

ON THE GLOBAL EXISTENCE FOR THE KURAMOTO-SIVASHINSKY EQUATION

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ABSTRACT. We address the global existence of solutions for the 2D Kuramoto-Sivashinsky equations in a periodic domain $[0, L_1] \times [0, L_2]$ with initial data satisfying $\|u_0\|_{L^2} \leq C^{-1}L_2^{-2}$, where C is a constant. We prove that the global solution exists under the condition $L_2 \leq 1/CL_1^{3/5}$, improving earlier results. The solutions are smooth and decrease energy until they are dominated by $CL_1^{3/2}L_2^{1/2}$, implying the existence of an absorbing ball in L^2 .

1. INTRODUCTION

In this paper, we address the global existence of the 2D periodic Kuramoto-Sivashinsky equation (KSE)

$$\partial_t \phi + \Delta \phi + \Delta^2 \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \quad (1.1)$$

with the initial data $\phi(0) = \phi_0$, in two space dimensions on the domain $\Omega = [0, L_1] \times [0, L_2]$ under the condition $L_2 \leq L_1^q$, where $q > 0$ is a certain exponent. As it is more common, we shall also consider the velocity formulation $u = (u_1, u_2) = \nabla \phi$, which reads

$$\begin{aligned} \partial_t u_1 + \Delta^2 u_1 + \Delta u_1 + u_1 \partial_x u_1 + u_2 \partial_x u_2 &= 0 \\ \partial_t u_2 + \Delta^2 u_2 + \Delta u_2 + u_1 \partial_y u_1 + u_2 \partial_y u_2 &= 0 \\ \partial_y u_1 &= \partial_x u_2, \end{aligned}$$

with the initial data $u(0) = u_0 = \nabla \phi(0)$. When $u_2 = 0$, the system reduces to the well-known 1D Kuramoto-Sivashinsky equation $u_t + u_{xxxx} + u_{xx} + uu_x = 0$ which is typically studied on a periodic domain $[0, L]$.

The Kuramoto-Sivashinsky equation arises in many important physical contexts [Kur,S,T]. In particular, it has been a model for instabilities of flame fronts and ion plasma. It also serves as a model for a flow down an inclined plane in the presence of an electric field. In addition, the KSE also serves as a model for low-dimensional chaos.

The global existence, dissipativity, and the existence of the global attractor for the one dimensional KSE has now been well-established, while in two space dimensions, there is a fundamental difficulty with global existence due to the lack of suitable energy conservation. In one space dimension, the primary problem for the instability, and thus the global existence, is the backward heat term u_{xx} . The first result establishing the global existence for the one-dimensional problem is that of Nicolaenko, Scheurer, and Temam [NST] for odd initial data. In [NST] it was also proven that $\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2} \leq CL^{5/2}$, where L denotes the size of the periodic domain. The oddness assumption was removed by Ilyashenko [I], and subsequently by Goodman [G] and Collet, Eckmann, Epstein, and Stubbe [CEES]. In particular, in [CEES], the authors proved that $\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2} \leq CL^{8/5}$ for arbitrary initial data. Finally, Giacomelli and Otto proved

in [GO], that $\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2} \leq CL^{3/2}$, which is currently the most precise upper bound for the size of the absorbing ball for the 1D KSE. More recently, Otto has found in [O] (cf. also [GJO]), estimates for the space time averages of solutions. For other results on regularity of solutions of the 1D model, see also [BG, BS, GK, M, RK, SS, T, TP, WH]

Considering the laminar flame front model, the two-dimensional KSE appears more physically interesting. However, the global existence in 2D is a long-standing open problem. The main issue mathematically is that the energy dissipation is not available. In addition, the methods treating the KSE as a perturbation of the Burgers equations [I, G, GJO], do not extend to higher space dimensions. There are however still several results available on the global existence of solutions. In [AM], Ambrose and Mazzucato obtained the global existence of solutions for small initial data for L_1 and L_2 less than 2π along with the decay and analyticity of solutions. Sell and Taboada proved in [ST] that the global solutions still exist if one of the scales, say L_2 , is sufficiently small compared to the other and with initial condition smaller than a function of L_1 . Note that the mechanism for the global existence is different here than when both spatial scales are small since the existence is obtained by using oscillations in the vertical direction rather than by damping. The paper [ST] used methods inspired by the work on the 3D Navier-Stokes equations by Raugel and Sell [RS1, RS2, RS3] (cf. also subsequent works [HS, KRZ, KZ, H1, H2]). Further, Molinet [M1, M2] showed local dissipation of the equation on $[0, L_1] \times [0, L_2]$ under the condition $L_2 \leq 1/CL_1^{67/35}$ and obtained a global solution provided also that the initial data is of a certain size as a function of L_1 and L_2 . The size of the domain was extended further in the paper [BKRZ] to $L_2 \leq 1/CL_1^{22/25}$. This last paper contains the best results reached at this point on the two-dimensional domain size relation to global in time solutions. The main result of this paper establishes the global existence under the condition $L_2 \leq 1/CL_1^{3/5}$, improving earlier results; as in [BKRZ], the initial data needs to satisfy the condition

$$\|u_0\|_{L^2} \leq \frac{1}{CL_2^2},$$

which agrees with the condition in [BKRZ]. In addition, we prove that

$$\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2} \leq CL_1^{3/2} L_2^{1/2}.$$

The bound agrees with the one in [GO] if the data is constant in the y direction.

In Sections 2–4, we introduce the approach used in analyzing the two dimensional KSE. In particular, we introduce the projection operators M and N , representing the average in the y -direction and the deviation from the average, respectively. The averaging method has been used in work on the Navier-Stokes system [A, KZ, TZ]. This method is expedient here as much of the inherent difficulties lie with the spatial average, which parallels the one dimensional problem. In Sections 5 and 6 we prove energy inequalities satisfied by the functions Mu_1 , Nu_1 , and $Nu_2 = u_2$ (denoted by M_1 , M_2 , and N_2 , respectively). It is important that the right-hand sides of these inequalities contain a factor L_2^2 . Lemma 6.1 in the next section contains a consequence of these energy inequalities on the increase of these quantities on a given time interval T . Next, in Lemma 6.3, we provide control of the space-time average of $M_1 = Mu_1$. The important aspect of this inequality is that it contains a higher power (4) of the variable M_1 on the left side than is the power of the quantities on the right. The idea of controlling the space-time average of the solution through its space-time L^4 norm is due to Giacomelli and Otto [GO]. A consequence of these preliminary results is Lemma 7.1, in which the time averages of the quantity M_1 are controlled by the time averages on a preceding interval.

Finally, in the last section, we present two barrier arguments, which provide control of the solution for all time. Note that we also obtain the existence of an absorbing ball, which reduces to bounds achieved for the 1D KSE due Giacomelli and Otto when the initial data are independent of y .

2. NOTATION AND THE MAIN THEOREM

Defining $u = \nabla\phi$ and observing that $\phi_{xy} = \phi_{yx}$, and thus also $\frac{1}{2}\nabla(|u|^2) = u \cdot \nabla u$, the KSE (1.1) becomes

$$\begin{aligned}\partial_t u + \Delta^2 u + \Delta u + u \cdot \nabla u &= 0 \\ \operatorname{curl} u &= 0\end{aligned}\tag{2.1}$$

on a periodic domain $\Omega = [0, L_1] \times [0, L_2]$, with the initial condition

$$u(\cdot, 0) = u_0.\tag{2.2}$$

Since the average of $u = (u_1, u_2)$ over Ω is preserved, we normalize it to zero, i.e., we assume

$$\int_{\Omega} u_0(x, y) \, dx dy = 0.\tag{2.3}$$

This in turn implies

$$\int_{\Omega} u(x, y, t) \, dx dy = 0, \quad t \geq 0,\tag{2.4}$$

for as long as the solution exists.

The following is the main result of the paper.

Theorem 2.1. *Let $L_1 \geq L_2 > 0$. There exists $\delta_0 \in (0, 1]$ such that if*

$$L_2 \leq \frac{\delta_0}{\max\{L_1^{3/5}, 1\}}\tag{2.5}$$

and

$$\|u(0)\|_{L^2} \leq \frac{\delta_0}{L_2^2},\tag{2.6}$$

then there exists a solution of u of (2.1) which is global in time. If $L_1 \geq 1$, it satisfies

$$\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2} \leq CL_1^{3/2} L_2^{1/2},\tag{2.7}$$

while if $L_1 \leq 1$, then

$$\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2} \leq CL_1 L_2^{1/2},\tag{2.8}$$

where C is a sufficiently large universal constant.

We emphasize that C and all the implicit constants in the symbol $\lesssim$ are universal; in particular, they do not depend on L_1 or L_2 . They may change from line to line. The numbered constant $C_0, C_1, C_2, \dots$ are also universal, but their value is fixed. We assume that all these constants are greater than 1.

The above theorem, however with a more restrictive assumptions $L_2 \leq \delta_0/L_1^{22/25}$ and $L_1 \geq 2\pi$, was proven in [BKRZ]. Note that when $L_1 \leq 1$, the above theorem extends [AM] since the initial data do not need to have a small L^2 norm, but only satisfy (2.6).

It is well-known that the KSE system is locally well-posed with the initial data in L^2 , which applies here due to the condition (2.6). Also, there exists the maximal time of existence $T_{\max} \in (0, \infty]$ such that the solution is smooth in (x, t) on $\mathbb{R}^2 \times (0, T_{\max})$. In addition, if $T_{\max} < \infty$ the solution satisfies

$\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L^2} = \infty$. Thus, from here on, we perform all the estimates on the time interval $(0, T_{\max})$. In order to obtain the global existence, we only need to prove boundedness of the L^2 norm on the interval $(0, T_{\max})$.

3. PRELIMINARIES

Expanding (2.1) out, we obtain the component-wise formulation

$$\begin{aligned} \partial_t u_1 + \Delta u_1 + \Delta^2 u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 &= 0 \\ \partial_t u_2 + \Delta u_2 + \Delta^2 u_2 + u_1 \partial_x u_2 + u_2 \partial_y u_2 &= 0 \\ \partial_y u_1 &= \partial_x u_2, \end{aligned} \tag{3.1}$$

with the initial condition $u(\cdot, 0) = u_0$.

Consider the average of a function $f \in L^1(\Omega)$ in the y direction

$$M(f) = \frac{1}{L_2} \int_0^{L_2} f(x, y, t) dy,$$

and the difference between the function and this average

$$N(f) = f(x, y, t) - M(f).$$

Observe that

$$M(u_2) = 0, \tag{3.2}$$

for all $t \geq 0$ such that the solution exists. This property results from applying the operator M to $\partial_y u_1 = \partial_x u_2$ which shows that $M(u_2)$ is constant in x . The constant then vanishes by (2.4).

For simplicity, denote by

$$M_1(x, t) = M(u_1) = \frac{1}{L_2} \int_0^{L_2} u_1(x, y, t) dy \tag{3.3}$$

the average of u_1 in the y direction, and

$$N_1(x, y, t) = u_1(x, y, t) - M_1(x, t)$$

its deviation from the average. Clearly, $u_1 = M_1 + N_1$. Analogously, we denote

$$N_2(x, y, t) = u_2(x, y, t), \tag{3.4}$$

since the average of u_2 in the y direction vanishes by (3.2).

4. PROPERTIES OF THE AVERAGE

First, we recall several projection identities involving the operators M and N .

Lemma 4.1. *The operators M and N satisfy the identities*

- (i) $M(M(f)) = M(f)$,
 - (ii) $M(N(f)) = 0$,
 - (iii) $N(M(f)) = 0$, and
 - (iv) $N(N(f)) = N(f)$,
- for all $f \in L^1(\Omega)$.

The identities above imply the product rules

$$M(fg) = MfMg + M(NfNg) \quad (4.1)$$

and

$$N(fg) = MfNg + NfMg + N(NfNg), \quad (4.2)$$

which hold for all sufficiently regular functions f and g . Both identities are obtained by writing $f = Mf + Nf$ and $g = Mg + Ng$ and using $M(M(f)N(g)) = M(f)M(N(g)) = 0$ and $M(N(f)M(g)) = M(g)M(N(f)) = 0$, as well as $N(M(f)M(g)) = 0$.

Proof of Lemma 4.1. The part (i) is immediate from the definition. For (ii), we have

$$M(N(f)) = M(f - M(f)) = M(f) - M(M(f)) = M(f) - M(f) = 0.$$

Similarly,

$$N(M(f)) = M(f) - M(M(f)) = M(f) - M(f) = 0$$

providing (iii), and

$$N(N(f)) = N(f - M(f)) = f - M(f) - (M(f) - M(M(f))) = f - M(f) = N(f),$$

establishing (iv). $\square$

Next, we rewrite the first equation in (3.1) in terms of M_1 and N_1 .

Lemma 4.2. *The average M_1 satisfies*

$$\partial_t M_1 + \partial_{xxxx} M_1 + \partial_{xx} M_1 + M_1 \partial_x M_1 + M(N_1 \partial_x N_1) + M(N_2 \partial_x N_2) = 0. \quad (4.3)$$

Observe that the equation for M_1 is a perturbation of the 1D KSE.

Proof of Lemma 4.2. Applying an average in y , and recalling that $u_1 = M_1 + N_1$, we get

$$\begin{aligned} & M(\partial_t u_1 + \Delta^2 u_1 + \Delta u_1 + u_1 \partial_x u_1 + u_2 \partial_x u_2) \\ &= \partial_t M_1 + \partial_{xxxx} M_1 + \partial_{xx} M_1 + M(u_1 \partial_x u_1) + M(u_2 \partial_x u_2) \\ &= \partial_t M_1 + \partial_{xxxx} M_1 + \partial_{xx} M_1 + M_1 \partial_x M_1 + M(N_1 \partial_x N_1) + M(N_2 \partial_x N_2), \end{aligned}$$

where in the last equality we used (3.4) and the product rule (4.1). $\square$

5. ENERGY ESTIMATES

Now we turn to the energy estimates. Throughout the paper, we abbreviate $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$.

Lemma 5.1. *The energy inequalities for the quantities M_1 , N_1 , and N_2 read*

- (i) $\frac{1}{2} \frac{d}{dt} \|M_1\|^2 + \|\partial_{xx} M_1\|^2 - \|\partial_x M_1\|^2 \lesssim L_2^2 (\|\Delta N_1\|^2 + \|\Delta N_2\|^2) \|M_1\|,$
- (ii) $\frac{1}{2} \frac{d}{dt} \|N_1\|^2 + \|\Delta N_1\|^2 - \|\nabla N_1\|^2 \lesssim L_2^2 (\|M_1\| + \|N_2\|) \|\Delta N_1\|^2 + L_2^2 \|N_2\| \|\Delta N_2\|^2,$ and
- (iii) $\frac{1}{2} \frac{d}{dt} \|N_2\|^2 + \|\Delta N_2\|^2 - \|\nabla N_2\|^2 \lesssim L_2^2 (\|M_1\| + \|N_1\|) \|\Delta N_2\|^2,$

where all the constants are universal.

Proof of Lemma 5.1. For the first energy inequality (i), we multiply (4.3) by M_1 and integrate over Ω , obtaining

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|M_1\|^2 + \|\partial_{xx} M_1\|^2 - \|\partial_x M_1\|^2 &= - \int_{\Omega} (M(N_1 \partial_x N_1) + M(N_2 \partial_x N_2)) M_1 \\ &= - \int_{\Omega} N_1 \partial_x N_1 M_1 - \int_{\Omega} N_2 \partial_x N_2 M_1 \\ &\leq \int_0^{L_1} \|N_1\|_{L_y^2[0, L_2]} \|\partial_x N_1\|_{L_y^2[0, L_2]} |M_1| dx + \int_0^{L_1} \|N_2\|_{L_y^2[0, L_2]} \|\partial_x N_2\|_{L_y^2[0, L_2]} |M_1| dx, \end{aligned} \quad (5.1)$$

where we applied the Cauchy-Schwarz inequality in the y variable. For the first term, we have

$$\begin{aligned} \int_0^{L_1} \|N_1\|_{L_y^2[0, L_2]} \|\partial_x N_1\|_{L_y^2[0, L_2]} |M_1| dx &\lesssim \|N_1\| \|\partial_x N_1\|_{L_x^\infty L_y^2} \|M_1\|_{L^2[0, L_1]} \\ &\lesssim \|N_1\| \|\partial_x N_1\|_{L_y^2 L_x^\infty} \|M_1\|_{L_x^2[0, L_1]} \lesssim \|N_1\| \|\partial_x N_1\|^{1/2} \|\partial_{xx} N_1\|^{1/2} \frac{1}{L_2^{1/2}} \|M_1\| \\ &\lesssim L_2^2 \|\partial_{yy} N_1\| L_2^{1/2} \|\partial_{xy} N_1\|^{1/2} \|\partial_{xx} N_1\|^{1/2} \frac{1}{L_2^{1/2}} \|M_1\| \lesssim L_2^2 \|\Delta N_1\|^2 \|M_1\|, \end{aligned} \quad (5.2)$$

where we applied Hölder's inequality in y in the first step, Agmon's inequality in x in the third, and the Poincaré inequality in y in the fourth. The last inequality in (5.2) follows from $\|v_{xx}\|^2 + 2\|v_{xy}\|^2 + \|v_{yy}\|^2 = \|\Delta v\|^2$, for all periodic $v \in H^2(\Omega)$. We emphasize that all constants in the symbol $\lesssim$ are universal, in particular independent of L_1 and L_2 .

The second term on the far right side of (5.1) is treated analogously.

For (ii), we start by applying N to the first equation in (3.1), then multiply by N_1 , and integrate over Ω . This yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} N_1^2 + \int_{\Omega} (\Delta N_1)^2 - \int_{\Omega} |\nabla N_1|^2 = - \int_{\Omega} N(u_1 \partial_x u_1) N_1 - \int_{\Omega} N(u_2 \partial_x u_2) N_1, \quad (5.3)$$

since N commutes with ∂_x and ∂_y . In order to complete the proof of (ii), we only need to estimate the nonlinear components. Using the product rule (4.2) and Lemma 4.1 (iv), the nonlinear term in (5.3) is rewritten as

$$\begin{aligned} & - \int_{\Omega} N(u_1 \partial_x u_1) N_1 - \int_{\Omega} N(u_2 \partial_x u_2) N_1 \\ &= - \int_{\Omega} M_1 \partial_x N_1 N_1 - \int_{\Omega} N_1 \partial_x M_1 N_1 - \int_{\Omega} N_1 \partial_x N_1 N_1 - \int_{\Omega} N_2 \partial_x N_2 N_1 \\ &= - \int_{\Omega} M_1 \partial_x N_1 N_1 + 2 \int_{\Omega} M_1 N_1 \partial_x N_1 - \int_{\Omega} N_1^2 \partial_x N_1 - \int_{\Omega} N_1 N_2 \partial_x N_2 \\ &= \int_{\Omega} M_1 N_1 \partial_x N_1 - \int_{\Omega} N_1 N_2 \partial_x N_2. \end{aligned} \quad (5.4)$$

We claim that the last expression is, up to a constant, less than or equal to

$$L_2^2 \|M_1\| \|\Delta N_1\|^2 + L_2^2 \|N_2\| \|\Delta N_1\| \|\Delta N_2\|.$$

Note that the first term on the far right side of (5.4) is identical to the term estimated in (5.2). For the second term in (5.4), we have

$$\begin{aligned} - \int_{\Omega} N_1 N_2 \partial_x N_2 &\leq \int_0^{L_1} \|N_1\|_{L_y^2[0, L_2]} \|N_2\|_{L_y^2[0, L_2]} \|\partial_x N_2\|_{L_y^\infty[0, L_2]} dx \\ &\lesssim \|N_1\|_{L_x^\infty L_y^2} \|N_2\| \|\partial_x N_2\|_{L_x^2 L_y^\infty} \lesssim \|N_1\|_{L_y^2 L_x^\infty} \|N_2\| \|\partial_x N_2\|_{L_x^2 L_y^\infty} \\ &\lesssim \|N_1\|^{1/2} \|\partial_x N_1\|^{1/2} \|N_2\| \|\partial_x N_2\|^{1/2} \|\partial_{xx} N_2\|^{1/2} \lesssim L_2^2 \|N_2\| \|\Delta N_1\| \|\Delta N_2\|, \end{aligned} \quad (5.5)$$

where we used $\int_0^{L_2} N f dy = 0$ for every $f \in L^1(\Omega)$ in the fourth inequality, by Lemma 4.1 (ii).

For (iii), we observe that $u_2 = N_2$, multiply the second equation in (3.1) by N_2 and integrate over Ω . We obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} N_2^2 + \int_{\Omega} (\Delta N_2)^2 - \int_{\Omega} |\nabla N_2|^2 = - \int_{\Omega} N(u_1 \partial_y u_1) N_2 - \int_{\Omega} N(u_2 \partial_y u_2) N_2, \quad (5.6)$$

where we also used $\partial_x u_2 = \partial_y u_1$. The nonlinear term in (5.6) is rewritten as

$$\begin{aligned} - \int_{\Omega} u_1 \partial_y u_1 N_2 - \int_{\Omega} u_2 \partial_y u_2 N_2 &= - \int_{\Omega} u_1 \partial_y N_1 N_2 - \int_{\Omega} N_2 \partial_y N_2 N_2 \\ &= - \int_{\Omega} u_1 \partial_y N_1 N_2 = \int_{\Omega} u_1 \partial_x N_2 N_2 = \int_{\Omega} (M_1 + N_1) \partial_x N_2 N_2 \\ &\lesssim L_2^2 (\|M_1\| + \|N_1\|) \|\Delta N_2\|^2, \end{aligned}$$

where the final inequality is obtained as in (5.2) and (5.5). $\square$

6. SPACE-TIME ENERGY ESTIMATES

We require control of the L^2 norms in space-time of various quantities involving M_1 , N_1 , and N_2 , which are then used in the barrier arguments in the next section.

Lemma 6.1. *For $s \geq 0$, denote*

$$\eta(t) = L_2^2 (\|M_1(t)\| + \|N_1(t)\| + \|N_2(t)\|). \quad (6.1)$$

Let $T \in [0, 1]$, and assume that for some $t_0 \geq 0$

$$\eta(t) \leq \frac{1}{C_0}, \quad t \in [t_0, t_0 + T], \quad (6.2)$$

where $C_0 \geq 2$ is sufficiently large universal constant. Then the quantity

$$A(t) = \|M_1(t)\|^2 + \|N_1(t)\|^2 + \|N_2(t)\|^2$$

satisfies

$$A'(t) \leq 2A(t), \quad t \in [t_0, t_0 + T]. \quad (6.3)$$

Also, we have

$$\begin{aligned} \sup_{t \in [t_0, t_0 + T]} \|M_1\|^2, \sup_{t \in [t_0, t_0 + T]} \|N_1\|^2, \sup_{t \in [t_0, t_0 + T]} \|N_2\|^2, \int_{t_0}^{t_0 + T} \|\partial_{xx} M_1\|^2, \int_{t_0}^{t_0 + T} \|\partial_x M_1\|^2, \\ \int_{t_0}^{t_0 + T} \|\Delta N_1\|^2, \int_{t_0}^{t_0 + T} \|\Delta N_2\|^2, \int_{t_0}^{t_0 + T} \|\nabla N_1\|^2, \int_{t_0}^{t_0 + T} \|\nabla N\|^2 \lesssim A(t_0) \end{aligned} \quad (6.4)$$

and

$$\int_{t_0}^{t_0+T} \int_{\Omega} |\partial_x M_1|^3 \lesssim \frac{1}{L_2^{1/2}} A(t_0)^{3/2}, \quad (6.5)$$

where the constants in (6.4) and (6.5) depend on T .

Proof of Lemma 6.1. Summing the three inequalities in Lemma 5.1, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|M_1\|^2 + \|N_1\|^2 + \|N_2\|^2) + \|\partial_{xx} M_1\|^2 \\ & \quad + (1 - CL_2^2(\|M_1\| + \|N_1\| + \|N_2\|)) (\|\Delta N_1\|^2 + \|\Delta N_2\|^2) \\ & \lesssim \|\partial_x M_1\|^2 + \|\nabla N_1\|^2 + \|\nabla N_2\|^2. \end{aligned} \quad (6.6)$$

Integrating by parts and using Young's inequality on the first derivatives, we find

$$\|\partial_x M_1\|^2 \leq \frac{1}{2} \|\partial_{xx} M_1\|^2 + \frac{1}{2} \|M_1\|^2$$

and

$$\|\nabla N_j\|^2 \leq \frac{1}{2} (1 - \eta) \|\Delta N_j\|^2 + \frac{1}{2(1 - \eta)} \|N_j\|^2, \quad j = 1, 2,$$

where $\eta(t)$ is defined in (6.1). Assuming (6.2) with $C \geq 2$ sufficiently large, we obtain from (6.6) that

$$\begin{aligned} & \frac{d}{dt} (\|M_1\|^2 + \|N_1\|^2 + \|N_2\|^2) + \|\partial_{xx} M_1\|^2 + \|\Delta N_1\|^2 + \|\Delta N_2\|^2 \\ & \leq 2(\|M_1\|^2 + \|N_1\|^2 + \|N_2\|^2), \end{aligned} \quad (6.7)$$

and the inequality (6.3) follows. In order to obtain (6.4), we then use (6.3) and (6.7).

It remains to establish (6.5). By the 1D Gagliardo-Nirenberg inequality we have

$$\int_{\Omega} |\partial_x M_1|^3 \lesssim \frac{1}{L_2^{1/2}} \|\partial_x M_1\|^{5/2} \|\partial_{xx} M_1\|^{1/2} \lesssim \frac{1}{L_2^{1/2}} \|M_1\| \|\partial_x M_1\|^{1/2} \|\partial_{xx} M_1\|^{3/2},$$

where the factor $L_2^{-1/2}$ results from the integration in the y variable. Therefore, integrating in time and applying Hölder's inequality, we obtain

$$\int_{t_0}^{t_0+T} \int_{\Omega} |\partial_x M_1|^3 \lesssim \frac{1}{L_2^{1/2}} \sup_{t \in (0, T)} \|M_1\| \left(\int_{t_0}^{t_0+T} \|\partial_x M_1\|^2 \right)^{1/4} \left(\int_{t_0}^{t_0+T} \|\partial_{xx} M_1\|^2 \right)^{3/4},$$

which by (6.4) implies (6.5). $\square$

Next, inspired by [GO], we express the space-time integral of M_1^4 in terms of quantities in Lemma 6.1. This in turn provides a means to bound $\|M_1\|^2$. We first consider the x -integral of (4.3) and thus denote

$$h(x, t) = \int_0^x M_1(\tilde{x}, t) d\tilde{x}.$$

By (2.4) and (3.3), we have

$$\int_0^{L_1} h dx = 0. \quad (6.8)$$

Integrating (4.3) in x , we obtain that the function h satisfies the equation

$$\partial_t h + \partial_{xxx} h + \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + \frac{1}{2} M(N_1^2 + N_2^2) + g(t) = 0, \quad (6.9)$$

where

$$g(t) = -\frac{1}{2L_1L_2} \int_{\Omega} (M_1^2 + N_1^2 + N_2^2). \quad (6.10)$$

The equation (6.10) follows by integrating (6.9) in x over $[0, L_1]$ and then using (6.8) and $\partial_x h = M_1$.

Lemma 6.2. *Under the assumptions of Lemma 6.1, i.e., assuming (6.2) for some $t_0 \geq 0$ and $T \in [0, 1]$, the function $h(x, t)$ satisfies*

$$\sup_{t \in [t_0, t_0+T]} \sup_x |h(x, t)| \lesssim \left(\frac{L_1}{L_2} \right)^{1/2} A(t_0)^{1/2}. \quad (6.11)$$

Proof of Lemma 6.2. For all $t \in [t_0, t_0 + T]$, we have

$$|h| \leq \int_0^{L_1} |\partial_x h| = \int_0^{L_1} |M_1| \leq L_1^{1/2} \|M_1\|_{L^2[0, L_1]} \leq \left(\frac{L_1}{L_2} \right)^{1/2} \|M_1\|,$$

which implies (6.11) by (6.4) in Lemma 6.1. $\square$

Next, we obtain the following integral identity for M_1 .

Lemma 6.3. *For the average M_1 , we have an identity*

$$\begin{aligned} \frac{1}{12} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^4 &= - \int_{t_0}^{t_0+T} \int_{\Omega} (\partial_x M_1)^3 + \int_{t_0}^{t_0+T} \int_{\Omega} ((\partial_{xx} M_1)^2 - (\partial_x M_1)^2) h \\ &+ \frac{1}{2} \int_{\Omega} (M_1(t_0 + T)^2 h(t_0 + T) - M_1(t_0)^2 h(t_0)) + \frac{1}{2} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^2 g(t) \\ &+ \frac{1}{4} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^2 (N_1^2 + N_2^2) + \int_{t_0}^{t_0+T} \int_{\Omega} (N_1 \partial_x N_1 + N_2 \partial_x N_2) M_1 h. \end{aligned} \quad (6.12)$$

Proof of Lemma 6.3. Multiplying (4.3) by $M_1 h$ and integrating over Ω , we obtain

$$\begin{aligned} \int_{\Omega} M_1 h \partial_t M_1 + \int_{\Omega} M_1 h \partial_{xxxx} M_1 + \int_{\Omega} M_1 h \partial_{xx} M_1 \\ + \int_{\Omega} M_1^2 h \partial_x M_1 + \int_{\Omega} M_1 (N_1 \partial_x N_1 + N_2 \partial_x N_2) M_1 h = 0. \end{aligned} \quad (6.13)$$

For the first term in (6.13), we have

$$\begin{aligned} \int_{\Omega} M_1 h \partial_t M_1 &= \frac{1}{2} \int_{\Omega} \partial_t (M_1^2 h) - \frac{1}{2} \int_{\Omega} M_1^2 \partial_t h \\ &= \frac{1}{2} \int_{\Omega} \partial_t (M_1^2 h) + \frac{1}{2} \int_{\Omega} M_1^2 (\partial_{xxx} M_1 + \partial_x M_1) \\ &\quad + \frac{1}{4} \int_{\Omega} (M_1^4 + M_1^2 M (N_1^2 + N_2^2)) + \frac{1}{2} \int_{\Omega} M_1^2 g(t) \\ &= \frac{1}{2} \int_{\Omega} \partial_t (M_1^2 h) - \int_{\Omega} M_1 \partial_x M_1 \partial_{xx} M_1 + \frac{1}{4} \int_{\Omega} M_1^4 \\ &\quad + \frac{1}{4} \int_{\Omega} M_1^2 (N_1^2 + N_2^2) + \frac{1}{2} \int_{\Omega} M_1^2 g(t), \end{aligned} \quad (6.14)$$

where we used (6.9), periodicity of M_1 , and $M(M_1^2) = M_1^2$. For the second term in (6.13), we have

$$\begin{aligned} \int_{\Omega} M_1 h \partial_{xxxx} M_1 &= - \int_{\Omega} \partial_x M_1 \partial_{xxx} M_1 h - \int_{\Omega} M_1^2 \partial_{xxx} M_1 \\ &= \int_{\Omega} (\partial_{xx} M_1)^2 h + \int_{\Omega} M_1 \partial_x M_1 \partial_{xx} M_1 + 2 \int_{\Omega} M_1 \partial_x M_1 \partial_{xx} M_1 \\ &= \int_{\Omega} (\partial_{xx} M_1)^2 h - \frac{3}{2} \int_{\Omega} (\partial_x M_1)^2, \end{aligned}$$

while for the third term in (6.13) we write

$$\int_{\Omega} M_1 h \partial_{xx} M_1 = - \int_{\Omega} (\partial_x M_1)^2 h - \int_{\Omega} M_1^2 \partial_x M_1 = - \int_{\Omega} (\partial_x M_1)^2 h.$$

For the fourth term in (6.13), we integrate by parts obtaining

$$\int_{\Omega} M_1^2 \partial_x M_1 h = - \frac{1}{3} \int_{\Omega} M_1^4. \quad (6.15)$$

Using (6.14)–(6.15), combined with $\int_{\Omega} M_1 \partial_x M_1 \partial_{xx} M_1 = -\frac{1}{2} \int_{\Omega} (\partial_x M_1)^3$ for the second term in (6.14), yields

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} \partial_t (M_1^2 h) + \left(\frac{1}{2} - \frac{3}{2} \right) \int_{\Omega} (\partial_x M_1)^3 + \left(\frac{1}{4} - \frac{1}{3} \right) \int_{\Omega} M_1^4 \\ &+ \frac{1}{4} \int_{\Omega} M_1^2 (N_1^2 + N_2^2) + \frac{1}{2} \int_{\Omega} M_1^2 g(t) \\ &+ \int_{\Omega} (\partial_{xx} M_1)^2 h - \int_{\Omega} (\partial_x M_1)^2 h + \int_{\Omega} M (N_1 \partial_x N_1 + N_2 \partial_x N_2) M_1 h = 0. \end{aligned} \quad (6.16)$$

Since $M(M_1 h) = M_1 h$, the last term equals $\int_{\Omega} (N_1 \partial_x N_1 + N_2 \partial_x N_2) M_1 h$. The argument is then completed by integrating (6.16) in t and combining fractions. $\square$

7. ESTIMATE ON M_1

Here we apply the energy inequalities in Lemma 6.1 and the integral identity (6.12) to estimate the space-time integral of M_1^2 . Suppose that (6.2) holds for some $t_0 \geq 0$ and $T \in [0, 1]$. The nonlinear components of the identity (6.12) are bounded as

$$\begin{aligned} \int_{t_0}^{t_0+T} \int_{\Omega} (N_1 \partial_x N_1 + N_2 \partial_x N_2) M_1 h &\lesssim \sup_{t \in [t_0, t_0+T]} \sup_x |h| \int_{t_0}^{t_0+T} \int_{\Omega} |N_1 \partial_x N_1 + N_2 \partial_x N_2| |M_1| \\ &\lesssim L_2^2 \left(\sup_{t \in [t_0, t_0+T]} \sup_x |h| \right) \sup_{t \in [t_0, t_0+T]} \|M_1\| \int_{t_0}^{t_0+T} (\|\Delta N_1\|^2 + \|\Delta N_2\|^2) \\ &\lesssim L_1^{1/2} L_2^{3/2} A(t_0)^{1/2} \sup_{t \in [t_0, t_0+T]} \|M_1\| \int_{t_0}^{t_0+T} (\|\Delta N_1\|^2 + \|\Delta N_2\|^2), \end{aligned}$$

where the first inequality is obtained as in (5.2), while the last follows from (6.11).

For the fifth term on the right-hand side of (6.12), we use the Cauchy-Schwarz inequality

$$\frac{1}{4} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^2 N_1^2 \leq \frac{1}{48} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^4 + 3 \int_{t_0}^{t_0+T} \int_{\Omega} N_1^4,$$

and for the L^4 norm of N_1 , we have

$$\begin{aligned} \int_{t_0}^{t_0+T} \int_{\Omega} N_1^4 &\lesssim \int_{t_0}^{t_0+T} \|N_1\|^2 \|\nabla N_1\|^2 \leq \sup_t \|N_1\|^2 \int_{t_0}^{t_0+T} \|\nabla N_1\|^2 \\ &\lesssim L_2^2 A(t_0) \int_{t_0}^{t_0+T} \|\Delta N_1\|^2 \lesssim L_2^2 A(t_0)^2. \end{aligned}$$

In the first inequality, we used

$$\|N(f)\|_{L^p(\Omega)} \lesssim p^{1/2} \|N(f)\|^{2/p} \|\nabla N(f)\|^{1-2/p}, \quad 2 \leq p < \infty$$

similarly to [KZ, Lemma 2]. (This is obtained as in [TZ] by the stacking principle and the 2D Gagliardo-Nirenberg inequality.) The argument for N_2 is the same. Thus, for the nonlinear terms in (6.12) we have

$$\int_{t_0}^{t_0+T} \int_{\Omega} (N_1 \partial_x N_1 + N_2 \partial_x N_2) M_1 h \lesssim L_1^{1/2} L_2^{3/2} A(t_0)^2 \quad (7.1)$$

and

$$\frac{1}{4} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^2 (N_1^2 + N_2^2) \leq C L_2^2 A(t_0)^2 + \frac{1}{24} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^4. \quad (7.2)$$

Using that $\int_{\Omega} M_1^2 g(t) \leq 0$ in (6.12) along with (7.1)–(7.2), we get

$$\begin{aligned} \frac{1}{24} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^4 &\lesssim \int_{t_0}^{t_0+T} \int_{\Omega} |\partial_x M_1|^3 + \sup_{t \in [t_0, t_0+T]} \sup_x |h(x, t)| \int_{t_0}^{t_0+T} \int_{\Omega} ((\partial_{xx} M_1)^2 + (\partial_x M_1)^2) \\ &\quad + \sup_{t \in [t_0, t_0+T]} \sup_x |h(x, t)| \int_{\Omega} (M_1^2(T) + M_1^2(0)) + L_1^{1/2} L_2^{3/2} A(t_0)^2 + L_2^2 A(t_0)^2 \\ &\lesssim \frac{1}{L_2^{1/2}} A(t_0)^{3/2} + \left(\frac{L_1}{L_2}\right)^{1/2} A(t_0)^{3/2} + L_1^{1/2} L_2^{3/2} A(t_0)^2 + L_2^2 A(t_0)^2 \\ &\lesssim \left(\frac{\max\{1, L_1\}}{L_2}\right)^{1/2} A(t_0)^{3/2} + (L_1^{1/2} L_2^{3/2} + L_2^2) A(t_0)^2, \end{aligned}$$

where we used (6.11) and Lemma 6.1 in the second inequality. Thus, by the Cauchy-Schwarz inequality, we obtain an estimate for the time integral of $\|M_1\|^2$, which reads

$$\begin{aligned} \int_{t_0}^{t_0+T} \int_{\Omega} M_1^2 &\leq L_1^{1/2} L_2^{1/2} \left(\int_{t_0}^{t_0+T} \int_{\Omega} M_1^4 \right)^{1/2} \\ &\lesssim L_1^{1/2} L_2^{1/2} \left(\left(\frac{\max\{1, L_1\}}{L_2}\right)^{1/2} A(t_0)^{3/2} + (L_1^{1/2} L_2^{3/2} + L_2^2) A(t_0)^2 \right)^{1/2} \\ &\lesssim \max\{L_1^{1/2}, L_1^{3/4}\} L_2^{1/4} A(t_0)^{3/4} + (L_1^{3/4} L_2^{5/4} + L_1^{1/2} L_2^{3/2}) A(t_0), \end{aligned} \quad (7.3)$$

recalling that $T \leq 1$.

Now, consider a solution which is defined on a time interval $[T_0 - 1, T_0 + 1]$, where $T_0 \geq 1$, and satisfies

$$\eta(t) \leq \frac{1}{2C_0}, \quad t \in [T_0 - 1, T_0 + 1].$$

By Lemma 6.1, we have

$$\int_t^{t+T} \int_{\Omega} (N_1^2 + N_2^2) \lesssim L_2^4 \int_t^{t+T} (\|\Delta N_1\|^2 + \|\Delta N_2\|^2) \lesssim L_2^4 A(t), \quad (7.4)$$

for $t \in [T_0 - 1, T]$ and $T \in [0, 1]$. Summing (7.3) with $t_0 = t$ and (7.4), we get

$$\int_t^{t+T} A(s) ds \lesssim \max\{L_1^{1/2}, L_1^{3/4}\} L_2^{1/4} A(t)^{3/4} + (L_1^{3/4} L_2^{5/4} + L_1^{1/2} L_2^{3/2} + L_2^4) A(t) \quad (7.5)$$

for $t \in [T_0 - 1, T_0 + 1]$ and $T \in [0, 1]$. In addition, if $t \geq 1$, Lemma 6.1 implies

$$A(t) \lesssim \int_{t-1}^t A(s) ds \quad (7.6)$$

since $A(t) \lesssim A(t-s)$ for $s \in (0, 1)$. The inequalities (7.5) and (7.6) then imply for $t \geq 1$

$$\begin{aligned} \int_t^{t+1} A(s) ds &\leq C_1 \max\{L_1^{1/2}, L_1^{3/4}\} L_2^{1/4} \left(\int_{t-1}^t A(s) ds \right)^{3/4} \\ &\quad + C_1 (L_1^{3/4} L_2^{5/4} + L_1^{1/2} L_2^{3/2} + L_2^4) \int_{t-1}^t A(s) ds, \end{aligned} \quad (7.7)$$

where $C_1 \geq 1$ is a fixed constant. The inequality (7.7) and the assumption (2.5) lead to the following lemma.

Lemma 7.1. *For every $\delta \in (0, 1/2]$, there exist $\delta_0 \in (0, 1]$ and a sufficiently large constant $C \geq 1$ with the following property: If (6.2) holds for $s \in [t-1, t+1]$, where $t \geq 1$, and if we have (2.5), then*

$$\int_t^{t+1} A(s) ds \leq C \max\{L_1^2, L_1^3\} L_2 + \delta \int_{t-1}^t A(s) ds \quad (7.8)$$

holds.

Note that at this stage, we are assuming the condition (6.2). The validity of this condition shall be established in the next section, alongside with the bound on $A(t)$.

Proof of Lemma 7.1. Applying Young's inequality on the first term in (7.7), we get

$$C_1 \max\{L_1^{1/2}, L_1^{3/4}\} L_2^{1/4} \left(\int_{t-1}^t A(s) ds \right)^{3/4} \leq \frac{2C_1^4}{\delta^3} \max\{L_1^2, L_1^3\} L_2 + \frac{\delta}{2} \int_{t-1}^t A(s) ds. \quad (7.9)$$

Also, by (2.5), we have for the second term in (7.7)

$$C_1 (L_1^{3/4} L_2^{5/4} + L_1^{1/2} L_2^{3/2} + L_2^4) \int_{t-1}^t A(s) ds \leq C_1 (\delta_0^{5/4} + \delta_0^{3/2} + \delta_0^4) \int_{t-1}^t A(s) ds. \quad (7.10)$$

Now, we restrict δ_0 so that

$$C_1 (\delta_0^{5/4} + \delta_0^{3/2} + \delta_0^4) \leq \frac{\delta}{2}. \quad (7.11)$$

Replacing the inequalities (7.9) and (7.10) in (7.7), with the help of (7.11), then gives (7.8). $\square$

8. TWO BARRIER ARGUMENTS

Let u be a solution on an interval $[0, T_{\max})$ as at the end of Section 2. Recalling the notation (6.1) for η and (8.5) for A , note that we need to prove that u satisfies the condition (6.2) in Lemma 6.1. If δ_0 is sufficiently small, we shall prove this inductively starting with the interval $[0, 2]$, stated next. The same lemma also shall be used to proceed from an interval $[t_0 + 1, t_0 + 2]$ to $[t_0, t_0 + 1]$, for $t_0 \in \mathbb{N}$.

Lemma 8.1. *Under the condition (2.5), if we have*

$$A(t_0) \leq \frac{\epsilon_0^2}{L_2^4} \quad (8.1)$$

for some sufficiently small $\epsilon_0 \in (0, 1]$ and some $t_0 \in [0, T_{\max})$, then $t_0 + 2 \leq T_{\max}$ and

$$A(t) \leq \frac{C_2 \epsilon_0^2}{L_2^4}, \quad (8.2)$$

for $t \in [t_0, t_0 + 2]$, where C_2 is a constant.

Observe that, by the discrete Cauchy-Schwarz inequality, we have

$$A(t) \leq \frac{\eta(t)^2}{L_2^4} \leq 3A(t), \quad t \in [0, T_{\max}), \quad (8.3)$$

for all $t \in [0, T_{\max})$.

Proof of Lemma 8.1. Let $\epsilon_0 \in (0, 1]$ be such that

$$3e^2 \epsilon_0^2 \leq \frac{1}{2C_0}. \quad (8.4)$$

By (8.1), we have

$$T = \sup \left\{ t_0 \leq t \leq \min\{T_{\max}, t_0 + 2\} : A(t) \leq \frac{e^4 \epsilon_0^2}{L_0^4} \right\} > t_0.$$

By the definition of T and (8.3), we have

$$\eta(t) \leq \sqrt{3}A(t)^{1/2}L_2^2 \leq 3e^2 \epsilon_0 \leq \frac{1}{2C_0}, \quad t \in [t_0, T],$$

where the last inequality holds by (8.4). Therefore, Lemma 4.1 applies on the interval $[t_0, T]$, and using (6.3) and (8.1), we get

$$A(t) \leq \frac{\epsilon_0^2}{L_0^4} e^{2t}. \quad (8.5)$$

Comparing this with the definition of T , we get that $T = 2$, and (8.2) is proven with $C_2 = e^4$. $\square$

Now, we proceed to the second barrier argument, leading to the conclusion of the proof of the main theorem.

Proof of Theorem 2.1. By the condition (2.6), we have

$$A(0) \leq \frac{3\delta_0^2}{L_0^4},$$

and thus, if δ_0 is sufficiently small, we may use Lemma 8.1 with $t_0 = 0$ and obtain

$$A(t) \leq \frac{3C_2 \delta_0^2}{L_2^4}, \quad t \in [0, 2]. \quad (8.6)$$

We proceed inductively with the inductive assumption

$$A(t) \leq \frac{3C_2 \delta_0^2}{L_2^4}, \quad t \in [0, t_0 + 1], \quad (8.7)$$

for some $t_0 \in \mathbb{N}$. By (8.6), this holds for $t_0 = 1$. Using Lemma 8.1, we get

$$A(t) \leq \frac{3C_2^2 \delta_0^2}{L_2^4}, \quad t \in [0, t_0 + 2],$$

and then, by (8.3),

$$\eta(t) \leq 3C_2\delta_0, \quad t \in [0, t_0 + 2]. \quad (8.8)$$

If $\delta_0 > 0$ is a sufficiently small constant (in particular, independent of t_0), the condition (6.2) is satisfied on $[t_0, t_0 + 2]$. Thus Lemma 7.1 applies and gives

$$\int_{t_0}^{t_0+1} A(s) ds \leq C \max\{L_1^2, L_1^3\} L_2 + \delta \frac{3C_2^2 \delta_0^2}{L_2^4},$$

where $\delta > 0$ is a small constant to be determined. By the mean value theorem, there exists $s_0 \in [t_0, t_0 + 1]$ such that

$$\begin{aligned} A(s_0) &= \int_{t_0}^{t_0+1} A(s) ds \leq C \max\{L_1^2, L_1^3\} L_2 + \delta \frac{3C_2^2 \delta_0^2}{L_2^4} \\ &\leq \frac{C\delta_0^5}{L_2^4} \frac{\max\{L_1^2, L_1^3\}}{\max\{L_1^{3/5}, 1\}^5} + \delta \frac{3C_2^2 \delta_0^2}{L_2^4} \leq C \frac{\delta_0^5 + \delta}{L_2^4}, \quad t \in [t_0, t_0 + 1]. \end{aligned}$$

If $\delta_0 > 0$ and $\delta > 0$ are sufficiently small, then the last expression is less than δ_0^2/L_2^4 , and the application of Lemma 6.1 gives (8.7) for $t \in [s_0, s_0 + 2]$, and thus, in particular, for $t \in [t_0 + 1, t_0 + 2]$. The induction step is thus complete, providing the global existence and validity of (8.7) for all $t \geq 0$.

Finally, we address (2.7) and (2.8). We have established that (8.7) holds for every $t \geq 0$, and thus also (8.8) is true for every $t \geq 0$. Therefore, if $\delta_0 > 0$ is a sufficiently small constant, Lemma 7.1 applies for every $t \geq 1$, and we get

$$\limsup_{t \rightarrow \infty} \int_t^{t+1} A(s) ds \leq C \max\{L_1^2, L_1^3\} L_2.$$

Therefore, using (6.3), the assertions (2.7) and (2.8) follow. $\square$

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