

Local existence of strong solutions to the stochastic Navier-Stokes equations with L^p data

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ABSTRACT. For the stochastic Navier-Stokes equations with a multiplicative white noise on $\mathbb{T}^3$, we prove that there exists a unique strong solution locally in time when the initial datum belongs to $L^p(\Omega; L^p)$ for $p > 3$.

1. Introduction

We consider the stochastic Navier-Stokes equation (SNSE)

$$du(t, x) = \Delta u(t, x) dt - \mathcal{P}((u(t, x) \cdot \nabla)u(t, x)) dt + \sigma(u(t, x)) d\mathbb{W}(t), \quad (1.1)$$

$$\nabla \cdot u = 0 \quad (1.2)$$

$$u(0, x) = u_0(x), \quad (t, x) \in (0, \infty) \times \mathbb{T}^3 \quad (1.3)$$

on the 3D torus $\mathbb{T}^3 = [0, 1]^3$, where we assume that the initial datum u_0 satisfies $\nabla \cdot u_0 = 0$ and $\int_{\mathbb{T}^3} u_0 = 0$. The variable u denotes the velocity and $\mathcal{P}$ stands for the Leray projector onto the mean zero divergence-free fields. The stochastic term $\sigma(u)d\mathbb{W}(t)$ represents a possibly infinite-dimensional multiplicative white noise and is interpreted in the Itô sense. Physically, it accounts for random velocity-dependent perturbations during the flow evolution.

In this paper, we obtain a pathwise unique strong solution of (1.1)–(1.3) in $L^p(\mathbb{T}^3)$ for the full range of exponents $p > 3$. This problem was previously addressed in the paper [KXZ], where we used the fixed point technique utilizing a multiplicative cut-off of the nonlinear term, which in turn led to the restriction $p > 5$ for the initial data. In this paper, we change the approach to using a spectral Galerkin type approximation. These approximations are generally well-suited for the Hilbert space setting but are not considered well-adapted for L^p type approximations. However, using a square cut-off in Fourier space, we show that the solutions of the approximation converge in suitable spaces to the sought-after local L^p solution of the stochastic Navier-Stokes system for $p > 3$.

When considering the approximating system and assuming an additive noise, one can write the equation as the difference of the stochastic Stokes equation and an auxiliary deterministic equation; cf. [F, MS] for applications of this approach to additive white noise and additive Lévy noise. When

having a general multiplicative noise, we lose such an advantage. Here, we reduce the finite-dimensional approximations of (1.1) to equations with an additive noise by a fixed point argument, and then we utilize stopping times to linearize $(u \cdot \nabla)u$ to derive energy estimates. However, introducing stopping times is known for bringing the potential issue of them being degenerate, which prevents one from claiming a limit on a nontrivial time interval for the sequence of approximating solutions. We circumvent this obstacle by employing the estimates in probability subspaces that monotonically expand to the whole probability space. In each of these subspaces, the sequence of stopping times has a positive limit almost surely, and thus we can extract a pathwise limit of approximating solutions in that space. Then, we extend the result to the whole probability space by using indicator functions and showing convergence in an appropriate sense. There are also some challenges when studying SNSE in non-Hilbert trajectory spaces. First, the usual Galerkin scheme does not converge in L^p when $p \neq 2$. In this paper, we use the rectangular partial sums instead (cf. (2.2) below) and prove their continuity in $L^p(l^2)$ by applying a vector analog of the Calderón-Zygmund theorem. Yet as a consequence, there is no available result ensuring the existence of approximating solutions for this finite scheme. Hence, we first construct approximating solutions in $H^1(\mathbb{T}^3)$. Then we derive estimates in $L^p(\mathbb{T}^3)$, where the one of the important ingredients is Lemma 3.2, which is proven in [KXZ] using [R].

A study of the SNSE dates back to the work of Bensoussan and Temam (cf. [BeT]) in the 1970's. Early investigations were usually conducted in Hilbert settings. Existing results on this include [MeS], where the SNSE with additive white noise was considered in 2D domains, and a global strong solution was shown to exist, and [F], where the consideration was in 3D. In [FS], the authors proved the same results for the SNSE in 2D unbounded domains with multiplicative noise. In [GZ], the authors constructed a maximal strong solution for the equation in 3D bounded domains assuming H^1 regularity of the initial data. If the SNSE is equipped with a non-degenerate noise and sufficiently small initial data in H^s , then by [Ki], a global strong solution exists with large probability. Development of the theory naturally leads to considerations in Banach spaces. An effort in this direction includes [AgV], where the authors studied the equation with multiplicative noise in critical Besov spaces, and [KXZ], where the SNSE with L^p initial data was considered. Under the condition that $p > 5$, a unique strong solution exists globally with large probability. There are also investigations on various notions of solutions to the SNSE or to relevant equations with different types of noise; cf. [BF, FG, MR] for results on martingale solutions of the SNSE, [BCF, CC, DZ, MS] for the mild formulation subject to white noise, and [BT, ZBL1, ZBL2, FRS] for mild formulation with Lévy noise. In [GV], the stochastic Euler equation with linear multiplicative noise was addressed in $W^{s,p}$, and in [BR], the vorticity equation of SNSE with a convolution-type noise was considered.

The paper is organized as follows. Section 2 is an introduction to the approximating finite dimensional scheme, while it also contains preliminaries on stochastic integration. In Section 3, we state our assumptions and the main result. In Section 4, we establish the local existence and the pathwise uniqueness of strong solutions. Furthermore, we prove that

$$\mathbb{E} \left[\sup_{0 \leq s \leq \tau} \|u(s, \cdot)\|_p^p \right] \leq C \mathbb{E} [\|u_0\|_p^p + 1]$$

and

$$\mathbb{E} \left[\int_0^\tau \sum_j \int_{\mathbb{T}^3} |\nabla(|u_j(s, x)|^{p/2})|^2 dx ds \right] \leq C \mathbb{E} [\|u_0\|_p^p + 1],$$

up to the maximal time of existence τ .

2. Notation and preliminaries

2.1. Basic Notation. The Fourier coefficients of an integrable function f are denoted by

$$\mathcal{F}f(m) = \hat{f}(m) = \int_{\mathbb{T}^3} f(x) e^{-2\pi i m \cdot x} dx, \quad m \in \mathbb{Z}^d,$$

where d is the space dimension, while the Fourier inversion formula reads

$$(\mathcal{F}^{-1}g)(x) = \sum_{m \in \mathbb{Z}^d} g(m) e^{2\pi i m \cdot x}.$$

We adopt the same notation for the Fourier transform of a distribution $f \in \mathcal{D}' = (C^\infty(\mathbb{T}^d))'$. As usual, $W^{s,p}(\mathbb{T}^d)$, where $p > 1$, denotes the class of functions $f \in \mathcal{D}'(\mathbb{T}^d)$ for which $\|f\|_{s,p} = \|J^s f\|_p < \infty$, where

$$J^s f(x) := \sum_{m \in \mathbb{Z}^d} (1 + 4\pi^2 |m|^2)^{s/2} \hat{f}(m) e^{2\pi i m \cdot x}, \quad x \in \mathbb{T}^d, \quad s \in \mathbb{R},$$

with $\|\cdot\|_p$ denoting the L^p norm. When $p = 2$, we also write $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$. The Leray projector

$$(\mathcal{P}u)_j(x) = \sum_{k=1}^d (\delta_{jk} + R_j R_k) u_k(x), \quad j = 1, 2, \dots, d$$

is defined for distributions that have mean zero over $\mathbb{T}^d$. Here,

$$R_j = -\frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}}, \quad j = 1, 2, \dots, d \quad (2.1)$$

are Riesz transforms. For convenience, we write

$$W_{\text{sol}}^{s,p} = \{\mathcal{P}f : f \in W^{s,p}\}.$$

When constructing solutions below, we use a finite approximation scheme to (1.1). For this purpose, we introduce the rectangular partial sums

$$\mathcal{T}_n f(x) = \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \hat{f}(k) e^{2\pi i k \cdot x} = \int_{\mathbb{T}^d} f(x-u) D_n(u) du, \quad (2.2)$$

for $f \in L^1(\mathbb{T}^d)$, where $n = (n_1, \dots, n_d)$ is a multi-index and $D_n(u) = \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} e^{2\pi i k \cdot u}$ is the rectangular Dirichlet kernel. Denote

$$\bar{n} = \min\{n_1, \dots, n_d\}, \quad n = (n_1, \dots, n_d) \in \mathbb{N}_0^d.$$

It was shown in [G] that $\mathcal{T}_n$ is a continuous operator on all $L^q(\mathbb{T}^d)$ spaces ($1 < q < \infty$), i.e.,

$$\|\mathcal{T}_n f\|_q \leq C_q \|f\|_q, \quad f \in L^q(\mathbb{T}^d), \quad (2.3)$$

and

$$\|\mathcal{T}_n f - f\|_q \rightarrow 0 \quad \text{as } \bar{n} \rightarrow \infty \quad (2.4)$$

The following lemma is one of our main devices for estimating the nonlinear terms.

LEMMA 2.1. *For all $q \in (1, \infty)$ and all $f \in W^{1,q}(\mathbb{T}^d)$ with x -zero mean, there exists $\alpha(q) \in (0, 1]$ such that*

$$\|(\mathcal{T}_n - \mathcal{T}_m)f\|_q \leq \frac{C_{q,d}}{\bar{m}^\alpha \wedge \bar{n}^\alpha} \|\nabla f\|_q, \quad (2.5)$$

for all multiindices $n = (n_1, \dots, n_d)$ and $m = (m_1, \dots, m_d)$.

PROOF OF LEMMA 2.1. Let $u \in (C^\infty(\mathbb{T}^d))^d$, and denote

$$T^{(n,m)} = \mathcal{T}_{nm} \Delta^{-1} \text{div},$$

where $\mathcal{T}_{nm} = \mathcal{T}_n - \mathcal{T}_m$ and

$$\Delta^{-1} \text{div } u = \sum_{k=(k_1, \dots, k_d) \neq 0} \left(\frac{-1}{4\pi^2 |k|^2} \sum_{l=1}^d 2\pi i k_l \hat{u}_l(k) \right) e^{2\pi i k \cdot x}.$$

(Below, we shall choose $u = \nabla f$.) Note that $\Delta^{-1} \operatorname{div} \nabla f = f$ since f has zero mean. By the orthogonality of $\{e^{2\pi i k \cdot x}\}$ and Parseval's identity, we have

$$\|T^{(n,m)}u\|_2^2 \leq \sum_{|k| \geq \bar{m} \wedge \bar{n}} \left| \frac{1}{4\pi^2 |k|^2} \sum_{l=1}^d 2\pi i k_l \hat{u}_l(k) \right|^2 \leq \frac{C_d}{\bar{m}^2 \wedge \bar{n}^2} \|u\|_2^2,$$

from where

$$\|T^{(n,m)}u\|_2 \leq \frac{C_d}{\bar{m} \wedge \bar{n}} \|u\|_2. \quad (2.6)$$

Substituting u by ∇f yields

$$\|\mathcal{T}_{nm}f\|_2 \leq \frac{C_2}{\bar{m} \wedge \bar{n}} \|\nabla f\|_2,$$

where C_2 is a constant depending only on the dimension d . Now, let $q \in (1, 2)$ be arbitrary and choose $r = (1 + q)/2 \in (1, q)$. Then we have

$$\|T^{(n,m)}u\|_r \leq C_r \|\Delta^{-1} \operatorname{div} u\|_r \leq C_r \|\nabla \Delta^{-1} \operatorname{div} u\|_r \leq C_r \|u\|_r, \quad (2.7)$$

where the constant C_r depends on r and thus on q ; in the second inequality in (2.7), we used the Poincaré inequality $\|f\|_r \leq C_r \|\nabla f\|_r$, which holds for $r \geq 1$ and $f \in W^{1,r}$ such that $\int_{\mathbb{T}^d} f = 0$. Using the Marcinkiewicz interpolation theorem on the inequalities (2.6) and (2.7) yields

$$\|T^{(n,m)}u\|_q \leq C \frac{C_d^\alpha C_r^{1-\alpha}}{\bar{m}^\alpha \wedge \bar{n}^\alpha} \|u\|_q, \quad (2.8)$$

where $\alpha \in (0, 1)$ is such that $1/q = \alpha/2 + (1 - \alpha)/r$, i.e., $\alpha = (1/r - 1/q)/(1/r - 1/2)$. Substituting $u = \nabla f$ in (2.8) yields (2.5) for $q \in (1, 2)$. The proof for $q \in (2, \infty)$ is the same except that we take $r = 2q$. $\square$

Above and in the rest of this paper, C denotes a generic positive constant, with additional dependence indicated when necessary.

2.2. Preliminaries on stochastic analysis. We denote by $\mathcal{H}$ a real separable Hilbert space with a complete orthonormal basis $\{\mathbf{e}_k\}_{k \geq 1}$, and by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a complete probability space with an augmented filtration $(\mathcal{F}_t)_{t \geq 0}$. With $\{W_k : k \in \mathbb{N}\}$ a family of independent $\mathcal{F}_t$ -adapted Brownian motions, $\mathbb{W}(t, \omega) := \sum_{k \geq 1} W_k(t, \omega) \mathbf{e}_k$ is an $\mathcal{F}_t$ -adapted and $\mathcal{H}$ -valued cylindrical Wiener process.

For a real separable Hilbert space $\mathcal{Y}$, we define $l^2(\mathcal{H}, \mathcal{Y})$ to be the set of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{Y}$ with the norm defined by

$$\|G\|_{l^2(\mathcal{H}, \mathcal{Y})}^2 := \sum_{k=1}^{\dim \mathcal{H}} |G \mathbf{e}_k|_{\mathcal{Y}}^2 < \infty, \quad G \in l^2(\mathcal{H}, \mathcal{Y}).$$

In this paper, we either regard (1.1) as a vector-valued equation or consider it componentwise. Correspondingly, $\mathcal{Y} = \mathbb{R}$ or $\mathbb{R}^d$. Let $G = (G_1, \dots, G_d)$ and $G \mathbf{e}_k := (G_1 \mathbf{e}_k, \dots, G_d \mathbf{e}_k)$. Then $G \in l^2(\mathcal{H}, \mathbb{R}^d)$ if and only if $G_i \in l^2(\mathcal{H}, \mathbb{R})$ for all $i \in \{1, \dots, d\}$. The Burkholder-Davis-Gundy (BDG) inequality

$$\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s G d\mathbb{W}_r \right|_{\mathcal{Y}}^p \right] \leq C \mathbb{E} \left[\left(\int_0^t \|G\|_{l^2(\mathcal{H}, \mathcal{Y})}^2 dr \right)^{p/2} \right]$$

holds for $p \in [1, \infty)$ and all $G \in l^2(\mathcal{H}, \mathcal{Y})$ such that the right hand side above is finite. For $s \geq 0$, introduce

$$\mathbb{W}^{s,p} := \left\{ f : \mathbb{T}^d \rightarrow l^2(\mathcal{H}, \mathcal{Y}) : f \mathbf{e}_k \in W^{s,p}(\mathbb{T}^d) \text{ for each } k, \text{ and } \int_{\mathbb{T}^d} \|J^s f\|_{l^2(\mathcal{H}, \mathcal{Y})}^p dx < \infty \right\},$$

which are Banach spaces with respect to the norm

$$\|f\|_{\mathbb{W}^{s,p}} := \left(\int_{\mathbb{T}^d} \|J^s f\|_{l^2(\mathcal{H}, \mathcal{Y})}^p dx \right)^{1/p}.$$

Above, we denoted $(J^s f)\mathbf{e}_k = J^s(f\mathbf{e}_k)$. Also, $\mathbb{W}^{0,p}$ is abbreviated as $\mathbb{L}^p$. If $f \in \mathbb{L}^2$, then $\int_0^t f d\mathbb{W}_t$ is an $L^2(\mathbb{T}^d)$ -valued Wiener process (cf. [DZ2]).

Letting $(\mathcal{P}f)\mathbf{e}_k = \mathcal{P}(f\mathbf{e}_k)$, where $\mathcal{P}$ is the Leray projector, we have $\mathcal{P}f \in \mathbb{W}^{s,p}$ if $f \in \mathbb{W}^{s,p}$. Write

$$\mathbb{W}_{\text{sol}}^{s,p} = \{\mathcal{P}f : f \in \mathbb{W}^{s,p}\}.$$

Letting $(\mathcal{T}_n f)\mathbf{e}_k = \mathcal{T}_n(f\mathbf{e}_k)$, where $\mathcal{T}_n$ is the rectangular partial sum in (2.2), we have the vector valued analog of (2.3)–(2.4), stated next.

LEMMA 2.2. *For every $q \in (1, \infty)$, we have*

$$\|\mathcal{T}_n f\|_{\mathbb{L}^q} \leq C_q \|f\|_{\mathbb{L}^q}, \quad n \in \mathbb{N}_0^d \quad (2.9)$$

and

$$\|\mathcal{T}_n f - f\|_{\mathbb{L}^q} \rightarrow 0 \quad \text{as } \bar{n} \rightarrow \infty, \quad (2.10)$$

for $f \in \mathbb{L}^q$ with x -zero mean.

PROOF OF LEMMA 2.2. It is sufficient to prove (2.9) when $d = 1$, in which case $n \in \mathbb{N}$. Consider the mapping

$$H \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{2\pi i m \cdot x} := \sum_{m \geq 1} \hat{f}(m) e^{2\pi i m \cdot x},$$

for l^2 -valued functions f such that $\hat{f}(0) = 0$. Using the vector valued Calderón-Zygmund theorem, we have

$$\|Hf\|_{\mathbb{L}^q} \leq C_q \|f\|_{\mathbb{L}^q},$$

where C_q is a constant, depending on q . Note that the same inequality is satisfied by the operators $H_n \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{2\pi i m \cdot x} = \sum_{m \leq n} \hat{f}(m) e^{2\pi i m \cdot x}$. The proof of (2.9) is then concluded by observing that $\mathcal{T}_n f = (I - H_{-(n+1)})H_n$, for all $n \in \mathbb{N}$.

First note that l^2 -valued trigonometric polynomials $\mathcal{T}$ are dense in $\mathbb{L}^q$. This holds since the Fejér kernel constitutes a partition of unity, implying the convergence of the l^2 -valued Fourier series in the mean. Assume that $f \in \mathbb{L}^q$, and let $\epsilon > 0$. Then there exists $P \in \mathcal{T}$ such that $\|f - P\|_{\mathbb{L}^q} \leq \epsilon$, whence

$$\|f - \mathcal{T}_n f\|_{\mathbb{L}^q} \leq \|f - P\|_{\mathbb{L}^q} + \|P - \mathcal{T}_n P\|_{\mathbb{L}^q} + \|\mathcal{T}_n P - \mathcal{T}_n f\|_{\mathbb{L}^q} \leq \epsilon + \|P - \mathcal{T}_n P\|_{\mathbb{L}^q} + C_q \epsilon.$$

Since, by $P \in \mathcal{T}$, the middle term vanishes for n sufficiently large, and we obtain $\limsup_{n \rightarrow \infty} \|f - \mathcal{T}_n f\|_{\mathbb{L}^q} \leq C_q \epsilon$. The proof of (2.10) is then concluded by sending $\epsilon \rightarrow 0$. $\square$

Lemma 2.2 confirms that $\int_0^t \mathcal{T}_n f d\mathbb{W}_t$ is an $\mathcal{T}_n L^2(\mathbb{T}^d)$ -valued Wiener process if $f \in \mathbb{L}^2$.

Let A be an operator, for instance a differential operator, and let σ and g be $(l^2(\mathcal{H}, \mathbb{R}))^d$ -valued operators. Given a cylindrical Wiener process $\mathbb{W}$ relative to a prescribed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, we define the local strong solution for the d -dimensional stochastic evolution partial differential equation

$$u(t, x) = u_0(x) + \int_0^t (Au(r, x) + f(r, x)) dr + \int_0^t (\sigma(u(r, x)) + g(r, x)) d\mathbb{W}(r) \quad (2.11)$$

in the following way.

DEFINITION 2.1 (Local strong solution). A pair (u, τ) is called a local strong solution of (2.11) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ if τ is a stopping time with $\mathbb{P}(\tau > 0) = 1$ and if $u \in L^p(\Omega; C([0, \tau \wedge T], L^p))$ is progressively measurable and it satisfies

$$(u(t), \phi) = (u_0, \phi) + \int_0^t (Au(r) + f(r), \phi) dr + \int_0^t (\sigma(u(r)) + g(r), \phi) d\mathbb{W}(r) \quad \mathbb{P}\text{-a.s.},$$

for all $\phi \in C^\infty(\mathbb{T}^d)$ and all $t \in [0, \tau \wedge T]$.

If A is a differential operator, we interpret $(Au(r), \phi)$ using integration by parts.

DEFINITION 2.2 (Pathwise uniqueness). The equation (2.11) is said to have a pathwise unique strong solution if for any pair of local strong solutions (u, τ) and (v, η) subject to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and the same Wiener process $\mathbb{W}$, we have $\mathbb{P}(u(t) = v(t), \forall t \in [0, \tau \wedge \eta]) = 1$.

3. Assumptions and main results

In this section, we summarize the assumptions and state the main result. We assume throughout that $d = 3$ and $p > 3$. We point out however that the results extend easily to any space dimension $d \geq 2$ with the condition $p > d$. On the noise coefficient σ in (1.1), we assume

(1) (sub-linear growth)

$$\sum_{i=1}^3 \|\sigma_i(u)\|_{\mathbb{L}^r} \leq C(\|u\|_r + 1), \quad r \in \{2, p, 3p\}, \quad (3.1)$$

(2) (Lipschitz continuity)

$$\sum_{i=1}^3 \|\sigma_i(u) - \sigma_i(v)\|_{\mathbb{L}^r} \leq C\|u - v\|_r, \quad r \in \{2, p\}, \quad (3.2)$$

(3) (preserved divergence and mean) $\sigma(W_{\text{sol}}^{s,p}) \subset \mathbb{W}_{\text{sol}}^{s,p}$, and $\int_{\mathbb{T}^3} \sigma(u) = 0$ if $\int_{\mathbb{T}^3} u = 0$, and

(4) (sublinearity of the gradient)

$$\sum_{i=1}^3 \|\nabla(\sigma_i(u))\|_{\mathbb{L}^2} \leq C\|u\|_2. \quad (3.3)$$

Under these conditions, we have the following statement.

THEOREM 3.1. (Local strong solution up to a stopping time) Let $p > 3$ and $u_0 \in L^p(\Omega; L^p(\mathbb{T}^3))$. Then there exists a pathwise unique local strong solution (u, τ) to (1.1)–(1.3) such that

$$\mathbb{E} \left[\sup_{0 \leq s \leq \tau} \|u(s, \cdot)\|_p^p + \int_0^\tau \sum_j \int_{\mathbb{T}^3} |\nabla(|u_j(s, x)|^{p/2})|^2 dx ds \right] \leq C\mathbb{E}[\|u_0\|_p^p + 1],$$

where $C > 0$ is a constant depending on p .

For the sake of completeness, we state two auxiliary results that were proven in [KXZ] (cf. Theorem 4.1 and Lemma 4.4 in [KXZ]).

LEMMA 3.2. Let $2 < p < \infty$ and $0 < T < \infty$. Suppose that $u_0 \in L^p(\Omega, L^p(\mathbb{T}^d))$, $f \in L^p(\Omega \times [0, T], W^{-1,q}(\mathbb{T}^d))$, and $g \in L^p(\Omega \times [0, T], \mathbb{L}^p(\mathbb{T}^d))$ are $\mathbb{R}^D$ -valued with x -mean zero (ω, t) a.s., and

$$\frac{dp}{p+d-2} < q \leq p,$$

provided $d \geq 2$, or $1 < q \leq p$ if $d = 1$. Then there exists a unique maximal solution $u \in L^p(\Omega; C([0, T], L^p))$ to

$$\begin{aligned} du(t, x) &= \Delta u(t, x) dt + f(t, x) dt + g(t, x) d\mathbb{W}(t), \\ u(0, x) &= u_0(x) \quad \text{a.s.,} \quad x \in \mathbb{T}^d. \end{aligned}$$

Moreover,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_p^p + \sum_{j=1}^D \int_0^T \int_{\mathbb{T}^d} |\nabla(|u_j(t, x)|^{p/2})|^2 dx dt \right] \\ & \leq C\mathbb{E} \left[\|u_0\|_p^p + \int_0^T \|f(t, \cdot)\|_{-1,q}^p dt + \sum_{j=1}^D \int_0^T \int_{\mathbb{T}^d} \|g_j(t, x)\|_{l^2(\mathcal{H}, \mathbb{R})}^p dx dt \right], \end{aligned} \quad (3.4)$$

where $C > 0$ depends on T and p .

The following lemma is essential when passing to the limit in $\| |u^{(n)}|^{p/2} \|_{H^1}$.

LEMMA 3.3. *Let $p \geq 2$. If*

$$u_n \rightarrow u \text{ in } L^p(\Omega; L^\infty([0, T], L^p(\mathbb{T}^d))) \text{ as } n \rightarrow \infty$$

and

$$\nabla(|u_n(\omega, t, x)|^{p/2}) \text{ are uniformly bounded in } L^2(\Omega \times [0, T], L^2(\mathbb{T}^d)),$$

then

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_n(\omega, t, x)|^{p/2})|^2 dx dt \right] \geq \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u(\omega, t, x)|^{p/2})|^2 dx dt \right]. \quad (3.5)$$

Note that in (3.4) and (3.5), T can be replaced by a stopping time by using indicator functions.

4. Existence of a local strong solution

Let $d = 3$, and fix $p > 3$; in particular, we allow all constants to depend on p without mention. Denote

$$\mathcal{S}_n f(x) = \sum_{|k_1| \leq n, |k_2| \leq n, |k_3| \leq n} \hat{f}(k) e^{2\pi i k \cdot x}, \quad f \in L^1(\mathbb{T}^3), \quad n \in \mathbb{N},$$

i.e.,

$$\mathcal{S}_n = \mathcal{T}_{(n,n,n)} \quad (4.1)$$

(cf. (2.2) and (4.1)). As pointed out in Section 2, $\mathcal{S}_n$ is a continuous linear operator in all L^q spaces ($1 < q < \infty$). To construct a solution of (1.1)–(1.3), we consider for $n \in \mathbb{N}$ a finite dimensional approximation

$$du^{(n)} = \Delta u^{(n)} dt - \mathcal{S}_n \mathcal{P}((u^{(n)} \cdot \nabla) u^{(n)}) dt + \mathcal{S}_n \sigma(u^{(n)}) d\mathbb{W}(t), \quad (4.2)$$

$$\nabla \cdot u^{(n)} = 0, \quad (4.3)$$

$$u^{(n)}(0, x) = \mathcal{S}_n u_0(x), \quad x \in \mathbb{T}^3. \quad (4.4)$$

Note that the cancellation property of the convective term remains true in $\mathcal{S}_n L^2$; namely, for $u, v \in (\mathcal{S}_n L^2(\mathbb{T}^3))^3 \subset (H^1(\mathbb{T}^3))^3$ with $\nabla \cdot u = \nabla \cdot v = 0$, we have

$$\begin{aligned} \int_{\mathbb{T}^d} u \cdot \mathcal{S}_n \mathcal{P}((v \cdot \nabla) u) dx &= \int_{\mathbb{T}^d} (u_j \mathcal{S}_n (v_i \partial_i u_j) + u_j \mathcal{S}_n R_j R_k \partial_i (v_i u_k)) dx \\ &= \int_{\mathbb{T}^d} (u_j v_i \partial_i u_j + u_j R_j R_k \partial_i (v_i u_k)) dx \\ &= \int_{\mathbb{T}^d} \left(-\frac{1}{2} (u_j u_j) \partial_i v_i - \partial_j u_j R_k R_i (v_i u_k) \right) dx = 0, \end{aligned} \quad (4.5)$$

where R_i are the Riesz transforms defined in (2.1); in the third inequality, we used $R_j \partial_i = \partial_j R_i$. If $\int_{\mathbb{T}^3} u = \int_{\mathbb{T}^3} v = 0$, then

$$\left| \int_{\mathbb{T}^d} u \cdot \mathcal{S}_n \mathcal{P}((v \cdot \nabla) w) dx \right| \leq C \|u\|_6 \|v\|_3 \|\nabla w\|_2 \leq C \|\nabla u\|_2 \|v\|_2^{1/2} \|\nabla v\|_2^{1/2} \|\nabla w\|_2. \quad (4.6)$$

Also, note that in $\mathcal{S}_n L^2(\mathbb{T}^3)$ we have the norm equivalence

$$\frac{1}{C} \|f\|_2 \leq \|\nabla f\|_2 \leq C_n \|f\|_2,$$

under the mean-zero condition, where the second constant depends on the dimension n of the projection $\mathcal{S}_n$, and the first constant is universal. Hence, we conclude from (4.6) that

$$\left| \int_{\mathbb{T}^d} v \cdot \mathcal{S}_n \mathcal{P}((v \cdot \nabla)w) dx \right| \leq C_n \|u\|_2 \|v\|_2 \|w\|_2, \quad (4.7)$$

for $u, v, w \in \mathcal{S}_n L^2(\mathbb{T}^3)$ with vanishing means.

The next result extends Theorem 3.1 and Lemma 3.1 in [F] from an additive to the multiplicative noise. The statement is proven by modifying the arguments in [F] and [KXZ]. For the sake of completeness and conciseness, we include a sketch of the proof.

LEMMA 4.1. *Let $T > 0$, and suppose $u_0 \in L^2(\Omega; L^2)$ satisfies $\nabla \cdot u_0 = 0$ and $\int_{\mathbb{T}^3} u_0 = 0$. Then for all $n \in \mathbb{N}$, the initial value problem (4.2)–(4.4) has a unique strong solution $u^{(n)} \in L^2(\Omega; C([0, T], L^2)) \cap L^2(\Omega; L^2([0, T], H^1))$. Moreover,*

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} \|u^{(n)}(s)\|_2^2 + \int_0^T \|\nabla u^{(n)}(s)\|_2^2 ds \right] \leq C, \quad (4.8)$$

for some positive constant C independent of n .

PROOF OF LEMMA 4.1. We fix $n \in \mathbb{N}$ and write $\mathcal{S}$ instead of $\mathcal{S}_n$ for simplicity. Let $u^{(0)}$ be the unique strong solution of the standard heat equation subject to the initial data $\mathcal{S}u_0$. Consider the iteration

$$du^{(k)} = \Delta u^{(k)} dt - (\phi_k^M)^2 \mathcal{S} \mathcal{P}((u^{(k)} \cdot \nabla)u^{(k)}) dt + (\phi_{k-1}^M)^2 \mathcal{S} \sigma(u^{(k-1)}) d\mathbb{W}(t), \quad (4.9)$$

$$\nabla \cdot u^{(k)} = 0, \quad (4.10)$$

$$u^{(k)}(0, x) = \mathcal{S}u_0(x), \quad x \in \mathbb{T}^3, \quad (4.11)$$

where

$$\phi_k^M = \phi^M(\|u^{(k)}(t, \cdot)\|_2)$$

with $\phi^M \in C^\infty(\mathbb{R})$, $\phi^M(x) = 1$ if $|x| \leq M/2$, $0 \leq \phi^M(x) \leq 1$ if $M/2 \leq |x| \leq M$, and $\phi^M(x) = 0$ if $|x| \geq M$. Since the noise is additive and the bilinear term is globally Lipschitz in $u^{(k)}$, due to the factor $(\phi_k^M)^2$, we can prove inductively as in [KXZ] that (4.9)–(4.11) has a pathwise unique strong solution in $L^2(\Omega; C([0, T], L^2))$ for all $k \in \mathbb{N}$. Furthermore, by (4.5), we conclude that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} \|u^{(k)}(s)\|_2^2 + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla u_j^{(k)}(s, x)|^2 dx ds \right] \leq C_{T,M} (\mathbb{E}[\|u_0\|_2^2] + 1). \quad (4.12)$$

Denoting $v^{(k)} = u^{(k)} - u^{(k-1)}$ and applying the Itô formula to the equation for the difference, we obtain

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \|v^{(k)}(s)\|_2^2 \right] \leq C_M t \mathbb{E} \left[\sup_{0 \leq s \leq t} \|v^{(k-1)}(s)\|_2^2 \right],$$

where we also used (4.6)–(4.7). This implies that there exists a fixed point u_M of (4.9)–(4.11) in $L^2(\Omega; C([0, t], L^2))$ for a sufficiently small $t \in (0, T]$. The square of the truncation function ϕ_k^M was needed for claiming that such an interval is uniform with respect to k and for previously establishing the existence of $u^{(k)}$. Passing to the limit in (4.12) by using Lemma 3.3, we conclude that (4.12) holds for u_M with T being replaced by t . Utilizing this estimate, we can show that u_M is a solution of the truncated finite-dimensional model

$$du = \Delta u dt - (\phi_u^M)^2 \mathcal{S} \mathcal{P}((u \cdot \nabla)u) dt + (\phi_u^M)^2 \mathcal{S} \sigma(u) d\mathbb{W}(t), \quad (4.13)$$

$$\nabla \cdot u = 0, \quad (4.14)$$

$$u(0, x) = \mathcal{S}u_0(x), \quad x \in \mathbb{T}^3. \quad (4.15)$$

The pathwise uniqueness of the solution follows from a contraction argument similarly to above. The extension of the existence and uniqueness results from a small time interval to $[0, T]$ is obtained by using the pathwise uniqueness, and (4.12) still holds.

To address the non-truncated model (4.2)–(4.4), we introduce the stopping time

$$\eta_n^M = \inf \left\{ t > 0 : \|u^{(n)}(s)\|_2 \geq \frac{M}{2} \right\}.$$

Clearly, (4.2)–(4.4) agrees with (4.13)–(4.15) on $[0, \eta_n^M \wedge T]$ and thus we have a pathwise unique strong solution at least up to $\eta_n^M \wedge T$. Applying the Itô formula to (4.2) on $[0, \eta_n^M \wedge T]$ and using (4.5), we obtain the energy estimate

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq \eta_n^M \wedge T} \|u^{(n)}(s)\|_2^2 + \sum_j \int_0^{\eta_n^M \wedge T} \int_{\mathbb{T}^3} |\nabla u_j^{(n)}(s, x)|^2 dx ds \right] \\ & \leq C \mathbb{E} \left[\|u_0\|_2^2 + \sum_j \int_0^{\eta_n^M \wedge T} \|\mathcal{S}\sigma_j(u^{(n)})(s)\|_{\mathbb{L}^2} ds \right], \end{aligned} \quad (4.16)$$

which by (3.1) and Lemma 2.2 leads to

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq \eta_n^M \wedge T} \|u^{(n)}(s)\|_2^2 \right] \leq C \left(\mathbb{E} \left[\|u_0\|_2^2 + \int_0^{\eta_n^M \wedge T} (\|u^{(n)}(s)\|_2^2 + 1) ds \right] \right) \\ & \leq C_T \mathbb{E}[\|u_0\|_2^2 + 1] + C \int_0^T \mathbb{E} \left[\sup_{0 \leq s \leq \eta_n^M \wedge t} \|u^{(n)}(s)\|_2^2 \right] dt. \end{aligned} \quad (4.17)$$

Then Grönwall's lemma yields

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} \|u^{(n)}(s \wedge \eta_n^M)\|_2^2 \right] = \mathbb{E} \left[\sup_{0 \leq s \leq \eta_n^M \wedge T} \|u^{(n)}(s)\|_2^2 \right] \leq C_T \mathbb{E}[\|u_0\|_2^2 + 1].$$

Note that

$$\mathbb{E} \left[\mathbb{1}_{\eta_n^M \leq T} \|u^{(n)}(\eta_n^M)\|_2^2 \right] \leq \mathbb{E} \left[\sup_{0 \leq s \leq T} \|u^{(n)}(s \wedge \eta_n^M)\|_2^2 \right].$$

Then, $\mathbb{P}(\eta_n^M \leq T) \leq C_T \mathbb{E}[\|u_0\|_2^2 + 1]/M^2$. Also, observe that η_n^M is an increasing function of M and that

$$\mathbb{P}(u_M = u_K, t \in [0, \eta_n^M]) = 1$$

if $M < K$. Then defining $\eta^\infty = \lim_{M \rightarrow \infty} \eta_n^M$, we may uniquely define a process u^∞ such that $u^\infty = u_M$ on $[0, \eta_n^M]$, and u^∞ solves (4.2)–(4.4) on $[0, \eta^\infty]$. Since $\mathbb{P}(\eta^\infty \leq T) = \lim_{M \rightarrow \infty} \mathbb{P}(\eta_n^M \leq T) = 0$ and T is arbitrary, the solution is global. Note that (4.16) also implies

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq \eta_n^M \wedge T} \|u^{(n)}(s)\|_2^2 + \sum_j \int_0^{\eta_n^M \wedge T} \int_{\mathbb{T}^3} |\nabla u_j^{(n)}(s, x)|^2 dx ds \right] \\ & \leq C_T \mathbb{E}[\|u_0\|_2^2 + 1] + C \int_0^T \mathbb{E} \left[\sup_{0 \leq s \leq \eta_n^M \wedge t} \|u^{(n)}(s)\|_2^2 + \sum_j \int_0^{\eta_n^M \wedge t} \int_{\mathbb{T}^3} |\nabla u_j^{(n)}(s, x)|^2 dx ds \right] dt. \end{aligned}$$

Applying Grönwall's lemma and sending $M \rightarrow \infty$ in (4.17), we obtain (4.8), concluding the proof. $\square$

We proceed by deriving an L^p estimate of $u^{(n)}$. For every $M > 0$, we introduce stopping times relative to solutions $\{u^{(n)}\}$ as

$$\tau_n^M = \inf \left\{ t > 0 : \sup_{0 \leq s \leq t} \|u^{(n)}(s)\|_p + \left(\int_0^t \|u^{(n)}(s)\|_{3p}^p ds \right)^{1/p} \geq M \right\}. \quad (4.18)$$

Adapting the proof in [KZ, Lemma 3] (by contradiction and passing to the limit) and assuming $\int_{\mathbb{T}^3} f = 0$, we obtain the Gagliardo-Nirenberg inequality on $\mathbb{T}^3$,

$$\| |f|^{p/2} \|_r \leq C_0 \| |f|^{p/2} \|_2^{1-\alpha} \| \nabla (|f|^{p/2}) \|_2^\alpha,$$

where $\alpha = 3(1/2 - 1/r)$ and $C_0 > 0$ is independent of f . In particular, setting $r = 6$ yields

$$\| f \|_{3p}^p \leq C_0 \| \nabla (|f|^{p/2}) \|_2^2, \quad (4.19)$$

provided f has x -mean zero. We need this inequality to prove the next lemma which asserts the boundedness of $\{u^{(n)}\}$ in $L^p(\Omega; C([0, S \wedge \tau_n^M], L^p))$ for some deterministic value S that is uniform with respect to n .

LEMMA 4.2. *Let $p > 3$ and $K \geq 1$. Suppose that $\nabla \cdot u_0 = 0$, $\int_{\mathbb{T}^3} u_0 = 0$, and $\|u_0\|_p \leq K$ a.s. Then there exist $M > K$ and $S > 0$ such that*

$$\mathbb{E} \left[\sup_{0 \leq s \leq \tau_n^M \wedge S} \|u^{(n)}(s, \cdot)\|_p^p + \int_0^{\tau_n^M \wedge S} \sum_j \int_{\mathbb{T}^3} |\nabla (|u_j^{(n)}(s, x)|^{p/2})|^2 dx ds \right] \leq C \mathbb{E} [\|u_0\|_p^p + 1]. \quad (4.20)$$

Moreover,

$$\lim_{S \rightarrow 0} \sup_n \mathbb{P} \left[\sup_{0 \leq s \leq \tau_n^M \wedge S} \|u^{(n)}(s, \cdot)\|_p^p + \int_0^{\tau_n^M \wedge S} \sum_j \int_{\mathbb{T}^3} |\nabla (|u_j^{(n)}(s, x)|^{p/2})|^2 dx ds \geq M^p \right] = 0, \quad (4.21)$$

for any fixed $K > 0$ and the corresponding M .

PROOF OF LEMMA 4.2. Let $M = 2 \sup_n \|\mathcal{S}_n u_0\|_p + 1$ (cf. (2.3)). The continuity of $u^{(n)}$ implies that $\tau_n^M > 0$. Since both $\mathcal{S}_n$ and $\mathcal{P}$ preserve the x -zero mean, we may apply Lemma 3.2 to (4.2) on

$$du_j^{(n)} - \Delta u_j^{(n)} dt = -(\mathcal{S}_n \mathcal{P}(u^{(n)} \cdot \nabla) u^{(n)})_j dt + \mathcal{S}_n \sigma_j(u^{(n)}) d\mathbb{W}_t, \quad j = 1, 2, 3, \quad (4.22)$$

on $[0, S \wedge \tau_n^M]$ for a fixed $j \in \{1, 2, 3\}$, writing the first term on the right side of (4.22) as

$$- \sum_i \partial_i \mathcal{S}_n (\mathcal{P}(u_i^{(n)} u^{(n)}))_j dt.$$

For q with $1/q \in [1/p, (p+1)/3p]$, we choose exponents r and l such that the three conditions

$$\frac{1}{r} + \frac{1}{l} = \frac{1}{q}, \quad r \leq p, \quad p < l < 3p \quad (4.23)$$

hold. Then,

$$\begin{aligned} \mathbb{E} \left[\int_0^{S \wedge \tau_n^M} \|\mathcal{S}_n \mathcal{P}(u_i^{(n)} u^{(n)})\|_q^p dt \right] &\leq C M^p \mathbb{E} \left[\int_0^{S \wedge \tau_n^M} \|u^{(n)}\|_l^p dt \right] \\ &\leq M^p \mathbb{E} \left[\int_0^{S \wedge \tau_n^M} (C_\varepsilon \|u^{(n)}\|_p^p + \varepsilon \|u^{(n)}\|_{3p}^p) dt \right]. \end{aligned} \quad (4.24)$$

For the stochastic term in (4.22) (cf. (3.4)), we also have

$$\mathbb{E} \left[\int_0^{S \wedge \tau_n^M} \int_{\mathbb{T}^d} \|\mathcal{S}_n \sigma(u^{(n)})\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx dt \right] \leq C \mathbb{E} \int_0^{S \wedge \tau_n^M} (\|u^{(n)}\|_p^p + 1) dt \quad (4.25)$$

due to (2.9) and the assumption (3.1). From (3.4), (4.24), and (4.25), we get

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq S \wedge \tau_n^M} \|u^{(n)}(t)\|_p^p + \sum_j \int_0^{S \wedge \tau_n^M} \int_{\mathbb{T}^d} |\nabla(|u_j^{(n)}(t, x)|^{p/2})|^2 dx dt \right] \\ & \leq C \mathbb{E} \left[\|u_0\|_p^p + \int_0^{S \wedge \tau_n^M} (M^p C_\varepsilon \|u^{(n)}(t)\|_p^p + M^p \varepsilon \|u^{(n)}(t)\|_{3p}^p + 1) dt \right]. \end{aligned} \quad (4.26)$$

Note that the right hand side of (4.26) is finite by the definition of the stopping time, which implies finiteness of the left-hand side. Choosing ε and then S sufficiently small so that $CM^p \varepsilon \ll 1$ and $CSM^p C_\varepsilon \ll 1$, we arrive at (4.20). The choice of S depends on M and the constants of embedding inequalities, but is independent of n .

To obtain (4.21), we fix $j \in \{1, 2, 3\}$ and utilize the trajectory Itô expansion on $[0, \tau_n^M \wedge S]$,

$$\begin{aligned} \|u_j^{(n)}(t)\|_p^p &= \|(\mathcal{S}_n u_0)_j\|_p^p - \frac{4(p-1)}{p} \int_0^t \int_{\mathbb{T}^d} |\nabla(|u_j^{(n)}|^{p/2})|^2 dx dr \\ &\quad + p \int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} u_j^{(n)} \mathcal{S}_n (\mathcal{P}((u^{(n)} \cdot \nabla) u^{(n)}))_j dx dr \\ &\quad + p \int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} u_j^{(n)} \mathcal{S}_n \sigma_j(u^{(n)}) dx d\mathbb{W}_r \\ &\quad + \frac{p(p-1)}{2} \int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} \|\mathcal{S}_n \sigma_j(u^{(n)})\|_{l^2}^2 dx dr. \end{aligned}$$

As in the proof of [KXZ, Theorem 4.1], we apply the Poincaré-type inequality, obtaining

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} u_j^{(n)} \mathcal{S}_n (\mathcal{P}((u^{(n)} \cdot \nabla) u^{(n)}))_j dx dr \\ & \leq \varepsilon \int_0^t \int_{\mathbb{T}^d} |\nabla(|u_j^{(n)}|^{p/2})|^2 dx dr + C_{M,p,\varepsilon} \int_0^t \|u^{(n)}\|_p^p dr \end{aligned}$$

for q in (4.23) and for an arbitrarily small ε . Under the assumption (3.1),

$$\int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} \|\mathcal{S}_n \sigma_j(u^{(n)})\|_{l^2}^2 dx dr \leq C \int_0^t (\|u^{(n)}\|_p^p + 1) dr.$$

Choosing a sufficiently small ε and summing in j , we arrive at

$$\begin{aligned} & \|u^{(n)}(t)\|_p^p + \sum_j \int_0^t \int_{\mathbb{T}^d} |\nabla(|u_j^{(n)}|^{p/2})|^2 dx dr - \|\mathcal{S}_n u_0\|_p^p \\ & \leq C_{p,M} \int_0^t (\|u^{(n)}\|_p^p + 1) dr + C_p \sum_j \int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} u_j^{(n)} \mathcal{S}_n \sigma_j(u^{(n)}) dx d\mathbb{W}_r \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Recall that M was set to satisfy $M \geq 2\|\mathcal{S}_n u_0\|_p$ and that $p > 3$. Hence, the inequality above implies

$$\begin{aligned} & \mathbb{P} \left[\sup_{0 \leq t \leq \tau_n^M \wedge S} \|u^{(n)}(t, \cdot)\|_p^p + \sum_j \int_0^{\tau_n^M \wedge S} \int_{\mathbb{T}^3} |\nabla(|u_j^{(n)}(t, x)|^{p/2})|^2 dx dt \geq M^p \right] \\ & \leq \mathbb{P} \left[\int_0^{\tau_n^M \wedge S} (\|u^{(n)}\|_p^p + 1) dr \geq \frac{M^p}{3C_{p,M}} \right] \\ & \quad + \mathbb{P} \left[\sum_j \sup_{0 \leq t \leq \tau_n^M \wedge S} \left| \int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} u_j^{(n)} \mathcal{S}_n \sigma_j(u^{(n)}) dx d\mathbb{W}_r \right| \geq \frac{M^p}{3C_p} \right]. \end{aligned} \quad (4.27)$$

Clearly,

$$\mathbb{E} \left[\int_0^{\tau_n^M \wedge S} (\|u^{(n)}\|_p^p + 1) dr \right] \leq (M^p + 1)S.$$

Also, by the BDG inequality and Minkowski's integral inequality,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq \tau_n^M \wedge S} \left| \int_0^t \int_{\mathbb{T}^d} |u_j^{(n)}|^{p-2} u_j^{(n)} \mathcal{S}_n \sigma_j(u^{(n)}) dx d\mathbb{W}_r \right| \right] \\ & \leq C \mathbb{E} \left[\left(\int_0^{\tau_n^M \wedge S} \left(\int_{\mathbb{T}^d} |u_j^{(n)}|^{p-1} \|\mathcal{S}_n \sigma_j(u^{(n)})\|_{l^2} dx \right)^2 dr \right)^{1/2} \right] \\ & \leq C \mathbb{E} \left[\sup_{0 \leq t \leq \tau_n^M \wedge S} \|u^{(n)}(t, \cdot)\|_p^{p-1} \left(\int_0^{\tau_n^M \wedge S} \int_{\mathbb{T}^d} \|\mathcal{S}_n \sigma_j(u^{(n)})\|_{l^2}^p dx dr \right)^{1/p} \right] \\ & \leq CM^{p-1} (M^p + 1)^{1/p} S^{1/p}. \end{aligned}$$

Using (4.27) and Chebyshev's inequality, we get

$$\mathbb{P} \left[\sup_{0 \leq t \leq \tau_n^M \wedge S} \|u^{(n)}(t, \cdot)\|_p^p + \sum_j \int_0^{\tau_n^M \wedge S} \int_{\mathbb{T}^3} |\nabla(|u_j^{(n)}(t, x)|^{p/2})|^2 dx dt \geq M^p \right] \leq C_M (S + S^{1/p}).$$

Then, for the fixed K and the associated M ,

$$\lim_{S \rightarrow 0} \sup_n \mathbb{P} \left[\sup_{0 \leq t \leq \tau_n^M \wedge S} \|u^{(n)}(t, \cdot)\|_p^p + \sum_j \int_0^{\tau_n^M \wedge S} \int_{\mathbb{T}^3} |\nabla(|u_j^{(n)}(t, x)|^{p/2})|^2 dx dt \geq M^p \right] = 0,$$

concluding the proof. $\square$

We next show that $\{u^{(n)}\}$ is Cauchy in $L_\omega^p L_t^\infty L_x^p \cap L_\omega^p L_t^p L_x^{3p}$ within a prescribed stopping time.

LEMMA 4.3. *Let $p > 3$ and $K \geq 1$. Assume that $\nabla \cdot u_0 = 0$, $\int_{\mathbb{T}^3} u_0 = 0$, and $\|u_0\|_p \leq K$ a.s. Then there exist $M > K$ and a positive constant S depending on M such that*

$$\lim_{m \rightarrow \infty} \sup_{n > m} \mathbb{E} \left[\sup_{0 \leq t \leq \tau_n^M \wedge \tau_m^M \wedge S} \|u^{(n)}(t) - u^{(m)}(t)\|_p^p + \int_0^{\tau_n^M \wedge \tau_m^M \wedge S} \sum_j \int_{\mathbb{T}^3} |\nabla(|u_j^{(n)} - u_j^{(m)}|^{p/2})|^2 dx dt \right] = 0. \quad (4.28)$$

PROOF OF LEMMA 4.3. We fix $M > 0$ as in the proof of Lemma 4.2. Then for every $S > 0$, $\tau_{nm} = \tau_n^M \wedge \tau_m^M \wedge S$ is a positive stopping time almost surely. Denote $w^{(n,m)} = u^{(n)} - u^{(m)}$ and $\mathcal{S}_{nm} = \mathcal{S}_n - \mathcal{S}_m$. Clearly on $[0, \tau_{nm}]$, the difference $w^{(n,m)}$ satisfies

$$\begin{aligned} dw^{(n,m)} - \Delta w^{(n,m)} dt &= \mathcal{S}_m \mathcal{P}((u^{(m)} \cdot \nabla) u^{(m)}) - \mathcal{S}_n \mathcal{P}((u^{(n)} \cdot \nabla) u^{(n)}) dt \\ &\quad + (\mathcal{S}_n \sigma(u^{(n)}) - \mathcal{S}_m \sigma(u^{(m)})) d\mathbb{W}_t, \end{aligned}$$

$$\nabla \cdot w^{(n,m)} = 0,$$

$$w^{(n,m)}(0) = \mathcal{S}_{nm} u_0(x) \quad \text{a.s.}$$

We rewrite the first equation as

$$dw_j^{(n,m)} - \Delta w_j^{(n,m)} dt = \sum_i \partial_i f_{ij} dt + g_j d\mathbb{W}_t, \quad j = 1, 2, 3,$$

where

$$\begin{aligned} f_{ij} &= \mathcal{S}_m(\mathcal{P}u_i^{(m)}u^{(m)})_j - \mathcal{S}_n(\mathcal{P}u_i^{(n)}u^{(n)})_j \\ &= -\mathcal{S}_m(\mathcal{P}u_i^{(m)}w^{(n,m)})_j - \mathcal{S}_m(\mathcal{P}w_i^{(n,m)}u^{(n)})_j - \mathcal{S}_{nm}(\mathcal{P}u_i^{(n)}u^{(n)})_j \\ &= f_{ij}^{(1)} + f_{ij}^{(2)} + f_{ij}^{(3)} \end{aligned}$$

and

$$g_j = \mathcal{S}_n(\sigma_j(u^{(n)}) - \sigma_j(u^{(m)})) + \mathcal{S}_{nm}\sigma_j(u^{(m)}) = g_j^{(1)} + g_j^{(2)}.$$

Now, choosing the exponents as in (4.23)–(4.24), we obtain

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau_{nm}} (\|f_{ij}^{(1)}\|_q^p + \|f_{ij}^{(2)}\|_q^p) dt \right] &\leq \mathbb{E} \left[\int_0^{\tau_{nm}} \|w^{(n,m)}\|_l^p (\|u_i^{(m)}\|_p^p + \|u_i^{(n)}\|_p^p) dt \right] \\ &\leq C_M \mathbb{E} \left[\int_0^{\tau_{nm}} (\varepsilon \|w^{(n,m)}\|_{3p}^p + C_\varepsilon \|w^{(n,m)}\|_p^p) dt \right]. \end{aligned} \quad (4.29)$$

Recall that $1/q \in [1/p, (p+1)/3p]$. For $0 < \delta \ll 1$ and $0 < \theta \ll 1$ such that $q + \delta < 4p/3$ and $1/q = \theta/(1+\delta) + (1-\theta)/(q+\delta)$, we have

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau_{nm}} \|f_{ij}^{(3)}\|_q^p dt \right] &\leq C \mathbb{E} \left[\int_0^{\tau_{nm}} \|\mathcal{S}_{nm}u_i^{(n)}u^{(n)}\|_{1+\delta}^{\theta p} \|\mathcal{S}_{nm}u_i^{(n)}u^{(n)}\|_{q+\delta}^{(1-\theta)p} dt \right] \\ &\leq \frac{C}{m^{\theta\alpha p} \wedge n^{\theta\alpha p}} \mathbb{E} \left[\int_0^{\tau_{nm}} \|\nabla(u_i^{(n)}u^{(n)})\|_{1+\delta}^{\theta p} \|u_i^{(n)}u^{(n)}\|_{q+\delta}^{(1-\theta)p} dt \right] \\ &\leq \frac{C}{m^{\theta\alpha p} \wedge n^{\theta\alpha p}} \mathbb{E} \left[\int_0^{\tau_{nm}} \|\nabla u^{(n)}\|_2^{\theta p} \|u^{(n)}\|_l^{\theta p} \|u^{(n)}\|_{2(q+\delta)}^{2(1-\theta)p} dt \right], \end{aligned}$$

where $1/(1+\delta) = 1/2 + 1/\tilde{l}$ and α is determined by Lemma 2.1. Note that δ can be sufficiently small so that $\theta p < 2$ and $2 < \tilde{l} < p$. By (4.8) and (4.20), we conclude

$$\mathbb{E} \left[\int_0^{\tau_{nm}} \|f_{ij}^{(3)}\|_q^p dt \right] \leq \frac{C_M}{m^{\theta\alpha p} \wedge n^{\theta\alpha p}} \mathbb{E} \left[\int_0^{\tau_{nm}} (\|\nabla u^{(n)}\|_2^2 + \|u^{(n)}\|_{3p}^p) dt \right] \leq \frac{C_M}{m^{\theta\alpha p} \wedge n^{\theta\alpha p}}.$$

Also, by Lemma 2.2 and the assumption (3.2), we have

$$\mathbb{E} \left[\int_0^{\tau_{nm}} \int_{\mathbb{T}^d} \|g_1\|_{l^2}^p dx dt \right] \leq C S \mathbb{E} \left[\sup_{s \in [0, \tau_{nm}]} \|w^{(n,m)}\|_p^p \right]. \quad (4.30)$$

Applying Lemmas 2.1 and 2.2, together with Fubini's theorem, we obtain

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau_{nm}} \int_{\mathbb{T}^d} \|g_2\|_{l^2}^p dx dt \right] &\leq C \mathbb{E} \left[\int_0^{\tau_{nm}} \left(\int_{\mathbb{T}^d} \|g_2\|_{l^2}^2 dx \right)^{1/2} \left(\int_{\mathbb{T}^d} \|g_2\|_{l^2}^{2(p-1)} dx \right)^{1/2} dt \right] \\ &\leq \frac{C}{m \wedge n} \mathbb{E} \left[\int_0^{\tau_{nm}} \|\nabla(\sigma(u^{(m)}))\|_{\mathbb{L}^2} \|\sigma(u^{(m)})\|_{\mathbb{L}^{2(p-1)}}^{p-1} dt \right] \\ &\leq \frac{C}{m \wedge n} \mathbb{E} \left[\int_0^{\tau_{nm}} \|u^{(m)}\|_{L^2} \|\sigma(u^{(m)})\|_{\mathbb{L}^{2(p-1)}}^{p-1} dt \right] \\ &\leq \frac{C_{M,S}}{m \wedge n}, \end{aligned} \quad (4.31)$$

where the third inequality follows from the assumptions (3.1) and (3.3). If $\varepsilon C_M \ll 1$ in (4.29) and if $S > 0$ is sufficiently small, then (4.29)–(4.31) lead to

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq \tau_{nm}} \|w^{(n,m)}(t, \cdot)\|_p^p + \int_0^{\tau_{nm}} \sum_{j,k} \int_{\mathbb{T}^3} |\partial_k(|w_j^{(n,m)}|^{p/2})|^2 dx dt \right] \\ & \leq C \mathbb{E} [\|\mathcal{S}_{nm} u_0(x)\|_p^p] + \frac{C_M}{m^{\theta\alpha p} \wedge n^{\theta\alpha p}} + \frac{C_{M,S}}{m \wedge n}, \end{aligned}$$

which completes the proof. $\square$

The following result asserts positivity of the stopping time. We fix $K > 1$ and choose a constant $M > K$ that fulfills the requirements in Lemmas 4.2 and 4.3.

LEMMA 4.4. *Let $p > 3$ and $K \geq 1$. Assume that $\nabla \cdot u_0 = 0$, $\int_{\mathbb{T}^3} u_0 = 0$, and $\|u_0\|_p \leq K$ a.s. Then there exist $M > K$, a stopping time τ_M with $\mathbb{P}(\tau_M > 0) = 1$, and a subsequence $\{u^{(n_k)}\}$ so that*

$$\lim_{k \rightarrow \infty} \left(\sup_{0 \leq t \leq \tau_M} \|u^{(n_k)}(t) - u(t)\|_p^p + \int_0^{\tau_M} \|u^{(n_k)}(t) - u(t)\|_{3p}^p dt \right) = 0 \quad \mathbb{P}\text{-a.s.},$$

for some adapted process $u \in L^p(\Omega, C([0, \tau_M], L^p)) \cap L^p(\Omega, L^p([0, \tau_M], L^{3p}))$.

PROOF OF LEMMA 4.4. Using (4.19) and (4.21), there exists M be sufficiently large relative to K so that

$$\lim_{\epsilon \rightarrow 0} \sup_n \mathbb{P} \left[\sup_{0 \leq s \leq \tau_n^M \wedge \epsilon} \|u^{(n)}(s, \cdot)\|_p + \left(\int_0^{\tau_n^M \wedge \epsilon} \|u^{(n)}(s, \cdot)\|_{3p}^p ds \right)^{1/p} \geq \frac{M}{2} \right] = 0. \quad (4.32)$$

Fix a corresponding value for the constant S in (4.20) and (4.28). From (4.28) we can infer the existence of a subsequence $\{n_k\}$ for which

$$\mathbb{E} \left[\sup_{0 \leq r \leq \tau_{n_k}^M \wedge \tau_{n_{k+1}}^M \wedge S} \|w^{(n_k, n_{k+1})}(r, \cdot)\|_p^p + \int_0^{\tau_{n_k}^M \wedge \tau_{n_{k+1}}^M \wedge S} \|w^{(n_k, n_{k+1})}(r, \cdot)\|_{3p}^p dr \right] \leq 4^{-kp}. \quad (4.33)$$

Now we need to prove that there exists a uniform time interval where this estimate can be applied. Inspired by [GZ], we introduce stopping times

$$\eta_k = \inf \left\{ t > 0 : \sup_{0 \leq r \leq t} \|u^{(n_k)}(r, \cdot)\|_p + \left(\int_0^t \|u^{(n_k)}(r, \cdot)\|_{3p}^p dr \right)^{1/p} > \frac{M}{2} + 2^{-k} \right\}$$

and probability events

$$\Omega_N = \bigcap_{k=N}^{\infty} \left\{ \omega : \sup_{0 \leq r \leq \eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_p + \left(\int_0^{\eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_{3p}^p dr \right)^{\frac{1}{p}} < 2^{-k-2} \right\}.$$

Note that $\eta_k \leq \tau_{n_k}^M$ (cf. (4.18)). Then, by Chebyshev's inequality,

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 \leq r \leq \eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_p + \left(\int_0^{\eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_{3p}^p dr \right)^{\frac{1}{p}} \geq 2^{-k-2} \right) \\ & \leq \mathbb{P} \left(\sup_{0 \leq r \leq \eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_p^p \geq 2^{-kp-3p} \right) \\ & \quad + \mathbb{P} \left(\int_0^{\eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_{3p}^p dr \geq 2^{-kp-3p} \right) \\ & \leq 2^{kp+3p+1} 4^{-kp} = 2^{-kp+3p+1}. \end{aligned}$$

Next, by the Borel-Cantelli lemma,

$$\mathbb{P} \left(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \left\{ \sup_{0 \leq r \leq \eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_p + \left(\int_0^{\eta_k \wedge \eta_{k+1} \wedge S} \|w^{(n_k, n_{k+1})}(r, \omega)\|_{3p}^p dr \right)^{\frac{1}{p}} \geq 2^{-k-2} \right\} \right) = 0.$$

This shows that $\mathbb{P}(\bigcup_{N=1}^{\infty} \Omega_N) = 1$. Note that $\eta_k \geq \eta_{k+1} \wedge S$ in Ω_N if $N \leq k$, because

$$\Omega_N \cap \{\eta_k < \eta_{k+1} \wedge S\} \subseteq \left\{ \omega : \sup_{0 \leq r \leq \eta_k} \|w^{(n_k, n_{k+1})}(r, \omega)\|_p + \left(\int_0^{\eta_k} \|w^{(n_k, n_{k+1})}(r, \omega)\|_{3p}^p dr \right)^{\frac{1}{p}} < 2^{-k-2} \right\},$$

and almost surely in $\Omega_N \cap \{\eta_k < \eta_{k+1} \wedge S\}$,

$$\begin{aligned} & \sup_{0 \leq r \leq \eta_k} \|u^{(n_{k+1})}(r, \cdot)\|_p + \left(\int_0^{\eta_k} \|u^{(n_{k+1})}(r, \cdot)\|_{3p}^p dr \right)^{1/p} \\ & \geq \sup_{0 \leq r \leq \eta_k} \|u^{(n_k)}(r, \cdot)\|_p - \sup_{0 \leq r \leq \eta_k} \|w^{(n_k, n_{k+1})}(r, \cdot)\|_p \\ & \quad + \left(\int_0^{\eta_k} \|u^{(n_k)}(r, \cdot)\|_{3p}^p dr \right)^{1/p} - \left(\int_0^{\eta_k} \|w^{(n_k, n_{k+1})}(r, \cdot)\|_{3p}^p dr \right)^{\frac{1}{p}} \\ & \geq \frac{M}{2} + 2^{-k} - 2^{-k-2} > \frac{M}{2} + 2^{-k-1}, \end{aligned}$$

which contradicts $\eta_k < \eta_{k+1} \wedge S$. Then, $\Omega_N \cap \{\eta_k < \eta_{k+1} \wedge S\} = \emptyset$, and $\{\eta_k(\omega) \wedge S\}$ is a non-increasing sequence in Ω_N . Also note that Ω_N monotonically expands to the whole probability space, and then $\tau_M = \lim_{k \rightarrow \infty} \eta_k \wedge S$ is well-defined almost everywhere in Ω . Furthermore,

$$\begin{aligned} \mathbb{P}(\tau_M < \epsilon) &= \mathbb{P} \left(\bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \{\eta_k \wedge S < \epsilon\} \right) = \sup_l \mathbb{P} \left(\bigcap_{k=l}^{\infty} \{\eta_k \wedge S < \epsilon\} \cap \Omega_l \right) \\ &\leq \sup_l \mathbb{P}(\eta_l \wedge S < \epsilon) \leq \sup_l \mathbb{P} \left(\sup_{0 \leq r \leq \epsilon} \|u^{(n_l)}(r, \cdot)\|_p + \left(\int_0^{\epsilon} \|u^{(n_l)}(r, \cdot)\|_{3p}^p dr \right)^{1/p} > \frac{M}{2} + 2^{-l} \right) \end{aligned}$$

if $\epsilon < S$, which by (4.32) yields

$$\mathbb{P}(\tau_M = 0) = \mathbb{P} \left(\bigcap_{m=1}^{\infty} \{\tau_M < 1/m\} \right) \leq \lim_{m \rightarrow \infty} \mathbb{P}(\tau_M < 1/m) = 0.$$

It remains to show that $\{u^{(n_k)}\}$ has an ω -pointwise limit and the limit belongs to $L^p(\Omega, C([0, \tau_M], L^p)) \cap L^p(\Omega, L^p([0, \tau_M], L^{3p}))$. Indeed, by (4.33) and [KXZ, Lemma 5.2],

$$u^{(n_k)} \mathbb{1}_{\Omega_N} \xrightarrow{k \rightarrow \infty} u_N \text{ in } C([0, \tau_M], L^p) \cap L^p([0, \tau_M], L^{3p}), \quad \mathbb{P}\text{-a.e.}$$

for some adapted process u_N and for all $N \in \mathbb{N}$. Then, $\{u^{(n_k)}\}$ converges in the same manner to $u = \lim_{N \rightarrow \infty} u_N$ on Ω . Using (4.20), we obtain that $\{u^{(n_k)} \mathbb{1}_{\Omega_N}\}$ are uniformly bounded in $L^p(\Omega, L^\infty([0, \tau_M], L^p)) \cap L^p(\Omega, L^p([0, \tau_M], L^{3p}))$. Thus, the same conclusion holds for u_N and u , which completes the proof. $\square$

Next, we prove the pathwise uniqueness of local strong solutions.

LEMMA 4.5. *Let $p > 3$. Assume that $\nabla \cdot u_0 = 0$, $\int_{\mathbb{T}^3} u_0 = 0$, and $u_0 \in L^p(\Omega, L^p)$. Then for any pair of local strong solutions $(v^{(1)}, \tau)$ and $(v^{(2)}, \tau)$ of (1.1)–(1.3) that satisfy*

$$\mathbb{E} \left[\sup_{0 \leq s \leq \tau} \|v(s)\|_p^p + \int_0^\tau \sum_j \int_{\mathbb{T}^3} |\nabla(|v_j(s, x)|^{p/2})|^2 dx ds \right] \leq C \mathbb{E}[\|u_0\|_p^p + 1], \quad (4.34)$$

we have $\mathbb{P}(v^{(1)}(t) = v^{(2)}(t), \forall t \in [0, \tau]) = 1$.

PROOF OF LEMMA 4.5. Let $M > 0$ and introduce stopping times

$$\eta_i^M = \inf \left\{ t > 0 : \sup_{0 \leq s \leq t} \|v(s)\|_p^p + \int_0^t \sum_j \int_{\mathbb{T}^3} |\nabla(|v_j(s, x)|^{p/2})|^2 dx ds \geq M \right\}.$$

If $\|u_0(\omega)\|_p < M$, then $\eta_i^M(\omega) > 0$, otherwise $\eta_i^M(\omega) = 0$ for $i = 1, 2$. Define $\eta^M = \eta_1^M \wedge \eta_2^M \wedge \tau$. Due to (4.34), $\lim_{M \rightarrow \infty} \mathbb{P}(\eta^M = \tau) = 1$. Let $S > 0$ and denote $w = v^{(1)} - v^{(2)}$. On $[0, \eta^M \wedge S]$, w satisfies

$$\begin{aligned} dw - \Delta w dt &= -\mathcal{P}((w \cdot \nabla)v^{(2)}) - \mathcal{P}((v^{(1)} \cdot \nabla)w) dt + (\sigma(v^{(1)}) - \sigma(v^{(2)})) d\mathbb{W}_t, \\ \nabla \cdot w &= 0, \\ w(0) &= 0 \quad \text{a.s.} \end{aligned}$$

The Itô formula yields, as in (4.29) and (4.30),

$$\mathbb{E} \left[\sup_{0 \leq s \leq \eta^M \wedge S} \|w(s)\|_p^p + \int_0^{\eta^M \wedge S} \sum_j \int_{\mathbb{T}^3} |\nabla(|w_j(s, x)|^{p/2})|^2 dx ds \right] \leq \mathbb{E} \left[\int_0^{\eta^M \wedge S} (\varepsilon \|w\|_{3p}^p + C_{M, \varepsilon} \|w\|_p^p) dt \right],$$

which can be further simplified to

$$\mathbb{E} \left[\sup_{0 \leq s \leq \eta^M \wedge S} \|w(s)\|_p^p \right] \leq C_M \int_0^S \mathbb{E} \left[\sup_{0 \leq s \leq \eta^M \wedge t} \|w\|_p^p \right] dt.$$

By Grönwall's lemma, we conclude that $w \equiv 0$ a.s. on $[0, \eta^M \wedge S]$. Since S is an arbitrary and positive constant independent of M , by sending both M and S to the positive infinity, we obtain the pathwise uniqueness on $[0, \tau]$. $\square$

PROOF OF THEOREM 3.1. Using the notation in the proof of Lemma 4.4, we first impose $\|u_0\|_p \leq K$ a.s. and show that u is a strong solution to (1.1)–(1.3) on $[0, \tau_M]$. Denote $\zeta_k = \inf_{l \geq k} \eta_l \wedge S$. By right-continuity of the filtration $\mathcal{F}_t$, $\{\zeta_k\}_{k \in \mathbb{N}}$ are also stopping times. Note that $\zeta_k \leq \zeta_l$ if $k < l$ and $\zeta_1 \leq \eta_1 \wedge S$.

Since each $\{(u^{(n_i)}, \eta_i)\}$ is a local solution to an approximating equation, then componentwise,

$$\begin{aligned} (\mathbb{1}_{[0, \zeta_m]}(s) u^{(n_i)}(s), \phi) &= \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (u^{(n_i)}, \Delta \phi) dr + \sum_j \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (\mathcal{S}_{n_i} \mathcal{P}(u_j^{(n_i)} u^{(n_i)}), \partial_j \phi) dr \\ &\quad + \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (\mathcal{S}_{n_i} \sigma(u^{(n_i)}), \phi) d\mathbb{W}_r + (\mathcal{S}_{n_i} u_0, \phi), \quad (s, \omega)\text{-a.e.}, \end{aligned} \quad (4.35)$$

for all $\phi \in C^\infty(\mathbb{T}^3)$, $s \in [0, S]$, and $m \leq l$. Also, (4.33) holds on $[0, \zeta_m]$ for $\{u_l\}_{l \geq m}$. Utilizing Lemma 4.4 and the boundedness of $\{u^{(n_k)}\}$ in $L_\omega^p L_t^\infty L_x^p$ on $[0, \zeta_m]$, we may pass to the limit in (4.35) and obtain

$$\begin{aligned} & \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (u^{(n_l)}, \Delta \phi) dr + \sum_j \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (\mathcal{S}_{n_l} \mathcal{P}(u_j^{(n_l)} u^{(n_l)}), \partial_j \phi) dr \\ & \rightarrow \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) ((u, \Delta \phi) + (\mathcal{P}(u_j u), \partial_j \phi)) dr, \end{aligned} \quad (4.36)$$

for a.e. (s, ω) as $l \rightarrow \infty$. Also, by the BDG inequality,

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, S]} \left| \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (\mathcal{S}_{n_l} \sigma(u^{(n_l)}) - \sigma(u), \phi) d\mathbb{W}_r \right| \right] \\ & \leq C \mathbb{E} \left[\left(\int_0^{\zeta_m} \|(\mathcal{S}_{n_l}(\sigma(u^{(n_l)}) - \sigma(u)), \phi)\|_{l^2}^2 dr \right)^{1/2} \right] + C \mathbb{E} \left[\left(\int_0^{\zeta_m} \|((\mathcal{S}_{n_l} - I)\sigma(u), \phi)\|_{l^2}^2 dr \right)^{1/2} \right], \end{aligned}$$

where

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^{\zeta_m} \|(\mathcal{S}_{n_l}(\sigma(u^{(n_l)}) - \sigma(u)), \phi)\|_{l^2}^2 dr \right)^{1/2} \right] \\ & \leq C \mathbb{E} \left[\left(\int_0^{\zeta_m} \left(\int_{\mathbb{T}^3} \|\mathcal{S}_{n_l}(\sigma(u^{(n_l)}) - \sigma(u))\|_{l^2}^2 dx \right) \|\phi\|_2^2 dr \right)^{1/2} \right] \\ & \leq C \|\phi\|_2 \mathbb{E} \left[\left(\int_0^{\zeta_m} \left(\int_{\mathbb{T}^3} \|\mathcal{S}_{n_l}(\sigma(u^{(n_l)}) - \sigma(u))\|_{l^2}^p dx \right)^{2/p} dr \right)^{1/2} \right] \\ & \leq C \|\phi\|_2 \mathbb{E} \left[\int_0^{\zeta_m} \|\mathcal{S}_{n_l}(\sigma(u^{(n_l)}) - \sigma(u))\|_{\mathbb{L}^p}^p dr \right] \end{aligned}$$

and

$$\mathbb{E} \left[\left(\int_0^{\zeta_m} \|((\mathcal{S}_{n_l} - I)\sigma(u), \phi)\|_{l^2}^2 dr \right)^{1/2} \right] \leq C \|\phi\|_2 \mathbb{E} \left[\int_0^{\zeta_m} \|(\mathcal{S}_{n_l} - I)\sigma(u)\|_{\mathbb{L}^p}^p dr \right].$$

Then by Lemma 2.2 and the assumptions on σ ,

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, S]} \left| \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (\mathcal{S}_{n_l} \sigma(u^{(n_l)}) - \sigma(u), \phi) d\mathbb{W}_r \right| \right] \\ & \leq CS \|\phi\|_2 \mathbb{E} \left[\sup_{s \in [0, \zeta_m]} \|u^{(n_l)} - u\|_p^p \right] + C \|\phi\|_2 \mathbb{E} \left[\int_0^{\zeta_m} \|(\mathcal{S}_{n_l} - I)\sigma(u)\|_{\mathbb{L}^p}^p dr \right]. \end{aligned}$$

The right-hand side approaches to zero as $l \rightarrow \infty$. Hence, we may infer the existence of a further subsequence, which for simplicity we still denote by $\{n_l\}$, such that

$$\int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (\mathcal{S}_{n_l} \sigma(u^{(n_l)}), \phi) d\mathbb{W}_r \xrightarrow{l \rightarrow \infty} \int_0^s \mathbb{1}_{[0, \zeta_m]}(r) (\sigma(u), \phi) d\mathbb{W}_r, \quad (s, \omega)\text{-a.e.} \quad (4.37)$$

Combining (4.36) and (4.37), we obtain

$$\begin{aligned} \mathbb{1}_{[0, \zeta_m]}(s)(u(s), \phi) &= (u_0, \phi) + \mathbb{1}_{[0, \zeta_m]}(s) \int_0^s (u, \Delta \phi) dr + \mathbb{1}_{[0, \zeta_m]}(s) \int_0^s (\sigma(u), \phi) d\mathbb{W}_r \\ &\quad + \sum_j \mathbb{1}_{[0, \zeta_m]}(s) \int_0^s (\mathcal{P}(u_j u), \partial_j \phi) dr, \quad (s, \omega)\text{-a.e.}, \end{aligned} \quad (4.38)$$

for all $m \in \mathbb{N}$. Sending $m \rightarrow \infty$ in (4.38) and noting that $\lim_{m \rightarrow \infty} \mathbb{1}_{[0, \zeta_m]}(t) = \mathbb{1}_{[0, \tau_M]}(t)$ $\mathbb{P}$ -a.s., we conclude that u is indeed a strong solution to (1.1)–(1.3) on $[0, \tau_M]$. Moreover, due to (4.20), we have

$$\mathbb{E} \left[\mathbb{1}_{\Omega_N} \sup_{0 \leq s \leq \tau_M} \|u^{(n_l)}(s, \cdot)\|_p^p + \mathbb{1}_{\Omega_N} \int_0^{\tau_M} \sum_j \int_{\mathbb{T}^3} |\nabla(|u_j^{(n_l)}(s, x)|^{p/2})|^2 dx ds \right] \leq C\mathbb{E}[\|u_0\|_p^p + 1]$$

if $N \leq l$. Then we use Lemmas 3.3 and 4.3, send $l \rightarrow \infty$ in above inequality, and send $N \rightarrow \infty$, arriving at

$$\mathbb{E} \left[\sup_{0 \leq s \leq \tau_M} \|u(s, \cdot)\|_p^p + \int_0^{\tau_M} \sum_j \int_{\mathbb{T}^3} |\nabla(|u_j(s, x)|^{p/2})|^2 dx ds \right] \leq C\mathbb{E}[\|u_0\|_p^p + 1]. \quad (4.39)$$

To remove the condition $\|u_0\|_p \leq K$ a.s., we denote the local strong solution corresponding to the initial data $u_0 \mathbb{1}_{k \leq \|u_0\|_p < k+1}$ by $(u(k), \tau_k)$, i.e.,

$$\begin{aligned} \mathbb{1}_{[0, \tau_k]}(s)(u(k)(s), \phi) &= (u_0 \mathbb{1}_{k \leq \|u_0\|_p < k+1}, \phi) + \mathbb{1}_{[0, \tau_k]}(s) \int_0^s (u(k), \Delta \phi) dr + \mathbb{1}_{[0, \tau_k]}(s) \int_0^s (\sigma(u(k)), \phi) d\mathbb{W}_r \\ &\quad + \sum_j \mathbb{1}_{[0, \tau_k]}(s) \int_0^s (\mathcal{P}((u(k))_j u(k)), \partial_j \phi) dr, \quad (s, \omega)\text{-a.e.}, \end{aligned} \quad (4.40)$$

for all $k \in \mathbb{N}$. Define

$$u = \sum_{k=0}^{\infty} u(k) \mathbb{1}_{k \leq \|u_0\|_p < k+1}, \quad \tau = \sum_{k=0}^{\infty} \tau_k \mathbb{1}_{k \leq \|u_0\|_p < k+1}.$$

Since $\mathbb{P}(\tau_k > 0) = 1$ for all τ_k , we have

$$\mathbb{P}(\tau > 0) = \sum_{k=0}^{\infty} \mathbb{P}(\tau_k > 0 | k \leq \|u_0\|_p < k+1) \mathbb{P}(k \leq \|u_0\|_p < k+1) = \sum_{k=0}^{\infty} \mathbb{P}(k \leq \|u_0\|_p < k+1) = 1.$$

Next, note $\mathbb{1}_{[0, \tau_k]} \mathbb{1}_{k \leq \|u_0\|_p < k+1} = \mathbb{1}_{[0, \tau]} \mathbb{1}_{k \leq \|u_0\|_p < k+1}$. Also, $\sum_{k=0}^{\infty} \sigma(u(k)) \mathbb{1}_{k \leq \|u_0\|_p < k+1}$ agrees with $\sigma(u)$ in $\mathbb{L}^p$. Multiplying both sides of (4.40) by $\mathbb{1}_{k \leq \|u_0\|_p < k+1}$ and summing over k , we obtain

$$\begin{aligned} \mathbb{1}_{[0, \tau]}(s)(u(s), \phi) &= (u_0, \phi) + \mathbb{1}_{[0, \tau]}(s) \int_0^s (u, \Delta \phi) dr + \mathbb{1}_{[0, \tau]}(s) \int_0^s (\sigma(u), \phi) d\mathbb{W}_r \\ &\quad + \sum_j \mathbb{1}_{[0, \tau]}(s) \int_0^s (\mathcal{P}(u_j u), \partial_j \phi) dr, \quad (s, \omega)\text{-a.e.}, \end{aligned}$$

namely, (u, τ) is a local solution associated with a general initial data $u_0 \in L^p(\Omega, L^p(\mathbb{T}^3))$.

Since $u_{(k)} \in C([0, \tau_k], L^p)$ almost surely, we have $u \in C([0, \tau], L^p)$ almost surely. In addition, using (4.39) and the pathwise uniqueness, we obtain

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq \tau} \|u(s, \cdot)\|_p^p + \sum_j \int_0^\tau \int_{\mathbb{T}^3} |\nabla(|u_j(s, x)|^{p/2})|^2 dx ds \right] \\ &= \lim_{k \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{0 \leq \|u_0\|_p < k+1} \left(\sup_{0 \leq s \leq \tau} \|u(s, \cdot)\|_p^p + \sum_j \int_0^\tau \int_{\mathbb{T}^3} |\nabla(|u_j(s, x)|^{p/2})|^2 dx ds \right) \right] \\ &\leq \lim_{k \rightarrow \infty} C \mathbb{E} [\mathbb{1}_{0 \leq \|u_0\|_p < k+1} \|u_0\|_p^p] + C \leq C \mathbb{E} [\|u_0\|_p^p] + C, \end{aligned}$$

concluding the proof. $\square$

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