On quantitative uniqueness for parabolic equations

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ABSTRACT. We consider the quantitative uniqueness properties for a parabolic type equation $u_t - \Delta u = w(x,t) \cdot \nabla u + v(x,t)u$, when $v \in L_t^{p_2} L_x^{p_1}$ and $w \in L_t^{q_2} L_x^{q_1}$, with a suitable range for exponents p_1, p_2, q_1, and q_2. We prove a strong unique continuation property and provide a pointwise in time observability estimate.

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CONTENTS

1. Introduction

We address quantitative uniqueness properties of solutions to the parabolic equation

$$u_t - \Delta u = w(x,t) \cdot \nabla u + v(x,t)u, \qquad (x,t) \in \Omega \times [T_0, T_0 + T],$$

where v and w satisfy $v \in L_t^{p_2} L_x^{p_1}(\Omega \times I)$ and $w \in L_t^{q_2} L_x^{q_1}(\Omega \times I)$, respectively, $\Omega \subseteq \mathbb{R}^n$, and $I = [T_0, T_0 + T]$. (In this paper, we consider $\Omega = \mathbb{T}^n$ and $\Omega = \mathbb{R}^n$.) In particular, we obtain bounds on the order of vanishing which are algebraic in the corresponding norms of v and w; recall that u has an order of vanishing d at a point (x_0, t_0) if d is the largest integer such that

$$|u(x,t)| \lesssim (|x - x_0|^2 + |t - t_0|)^{d/2}$$

in a neighborhood of (x_0, t_0). Additionally, we obtain pointwise in time observability estimates, i.e., inequalities of the form

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq M_\delta \|u(\cdot,t)\|_{L^2(B_\delta(x_0))},$$

for an arbitrary $\delta > 0$ under the Lebesgue conditions on v and w.

The unique continuation for PDEs has a rich history (see the review papers by Kenig [**K1, K2**] and Vessella [**V**]), so we only mention several results pertaining to this paper. In [**JK**], Jerison and Kenig proved that the second order elliptic equation has the strong unique continuation property (i.e., is identically zero if it vanishes to an infinite order at a point) if $w = 0$ and $v \in L^{n/2}$, with a sufficiently small $L^{n/2}$ norm, which is a sharp result. The parabolic counterpart was obtained by Escauriaza and Vega [**EV**] (see also [**E**] for a previous unique continuation result when $v \in L_t^\infty L_x^p$ with $p > n/2$). The difficult case when w is nonzero was addressed by Koch and Tataru for the elliptic case in [**KT1**] and the parabolic case in [**KT2**]. In particular, they obtained the strong unique continuation for $v \in L_t^1 L_x^\infty + L_t^\infty L_x^{n/2}$ with the norm sufficiently small and $w \in L_{x,t}^{n+2}$. All the mentioned works rely on suitable Carleman type estimates and lead to observability type estimates on the space-time rectangles. For other works on the frequency approach to the unique continuation, see [**A, AN, Ch, Ku1, Ku2**], for the works related to Dirichlet quotients, see [**A, CFNT, FS**] and for some other related works, see [**AE, AEWZ, ApE, AMRV, An, AV, BC, B, BK, CRV, D, DF1, DF2, DZ1, DZ2, EF, EFV, EVe, F, H1, H2, K3, KSW, L, M, Z, Zh**].

In this paper (see Theorems 2.1, 2.3, and 2.5 below), we obtain explicit algebraic observability estimates for a fixed time (i.e., not only on space-time rectangles) under the assumptions on the coefficients $v \in L_t^\infty L_x^p$ and $w \in L_t^\infty L_x^q$, where $p > 2n/3$ and $q > 2n$. More general conditions $v \in L_t^{p_2} L_x^{p_1}$ and $w \in L_t^{q_2} L_x^{q_1}$ under certain assumptions on p_1, p_2, q_1, and q_2 are addressed in Theorems 6.1. While the results cited in the second paragraph use the Carleman estimates, we rely on the frequency function approach, developed in [**Al, GL**] for the elliptic and [**Kur, P**] for the parabolic equations. The main idea of this approach for parabolic equations is the logarithmic convexity of the frequency function

$$Q(t) = \frac{|t| \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 G(x,t)\, dx}{\int_{\mathbb{R}^n} u(x,t)^2 G(x,t)\, dx} \tag{1.1}$$

for the heat equation. In (1.1), G is the $(4\pi)^{n/2}$-multiple of the backward Gaussian kernel, i.e.,

$$G(x,t) = \frac{1}{|t|^{n/2}} e^{|x|^2/4t}.$$

Another reformulation of the idea is to use the similarity variables (see (3.11) and (3.12) below) and obtain a logarithmic convexity of the unweighted norm [**C, Ku3**]. In [**CK**], the approach was used to obtain an estimate for an order of vanishing $C(\|w\|_{L_{x,t}^\infty}^2 + \|v\|_{L_{x,t}^\infty}^{2/3})$, which is, at least for complex valued coefficients, sharp [**CKW1, CKW2**]. Poon and Kurata showed that the frequency approach leads to the unique continuation property for $p > n$ and $q = \infty$.

In this paper, we deduce the quantitative unique continuation statement and the observability estimate for $p > 2n/3$ and $q > 2n$. The improved range is obtained by three main devices. The first is to find the point in space where the frequency function is the smallest and translate the equation so it starts at that point (see Lemma 3.1 below); this idea has been introduced in [**Ku4, CK**]. The second device is to use the embedding theorems with Laplacian and use (3.40) below to bound the parts containing v and w. The third is to use the finiteness of the integral in (3.69) below, which then allows us to show the convergence of the quantity under the integral. Note that we obtain an explicit algebraic bound on the order of vanishing, which is a constant multiple of

$$\|v\|_{L_t^\infty L_x^p}^{2/(3-2n/p)} + \|w\|_{L_t^\infty L_x^q}^{2/(1-2n/q)} + 1.$$

When setting $p = q = \infty$, the estimate reduces to the sharp bound from [**CK**]. We also provide a pointwise estimate in time for a better understanding of the behavior of solution u. In particular, for all $\delta_0 \in (0, 1]$, we have

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \lesssim e^P \|u(\cdot,t)\|_{L^2(B_{\delta_0}(0))}, \tag{1.2}$$

for all $t \in [T_0 + T/2, T_0 + T]$, where P is a polynomial depending only on n, q, p, δ_0, $\|v\|_{L_t^\infty L_x^p}$, and $\|w\|_{L_t^\infty L_x^q}$. The explicit formula for P can be found in Lemma 4.1 below. Note that the estimates of the type (1.2) are an essential ingredient when considering qualitative properties of solutions of evolutionary parabolic PDE. For instance, they are needed when considering the size of the zero set of solution at time t or, more generally, complexity of a graph of a function at a time t ([**Ku3, Ku4**]). Note finally that compared to [**CK**], we reduce the necessary regularity for the solution u to a simple boundedness, so we do not require differentiability.

We conclude the introduction by several general comments about the presented results. We believe that the restrictions $p > 2n/3$ and $q > 2n$ are sharp from the perspective of the frequency approach (see however Theorem 2.4 and Sections 6.2–6.3 for extensions); it would be interesting to know if they are optimal for obtaining the inequality of the type (2.7) pointwise in time. The restriction on p and q results from the Gronwall-type argument applied to (3.54) below. It is not clear if the approaches related to the Carleman estimates (as those in [**KT2**]) can be adapted to obtain pointwise in time observability estimates with the low regularity of v and w. It seems that approaches using the frequency require all Lebesgue exponents to be greater than or equal to 2 (cf. [**DZ1, DZ2**] where the exponents lower than two were obtained in the elliptic case when $n = 2$).

The paper is structured as follows. In Section 2, we provide the setup of the problem and state the two main results, Theorems 2.1 and 2.3, on the order of vanishing and pointwise-in-time observability with periodic boundary conditions, respectively. The case of $\mathbb{R}^n$ under a natural non-growth condition on the solution (see (2.11) below) is considered in Theorem 2.5. The three theorems are proven in Sections 3, 4, and 5, respectively. In Section 6, we address several extensions. First, in Section 6.1 (Theorem 6.1), we consider the situation when the time exponent is finite, i.e., we consider $v \in L_t^{p_2} L_x^{p_1}$ and $w \in L_t^{q_2} L_x^{q_1}$ when $p_2 < \infty$ or $q_2 < \infty$. In Sections 6.2 and 6.3 we show that the exponents $p \in (n/2, 3n/2)$ and $q \in (n, 2n)$ can be considered. Namely, Theorem 6.2 extends the results by assuming that the L^p norm of v or L^q norm of w vanish at an algebraic rate as $t \to T_0 + T$. Finally, in Theorem 2.4 stated in Section 2 and proven in Section 6.3, we extend the results to the case when $p > n/2$ or $q > n$ and the norm of v or w, respectively, is sufficiently small compared to the Dirichlet quotient.

2. The main results

We address the quantitative uniqueness of a nontrivial solution $u \in L^\infty(I, L^2(\mathbb{T}^n)) \cap L^2(I, H^1(\mathbb{T}^n))$ of the problem

$$\begin{aligned} & u_t - \Delta u = w(x,t) \cdot \nabla u + v(x,t) u \\ & u(\cdot, T_0) = u_0, \end{aligned} \tag{2.1}$$

with the first equation defined for $(x,t) \in \mathbb{R}^n \times I$ where $I = [T_0, T_0 + T]$ is a given time interval, assuming $T, T_0 > 0$ and $n \geq 2$. The results are also valid for $n = 1$ with minor changes; see Remark 2.6 below. We assume that u, v, and w are 1-periodic (in all n directions) and that they satisfy

$$\|v(\cdot,t)\|_{L^p(\mathbb{T}^n)} \leq M_0 \tag{2.2}$$

and

$$\|w(\cdot,t)\|_{L^q(\mathbb{T}^n)} \leq M_1, \tag{2.3}$$

for all $t \in I$. When we consider the periodic boundary conditions, we use the notation Ω for the set $[-1/2, 1/2]^n$, while $\mathbb{T}^n$ means $\mathbb{R}^n/\mathbb{Z}^n$, i.e., $\mathbb{T}^n$ is the set of equivalence classes of points which are identified if the difference belongs to $\mathbb{Z}^n$. Let $O_{(x_0,t_0)}(u)$ be the vanishing order of u at (x_0, t_0), which is defined as the largest integer d such that

$$\|u\|_{L^2(Q_r(x_0,t_0))} = O(r^{d+(n+2)/2}) \quad \text{as } r \to 0,$$

where

$$Q_r(x_0,t_0) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_0| < r, -r^2 < t - t_0 < 0\} \tag{2.4}$$

stands for the parabolic cylinder centered at (x_0, t_0) with the radius r. Note that, by the parabolic regularity and Hölder's inequality, this definition of vanishing order is equivalent to the one stated in the introduction. For $t \in I$, denote by

$$q_{\mathrm{D}}(t) = \frac{\|\nabla u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2}{\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2} \tag{2.5}$$

the Dirichlet quotient of u at the time $t \in I$. We assume that $\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}$ is nonzero for all $t \in I$. We also suppose that

$$q_0 = \sup_{t \in I} q_{\mathrm{D}}(t) < \infty. \tag{2.6}$$

The following is the main result of this paper; see also Theorems 6.1, and 6.2 (as well as Remark 6.3) for extensions. Here and in the sequel, we denote $L_t^q L_x^p(\mathbb{T}^n \times I) = L^q(I, L^p(\mathbb{T}^n))$ and $L_{x,t}^p = L_t^p L_x^p$, for $p, q \in [1, \infty]$.

THEOREM 2.1. *Let $u \in L_{x,t}^\infty(\mathbb{T}^n \times I)$ be a solution of (2.1) with $v \in L_t^\infty L_x^p(\mathbb{T}^n \times I)$ and $w \in L_t^\infty L_x^q(\mathbb{T}^n \times I)$ such that (2.2) and (2.3) hold where*

$$p > \frac{2n}{3}$$

and

$$q > 2n.$$

Then, for all $(x_0, t_0) \in \mathbb{T}^n \times [T_0 + T/2, T_0 + T]$, the vanishing order of u at (x_0, t_0) satisfies

$$O_{(x_0,t_0)}(u) \lesssim M_0^a + M_1^b + 1, \tag{2.7}$$

where

$$a = \frac{2}{3 - 2n/p} \quad \textit{and} \quad b = \frac{2}{1 - 2n/q}, \tag{2.8}$$

with the implicit constant in (2.7) *depending on* q_0 *and* T.

Note that the dimension n is considered fixed, so all constants and polynomials may depend on n without mention.

REMARK 2.2. The assumption (2.6) is necessary as the Dirichlet quotient controls oscillations of solutions. For instance, let ϕ_n be an λ_n-eigenfunction of $-\Delta$ with periodic boundary conditions, which vanishes of order n at 0. Then the solution $u = \phi_n e^{-\lambda_n t}$ of the heat equation does not satisfy (2.7) for n sufficiently large as $v = w = 0$. Note that the Dirichlet quotient for this solution equals λ_n. The eigenfunctions with an arbitrarily high order of vanishing can easily be constructed in bounded domains with Dirichlet boundary conditions; however we expect that it holds for periodic boundary conditions as well; cf. also [**CKW1, CKW2**] for constructions of solutions of elliptic equations with a high order of vanishing, including the periodic boundary conditions.

In the next statement, we provide a pointwise in time observability property of solutions.

THEOREM 2.3. *Under the conditions of Theorem 2.1, there exists a polynomial* P *such that*

$$\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)} \le e^{P(\delta_0,M_0,M_1)}\|u(\cdot,t)\|_{L^2(B(0,\delta_0))}, \tag{2.9}$$

for all $t \in [T_0 + T/2, T_0 + T]$ *and* $\delta_0 \in (0, 1/2]$, *where the coefficients depend on* p, q, T, *and* q_0.

Here and in the sequel, P denotes a generic nonnegative polynomial. Although in the proof we do not follow the dependence on q_0 and T, it is easy to check that the dependence on these quantities is also polynomial.

Theorems 2.1 and 2.3 allow extensions, some of which are stated in Section 6. Here we point out one, which allows the exponents p and q to belong to extended ranges $p > n/2$ and $q > n$, considered critical for unique continuation.

THEOREM 2.4. *Let* $u \in L^\infty_{x,t}(\mathbb{T}^n \times I)$ *be a solution of* (2.1) *with* $v \in L^\infty_t L^p_x(\mathbb{T}^n \times I)$ *and* $w \in L^\infty_t L^q_x(\mathbb{T}^n \times I)$ *such that* (2.2) *and* (2.3) *hold where*

$$p > \frac{n}{2}$$

and

$$q > 2n.$$

If M_0 *is less than a constant depending on* q_0, *then for all* $(x_0, t_0) \in \mathbb{T}^n \times [T_0 + T/2, T_0 + T]$, *the vanishing order of* u *at* (x_0, t_0) *satisfies*

$$O_{(x_0,t_0)}(u) \lesssim M_1^b + 1, \tag{2.10}$$

where b *is as in* (2.8). *Similarly, if*

$$p > \frac{2n}{3}$$

and

$$q > n,$$

and if M_1 *is less than a constant depending on* q_0, *then for all* $(x_0, t_0) \in \mathbb{T}^n \times [T_0 + T/2, T_0 + T]$, *the vanishing order of* u *at* (x_0, t_0) *satisfies*

$$O_{(x_0,t_0)}(u) \lesssim M_0^a + 1,$$

where a *is as in* (2.8). *If* $p > n/2$ *and* $q > n$, *and if* M_0 *and* M_1 *are less than a constant depending on* q_0, *then the same conclusion holds with* (2.10) *replaced by* $O_{(x_0,t_0)}(u) \lesssim 1$, *with all the implicit constants depending on* q_0 *and* T.

Now, consider u, v, and w defined on $\mathbb{R}^n$ instead of $\mathbb{T}^n$. Suppose that u satisfies a doubling type (or mild-growth) condition

$$\int_{\mathbb{R}^n} u(x,t)^2\,dx \le K \int_{B_1} u(x,t)^2\,dx, \qquad t \in [T_0, T_0 + T], \tag{2.11}$$

for some constant K. In this case, we obtain the following analogue of Theorems 2.1 and 2.3.

THEOREM 2.5. *Let* $u \in L^\infty_{x,t}(\mathbb{R}^n \times I)$, *where* $I = [T_0, T_0 + T]$ *and* $T_0, T > 0$, *be a solution of* (2.1) *satisfying* (2.11), *with the coefficients verifying* $v \in L^\infty_t L^p_x(\mathbb{R}^n \times I)$ *and* $w \in L^\infty_t L^q_x(\mathbb{R}^n \times I)$ *with*

$$\|v(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le M_0$$

and

$$\|w(\cdot,t)\|_{L^q(\mathbb{R}^n)} \le M_1,$$

for $t \in I$. Assume additionally that

$$p > \frac{2n}{3} \tag{2.12}$$

and

$$q > 2n. \tag{2.13}$$

Then, for all $(x_0, t_0) \in B_R \times [T_0 + T/2, T_0 + T]$, where $R > 0$, the vanishing order of u at (x_0, t_0) satisfies

$$O_{(x_0,t_0)}(u) \lesssim M_0^a + M_1^b + 1, \tag{2.14}$$

where $a = 2/(3 - 2n/p)$ and $b = 2/(1 - 2n/q)$, with the implicit constant in (2.14) depending on q_0, K, T, and R. Moreover, for $\delta_0 \in (0, 1/2]$, we have

$$\|u(\cdot, t)\|_{L^2(\mathbb{T}^n)} \leq e^{P(K,\delta_0,M_0,M_1)} \|u(\cdot, t)\|_{L^2(B_{\delta_0})},$$

for all $t \in [T_0 + T/2, T_0 + T]$, where P is a polynomial with coefficients depending on q_0.

The theorem is proven in Section 5 below.

REMARK 2.6. In the theorems above, we assumed $n \geq 2$. For the case $n = 1$, we suppose additionally that $p \geq 2$ and $q \geq 4$. The reason for the restriction $p \geq 2$ is that the methods are L^2-based requiring the exponents to be at least 2. The reason for $q \geq 4$ is technical; see the comment below (3.48).

3. Proof of the statement on quantitative uniqueness

This section is devoted to the proof of Theorem 2.1. We first start with the case when v, w, and u_0 are smooth and then use an approximation argument to prove the theorem under the general conditions of Theorem 2.1. Thus assume for now that v, w, and u_0 are smooth.

3.1. Frequency smallness lemma. By a translation and rescaling, we may restrict ourselves, throughout the section, to $I = [-1, 0]$ and $(x_0, t_0) = (0, 0)$. The following lemma allows us to find a point $-x_\epsilon$ where the frequency

$$Q(t) = \frac{|t| \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 G(x,t)\, dx}{\int_{\mathbb{R}^n} u(x,t)^2 G(x,t)\, dx},$$

after being translated so it is centered at $-x_\epsilon$, is small, at a small time $t = -\epsilon$, where $\epsilon \in (0, 1]$.

LEMMA 3.1. *Let $u \in L^\infty_{x,t}(\Omega \times I)$ be smooth and 1-periodic in x, for $t \in I = [-1, 0]$. For any $\epsilon \in (0, 1]$ such that $u(\cdot, -\epsilon)$ is not identically zero, there exists $x_\epsilon \in \Omega$ such that*

$$\frac{\epsilon \int_{\mathbb{R}^n} |\nabla u(x_\epsilon + y, -\epsilon)|^2 G(y, -\epsilon)\, dy}{\int_{\mathbb{R}^n} u(x_\epsilon + y, -\epsilon)^2 G(y, -\epsilon)\, dy} \leq \epsilon q_D(-\epsilon),$$

where

$$G(x,t) = \frac{1}{|t|^{n/2}} e^{|x|^2/4t}, \qquad x \in \mathbb{R}^n, \quad t < 0. \tag{3.1}$$

The lemma is proven in [**CK, Ku4**]; we provide a short proof for the sake of completeness.

PROOF OF LEMMA 3.1. Assume, contrary to the assertion, that we have

$$q_D(-\epsilon) \int_{\mathbb{R}^n} u(x+y, -\epsilon)^2 G(y, -\epsilon)\, dy < \int_{\mathbb{R}^n} |\nabla u(x+y, -\epsilon)|^2 G(y, -\epsilon)\, dy, \tag{3.2}$$

for all $x \in \Omega$, which by a simple change of variable reads

$$\int_{\mathbb{R}^n} u(y, -\epsilon)^2 G(y - x, -\epsilon)\, dy < \frac{1}{q_D(-\epsilon)} \int_{\mathbb{R}^n} |\nabla u(y, -\epsilon)|^2 G(y - x, -\epsilon)\, dy, \qquad x \in \Omega.$$

We shall integrate both sides in x over Ω, obtaining a contradiction. The integral of the left-hand side over Ω equals

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^n} u(y,-\epsilon)^2 G(y-x,-\epsilon)\,dy\,dx &= \int_\Omega \sum_{j\in\mathbb{Z}^n} \int_{j+\Omega} u(y,-\epsilon)^2 G(y-x,-\epsilon)\,dy\,dx \\ &= \int_\Omega \sum_{j\in\mathbb{Z}^n} \int_\Omega u(y,-\epsilon)^2 G(y+j-x,-\epsilon)\,dy\,dx \\ &= \int_\Omega u(y,-\epsilon)^2 \left(\sum_{j\in\mathbb{Z}^n} \int_\Omega G(y+j-x,-\epsilon)\,dx \right) dy \\ &= \int_\Omega u(y,-\epsilon)^2 \int_{\mathbb{R}^n} G(y-x,-\epsilon)\,dx\,dy = (4\pi)^{n/2} \int_\Omega u(y,-\epsilon)^2\,dy,\end{aligned} \tag{3.3}$$

where we used $\int_{\mathbb{R}^n} G(y-x,-\epsilon)\,dx = (4\pi)^{n/2}$ in the last equality. Similarly, we have

$$\int_\Omega \int_{\mathbb{R}^n} |\nabla u(y,-\epsilon)|^2 G(y-x,-\epsilon)\,dy\,dx = (4\pi)^{n/2} \int_\Omega |\nabla u(y,-\epsilon)|^2\,dy. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$q_{\mathrm{D}}(-\epsilon) \int_\Omega u(y,-\epsilon)^2\,dy < \int_\Omega |\nabla u(y,-\epsilon)|^2\,dy,$$

which is a contradiction with the definition (2.5) of the Dirichlet quotient. Therefore, (3.2) cannot hold for all $x \in \Omega$, and the lemma follows. $\square$

Note that the argument above does not require u to solve (2.1).

3.2. Setting and notation. Let $\epsilon \in (0,1/2]$ be a fixed parameter, to be chosen in the proof of Lemma 3.2 below; see (3.24). We proceed with a change of variables

$$\bar{u}(x,t) = u\left(x - \frac{x_\epsilon}{\epsilon} t, t\right), \tag{3.5}$$

where x_ϵ is as in Lemma 3.1, so that

$$\bar{u}(x,-\epsilon) = u(x + x_\epsilon, -\epsilon) \tag{3.6}$$

and

$$\bar{u}(x,0) = u(x,0), \tag{3.7}$$

for all $x \in \mathbb{T}^n$. By Lemma 3.1, we have

$$\frac{\epsilon \int_{\mathbb{R}^n} |\nabla \bar{u}(y,-\epsilon)|^2 G(y,-\epsilon)\,dy}{\int_{\mathbb{R}^n} \bar{u}(y,-\epsilon)^2 G(y,-\epsilon)\,dy} \le \epsilon q_0,$$

i.e., the frequency of $\bar{u}$ at $t=-\epsilon$ is small. It is not difficult to check that $\bar{u}$ solves the equation

$$\partial_t \bar{u} - \Delta \bar{u} = -\frac{x_\epsilon}{\epsilon} \cdot \nabla \bar{u} + w \cdot \nabla \bar{u} + v \bar{u}. \tag{3.8}$$

Since $\bar{u}$ and u have the same order of vanishing at $(0,0)$, we write u instead of $\bar{u}$ until the end of Section 3. Denoting

$$r = -\frac{x_\epsilon}{\epsilon} \tag{3.9}$$

throughout, the equation (3.8) becomes

$$\partial_t u - \Delta u = r \cdot \nabla u + w \cdot \nabla u + v u. \tag{3.10}$$

We now proceed with a change of variable

$$U(y,\tau) = e^{-|y|^2/8} u(y e^{-\tau/2}, -e^{-\tau}), \qquad (y,\tau) \in \mathbb{R}^n \times [\tau_0, \infty), \tag{3.11}$$

that is,

$$u(x,t) = e^{|x|^2/8(-t)} U\left(\frac{x}{\sqrt{-t}}, -\log(-t)\right), \qquad (x,t) \in \mathbb{R}^n \times [-\epsilon, 0), \tag{3.12}$$

with $y = x/\sqrt{-t}$ and $\tau = -\log(-t)$, where

$$\tau_0 = \log \frac{1}{\epsilon}. \tag{3.13}$$

Also, let

$$V(y,\tau) = v(ye^{-\tau/2}, -e^{-\tau}), \qquad (y,\tau) \in \mathbb{R}^n \times [\tau_0, \infty) \tag{3.14}$$

and

$$W(y,\tau) = w(ye^{-\tau/2}, -e^{-\tau}), \qquad (y,\tau) \in \mathbb{R}^n \times [\tau_0, \infty). \tag{3.15}$$

Then (3.10) becomes

$$\partial_\tau U + HU = e^{-\tau/2}\left(\frac{1}{4} r_j y_j U + r_j \partial_j U\right) + e^{-\tau/2}\left(\frac{1}{4} y_j W_j U + W_j \partial_j U\right) + e^{-\tau} VU, \tag{3.16}$$

where

$$HU = -\Delta U + \left(\frac{|y|^2}{16} - \frac{n}{4}\right) U, \tag{3.17}$$

with the initial data

$$U(y,\tau_0) = U\left(y, \log\frac{1}{\epsilon}\right) = e^{-|y|^2/8} u(y\sqrt{\epsilon}, -\epsilon).$$

A short computation shows that

$$\|U(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x,t)^2 G(x,t)\, dx, \tag{3.18}$$

where $\tau = -\log(-t)$ throughout, and

$$(HU, U)_{L^2(\mathbb{R}^n)} = |t| \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 G(x,t)\, dx. \tag{3.19}$$

Thus also

$$Q(\tau) = \frac{(HU,U)_{L^2(\mathbb{R}^n)}}{\|U\|^2} = \frac{|t| \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 G(x,t)\, dx}{\int_{\mathbb{R}^n} u(x,t)^2 G(x,t)\, dx},$$

where we write

$$\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}; \tag{3.20}$$

also, if the domain of integration is not indicated, it is assumed to be $\mathbb{R}^n$. Denoting

$$A(\tau)U = HU - \frac{1}{4} e^{-\tau/2} r_j y_j U = -\Delta U + \left(\frac{|y|^2}{16} - \frac{n}{4}\right) U - \frac{1}{4} e^{-\tau/2} r_j y_j U \tag{3.21}$$

and

$$\bar{Q}(\tau) = \frac{(A(\tau)U, U)_{L^2(\mathbb{R}^n)}}{\|U\|^2} = Q(\tau) - \frac{e^{-\tau/2} r_j}{4\|U\|^2} \int y_j U^2\, dy, \tag{3.22}$$

we may rewrite (3.16) as

$$\partial_\tau U + A(\tau)U = F(U), \tag{3.23}$$

where

$$F(U) = e^{-\tau/2} r_j \partial_j U + e^{-\tau/2}\left(\frac{1}{4} y_j W_j U + W_j \partial_j U\right) + e^{-\tau} VU.$$

For simplicity, denote

$$\tilde{U} = \frac{U}{\|U\|},$$

so that $\|\tilde{U}\| = 1$.

3.3. The main frequency lemma. We now show that the modified frequency function $\bar{Q}$ is bounded with an expression on the right-hand side of (2.7) for a suitable choice of ϵ.

LEMMA 3.2. *Let*

$$\epsilon = \frac{1}{C(M_0^a + M_1^b + 1)}, \tag{3.24}$$

where a and b are given in (2.8), and C is a sufficiently large constant depending on q_0. Under the assumptions of Theorem 2.1, and assuming that u, v, and w are smooth, the modified frequency function satisfies

$$\bar{Q}(\tau) \lesssim M_0^a + M_1^b + 1, \qquad \tau \geq \tau_0, \tag{3.25}$$

where τ_0 is given in (3.13), with the implicit constant in (3.25) depending on q_0.

Note that $\epsilon \in (0, 1/2]$ by (3.24).

PROOF OF LEMMA 3.2. Let $\epsilon \in (0, 1/2]$ first be arbitrary, with the choice (3.24) made before (3.58) below. Also, we use the notation from Section 3.2. With $\mathcal{I}$ denoting the identity matrix, we claim that

$$\frac{1}{2}\bar{Q}'(\tau) + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2 = \frac{1}{2}(A'(\tau)\tilde{U}, \tilde{U}) + \left(\frac{F(U)}{\|U\|}, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\right). \tag{3.26}$$

To establish (3.26), we first divide (3.23) by $\|U\|$ and take the inner product of the resulting equation with $(A(\tau) - \bar{Q}(\tau)I)U/\|U\|$ to obtain

$$\begin{aligned} &\frac{1}{\|U\|^2}(\partial_\tau U, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}) + \left\|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\right\|^2 + \bar{Q}(\tau)\frac{(U, A(\tau)U)}{\|U\|^2} \\ &\quad = \bar{Q}^2(\tau) + \left(\frac{F(U)}{\|U\|}, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\right), \end{aligned} \tag{3.27}$$

where $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\mathbb{R}^n)}$. By (3.22), the last term on the left-hand side cancels with the first term on the right-hand side of (3.27), so the equation (3.27) becomes

$$\left(\frac{\partial_\tau U}{\|U\|}, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\right) + \left\|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\right\|^2 = \left(\frac{F(U)}{\|U\|}, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\right). \tag{3.28}$$

On the other hand, by differentiating (3.22), we have

$$\frac{1}{2}\bar{Q}'(\tau) = \frac{(A'(\tau)U, U)}{2\|U\|^2} + \frac{(\partial_\tau U, A(\tau)U)}{\|U\|^2} - \frac{(\partial_\tau U, U)(A(\tau)U, U)}{\|U\|^4},$$

and thus

$$\frac{1}{2}\bar{Q}'(\tau) = \frac{1}{2}(A'(\tau)\tilde{U}, \tilde{U}) + \frac{(\partial_\tau U, (A(\tau) - \bar{Q}(\tau)\mathcal{I})U)}{\|U\|^2}. \tag{3.29}$$

The identity (3.26) then follows by adding (3.28) and (3.29). Since $A'(\tau)\tilde{U} = \frac{1}{8}e^{-\tau/2}r_j y_j \tilde{U}$, we obtain

$$\begin{aligned} \frac{1}{2}\bar{Q}'(\tau) + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2 &= \frac{1}{16}e^{-\tau/2}(r_j y_j \tilde{U}, \tilde{U}) + e^{-\tau/2}r_j\big(\partial_j \tilde{U}, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\big) \\ &\quad + \big(e^{-\tau/2}(y_j W_j \tilde{U} + W_j \partial_j \tilde{U}) + e^{-\tau} V \tilde{U}, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\big). \end{aligned} \tag{3.30}$$

For the second term on the right-hand side of (3.30), we have

$$\begin{aligned} &e^{-\tau/2}r_j\big(\partial_j \tilde{U}, (A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\big) \\ &\quad = -e^{-\tau/2}r_j \int \Delta\tilde{U}\partial_j\tilde{U} + e^{-\tau/2}r_j \int \frac{|y|^2}{16}\tilde{U}\partial_j\tilde{U} - \frac{n}{4}e^{-\tau/2}r_j \int \tilde{U}\partial_j\tilde{U} - \frac{1}{4}e^{-\tau}r_j r_k \int y_k \tilde{U}\partial_j\tilde{U} \\ &\quad = \frac{1}{16}e^{-\tau/2}r_j \int |y|^2\tilde{U}\partial_j\tilde{U} - \frac{1}{4}e^{-\tau}r_j r_k \int y_k\tilde{U}\partial_j\tilde{U} \\ &\quad = -\frac{1}{16}e^{-\tau/2}r_j \int y_j \tilde{U}^2 + \frac{1}{8}e^{-\tau}|r|^2, \end{aligned} \tag{3.31}$$

where we used $\int \tilde{U}\partial_j\tilde{U} = 0$ and $\int \Delta\tilde{U}\partial_j\tilde{U} = 0$ in the second equality (since v, w, and u_0 are assumed smooth, $\tilde{U}$ and its derivatives are smooth and decaying fast in the spatial variable) and $\|\tilde{U}\| = 1$ in the last. Recall that all the integrals with the domain not indicated are understood to be over $\mathbb{R}^n$. Note that the first term in the last line of (3.31) cancels with the first term on the right-hand side of (3.30). Therefore,

$$\begin{aligned}&\frac{1}{2}\bar{Q}'(\tau)+\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2\\&\quad=\frac{1}{8}e^{-\tau}|r|^2+e^{-\tau/2}\int(y_jW_j\tilde{U}+W_j\partial_j\tilde{U}+e^{-\tau/2}V\tilde{U})(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\,dy=I_1+I_2.\end{aligned}\tag{3.32}$$

In order to estimate I_2, we first claim

$$\|D^2U\|\lesssim\|HU\|+\|U\|.\tag{3.33}$$

To prove (3.33), we expand $\|HU\|^2$ as

$$\begin{aligned}\|HU\|^2&=\int\left(-\Delta U+\left(\frac{|y|^2}{16}-\frac{n}{4}\right)U\right)^2dy\\&=\int\left((\Delta U)^2+\left(\frac{|y|^2}{16}-\frac{n}{4}\right)^2U^2\right)dy-2\int\left(\frac{|y|^2}{16}-\frac{n}{4}\right)U\Delta U\,dy\\&=\int\left((\Delta U)^2+\left(\frac{|y|^2}{16}-\frac{n}{4}\right)^2U^2\right)dy+2\int\partial_jU\partial_jU\left(\frac{|y|^2}{16}-\frac{n}{4}\right)dy+\frac{1}{4}\int y_jU\partial_jU\,dy.\end{aligned}$$

Since the last term equals $(-n/8)\int U^2\,dy$, we get

$$\|HU\|^2\ge\int\left((\Delta U)^2+\left(\frac{|y|^2}{16}-\frac{n}{4}\right)^2U^2\right)dy-\frac{n}{2}\int|\nabla U|^2\,dy-\frac{n}{8}\int U^2\,dy,$$

from where

$$\|\Delta U\|^2\lesssim\|HU\|^2+\|U\|^2+\|\nabla U\|^2.$$

By Sobolev's and the Cauchy-Schwarz inequalities, we get

$$\|\nabla U\|^2\le\|U\|\|\Delta U\|\le\frac{\|U\|^2+\|\Delta U\|^2}{2},$$

and then using

$$\|D^2U\|\lesssim\|\Delta U\|\tag{3.34}$$

we obtain (3.33); note that the inequality (3.34) follows by $\int\Delta U\Delta U=\int\partial_{ii}U\partial_{jj}U=\int\partial_{ij}U\partial_{ij}U$, due to fast decay of U and its spatial derivatives (note that u is smooth and periodic).

To treat I_2 in (3.32), we first estimate $\|V\tilde{U}\|$. Observe that we cannot apply the Gagliardo-Nirenberg inequalities directly since $\|V(\cdot,\tau)\|_{L^p}$ is infinite whenever v is not identically zero, due to periodicity of v. Noting that V is $e^{\tau/2}$-periodic, we tile $\mathbb{R}^n$ as

$$\mathbb{R}^n=\bigcup_{j\in\mathbb{Z}^n}\Omega_{j,\tau},\tag{3.35}$$

where $\Omega_{j,\tau}=je^{\tau/2}+e^{\tau/2}\Omega$. Then we have

$$\|VU\|^2=\sum_{j\in\mathbb{Z}^n}\|VU\|^2_{L^2(\Omega_{j,\tau})}\lesssim\sum_{j\in\mathbb{Z}^n}\|V\|^2_{L^p(\Omega_{j,\tau})}\|U\|^2_{L^{2p/(p-2)}(\Omega_{j,\tau})}.\tag{3.36}$$

Note that

$$\|V\|_{L^p(\Omega_{j,\tau})}\le M_0e^{\alpha\tau},\qquad j\in\mathbb{Z}^n,$$

where

$$\alpha=\frac{n}{2p},\tag{3.37}$$

by periodicity and using a substitution. Hence, using the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}\|VU\|^2 &\lesssim \sum_{j\in\mathbb{Z}^n} M_0^2 e^{2\alpha\tau}\|U\|_{L^2(\Omega_{j,\tau})}^{2-2\alpha}(\|D^2U\|_{L^2(\Omega_{j,\tau})}+\|U\|_{L^2(\Omega_{j,\tau})})^{2\alpha}\\ &\lesssim M_0^2e^{2\alpha\tau}\left(\sum_{j\in\mathbb{Z}^n}\|U\|_{L^2(\Omega_{j,\tau})}^2\right)^{1-\alpha}\left(\sum_{j\in\mathbb{Z}^n}\|D^2U\|_{L^2(\Omega_{j,\tau})}^2+\sum_{j\in\mathbb{Z}^n}\|U\|_{L^2(\Omega_{j,\tau})}^2\right)^{\alpha}\\ &= M_0^2e^{2\alpha\tau}\|U\|^{2-2\alpha}(\|D^2U\|+\|U\|)^{2\alpha},\end{aligned} \tag{3.38}$$

where we used the discrete Hölder inequality in the second step. Therefore, taking the square root of (3.38) and dividing by $\|U\|$, we get

$$\|V\tilde{U}\| \lesssim M_0e^{\alpha\tau}(\|D^2\tilde{U}\|+1)^{\alpha}, \tag{3.39}$$

where we used $\|\tilde{U}\|=1$. By (3.33), we have

$$\begin{aligned}\|D^2\tilde{U}\| &\lesssim \|H\tilde{U}\|+1 \lesssim \|H\tilde{U}-A(\tau)\tilde{U}\|+\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|+\|\bar{Q}(\tau)\tilde{U}\|+1\\ &\lesssim e^{-\tau/2}|r|\|y\tilde{U}\|+\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|+|\bar{Q}(\tau)|+1,\end{aligned} \tag{3.40}$$

applying (3.21) in the last step. Since

$$\|\nabla\tilde{U}\|+\|y\tilde{U}\| \lesssim |\bar{Q}(\tau)|^{1/2}+e^{-\tau/2}|r|+1, \tag{3.41}$$

([**Ku4**, p. 780]), as one may readily check, we get

$$\|D^2\tilde{U}\| \lesssim e^{-\tau/2}|r|\big(|\bar{Q}(\tau)|^{1/2}+e^{-\tau/2}|r|+1\big)+\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|+|\bar{Q}(\tau)|+1. \tag{3.42}$$

Using (3.39) and (3.42), we obtain

$$\|V\tilde{U}\| \lesssim M_0e^{\alpha\tau}\Big(e^{-\alpha\tau/2}|r|^{\alpha}\left(|\bar{Q}(\tau)|^{\alpha/2}+e^{-\alpha\tau/2}|r|^{\alpha}+1\right)+\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|^{\alpha}+|\bar{Q}(\tau)|^{\alpha}+1\Big). \tag{3.43}$$

Next, we proceed to estimate $\|y_jW_j\tilde{U}+W_j\partial_j\tilde{U}\|$ by first bounding $\|W_j\partial_j\tilde{U}\|$ and then $\|y_jW_j\tilde{U}\|$. Analogously to (3.36)–(3.38), we have

$$\|W_j\partial_jU\| \lesssim M_1e^{(\beta-1/2)\tau}\|U\|^{1-\beta}(\|D^2U\|+\|U\|)^{\beta} \tag{3.44}$$

with

$$\beta = \frac{n}{2q}+\frac{1}{2}, \tag{3.45}$$

where we also used

$$\|W\|_{L^q(\Omega_{j,\tau})} \lesssim M_1e^{n\tau/2q} = M_1e^{(\beta-1/2)\tau}, \qquad j\in\mathbb{Z}^n.$$

Note that the exponents a and b in (2.8) satisfy

$$a = \frac{2}{3-4\alpha}$$

and

$$b = \frac{2}{3-4\beta}.$$

Dividing (3.44) by $\|U\|$, we obtain, similarly to (3.43), that

$$\begin{aligned}\|W_j\partial_j\tilde{U}\| &\lesssim M_1e^{(\beta-1/2)\tau}\left(\|H\tilde{U}\|^{\beta}+1\right)\\ &\lesssim M_1e^{(\beta-1/2)\tau}\left(e^{-\beta\tau/2}|r|^{\beta}(|\bar{Q}(\tau)|^{\beta/2}+e^{-\beta\tau/2}|r|^{\beta}+1)+\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|^{\beta}+|\bar{Q}(\tau)|^{\beta}+1\right).\end{aligned} \tag{3.46}$$

Next,

$$
\begin{aligned}
\|y_j W_j U\|^2 &= \sum_{j\in\mathbb{Z}^n} \|y_j W_j U\|^2_{L^2(\Omega_{j,\tau})} \lesssim \sum_{j\in\mathbb{Z}^n} \|W\|^2_{L^q(\Omega_{j,\tau})} \|yU\|^2_{L^{2q/(q-2)}(\Omega_{j,\tau})} \\
&\lesssim \sum_{j\in\mathbb{Z}^n} M_1^2 e^{(2\beta-1)\tau} \|y|U|^{1/2}\|^2_{L^4(\Omega_{j,\tau})} \||U|^{1/2}\|^2_{L^{4q/(q-4)}(\Omega_{j,\tau})} \\
&= \sum_{j\in\mathbb{Z}^n} M_1^2 e^{(2\beta-1)\tau} \||y|^2 U\|_{L^2(\Omega_{j,\tau})} \|U\|_{L^{2q/(q-4)}(\Omega_{j,\tau})} \\
&\lesssim \sum_{j\in\mathbb{Z}^n} M_1^2 e^{(2\beta-1)\tau} \||y|^2 U\|_{L^2(\Omega_{j,\tau})} \|U\|^{1-n/2q}_{L^{2q/(q-4)}(\Omega_{j,\tau})} (\|D^2 U\|_{L^2(\Omega_{j,\tau})} + \|U\|_{L^2(\Omega_{j,\tau})})^{n/2q}.
\end{aligned}
\tag{3.47}
$$

Applying the discrete Hölder inequality, taking a square root, and dividing by $\|U\|$ leads to

$$
\|y_j W_j \tilde{U}\| \lesssim M_1 e^{(\beta-1/2)\tau} \||y|^2 \tilde{U}\|^{1/2} (\|\Delta \tilde{U}\|^{n/2q} + 1).
\tag{3.48}
$$

Observe that in (3.47) we need $q \geq 4$. (Note that for $n = 1$ we have $q \geq 4$, needed in (3.47), by Remark 2.6.) Recall that $HU = -\Delta U + (|y|^2/16 - n/4)U$, from where

$$
\||y|^2 U\| \lesssim \|HU\| + \|\Delta U\| + \|U\| \lesssim \|HU\| + \|U\|,
\tag{3.49}
$$

by using (3.33) in the last inequality. Applying (3.49) in (3.48), we get

$$
\begin{aligned}
\|y_j W_j \tilde{U}\| &\lesssim M_1 e^{(\beta-1/2)\tau} (\|\Delta \tilde{U}\|^{n/2q} + 1)\big(\|H\tilde{U}\|^{1/2} + 1\big) \\
&\lesssim M_1 e^{(\beta-1/2)\tau} \big(\|H\tilde{U}\|^{n/2q} + 1\big)\big(\|H\tilde{U}\|^{1/2} + 1\big) \\
&\lesssim M_1 e^{(\beta-1/2)\tau} (\|H\tilde{U}\|^{\beta} + 1),
\end{aligned}
$$

where we used (3.45). With (3.42), we then obtain

$$
\|y_j W_j \tilde{U}\| \lesssim M_1 e^{(\beta-1/2)\tau} \Big(e^{-\beta\tau/2}|r|^\beta (|\bar{Q}(\tau)|^{\beta/2} + e^{-\beta\tau/2}|r|^\beta + 1) + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^\beta + |\bar{Q}(\tau)|^\beta + 1\Big).
\tag{3.50}
$$

By (3.43), (3.46), and (3.50), we get an estimate for I_2 from (3.32) which reads

$$
\begin{aligned}
I_2 \lesssim\ & \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\| \Big(e^{(\beta-1)\tau} M_1 \big(e^{-\beta\tau/2}|r|^\beta (|\bar{Q}(\tau)|^{\beta/2} + e^{-\beta\tau/2}|r|^\beta + 1) \\
&\qquad + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^\beta + |\bar{Q}(\tau)|^\beta + 1\big)\Big) \\
&+ \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\| \Big(e^{(\alpha-1)\tau} M_0 \big(e^{-\alpha\tau/2}|r|^\alpha (|\bar{Q}(\tau)|^{\alpha/2} + e^{-\alpha\tau/2}|r|^\alpha + 1) \\
&\qquad + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^\alpha + |\bar{Q}(\tau)|^\alpha + 1\big)\Big).
\end{aligned}
\tag{3.51}
$$

Using (3.51) in (3.32), we obtain

$$
\begin{aligned}
&\frac{1}{2}\bar{Q}'(\tau) + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2 \\
&\quad\lesssim e^{-\tau}|r|^2 + e^{(\beta-1)\tau} M_1 \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\| \Big(e^{-\beta\tau/2}|r|^\beta |\bar{Q}(\tau)|^{\beta/2} + e^{-\beta\tau}|r|^{2\beta} + e^{-\beta\tau/2}|r|^\beta \\
&\qquad\qquad + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^\beta + |\bar{Q}(\tau)|^\beta + 1\Big) \\
&\qquad + e^{(\alpha-1)\tau} M_0 \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\| \Big(e^{-\alpha\tau/2}|r|^\alpha |\bar{Q}(\tau)|^{\alpha/2} + e^{-\alpha\tau}|r|^{2\alpha} + e^{-\alpha\tau/2}|r|^\alpha \\
&\qquad\qquad + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^\alpha + |\bar{Q}(\tau)|^\alpha + 1\Big).
\end{aligned}
\tag{3.52}
$$

We now apply Young's inequality to the terms involving $\|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|$ on the right-hand side so that we can take advantage of the second term in the left hand side; namely, we use

$$
N\|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^\gamma \leq \epsilon_0 \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2 + C_{\epsilon_0} N^{2/(2-\gamma)},
$$

where $\epsilon_0 \in (0,1]$ is arbitrarily small, with $\gamma = 1, \alpha+1, \beta+1$ and corresponding expressions for N. Thus, we get

$$\begin{aligned}
&\bar{Q}'(\tau) + \|(A(\tau) - \bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2 \\
&\quad\lesssim e^{-\tau}|r|^2 + M_1^2 e^{(\beta-2)\tau}|r|^{2\beta}|\bar{Q}(\tau)|^{\beta} + M_1^2 e^{(2\beta-2)\tau}|\bar{Q}(\tau)|^{2\beta} \\
&\qquad + M_1^2 e^{-2\tau}|r|^{4\beta} + M_1^2 e^{(\beta-2)\tau}|r|^{2\beta} + M_1^{2/(1-\beta)} e^{-2\tau} + M_1^2 e^{(2\beta-2)\tau} \\
&\qquad + M_0^2 e^{(\alpha-2)\tau}|r|^{2\alpha}|\bar{Q}(\tau)|^{\alpha} + M_0^2 e^{(2\alpha-2)\tau}|\bar{Q}(\tau)|^{2\alpha} \\
&\qquad + M_0^2 e^{-2\tau}|r|^{4\alpha} + M_0^2 e^{(\alpha-2)\tau}|r|^{2\alpha} + M_0^{2/(1-\alpha)} e^{-2\tau} + M_0^2 e^{(2\alpha-2)\tau}.
\end{aligned} \tag{3.53}$$

The second term on the right-hand side of (3.53) may be absorbed into the third and the fourth by the Cauchy-Schwarz inequality. Similarly, the eighth term is absorbed by the ninth and tenth. Using also $|r| \lesssim 1/\epsilon$, the last inequality implies

$$\begin{aligned}
\bar{Q}'(\tau) &\lesssim e^{-\tau}\epsilon^{-2} + M_1^2 e^{(2\beta-2)\tau}|\bar{Q}(\tau)|^{2\beta} + M_1^2 e^{-2\tau}\epsilon^{-4\beta} + M_1^2 e^{(\beta-2)\tau}\epsilon^{-2\beta} + M_1^{2/(1-\beta)} e^{-2\tau} + M_1^2 e^{(2\beta-2)\tau} \\
&\quad + M_0^2 e^{(2\alpha-2)\tau}|\bar{Q}(\tau)|^{2\alpha} + M_0^2 e^{-2\tau}\epsilon^{-4\alpha} + M_0^2 e^{(\alpha-2)\tau}\epsilon^{-2\alpha} + M_0^{2/(1-\alpha)} e^{-2\tau} + M_0^2 e^{(2\alpha-2)\tau}.
\end{aligned} \tag{3.54}$$

To estimate $\bar{Q}(\tau_0)$, we now compare $\bar{Q}(\tau)$ and $Q(\tau)$ for any $\tau \geq \tau_0$. Using (3.41), we have

$$\||y|^{1/2}\tilde{U}\| \leq \||y|\tilde{U}\|^{1/2} \lesssim |\bar{Q}(\tau)|^{1/4} + e^{-\tau/4}|r|^{1/2} + 1, \tag{3.55}$$

where we used the Cauchy-Schwarz inequality in the first step. By (3.22) and (3.55), we get

$$\begin{aligned}
\bar{Q}(\tau) &\leq Q(\tau) + e^{-\tau/2}\left| r_j \int y_j \tilde{U}^2\, dy \right| \lesssim Q(\tau) + e^{-\tau/2}|r|\||y|^{1/2}\tilde{U}\|^2 \\
&\lesssim Q(\tau) + e^{-\tau/2}|r|\big(|\bar{Q}(\tau)|^{1/2} + e^{-\tau/2}|r| + 1\big),
\end{aligned}$$

from where, after absorbing the second term on the far right side,

$$\bar{Q}(\tau) \lesssim Q(\tau) + e^{-\tau}|r|^2 + 1. \tag{3.56}$$

Note, in passing, that a similar derivation also leads to

$$Q(\tau) \lesssim \bar{Q}(\tau)_+ + e^{-\tau}|r|^2 + 1.$$

Using Lemma 3.1 and (3.56), we have

$$\bar{Q}(\tau_0) \leq C_0\big(\epsilon q_{\mathrm{D}}(-\epsilon) + \epsilon^{-1} + 1\big) \leq \frac{C_1}{\epsilon}, \tag{3.57}$$

where $C_1 = C_0(q_0 + 2)$ and $C_0 \geq 1$ is the constant in the inequality (3.56). Denote by $C_2 \geq 1$ the implicit constant in the inequality (3.54).

Up to this point, all the estimates hold for any fixed $\epsilon \in (0, 1/2]$. Now, fix $\epsilon \in (0, 1/2]$ as in (3.24), denoting the constant in (3.24) by $\bar{C}$. We claim that (3.54) and (3.57) imply

$$\bar{Q}(\tau) < 2\frac{C_1 + C_2}{\epsilon}, \qquad \tau \geq \tau_0 = -\log\epsilon. \tag{3.58}$$

Assume, contrary to the assertion, that there exists $\tau_1 \geq \tau_0$ such that $\bar{Q}(\tau_1) = 2(C_1 + C_2)/\epsilon$, and assume that τ_1 is the first such time. Also, let

$$\tau_0' = \sup\left\{\tau \in [\tau_0, \tau_1] : \bar{Q}(\tau) = \frac{C_1}{\epsilon}\right\} \tag{3.59}$$

so that

$$\bar{Q}(\tau) \in \left[\frac{C_1}{\epsilon}, \frac{2(C_1 + C_2)}{\epsilon}\right), \qquad \tau \in [\tau_0', \tau_1). \tag{3.60}$$

(The purpose of introducing τ_0' is to remedy the fact that $\bar{Q}$ may be negative with a possibly large absolute value.) Integrating (3.54) between τ_0' and τ_1 and using (3.60), we arrive at

$$\begin{aligned}
\bar{Q}(\tau_1) &\leq \bar{Q}(\tau_0') + C_2\epsilon^{-1} + 2C_2 M_1^2 2^{2\beta}(C_1 + C_2)^{2\beta}\epsilon^{2-4\beta} + C_2 M_1^2 \epsilon^{2-4\beta} \\
&\quad + C_2 M_1^2 \epsilon^{2-3\beta} + C_2 M_1^{2/(1-\beta)}\epsilon^2 + 2C_2 M_1^2 \epsilon^{2-2\beta} + 2C_2 M_0^2 2^{2\alpha}(C_1 + C_2)^{2\alpha}\epsilon^{2-4\alpha} \\
&\quad + C_2 M_0^2 \epsilon^{2-4\alpha} + C_2 M_0^2 \epsilon^{2-3\alpha} + C_2 M_0^{2/(1-\alpha)}\epsilon^2 + 2C_2 M_0^2 \epsilon^{2-2\alpha},
\end{aligned} \tag{3.61}$$

where we used $0 \leq \alpha, \beta \leq 3/4$.

Suppose first that (2.12) and (2.13) hold. The third term on the right-hand side of (3.61) satisfies

$$\begin{aligned}
2C_2M_1^2 2^{2\beta}(C_1+C_2)^{2\beta}\epsilon^{2-4\beta} &= 2C_2M_1^2 2^{2\beta}(C_1+C_2)^{2\beta}\epsilon^{3-4\beta}\epsilon^{-1} \\
&\le \frac{2C_2 2^{2\beta}(C_1+C_2)^{2\beta}}{\bar{C}^{3-4\beta}\epsilon} \le \frac{C_1+C_2}{20\epsilon},
\end{aligned}$$

where in the second step we used $M_1^2\epsilon^{3-4\beta} \le 1/\bar{C}^{3-4\beta}$, and this results from (3.24), while the last inequality holds if $\bar{C}$ is sufficiently large. We proceed similarly for the rest of the terms in (3.61) and obtain

$$\bar{Q}(\tau_1) \le \bar{Q}(\tau_0') + \frac{C_1+C_2}{\epsilon} \le \frac{C_1}{\epsilon} + \frac{C_1+C_2}{\epsilon} < 2\frac{C_1+C_2}{\epsilon}.$$

This is a contradiction with a choice of τ_1, and thus we conclude that (3.58) holds for all $\tau \ge \tau_0$. Finally, by (3.24) and (3.58), we get

$$\bar{Q}(\tau) \lesssim M_0^a + M_1^b + 1,$$

as desired. □

3.4. The order of vanishing. Now, we show that the modified frequency function $\bar{Q}$ controls the vanishing order of u. We first prove the following lemma, which shows the convergence of $\bar{Q}(\tau)$ as $\tau \to \infty$ and provides the connection between the order of vanishing of u and the quantity $\int u(x,t)^2 G(x,t)\,dx$, where G is defined in (3.1).

LEMMA 3.3. *Under the assumptions of Theorem 2.1, and assuming that u, v, and w are smooth, the modified frequency function satisfies $\bar{Q}(\tau) \to m/2$ as $\tau \to \infty$ for some $m \in \mathbb{N}$ such that $m \lesssim M_0^a + M_1^b + 1$, where a and b are as in (2.8). Also, with ϵ as in (3.24), for all $\delta > 0$, there exist $t_1 \in (-\log(1/\epsilon), 0)$ and $A_1(\delta), A_2(\delta) > 0$ such that*

$$A_1(\delta)|t|^{m+\delta} \le \int_{\mathbb{R}^n} u(x,t)^2 G(x,t)\,dx \le A_2(\delta)|t|^{m-\delta}, \tag{3.62}$$

for all $t \in [t_1, 0)$.

We emphasize that the constants $A_1(\delta)$ and $A_2(\delta)$ are allowed to depend on u, but not on t.

PROOF OF LEMMA 3.3. We start the proof by establishing the connection between $\bar{Q}$ and the behavior of $\|U(\tau)\|^2$. By Lemma 3.2, we have (3.25). Also, let x_ϵ be as in Lemma 3.1 and r as in (3.9). Taking the inner product of (3.23) with U, we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|U\|^2 + (A(\tau)U,U) = f(\tau), \tag{3.63}$$

where we denoted

$$f(\tau) = e^{-\tau}(VU,U) + \frac{1}{4}e^{-\tau/2}(y_jW_jU,U) + e^{-\tau/2}(W_j\partial_jU,U); \tag{3.64}$$

note that we used $(e^{-\tau/2}r_j\partial_jU,U) = 0$. Now, we bound the terms as in the proof of Lemma 3.2. For the first term, we have, as in (3.36),

$$\begin{aligned}
e^{-\tau}(VU,U) &\lesssim e^{-\tau}\sum_{j\in\mathbb{Z}^n}\|V\|_{L^p(\Omega_{j,\tau})}\|U\|^2_{L^{2p/(p-1)}(\Omega_{j,\tau})} \\
&\lesssim e^{(\alpha-1)\tau}M_0\sum_{j\in\mathbb{Z}^n}\|U\|^{2-n/p}_{L^2(\Omega_{j,\tau})}(\|\nabla U\|^{n/p}_{L^2(\Omega_{j,\tau})} + \|U\|^{n/p}_{L^2(\Omega_{j,\tau})}) \\
&\lesssim e^{(\alpha-1)\tau}M_0(\|U\|^{2-n/p}\|\nabla U\|^{n/p} + \|U\|^2).
\end{aligned}$$

Similarly, for the second term in (3.64)

$$\begin{aligned}
\frac{1}{4}e^{-\tau/2}(y_jW_jU,U) &\lesssim e^{-\tau/2}\sum_{j\in\mathbb{Z}^n}\|W\|_{L^q(\Omega_{j,\tau})}\|yU^2\|_{L^{q/(q-1)}(\Omega_{j,\tau})} \\
&\lesssim e^{(\beta-1)\tau}M_1\sum_{j\in\mathbb{Z}^n}\|yU\|_{L^2(\Omega_{j,\tau})}\|U\|_{L^{2q/(q-2)}(\Omega_{j,\tau})} \\
&\lesssim e^{(\beta-1)\tau}M_1\sum_{j\in\mathbb{Z}^n}\|yU\|_{L^2(\Omega_{j,\tau})}\|U\|^{1-n/q}_{L^2(\Omega_{j,\tau})}(\|\nabla U\|^{n/q}_{L^2(\Omega_{j,\tau})} + \|U\|^{n/q}_{L^2(\Omega_{j,\tau})}) \\
&\lesssim e^{(\beta-1)\tau}M_1\|yU\|\|U\|^{1-n/q}(\|\nabla U\|^{n/q} + \|U\|^{n/q}),
\end{aligned}$$

while for the third term in (3.64), we have

$$\begin{aligned} e^{-\tau/2}(W_j\partial_j U, U) &\lesssim e^{(\beta-1)\tau}M_1 \sum_{j\in\mathbb{Z}^n} \|\nabla U\|_{L^2(\Omega_{j,\tau})}\|U\|^{1-n/q}_{L^2(\Omega_{j,\tau})}(\|\nabla U\|^{n/q}_{L^2(\Omega_{j,\tau})}+\|U\|^{n/q}_{L^2(\Omega_{j,\tau})})\\ &\lesssim e^{(\beta-1)\tau}M_1\|\nabla U\|\|U\|^{1-n/q}(\|\nabla U\|^{n/q}+\|U\|^{n/q}),\end{aligned}$$

where α and β are as in (3.37) and (3.45). Therefore, by (3.41), we may bound

$$\frac{|f(\tau)|}{\|U(\cdot,\tau)\|^2} \lesssim e^{(\beta-1)\tau}M_1\big(|\bar{Q}(\tau)|+e^{-\tau}|r|^2+1\big)^{\beta}+e^{(\alpha-1)\tau}M_0\big(|\bar{Q}(\tau)|+e^{-\tau}|r|^2+1\big)^{\alpha}, \tag{3.65}$$

and thus, allowing all constants in this proof to depend on M_0 and M_1 (and thus also on ϵ and r), we obtain

$$\int_{\tau_1}^{\tau}\frac{|f(s)|}{\|U(\cdot,s)\|^2}\,ds \lesssim e^{-\tau/4}, \qquad \tau\geq\tau_1,$$

where $\tau_1\geq 0$ is arbitrary. Integrating the equation

$$\frac{1}{2\|U\|^2}\frac{d}{d\tau}\|U\|^2+\bar{Q}(\tau)=\frac{f(\tau)}{\|U\|^2},$$

from τ_1 to τ, we get

$$\frac12\log\|U(\cdot,\tau)\|^2-\frac12\log\|U(\cdot,\tau_1)\|^2=-\int_{\tau_1}^{\tau}\bar{Q}(s)\,ds+\int_{\tau_1}^{\tau}\frac{f(s)}{\|U(\cdot,s)\|^2}\,ds, \qquad \tau\geq\tau_1. \tag{3.66}$$

Note that (3.66) shows that $\bar{Q}$ controls the exponential decay of $\|U\|^2$.

In order to prove the first assertion in the statement of the lemma, it is sufficient to prove that $\bar{Q}$ converges to a number in $2^{-1}\mathbb{N}$. Thus, for the rest of the proof, we allow all the constants to depend on M_0 and M_1 (and thus also on ϵ and r). Using $\bar{Q}(\tau)\lesssim 1$ in (3.53) and integrating between τ_1 and τ, where $0\leq\tau_1\leq\tau$, we obtain

$$\bar{Q}(\tau)-\bar{Q}(\tau_1)+\int_{\tau_1}^{\tau}\|\big(A(s)-\bar{Q}(s)\mathcal{I}\big)\tilde{U}\|^2\,ds\lesssim e^{-\tau_1/4}+e^{-\tau/4},$$

from where, by $\bar{Q}(\tau_1)\lesssim 1$,

$$\int_{\tau_1}^{\tau}\|\big(A(s)-\bar{Q}(s)\mathcal{I}\big)\tilde{U}\|^2\,ds\lesssim 1+e^{-\tau_1/4}+e^{-\tau/4}, \qquad \tau\geq\tau_1. \tag{3.67}$$

We also have

$$\begin{aligned}\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2 &\geq \frac12\|(H-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2-Ce^{-\tau}|r|^2\||y|\tilde{U}\|^2\\ &\geq\frac12\|(H-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2-Ce^{-\tau},\end{aligned} \tag{3.68}$$

where we used (3.22) and $\int|y|^2\tilde{U}^2\leq\bar{Q}+C|r|^2e^{-\tau}+C$, which in turn follows from (3.17). Therefore, applying (3.68) in (3.67),

$$\int_{\tau_1}^{\tau}\|(H-\bar{Q}(s)\mathcal{I})\tilde{U}\|^2\,ds\lesssim\int_{\tau_1}^{\tau}\|\big(A(s)-\bar{Q}(s)\mathcal{I}\big)\tilde{U}\|^2\,ds+e^{-\tau_1}\lesssim 1+e^{-\tau_1/4}+e^{-\tau/4}<\infty, \qquad \tau\geq\tau_1, \tag{3.69}$$

where we used the boundedness of $\bar{Q}$ in the first inequality. Combining (3.69) with

$$\operatorname{dist}(\bar{Q}(\tau),\operatorname{sp}(H))\lesssim\|(H-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|, \tag{3.70}$$

recalling that $\|\tilde{U}\|=1$, we get

$$\int_0^{\infty}\operatorname{dist}(\bar{Q}(s),\operatorname{sp}(H))^2\,ds<\infty;$$

note that the inequality (3.70) follows since H is a self-adjoint positive operator whose inverse is a bounded compact operator. It is well-known that

$$\operatorname{sp}(H)=\left\{\frac{m}{2}:m\in\mathbb{N}_0\right\}, \tag{3.71}$$

(see [**CK**, p. 664]), whence

$$\int_0^{\infty}\operatorname{dist}(\bar{Q}(s),2^{-1}\mathbb{N}_0)^2\,ds<\infty. \tag{3.72}$$

An elementary argument shows that (3.72) combined with $\bar{Q}'(\tau) \lesssim 1$ for $\tau \geq \tau_1$ yields $\operatorname{dist}(\bar{Q}(\tau), 2^{-1}\mathbb{N}_0) \to 0$ as $\tau \to \infty$, i.e.,

$$\lim_{\tau\to\infty} \bar{Q}(\tau) \to \frac{m}{2}, \tag{3.73}$$

for some $m \in \mathbb{N}_0$ as in the statement.

It remains to prove the last assertion in the statement of the lemma. From (3.66) and (3.73), we deduce that for every $\delta > 0$ there exists $\tau_1 > 0$, depending on δ, such that

$$-\delta(\tau - \tau_1) \leq \log \|U(\tau)\|^2 - \log \|U(\tau_1)\|^2 + m(\tau - \tau_1) \leq \delta(\tau - \tau_1), \qquad \tau \geq \tau_1,$$

from where

$$e^{-\delta(\tau-\tau_1)} \leq \frac{e^{\tau m}\|U(\tau)\|^2}{e^{\tau_1 m}\|U(\tau_1)\|^2} \leq e^{\delta(\tau-\tau_1)}.$$

Therefore, there exist $A_1(\delta), A_2(\delta) > 0$ such that

$$A_1(\delta)e^{-\delta\tau} \leq e^{\tau m}\|U(\tau)\|^2 \leq A_2(\delta)e^{\delta\tau}.$$

Finally, recalling (3.18), we obtain (3.62). $\square$

The following lemma provides a control on $\int_{B_R} P(x,t)G(x,t)\,dx$, with P a homogeneous polynomial of degree d. We use this lemma in the proof of Theorem 2.1.

LEMMA 3.4. *Let $P(x,t) = \Sigma_{|\mu|+2l=d} C_{\mu,l} x^\mu t^l$ be a homogeneous polynomial of degree $d \in \mathbb{N}$. Then,*

$$\int_{\mathbb{R}^n} P(x,t)G(x,t)\,dx \lesssim |t|^{d/2}, \qquad t < 0, \tag{3.74}$$

where the constant in (3.74) depends on the polynomial only. Moreover, if all the coordinates of $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ are even, then for all $R > 0$,

$$\int_{B_R} x^\mu t^l G(x,t)\,dx \lesssim |t|^{l+|\mu|/2}, \qquad t < 0,$$

as $t \to 0^-$. If μ_i is an odd integer for some $i \in \{1, \ldots, n\}$, then

$$\int_{B_R} x^\mu t^l G(x,t)\,dx = 0,$$

for $t < 0$.

For the proof of Lemma 3.4, see [**CK**, p. 670].

PROOF OF THEOREM 2.1. Without loss of generality, let $I = [-1, 0]$ and $(x_0, t_0) = (0, 0)$. First, we assume that u, v, and w are smooth. Let ϵ be as in (3.24), and let m be as in the statement of Lemma 3.3. Denote by d the vanishing order of u at $(0, 0)$. We claim that $d \leq m$. Since the degree of vanishing of u at $(0, 0)$ is d, we have

$$|u(x,t)| \lesssim (|x|^2 + |t|)^{d/2}, \tag{3.75}$$

for all $(x,t) \in Q_1(0,0)$ with $Q_1(0,0)$ defined in (2.4); from (3.75) to (3.80) below, the constants are allowed to depend on u. Let $\delta \in (0, 1]$ be arbitrary. By Lemma 3.3, there exist $t_1 \in (-\log(1/\epsilon), 0)$ and $A_1(\delta), A_2(\delta) > 0$ such that

$$A_1(\delta)|t|^{m+\delta} \leq \int u(x,t)^2 G(x,t)\,dx \leq A_2(\delta)|t|^{m-\delta}, \qquad t \in [t_1, 0). \tag{3.76}$$

Let $R = 1/4$. Note that we have

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_R} u(x,t)^2 G(x,t)\,dx &\lesssim \|u\|^2_{L^\infty(\mathbb{T}^n)} \int_R^\infty \frac{e^{-\rho^2/4|t|}}{|t|^{n/2}} \rho^{n-1}\,d\rho \\
&\leq \|u\|^2_{L^\infty(\mathbb{T}^n)} e^{-R^2/8|t|} \int_R^\infty \frac{e^{-\rho^2/8|t|}}{|t|^{n/2}} \rho^{n-1}\,d\rho \lesssim \|u\|^2_{L^\infty(\mathbb{T}^n)} e^{-R^2/8|t|} \int_0^\infty \frac{e^{-\rho^2/8|t|}}{|t|^{n/2}} \rho^{n-1}\,d\rho \\
&\lesssim \|u\|^2_{L^\infty(\mathbb{T}^n)} e^{-R^2/8|t|}.
\end{aligned} \tag{3.77}$$

Using (3.77) in (3.76), we may increase $t_1 < 0$ to obtain

$$\frac{1}{2}A_1(\delta)|t|^{m+\delta} \leq \int_{B_R} u(x,t)^2 G(x,t)\,dx \leq 2A_2(\delta)|t|^{m-\delta}, \qquad t \in [t_1, 0). \tag{3.78}$$

Moreover, by (3.75) and Lemma 3.4,

$$\int_{B_R} u(x,t)^2 G(x,t)\,dx \lesssim \int_{B_R} (|x|^2+|t|)^d G(x,t)\,dx \lesssim |t|^d. \tag{3.79}$$

Combining (3.78) with (3.79), we get

$$A_1(\delta)|t|^{m+\delta} \lesssim |t|^d, \qquad t\in[t_1,0),$$

which yields

$$d \le m+\delta.$$

Letting $\delta\to 0$, we conclude that

$$d \le m, \tag{3.80}$$

as desired. Thus we have proven Theorem 2.1 under the additional assumption that v, w, and u_0 are smooth.

Now, consider the general case. Recall that u (which still represents $\bar{u}$ from (3.5)) satisfies the equation (3.10), and it is defined for all $t\in[-1,0]$, even though its frequency was studied only for $t\ge\log\epsilon$. Note that by the parabolic regularity u is locally Hölder continuous and $u\in L_t^\infty H_x^1\cap L_t^2H_x^2$ on $(-1+\delta,0)\times\mathbb{T}^n$. Therefore, we may assume, without loss of generality, that $u_0\in C(\mathbb{T}^n)\cap H^1(\mathbb{T}^n)$, just by adjusting the initial time, and that $u\in L_t^\infty H_x^1\cap L_t^2H_x^2$ on $(-1,0)\times\mathbb{T}^n$. Recall that $n\ge 2$; cf. Remark 2.6 for $n=1$.

We approximate v, w, and u_0 by smooth functions v^η, w^η, and u_0^η, where $\eta\in(0,1]$, so that v^η, w^η, and u_0^η converge in $L_t^sL_x^p$, $L_t^sL_x^q$, and $L_x^\infty(\mathbb{T}^2)\cap H_x^1(\mathbb{T}^2)$ to v, w, and u_0, respectively, for any $s\in[1,\infty)$ as $\eta\to 0$. In other words, we have

$$\lim_{\eta\to 0}(\|v-v^\eta\|_{L_t^sL_x^p}+\|w-w^\eta\|_{L_t^sL_x^q}+\|u_0-u_0^\eta\|_{H^1(\mathbb{T}^n)}+\|u_0-u_0^\eta\|_{L^\infty(\mathbb{T}^n)})=0, \qquad s\in[1,\infty). \tag{3.81}$$

In the rest of the proof, the space-time Lebesgue spaces are understood to be over $\mathbb{T}^n\times(-1,0)$; also, we may assume that $\|v^\eta\|_{L_t^\infty L_x^p}$, $\|w^\eta\|_{L_t^\infty L_x^q}$, $\|u_0^\eta\|_{L^\infty(\mathbb{T}^n)}$, and $\|u_0^\eta\|_{H^1(\mathbb{T}^n)}$ are uniformly bounded by constant multiples of $\|v\|_{L_t^\infty L_x^p}$, $\|w\|_{L_t^\infty L_x^q}$, $\|u_0\|_{L^\infty(\mathbb{T}^n)}$, and $\|u_0\|_{H^1(\mathbb{T}^n)}$, respectively. For convenience, we allow all constants until (3.96) below to depend on these four quantities, as well as on $\|u\|_{L_t^\infty H_x^1}$ and $\|u\|_{L_t^2L_x^2}$.

For $\eta\in(0,1]$, let u^η be a solution of the equation

$$\begin{aligned}&\partial_t u^\eta-\Delta u^\eta = r\cdot\nabla u^\eta + w^\eta\cdot\nabla u^\eta + v^\eta u^\eta\\ &u^\eta(\cdot,-1)=u_0^\eta.\end{aligned} \tag{3.82}$$

Subtracting (3.82) from (2.1), we get

$$\begin{aligned}&\partial_t\tilde{u}-\Delta\tilde{u} = r\cdot\nabla\tilde{u} + w^\eta\cdot\nabla\tilde{u} + \tilde{w}\cdot\nabla u + v^\eta\tilde{u}+\tilde{v}u\\ &\tilde{u}(\cdot,-1)=u_0^\eta-u_0,\end{aligned} \tag{3.83}$$

where $\tilde{u}=u^\eta-u$, $\tilde{v}=v^\eta-v$, and $\tilde{w}=w^\eta-w$. First, we have $u,\tilde{u}\in L_t^\infty L_x^2\cap L_t^2H_x^1$. Using $v\in L_t^\infty L_x^{3n/2}$ and $w\in L_t^\infty L_x^{2n}$, we also get

$$D^2u, D^2\tilde{u}\in L_t^2L_x^2. \tag{3.84}$$

Taking the inner product of (3.83) with $\tilde{u}$, we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|\tilde{u}\|_{L^2(\mathbb{T}^n)}^2+\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 &= \int_{\mathbb{T}^n} r_j\tilde{u}\partial_j\tilde{u}+\int_{\mathbb{T}^n} w_j^\eta\tilde{u}\partial_j\tilde{u}+\int_{\mathbb{T}^n}\tilde{w}_j\tilde{u}\partial_j u+\int_{\mathbb{T}^n} v^\eta\tilde{u}^2+\int_{\mathbb{T}^n}\tilde{v}u\tilde{u}\\ &\lesssim |r|\|\tilde{u}\|_{L^2(\mathbb{T}^n)}\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}+\|w^\eta\|_{L^q(\mathbb{T}^n)}\|\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad+\|\tilde{w}\|_{L^q(\mathbb{T}^n)}\|\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\nabla u\|_{L^2(\mathbb{T}^n)}+\|v^\eta\|_{L^p(\mathbb{T}^n)}\|\tilde{u}\|_{L^{2p/(p-1)}(\mathbb{T}^n)}^2\\ &\quad+\|\tilde{v}\|_{L^p(\mathbb{T}^n)}\|u\|_{L^{2p/(p-1)}(\mathbb{T}^n)}\|\tilde{u}\|_{L^{2p/(p-1)}(\mathbb{T}^n)},\end{aligned}$$

whence

$$\begin{aligned}&\frac12\frac{d}{dt}\|\tilde{u}\|_{L^2(\mathbb{T}^n)}^2+\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^2\\ &\quad\lesssim \|\tilde{u}\|_{L^2(\mathbb{T}^n)}\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}+\|\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}+\|\tilde{w}\|_{L^q(\mathbb{T}^n)}\|\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\\ &\qquad+\|\tilde{u}\|_{L^{2p/(p-1)}(\mathbb{T}^n)}^2+\|\tilde{v}\|_{L^p(\mathbb{T}^n)}\|u\|_{L^{2p/(p-1)}(\mathbb{T}^n)}\|\tilde{u}\|_{L^{2p/(p-1)}(\mathbb{T}^n)}.\end{aligned}$$

Using $\|\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)} \lesssim \|\tilde{u}\|_{L^2(\mathbb{T}^n)}^{1-n/q}\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^{n/q} + \|\tilde{u}\|_{L^2(\mathbb{T}^n)}$ and $\|\tilde{u}\|_{L^{2p/(p-1)}(\mathbb{T}^n)} \lesssim \|\tilde{u}\|_{L^2(\mathbb{T}^n)}^{1-n/2p}\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^{n/2p} + \|\tilde{u}\|_{L^2(\mathbb{T}^n)}$, we get

$$\frac{d}{dt}\|\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 \lesssim \|\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 + \|\tilde{w}\|_{L^q(\mathbb{T}^n)} + \|\tilde{v}\|_{L^p(\mathbb{T}^n)}. \tag{3.85}$$

Let $\epsilon_0 \in (0,1]$. Then, for η sufficiently small, we have

$$\|\tilde{u}(\cdot,-1)\|_{H^1(\mathbb{T}^n)}^2, \|\tilde{w}\|_{L_t^1L_x^q}, \|\tilde{v}\|_{L_t^1L_x^p} \leq \epsilon_0, \tag{3.86}$$

where, recall, the mixed space-time norms are taken over $\mathbb{T}^n \times (-1,0)$. Applying (3.86) and the Gronwall inequality to (3.85) leads to

$$\|\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 \lesssim \epsilon_0, \qquad t \in [-1,0].$$

Since $\epsilon_0 \in (0,1]$ was arbitrary, we get

$$\lim_{\eta\to 0}\sup_{t\in[-1,0]}\|\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)} = 0. \tag{3.87}$$

In order to obtain the analog of (3.87) for the H^1-norm, we test the first equation in (3.83) with $-\Delta\tilde{u}$, which we may by (3.84), and obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2 + \|\Delta\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2\\ &\quad = -\int_{\mathbb{T}^n} r_j\partial_j\tilde{u}\Delta\tilde{u} - \int_{\mathbb{T}^n} w_j^\eta\partial_j\tilde{u}\Delta\tilde{u} - \int_{\mathbb{T}^n}\tilde{w}_j\partial_j u\Delta\tilde{u} - \int_{\mathbb{T}^n} v^\eta\tilde{u}\Delta\tilde{u} - \int_{\mathbb{T}^n}\tilde{v}u\Delta\tilde{u}.\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2 + \|\Delta\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2\\ &\quad\lesssim |r|\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} + \|w^\eta\|_{L^q(\mathbb{T}^n)}\|\nabla\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\quad + \|\tilde{w}\|_{L^q(\mathbb{T}^n)}\|\nabla u\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} + \|v^\eta\|_{L^p(\mathbb{T}^n)}\|\tilde{u}\|_{L^{2p/(p-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\quad + \|\tilde{v}\|_{L^p(\mathbb{T}^n)}\|u\|_{L^{2p/(p-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}.\end{aligned} \tag{3.88}$$

For the second term on the right-hand side, we have, using the agreement on constants from above (3.82),

$$\begin{aligned}&\|w^\eta\|_{L^q(\mathbb{T}^n)}\|\nabla\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} \leq \|\nabla\tilde{u}\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\lesssim \|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^{(q-n)/q}(\|D^2\tilde{u}\|_{L^2(\mathbb{T}^n)} + \|\tilde{u}\|_{L^2})^{n/q}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\lesssim \|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^{(q-n)/q}(\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} + \|\tilde{u}\|_{L^2})^{n/q}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\lesssim \epsilon_0\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 + C_{\epsilon_0}(\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 + \|\tilde{u}\|_{L^2(\mathbb{T}^n)}^2),\end{aligned} \tag{3.89}$$

where $\epsilon_0 \in (0,1]$ is arbitrary. An analogous estimate also holds for the fourth term on the right-hand side of (3.88). For the third term, we have similarly to (3.89)

$$\begin{aligned}&\|\tilde{w}\|_{L^q(\mathbb{T}^n)}\|\nabla u\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} \lesssim \|\tilde{w}\|_{L^q(\mathbb{T}^n)}\|\nabla u\|_{L^2(\mathbb{T}^n)}^{(q-n)/q}(\|D^2u\|_{L^2(\mathbb{T}^n)} + \|u\|_{L^2(\mathbb{T}^n)})^{n/q}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\lesssim \|\tilde{w}\|_{L^q(\mathbb{T}^n)}(\|\Delta u\|_{L^2(\mathbb{T}^n)} + \|u\|_{L^2(\mathbb{T}^n)})^{n/q}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} \lesssim \|\tilde{w}\|_{L^q(\mathbb{T}^n)}(\|\Delta u\|_{L^2(\mathbb{T}^n)} + 1)^{n/q}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\lesssim \epsilon_0\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 + \epsilon_0\|\Delta u\|_{L^2(\mathbb{T}^n)}^2 + C_{\epsilon_0}\|\tilde{w}\|_{L^q(\mathbb{T}^n)}^{2q/(q-n)},\end{aligned}$$

while for the last term in (3.88), we estimate similarly

$$\begin{aligned}&\|\tilde{v}\|_{L^p(\mathbb{T}^n)}\|u\|_{L^{2p/(p-2)}(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} \lesssim \|\tilde{v}\|_{L^p(\mathbb{T}^n)}\|u\|_{L^2(\mathbb{T}^n)}^{(p-n)/p}(\|Du\|_{L^2} + \|u\|_{L^2})^{n/p}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}\\ &\quad\lesssim \|\tilde{v}\|_{L^p(\mathbb{T}^n)}\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)} \lesssim \epsilon_0\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 + C_{\epsilon_0}\|\tilde{v}\|_{L^p(\mathbb{T}^n)}^2.\end{aligned}$$

Using all the bounds on the terms on the right hand side of (3.88) and absorbing the terms involving $\|\Delta\tilde{u}\|_{L^2(\mathbb{T}^n)}^2$, we obtain

$$\frac{d}{dt}\|\nabla\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2 \lesssim \epsilon_0\|\Delta u\|_{L^2(\mathbb{T}^n)}^2 + C_{\epsilon_0}\big(\|\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 + \|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}^2 + \|\tilde{v}\|_{L^q(\mathbb{T}^n)}^2 + \|\tilde{w}\|_{L^q(\mathbb{T}^n)}^2\big).$$

Now, applying the Gronwall inequality, along with (3.81) and (3.87), we get

$$\limsup_{\eta\to 0}\sup_{t\in[-1,0]}\|\nabla\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)} \lesssim \epsilon_0\int_{\mathbb{T}^n}\|\Delta u\|_{L^2(\mathbb{T}^n)}^2 \lesssim \epsilon_0. \tag{3.90}$$

Denote by $\bar{Q}^\eta(\tau)$ and $\bar{Q}(\tau)$ the modified frequency functions corresponding to u^η and u, respectively. Recalling that

$$\bar{Q}(\tau) = \frac{(HU,U)}{\|U\|^2} - \frac{e^{-\tau/2}r_j}{\|U\|^2}\int y_j U^2\,dy, \tag{3.91}$$

and similarly

$$\bar{Q}^\eta(\tau) = \frac{(HU^\eta,U^\eta)}{\|U^\eta\|^2} - \frac{e^{-\tau/2}r_j}{\|U^\eta\|^2}\int y_j (U^\eta)^2\,dy, \tag{3.92}$$

we now claim that $\bar{Q}^\eta(\tau) \to \bar{Q}(\tau)$ as $\eta \to 0$ for all $\tau \in [0,\infty)$; here, U is given in (3.11), while $U^\eta(y,\tau) = e^{-|y|^2/8}u^\eta(ye^{-\tau/2}, -e^{-\tau})$ for $(y,\tau) \in \mathbb{R}^n \times [\tau_0,\infty)$. Note that

$$\begin{aligned}\big|\|U^\eta(\tau)\|^2 - \|U(\tau)\|^2\big| &= \left|\int ((u^\eta)^2 - u^2)G(x,t)\,dx\right| \lesssim \frac{1}{t^{n/2}}\int |(u^\eta)^2 - u^2|\,dx\\ &\lesssim \frac{1}{t^{n/2}}\|\tilde{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)}(\|u^\eta(\cdot,t)\|_{L^2(\mathbb{T}^n)} + \|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}),\end{aligned}$$

and thus

$$\lim_{\eta\to 0}\|U^\eta(\tau)\|^2 = \|U(\tau)\|^2, \qquad \tau \geq 0, \tag{3.93}$$

by (3.87). Also, we have

$$\begin{aligned}|(HU^\eta,U^\eta) - (HU,U)| &\leq |t|\int \big||\nabla u^\eta|^2 - |\nabla u|^2\big|G(x,t)\,dx\\ &\leq \frac{1}{|t|^{n/2-1}}\|\nabla\tilde{u}\|_{L^2(\mathbb{T}^n)}\|\nabla(u^\eta + u)\|_{L^2(\mathbb{T}^n)},\end{aligned} \tag{3.94}$$

from where

$$\lim_{\eta\to 0}(HU^\eta,U^\eta) = (HU,U), \qquad \tau \geq 0, \tag{3.95}$$

by (3.90). Lastly, we examine the convergence of $\int y_j(U^\eta)^2\,dy$. For this purpose, we estimate, for every $\tau \geq 0$,

$$\begin{aligned}\left|\int y_j(U^\eta)^2\,dy - \int y_jU^2\,dy\right| &\leq \int |y||(U^\eta)^2 - U^2| \lesssim \|y(U^\eta - U)\|\|U^\eta + U\|\\ &\lesssim \big((H(U^\eta - U), U^\eta - U)^{1/2} + \|U^\eta - U\|\big)\|U^\eta + U\|\\ &= \left(|t|\int_{\mathbb{R}^n}|\nabla\tilde{u}(x,t)|^2G(x,t)\,dx + \|U^\eta - U\|\right)\|U^\eta + U\|\\ &\lesssim \big((|t|\|\nabla\tilde{u}\|\|\nabla\tilde{u}G\|)^{1/2} + \|U^\eta - U\|\big)\|U^\eta + U\|,\end{aligned} \tag{3.96}$$

where we used $\|yU\|^2 \lesssim (HU,U) + \|U\|^2$ in the third inequality, which in turn follows from the identity

$$(HU,U) = \|\nabla U\|^2 + \frac{1}{16}\|yU\|^2 - \frac{n}{4}\|U\|^2;$$

also, in the fourth step of (3.96), we applied (3.19). By (3.93), (3.94), and (3.96), we get

$$\lim_{\eta\to 0}\int y(U^\eta)^2\,dy = \int yU^2\,dy. \tag{3.97}$$

Using (3.93), (3.95), and (3.97) in (3.91) and (3.92), we obtain that $\bar{Q}^\eta(\tau) \to \bar{Q}(\tau)$ as $\eta \to 0$ for all $\tau \in [0,\infty)$. Note that we have

$$\bar{Q}^\eta(\tau) \lesssim 1, \tag{3.98}$$

uniformly in τ and $\eta \in (0,1]$, where the constant depends only on q_0. Passing to the limit, we get

$$\bar{Q}(\tau) \lesssim 1, \qquad \tau \geq 0. \tag{3.99}$$

Next, by (3.69) and (3.70), we obtain, for all $\eta \in (0,1]$,

$$\int_{\tau_1}^{\tau}\mathrm{dist}(\bar{Q}^\eta(\tau), \mathrm{sp}(H))^2 \lesssim 1 + e^{-\tau_1/4} + e^{-\tau/4}, \qquad \tau \geq \tau_1 \geq 0, \tag{3.100}$$

allowing the constants to depend on M_0 and M_1 as above. Observe that by (3.98) we have $\bar{Q}^\eta(\tau) \in [0, C]$, where the constant is independent of η and τ. Thus letting $\eta \to 0$ in (3.100) yields

$$\int_{\tau_1}^{\tau} \operatorname{dist}(\bar{Q}(\tau), \operatorname{sp}(H))^2 \lesssim 1 + e^{-\tau_1/4} + e^{-\tau/4}, \qquad \tau \geq \tau_1 \geq 0. \tag{3.101}$$

Combining (3.71), (3.99), and (3.101), we may use the same argument as in the proof of Lemma 3.3, to obtain $\bar{Q}(\tau) \to m/2$ for some $m \in \mathbb{N}_0$. The rest of the proof is similar to the case when v, w, and u_0 are smooth. ☐

4. Pointwise in time observability

In this section, we use the notation from Section 3.2. We fix ϵ as in (3.24), where a and b are given in (2.8). Since we are interested in observability, it is advantageous to slightly generalize the function $\bar{u}$ from (3.5) as follows. Let $t_1 \in [-\epsilon/2, 0]$ be arbitrary but fixed, and let $\tau_1 = -\log t_1$. Then, define

$$\bar{u}(x,t) = u\left(x - \frac{x_\epsilon}{t_1+\epsilon}(t - t_1), t\right), \tag{4.1}$$

where x_ϵ is as in Lemma 3.1, so that, instead of (3.6) and (3.7), we have

$$\bar{u}(x, -\epsilon) = u(x + x_\epsilon, -\epsilon)$$

and

$$\bar{u}(x, t_1) = u(x, t_1),$$

for all $x \in \mathbb{T}^n$. The equation (3.10) continues to hold with (3.9) replaced by

$$r = -\frac{x_\epsilon}{-t_1+\epsilon}.$$

Since $|r| \lesssim \epsilon^{-1}$, due to $t_1 \in [-\epsilon/2, 0]$, all the estimates from Section 3 continue to hold for $\bar{u}$ in (4.1).

The following lemma provides a comparison between the unweighted and weighted L^2 norms of $\bar{u}$.

LEMMA 4.1. *Under the assumptions of Theorem 2.1, with $\bar{u}$ defined in (4.1), and ϵ in (3.24), we have*

$$\|\bar{u}(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2 \lesssim \frac{e^K}{|t|^M} \int_{\mathbb{R}^n} \bar{u}(x,t)^2 G(x,t)\,dx, \tag{4.2}$$

for all $t \in [-\epsilon, 0)$, where $K = C(M_1^2 + M_0 + M_1 M^{2\beta-1} + M_0 M^{2\alpha-1})$ and $M = M_0^a + M_1^b + 1$ with a and b as in (2.8).

PROOF OF LEMMA 4.1. In this proof, we use the convention from the proof of Lemma 3.2 by writing u instead of $\bar{u}$. We start with the identity (3.63), where f is defined in (3.64). From (3.65), we get

$$\frac{|f(\tau)|}{\|U\|^2} \lesssim e^{(\beta-1)\tau} M_1 \big(|\bar{Q}(\tau)| + e^{-\tau}|r|^2 + 1\big)^\beta + e^{(\alpha-1)\tau} M_0 \big(|\bar{Q}(\tau)| + e^{-\tau}|r|^2 + 1\big)^\alpha,$$

where α and β are as in (3.37) and (3.45). Therefore, as in the first equality in (3.66), we have

$$\frac{1}{2}\log\|U(\cdot,\tau)\|^2 - \frac{1}{2}\log\|U(\cdot,\tau_0)\|^2 = -\int_{\tau_0}^{\tau} \bar{Q}(s)\,ds + \int_{\tau_0}^{\tau} \frac{f(s)}{\|U(\cdot,s)\|^2}\,ds, \qquad \tau \geq \tau_0,$$

from where we obtain, using $\bar{Q}(\tau) \lesssim M = M_0^a + M_1^b + 1$ for $\tau \geq \tau_0$, that

$$\begin{aligned} &\log\frac{\|U(\cdot,\tau_0)\|^2}{\|U(\cdot,\tau)\|^2} \\ &\quad\lesssim (\tau-\tau_0)M + e^{(\beta-1)\tau_0} M_1 (M + e^{-\tau_0}|r|^2 + 1)^\beta + e^{(\alpha-1)\tau_0} M_0 (M + e^{-\tau_0}|r|^2+1)^\alpha \\ &\quad\lesssim (\tau-\tau_0)M + \epsilon^{1-\beta} M_1 (M + \epsilon|r|^2 + 1)^\beta + \epsilon^{1-\alpha} M_0 (M + \epsilon|r|^2 + 1)^\alpha, \qquad \tau \geq \tau_0, \end{aligned} \tag{4.3}$$

where we used $\epsilon = e^{-\tau_0}$ from (3.13).

We now derive a similar estimate for $\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2/\|u(\cdot,-\epsilon)\|_{L^2(\mathbb{T}^n)}^2$. Taking the inner product of (3.10) with u and using $\|\nabla u\|_{L^2(\mathbb{T}^n)}^2 = q_{\rm D}(t)\|u\|_{L^2(\mathbb{T}^n)}^2$, where $q_{\rm D}$ is defined in (2.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_{L^2(\mathbb{T}^n)}^2 + q_{\rm D}(t)\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2\\
&\quad\lesssim \|w\|_{L^q(\mathbb{T}^n)}\|u\|_{L^{2q/(q-2)}(\mathbb{T}^n)}\|u\|_{L^2(\mathbb{T}^n)}q_{\rm D}(t)^{1/2} + \|v\|_{L^p(\mathbb{T}^n)}\|u\|_{L^{2p/(p-1)}(\mathbb{T}^n)}^2\\
&\quad\lesssim M_1\|u\|_{L^2(\mathbb{T}^n)}^{2-2\beta}\|\nabla u\|_{L^2(\mathbb{T}^n)}^{2\beta-1}\|u\|_{L^2(\mathbb{T}^n)}q_{\rm D}(t)^{1/2} + M_1\|u\|_{L^2(\mathbb{T}^n)}^2 q_{\rm D}(t)^{1/2}\\
&\qquad + M_0\|u\|_{L^2(\mathbb{T}^n)}^{2-2\alpha}\|\nabla u\|_{L^2(\mathbb{T}^n)}^{2\alpha} + M_0\|u\|_{L^2(\mathbb{T}^n)}^2,
\end{aligned} \tag{4.4}$$

since $\int_{\mathbb{T}^n} r_j u\partial_j u\,dx = 0$, where the quantities are evaluated at t. Dividing both sides of (4.4) by $\|u\|_{L^2(\mathbb{T}^n)}^2$, we get

$$\frac{1}{\|u\|_{L^2(\mathbb{T}^n)}^2}\frac{d}{dt}\|u\|_{L^2(\mathbb{T}^n)}^2 + q_{\rm D}(t) \lesssim M_1 q_{\rm D}(t)^\beta + M_1 q_{\rm D}(t)^{1/2} + M_0 q_{\rm D}(t)^\alpha + M_0 \lesssim M_1 + M_0, \tag{4.5}$$

allowing the last implicit constant to depend on q_0. Integrating (4.5) from $-\epsilon$ to t leads to

$$\log\frac{\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2}{\|u(\cdot,-\epsilon)\|_{L^2(\mathbb{T}^n)}^2} \lesssim (M_1+M_0)\epsilon, \qquad t\in(-\epsilon,0). \tag{4.6}$$

By the definition of U in (3.11), we have

$$\frac{\|U(\cdot,\tau_0)\|^2}{\|u(\cdot,-\epsilon)\|_{L^2(\mathbb{T}^n)}^2} = \frac{\int_{\mathbb{R}^n} u(x,-\epsilon)^2 e^{-|x|^2/4\epsilon}\,dx}{\epsilon^{n/2}\int_{\mathbb{T}^n} u(x,-\epsilon)^2\,dx} \geq \frac{e^{-n/4\epsilon}}{\epsilon^{n/2}}, \tag{4.7}$$

since $|x|\leq\sqrt{n}$ for $x\in\Omega$. We then take the logarithm of (4.7) to get

$$\log\frac{\|u(\cdot,-\epsilon)\|_{L^2(\mathbb{T}^n)}^2}{\|U(\cdot,\tau_0)\|^2} \leq -\frac{n}{2}\log\frac{1}{\epsilon} + \frac{n}{4\epsilon} \lesssim \frac{1}{\epsilon}. \tag{4.8}$$

Adding (4.3), (4.6), and (4.8), we obtain

$$\begin{aligned}
\log\frac{\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2}{\|U(\cdot,\tau)\|^2} &\lesssim (M_1+M_0)\epsilon + \frac{1}{\epsilon} + (\tau - \log\epsilon^{-1})M\\
&\quad + \epsilon^{1-\beta}M_1(M+\epsilon|r|^2+1)^\beta + \epsilon^{1-\alpha}M_0(M+\epsilon|r|^2+1)^\alpha\\
&\lesssim \frac{M_0+M_1}{M} + M + \tau M + (M_0+M_1)M^{2\beta-1},
\end{aligned}$$

where we used $\epsilon\lesssim 1/M$ and $|r|\lesssim\epsilon^{-1}\lesssim M$, and thus

$$\log\frac{\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2}{\|U(\cdot,\tau)\|^2} \lesssim M_1^2 + M_0 + M\tau + M_1M^{2\beta-1} + M_0M^{2\alpha-1},$$

whence

$$\|u(\cdot,t)\|_{L^2(\mathbb{T}^n)}^2 \lesssim \exp(C(M_1^2 + M_0 + M_1M^{2\beta-1} + M_0M^{2\alpha-1}))\frac{1}{|t|^M}\int_{\mathbb{R}^n} u(x,t)^2G(x,t)\,dx,$$

as claimed. $\square$

Lemma 4.1 is combined below with the next statement from [**Ku4**].

LEMMA 4.2. *Let $K\geq 0$ and $t_0\in[-1/4,0)$. If a 1-periodic function f satisfies*

$$\|f\|_{L^2(\mathbb{T}^n)}^2 \leq e^K\int_{\mathbb{R}^n} f(x)^2G(x,t)\,dx$$

with $t\in[t_0,0)$ such that

$$\frac{1}{|t|} \geq \frac{C}{|t_0|}\log\frac{1}{|t|} + \frac{C(K+1)}{|t_0|},$$

for a sufficiently large constant $C>0$, then

$$\|f\|_{L^2(\mathbb{T}^n)}^2 \leq \frac{Ce^K}{|t|^{n/2}}\|f\|_{L^2(B_{\sqrt{|t_0|}})}^2.$$

For the proof, see [**Ku4**, p. 775].

PROOF OF THEOREM 2.3. We first assume that u_0, v, and w are smooth as in the proof of Theorem 2.1. By Lemma 4.1, we have (4.2), where K and M are given in the statement. In order to apply Lemma 4.2, we need $t \in [-\epsilon/2, 0)$, along with

$$\frac{1}{|t|} \geq \frac{C}{\epsilon} \log \frac{1}{|t|} + \frac{C(K+1)}{\epsilon} M \log \frac{1}{|t|}. \tag{4.9}$$

One may readily check that the sufficient condition for (4.9) is

$$|t| \leq \frac{\epsilon}{C(K+1)M(\log(1/\epsilon))^2(\log((K+2)(M+1)))^2}. \tag{4.10}$$

Let $t_1 \in [-\epsilon/2, 0]$ be such that $|t_1|$ equals the right-hand side of (4.10). Using this t_1 in (4.1) and applying Lemma 4.2 leads to

$$\|u(\cdot, t_1)\|_{L^2(\mathbb{T}^n)} \leq e^{P(M)} \|u(\cdot, t_1)\|_{L^2(B_{\sqrt{|t_1|}})},$$

where P is a polynomial, and we obtain the conclusion of the theorem for the time t_1. For other times, we simply translate in time. (Note that it sufficient to obtain the observability estimate (2.9) for a sufficiently small δ_0, as it is then automatic for larger values.)

For the general case, we approximate u_0, v, and w by smooth functions u_0^η, v^η, and w^η respectively. We then have $\|u^\eta(\cdot, t)\|^2_{L^2(\mathbb{T}^n)} \to \|u(\cdot, t)\|^2_{L^2(\mathbb{T}^n)}$ and $\|u^\eta(\cdot, t)\|^2_{L^2(B_{\delta_0})} \to \|u(\cdot, t)\|^2_{L^2(B_{\delta_0})}$ as $\eta \to 0$ by (3.87). $\square$

5. The case $\mathbb{R}^n$

In this section, we prove the theorem concerning the case of $\mathbb{R}^n$.

PROOF OF THEOREM 2.5. Note that we now assume the growth condition (2.11) instead of periodicity. Again, without loss of generality, we may consider $I = [-1, 0]$ and $(x_0, t_0) = (0, 0)$. The proof is similar to the periodic case with small modifications. With $\epsilon \in [0, 1/2]$, we have

$$\frac{\epsilon \int_{\mathbb{R}^n} |\nabla u(x_\epsilon + y, -\epsilon)|^2 G(y, -\epsilon)\, dy}{\int_{\mathbb{R}^n} u(x_\epsilon + y, -\epsilon)^2 G(y, -\epsilon)\, dy} \leq K \epsilon q_{\mathrm{D}}(-\epsilon), \tag{5.1}$$

for some $x_\epsilon \in B_2$. This was proven in [**CK**], but since the argument is short, we present it here. Assume that

$$\lambda \int_{\mathbb{R}^n} u(x+y, -\epsilon)^2 G(y, -\epsilon)\, dy < \int_{\mathbb{R}^n} |\nabla u(x+y, -\epsilon)|^2 G(y, -\epsilon)\, dy, \qquad x \in B_2, \tag{5.2}$$

where $\lambda = C_0 K q_{\mathrm{D}}(-\epsilon)$ and $C_0 \geq 1$ is sufficiently large constant, to be determined. Integrating (5.2) over B_2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} G(y, -\epsilon)\, dy \int_{B_2} u(x+y, -\epsilon)^2\, dx &< \frac{1}{\lambda} \int_{\mathbb{R}^n} G(y, -\epsilon)\, dy \int_{B_2} |\nabla u(x+y, -\epsilon)|^2\, dx \\ &\leq \frac{1}{\lambda} \|\nabla u(\cdot, -\epsilon)\|^2 \int_{\mathbb{R}^n} G(y, -\epsilon)\, dy \leq \frac{(2\pi)^{n/2}}{\lambda} \|\nabla u(\cdot, -\epsilon)\|^2, \end{aligned} \tag{5.3}$$

where we continue using the convention (3.20). On the other hand, we have a lower bound for the far-left side of (5.3), which reads

$$\begin{aligned} \int_{\mathbb{R}^n} G(y, -\epsilon)\, dy \int_{B_2} u(x+y, -\epsilon)^2\, dx &= \int_{\mathbb{R}^n} G(y, -\epsilon)\, dy \int_{B_2(y)} u(x, -\epsilon)^2\, dx \\ &\geq \int_{B_{1/2}} G(y, -\epsilon)\, dy \int_{B_2(y)} u(x, -\epsilon)^2\, dx \geq \int_{B_{1/2}} G(y, -\epsilon) dy \int_{B_1} u(x, -\epsilon)^2\, dx \\ &\geq \int_{B_{1/2}} G\left(y, -\frac{1}{2}\right) dy \int_{B_1} u(x, -\epsilon)^2\, dx \geq \frac{1}{C_0 K} (2\pi)^{n/2} \int_{\mathbb{R}^n} u(x, -\epsilon)^2\, dx, \end{aligned}$$

fixing $C_0^{-1} = (2\pi)^{-n/2} \int_{B_{1/2}} G(y, -\epsilon) dy$, where the last inequality holds by the doubling type condition (2.11). Thus we obtain

$$\|u(x, -\epsilon)\|^2 < \frac{C_0 K}{\lambda} \|\nabla u(\cdot, -\epsilon)\|^2 \leq \frac{1}{q_{\mathrm{D}}(-\epsilon)} \|\nabla u(\cdot, -\epsilon)\|^2,$$

which contradicts $q_{\mathrm{D}}(-\epsilon) = \|\nabla u(\cdot,-\epsilon)\|^2/\|u(\cdot,-\epsilon)\|^2$. Therefore, (5.2) cannot hold for all $x \in B_2$, i.e., there exists $x_\epsilon \in B_2$ such that (5.1) holds. Theorem 2.5 then follows as in the proofs of Theorems 2.1 and 2.3. $\square$

6. Extensions

In this section, we extend Theorems 2.1 and 2.3 to two settings. In the first (Theorem 6.1 below), we allow v and w to have a lower integrability in time, while in the second, the space exponents for v and w are lowered below $2n/3$ and $2n$ requiring a degree of vanishing of the norms. Here we adopt the notation from Section 2; in particular, $I = [T_0, T + T_0]$.

6.1. The case $L_t^q L_x^p$. The methods in this paper allow us to consider the case $v \in L_t^{p_2} L_x^{p_1}(\mathbb{T}^n \times I)$ and $w \in L_t^{q_2} L_x^{q_1}(\mathbb{T}^n \times I)$ with p_2 and q_2 finite. For simplicity, we restrict ourselves to the case of periodic boundary conditions; however, the same holds for the case of $\mathbb{R}^n$ using the approach from the previous section. When limiting $p_2 \to \infty$ and $q_2 \to \infty$, the next theorem reduces to the results in Section 2.

THEOREM 6.1. *Let $n \geq 2$ and $I = [T_0, T_0 + T]$. Assume that $u \in L_{x,t}^\infty(\mathbb{T}^n \times I)$ is a solution of (2.1) where $v \in L_t^{p_2} L_x^{p_1}(\mathbb{T}^n \times I)$ and $w \in L_t^{q_2} L_x^{q_1}(\mathbb{T}^n \times I)$ such that $p_1 > 2n/3$ and $q_1 > 2n$ with*

$$p_2 > \max\left\{\frac{2}{1-\alpha}, \frac{2}{3-4\alpha}\right\} \tag{6.1}$$

and

$$q_2 > \max\left\{\frac{2}{1-\beta}, \frac{2}{3-4\beta}\right\}$$

where $\alpha = n/2p_1$ and $\beta = 1/2 + n/2q_1$. Then, for all $(x_0, t_0) \in \mathbb{T}^n \times [T_0 + T/2, T_0 + T]$, the vanishing order of u at (x_0, t_0) satisfies

$$O_{(x_0,t_0)}(u) \lesssim \|v\|^a_{L_t^{p_2} L_x^{p_1}(\mathbb{T}^n \times I)} + \|w\|^b_{L_t^{q_2} L_x^{q_1}(\mathbb{T}^n \times I)} + 1,$$

where

$$a = \frac{2}{3 - 4\alpha - 2/p_2} \tag{6.2}$$

and

$$b = \frac{2}{3 - 4\beta - 2/q_2}.$$

PROOF OF THEOREM 6.1. Without loss of generality, we may assume $I = [-1, 0]$ and $(x_0, t_0) = (0, 0)$. For simplicity, we only consider the case $w = 0$ and $v \in L_t^{p_2} L_x^{p_1}(\mathbb{T}^n \times I)$, as for a nonzero w the proof is similar. Here we define

$$M_0(\tau) = \|v(\cdot, t)\|_{L^{p_1}(\mathbb{T}^n)},$$

where $\tau = -\log(-t)$. As in (3.54), we have

$$\begin{aligned} \bar{Q}'(\tau) &\lesssim e^{-\tau}\epsilon^{-2} + M_0(\tau)^2 e^{(2\alpha-2)\tau} |\bar{Q}(\tau)|^{2\alpha} + M_0(\tau)^2 e^{-2\tau}\epsilon^{-4\alpha} + M_0(\tau)^2 e^{(\alpha-2)\tau}\epsilon^{-2\alpha} \\ &\quad + M_0(\tau)^{2/(1-\alpha)} e^{-2\tau} + M_0(\tau)^2 e^{(2\alpha-2)\tau}, \end{aligned} \tag{6.3}$$

where $\alpha = n/2p_1$, i.e., (3.54) holds with M_0 replaced by $M_0(\tau)$ and M_1 set to zero as we assumed that $w = 0$. Let

$$\epsilon = \frac{1}{\bar{C}(\|v\|^a_{L_t^{p_2} L_x^{p_1}(\mathbb{T}^n \times I)} + 1)}, \tag{6.4}$$

with a as in (6.2) and $\bar{C}$ sufficiently large determined in the Gronwall argument below. Denote by $C_2 \geq 1$ the implicit constant in (6.3), and we have (3.57). Under the condition (3.57), we claim that

$$\bar{Q}(\tau) < 2\frac{C_1 + C_2}{\epsilon}, \qquad \tau \geq \tau_0, \tag{6.5}$$

where $C_1 = C_0(q_0 + 2)$ and $C_0 \geq 1$ is the constant in (3.56) and where τ_0 is given in (3.13). Assume, contrary to the assertion, that there exists $\tau_1 \geq \tau_0$ such that $\bar{Q}(\tau_1) = 2(C_1 + C_2)/\epsilon$, and suppose that τ_1 is the first time with this property. Also let τ_0' be as in (3.59) so that, in particular, (3.60) holds. Then we have

$$\begin{aligned} \bar{Q}'(\tau) &\leq C_2 e^{-\tau}\epsilon^{-2} + C_2 M_0(\tau)^2 2^{2\alpha} e^{(2\alpha-2)\tau}(C_1 + C_2)^{2\alpha}\epsilon^{-2\alpha} \\ &\quad + C_2 M_0(\tau)^2 e^{-2\tau}\epsilon^{-4\alpha} + C_2 M_0(\tau)^2 e^{(\alpha-2)\tau}\epsilon^{-2\alpha} + C_2 M_0(\tau)^{2/(1-\alpha)} e^{-2\tau} + C_2 M_0(\tau)^2 e^{(2\alpha-2)\tau}, \end{aligned} \tag{6.6}$$

for $\tau \in [\tau_0', \tau_1]$. We integrate (6.6) in τ from τ_0' to τ_1 and obtain

$$\begin{aligned}\bar{Q}(\tau_1) \le \bar{Q}(\tau_0') + C_2\epsilon^{-1} + C_2 2^{2\alpha}(C_1+C_2)^{2\alpha}\epsilon^{-2\alpha}\int_{\tau_0}^{\tau_1} M_0(\tau)^2 e^{(2\alpha-2)\tau} d\tau + C_2\epsilon^{-4\alpha}\int_{\tau_0}^{\tau_1} M_0(\tau)^2 e^{-2\tau}\,d\tau\\ + C_2\epsilon^{-2\alpha}\int_{\tau_0}^{\tau_1} M_0(\tau)^2 e^{(\alpha-2)\tau}\,d\tau + C_2\int_{\tau_0}^{\tau_1} M_0(\tau)^{2/(1-\alpha)} e^{-2\tau}\,d\tau + C_2\int_{\tau_0}^{\tau_1} M_0(\tau)^2 e^{(2\alpha-2)\tau}\,d\tau,\end{aligned} \tag{6.7}$$

since $\tau_0 \le \tau_0'$. By Hölder's inequality, the third term on the right-hand side satisfies

$$\begin{aligned}&C_2 2^{2\alpha}(C_1+C_2)^{2\alpha}\epsilon^{-2\alpha}\int_{\tau_0}^{\tau_1} M_0(\tau)^2 e^{(2\alpha-2)\tau}\,d\tau\\ &\le C_2 2^{2\alpha}(C_1+C_2)^{2\alpha}\epsilon^{-2\alpha}\|M_0(\tau)^2 e^{-2\tau/p_2}\|_{L^{p_2/2}(\tau_0,\infty)}\|e^{(2\alpha-2+2/p_2)\tau}\|_{L^{p_2/(p_2-2)}(\tau_0,\infty)}\\ &\le C_2 2^{2\alpha}(C_1+C_2)^{2\alpha}\epsilon^{-2\alpha}\left(\int_{\tau_0}^{\infty} M_0(\tau)^{p_2}e^{-\tau}\,d\tau\right)^{2/p_2}\epsilon^{2-2\alpha-2/p_2}\\ &\le C_2 2^{2\alpha}(C_1+C_2)^{2\alpha}\epsilon^{3-4\alpha-2/p_2}\|v\|^2_{L_t^{p_2}L_x^{p_1}(\mathbb{T}^n\times I)}\epsilon^{-1} \le \frac{C_1+C_2}{20\epsilon};\end{aligned} \tag{6.8}$$

in the last step, we used $p_2 > 2/(3-4\alpha)$ from (6.1) in addition to $\epsilon^{3-4\alpha-2/p_2}\|v\|^2_{L_t^{p_2}L_x^{p_1}(\mathbb{T}^n\times I)} \le 1/\bar{C}^{3-4\alpha-2/p_2}$ which is due to (6.4) with $\bar{C}$ sufficiently large. Other terms in (6.7) are estimated similarly, except for the sixth one, for which we write

$$\begin{aligned}&C_2\int_{\tau_0}^{\tau_1} M_0(\tau)^{2/(1-\alpha)}e^{-2\tau}\,d\tau\\ &\le C_2\|M_0(\tau)^{2/(1-\alpha)}e^{-2\tau/(1-\alpha)p_2}\|_{L^{p_2(1-\alpha)/2}(\tau_0,\infty)}\|e^{(-2+2/(1-\alpha)p_2)\tau}\|_{L^{p_2(1-\alpha)/((1-\alpha)p_2-2)}(\tau_0,\infty)}\\ &\le C_2\epsilon^{2-2/(1-\alpha)p_2}\|M_0(\tau)e^{-\tau/p_2}\|^{2/(1-\alpha)}_{L^{p_2}(\tau_0,\infty)}\\ &\le C_2\epsilon^{3-2/(1-\alpha)p_2}\|v\|^{2/(1-\alpha)}_{L_t^{p_2}L_x^{p_1}(\mathbb{T}^n\times I)}\epsilon^{-1} \le \frac{C_1+C_2}{20\epsilon}.\end{aligned}$$

In order to use Hölder's inequality, we need $p_2(1-\alpha)/2 \ge 1$, which is guaranteed by (6.1). Also, the last step requires $\epsilon^{3-2/(1-\alpha)p_2}\|v\|^{2/(1-\alpha)}_{L_t^{p_2}L_x^{p_1}(\mathbb{T}^n\times I)} \le 1/\bar{C}^{3-2/(1-\alpha)p_2}$, see (6.4), in addition to $p_2 > 2/3(1-\alpha)$, which is satisfied due to (6.1). We proceed similarly estimating all the terms in (6.7) from the far right side of (6.8), obtaining

$$\bar{Q}(\tau_1) \le \bar{Q}(\tau_0') + \frac{C_1+C_2}{2\epsilon} \le \frac{3}{2}\frac{C_1+C_2}{\epsilon}.$$

This is a contradiction since $\bar{Q}(\tau_1) = 2(C_1+C_2)/\epsilon$, showing that (6.5) indeed holds for all $\tau \ge \tau_0$. The general case, when both $v \in L_t^{p_2}L_x^{p_1}(\mathbb{T}^n\times I)$ and $w \in L_t^{q_2}L_x^{q_1}(\mathbb{T}^n\times I)$ are present, is obtained analogously. ☐

6.2. The cases $L_t^\infty L_x^{n/2}$ and $L_t^\infty L_x^n$. In this section, we assume $t^{-\alpha_0}v \in L^\infty L^p(\mathbb{T}^n\times I)$ and $t^{-\beta_0}w \in L^\infty L^q(\mathbb{T}^n\times I)$ in the interval $n/2 \le p \le 2n/3$ and $n \le q \le 2n$ with α_0 and β_0 indicated in the statement. We assume that $n \ge 2$ throughout this section and that M_0 and M_1 are constants such that

$$\|(t-(T+T_0))^{-\alpha_0}v(\cdot,t)\|_{L^p(\mathbb{T}^n)} \le M_0 \tag{6.9}$$

and

$$\|(t-(T+T_0))^{-\beta_0}w(\cdot,t)\|_{L^q(\mathbb{T}^n)} \le M_1, \tag{6.10}$$

for $t \in I$.

THEOREM 6.2. *Let $n/2 \le p \le 2n/3$ and $n \le q \le 2n$. Assume that $u \in L_t^\infty L_x^\infty(\mathbb{T}^n\times I)$ is a solution of (2.1) with $t^{-\alpha_0}v \in L_t^\infty L_x^p(\mathbb{T}^n\times I)$ and $t^{-\beta_0}w \in L_t^\infty L_x^q(\mathbb{T}^n\times I)$ satisfying (6.9) and (6.10) for $t \in I$ such that*

$$\alpha_0 > \frac{2n/p-3}{2} \tag{6.11}$$

and

$$\beta_0 > \frac{2n/q-1}{2}. \tag{6.12}$$

Then, for all $(x_0,t_0) \in \mathbb{T}^n \times [T_0+T/2, T_0+T]$, the vanishing order of u at (x_0,t_0) satisfies

$$O_{(x_0,t_0)}(u) \lesssim M_0^a + M_1^b + 1, \tag{6.13}$$

where $a = 2/(3 + 2\alpha_0 - 2n/p)$ and $b = 2/(1 + 2\beta_0 - 2n/q)$, with the implicit constant in (6.13) depending on q_0 and T.

PROOF OF THEOREM 6.2. Without loss of generality, $I = [-1, 0]$ and $(x_0, t_0) = (0, 0)$. Using the notation (3.37) and (3.45), the conditions (6.11) and (6.12) read

$$\alpha_0 > \frac{4\alpha - 3}{2}$$

and

$$\beta_0 > \frac{4\beta - 3}{2}.$$

We also have $a = 2/(3 + 2\alpha_0 - 4\alpha)$, and $b = 2/(3 + 2\beta_0 - 4\beta)$. It suffices to show that $\bar{Q}(\tau) \lesssim M_0^a + M_1^b + 1$, where $\bar{Q}$ and τ are defined in the proof of Theorem 2.1. Let V and W be as in (3.14) and (3.15), and note that V and W are $e^{\tau/2}$-periodic. As in (3.35), we write

$$\mathbb{R}^n = \bigcup_{j \in \mathbb{Z}^n} \Omega_{j,\tau},$$

where $\Omega_{j,\tau} = je^{\tau/2} + e^{\tau/2}\Omega$. After a simple change of variable, we then have

$$\|V\|_{L^p(\Omega_{j,\tau})} \le M_0 e^{\alpha\tau} e^{-\tau\alpha_0} = M_0 e^{(\alpha-\alpha_0)\tau} \tag{6.14}$$

and

$$\|W\|_{L^q(\Omega_{j,\tau})} \le M_1 e^{(\beta-1/2)\tau} e^{-\tau\beta_0} \le M_1 e^{(\beta-\beta_0-1/2)\tau},$$

from where, similarly to (3.38) (but with $\tilde{U} = U/\|U\|$ instead of U),

$$\begin{aligned}
\|V\tilde{U}\|^2 &= \sum_{j\in\mathbb{Z}^n} \|V\tilde{U}\|^2_{L^2(\Omega_{j,r})} \lesssim \sum_{j\in\mathbb{Z}^n} \|V\|^2_{L^p(\Omega_{j,r})} \|\tilde{U}\|_{L^{2p/(p-2)}(\Omega_{j,r})} \\
&\lesssim M_0^2 e^{2(\alpha-\alpha_0)\tau} \|U\|^{2-2\alpha} (\|D^2\tilde{U}\| + 1)^{2\alpha},
\end{aligned}$$

where we used (6.14) in the last inequality. Similarly, we have the modified version of (3.47), which is

$$\begin{aligned}
\|y_j W_j \tilde{U}\|^2 &= \sum_{j\in\mathbb{Z}^n} \|y_j W_j \tilde{U}\|^2_{L^2(\Omega_{j,r})} \lesssim \sum_{j\in\mathbb{Z}^n} \|W\|^2_{L^q(\Omega_{j,r})} \|y\tilde{U}\|^2_{L^{2q/(q-2)}(\Omega_{j,r})} \\
&\lesssim \sum_{j\in\mathbb{Z}^n} M_1^2 e^{(2\beta-2\beta_0-1)\tau} \||y|^2\tilde{U}\| \|\tilde{U}\|^{1-n/2q}_{L^{2q/(q-4)}(\Omega_{j,r})} (\|D^2\tilde{U}\|_{L^2(\Omega_{j,r})} + \|\tilde{U}\|_{L^2(\Omega_{j,r})})^{n/2q},
\end{aligned}$$

and then, applying Hölder's inequality,

$$\|y_j W_j \tilde{U}\| \lesssim M_1 e^{(\beta-\beta_0-1/2)\tau} \||y|^2\tilde{U}\|^{1/2} (\|\Delta\tilde{U}\|^{n/2q} + 1).$$

Other estimates are the same as in the proof of Lemma 3.2, leading to a bound for $\bar{Q}'(\tau)$, which reads

$$\begin{aligned}
&\frac{1}{2}\bar{Q}'(\tau) + \|(A(\tau) - \bar{Q}(\tau)I)\tilde{U}\|^2 \\
&\quad\lesssim e^{-\tau}|r|^2 + \|(A(\tau) - \bar{Q}(\tau)I)\tilde{U}\| \Big(e^{(\beta-\beta_0-1)\tau} M_1 \big(e^{-\beta\tau/2}|r|^\beta (|\bar{Q}(\tau)|^{\beta/2} + e^{-\beta\tau/2}|r|^\beta + 1) \\
&\qquad\qquad + \|(A(\tau) - \bar{Q}(\tau)I)\tilde{U}\|^\beta + \bar{Q}(\tau)^\beta + 1 \big) \Big) \\
&\qquad + \|(A(\tau) - \bar{Q}(\tau)I)\tilde{U}\| \Big(e^{(\alpha-\alpha_0-1)\tau} M_0 \big(e^{-\alpha\tau/2}|r|^\alpha (|\bar{Q}(\tau)|^{\alpha/2} + e^{-\alpha\tau/2}|r|^\alpha + 1) \\
&\qquad\qquad + \|(A(\tau) - \bar{Q}(\tau)I)\tilde{U}\|^\alpha + \bar{Q}(\tau)^\alpha + 1 \big) \Big);
\end{aligned} \tag{6.15}$$

see (3.52). Using Young's inequality, we absorb the terms involving $\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|$ on the right-hand side of (6.15) into $\|(A(\tau)-\bar{Q}(\tau)\mathcal{I})\tilde{U}\|^2$ on the left-hand side to obtain

$$\begin{aligned}\bar{Q}'(\tau) &\lesssim e^{-\tau}|r|^2 + e^{(\beta-2\beta_0-2)\tau}M_1^2|r|^{2\beta}|\bar{Q}(\tau)|^{\beta} + e^{(2\beta-2\beta_0-2)\tau}M_1^2|\bar{Q}(\tau)|^{2\beta}\\ &\quad + M_1^2e^{(-2\beta_0-2)\tau}|r|^{4\beta} + M_1^2e^{(\beta-2\beta_0-2)\tau}|r|^{2\beta} + M_1^{2/(1-\beta)}e^{2(\beta-\beta_0-1)\tau/(1-\beta)}\\ &\quad + M_1^2e^{(2\beta-2\beta_0-2)\tau} + M_0^2e^{(\alpha-2\alpha_0-2)\tau}|r|^{2\alpha}|\bar{Q}(\tau)|^{\alpha} + M_0^2e^{(2\alpha-2\alpha_0-2)\tau}|\bar{Q}(\tau)|^{2\alpha}\\ &\quad + M_0^2e^{(-2\alpha_0-2)\tau}|r|^{4\alpha} + M_0^2e^{(\alpha-2\alpha_0-2)\tau}|r|^{2\alpha} + M_0^{2/(1-\alpha)}e^{2(\alpha-\alpha_0-1)\tau/(1-\alpha)}\\ &\quad + M_0^2e^{(2\alpha-2\alpha_0-2)\tau}.\end{aligned} \tag{6.16}$$

Using $|r|\lesssim\epsilon^{-1}$ and absorbing the second and eighth terms, the inequality (6.16) implies

$$\begin{aligned}\bar{Q}'(\tau) &\lesssim e^{-\tau}\epsilon^{-2} + e^{(2\beta-2\beta_0-2)\tau}M_1^2|\bar{Q}(\tau)|^{2\beta} + M_1^2e^{(-2\beta_0-2)\tau}\epsilon^{-4\beta}\\ &\quad + M_1^2e^{(\beta-2\beta_0-2)\tau}\epsilon^{-2\beta} + M_1^{2/(1-\beta)}e^{2(\beta-\beta_0-1)\tau/(1-\beta)} + M_1^2e^{(2\beta-2\beta_0-2)\tau}\\ &\quad + M_0^2e^{(2\alpha-2\alpha_0-2)\tau}|\bar{Q}(\tau)|^{2\alpha} + M_0^2e^{(-2\alpha_0-2)\tau}\epsilon^{-4\alpha} + M_0^2e^{(\alpha-2\alpha_0-2)\tau}\epsilon^{-2\alpha}\\ &\quad + M_0^{2/(1-\alpha)}e^{2(\alpha-\alpha_0-1)\tau/(1-\alpha)} + M_0^2e^{(2\alpha-2\alpha_0-2)\tau}.\end{aligned} \tag{6.17}$$

Let C_2 be the implicit constant in (6.17) and C_1 be defined as in (3.57). We can now use the barrier argument to prove that

$$\bar{Q}(\tau) \le 2\frac{C_1+C_2}{\epsilon}, \qquad \tau\ge\tau_0=-\log\epsilon,$$

where

$$\epsilon = \frac{1}{\bar{C}(M_0^a+M_1^b+1)},$$

with $\bar{C}$ sufficiently large. The concluding Gronwall argument is identical to that in the proof of Lemma 3.2, and thus we omit the details. ☐

REMARK 6.3. A minor modification in the proof allows us to consider also $L_t^{p_2}L_x^{p_1}(\mathbb{R}^n\times I)$ and $L_t^{q_2}L_x^{q_1}(\mathbb{R}^n\times I)$ where $n/2\le p_1\le 2n/3$ and $n\le q_1\le 2n$. To avoid repetition, we only state the result, which is as follows. Let $\alpha = n/2p_1$ and $\beta=1/2+n/2q_1$. Assume that $u\in L_t^\infty L_x^\infty(\mathbb{T}^n\times I)$ is a solution of (2.1) with $t^{-\alpha_0}v\in L_t^{p_2}L_x^{p_1}(\mathbb{T}^n\times I)$ and $t^{-\beta_0}w\in L_t^{q_2}L_x^{q_1}(\mathbb{T}^n\times I)$ where $p_2>2/(1-\alpha)$ and $q_2>2/(1-\beta)$ are such that

$$\alpha_0 > \max\left\{\frac{2/p_2+4\alpha-3}{2}, \frac{2/p_2+2\alpha-2}{2}\right\}$$

and

$$\beta_0 > \max\left\{\frac{2/q_2+4\beta-3}{2}, \frac{2/q_2+2\beta-2}{2}\right\}.$$

Then, for all $(x_0,t_0)\in\mathbb{T}^n\times I$, the vanishing order of u at (x_0,t_0) satisfies

$$O_{(x_0,t_0)}(u) \lesssim \|t^{-\alpha_0}v\|^a_{L_t^{p_2}L_x^{p_1}(\mathbb{T}^n\times I)} + \|t^{-\beta_0}w\|^b_{L_t^{q_2}L_x^{q_1}(\mathbb{T}^n\times I)} + 1 \tag{6.18}$$

where

$$a = \frac{2}{3+2\alpha_0-2/p_2-4\alpha}$$

and

$$b = \frac{2}{3+2\beta_0-2/q_2-4\beta},$$

with the implicit constant in (6.18) depending only on q_0 and T.

6.3. The cases $L^\infty_t L^p_x$ and $L^\infty_t L^q_x$ where $2n/3 > p > n/2$ and $q > 2n$. In this section, we prove Theorem 2.4.

PROOF OF THEOREM 2.4. To avoid repetition, we only provide the estimate (2.10) under the smallness assumption on M_0. Note that the proof in Section 3 holds until (3.58), but we use

$$\epsilon = \frac{1}{\bar{C}(M_1^b + 1)},$$

for $\bar{C}$ sufficiently large. We use the barrier argument to obtain

$$\bar{Q}(\tau) \leq 2\frac{C_1 + C_2}{\epsilon}, \qquad \tau \geq \tau_0. \tag{6.19}$$

Assume that there exists $\tau \geq \tau_0$ such that (6.19) does not hold and let τ_1 be the first such time. Define τ_0' by (3.59), and observe that (3.60) holds. Integrating (3.54) in τ between τ_0' and τ_1 yields

$$\begin{aligned}
\bar{Q}(\tau_1) &\leq \bar{Q}(\tau_0') + C_2\epsilon^{-1} + C_2 M_1^2 2^\beta (C_1 + C_2)^\beta \epsilon^{2-4\beta} + \frac{C_2}{2-2\beta} M_1^2 2^{2\beta}(C_1+C_2)^{2\beta}\epsilon^{2-4\beta} \\
&\quad + C_2 M_1^2 \epsilon^{2-4\beta} + C_2 M_1^2 \epsilon^{2-3\beta} + C_2 M_1^{2/(1-\beta)}\epsilon^2 + \frac{C_2}{2-2\beta} M_1^2 \epsilon^{2-2\beta} \\
&\quad + C_2 M_0^2 2^\alpha (C_1+C_2)^\alpha \epsilon^{2-4\alpha} + \frac{C_2}{2-2\alpha} M_0^2 2^{2\alpha}(C_1+C_2)^{2\alpha}\epsilon^{2-4\alpha} \\
&\quad + C_2 M_0^2 \epsilon^{2-4\alpha} + C_2 M_0^2 \epsilon^{2-3\alpha} + C_2 M_0^{2/(1-\alpha)}\epsilon^2 + \frac{C_2}{2-2\alpha} M_0^2 \epsilon^{2-2\alpha},
\end{aligned} \tag{6.20}$$

where we use $0 \leq \alpha, \beta \leq 1$. We now claim that each term on the right-hand side of (6.20) is bounded by $(C_1 + C_2)/20\epsilon$. We only estimate the terms involving M_0 since others are estimated same as (3.61) and since the condition on q is the same as in Theorem 2.5. Starting with the ninth term, we have

$$C_2 M_0^2 2^\alpha (C_1+C_2)^\alpha \epsilon^{2-4\alpha} \leq (C_1+C_2) M_0^2 \epsilon^{2-4\alpha} \leq (C_1+C_2) M_0^2 C^{4\alpha-3}(1+M_1^b)^{4\alpha-3}\epsilon^{-1} \leq \frac{C_1+C_2}{20\epsilon},$$

where we use C_1 is sufficiently large in the first inequality and M_0 is sufficiently small in the last inequality. Note that $-1 < 3 - 4\alpha < 0$ since $n/2 < p < 2n/3$. Similarly,

$$\frac{C_2}{2-2\alpha} M_0^2 2^{2\alpha}(C_1+C_2)^{2\alpha}\epsilon^{2-4\alpha} \leq \frac{C_2}{2-2\alpha} M_0^2 2^{2\alpha}(C_1+C_2)^{2\alpha} C^{4\alpha-3}(1+M_1^b)^{4\alpha-3}\epsilon^{-1} \leq \frac{C_1+C_2}{20\epsilon},$$

given M_0 is sufficiently small. Proceeding similarly with the other terms, we conclude that $\bar{Q}(\tau) \leq 2(C_1 + C_2)/\epsilon$, which is a contradiction. Therefore, (6.19) holds for all $\tau \geq \tau_0$, i.e., $\bar{Q}(\tau) \lesssim M_1^b + 1$, for all $\tau \geq \tau_0$. ☐

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References

[A] S. Agmon, *Unicité et convexité dans les problèmes différentiels*, Séminaire de Mathématiques Supérieures, No. 13 (Été, 1965), Les Presses de l'Université de Montréal, Montreal, Que., 1966.

[AN] S. Agmon and L. Nirenberg, *Lower bounds and uniqueness theorems for solutions of differential equations in a Hilbert space*, Comm. Pure Appl. Math. **20** (1967), 207–229.

[AE] G. Alessandrini and L. Escauriaza, *Null-controllability of one-dimensional parabolic equations*, ESAIM Control Optim. Calc. Var. **14** (2008), no. 2, 284–293.

[AEWZ] J. Apraiz, L. Escauriaza, G. Wang, and C. Zhang, *Observability inequalities and measurable sets*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 11, 2433–2475.

[ApE] J. Apraiz and L. Escauriaza, *Null-control and measurable sets*, ESAIM Control Optim. Calc. Var. **19** (2013), no. 1, 239–254.

[AMRV] G. Alessandrini, A. Morassi, E. Rosset, and S. Vessella, *On doubling inequalities for elliptic systems*, J. Math. Anal. Appl. **357** (2009), no. 2, 349–355.

[Al] F.J. Almgren, Jr., *Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents*, Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977), North-Holland, Amsterdam-New York, 1979, pp. 1–6.

[An] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79–96.

[AV] G. Alessandrini and S. Vessella, *Local behaviour of solutions to parabolic equations*, Comm. Partial Differential Equations **13** (1988), no. 9, 1041–1058.

[BC] L. Bakri and J.-B. Casteras, *Quantitative uniqueness for Schrödinger operator with regular potentials*, Math. Methods Appl. Sci. **37** (2014), no. 13, 1992–2008.

[B] L. Bakri, *Carleman estimates for the Schrödinger operator. Applications to quantitative uniqueness*, Comm. Partial Differential Equations **38** (2013), no. 1, 69–91.

[BK] J. Bourgain and C.E. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. **161** (2005), no. 2, 389–426.

[CRV] B. Canuto, E. Rosset, and S. Vessella, *Quantitative estimates of unique continuation for parabolic equations and inverse initial-boundary value problems with unknown boundaries*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 491–535.

[C] T. Carleman, *Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat., Astr. Fys. **26** (1939), no. 17, 9.

[Ch] X.-Y. Chen, *A strong unique continuation theorem for parabolic equations*, Math. Ann. **311** (1998), no. 4, 603–630.

[CFNT] P. Constantin, C. Foias, B. Nicolaenko, and R. Temam, *Spectral barriers and inertial manifolds for dissipative partial differential equations*, J. Dynam. Differential Equations **1** (1989), no. 1, 45–73.

[CK] G. Camliyurt and I. Kukavica, *Quantitative unique continuation for a parabolic equation*, Indiana Univ. Math. J. **67** (2018), no. 2, 657–678.

[CKW1] G. Camliyurt, I. Kukavica, and F. Wang, *On quantitative uniqueness for elliptic equations*, Math. Z. **291** (2019), no. 1-2, 227–244.

[CKW2] G. Camliyurt, I. Kukavica, and F. Wang, *On localization and quantitative uniqueness for elliptic partial differential equations*, Partial differential equations in fluid mechanics, London Math. Soc. Lecture Note Ser., vol. 452, Cambridge Univ. Press, Cambridge, 2018, pp. 68–96.

[D] B. Davey, *Quantitative unique continuation for Schrödinger operators*, J. Funct. Anal. **279** (2020), no. 4, 108566, 23.

[DZ1] B. Davey and J. Zhu, *Quantitative uniqueness of solutions to second order elliptic equations with singular potentials in two dimensions*, Calc. Var. Partial Differential Equations **57** (2018), no. 3, Paper No. 92, 27.

[DZ2] B. Davey and J. Zhu, *Quantitative uniqueness of solutions to second-order elliptic equations with singular lower order terms*, Comm. Partial Differential Equations **44** (2019), no. 11, 1217–1251.

[DF1] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), no. 1, 161–183.

[DF2] H. Donnelly and C. Fefferman, *Nodal sets for eigenfunctions of the Laplacian on surfaces*, J. Amer. Math. Soc. **3** (1990), no. 2, 333–353.

[EF] L. Escauriaza and F.J. Fernández, *Unique continuation for parabolic operators*, Ark. Mat. **41** (2003), no. 1, 35–60.

[E] L. Escauriaza, *Carleman inequalities and the heat operator*, Duke Math. J. **104** (2000), no. 1, 113–127.

[EFV] L. Escauriaza, F. J. Fernández, and S. Vessella, *Doubling properties of caloric functions*, Appl. Anal. **85** (2006), no. 1-3, 205–223.

[EV] L. Escauriaza and L. Vega, *Carleman inequalities and the heat operator. II*, Indiana Univ. Math. J. **50** (2001), no. 3, 1149–1169.

[EVe] L. Escauriaza and S. Vessella, *Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients*, Inverse problems: theory and applications (Cortona/Pisa, 2002), Contemp. Math., vol. 333, Amer. Math. Soc., Providence, RI, 2003, pp. 79–87.

[F] F.J. Fernandez, *Unique continuation for parabolic operators. II*, Comm. Partial Differential Equations **28** (2003), no. 9-10, 1597–1604.

[FS] C. Foias and J.-C. Saut, *Asymptotic behavior, as $t \to \infty$, of solutions of the Navier-Stokes equations*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982), Res. Notes in Math., vol. 84, Pitman, Boston, MA, 1983, pp. 74–86.

[GL] N. Garofalo and F.-H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. **35** (1986), no. 2, 245–268.

[H1] Q. Han, *Schauder estimates for elliptic operators with applications to nodal sets*, J. Geom. Anal. **10** (2000), no. 3, 455–480.

[H2] Q. Han, *On the Schauder estimates of solutions to parabolic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **27** (1998), no. 1, 1–26 (1998).

[JK] D. Jerison and C.E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. (2) **121** (1985), no. 3, 463–494, With an appendix by E. M. Stein.

[K1] C.E. Kenig, *Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation*, Harmonic analysis and partial differential equations (El Escorial, 1987), Lecture Notes in Math., vol. 1384, Springer, Berlin, 1989, pp. 69–90.

[K2] C.E. Kenig, *Some recent applications of unique continuation*, Recent developments in nonlinear partial differential equations, Contemp. Math., vol. 439, Amer. Math. Soc., Providence, RI, 2007, pp. 25–56.

[K3] C.E. Kenig, *Quantitative unique continuation, logarithmic convexity of Gaussian means and Hardy's uncertainty principle*, Perspectives in partial differential equations, harmonic analysis and applications, Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 207–227.

[KSW] C. Kenig, L. Silvestre, and J.-N. Wang, *On Landis' conjecture in the plane*, Comm. Partial Differential Equations **40** (2015), no. 4, 766–789.

[Ku1] I. Kukavica, *Quantitative uniqueness for second-order elliptic operators*, Duke Math. J. **91** (1998), no. 2, 225–240.

[Ku2] I. Kukavica, *Quantitative uniqueness and vortex degree estimates for solutions of the Ginzburg-Landau equation*, Electron. J. Differential Equations (2000), No. 61, 15 pp. (electronic).

[Ku3] I. Kukavica, *Self-similar variables and the complex Ginzburg-Landau equation*, Comm. Partial Differential Equations **24** (1999), no. 3-4, 545–562.

[Ku4] I. Kukavica, *Length of vorticity nodal sets for solutions of the 2D Navier-Stokes equations*, Comm. Partial Differential Equations **28** (2003), no. 3-4, 771–793.

[Kur] K. Kurata, *On a backward estimate for solutions of parabolic differential equations and its application to unique continuation*, Spectral and scattering theory and applications, Adv. Stud. Pure Math., vol. 23, Math. Soc. Japan, Tokyo, 1994, pp. 247–257.

[KT1] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Comm. Pure Appl. Math. **54** (2001), no. 3, 339–360.

[KT2] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second order parabolic equations with nonsmooth coefficients*, Comm. Partial Differential Equations **34** (2009), no. 4-6, 305–366.

[L] F.-H. Lin, *Nodal sets of solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math. **44** (1991), no. 3, 287–308.

[M] V.Z. Meshkov, *On the possible rate of decrease at infinity of the solutions of second-order partial differential equations*, Mat. Sb. **182** (1991), no. 3, 364–383.

[P] C.-C. Poon, *Unique continuation for parabolic equations*, Comm. Partial Differential Equations **21** (1996), no. 3-4, 521–539.

[V] S. Vessella, *Carleman estimates, optimal three cylinder inequalities and unique continuation properties for parabolic operators*, Progress in analysis, Vol. I, II (Berlin, 2001), World Sci. Publ., River Edge, NJ, 2003, pp. 485–492.

[Z] S. Zelditch, *Hausdorff measure of nodal sets of analytic Steklov eigenfunctions*, Math. Res. Lett. **22** (2015), no. 6, 1821–1842.

[Zh] J. Zhu, *Quantitative uniqueness of solutions to parabolic equations*, J. Funct. Anal. **275** (2018), no. 9, 2373–2403.

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