Local-in-time existence of free-surface 3D Euler flow with $H^{2+\delta}$ initial vorticity in a neighborhood of the free boundary

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Abstract

We consider the three-dimensional Euler equations in a domain with a free boundary with no surface tension. We assume that $u_0 \in H^{2.5+\delta}$ is such that $\operatorname{curl} u_0 \in H^{2+\delta}$ in an arbitrarily small neighborhood of the free boundary, and we use Lagrangian approach to derive an a priori estimate that can be used to prove local-in-time existence and uniqueness of solutions under the Rayleigh-Taylor stability condition.

1 Introduction

In this note we address the local existence of solutions to the free-surface Euler equations

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0 \end{aligned} \tag{1.1}$$

in $\Omega(t)$, where $\Omega(t)$ is the region that is periodic in x_1, x_2, and x_3 lies between $\Gamma_0 := \{x_3 = 0\}$ and a free surface $\Gamma_1(t)$ such that $\Gamma_1(0) := \{x_3 = 1\}$. The Euler equations (1.1) are supplemented with an initial condition $u(0) = u_0$. We consider the impermeable boundary condition on the fixed bottom boundary Γ_0 and no surface tension on the free boundary, that is

$$\begin{aligned} u_3 = 0 \qquad & \text{on } \Gamma_0, \\ p = 0 \qquad & \text{on } \Gamma_1. \end{aligned} \tag{1.2}$$

Furthermore the initial pressure is assumed to satisfy the Rayleigh-Taylor condition

$$\partial_{x_3} p(x, 0) \le -b < 0 \qquad \text{for } x \in \Gamma_1(0), \tag{1.3}$$

where $b > 0$ is a constant.

The problem of local existence of solutions to the free boundary Euler equations has been initially considered in [Sh, Shn, N, Y1, Y2] under the assumption of irrotationality of the initial data, i.e., with $\operatorname{curl} u_0 = 0$. The local existence of solutions in such case with u_0 from a Sobolev space was established by Wu in [W1, W2]. Ebin showed in [E] that for general data in a Sobolev space, the problem is unstable without assuming the Rayleigh-Taylor sign condition (1.3), which in essence requires that next to the free boundary the pressure in the fluid is higher than the pressure of air.

With the Rayleigh-Taylor condition, the a priori bounds for the existence of solutions for initial data in a Sobolev space were provided in [ChL]. New energy estimates in the Lagrangian coordinates

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along with the construction of solutions were provided in [CS1] (see also [CS2]). We refer the reader to [ABZ1, B, CL, DE1, DE2, DK, KT1, KT2, KT3, S, T, ZZ] for other works concerned with local or global existence results of the Euler equations with or without surface tension, and to [ABZ2, AD, AM1, AM2, EL, GMS, HIT, I, IT, IK, IP, KT3, L, Lin1, Lin2, MR, OT, P, W3] for other related works on the Euler equations with a free-evolving boundary.

In this paper, we are concerned with the problem of local existence of solutions, and we aim to impose minimal regularity assumptions on the initial data u_0 that gives local-in-time existence of solutions in the Lagrangian setting of the problem. It is well-known that the threshold of regularity for the Euler equations in $\mathbb{R}^n$ ([KP]) or a fixed domain is $u_0 \in H^{2.5+\delta}$, where $\delta > 0$ is arbitrarily small. In the case of the domain with a free boundary, this has recently been proven in [WZZZ] in the Eulerian candidates (cf. [SZ1, SZ2] for the local existence in H^3). However, the same result in the Lagrangian coordinates is not known. One of the main difficulties is that it is not clear whether an assumption that $f, g \in H^{2.5+\delta}$ implies that the same is true of $f \circ g$ (the composition of f and g) when δ is not an integer. The main result of this paper is to obtain a priori estimates for the local existence in $H^{2.5+\delta}$ with an additional regularity assumption on initial vorticity $\omega_0 := \operatorname{curl} u_0$ in an arbitrarily small neighborhood of the free boundary.

We note that a related result was obtained in [KTVW] in the 2D case, but the coordinates used in [KTVW] are not Lagrangian; they are in a sense a concatenation of Eulerian and Lagrangian variables. Moreover, the proof in [KTVW] uses in an essential way the preservation of Lagrangian vorticity, which is a property that does not hold in the 3D case. Another paper [KT4] considers the problem in ALE coordinates (cf. [MC] for the ALE coordinates in the fluid-structure interaction problem), but the proof requires an additional assumption on the initial data, due to additional regularity required by the Lagrangian variable on the boundary; for instance, the present paper shows that the conditions of Theorem 1 (see below) suffice.

The main feature of the present paper is that the particle map preserves additional ($H^{3+\delta}$) regularity for a short time in a small neighborhood of the boundary, a feature not available in the Eulerian approach. The proof of our main result, which we state in Theorem 1 below, after a brief introduction of the Lagrangian coordinates, is direct and short. It is inspired by earlier works [KT2, KTV] and previously by [ChL, CS1, CS2]. Another new idea of our approach is a localization of the tangential estimates (i.e., estimates concerned with differential operators with respect to x_1, x_2 only) which, due to the additional regularity of the particle map, can be performed in a domain close to the boundary. Moreover, we formulate our a priori estimates in terms of three quantities (see (3.1)) that control the system, and we make use of the fact that all our estimates, except for the tangential estimates, are not of evolution type. Namely, they are static.

We note that all the results in the paper also hold in the 2D case with all the Sobolev exponents reduced by 1/2.

In the next section we introduce our notation regarding the Euler equations in the Lagrangian coordinates, state the main result, and recall some known facts and inequalities. The proof of Theorem 1 is presented in Section 3, where we first give a heuristic argument about our strategy. The proof is then divided into several parts: The div-curl-type estimates are presented in Section 3.1, the pressure estimates are discussed in Sections 3.2–3.3, and the tangential estimates are presented in Section 3.4. The proof of the theorem is concluded with the final estimates in Section 3.5.

2 Preliminaries

2.1 Lagrangian setting of the Euler equations and the main theorem

We use the summation convention of repeated indices. We denote the time derivative by ∂_t, and a derivative with respect to x_j by ∂_j. We denote by $v(x,t) = (v_1, v_2, v_3)$ the velocity field in the Lagrangian coordinates and by $q(x,t)$ the Lagrangian pressure function. The Euler equations then become

$$\begin{aligned} \partial_t v_i &= -a_{ki}\partial_k q, \qquad i=1,2,3,\\ a_{ik}\partial_i v_k &= 0 \end{aligned} \tag{2.1}$$

in $\Omega \times (0,T)$, where $\Omega := \Omega(0) = \mathbb{T}^2 \times (0,1)$, and a_{ij} denotes the (i,j)-th entry of the matrix $a = (\nabla\eta)^{-1}$. Here, η stands for the particle map, i.e., the solution of the system

$$\begin{aligned} \partial_t \eta(x,t) &= v(x,t)\\ \eta(x,0) &= x \end{aligned} \tag{2.2}$$

in $\Omega \times [0,T)$. (Note that the Lagrangian variable is denoted by x.) Due to the incompressibility condition in (2.1), we have $\det \nabla\eta = 1$ for all times, which shows that a is the corresponding cofactor matrix. Therefore,

$$a_{ij} = \frac{1}{2}\epsilon_{imn}\epsilon_{jkl}\partial_m\eta_k\partial_n\eta_l, \tag{2.3}$$

where ϵ_{ijk} denotes the permutation symbol. As for the boundary conditions (1.2), in the Lagrangian coordinates the impermeable condition for u at the stationary bottom boundary Γ_0 becomes

$$v_3 = 0 \qquad \text{on } \Gamma_0, \tag{2.4}$$

while the zero surface tension condition at the top boundary $\Gamma_1 := \Gamma_1(0) = \{x_3 = 1\}$ reads

$$q = 0 \qquad \text{on } \Gamma_1 \times (0,T). \tag{2.5}$$

Note that the initial conditions for v is the same as for u, i.e., $v(0) = v_0 := u_0$. Moreover, observe that, in the Lagrangian coordinates, both Γ_0 and Γ_1 do not depend on time.

Now, consider a localization of u_0 given by χu_0, where $\chi \equiv \chi(x_3) \in C^\infty(\mathbb{R};[0,1])$ is such that $\chi(x_3) = 1$ in a neighborhood of $\Gamma_1(0)$ and $\chi(x_3) = 0$ outside of a larger neighborhood. The following is our main result establishing a priori estimates for the local existence of solutions of the free boundary Euler equations in the Lagrangian formulation.

Theorem 1. *Let $\delta > 0$. Assume that (v,q,a,η) is a C^∞ solution of the Euler system in the Lagrangian setting (2.1)–(2.5), and assume that v_0 satisfies the Rayleigh-Taylor condition (1.3). Then there exists a time $T > 0$ depending on $\|v_0\|_{2.5+\delta}$ and $\|\chi(\operatorname{curl} v_0)\|_{2+\delta}$ such that the norms $\sup_t \|v\|_{2.5+\delta}$, $\sup_t \|q\|_{2.5+\delta}$, $\sup_t \|q\chi\|_{3+\delta}$, and $\sup_t \|\chi\eta\|_{3+\delta}$ on $[0,T]$ are bounded from above by a constant depending only on $\|v_0\|_{2.5+\delta}$ and $\|\chi(\operatorname{curl} v_0)\|_{2+\delta}$.*

The rest of the paper is devoted to the proof of this theorem.

2.2 Product and commutator estimates

We use the standard notions of Lebesgue spaces, L^p, and Sobolev spaces, $W^{k,p}$, H^s, and we reserve the notation $\|\cdot\|_s := \|\cdot\|_{H^s(\Omega)}$ for the H^s norm. We recall the multiplicative Sobolev inequality

$$\|fg\|_\sigma \lesssim \|f\|_a \|g\|_b, \tag{2.6}$$

for any $a, b \geq \sigma \geq 0$ such that either $a + b > \sigma + 1.5$ or $a + b = \sigma + 1.5$ and $a, b > \sigma$. In particular,

$$\|fg\|_s \lesssim \|f\|_s \|g\|_{1.5+\delta}, \qquad s \in [0, 1.5+\delta]. \tag{2.7}$$

We shall also use the commutator estimates

$$\|J(fg) - fJg\|_{L^2} \lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{W^{1,q_1}}\|g\|_{W^{s-1,q_2}} \tag{2.8}$$

for $s \geq 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$, and

$$\|J(fg) - fJg - gJf\|_{L^p} \lesssim \|f\|_{W^{1,p_1}} \|g\|_{W^{s-1,p_2}} + \|f\|_{W^{s-1,q_1}} \|g\|_{W^{1,q_2}} \tag{2.9}$$

for $s \geq 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$, $p \in (1, p_1)$, and $p_2, q_1, q_2 < \infty$, where J is a nonhomogeneous differential operator in (x_1, x_2) of order $s \geq 0$. We refer the reader to [KP, Li] for the proofs. We set

$$\Lambda := (1 - \Delta_2)^{\frac{1}{2}}$$

and

$$S := \Lambda^{\frac{5}{2}+\delta}, \tag{2.10}$$

where Δ_2 denotes the Laplacian in (x_1, x_2).

2.3 Properties of the particle map η and the cofactor matrix a

Note that applying the product estimate (2.7) to the representation formula (2.3) for a, we get

$$\|a\|_{1.5+\delta} \lesssim \|\eta\|_{2.5+\delta}^2. \tag{2.11}$$

Moreover, by writing $\chi \nabla \eta = \nabla(\chi \eta) - \eta \nabla \chi$, where χ is as above, we obtain

$$\|\chi^2 a\|_{2+\delta} \lesssim \|\chi \eta\|_{3+\delta}^2 + \|\eta\|_{2.5+\delta}^2. \tag{2.12}$$

Finally, we recall the Brezis-Bourgonion inequality

$$\|f\|_s \lesssim \|f\|_{L^2} + \|\operatorname{curl} f\|_{s-1} + \|\operatorname{div} f\|_{s-1} + \|\nabla_2 f_3\|_{H^{s-1.5}(\partial\Omega)}, \qquad s \geq 1 \tag{2.13}$$

cf. [BB].

3 Proof of the main result

The main idea of the proof of Theorem 1 is to simplify the estimates introduced in [KT2] and [KTV] by localizing the analysis to an arbitrarily small region near the free boundary Γ_1 and showing that all the important quantities can be controlled by

$$\|\chi \eta\|_{3+\delta}, \|v\|_{2.5+\delta}. \tag{3.1}$$

First, we employ the div-curl estimates to bound both key quantities (3.1) at the time t using a time integral from 0 until t of a polynomial expression involving the same quantities, as well as terms concerned with initial data and two other terms. The two terms involve derivatives of η and v only in the variables x_1, x_2, and only very close to the free boundary Γ_1, namely $\|S\eta_3\|_{L^2(\Gamma_1)}$ and $\|S(\psi v)\|_{L^2}$, where ψ is a cutoff supported in a neighborhood of Γ_1 with $\operatorname{supp} \psi \subset \{\chi = 1\}$; see (3.9),(3.10) below for details. The cutoff ψ is introduced at the beginning of Section 3.1. These two terms, however, can also be controlled by a time integral of a polynomial expression involving (3.1) only, which we show in Section 3.4, after a brief discussion of some estimates, at each fixed time, of the pressure function and its time derivative in Sections 3.2–3.3. Finally, Section 3.5 combines the div-curl estimates with the tangential estimates to give an a priori bound that enables local-in-time existence and uniqueness.

Before proceeding to the proof, we note that, by (2.2), the particle map η satisfies

$$\nabla \eta - I = \int_0^t \nabla v \, ds, \tag{3.2}$$

where, for brevity, we have omitted the time argument t on the left-hand side. We continue this convention throughout. Moreover, observing that $a(0) = I$, where I denotes the three-dimensional identity matrix, we see from (2.3) that

$$a - I = \int_0^t \partial_t a \,\mathrm{d}s = \int_0^t \nabla\eta\nabla v \,\mathrm{d}s. \tag{3.3}$$

Here and in the sequel, we use the convention of omitting writing various indices when only the product structure matters; for instance, the expression on the far right side of (3.3) stands for $\epsilon_{imn}\epsilon_{jkl}\int_0^t \partial_m v_k \partial_n \eta_l$. The equations (3.2) and (3.3) demonstrate an important property that, as long as the key quantities (3.1) stay bounded, both a and $\nabla\eta$ remain close to I in the $H^{1.5+\delta}$ norm for sufficiently small times. In other words we obtain the following lemma.

Lemma 2 (Stability of a and η at initial time)**.** *Let $M, T_0 > 0$, and suppose that $\|v\|_{2.5+\delta}$, $\|\eta\|_{2.5+\delta}$, $\|\chi\eta\|_{3+\delta} \leq M$ for $t \in [0, T_0]$. Given $\varepsilon > 0$, there exists $T = T(M, \varepsilon) \in (0, T_0)$ such that*

$$\|I - a\|_{1.5+\delta}, \|I - aa^T\|_{1.5+\delta}, \|I - \nabla\eta\|_{1.5+\delta} \leq \epsilon \tag{3.4}$$

and

$$\|\eta\|_{2.5+\delta} \lesssim 1, \tag{3.5}$$

for $t \in [0, T]$. In particular, we also have $\|a\|_{1.5+\delta} \lesssim 1$.

Proof. By (3.2) and (3.3), we have $\|I - a\|_{1.5+\delta} \lesssim M^2 t$ and $\|\nabla\eta - I\|_{1.5+\delta} \lesssim Mt$. Moreover, the triangle inequality and (2.11) give $\|I - aa^T\|_{1.5+\delta} \leq \|I - a\|_{1.5+\delta}(1 + \|a\|_{1.5+\delta}) \lesssim M^2(1 + M^2)t$, and the claim follows by taking T sufficiently small. For (3.5), we write $\|\eta\|_{2.5+\delta} \lesssim 1 + Mt$, and the same T works as well. □

Corollary 3 below provides a similar estimate for the pressure function, enabling us to extend the Rayleigh-Taylor condition (1.3) for small $t > 0$.

Moreover we can use (3.5) to obtain further estimates of cofactor map a, introduced in Section 2.3. For example

$$\|\partial_t a\|_r \lesssim \|\nabla v\|_r, \qquad r \in [0, 1.5 + \delta] \tag{3.6}$$

and $\|\partial_{tt} a\|_r \lesssim \|\nabla v\|_{1.5+\delta}\|\nabla v\|_r + \|\nabla\partial_t v\|_r$, from where

$$\|\partial_{tt} a\|_r \lesssim \|\nabla v\|_{1.5+\delta}\|\nabla v\|_r + \|a\|_{1.5+\delta}\|q\|_{2+r}, \qquad r \in [0, 1.5 + \delta]. \tag{3.7}$$

We shall use this inequality in the range $r \in [0, 0.5 + \delta]$.

3.1 Div-curl estimates

Let $\psi(x_3) \in C^\infty(\mathbb{R}; [0, 1])$ be such that supp $\psi \Subset \{\chi = 1\}$ and $\psi = 1$ in a neighborhood of Γ_1. Note that both χ and ψ commute with any differential operator in the variables x_1, x_2, and that, provided ψ is present in any given expression involving classical derivatives or Λ, we can insert χ in at any other place. For example

$$\nabla f T g \nabla(\psi w) = \nabla f T(\chi g)\nabla(\psi w), \tag{3.8}$$

for all functions f, g, w and differential operators T in x_1, x_2.

In this section, we provide estimates that allow us to control the key quantities $\|v\|_{2.5+\delta}$ and $\|\chi\eta\|_{3+\delta}$ from (3.1). Namely, denoting by P any polynomial depending on these two quantities, we show that

$$\|\chi\eta\|_{3+\delta} \lesssim t\|\chi\nabla\omega_0\|_{1+\delta} + 1 + \|\Lambda^{2.5+\delta}\eta_3\|_{L^2(\Gamma_1)} + \int_0^t P \,\mathrm{d}s \tag{3.9}$$

and

$$\|v\|_{2.5+\delta} \lesssim \|v\|_{L^2} + \|S(\psi v)\|_{L^2} + \|\omega_0\|_{1.5+\delta}. \tag{3.10}$$

As pointed out above, we simplify the notation by omitting any indices in the cases where the exact value of the index becomes irrelevant. In those cases we only keep track of the power of the term and the order of any derivatives, as such terms are estimated using a Hölder, Sobolev, interpolation, or commutator inequality.

We start with the proof of (3.9). We use (2.13) to get

$$\|\chi\eta\|_{3+\delta} \lesssim \|\chi\eta\|_{L^2} + \|\mathrm{curl}\,(\chi\eta)\|_{2+\delta} + \|\mathrm{div}\,(\chi\eta)\|_{2+\delta} + \|\Lambda^{2.5+\delta}\eta_3\|_{L^2(\Gamma_1)}. \tag{3.11}$$

For the term involving curl, we first recall the Cauchy invariance

$$\epsilon_{ijk}\partial_j v_m \partial_k \eta_m = (\omega_0)_i, \tag{3.12}$$

cf. Appendix of [KTV] for a proof. For $i = 1, 2, 3$, we have

$$\begin{aligned}
\nabla((\mathrm{curl}\,(\chi\eta))_i) &= \epsilon_{ijk}\partial_j \nabla(\chi\eta_k) \\
&= \epsilon_{ijk}\delta_{km}\partial_j \nabla(\chi\eta_m) \\
&= \epsilon_{ijk}(\delta_{km} - \partial_k\eta_m)\partial_j\nabla(\chi\eta_m) + 2\chi\int_0^t \epsilon_{ijk}\partial_k v_m \partial_j \nabla\eta_m \,\mathrm{d}s + t\chi\nabla\omega_0^i \\
&\qquad + \underbrace{\nabla\eta(D^2\chi\eta + 2\nabla\chi\nabla\eta)}_{=:\mathrm{LOT}_1} \\
&= \epsilon_{ijk}(\delta_{km} - \partial_k\eta_m)\partial_j\nabla(\chi\eta_m) + 2\int_0^t \epsilon_{ijk}\partial_k v_m \partial_j \nabla(\chi\eta_m) \,\mathrm{d}s + t\chi\nabla\omega_0^i + \mathrm{LOT}_1 \\
&\qquad + \underbrace{2\int_0^t \nabla v(D^2\chi\eta + 2\nabla\chi\nabla\eta)\,\mathrm{d}s}_{=:\mathrm{LOT}_2},
\end{aligned}$$

where we used

$$0 = -\epsilon_{ijk}\partial_k\eta_m\partial_j\nabla\eta_m + 2\int_0^t \epsilon_{ijk}\partial_k v_m \partial_j\nabla\eta_m \,\mathrm{d}s + t\nabla(\omega_0)_i$$

in the third equality, which in turn is a consequence of the Cauchy invariance (3.12). Thus

$$\begin{aligned}
\|\nabla(\mathrm{curl}\,(\chi\eta))\|_{1+\delta} &\lesssim \|\chi\eta\|_{3+\delta}\|I - \nabla\eta\|_{1.5+\delta} + 2\int_0^t \underbrace{\|v\|_{2.5+\delta}\|\chi\eta\|_{3+\delta}}_{\le P} \,\mathrm{d}s \\
&\qquad + t\|\chi\nabla\omega_0\|_{1+\delta} + \|\mathrm{LOT}_1 + \mathrm{LOT}_2\|_{1+\delta},
\end{aligned} \tag{3.13}$$

applying (2.7). Note that

$$\|\mathrm{LOT}_1\|_{1+\delta}, \|\mathrm{LOT}_2\|_{1+\delta} \lesssim 1 + \int_0^t \|v\|_{2.5+\delta}\|\eta\|_{2+\delta}\,\mathrm{d}s \lesssim 1 + \int_0^t P\,\mathrm{d}s,$$

where we used $\eta(x) = x + \int_0^t v(x,s)\,\mathrm{d}s$ and $T \le 1/CM$ with $\|v\|_{2.5+\delta} \le M$ to estimate LOT_1.

As for the divergence, we use $\partial_t\eta = v$ to write

$$\begin{aligned}
\mathrm{div}\ \partial_l(\chi\eta) &= \delta_{kj}\partial_l\partial_k(\chi\eta_j) = (\delta_{kj} - a_{kj})\partial_l\partial_k(\chi\eta_j) + a_{kj}\partial_l\partial_k(\chi\eta_j) \\
&= (\delta_{kj} - a_{kj})\partial_l\partial_k(\chi\eta_j) + \int_0^t \big(\partial_t a_{kj}\partial_l\partial_k(\chi\eta_j) + a_{kj}\partial_l\partial_k(\chi v_j)\big)\,\mathrm{d}s + \delta_{l3}(\partial_3\partial_3\chi x_2 + 2\partial_3\chi) \\
&= (\delta_{kj} - a_{kj})\partial_l\partial_k(\chi\eta_j) \\
&\quad + \int_0^t \big(\partial_t a_{kj}\partial_l\partial_k(\chi\eta_j) + \underbrace{a_{kj}(\partial_l\partial_k\chi v_j + \partial_l\chi\partial_k v_j + \partial_k\chi\partial_l v_j)}_{=:\mathrm{LOT}_3} - \chi\partial_l a_{kj}\partial_k v_j\big)\,\mathrm{d}s \\
&\quad + \delta_{l3}(\partial_3\partial_3\chi x_2 + 2\partial_3\chi),
\end{aligned}$$

where in the last step we used $a_{kj}\partial_l\partial_k v_j = -\partial_l a_{kj}\partial_k v_j$, a consequence of the divergence-free condition $a_{kj}\partial_k v_j = 0$. Therefore,

$$\begin{aligned}
\|\nabla\,\mathrm{div}\,(\chi\eta)\|_{1+\delta} &\lesssim \|I-a\|_{1.5+\delta}\|\chi\eta\|_{3+\delta} + 1 \\
&\quad + \int_0^t \big(\underbrace{\|\partial_t a\|_{1.5+\delta/2}\|\chi\eta\|_{3+\delta}}_{\le P} + \|\mathrm{LOT}_3\|_{1+\delta} + \|\chi\partial_l a\|_{1+\delta}\|v\|_{2.5+\delta}\big)\,\mathrm{d}s.
\end{aligned} \tag{3.14}$$

Since $\partial_l a$ consists of sums of the terms of the form $\partial_a\eta_m\partial_l\partial_b\eta_n$, for $m,n,a,b = 1,2,3$, we have

$$\|\chi\partial_l a\|_{1+\delta} \lesssim \|\eta\|_{2.5+\delta}(\|\chi\eta\|_{3+\delta} + \|\eta\|_{2+\delta}).$$

Moreover,

$$\|\mathrm{LOT}_3\|_{1+\delta} \lesssim \|a\|_{1.5+\delta}\|v\|_{2+\delta} \lesssim P,$$

and thus, from (3.14)

$$\|\nabla\,\mathrm{div}\,(\chi\eta)\|_{1+\delta} \lesssim \|I-a\|_{1.5+\delta}\|\chi\eta\|_{3+\delta} + 1 + \int_0^t P\,\mathrm{d}s.$$

Applying this inequality and (3.13) into (3.11) gives

$$\begin{aligned}
\|\chi\eta\|_{3+\delta} &\lesssim \|\chi\eta\|_{L^2} + \|\mathrm{curl}\,(\chi\eta)\|_{1+\delta} + \|\mathrm{div}\,(\chi\eta)\|_{1+\delta} + \|\chi\eta\|_{3+\delta}\,(\|I-a\|_{1.5+\delta} + \|I-\nabla\eta\|_{1.5+\delta}) \\
&\quad + \int_0^t P\,\mathrm{d}s + t\|\chi\nabla\omega_0\|_{1+\delta} + \|\Lambda^{2.5+\delta}\eta_3\|_{L^2(\Gamma_1)} + 1.
\end{aligned}$$

Recalling (3.4), we may estimate the fourth term on the right-hand side by $\varepsilon\|\chi\eta\|_{3+\delta}$, and so we may absorb it on the left-hand side. Furthermore, using (3.5), the first three terms are bounded by a constant, concluding the proof of (3.9).

As for the estimate (3.10) on v, we note that the Cauchy invariance (3.12) implies

$$(\mathrm{curl}\,v)_i = \epsilon_{ijk}\partial_j v_k = \epsilon_{ijk}(\delta_{km} - \partial_k\eta_m)\partial_j v_m + (\omega_0)_i,$$

while the divergence-free condition, $a_{ji}\partial_j v_i = 0$, gives

$$\mathrm{div}\,v = \delta_{ji}\partial_j v_i = (\delta_{ji} - a_{ji})\partial_j v_i.$$

Thus, using (2.13), we obtain

$$\begin{aligned}
\|v\|_{2.5+\delta} &\lesssim \|v\|_{L^2} + \|\epsilon_{ijk}(\delta_{km} - \partial_k\eta_m)\partial_j v_m\|_{1.5+\delta} \\
&\quad + \|(\delta_{ji} - a_{ji})\partial_j v_i\|_{1.5+\delta} + \|\nabla_2 v_3\|_{H^{1+\delta}(\Gamma_1)} + \|\omega_0\|_{1.5+\delta}.
\end{aligned} \tag{3.15}$$

For the boundary term, applying the Sobolev interpolation and trace estimates gives

$$\|\nabla_2 v_3\|_{H^{1+\delta}(\Gamma_1)} \lesssim \|v\|_{L^2} + \|\Lambda^{1.5+\delta}v_3\|_{H^{0.5}(\Gamma_1)} \lesssim \|v\|_{L^2} + \|\Lambda^{1.5+\delta}\nabla(\psi v_3)\|_{L^2}.$$

As for the last term, since

$$\begin{aligned}
\partial_3(\psi v_3) &= \psi\,\mathrm{div}\;v - \psi\partial_1 v_1 - \psi\partial_2 v_2 + v_3\partial_3\psi \\
&= \psi(\delta_{ji} - a_{ji})\partial_j v_i - \partial_1(\psi v_1) - \partial_2(\psi v_2) + v_3\partial_3\psi,
\end{aligned}$$

we have

$$\begin{aligned}
\|\Lambda^{1.5+\delta}\nabla(\psi v_3)\|_{L^2} &\lesssim \|\psi(I-a)\nabla v\|_{1.5+\delta} + \|S(\psi v)\|_{L^2} + \|v_3\partial_3\psi\|_{1.5+\delta} \\
&\lesssim \epsilon\|v\|_{2.5+\delta} + C_\epsilon\|v\|_{L^2} + \|S(\psi v)\|_{L^2},
\end{aligned}$$

for any $\epsilon > 0$, where we used (3.4) and Sobolev interpolation of between L^2 and $H^{2.5}$ for the last term. Applying this in (3.15) and using (3.4) again we obtain

$$\|v\|_{2.5+\delta} \lesssim \epsilon\|v\|_{2.5+\delta} + C_\epsilon\|v\|_{L^2} + \|S(\psi v)\|_{L^2} + \|\omega_0\|_{1.5+\delta},$$

which, after absorbing the first term on the right-hand side, gives (3.10), as required.

3.2 The pressure estimate

In this section we show that if $\|q\|_{2.5+\delta}$ and $\|\psi q\|_{3+\delta}$ are finite, then

$$\|q\|_{2.5+\delta} \lesssim \|v\|_{2+\delta}^2 \tag{3.16}$$

and

$$\|\psi q\|_{3+\delta} \leq P(\|\chi\eta\|_{3+\delta}, \|v\|_{2.5+\delta}). \tag{3.17}$$

In the remainder of this section we denote any polynomial of the form of the right-hand side by P, for simplicity. We also continue the convention of omitting the irrelevant indices.

We first prove (3.17) assuming that (3.16) holds. Multiplying the Euler equation, $\partial_t v_i = -a_{ki}\partial_k q$ by ψ we obtain $\partial_t(\psi v_i) = -a_{ki}\partial_k(\psi q) + a_{ki}\partial_k\psi\, q$. Now applying $a_{ji}\partial_j$, summing over i, j, and using the Piola identity $\partial_j a_{ji} = 0$ we get

$$\begin{aligned}
&- \partial_j(a_{ji}a_{ki}\partial_k(\psi q)) + \partial_j(a_{ji}a_{ki}\partial_k\psi\, q) = a_{ji}\partial_j\partial_t(\psi v_i)\\
&\quad = \psi a_{ji}\partial_j\partial_t v_i + a_{ji}\partial_j\psi\partial_t v_i = -\psi\partial_t a_{ji}\partial_j v_i - a_{ji}\partial_j\psi a_{ki}\partial_k q,
\end{aligned}$$

where we have applied the product rule for ∂_j in the second equality, and, in the third equality, we used $\partial_t(a_{ji}\partial_j v_i) = 0$ for the first term. Thus

$$\begin{aligned}
\Delta(\psi q) &= \partial_j\big((\delta_{jk} - a_{ji}a_{ki})\partial_k(\psi q)\big) + \partial_j(a_{ji}a_{ki}\partial_k(\psi q))\\
&= \partial_j\big((\delta_{jk} - a_{ji}a_{ki})\partial_k(\psi q)\big) + \partial_j(a_{ji}a_{ki}\partial_k\psi q) + \psi\partial_t a_{ji}\partial_j v_i + a_{ji}\partial_j\psi a_{ki}\partial_k q,
\end{aligned} \tag{3.18}$$

from where, by the elliptic regularity, noting that $\psi q = 0$ on $\Gamma_0 \cup \Gamma_1$,

$$\begin{aligned}
\|\psi q\|_{3+\delta} &\lesssim \|(I - aa^T)\nabla(\psi q)\|_{2+\delta} + \|a^2\nabla\psi q\|_{2+\delta} + \|\psi\partial_t a\nabla v\|_{1+\delta} + \|a\nabla\psi a\nabla q\|_{1+\delta}\\
&\lesssim \|I - aa^T\|_{L^\infty}\|\psi q\|_{3+\delta} + \|\chi^4(I - aa^T)\|_{2+\delta}\|\psi q\|_{2.5+\delta}\\
&\quad + \|\chi^2 a\|_{2+\delta}^2\|q\|_{2+\delta} + \|\partial_t a\|_{1.2+\delta}\|v\|_{2.3+\delta} + \|a\|_{1.5+\delta}^2\|q\|_{2+\delta}\\
&\lesssim \varepsilon\|\psi q\|_{3+\delta} + (1 + \|\chi\eta\|_{3+\delta}^4)\|\psi q\|_{2.5+\delta} + P,
\end{aligned} \tag{3.19}$$

where we used (3.8), the estimate $\|fg\|_{2+\delta} \lesssim \|f\|_{L^\infty}\|g\|_{2+\delta} + \|f\|_{2+\delta}\|g\|_{L^\infty}$, as well as the embedding $\|g\|_{L^\infty} \lesssim \|g\|_{1.5+\delta}$ and (2.7) in the second inequality. In the third inequality we used (3.6), (3.4), and (3.5). In the second term on the far right side of (3.19), we use (3.16), proven below, to show that it is dominated by P. Thus we have obtained (3.17), given (3.16).

In order to estimate q in $H^{2.5+\delta}$, we note that, as in (3.18), q satisfies the Poisson equation

$$\partial_{kk} q = \partial_t a_{ji}\partial_j v_i + \partial_j((\delta_{jk} - a_{ji}a_{ki})\partial_k q)$$

in Ω, together with homogeneous boundary condition $q = 0$ on Γ_1 (by (2.5)), and nonhomogeneous Neumann boundary condition $\partial_3 q = (\delta_{k3} - a_{k3})\partial_k q$ on Γ_0, since taking ∂_t of the boundary condition $v_3 = 0$ on Γ_0 gives $a_{k3}\partial_k q = 0$. Thus, the elliptic estimates imply

$$\begin{aligned}
\|q\|_{2.5+\delta} &\lesssim \|\partial_t a\nabla v\|_{0.5+\delta} + \|(I - aa^T)\nabla q\|_{1.5+\delta} + \|(I - a)\nabla q\|_{H^{1+\delta}(\Gamma_0)}\\
&\lesssim \|\partial_t a\|_{1+\delta}\|v\|_{2+\delta} + \|I - aa^T\|_{1.5+\delta}\|q\|_{2.5+\delta} + \|I - a\|_{1.5+\delta}\|q\|_{2.5+\delta}\\
&\lesssim \|v\|_{2+\delta}^2 + \varepsilon\|q\|_{2.5+\delta},
\end{aligned}$$

where we used (3.6) and (3.4) in the last step. Taking a sufficiently small $\varepsilon > 0$ proves (3.16), as required.

3.3 Time derivative of q

In this section, we supplement the pressure estimates (3.16) and (3.17) from the previous section with

$$\|\partial_t q\|_{2+\delta} \le P \tag{3.20}$$

and

$$\|\partial_t(\psi q)\|_{2.5+\delta} \le P, \tag{3.21}$$

where P denotes a polynomial in terms of $\|\chi\eta\|_{3+\delta}$ and $\|v\|_{2.5+\delta}$. We note that (3.21) implies that the Rayleigh-Taylor condition (1.3) holds for sufficiently small t provided our key quantities (3.1) remain bounded.

Corollary 3 (Rayleigh-Taylor condition for small times)**.** *Let $M, T_0 > 0$, and suppose that $\|v\|_{2.5+\delta}$, $\|\eta\|_{2.5+\delta}$, $\|\chi\eta\|_{3+\delta} \le M$ for $t \in [0, T_0]$. There exists $T = T(M, b) \in (0, T_0)$ such that*

$$\partial_3 q(x,t) \le -\frac{b}{2}, \tag{3.22}$$

for $x \in \Gamma_1$ and $t \in [0, T]$.

Proof. The proof is analogous to the proof of Lemma 2, by noting that for $x \in \Gamma_1$

$$\partial_3 q(x,t) - \partial_3 q(x,0) = \int_0^t \partial_t \partial_3 q \,\mathrm{d}s \le \int_0^t \|\partial_t \nabla q\|_{L^\infty(\Gamma_1)} \,\mathrm{d}s \le \int_0^t \|\partial_t(q\psi)\|_{2.5+\delta} \,\mathrm{d}s \le P(M)t,$$

which is bounded by $b/2$ for sufficiently small t. □

We prove (3.21) first. Applying ∂_t to (3.18) we obtain

$$\begin{aligned}
\Delta \partial_t(\psi q) &= \partial_j\big((\delta_{jk} - a_{ji}a_{ki})\partial_k \partial_t(\psi q)\big) - 2\nabla(a\partial_t a \psi \nabla q) + \nabla(2a\partial_t a \nabla\psi q + a^2 \nabla\psi \partial_t q) \\
&\quad + \psi \partial_{tt} a \nabla v + \psi \partial_t a \nabla \partial_t v + 2a\partial_t a \nabla\psi \nabla q + a^2 \nabla\psi \nabla \partial_t q.
\end{aligned}$$

Noting that $\partial_t(\psi q)$ satisfies the homogeneous Dirichlet boundary conditions at both Γ_1 and Γ_0 and is periodic in x_1 and x_2, the elliptic regularity with $\sigma = 0.5 + \delta$ gives

$$\begin{aligned}
\|\partial_t(\psi q)\|_{2+\sigma} &\lesssim \|(I - aa^T)\nabla \partial_t(\psi q)\|_{1+\sigma} + \|a\partial_t a \psi \nabla q\|_{1+\sigma} + \|a\partial_t a \nabla\psi q\|_{1+\sigma} \\
&\quad + \|a^2 \nabla\psi \partial_t q\|_{1+\sigma} + \|\partial_{tt} a \nabla v\|_\sigma + \|\partial_t a \nabla(a\nabla q)\|_\sigma + \|a\partial_t a \nabla q\|_\sigma + \|a^2 \nabla \partial_t q\|_\sigma \\
&\lesssim \varepsilon \|\partial_t(\psi q)\|_{2+\sigma} + \|a\|_{1.5+\delta}\|\partial_t a\|_{1.5+\delta}\|q\|_{2+\sigma} + \|a\|_{1.5+\delta}^2 \|\nabla\psi \partial_t q\|_{1+\sigma} \\
&\quad + \|\partial_{tt} a\|_\sigma \|v\|_{2.5+\delta} + \|\partial_t a\|_{1.5+\delta}\|a\|_{1.5+\delta}\|q\|_{2+\sigma} \\
&\quad + \|a\|_{1.5+\delta}\|\partial_t a\|_{1.5+\delta}\|q\|_{1+\sigma} + \|a\|_{1.5+\delta}^2 \|\nabla \partial_t q\|_\sigma,
\end{aligned} \tag{3.23}$$

where we used (2.1) in the first and (3.4) in the second inequality. Taking $\varepsilon > 0$ sufficiently small, and absorbing the first term on the left-hand side, and using (3.16), (3.6), and (3.7), we obtain

$$\|\partial_t(\psi q)\|_{2+\sigma} \lesssim (1 + \|\partial_t q\|_{1+\sigma})P. \tag{3.24}$$

Thus, given (3.20), we have obtained (3.21).

It remains to show (3.20). Although it might seem that taking $\psi := 1$ and $\sigma := \delta$ in (3.24) is equivalent to (3.20), we must note that in such case $\partial_t q$ does not satisfy the homogeneous Dirichlet boundary condition at Γ_0. Instead, we have the Neumann condition

$$\partial_3 \partial_t q = (\delta_{k3} - a_{k3})\partial_k \partial_t q - \partial_t a_{k3} \partial_k q,$$

by applying ∂_t to $a_{k3}\partial_k q = 0$ on Γ_1. Hence, in the case $\psi = 1$ and $\sigma := \delta$, the estimate (3.24) does include the last two terms inside the parentheses, as $\nabla\psi = \nabla 1 = 0$, but instead (3.23) needs to be amended by the boundary term

$$\|(I-a)\nabla\partial_t q - \partial_t a \nabla q\|_{H^{0.5+\delta}(\Gamma_0)} \lesssim \|I-a\|_{1.5+\delta}\|\partial_t q\|_{2+\delta} + \|\partial_t a\|_{1.5+\delta}\|q\|_{2+\delta} \lesssim \varepsilon\|\partial_t q\|_{2+\delta} + P,$$

(just avoid Γ_0); that's all we need for (3.21) where we used (3.4) again and (3.6), (3.16). Thus choosing a sufficiently small $\varepsilon > 0$ and absorbing the first term on the right-hand side we obtain (3.20), as required.

3.4 Tangential estimates

In this section we show that

$$\|S(\psi v)\|_{L^2}^2 + \|a_{3l} S\eta_l\|_{L^2(\Gamma_1)}^2 \lesssim \|\psi v_0\|_{2.5+\delta}^2 + \|v_0\|_{2+\delta}^2 + \int_0^t P \, \mathrm{d}s, \tag{3.25}$$

where, as above, P denotes a polynomial in $\|v(s)\|_{2.5+\delta}$ and $\|\chi\eta(s)\|_{3+\delta}$.

Our estimate follows a similar scheme as [KT2, Lemma 6.1], except that the argument is shorter and sharper in the sense that we eliminate the dependence on $\partial_t v$, q and $\partial_t q$. Another essential change results from the appearance of the cutoff ψ. This localizes the estimate and causes minor changes to the scheme. Note that the appearance of ψ is essential for localizing the highest order dependence on η. Namely, it allows us to use the $H^{3+\delta}$ norm of merely $\chi\eta$, rather than η itself, which we only need to control in the $H^{2.5+\delta}$ norm.

In the remainder of Section 3.4, we prove (3.25). Note that $S\partial_t(\psi v_i) = -S(a_{ki}\psi\partial_k q)$, which gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|S(\psi v)\|_{L^2}^2 = -\int S(a_{ki}\psi\partial_k q) S(\psi v_i). \tag{3.26}$$

In what follows, we estimate the right-hand side by $P + I_{113}$, where I_{113} is defined in (3.28) below and is such that

$$\int_0^t I_{113} \, \mathrm{d}s \lesssim -\|a_{3l} S\eta_l(t)\|_{L^2(\Gamma)}^2 + \|\partial_3 q(0)\|_{1.5+\delta} + \int_0^t P \, \mathrm{d}s. \tag{3.27}$$

The inequality (3.25) then follows by integrating (3.26) in time on $(0,t)$ and recalling (3.16).

First we write

$$\begin{aligned}
-\int S(a_{ki}\psi\partial_k q)S(\psi v_i) &= -\int S a_{ki}\psi\partial_k q \, S(\psi v_i) - \int a_{ki}\partial_k S(\psi q)\, S(\psi v_i) + \int a S(\nabla\psi q) S(\psi v) \\
&\quad - \int \big(S(a\psi\nabla q) - S a \psi \nabla q - a S(\psi\nabla q)\big) S(\psi v) \\
&=: I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where we used $\psi\partial_k q = \partial_k(\psi q) - \partial_k\psi q$ to obtain the second and third terms. By (3.16), we obtain

$$I_3 \lesssim \|a\|_{L^\infty}\|q\|_{2.5+\delta}\|\psi v\|_{2.5+\delta} \le P.$$

For I_2, we integrate by parts in x_k and use the Piola identity $\partial_k a_{ki} = 0$ to get

$$I_2 = \int a_{ki} S(\psi q)\partial_k S(\psi v_i) = \int a_{ki} S(\psi q) S(\psi \partial_k v_i) + \underbrace{\int a S(\psi q) S(\nabla\psi v)}_{\lesssim \|a\|_{L^\infty}\|q\|_{2.5+\delta}\|v\|_{2.5+\delta}}.$$

Note that the boundary terms in the first step vanish, as $q = 0$ on Γ_1 and ψ vanishes in a neighborhood of Γ_0. Moving S away from $\psi\partial_k v$ in the first term above, and recalling the divergence-free condition $a_{ki}\partial_k v_i = 0$, we obtain

$$a_{ki}S(\psi\partial_k v_i) = -Sa_{ki}\psi\partial_k v_i - (S(a_{ki}\psi\partial_k v_i) - a_{ki}S(\psi\partial_k v_i) - Sa_{ki}\psi\partial_k v_i),$$

and thus

$$\begin{aligned}
I_2 &\lesssim -\int SaS(\psi q)\psi\nabla v + \|S(\psi q)\|_{L^3}\|S(a\psi\nabla v) - aS(\psi\nabla v) - Sa\psi\nabla v\|_{L^{\frac{3}{2}}} + P \\
&\lesssim -\int \Lambda^{2+\delta}a\Lambda^{\frac{1}{2}}\left(S(\psi q)\psi\nabla v\right) + \|\psi q\|_{3+\delta}(\|\chi^2 a\|_{W^{1,6}}\|v\|_{2.5+\delta} + \|\chi^2 a\|_{W^{1.5+\delta,3}}\|v\|_{W^{2,3}}) + P \\
&\lesssim \|\chi^2 a\|_{2+\delta}(\|S(\psi q)\psi\nabla v\|_{0.5} + P) \lesssim \|\chi\eta\|^2_{3+\delta}(\|\psi q\|_{3+\delta}\|v\|_{2.5+\delta} + P) \lesssim P,
\end{aligned}$$

where we used the embeddings $H^{0.5} \subset L^3$ and $H^1 \subset L^6$ as well as the commutator estimate (2.9); we also applied (3.17) and (2.12) in the fourth and fifth inequalities respectively. As for I_4 we have

$$\begin{aligned}
I_4 &\lesssim \|\psi v\|_{2.5+\delta}\|S(a\psi\nabla q) - Sa\psi\nabla q - aS(\psi\nabla q)\|_{L^2} \\
&\lesssim P\|S(a\psi\nabla q) - Sa\psi\nabla q - aS(\psi\nabla q)\|_{L^2} \\
&\lesssim P(\|\chi^2 a\|_{W^{1,6}}\|\psi\nabla q\|_{W^{1.5+\delta,3}} + \|\chi^2 a\|_{W^{1.5+\delta,3}}\|\psi\nabla q\|_{W^{1,6}}) \lesssim P,
\end{aligned}$$

where we have applied (2.9), (2.12), (3.8), (3.16), and (3.17) in the last line.

For I_1, we write

$$S = \sum_{m=1,2} S_m\partial_m + S_0,$$

where $S_m := -\Lambda^{\frac{1}{2}+\delta}\partial_m$ and $S_0 := \Lambda^{\frac{1}{2}+\delta}$. Furthermore, differentiating the identity $a\nabla\eta = I$ with respect to x_m, where $m = 1, 2$, we get $\partial_m a\nabla\eta = -a\partial_m\nabla\eta$, and then multiplying this matrix equation by a on the right-side gives $\partial_m a = -a\partial_m\nabla\eta a$, which reads component-wise

$$\partial_m a_{ki} = -a_{kj}\partial_m\partial_l\eta_j a_{li}.$$

Thus we may write

$$Sa_{ki} = S_0 a_{ki} - \sum_{m=1,2} S_m(a_{kj}\partial_m\partial_l\eta_j a_{li}),$$

and consequently

$$\begin{aligned}
I_1 &= \sum_{m=1,2}\int S_m(a_{kj}\partial_m\partial_l\eta_j a_{li})\psi\partial_k q\, S(\psi v_i) - \int S_0 a\psi\nabla q S(\psi v) \\
&= \sum_{m=1,2}\int a_{kj}S_m\partial_m\partial_l\eta_j a_{li}\psi\partial_k q S(\psi v_i) + \sum_{m=1,2}\int \big(S_m(aD^2\eta a) - aS_m D^2\eta a\big)\psi\nabla q S(\psi v) \\
&\quad - \int S_0 a\psi\nabla q S(\psi v) \\
&=: I_{11} + I_{12} + I_{13}.
\end{aligned}$$

We have

$$I_{13} \lesssim \|S_0 a\|_{L^2}\|\nabla q\|_{L^\infty}\|S(\psi v)\|_{L^2} \lesssim \|a\|_{0.5+\delta}\|q\|_{2.5+\delta}\|v\|_{2.5+\delta} \le P,$$

and

$$\begin{aligned}
I_{12} &\lesssim \|\nabla q\|_{L^\infty}\|\psi v\|_{2.5+\delta}\|S_m((\chi^2 a)^2 D^2(\chi\eta)) - (\chi^2 a)^2 S_m D^2(\chi\eta)\|_{L^2} \\
&\lesssim \big(\|(\chi^2 a)^2\|_{W^{1.5+\delta,3}}\|\chi\eta\|_{W^{2,6}} + \|(\chi^2 a)^2\|_{W^{1,6}}\|\chi\eta\|_{W^{2.5+\delta,3}}\big) P \le P,
\end{aligned}$$

as claimed, where we used (3.8) in the first inequality, and (2.8), (2.12) in the second.

For I_{11} we write $\sum_{m=1,2} S_m\partial_m = -S_0 + S$, integrate by parts in x_l and use the Piola identity $\partial_l a_{li} = 0$ to get

$$\begin{aligned}
I_{11} &= -\int aS_0\nabla\eta a\psi\nabla qS(\psi v) - \int \nabla aS\eta a\psi\nabla qS(\psi v) - \int aS\eta a\nabla\psi\nabla qS(\psi v) \\
&\quad \underbrace{-\int aS\eta a\psi D^2qS(\psi v)}_{=:I_{111}}\underbrace{-\int a_{kj}S\eta_j a_{li}\psi\partial_k q\partial_l S(\psi v_i)}_{=:I_{112}} + \underbrace{\int_{\Gamma_1} a_{kj}S\eta_j a_{3i}\partial_k qS(\psi v_i)\,\mathrm{d}\sigma}_{=:I_{113}} \\
&\lesssim \|a\|_{L^\infty}^2\|q\|_{W^{1,\infty}}\|\eta\|_{2.5+\delta}\|v\|_{2.5+\delta} \\
&\quad + \|\nabla(\chi^2 a)\|_{L^6}\|S(\chi\eta)\|_{L^3}\|a\|_{L^\infty}\|q\|_{W^{1,\infty}}\|v\|_{2.5+\delta} + I_{111} + I_{112} + I_{113} \\
&\lesssim P + I_{111} + I_{112} + I_{113},
\end{aligned} \tag{3.28}$$

where we used (3.8) in the first inequality, and inequalities $\|f\|_{L^\infty} \lesssim \|f\|_{1.5+\delta}$, and (2.12) to obtain $\|\nabla(\chi^2 a)\|_{L^6} \lesssim \|\chi^2 a\|_2 \lesssim \|\chi\eta\|_{3+\delta}$ in the second inequality. Note that there is no boundary term at Γ_0 as $\psi = 0$ on Γ_0.

For I_{111} we write $\psi D^2 q = D^2(\psi q) - 2\nabla\psi\nabla q - D^2\psi q$ to obtain

$$\begin{aligned}
I_{111} &= -\int aS\eta aD^2(\psi q)S(\psi v) + \int aS\eta a(2\nabla\psi\nabla q + D^2\psi q)S(\psi v) \\
&\lesssim \|a\|_{L^\infty}^2\|S(\chi\eta)\|_{L^3}\|v\|_{2.5+\delta}(\|\psi q\|_{W^{2,6}} + \|q\|_{W^{1,6}}) \lesssim P,
\end{aligned}$$

where we used (2.12), (3.17), and the embedding $H^{0.5} \subset L^3$ in the last inequality. As for I_{112}, we note that the divergence-free condition, $a_{li}\partial_l v_i = 0$, gives

$$S(a_{li}\partial_l(\psi v_i)) = S(a_{li}\partial_l\psi v_i).$$

In order to use this fact, we put a_{li} inside the second S in I_{112} and extract the resulting commutator. Namely, we denote $f := -a_{kj}S\eta_j\psi\partial_k q$ for brevity, and write

$$\begin{aligned}
I_{112} &= \int a_{li}S\partial_l(\psi v_i)f \\
&= -\int \left(S(a_{li}\partial_l(\psi v_i)) - Sa_{li}\partial_l(\psi v_i) - a_{li}S\partial_l(\psi v_i)\right)f \\
&\quad + \int \Lambda^{2+\delta}(\chi^2 a_{li}\partial_l\psi v_i)\Lambda^{0.5}f - \int \Lambda^{2+\delta}(\chi^2 a_{li})\Lambda^{0.5}(\partial_l(\psi v_i)f) \\
&\lesssim \left\|S(\chi^2 a_{li}\partial_l(\psi v_i)) - S(\chi^2 a_{li})\partial_l(\psi v_i) - \chi^2 a_{li}S\partial_l(\psi v_i)\right\|_{L^2}\|f\|_{L^2} \\
&\quad + \|\chi^2 a\nabla\psi v\|_{2+\delta}\|f\|_{0.5} + \|\chi^2 a\|_{2+\delta}\|\nabla(\psi v)f\|_{0.5} \lesssim P,
\end{aligned}$$

where, in the second equality, we recalled the fact that $S = \Lambda^{2+\delta}\Lambda^{0.5}$ (recall (2.10)) and (3.8). We also used (3.8) in the first inequality and (2.12) in the last; we have also noted that (3.4) and (3.16) give $\|f\|_{0.5} \lesssim \|a\|_{1.5+\delta}\|\chi\eta\|_{3+\delta}\|q\|_{2.5+\delta} \lesssim P$, and we estimated the commutator term by

$$\|\chi^2 a\|_{W^{1,3}}\|v\|_{W^{1.5+\delta,6}} + \|v\|_{W^{1,6}}\|\chi^2 a\|_{W^{1.5+\delta,3}} \le \|\chi^2 a\|_{2+\delta}\|v\|_{2.5+\delta} \le P,$$

using (2.12) and the Kato-Ponce inequality (2.9).

It remains to estimate the boundary term I_{113}, as claimed in (3.27). We note that $\partial_k q = 0$, for

$k = 1,2$, on Γ_1, and $v_i = \partial_t \eta_i$, which gives $a_{3i} S v_i = \partial_t (a_{3i} S \eta_i) - \partial_t a_{3i} S \eta_i$. Thus

$$\begin{aligned}
I_{113} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1} |a_{3i} S\eta_i|^2 \partial_3 q \,\mathrm{d}\sigma - \int_{\Gamma_1} a_{3j} S\eta_j \partial_3 q \partial_t a_{3i} S\eta_i \,\mathrm{d}\sigma - \frac{1}{2}\int_{\Gamma_1} |a_{3i} S\eta_i|^2 \partial_3 \partial_t q \,\mathrm{d}\sigma \\
&\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1} |a_{3i} S\eta_i|^2 \partial_3 q \,\mathrm{d}\sigma + \|a\|_{L^\infty} \|\partial_t a\|_{L^\infty} \|q\|_{W^{1,\infty}} \|S(\chi\eta)\|^2_{L^2(\Gamma_1)} \\
&+ \|a\|^2_{L^\infty} \|S(\chi\eta)\|^2_{L^2(\Gamma_1)} \|\partial_t(\psi q)\|_{W^{1,\infty}} \\
&\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1} |a_{3i} S\eta_i|^2 \partial_3 q \,\mathrm{d}\sigma + P,
\end{aligned}$$

where we used (2.11), (3.6), the facts $\|S(\chi\eta)\|_{L^2(\Gamma_1)} \lesssim \|\chi\eta\|_{3+\delta} \le P$, and $\|q\|_{W^{1,\infty}} \lesssim \|q\|_{2.5+\delta} \le P$, as well as the pressure estimate (3.21) to get $\|\partial_t(\psi q)\|_{W^{1,\infty}} \lesssim \|\partial_t(\psi q)\|_{2.5+\delta} \le P$.

Consequently, the Rayleigh-Taylor condition for small times (3.22) and the fact that $a_{ki} = \delta_{ki}$ at time 0 give

$$\begin{aligned}
\int_0^t I_{113} \,\mathrm{d}s &\le \frac{1}{2}\int_{\Gamma_1} |a_{3i} S\eta_i|^2 \partial_3 q \,\mathrm{d}\sigma\Big|_t - \frac{1}{2}\int_{\Gamma_1} |S\eta_3|^2 \partial_3 q \,\mathrm{d}\sigma\Big|_0 + \int_0^t P \,\mathrm{d}s \\
&\le -\frac{b}{C}\|a_{3i} S\eta_i\|^2_{L^2} + C\|\partial_3 q(0)\|_{L^\infty} + \int_0^t P \,\mathrm{d}s,
\end{aligned}$$

where we used $\eta(x,0) = x$ in the last step. This concludes the proof of (3.25).

3.5 Final estimates

In this section, we collect the estimates, thus completing the proof of the main theorem.

Proof of Theorem 1. From the inequalities (3.9) and (3.10), combined with the tangential estimate (3.25), we obtain the inequality

$$\|v\|_{2.5+\delta}, \|\chi\eta\|_{3+\delta} \lesssim \int_0^t P \,\mathrm{d}s + \|\psi v_0\|^2_{2.5+\delta} + \|v_0\|^2_{2+\delta} + t\|\chi\omega_0\|_{2+\delta} + \|\omega_0\|_{1.5+\delta} + 1 + \|v_0\|_{L^2}, \quad (3.29)$$

where P is a polynomial in $\|v\|_{2.5+\delta}$ and $\|\chi\eta\|_{3+\delta}$. Note that, in order to estimate the boundary terms on the right hand side in (3.9), we have used

$$\|S\eta_3\|_{L^2(\Gamma_1)} \le \|a_{3l} S\eta_l\|_{L^2(\Gamma_1)} + \|(\delta_{3l} - a_{3l}) S\eta_l\|_{L^2(\Gamma_1)} \le \|a_{3l} S\eta_l\|_{L^2(\Gamma_1)} + \varepsilon\|\chi\eta\|_{3+\delta},$$

where we applied (3.4) and a trace estimate in the second step, and we absorbed the last term by the left hand side above. Moreover, in order to obtain the initial kinetic energy $\|v_0\|_{L^2}$ in (3.29), instead of $\|v\|_{L^2}$ from (3.10), we note that

$$\|v\|_{L^2} \le \|v_0\|_{L^2} + \int_0^t \|\partial_t v\|_{L^2} \,\mathrm{d}s \le \|v_0\|_{L^2} + \int_0^t \|a\|_{1.5+\delta} \|q\|_1 \,\mathrm{d}s \lesssim \|v_0\|_{L^2} + \int_0^t P \,\mathrm{d}s,$$

where we used $\partial_t v = -a\nabla q$, in the second inequality and (2.11), (3.16) in the last.

The a priori estimate (3.29) allows us to apply a standard Gronwall argument, concluding the proof. $\square$

References

[ABZ1] T. Alazard, N. Burq, and C. Zuily, *On the water-wave equations with surface tension*, Duke Math. J. **158** (2011), no. 3, 413–499.

[ABZ2] T. Alazard, N. Burq, and C. Zuily, *Low regularity Cauchy theory for the water-waves problem: canals and wave pools*, Lectures on the analysis of nonlinear partial differential equations. Part 3, Morningside Lect. Math., vol. 3, Int. Press, Somerville, MA, 2013, pp. 1–42.

[AD] T. Alazard and J.-M. Delort, *Global solutions and asymptotic behavior for two dimensional gravity water waves*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 5, 1149–1238.

[AM1] D.M. Ambrose and N. Masmoudi, *The zero surface tension limit of two-dimensional water waves*, Comm. Pure Appl. Math. **58** (2005), no. 10, 1287–1315.

[AM2] D.M. Ambrose and N. Masmoudi, *The zero surface tension limit of three-dimensional water waves*, Indiana Univ. Math. J. **58** (2009), no. 2, 479–521.

[B] J.T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure Appl. Math. **34** (1981), no. 3, 359–392.

[BB] J.P. Bourguignon and H. Brezis, *Remarks on the Euler equation*, J. Functional Analysis **15** (1974), 341–363.

[CL] A. Castro and D. Lannes, *Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity*, Indiana Univ. Math. J. **64** (2015), no. 4, 1169–1270.

[ChL] D. Christodoulou and H. Lindblad, *On the motion of the free surface of a liquid*, Comm. Pure Appl. Math. **53** (2000), no. 12, 1536–1602.

[CS1] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc. **20** (2007), no. 3, 829–930.

[CS2] D. Coutand and S. Shkoller, *A simple proof of well-posedness for the free-surface incompressible Euler equations*, Discrete Contin. Dyn. Syst. Ser. S **3** (2010), no. 3, 429–449.

[DE1] M.M. Disconzi and D.G. Ebin, *The free boundary Euler equations with large surface tension*, J. Differential Equations **261** (2016), no. 2, 821–889.

[DE2] M.M. Disconzi and D.G. Ebin, *On the limit of large surface tension for a fluid motion with free boundary*, Comm. Partial Differential Equations **39** (2014), no. 4, 740–779.

[DK] M.M. Disconzi and I. Kukavica, *A priori estimates for the free-boundary Euler equations with surface tension in three dimensions* (submitted).

[E] D.G. Ebin, *The equations of motion of a perfect fluid with free boundary are not well posed*, Comm. Partial Differential Equations **12** (1987), no. 10, 1175–1201.

[EL] T. Elgindi and D. Lee, *Uniform regularity for free-boundary Navier-Stokes equations with surface tension*, arXiv:1403.0980 (2014).

[GMS] P. Germain, N. Masmoudi, and J. Shatah, *Global solutions for the gravity water waves equation in dimension 3*, Ann. of Math. (2) **175** (2012), no. 2, 691–754.

[HIT] J.K. Hunter, M. Ifrim, and D. Tataru, *Two dimensional water waves in holomorphic coordinates*, Comm. Math. Phys. **346** (2016), no. 2, 483–552.

[IT] M. Ifrim and D. Tataru, *Two dimensional water waves in holomorphic coordinates II: Global solutions*, Bull. Soc. Math. France **144** (2016), no. 2, 369–394.

[I] T. Iguchi, *Well-posedness of the initial value problem for capillary-gravity waves*, Funkcial. Ekvac. **44** (2001), no. 2, 219–241.

[IK] M. Ignatova and I. Kukavica, *On the local existence of the free-surface Euler equation with surface tension*, Asymptot. Anal. **100** (2016), no. 1-2, 63–86.

[IP] A.D. Ionescu and F. Pusateri, *Global solutions for the gravity water waves system in 2d*, Invent. Math. **199** (2015), no. 3, 653–804.

[KP] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 7, 891–907.

[KT1] I. Kukavica and A. Tuffaha, *On the 2D free boundary Euler equation*, Evol. Equ. Control Theory **1** (2012), no. 2, 297–314.

[KT2] I. Kukavica and A. Tuffaha, *A regularity result for the incompressible Euler equation with a free interface*, Appl. Math. Optim. **69** (2014), no. 3, 337–358.

[KT3] I. Kukavica and A. Tuffaha, *Well-posedness for the compressible Navier-Stokes-Lamé system with a free interface*, Nonlinearity **25** (2012), no. 11, 3111–3137.

[KT4] I. Kukavica and A. Tuffaha, *A sharp regularity result for the Euler equation with a free interface*, Asymptot. Anal. **106** (2018), no. 2, 121–145.

[KTV] I. Kukavica, A. Tuffaha, and V. Vicol, *On the local existence and uniqueness for the 3D Euler equation with a free interface*, Appl. Math. Optim. (2016), doi:10.1007/s00245-016-9360-6.

[KTVW] I. Kukavica, A. Tuffaha, V. Vicol, and F. Wang, *On the existence for the free interface 2D Euler equation with a localized vorticity condition*, Appl. Math. Optim. **73** (2016), no. 3, 523–544.

[L] D. Lannes, *Well-posedness of the water-waves equations*, J. Amer. Math. Soc. **18** (2005), no. 3, 605–654 (electronic).

[Li] D. Li, *On Kato-Ponce and fractional Leibniz*, Rev. Mat. Iberoam. **35** (2019), no. 1, 23–100.

[Lin1] H. Lindblad, *Well-posedness for the linearized motion of an incompressible liquid with free surface boundary*, Comm. Pure Appl. Math. **56** (2003), no. 2, 153–197.

[Lin2] H. Lindblad, *Well-posedness for the motion of an incompressible liquid with free surface boundary*, Ann. of Math. (2) **162** (2005), no. 1, 109–194.

[MC] B. Muha and S. Čanić, *Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy*, Interfaces Free Bound. **17** (2015), no. 4, 465–495.

[MR] N. Masmoudi and F. Rousset, *Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations*, Arch. Ration. Mech. Anal. **223** (2017), no. 1, 301–417.

[N] V.I. Nalimov, *The Cauchy-Poisson problem*, Dinamika Splošn. Sredy (1974), no. Vyp. 18 Dinamika Zidkost. so Svobod. Granicami, 104–210, 254.

[OT] M. Iowa and A. Tani, *Free boundary problem for an incompressible ideal fluid with surface tension*, Math. Models Methods Appl. Sci. **12** (2002), no. 12, 1725–1740.

[P] F. Pusateri, *On the limit as the surface tension and density ratio tend to zero for the two-phase Euler equations*, J. Hyperbolic Differ. Equ. **8** (2011), no. 2, 347–373.

[S] B. Schweizer, *On the three-dimensional Euler equations with a free boundary subject to surface tension*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 6, 753–781.

[Sh] M. Shinbrot, *The initial value problem for surface waves under gravity. I. The simplest case*, Indiana Univ. Math. J. **25** (1976), no. 3, 281–300.

[Shn] A.I. Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Mat. Sb. (N.S.) **128(170)** (1985), no. 1, 82–109, 144.

[SZ1] J. Shatah and C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math. **61** (2008), no. 5, 698–744.

[SZ2] J. Shatah and C. Zeng, *Local well-posedness for fluid interface problems*, Arch. Ration. Mech. Anal. **199** (2011), no. 2, 653–705.

[T] A. Tani, *Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface*, Arch. Rational Mech. Anal. **133** (1996), no. 4, 299–331.

[WZZZ] C. Wang, Z. Zhang, W. Zhao, and Y. Zheng, *Local well-posedness and break-down criterion of the incompressible Euler equations with free boundary*, arXiv:1507.02478, 2015.

[W1] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in* 2*-D*, Invent. Math. **130** (1997), no. 1, 39–72.

[W2] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. **12** (1999), no. 2, 445–495.

[W3] S. Wu, *Global wellposedness of the 3-D full water wave problem*, Invent. Math. **184** (2011), no. 1, 125–220.

[Y1] H. Yosihara, *Gravity waves on the free surface of an incompressible perfect fluid of finite depth*, Publ. Res. Inst. Math. Sci. **18** (1982), no. 1, 49–96.

[Y2] H. Yosihara, *Capillary-gravity waves for an incompressible ideal fluid*, J. Math. Kyoto Univ. **23** (1983), no. 4, 649–694.

[ZZ] P. Zhang and Z. Zhang, *On the free boundary problem of three-dimensional incompressible Euler equations*, Comm. Pure Appl. Math. **61** (2008), no. 7, 877–940.