

Using the same idea and estimates for p_N we have the following.

Proposition 2. *Fix* N*. Let* $x = 2\cos\frac{\pi}{N}$*. Suppose that*

$$\left|\sum_{k=1}^{N-1} a_{k+1}U_k^2\right| \leq \frac{C}{N} \quad \text{and} \quad |a_k| \leq \frac{C}{N^3} \tag{12}$$

for all $1 \leq k \leq N$ *and a fixed constant* C *independent of* N*. Then:*

$$|\widetilde{p}_N| \leq C'/N, |\widetilde{p}_{N-1} - 1| \leq C'/N$$

for some C' *independent of* N*.*

Proof. The proof is exactly the same as the proof of Proposition 1; the only difference pertains to the shifted terms which involve the $a_i's$. □

We are ready now to combine all the lemmas above and finish the proof of Theorem 3.

We give a sequence $\{\epsilon_k\}$ such that:

$$\left|\sum_{k=1}^{N}\left(\frac{\pi^2}{N^2} - \epsilon_k^2\right)\frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2}\right| < \frac{A}{N}$$

and

$$\left|\frac{\pi^2}{N^2} - \epsilon_k^2\right| < \frac{A}{N^3}$$

for all $1 \leq k \leq N$ and a fixed constant A independent of N. Notice that we can write this in terms of $x = 2\cos(\pi/N)$ and $a_k = t_k - x$ where $t_k = 2 - \epsilon_k^2$, so we obtain:

$$\left|\sum_{k=1}^{N} a_k U_k^2\right| < \frac{B}{N}$$

and

$$|a_k| < \frac{C}{N^3}$$

for all $1 \leq k \leq N$. Using Lemma 1 and Propositions 1 and 2, we see that $A_N = D_N = -1 + O(1/N)$ and $B_N = C_N = O(1/N)$, which translating back into F_N implies that $F_N(z) \to \text{Id}$ when $N \to \infty$.

2.3. **Proof of Theorem 2.** All that is left to prove is that the conditions on ϵ_k in Theorem 2 are satisfied for Theorem 3. Given ϵ_k such that

$$\epsilon_k = \frac{\pi}{N} + \frac{\alpha(k)}{N^2},$$

where $\alpha(k)$ are bounded, we immediately have:

$$\left|\frac{\pi^2}{N^2} - \epsilon_k^2\right| < \frac{A}{N^3}.$$

Also

$$\frac{\pi^2}{N^2} - \epsilon_k^2 = \frac{-2\pi\alpha(k)}{N^3} + O\left(\frac{1}{N^4}\right).$$

Therefore

$$\begin{aligned} S &= \left|\sum_{k=1}^{N}\left(\frac{\pi^2}{N^2} - \epsilon_k^2\right)\frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2}\right| \\ &= \left|\sum_{k=1}^{[N/2]}\left(\frac{-2\pi(\alpha(k)+\alpha(N-k))}{N^3} + O\left(\frac{1}{N^4}\right)\right)\frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2}\right|. \end{aligned}$$

Since we have the condition $\alpha(k) + \alpha(N-k) = O(1/N)$ we have

$$S = \left|\sum_{k=1}^{[N/2]} O\left(\frac{1}{N^4}\right)\frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2}\right| < \frac{C}{N^4}.[N/2].N^2 = \frac{C'}{N},$$

where we are using the trivial bounds on each $\frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2} < N^2$ and adding the $N/2$ factors. We have that both conditions of Theorem 3 are satisfied and the conclusion follows.

2.4. **Conditions are necessary.** The conditions in Theorem 2 on $\{\epsilon_k\}, 1 \le k \le N$ are

$$\epsilon_k = \frac{\pi}{N} + \frac{\alpha(k)}{N^2}, \tag{13}$$

where $\alpha(k)$ are bounded and $\alpha(k) + \alpha(N-k) = O(1/N)$.

Note that the autonomous case $N - \sigma - \frac{\pi}{\epsilon} \to 0$, or equivalently $\epsilon = \frac{\pi}{N} + \frac{\pi\sigma}{N^2} + o(1/N^2)$, implies the convergence to the phase σ Lavaurs map of $z/(1-z)$ which is precisely $z/(1-\sigma z)$. So clearly $\alpha(k)$ bounded is not enough to conclude the convergence of the perturbations by ϵ.

It might be tempting to suppose that the following conditions on $\alpha(k)$:

$$\sum_{k=1}^{N} \alpha(k) = O(1)$$

which is satisfied when $\alpha(k) + \alpha(N-k) = O(1/N)$ is enough to prove the result in Theorem 2. We prove below that this is not the case.

Lemma 7. *There exists $\{\epsilon_k\}, 1 \le k \le N$ a sequence such that:*

$$\epsilon_k = \frac{\pi}{N} + \frac{\alpha(k)}{N^2},$$

where $\alpha(k)$ are bounded and $\sum_{k=1}^{N}\alpha(k) = O(1)$. However the condition (4) *below is not satisfied:*

$$\left|\sum_{k=1}^{N}\left(\frac{\pi^2}{N^2}-\epsilon_k^2\right)\frac{\sin{(k\pi/N)^2}}{\sin(\pi/N)^2}\right| < \frac{A}{N}.$$

Proof. Fix N even for simplicity. We use the following choice for each $\alpha(k)$:

$$\alpha(k) = \begin{cases} 4k/N-1 & \text{when } 1\le k\le N/2,\\ 0 & \text{when } N/2+1\le k\le N,\end{cases}$$

that satisfies the condition $\sum_{k=1}^{N}\alpha(k) = O(1)$. However when we compute

$$\left|\sum_{k=1}^{N}\left(\frac{\pi^2}{N^2}-\epsilon_k^2\right)\frac{\sin{(k\pi/N)^2}}{\sin(\pi/N)^2}\right| = \left|\sum_{k=1}^{N}\left(\frac{2\pi\alpha(k)}{N^3}+O\left(\frac{1}{N^4}\right)\right)\frac{\sin{(k\pi/N)^2}}{\sin(\pi/N)^2}\right|$$
$$= \left|\sum_{k=1}^{N/2}\left(\frac{2\pi(4k/N-1)}{N^3}\right)\frac{\sin{(k\pi/N)^2}}{\sin(\pi/N)^2}\right| + O\left(\frac{1}{N}\right) \sim 2\pi\left|\sum_{k=1}^{N/2}\left(\frac{(4k/N-1)}{N^3}\right)k^2\right|$$
$$+O\left(\frac{1}{N}\right)\sim O(1)$$

and therefore the hypotheses of Theorem 3 are not satisfied. □

Numerical experiments using the choice of $\alpha(k)$ above do show that $p_k \not\to 0$ and therefore $f_N \circ f_{N-1}\circ \dots f_2\circ f_1 \not\to \mathrm{Id}$.

Remark 2. A similar condition as the one in (4) can be formulated for phase σ, however, the condition is equivalent to $\left|\sum_{k=1}^{M}\left(\frac{\pi^2}{M^2}-\epsilon_k^2\right)\frac{\sin{(k\pi/M)^2}}{\sin(\pi/M)^2}\right| < \frac{A}{M}$ for $M = N-[\sigma]$.

3. Special examples

3.1. Perturbations of the autonomous case.

Theorem 4. *Fix $N>0$ and a sequence of positive real numbers $\{\epsilon_k, 1\le k\le N\}$ satisfying the following condition:*

$$\epsilon_k = \frac{\pi}{N} + A\left(-\frac{1}{N^2}+\frac{2k}{N^3}\right)+O\left(\frac{1}{N^3}\right)$$

for $1\le k\le N$, and a constant A independent of N. Then we have that the following holds:

$$F_N = f_N\circ f_{N-1}\circ\dots f_2\circ f_1 = z+\frac{B(z)}{N},$$

where $f_k(z) = f_{\epsilon_k}(z) = \frac{z}{1-z}+\epsilon_k^2$ for $k\ge 1$.

Proof. Note that

$$a_k = 2-\epsilon_k^2 - 2\cos\left(\frac{\pi}{N}\right) = \frac{\pi^2}{N^2}-\epsilon_k^2+O(1/N^4)$$

and given the condition on ϵ_k we have therefore that

$$a_k = \frac{2A\pi}{N}\left(\frac{1}{N^2}-\frac{2k}{N^3}\right)+O\left(\frac{1}{N^4}\right); |a_k|\le\frac{C'}{N^3}.$$

Also

$$\left|\sum_{k=1}^{N}\left(2-\epsilon_k^2-2\cos\frac{\pi}{N}\right)\frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2}\right| = \left|\sum_{k=1}^{N} a_k U_k^2\right|$$
$$= \left|\sum_{k=1}^{\lfloor N/2\rfloor}(a_k+a_{N-k})U_k^2\right|,$$

where we use $U_k = U_{N-k}$. Since

$$a_k = \frac{2A\pi}{N^4}(N-2k)+O\left(\frac{1}{N^4}\right);$$

then $a_k + a_{N-k} = O\left(\frac{1}{N^4}\right)$ therefore

$$\left|\sum_{k=1}^{\lfloor N/2\rfloor}(a_k+a_{N-k})U_k^2\right| \le \frac{C'}{N}$$

and both conditions of Theorem 3 are satisfied. □

Example 1. Given $m \in \mathbb{N}$, consider the following sequence:

$$\epsilon_k = \frac{\pi}{2\sqrt{m^2+k}} \tag{14}$$

for $1 \le k \le 2m+1 = N$. Then $N-1 = 2m$ and:

$$\epsilon_k = \frac{\pi}{\sqrt{(N-1)^2+4k}} = \frac{\pi}{N}\left(1-\frac{(2-4k/N)}{N}+\frac{1}{N^2}\right)^{-1/2}$$
$$= \frac{\pi}{N} - \frac{\pi(2k/N-1)}{N^2} + O(1/N^3).$$

So we have $|a_k + a_{N-k}| < \frac{C}{N^4}$ for all $1 \le k \le N$; equivalently, Theorem 4 with $A = -\pi$ applies.

Example 2. Given $m \in \mathbb{N}$, consider the following sequence:

$$\epsilon_k = \frac{\pi}{2\sqrt{4m^2+2k}} \tag{15}$$

for $1 \le k \le 4m+2 = N$. Then, a similar computation as above shows that:

$$\epsilon_k = \frac{\pi}{\sqrt{16m^2+8k}} = \frac{\pi}{\sqrt{(N-2)^2+8k}} = \frac{\pi}{N} - \frac{2\pi(2k/N-1)}{N^2} + O(1/N^3)$$

and we can apply Theorem 4 again.

3.2. **Very close perturbations.**

Theorem 5. *Fix $N > 0$ and a sequence of positive real numbers $\{\epsilon_k, 1 \le k \le N\}$ satisfying the following condition:*

$$\left|\epsilon_k - \frac{\pi}{N}\right| \le \frac{C}{N^3}$$

for a constant C independent of N. Then we have that the following holds:

$$F_N = f_N \circ f_{N-1} \circ \dots f_2 \circ f_1 = z + \frac{A(z)}{N},$$

where $f_k(z) = f_{\epsilon_k}(z) = \frac{z}{1-z} + \epsilon_k^2$ for $k \ge 1$.

Proof. Note that

$$a_k = 2 - \epsilon_k^2 - 2\cos\left(\frac{\pi}{N}\right) = \frac{\pi^2}{N^2} - \epsilon_k^2$$

and given the condition on ϵ_k we have therefore that

$$|a_k| \le \frac{C'}{N^4}.$$

So both conditions on Theorem 3 are satisfied. Indeed, the second condition is clear, and the first one follows since each sine term is bounded by 1 above. Then

$$\begin{aligned}\left|\sum_{k=1}^{N}\left(2-\epsilon_k^2-2\cos\frac{\pi}{N}\right)\frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2}\right| &= \left|\sum_{k=1}^{N} a_k \frac{\sin(k\pi/N)^2}{\sin(\pi/N)^2}\right| \\ &\le \left|\sum_{k=1}^{N} a_k \frac{1}{\sin(\pi/N)^2}\right| \\ &\le \left|N\frac{C'}{N^4}\frac{4N^2}{\pi^2}\right| \\ &< \frac{A}{N}.\end{aligned}$$

□

Example 3. Given $N \in \mathbb{N}$, consider the following sequence:

$$\epsilon_k = \frac{\pi}{(N^3+k)^{1/3}}$$

for $1 \le k \le N$. Then

$$\epsilon_k = \frac{\pi}{N}\left(1+\frac{k}{N^3}\right)^{-1/3} \sim \frac{\pi}{N} - \frac{\pi k}{3N^4} + O\left(\frac{k^2}{N^7}\right).$$

Then Theorem 5 applies and we have the result for this specific choice of ϵ_k.

4. Bifurcations for two dimensional maps

Much of this work was inspired by the recent paper by Astorg, Buff, Dujardin, Peters, and Raissy [1] on bifurcations for a specific map on two dimensions. Let us recall one part of their result. Given the map:

$$F(z,w) = (z + z^2 + az^3 + \frac{\pi^2}{4}w, w - w^2 + w^3) = (f_w(z), g(w))$$

they prove that the following holds: the sequence of maps $F^{\circ 2n+1}(z, g^{\circ n^2}(w))$ converges locally uniformly to the map $(\mathcal{L}_f(z), 0)$. Here $\mathcal{L}_f$ is the Lavaurs map corresponding to the map f where $F(z,0) = (f(z), 0)$.

We see now that by applying the same idea we can prove the following.

Corollary 1. *For the map*

$$H(z,w) = \left(\frac{z}{1-z} + \frac{\pi^2}{4}w, w - w^2 + w^3\right) = (h_w(z), w - w^2 + w^3)$$

the sequence of maps $H^{\circ 2n+1}(z, g^{\circ n^2}(w))$ converges locally uniformly to the map $(z, 0)$. As a consequence, the sequence $(H^{\circ n^2})_{n \ge 0}$ converges locally uniformly to $(\pi_z, 0)$ on $\mathbb{C} \times \mathcal{B}_g$, where π_z is the projection to the first coordinate and $\mathcal{B}_g$ is the parabolic basin of g.

Proof. Note that the w_k term depends only on w_0 (and not on z_0). Denote by ϕ_g the Fatou Coordinate for the map g that conjugates g to a translation by 1 in the attracting basin $\mathcal{B}_g$. Then we obtain $\phi_g(w_k) = \frac{1}{w_k} + o(1) = \phi_g(w_0) + k$. From this it follows that $w_k = \frac{1}{k} + O\left(\frac{1}{k^2}\right)$. Let $w_{n^2} = g^{\circ n^2}(w)$ and $h_j := h_{w_j}$; then:

$$H^{\circ 2n+1}(z, g^{\circ n^2}(w)) = H^{\circ 2n+1}(z, w_{n^2}) = (h_{n^2+2n} \circ \ldots \circ h_{n^2+1} \circ h_{n^2}(z), w_{n^2+2n+1}),$$

where each $h_k(z)$ is as follows:

$$h_k(z) = \frac{z}{1-z} + \frac{\pi^2}{4} w_k = \frac{z}{1-z} + \frac{\pi^2}{4k} + O\left(\frac{1}{k^2}\right).$$

If we rename $f_1 = h_{n^2}, f_2 = h_{n^2+1}, \ldots, f_{2n+1} = h_{n^2+2n}$, then:

$$h_{n^2+2n} \circ \ldots \circ h_{n^2+1} \circ h_{n^2}(z) = f_{2n+1} \circ \ldots \circ f_2 \circ f_1(z)$$

and

$$f_k(z) = h_{n^2+k-1}(z) = \frac{z}{1-z} + \frac{\pi^2}{4(n^2+k-1)} + O\left(\frac{1}{n^4}\right),$$

and we see that this reduces to our Example 1. Indeed, each ϵ_k is precisely chosen to be so that $\epsilon_k^2 = \frac{\pi^2}{4(n^2+k-1)} + O(1/n^4)$. □

Now, we use Example 2 to prove that a similar construction applies when we change the coefficient in front of the w term on the first coordinate.

Corollary 2. *For the map*

$$L(z, w) = \left(\frac{z}{1-z} + \frac{\pi^2}{8} w, w - w^2 + w^3\right) = (l_w(z), w - w^2 + w^3)$$

the sequence of maps $L^{\circ 4n+2}(z, g^{\circ 2n^2}(w))$ converges locally uniformly to the map $(z, 0)$. As a consequence, the sequence $(L^{\circ 2n^2})_{n \geq 0}$ converges locally uniformly to $(\pi_z, 0)$ on $\mathbb{C} \times \mathcal{B}_g$, where π_z is the projection to the first coordinate and $\mathcal{B}_g$ is the parabolic basin of g.

Proof. The proof follows exactly as before. The ϵ_k in this case will be as chosen in (15). □

5. Final remarks and questions

Remark 3. McMullen also studied bifurcations for general maps by focusing on Möbius transformations in [7]. McMullen studied radial and horocyclic perturbations of parabolic maps. These types of perturbations are ones for which no parabolic implosion occurs.

Remark 4. Notice that our starting point for estimates was the estimate in Lemma 3. That lemma holds for more general Chebyshev generalized polynomials. Let us expand a little more here. Suppose we are given the sequence $p_0 = 0, p_1 = 1$ and $p_k = (x + a_k)p_k - \frac{b_k}{b_{k-1}} p_{k-1}$. Then similar estimates (as in Lemma 3) are obtained for this sequence. Notice that the case $b_k = 1$ is the one studied here. However the case of b_k not necessarily equal to 1 also has similar estimates that will allow us to conclude that p_N and U_N are $O(1/N)$ distant from each other. Those more general sequences correspond to more general matrix products, which in principle would allow us to have parabolic bifurcations not only for additive perturbations but also for multiplicative perturbations. We hope to study this case in the near future.

Acknowledgments

The author would like to thank Han Peters for comments on a preliminary version of the paper. Part of this work was done while the author visited Oberwolfach for the workshop Geometric Methods of Complex Analysis in August 2018. Thanks to this institution and the organizers for great working conditions. Many thanks to the referee for very useful remarks and suggestions.

References

[1] Matthieu Astorg, Xavier Buff, Romain Dujardin, Han Peters, and Jasmin Raissy, *A two-dimensional polynomial mapping with a wandering Fatou component*, Ann. of Math. (2) **184** (2016), no. 1, 263–313, DOI 10.4007/annals.2016.184.1.2. MR3505180

[2] Fabrizio Bianchi, *Parabolic implosion for endomorphisms of* $\mathbb{C}^2$, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 12, 3709–3737, DOI 10.4171/JEMS/911. MR4022713

[3] Eric Bedford, John Smillie, and Tetsuo Ueda, *Semi-parabolic bifurcations in complex dimension two*, Comm. Math. Phys. **350** (2017), no. 1, 1–29, DOI 10.1007/s00220-017-2832-y. MR3606468

[4] Adrien Douady, *Does a Julia set depend continuously on the polynomial?*, Complex dynamical systems (Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., vol. 49, Amer. Math. Soc., Providence, RI, 1994, pp. 91–138, DOI 10.1090/psapm/049/1315535. MR1315535

[5] Romain Dujardin and Mikhail Lyubich, *Stability and bifurcations for dissipative polynomial automorphisms of* $\mathbb{C}^2$, Invent. Math. **200** (2015), no. 2, 439–511, DOI 10.1007/s00222-014-0535-y. MR3338008

[6] P. Lavaurs, *Systemes dynamiques holomorphes: explosion de points périodiques paraboliques*, Ph.D. thesis, Paris 11, 1989.

[7] Curtis T. McMullen, *Hausdorff dimension and conformal dynamics. II. Geometrically finite rational maps*, Comment. Math. Helv. **75** (2000), no. 4, 535–593, DOI 10.1007/s000140050140. MR1789177

[8] Paul G. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. **18** (1979), no. 213, v+185, DOI 10.1090/memo/0213. MR519926

[9] Mitsuhiro Shishikura, *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets*, Ann. of Math. (2) **147** (1998), no. 2, 225–267, DOI 10.2307/121009. MR1626737

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210

Email address: vivas@math.osu.edu