ASYMPTOTIC PROPERTIES OF THE BOUSSINESQ EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. We address the asymptotic properties for the Boussinesq equations with vanishing thermal diffusivity in a bounded domain with no-slip boundary conditions. We show the dissipation of the L^2 norm of the velocity and its gradient, convergence of the L^2 norm of Au, and an $o(1)$-type exponential growth for $\|A^{3/2}u\|_{L^2}$. We also obtain that in the interior of the domain the gradient of the vorticity is bounded by a polynomial function of time.

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1. Introduction

In this paper, we address the asymptotic behavior of the Boussinesq equations

$$\begin{aligned} &u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = \rho e_2 \\ &\rho_t + u\cdot\nabla\rho = 0 \\ &\nabla\cdot u = 0 \end{aligned} \tag{1.1}$$

EQ58

with vanishing thermal/density diffusivity, in a smooth bounded domain $\Omega\subseteq\mathbb{R}^2$ with the Dirichlet boundary condition

$$u\big|_{\partial\Omega} = 0 \tag{1.2}$$

EQ126

and subject to the initial condition $(u(0),\rho(0)) = (u_0,\rho_0)$. Here, u represents the velocity, p the pressure, and ρ the density or the temperature, depending on the physical context. The 2D Boussinesq system of equations is used in a wide range of physical contexts, from large scale oceanic and atmospheric flows where rotation and stratification are significant to microfluids and biophysics. It also relates closely to fundamental models in fluid dynamics. In particular, the vorticity formulation of the incompressible Euler

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equations away from the singularity can be described by the 2D Boussinesq equations (see [DWZZ]). For simplicity of exposition, we shall refer to the variable ρ as the density, although it may also represent a temperature.

While global existence results have been well-known in the case of positive viscosity and positive thermal diffusivity, i.e., when adding the term $-\kappa\Delta\rho$ in the equation for the density/temperature, we address here the case of vanishing thermal diffusivity. In the case when both viscosity ν and diffusion coefficients κ vanish, the global existence and uniqueness remain open questions, although results on the local existence, blow-up criteria, explicit solutions, and finite time singularities have been proven; see the blow-up results in [CH, EJ], based on the singularity creation theorem for the Euler equations by Elgindi [E]. The case $\nu > 0$ and $\kappa = 0$, considered here, was initially considered by Chae [C] and Hou and Li [HL]. In particular, Hou and Li obtained the global existence and persistence of regularity in $H^s \times H^{s-1}$ for integer valued $s \geq 3$ in the case of periodic boundary conditions. The paper [LLT] by Lai et al extended the result in [HL] to the Dirichlet boundary conditions. The persistence of regularity for the lower value $s = 2$ in the case of Dirichlet or periodic boundary conditions was addressed in [HKZ1]. Subsequently, Ju obtained in [J] that Ce^{Ct^2} is an upper bound for the H^1 norm for the density, also for the Dirichlet boundary conditions. The bound was lowered to e^{Ct} in [KW2], where also more precise results were obtained for periodic boundary conditions. In particular, [KW2, Theorem 2.1] contains a uniform in time upper bound for the quantity $\|D^2u\|_{L^p}$ for all $p \geq 2$ in the periodic case. In a recent paper by Doering et al [DWZZ], the global existence, uniqueness, and regularity for the Boussinesq for the Lions boundary condition on a Lipschitz domain Ω, was proven along with the dissipation of the L^2 norm of the velocity and its gradient. For other papers on the global existence and the regularity in Sobolev and Besov spaces, see [ACW, ACS.., BFL, BS, BrS, CD, CG, CN, CW, DP, HK1, HK2, HKR, HKZ2, HS, JMWZ, KTW, KW2, KWZ, LPZ, SW].

In this paper, we prove several results on the asymptotic behavior of solutions of the Boussinesq system (1.1) with the Dirichlet boundary conditions (1.2). In our first main theorem, Theorem 2.1, we show that the H^1 norm of the velocity dissipates. We also establish a balanced convergence of Au (see (2.7) below), where A is the Stokes operator. Regarding the growth of the density, we prove that the first Sobolev norm of the density is bounded, up to a constant, by $e^{\epsilon t}$ for an arbitrarily small $\epsilon > 0$, thus improving a result from [KW2] where the bound of the type e^{Ct} was proven. Since the growth of the Sobolev norms of the density is controlled by the time integral of $\|\nabla u\|_{L^\infty}$, it is reasonable to expect that the bound was optimal; however, here we prove that the optimal bound is in fact $e^{\epsilon t}$. It remains an open problem if one can achieve the estimate of the type e^{Ct^α}, where $\alpha \in [0, 1)$; it seems that such an improvement would require the Lipschitz norm of u to decay, which may not be reasonable to expect. The theorem holds under the assumption that (u_0, ρ_0) belongs to $H^2 \times H^1$. The ideas for the proof of Theorems 2.1 draw from the approaches in [DWZZ], [HKZ1], [LLT], [HKZ1], [J], [KW1], and [KW2].

In the second main theorem, Theorem 2.2, we address the behavior of the solution in a higher regularity norm. We prove that, under the $H^3 \times H^2$ assumption on the initial data, that for every $\epsilon > 0$ the norm of (u, ρ) in the $H^3 \times H^2$ norm is bounded by $e^{\epsilon t}$, up to a constant depending on $\epsilon > 0$. This holds under the $H^3 \times H^2$ regularity of the initial data (u_0, ρ_0).

In the last main theorem, Theorem 2.3, we consider the upper bound for the L^p norm of the second derivatives of the velocity. As shown in [HKZ1], one may obtain a uniform bound when $p = 2$. When $p > 2$, this is not known except in the case of periodic boundary condition, which is a result obtained in [KW1]. Here, we prove that we can obtain a polynomial in time bound in the interior of a domain when considering the Dirichlet boundary condition, which is considerably lower than $e^{\epsilon t}$ type bound that would result from applying the Gagliardo-Sobolev inequality on the conclusions of Theorem 2.2. The proof is obtained by the change of variable from [KW1] combined with new localization arguments controlling the nonlocal nature of the transformation in [KW1] (see the double cut-off strategy in the proof of Theorem 2.3 below).

We emphasize that all our results extend also in the often-studied problem of the channel with Dirichlet boundary conditions on top and the bottom and periodic boundary conditions on the sides. Also, our proofs are completely self-contained.

sec02

2. Main theorems

We consider the asymptotic behavior of the Boussinesq equations

$$\begin{aligned} &u_t - \Delta u + u\cdot\nabla u + \nabla p = \rho e_2 \\ &\rho_t + u\cdot\nabla\rho = 0 \\ &\nabla\cdot u = 0 \end{aligned} \tag{2.1}$$

EQ01

and

$$u\big|_{\partial\Omega} = 0, \tag{2.2}$$

EQ02

coupling the Navier-Stokes equations [CF, DG, R, K1, K2, T1–T3] for the velocity $u = (u_1, u_2)$ and the pressure p with the equation for the density ρ. The system is set on a smooth, bounded, and connected domain $\Omega \subseteq \mathbb{R}^2$ and supplemented with the initial condition

$$(u,\rho)(0) = (u_0,\rho_0) \qquad \text{in } \Omega.$$

(EQ03)

Here, u denotes the velocity, p the pressure, and ρ the density. Note that we set $\nu = 1$ for simplicity of exposition; all the results extend to other values of ν with constants depending additionally on ν.

From [CF, T1], we recall the classical spaces

$$H = \{u \in L^2(\Omega) : \nabla\cdot u = 0 \text{ in } \Omega, u\cdot n = 0 \text{ on } \partial\Omega\},$$

(EQ04)

where n denotes the outward unit normal, and

$$V = \{u \in H_0^1(\Omega) : \nabla\cdot u = 0 \text{ in } \Omega\},$$

(EQ05)

utilized in the study of the Navier-Stokes equations. With $\mathbb{P}\colon L^2 \to H$ the Leray projector, denote by

$$A = -\mathbb{P}\Delta,$$

(EQ06)

the Stokes operator with the domain $D(A) = H^2(\Omega)\cap V$.

It is known that for a sufficiently regular initial condition there exists a unique, global in time solution for (2.1)–(2.2) (see [C, HL]). In the first theorem, we obtain the asymptotic properties of $A^{1/2}u$ and Au in the L^2 norm.

T01 **Theorem 2.1.** *Let $(u_0,\rho_0) \in (H^2(\Omega)\cap V)\times H^1(\Omega)$. Then the solution*

$$(u,\rho) \in (C([0,\infty);V)\cap L^2_{loc}([0,\infty);D(A)))\times L^\infty_{loc}([0,\infty),H^1(\Omega))$$

(EQ07)

of (2.1)–(2.2) *satisfies*

$$\|Au(t)\|_{L^2} \le C, \qquad t \ge 0, \tag{2.3}$$

where C depends on the size of the initial data, i.e., on the norms $\|Au_0\|_{L^2}$ and $\|\rho_0\|_{H^1}$, as well as

$$\int_t^{t+t_0} \|u\|_{H^3}^2 \le C, \qquad t \ge 0 \tag{2.4}$$

for every $t_0 > 0$, where C depends additionally on t_0 and

$$\lim_{t\to\infty} \int_t^{t+t_0} \|u\|_{H^3}^2 = 0, \qquad t_0 > 0. \tag{2.5}$$

Moreover,

$$\|A^{1/2}u(t)\|_{L^2} = \|\nabla u(t)\|_{L^2} \to 0 \quad \text{as } t \to \infty, \tag{2.6}$$

and

$$\|Au(t) - \mathbb{P}(\rho(t)e_2)\|_{L^2} \to 0 \quad \text{as } t \to \infty, \tag{2.7}$$

and for every $\epsilon > 0$ we have

$$\|\rho(t)\|_{H^1} \le C_\epsilon e^{\epsilon t}, \qquad t \ge 0, \tag{2.8}$$

where C_ϵ is a constant depending on ϵ and the size of initial data.

Above and in the sequel, we allow all constants to depend on Ω.

One of the main conclusions of the above theorem is that the persistence holds for initial data in $D(A) \times H^1$. It is not known whether persistence holds in the space $D(A^{1/2}) \times H^1$. Note that in the case of Navier boundary conditions, this is possible due to better continuity properties of the bilinear term. The persistence in $L^\infty \times H^1$ in the case of Navier boundary conditions was established in [HWW+] (see also [HW]).

In the next statement, we obtain the asymptotic behavior of the $H^3 \times H^2$ norm of the solution (u, ρ). By [LLT, T5], the local existence requires the initial data to satisfy the compatibility condition

$$(-\Delta u_0 - \nabla p_0 - \rho_0 e_2)|_{\partial\Omega} = 0, \tag{2.9}$$

where p_0 denotes the initial pressure, which solves the Neumann boundary problem

$$\begin{aligned} \Delta p_0 &= \nabla \cdot (\rho_0 e_2 - u_0 \cdot \nabla u_0) \qquad \text{in } \partial\Omega, \\ -\nabla p_0 \cdot n\big|_{\partial\Omega} &= (\Delta u_0 + \rho_0 e_2) \cdot n\big|_{\partial\Omega} \end{aligned}$$

with n denoting the outward unit normal.

Theorem 2.2. *Assume that $(u_0, \rho_0) \in (H^3(\Omega) \cap V) \times H^2(\Omega)$ satisfies the compatibility condition* (2.9), *and let (u, ρ) be the corresponding solution of* (2.1)–(2.2). *Then for every $\epsilon > 0$, we have*

$$\|u(t)\|_{H^3} \le C_\epsilon e^{\epsilon t}, \qquad t \ge 0$$

and

$$\|\rho(t)\|_{H^2} \le C_\epsilon e^{\epsilon t}, \qquad t \ge 0, \tag{2.10}$$

where C_ϵ is a constant depending on ϵ and the size of initial data.

In the next theorem, we obtain the interior bounds for the L^p norm of the Hessian $D^2 u$ of the velocity in the interior, for any $p \ge 2$.

T04 **Theorem 2.3.** *Let* $(u_0, \rho_0) \in (H^2(\Omega) \cap V) \times H^1(\Omega)$ *and* $p \in [2, \infty)$*, and suppose that* $\Omega' \subseteq \Omega$ *is open and relatively compact. Then for the corresponding solution* (u, ρ) *of* (2.1)–(2.2) *and all* $t_0 > 0$ *we have a space-time bound*

$$\|D^2 u\|_{L^p([t_0,T];L^p(\Omega'))} \le C T^{1/p}, \tag{2.11}$$ EQ15

for $T \ge t_0 > 0$*, while in addition we have a pointwise in time bound*

$$\|D^2 u(t)\|_{L^p(\Omega')} \le C t^{(p+2)/4p}, \qquad t \ge t_0, \tag{2.12}$$ EQ16

where the constants in (2.11) *and* (2.12) *depend on* t_0*,* p*, and* $\operatorname{dist}(\Omega', \partial\Omega)$.

sec03

3. Proofs for the global bounds

First, we recall prior results on the L^2 norms corresponding to Theorem 2.1. Let $(u_0, \rho_0) \in (H^2(\Omega) \cap V) \times H^1(\Omega)$. Then there exists a unique global solution (u, ρ) such that $u \in L^\infty((0,\infty), H^2(\Omega)) \cap L^2_{\mathrm{loc}}((0,\infty), H^3(\Omega))$ and $\rho \in L^\infty_{\mathrm{loc}}((0,\infty), H^1(\Omega))$ of (2.1)–(2.2). Furthermore, the solution (u, ρ) satisfies

$$\|u(t)\|_{L^2} + \|\rho(t)\|_{L^2} \lesssim 1, \qquad t \ge 0. \tag{3.1}$$ EQ17

Here and below, the notation $a \lesssim b$ means $a \le Cb$, where C is a constant, which is allowed to depend on the size of the initial data in the pertinent norms. We denote by

$$B(u, v) = \mathbb{P}(u \cdot \nabla v) \qquad u, v \in V$$ (EQ18)

the bilinear term corresponding to the Navier-Stokes equations. This allows us to rewrite (2.1) as

$$\begin{aligned} &u_t + Au + B(u, u) = \mathbb{P}(\rho e_2) \\ &\rho_t + u \cdot \nabla \rho = 0. \end{aligned} \tag{3.2}$$ EQ19

We now turn to the proof of the first theorem.

Proof of Theorem 2.1. We begin by proving that $\|u\|_{L^2}$ dissipates. Inspired by [DWZZ], we shift the density by x_2, i.e., introduce

$$\theta(x_1, x_2, t) = \rho(x_1, x_2, t) - x_2, \tag{3.3}$$ EQ123

and compensate with $P = p(x_1, x_2, t) - x_2^2/2$ to derive an equivalent system of equations

$$\begin{aligned} &u_t - \Delta u + u \cdot \nabla u + \nabla P = \theta e_2 \\ &\theta_t + u \cdot \nabla \theta = -u \cdot e_2 \\ &\nabla \cdot u = 0, \end{aligned} \tag{3.4}$$ EQ20

with $u\big|_{\partial\Omega} = 0$. Multiplying the first equation of (3.4) with u and the second by θ, integrating, and applying the Dirichlet boundary conditions and incompressibility, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 = 0. \tag{3.5}$$ EQ21

Observe that the norm $\|\theta\|_{L^2}$ may increase, and thus no direct conclusion on decay rates can be reached from (3.5). The identity (3.5) implies $\|u\|_{L^2}^2$ and $\|\theta\|_{L^2}^2$ are uniformly bounded in time and

$$\int_0^\infty \|\nabla u\|_{L^2}^2 \lesssim 1,$$ (EQ22)

where we allow all constants to depend on $\|u_0\|_{H^2}$ and $\|\rho_0\|_{H^1}$. Also, $\|\rho(t)\|_{L^p} = \|\rho_0\|_{L^p}$ for $t \geq 0$ for any $p \in [1, \infty)$; note that $\|\rho_0\|_{L^p} < \infty$ by $\rho_0 \in H^1(\Omega)$. Utilizing the Poincaré inequality, we also get

$$\int_0^\infty \|u\|_{L^2}^2 \lesssim 1. \tag{3.6}$$

EQ23

To prove the uniform continuity from above of the L^2 norm of u, we multiply the first equation in (3.4) with u and integrate by parts to find that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int_\Omega \theta u \cdot e_2 \leq \|u\|_{L^2}\|\theta\|_{L^2} \lesssim \|u\|_{L^2},$$

(EQ24)

which, by Poincaré and Young's inequalities, implies

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \lesssim 1. \tag{3.7}$$

EQ139

It is elementary to show that if a differentiable function $f\colon [0,\infty) \to [0,\infty)$ satisfies $\int_0^\infty f(s)\,ds < \infty$ and $f'(t) \lesssim 1$, then $\lim_{t\to\infty} f(t) = 0$. Applying the statement with $f(t) = \|u\|_{L^2}^2$, the inequalities (3.6) and (3.7) imply

$$\|u\|_{L^2} \to 0 \qquad \text{as } t \to \infty. \tag{3.8}$$

EQ26

Next, we aim to prove that $\|\nabla u\|_{L^2}^2 \to 0$. We take the L^2 inner product of $(3.2)_1$ with Au to find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 &= -\langle B(u,u), Au\rangle_{L^2} + \langle \mathbb{P}(\theta e_2), Au\rangle_{L^2} \\
&\leq \|B(u,u)\|_{L^2}\|Au\|_{L^2} + \|\theta\|_{L^2}\|Au\|_{L^2} \lesssim \|u\|_{L^2}^{1/2}\|A^{1/2}u\|_{L^2}\|Au\|_{L^2}^{3/2} + \|Au\|_{L^2},
\end{aligned} \tag{3.9}$$

EQ28

where we used

$$\|B(u,u)\|_{L^2} \lesssim \|u\|_{L^4}\|\nabla u\|_{L^4} \lesssim \|u\|_{L^2}^{1/2}\|u\|_{H^1}\|u\|_{H^2}^{1/2} \lesssim \|u\|_{L^2}^{1/2}\|A^{1/2}u\|_{L^2}\|Au\|_{L^2}^{1/2}. \tag{3.10}$$

EQ131

In (3.9), we apply Young's inequality and absorb the factors $\|Au\|_{L^2}$ into the second term on the left side, obtaining

$$\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \lesssim \|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^4 + 1 \lesssim \|A^{1/2}u\|_{L^2}^4 + 1.$$

(EQ29)

Utilizing Lemma A.1 in the Appendix, we obtain

$$\|A^{1/2}u(t)\|_{L^2} \lesssim 1, \qquad t \geq 0 \tag{3.11}$$

EQ122

and

$$\|A^{1/2}u(t)\|_{L^2} \to 0 \qquad \text{as } t \to \infty,$$

(EQ138)

giving (2.6). In addition, by the same lemma,

$$\int_t^{t+t_0} \|Au\|_{L^2}^2 \lesssim 1, \qquad t, t_0 \geq 0, \tag{3.12}$$

EQ173

where the constant depends on t_0 and

$$\limsup_{t\to\infty} \int_t^{t+t_0} \|Au\|_{L^2}^2 \lesssim t_0, \qquad t_0 \geq 0. \tag{3.13}$$

EQ30

We note in passing, and since it is needed in the proof of Theorem 2.3, that the inequality of type (3.13) also holds with Au replaced with u_t. To show that u_t dissipates in the L^2 norm, we take the time derivative of $(3.4)_1$, multiply by u_t, and integrate by parts, to get the equation

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 = \langle \theta_t e_2, u_t\rangle_{L^2} - \langle u_t \cdot \nabla u, u_t\rangle_{L^2}. \tag{3.14}$$ EQ33

For the first term on the right, we apply $(3.4)_2$ to obtain

$$\begin{aligned}\langle \theta_t e_2, u_t\rangle_{L^2} &= -\int_\Omega (u\cdot\nabla\theta)(\partial_t u_2) - \int_\Omega u_2\partial_t u_2 = \int_\Omega \theta u\cdot\nabla\partial_t u_2 - \int_\Omega u_2\partial_t u_2\\ &\lesssim \|\theta\|_{L^4}\|u\|_{L^2}^{1/2}\|A^{1/2}u\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2} + \|u\|_{L^2}\|u_t\|_{L^2}\\ &\lesssim \|A^{1/2}u\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2} + \|u\|_{L^2}\|\nabla u_t\|_{L^2},\end{aligned} \tag{3.15}$$ EQ34

where we used $\|\theta\|_{L^4} \lesssim 1$, by $\|\rho\|_{L^4} \lesssim 1$, and $\|u_t\|_{L^2} \lesssim \|\nabla u_t\|_{L^2}$ in the last inequality. For the second term on the right-hand side of (3.14), we write

$$-\langle u_t\cdot\nabla u, u_t\rangle_{L^2} \lesssim \|u_t\|_{L^4}^2\|\nabla u\|_{L^2} \lesssim \|u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|A^{1/2}u\|_{L^2}. \tag{3.16}$$ EQ35

Using (3.15) and (3.16) in (3.14) and then absorbing the factors $\|\nabla u_t\|_{L^2}$ by Young's inequality, we get

$$\frac{d}{dt}\|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \lesssim \|A^{1/2}u\|_{L^2} + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2 \lesssim \phi(t)(1+\|u_t\|_{L^2}^2), \tag{3.17}$$ EQ36

where $\phi\colon [0,\infty) \to [0,\infty)$ is a bounded function, which satisfies $\lim_{t\to\infty}\phi(t) = 0$. Also, note that $u_0(0)$, defined as $u_t(0) = -Au_0 - B(u_0,u_0) + \mathbb{P}(\rho_0 x_2)$, satisfies

$$\|u_t(0)\|_{L^2} \lesssim \|Au_0\|_{L^2} + \|u_0\|_{L^2}^{1/2}\|Au_0\|_{L^2}^{1/2}\|A^{1/2}u_0\|_{L^2} \lesssim 1. \tag{3.18}$$ EQ91

By Lemma A.2, we get

$$\|u_t(t)\|_{L^2} \lesssim 1, \qquad t\in[0,\infty) \tag{3.19}$$ EQ163

and

$$\|u_t(t)\|_{L^2} \to 0 \qquad \text{as } t\to\infty \tag{3.20}$$ EQ37

as well as

$$\int_t^{t+t_0} \|\nabla u_t\|_{L^2}^2 \le C, \qquad t, t_0 \ge 0 \tag{3.21}$$ EQ82

and

$$\limsup_{t\to\infty}\int_t^{t+t_0} \|\nabla u_t\|_{L^2}^2 = 0, \qquad t_0 \ge 0, \tag{3.22}$$ EQ38

where the constant in (3.21) depends on t_0. Using the H^3 regularity on the stationary Stokes formulation $Au = -u_t - B(u,u) + \mathbb{P}(\rho e_2)$, we obtain

$$\|u\|_{H^3} \lesssim \|u_t\|_{H^1} + \|B(u,u)\|_{H^1} + \|\mathbb{P}(\rho e_2)\|_{H^1} \lesssim \|u_t\|_{H^1} + \|u\cdot\nabla u\|_{H^1} + \|\rho\|_{H^1},$$ (EQ174)

where we used the continuity of $\mathbb{P}$ on $\mathbb{H}^1$. Since

$$\|u\cdot\nabla u\|_{H^1} \lesssim \|u\|_{L^2}^{1/2}\|Au\|_{H^2}^{3/2},$$ (EQ175)

we get

$$\|u\|_{H^3} \lesssim \|\nabla u_t\|_{L^2} + \|u\|_{L^2}^{1/2} + \|u\|_{L^2}^{1/2}\|Au\|_{L^2}^{3/2} + 1, \tag{3.23}$$ EQ176

and the assertions (3.21) and (2.5) follow from (3.12) and (3.13), respectively.

Next, from $(3.2)_1$, we obtain

$$\|Au\|_{L^2} \lesssim \|u_t\|_{L^2} + \|B(u,u)\|_{L^2} + \|\rho\|_{L^2} \lesssim \|u_t\|_{L^2} + \|u\|_{L^2}^{1/2}\|A^{1/2}u\|_{L^2}\|Au\|_{L^2}^{1/2} + 1.$$
(EQ39)

Absorbing the factor $\|Au\|_{L^2}^{1/2}$ in the left-hand side by using Young's inequality, we get

$$\|Au\|_{L^2} \lesssim \|u_t\|_{L^2} + \|u\|_{L^2}\|A^{1/2}u\|_{L^2}^2 + 1,$$
(EQ40)

from where, by (2.6) and (3.20), we get (2.3). Note, in passing, that (2.3) and (3.8) imply

$$\|u(t)\|_{L^\infty} \to 0 \qquad \text{as } t \to \infty, \tag{3.24}$$
EQ41

by Agmon's inequality. From $(3.2)_1$, we get

$$\|Au - \mathbb{P}(\rho e_2)\|_{L^2} \lesssim \|u_t\|_{L^2} + \|B(u,u)\|_{L^2} \lesssim \|u_t\|_{L^2} + \|u\|_{L^\infty}\|A^{1/2}u\|_{L^2}. \tag{3.25}$$
EQ42

By (3.20) and (3.24), the right-hand side of (3.25) converges to 0 as $t \to \infty$, and we obtain (2.7).

We lastly proceed to prove the $o(1)$-type exponential estimate on the growth of $\|\nabla\theta\|_{L^2}$. For this, we first need to prove the local in time boundedness of $\|\theta\|_{H^1}$, which in turn requires us to first bound $\int_0^T \|\nabla u\|_{L^\infty}$ for some $T > 0$. From (3.17) and (3.18), we have

$$\int_0^T \|\nabla u_t\|_{L^2}^2 \lesssim 1,$$
(EQ155)

for all $T > 0$, where the constant depends on T. Now, consider the Stokes problem

$$\begin{aligned} &u_t - \Delta u + \nabla p = -u\cdot\nabla u + \rho e_2 \\ &\nabla\cdot u = 0 \\ &u|_{\partial\Omega} = 0. \end{aligned}$$
(EQ25)

By [SvW, Theorem 2.7] (see also [GS]) applied with $s = p = 3$, we obtain that for any $\tilde\epsilon > 0$

$$\int_0^T \|u\|_{W^{2,3}}^3 \lesssim \|A_3^{2/3+\tilde\epsilon}u_0\|_{L^3}^3 + \int_0^T \|u\cdot\nabla u - \rho e_2\|_{L^3}^3, \tag{3.26}$$
EQ167

for all $T > 0$, where the constant depends on T and $\tilde\epsilon$. In (3.26), A_3 denotes the L^3 version of the Stokes operator (see [GS, SvW]). For the first term on the right-hand side in (3.26), we use

$$\|A_3^{2/3+\tilde\epsilon}u_0\|_{L^3} \lesssim \|Au_0\|_{L^2} \lesssim 1 \tag{3.27}$$
EQ169

with $\tilde\epsilon = 1/6$ from the embedding property on [SvW, p. 430], while for the second term we estimate

$$\|u\cdot\nabla u - \rho e_2\|_{L^3}^3 \lesssim \|u\|_{L^6}^3\|\nabla u\|_{L^6}^3 + \|\rho\|_{L^3}^3 \lesssim \|u\|_{L^2}\|A^{1/2}u\|_{L^2}^3\|Au\|_{L^2}^2 + 1 \lesssim 1. \tag{3.28}$$
EQ168

Applying (3.27) and (3.28) in (3.26), we get

$$\int_0^T \|D^2u\|_{L^3}^3 \lesssim 1, \tag{3.29}$$
EQ170

where the constant depends on T and consequently

$$\int_0^T \|\nabla u\|_{L^\infty} \lesssim 1 \tag{3.30}$$
EQ159

for all $T > 0$, where the constant depends on T, due to the Gagliardo-Nirenberg type inequality

$$\|v\|_{L^\infty} \lesssim \|v\|_{L^2}^{1/4}\|\nabla v\|_{L^3}^{3/4} + \|v\|_{L^2}. \tag{3.31}$$
EQ32

By applying the gradient to $(3.4)_2$ and taking the inner product with $\nabla\theta$, we find that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 = -\langle\nabla(u\cdot\nabla\theta),\nabla\theta\rangle_{L^2} - \langle\nabla(u\cdot e_2),\nabla\theta\rangle_{L^2}. \tag{3.32}$$
EQ44

The second term is estimated by $C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}$, while the first term is likewise bounded as

$$-\langle\nabla(u\cdot\nabla\theta),\nabla\theta\rangle_{L^2} = -\int_\Omega \partial_j(u_i\partial_i\theta)\partial_j\theta = -\int_\Omega \partial_j u_i \partial_i\theta\partial_j\theta - \frac{1}{2}\int_\Omega u_i\partial_i|\nabla\theta|^2 \lesssim \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2,$$
(EQ45)

by $(3.4)_3$ and $u\big|_{\partial\Omega}=0$. Thus, estimating the two terms in (3.32) as indicated and canceling a factor of $\|\nabla\theta\|_{L^2}$, we conclude that

$$\frac{d}{dt}\|\nabla\theta\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2} + \|\nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}(\|\nabla\theta\|_{L^2}+1), \tag{3.33}$$
EQ46

which implies that the exponential growth of $\|\nabla\theta\|_{L^2}$ is determined by the time integral of $\|\nabla u\|_{L^\infty}$. In particular, applying (3.30) to (3.33) yields

$$\|\theta\|_{H^1} \lesssim 1, \qquad t\in[0,T], \tag{3.34}$$
EQ160

for all $T>0$, where the constant depends on T.

Next, we fix $\epsilon\in(0,1]$ and claim that

$$\|\theta(t)\|_{H^1} \lesssim e^{\epsilon t}, \qquad t\geq 0, \tag{3.35}$$
EQ43

where we allow all constants to depend on ϵ. Note that (3.35) directly implies (2.8) by the definition (3.3). To prove (3.35), we need to estimate the time integral of $\|\nabla u\|_{L^\infty}$. Let $0<t_0\leq t_1$, where $t_0\geq 2$ is a large time to be determined based on ϵ. By the Gagliardo-Nirenberg in space and Hölder's inequalities in time, we have, using (3.31)

$$\begin{aligned}\int_{t_1}^{t_1+1}\|\nabla u\|_{L^\infty} &\leq \int_{t_1}^{t_1+1}\Big(\|\nabla u\|_{L^2}^{1/4}\|\Delta u\|_{L^3}^{3/4} + \|\nabla u\|_{L^2}\Big)\\ &\leq C\left(\int_{t_1}^{t_1+1}\|\nabla u\|_{L^2}^{1/3}\right)^{3/4}\left(\int_{t_1}^{t_1+1}\|\Delta u\|_{L^3}^3\right)^{1/4} + \frac{1}{2}\epsilon,\end{aligned} \tag{3.36}$$
EQ47

provided t_0 is sufficiently large (recall that $t_1\geq t_0$ is arbitrary). To bound the L^3L^3 norm of Δu, we introduce a smooth cut-off function $\phi\colon[0,\infty)\to[0,1]$, where $\phi(t)=0$ on $[0,t_1-1]$ and $\phi(t)=1$ on $[t_1,\infty]$ with $|\phi'|\lesssim 1$. Now we consider the equation

$$(\phi u)_t - \Delta(\phi u) + \nabla(\phi p) = \phi' u - u\cdot\nabla(\phi u) + \phi\rho e_2 \tag{3.37}$$
EQ164

which follows from $(2.1)_1$; note that $\nabla\cdot(\phi u)=0$ since ϕ is a function of time only, and also, for the same reason, there are no additional terms resulting from the nonlinearity. Using the $W^{2,3}$ estimate due to Sohr and Von Wahl [SvW] we have, similarly to (3.26)–(3.29),

$$\begin{aligned}\int_{t_1}^{t_1+1}\|D^2u\|_{L^3}^3 &\lesssim \int_{t_1-1}^{t_1+1}\|u\cdot\nabla(\phi u)\|_{L^3}^3 + \int_{t_1-1}^{t_1+1}\|\phi' u\|_{L^3}^3 + \int_{t_1-1}^{t_1+1}\|\rho\|_{L^3}^3\\ &\lesssim \int_{t_1-1}^{t_1+1}\|u\|_{L^6}^3\|\nabla u\|_{L^6}^3 + \int_{t_1-1}^{t_1+1}\|u\|_{L^3}^3 + 1\\ &\lesssim \int_{t_1-1}^{t_1+1}\|u\|_{L^2}\|\nabla u\|_{L^2}^3\|Au\|_{L^2}^2 + \int_{t_1-1}^{t_1+1}\|u\|_{L^2}^2\|\nabla u\|_{L^2} + 1 \lesssim 1,\end{aligned} \tag{3.38}$$
EQ48

where we used (2.3), (3.1), and (3.11). Also, for the first factor of the first term in (3.36), we use (2.6) to obtain that for any $\epsilon_0 > 0$ there exists $t_0 \geq 1$ sufficiently large so that

$$\left(\int_{t_1-1}^{t_1+1} \|\nabla u\|_{L^2}^{1/3} \right)^{3/4} \leq \epsilon_0 \epsilon. \tag{3.39}$$

EQ50

Thus, using (3.38) and (3.39) in (3.36), we obtain

$$\int_{t_1}^{t_1+1} \|\nabla u\|_{L^\infty} \leq C\epsilon_0 \epsilon + \frac{1}{2}\epsilon,$$

(EQ51)

for $t_0 \geq 1$ sufficiently large, which in turn implies

$$\int_{t_0}^{t} \|\nabla u\|_{L^\infty} \leq \epsilon(t - t_0), \qquad t \geq t_0 + 1 \tag{3.40}$$

EQ165

if we choose ϵ_0 to be a sufficiently small constant. Note that (3.40) is obtained by adding the integrals of unit length. Returning to (3.33), we find that Gronwall's inequality implies

$$\|\nabla\theta(t)\|_{L^2} \leq (\|\nabla\theta(t_0)\|_{L^2} + 1)e^{\epsilon(t-t_0)}. \tag{3.41}$$

EQ52

Finally, we use (3.34) implying

$$\|\theta(t_0)\|_{H^1} \lesssim 1, \tag{3.42}$$

EQ162

where the constant depends on t_0, which in turn only depends on ϵ. Combining (3.41) and (3.42) leads to the claimed inequality (3.35). $\square$

Next, we address a higher regularity norm.

Proof of Theorem 2.2. We start with a priori estimates and at the end of the proof we provide a sketch of the justification. Taking a time derivative of $(2.1)_1$, we obtain

$$u_{tt} - \Delta u_t + u_t \cdot \nabla u + u \cdot \nabla u_t + \nabla p_t = \rho_t e_2,$$

(EQ59)

which, after testing with u_{tt} gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 \\
&\quad = -\int_\Omega u_t \cdot \nabla u_j \partial_{tt} u_j - \int_\Omega u \cdot \nabla \partial_t u_j \partial_{tt} u_j + \int_\Omega (\rho_t e_2) \cdot u_{tt} \\
&\quad \lesssim \|u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \|Au\|_{L^2}^{1/2} \|u_{tt}\|_{L^2} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|u_{tt}\|_{L^2} + \|\rho_t\|_{L^2} \|u_{tt}\|_{L^2}.
\end{aligned}$$

(EQ60)

Now we apply $\|u\|_{L^\infty} \lesssim \|u\|_{L^2}^{1/2} \|Au\|_{L^2}^{1/2}$ for the second term and

$$\|\rho_t\|_{L^2} = \|u \cdot \nabla\rho\|_{L^2} \lesssim \|u\|_{L^\infty} \|\nabla\rho\|_{L^2},$$

(EQ55)

by $(2.1)_2$, on the last. Absorbing the factors of $\|u_{tt}\|_{L^2}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 \\
&\qquad \lesssim \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \|Au\|_{L^2} + \|u\|_{L^\infty}^2 \|\nabla u_t\|_{L^2}^2 + \|u\|_{L^\infty}^2 \|\nabla\rho\|_{L^2}^2 \\
&\qquad \lesssim 1 + \|\nabla u_t\|_{L^2}^2 + C_\epsilon e^{2\epsilon t},
\end{aligned} \tag{3.43}$$

EQ61

where $\epsilon > 0$ is arbitrarily small. In (3.43), we also used (2.3). For the initial norm, we have

$$\begin{aligned}\|\nabla u_t(0)\|_{L^2} &\lesssim \|Au_0\|_{H^1} + \|B(u_0,u_0)\|_{H^1} + \|\mathbb{P}(\rho_0 x_2)\|_{H^1} \lesssim \|u_0\|_{H^3} + \|u_0\cdot\nabla u_0\|_{H^1} + \|\rho_0\|_{H^1} \\ &\lesssim \|u_0\|_{L^2}^{1/2}\|Au_0\|_{L^2}^{3/2} + \|\nabla u_0\|_{L^4}^2\|Au_0\|_{L^2}\|Au_0\|_{L^2} \lesssim 1,\end{aligned} \tag{3.44}$$
EQ116

where we used the continuity of $\mathbb{P}$ on H^1. Combining (3.22), (3.44), and (3.43) with a uniform Gronwall argument, we get

$$\|\nabla u_t(t)\|_{L^2} \lesssim e^{\epsilon t}, \qquad t \geq 0 \tag{3.45}$$
EQ62

and

$$\int_0^t \|u_{tt}\|_{L^2}^2 \lesssim e^{\epsilon t}, \qquad t \geq 0,$$
(EQ63)

where we allow constants to depend on ϵ. Now, consider the stationary, i.e., pointwise in time, Stokes problem

$$\begin{aligned}&-\Delta u + \nabla p = -u\cdot\nabla u - u_t + \rho e_2 \\ &u|_{\partial\Omega} = 0.\end{aligned} \tag{3.46}$$
EQ156

Note that

$$\begin{aligned}\|u\cdot\nabla u + u_t - \rho e_2\|_{H^1} &\lesssim \|D(u\cdot\nabla u - u_t)\|_{L^2} + \|\rho\|_{H^1} \\ &\lesssim \|Du\|_{L^4}^2 + \|u\|_{L^\infty}\|D^2u\|_{L^2} + \|\nabla u_t\|_{L^2} + e^{\epsilon t} \\ &\lesssim \|A^{1/2}u\|_{L^2}\|Au\|_{L^2} + \|u\|_{L^\infty}\|Au\|_{L^2} + \|\nabla u_t\|_{L^2} + e^{\epsilon t} \lesssim e^{\epsilon t},\end{aligned}$$
(EQ157)

using (3.45) in the last step. Applying the H^3 regularity for the Stokes problem (3.46) (see [T4, Proposition 3.3]), leads to

$$\|u\|_{H^3} + \|\nabla p\|_{H^1} \lesssim e^{\epsilon t}. \tag{3.47}$$
EQ166

In order to obtain (2.10), we apply ∂_{ij}, for $i,j = 1,2$, to $(2.1)_2$, test with $\partial_{ij}\rho$, and sum which leads to

$$\frac{1}{2}\frac{d}{dt}\|\partial_{ij}\rho\|_{L^2}^2 = \langle\partial_{ij}(u\cdot\nabla\rho), \partial_{ij}\rho\rangle_{L^2} = \int_\Omega \partial_{ij}u_k\partial_k\rho\partial_{ij}\rho + 2\int_\Omega \partial_i u_k\partial_{jk}\rho\partial_{ij}\rho + \int_\Omega u_k\partial_{ijk}\rho\partial_{ij}\rho, \tag{3.48}$$
EQ67

which holds for all $t \geq 0$. The last term vanishes by the incompressibility of u, while the second is bounded by $C\|\nabla u\|_{L^\infty}\|D^2\rho\|_{L^2}^2$. For the first term on the far right side of (3.48), we write

$$\begin{aligned}\int_\Omega \partial_{ij}u_k\partial_k\rho\partial_{ij}\rho &\lesssim \|\Delta u\|_{L^4}\|\nabla\rho\|_{L^4}\|D^2\rho\|_{L^2} \\ &\lesssim (\|\Delta u\|_{L^2}^{1/2}\|D^3u\|_{L^2}^{1/2} + \|\Delta u\|_{L^2})(\|\nabla\rho\|_{L^2}^{1/2}\|D^2\rho\|_{L^2}^{1/2} + \|\nabla\rho\|_{L^2})\|D^2\rho\|_{L^2},\end{aligned} \tag{3.49}$$
EQ68

where we utilized the Gagliardo-Nirenberg inequalities. Now, we use (2.8) and (3.47) in (3.49), sum in i and j, and cancel a factor of $\|D^2\rho\|_{L^2}$ on both sides to obtain

$$\frac{1}{2}\frac{d}{dt}\|D^2\rho\|_{L^2} \lesssim C_\epsilon e^{3\epsilon t/2} + C_\epsilon e^{\epsilon t}\|D^2\rho\|_{L^2}^{1/2} + \|\nabla u\|_{L^\infty}\|D^2\rho\|_{L^2},$$
(EQ66)

whence, applying Young's inequality

$$\frac{1}{2}\frac{d}{dt}\|D^2\rho\|_{L^2} \lesssim e^{2\epsilon t} + (\epsilon + \|\nabla u\|_{L^\infty})\|D^2\rho\|_{L^2},$$
(EQ158)

for all $t \geq 0$. Applying a Gronwall argument and using (3.40), which holds for $t_0 > 0$ sufficiently large depending on ϵ, we get

$$\|D^2\rho(t)\|_{L^2} \leq C_\epsilon e^{C\epsilon t}(\|D^2\rho(t_0)\|_{L^2} + 1), \qquad t \geq t_0. \tag{3.50}$$
EQ171

On the other hand, using Gronwall's argument on $[0,t_0]$ with (3.30) for $T = t_0$, we get

$$\|D^2\rho(t)\|_{L^2} \lesssim \|D^2\rho_0\|_{L^2} + 1, \qquad t \in [0,t_0], \tag{3.51}$$

EQ172

where the constant depends on t_0 and thus on ϵ. Combining (3.50) and (3.51), we finally obtain (2.10) with $C\epsilon$ replacing ϵ.

To justify the a priori bounds above, we consider the sequence of solutions

$$\begin{aligned}
&u_t^{(n+1)} - \Delta u^{(n+1)} + u^{(n)} \cdot \nabla u^{(n+1)} + \nabla P^{(n+1)} = \theta^{(n+1)} e_2\\
&\theta_t^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} = -u^{(n+1)} \cdot e_2\\
&\nabla \cdot u^{(n+1)} = 0,
\end{aligned} \tag{3.52}$$

EQ71

with the boundary condition $u^{(n+1)}|_{\partial\Omega} = 0$ and with the initial data

$$(u^{(n+1)}(0), \theta^{(n+1)}(0)) = (u_0, \rho_0 - x_2),$$

(EQ72)

for $n \in \mathbb{N}_0$. For $n = 0$, we define

$$\begin{aligned}
&u_t^{(0)} - \Delta u^{(0)} + \nabla P^{(0)} = \theta^{(0)} e_2\\
&\theta_t^{(0)} = -u^{(0)} \cdot e_2\\
&\nabla \cdot u^{(0)} = 0,
\end{aligned}$$

(EQ73)

with the boundary condition $u^{(0)}|_{\partial\Omega} = 0$ and with the initial data

$$(u^{(0)}(0), \theta^{(0)}(0)) = (u_0, \rho_0 - x_2).$$

(EQ74)

Since the system (3.52) is linear in $(u^{(n+1)}, \theta^{(n+1)})$, it is easy to construct a local solution $(u^{(n+1)}, \theta^{(n+1)})$. Also, our a priori estimates apply to the sequence and one may pass uniform bounds to the limit. The detailed construction of solutions will be provided in [AKZ]. $\square$

4. Interior bounds

sec04

In this section, we establish the final result on the interior regularity of the second order derivatives.

Proof of Theorem 2.3. In the proof, we work in the interior of the domain and thus localize the vorticity equation using a smooth cut-off function. Fix $T \geq 1 \geq t_0 > 0$, and with Ω' as in the statement, consider a smooth function $\eta\colon \mathbb{R}^2 \times [0,\infty) \to [0,1]$ such that $\operatorname{supp} \eta \subseteq \Omega \times (t_0/2, 4T)$ with $\eta = 1$ on $\Omega'' \times [3t_0/4, 2T]$, where Ω'' is an open set such that $\Omega' \Subset \Omega'' \Subset \Omega$, with $\operatorname{dist}(\Omega', \partial\Omega)$ and $\operatorname{dist}(\Omega'', \partial\Omega')$ comparable with a multiplicative constant. In order to prove (2.11), we first claim that the vorticity $\omega = \operatorname{curl} u$ satisfies

$$\|\nabla\omega\|_{L^p([t_0,T]:L^p(\Omega'))} \lesssim T^{1/p}, \tag{4.1}$$

EQ75

where we allow all the constants in the proof to depend on t_0, p, and $\operatorname{dist}(\Omega', \partial\Omega)$. Since (2.11) and (2.12) for $p = 2$ follow from (2.3), we fix $p > 2$. We allow all constants to depend on p and t_0, where $t_0 \in (0,1]$ should be considered small.

As in [KW2], we introduce the operator

$$R = \partial_1 (I - \Delta)^{-1}$$

(EQ76)

and a change of variable

$$\zeta = \omega\eta - R(\rho\eta). \tag{4.2}$$ EQ77

We shall apply R to functions which are compactly supported in Ω, and we consider such functions extended to $\mathbb{R}^2$ by setting them identically to zero on Ω^c. Recalling the vorticity formulation for (2.1),

$$\omega_t - \Delta\omega + u\cdot\nabla\omega = \partial_1\rho,$$ (EQ78)

we have, as in [KW2], that

$$\begin{aligned}\zeta_t - \Delta\zeta + u\cdot\nabla\zeta = [R, u\cdot\nabla](\rho\eta) - N(\rho\eta) - \rho\partial_1\eta - 2\partial_j(\omega\partial_j\eta)\\ + \omega(\eta_t + \Delta\eta + u\cdot\nabla\eta) - R(\rho(u\cdot\nabla\eta)),\end{aligned} \tag{4.3}$$ EQ79

where

$$N = -((I-\Delta)^{-1}\Delta + I)\partial_1, \tag{4.4}$$ EQ80

which has the property that ∇N is in the Calderón-Zygmund class. The equation (4.3) is obtained by a direct computation from

$$(\omega\eta)_t - \Delta(\omega\eta) + u\cdot\nabla(\omega\eta) = \omega\eta_t + \omega\Delta\eta - 2\partial_j(\omega\partial_j\eta) + \omega u\cdot\nabla\eta + \partial_1(\rho\eta) - \rho\partial_1\eta$$ (EQ69)

and

$$(\rho\eta)_t + u\cdot\nabla(\rho\eta) = \rho u\cdot\nabla\eta$$ (EQ70)

and then using the identity $\partial_1 - \Delta R = -N$. Note that both operators R and N commute with translations (and hence derivatives) and they are smoothing of order one, i.e., they satisfy

$$\|Rf\|_{W^{1,p}}, \|Nf\|_{W^{1,p}} \lesssim \|f\|_{L^p}, \qquad f\in L^p(\mathbb{R}^2), \tag{4.5}$$ EQ64

for $p\in(1,\infty)$, where the constant depends on p; the property (4.5) can be verified by computing the Fourier multiplier symbols corresponding to R and N (or see [KW2]). Since u is divergence free, we may rewrite

$$[R, u_j\partial_j](\rho\eta) = R(u_j\partial_j(\rho\eta)) - u_j\partial_j R(\rho\eta) = \partial_j R(u_j\rho\eta) - u_j\partial_j R(\rho\eta).$$ (EQ81)

To acquire L^p space-time estimates, we rewrite our solution as $\zeta = \zeta^{(1)} + \zeta^{(2)}$, where $\zeta^{(1)}$ satisfies

$$\begin{aligned}&\zeta^{(1)}_t - \Delta\zeta^{(1)} = f\\ &\zeta^{(1)}\big|_{t=0} = 0\end{aligned}$$ (EQ83)

with

$$f = \omega(\eta_t + \Delta\eta + u\cdot\nabla\eta) - R(\rho(u\cdot\nabla\eta)) - N(\rho\eta) - u\cdot\nabla R(\rho\eta) - \rho\partial_1\eta,$$ (EQ84)

while for $\zeta^{(2)}$ we have

$$\begin{aligned}&\zeta^{(2)}_t - \Delta\zeta^{(2)} = \nabla\cdot g\\ &\zeta^{(2)}\big|_{t=0} = 0,\end{aligned}$$ (EQ85)

where

$$g = -u\zeta - 2\omega\nabla\eta + R(u\rho\eta).$$ (EQ86)

Using the $L^pW^{2,p}$ regularity for the nonhomogeneous heat equation and the Gagliardo-Nirenberg inequality, we have

$$\|D\zeta^{(1)}\|_{L^pL^p(\mathbb{R}^2\times(0,\infty))} \lesssim \|D^2\zeta^{(1)}\|_{L^pL^{2p/(p+2)}(\mathbb{R}^2\times(0,\infty))} \lesssim \|f\|_{L^pL^{2p/(p+2)}(\mathbb{R}^2\times(0,\infty))}; \tag{4.6}$$ EQ27

observe that $2p/(p+2) > 1$ since $p > 2$. Similarly, using the $L^p W^{1,p}$ regularity for the nonhomogeneous heat equation in divergence form, we have

$$\|D\zeta^{(2)}\|_{L^pL^p(\mathbb{R}^2\times(0,\infty))} \lesssim \|g\|_{L^pL^p(\mathbb{R}^2\times(0,\infty))}. \tag{4.7}$$ EQ49

For the right-hand side of (4.6), we use (4.5) to obtain

$$\begin{aligned}\|f\|_{L^{2p/(p+2)}} &\lesssim \|\omega\|_{L^2}(\|\eta_t\|_{L^p} + \|\Delta\eta\|_{L^p} + \|u\|_{L^\infty}\|\nabla\eta\|_{L^p}) + \|\rho\|_{L^2}\|u\|_{L^\infty}\|\nabla\eta\|_{L^p} \\ &\quad + \|\rho\|_{L^2}\|\eta\|_{L^p} + \|u\|_{L^\infty}\|\rho\|_{L^2}\|\eta\|_{L^p} + \|\rho\|_{L^2}\|\partial_1\eta\|_{L^p} \\ &\lesssim \|\omega\|_{L^2} + 1 \lesssim 1,\end{aligned} \tag{4.8}$$ EQ88

for every $t \geq 0$, where the domains are understood to be $\mathbb{R}^2$. For the right-hand side in (4.7), we estimate

$$\begin{aligned}\|g\|_{L^p} &\lesssim \|u\|_{L^{2p}}\|\zeta\|_{L^{2p}} + \|\omega\|_{L^{2p}}\|\nabla\eta\|_{L^{2p}} + \|u\|_{L^\infty}\|\rho\|_{L^p}\|\eta\|_{L^\infty} \\ &\lesssim \|u\|_{L^{2p}}\|\zeta\|_{L^{2p}} + \|\omega\|_{L^{2p}} + 1\end{aligned} \tag{4.9}$$ EQ90

for every $t \geq 0$, by (4.5). To bound the right-hand side of (4.9), we write

$$\|\zeta\|_{L^q} \lesssim \|\omega\eta\|_{L^q} + \|R(\rho\eta)\|_{L^q} \lesssim \|\omega\eta\|_{L^q} + \|\rho\|_{L^2} \lesssim 1, \qquad q \in [2,\infty). \tag{4.10}$$ EQ92

Therefore, we obtain $\|g\|_{L^p} \lesssim 1$ for all $t \geq 0$. This fact and (4.8) imply by integration that the left-hand sides of (4.6) and (4.7) are bounded by $T^{1/p}$ for $T \geq t_0$, from where

$$\|D\zeta\|_{L^pL^p(\mathbb{R}^2\times(0,\infty))} \lesssim T^{1/p} \tag{4.11}$$ EQ65

and thus

$$\|\nabla(\omega\eta)\|_{L^pL^p(\mathbb{R}^2\times(0,\infty))} \lesssim \|\nabla\zeta\|_{L^pL^p(\mathbb{R}^2\times(0,\infty))} + \|R\nabla(\rho\eta)\|_{L^pL^p(\mathbb{R}^2\times(0,\infty))} \lesssim T^{1/p},$$ (EQ93)

which proves (4.1). The bound (2.11) then follows by a simple application of the interior elliptic estimate connecting u and ω.

The pointwise in time bound in (2.12) follows once we obtain

$$\|\nabla\omega(t)\|_{L^p(\Omega')} \lesssim t^{(p+2)/4p}, \qquad t_0 \leq t \leq T, \tag{4.12}$$ EQ94

with an agreement that all the constants depend on t_0, p, and $\operatorname{dist}(\Omega', \partial\Omega)$, and $\operatorname{dist}(\Omega'', \partial\Omega')$; we assume $T \geq \max\{2t_0, 1\}$ that is arbitrary. To prove (4.12), we begin by introducing a second smooth cut-off function $\phi\colon \mathbb{R}^2 \times [0,\infty) \to [0,1]$ for which $\operatorname{supp}\phi \subseteq \Omega'' \times (t_0, 2T)$ with $\phi = 1$ on $\Omega' \times [3t_0/4, T]$, Denote

$$\tilde{\zeta} = \zeta\phi.$$ (EQ96)

Using (4.3), we find that

$$\begin{aligned}&\tilde{\zeta}_t - \Delta\tilde{\zeta} + u\cdot\nabla\tilde{\zeta} \\ &\quad = \big([R, u\cdot\nabla](\rho\eta) - N(\rho\eta)\big)\phi - R(\rho(u\cdot\nabla\eta))\phi - 2\nabla\zeta\cdot\nabla\phi + \zeta(\phi_t - \Delta\phi + u\cdot\nabla\phi);\end{aligned} \tag{4.13}$$ EQ97

note that the terms in (4.3) containing derivatives of η vanish after multiplication with ϕ, except for the term involving R, which is a non-local operator. The main reason for introducing the second cut-off function ϕ is that ζ does not vanish on the boundary $\partial\Omega$ due to nonlocality of R; see the definition (4.2). In order to estimate $\nabla\tilde{\zeta}$, we apply ∂_k to (4.13) for $k = 1,\ 2$, multiply by $|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta}$, integrate, and sum in k to acquire

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\sum_k \|\partial_k\tilde{\zeta}\|_{L^{2p}}^{2p} - \sum_k\int \Delta\partial_k\tilde{\zeta}|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta}\\
&\quad = -\sum_k\int \partial_k(u_j\partial_j\tilde{\zeta})|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} + \sum_k\int \partial_k(\phi[R,u\cdot\nabla](\rho\eta))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta}\\
&\qquad - \sum_k\int \partial_k(\phi N(\rho\eta))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} - \sum_k\int \partial_k(\phi R(\rho(u\cdot\nabla\eta)))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta}\\
&\qquad - 2\sum_k\int \partial_k(\partial_j\zeta\partial_j\phi)|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta}\\
&\qquad + \sum_k\int \partial_k(\zeta(\phi_t - \Delta\phi + u\cdot\nabla\phi))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta}.
\end{aligned}\tag{4.14}$$
EQ98

The second term on the left-hand side of (4.14) is estimated as

$$-\sum_k\int \Delta\partial_k\tilde{\zeta}|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} = \frac{2p-1}{p^2}\sum_k\int \partial_j(|\partial_k\tilde{\zeta}|^p)\partial_j(|\partial_k\tilde{\zeta}|^p) \geq \frac{1}{p}\bar{D},\tag{4.15}$$
EQ99

where we denoted $\bar{D} = \sum_k \|\nabla(|\partial_k\tilde{\zeta}|^p)\|_{L^2}^2$. For the first term on the right-hand side of (4.14), we use the incompressibility of u to determine that

$$\begin{aligned}
-\sum_k\int \partial_k(u_j\partial_j\tilde{\zeta})|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} &= -\sum_k\int \partial_k u_j\partial_j\tilde{\zeta}|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta}\\
&\lesssim \|\nabla u\|_{L^2}\|\nabla\tilde{\zeta}\|_{L^{4p}}^{2p} \lesssim o(1)\|\nabla\tilde{\zeta}\|_{L^{4p}}^{2p},
\end{aligned}\tag{4.16}$$
EQ100

where $o(1)$ denotes a function which is bounded on $[0,\infty)$ and converges to 0 as $t\to\infty$. Applying the estimate

$$\|\partial_k\tilde{\zeta}\|_{L^{4p}}^{2p} = \||\partial_k\tilde{\zeta}|^p\|_{L^4}^2 \lesssim \||\partial_k\tilde{\zeta}|^p\|_{L^2}\|\nabla(|\partial_k\tilde{\zeta}|^p)\|_{L^2} \lesssim \bar{D}^{1/2}\|\partial_k\tilde{\zeta}\|_{L^{2p}}^p$$
(EQ101)

for $k = 1, 2$ in (4.16), we obtain

$$-\sum_k\int \partial_k(u_j\partial_j\tilde{\zeta})|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} \leq o(1)\bar{D}^{1/2}\sum_k\|\partial_k\tilde{\zeta}\|_{L^{2p}}^p \leq \frac{\bar{D}}{8} + o(1)\|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p}.\tag{4.17}$$
EQ102

For the second term on the right-hand side of (4.14), we use integration by parts and write

$$\begin{aligned}
&\sum_k\int \partial_k(\phi[R,u\cdot\nabla](\rho\eta))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} = -(2p-1)\sum_k\int \phi[R,u\cdot\nabla](\rho\eta)|\partial_k\tilde{\zeta}|^{2p-2}\partial_{kk}\tilde{\zeta}\\
&\quad = -\frac{2p-1}{p}\sum_k\int \phi[R,u\cdot\nabla](\rho\eta)|\partial_k\tilde{\zeta}|^{p-2}\partial_k\tilde{\zeta}\partial_k(|\partial_k\tilde{\zeta}|^p)\\
&\quad \lesssim \|\phi[R,u\cdot\nabla](\rho\eta)\|_{L^{2p}}\sum_k\||\partial_k\tilde{\zeta}|^{p-1}\|_{L^{2p/(p-1)}}\|\nabla(|\partial_k\tilde{\zeta}|^p)\|_{L^2}\\
&\quad \lesssim \bar{D}^{1/2}\|\phi[R,u\cdot\nabla](\rho\eta)\|_{L^{2p}}\|\nabla\tilde{\zeta}\|_{L^{2p}}^{p-1}.
\end{aligned}\tag{4.18}$$
EQ103

For the second factor in the last expression, we have

$$\begin{aligned}
\|\phi[R,u\cdot\nabla](\rho\eta)\|_{L^{2p}} &\lesssim \|\phi\|_{L^\infty}\|R(u_j\partial_j(\rho\eta)) - u_j\partial_jR(\rho\eta)\|_{L^{2p}}\\
&\lesssim \|\partial_jR(u_j(\rho\eta))\|_{L^{2p}} + \|u\|_{L^\infty}\|\partial_jR(\rho\eta)\|_{L^{2p}} \lesssim \|\rho u\eta\|_{L^{2p}} + \|\rho\eta\|_{L^{2p}} \lesssim 1,
\end{aligned}\tag{4.19}$$
EQ104

using the incompressibility of u and

$$\|\rho(t)\|_{L^{2p}} \lesssim 1, \tag{4.20}$$ EQ89

which follows from $\|\rho_0\|_{L^{2p}} \lesssim \|\rho_0\|_{H^1} \lesssim 1$ and the L^p conservation for ρ. (Recall that all constants depend on p.) Thus, by (4.18)–(4.19), we have

$$\sum_k \int \partial_k(\phi[R, u\cdot\nabla](\rho\eta))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} \le C\bar{D}^{1/2}\|\nabla\tilde{\zeta}\|_{L^{2p}}^{p-1} \le \frac{\bar{D}}{8} + C\|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-2}. \tag{4.21}$$ EQ105

For the third term on the right-hand side of (4.14), we obtain

$$\begin{aligned}
-\sum_k \int \partial_k(\phi N(\rho\eta))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} &\lesssim \sum_k \|\partial_k(\phi N(\rho\eta))\|_{L^{2p}}\||\partial_k\tilde{\zeta}|^{2p-1}\|_{L^{2p/(2p-1)}} \\
&\lesssim (\|\nabla\phi\|_{L^\infty}\|N(\rho\eta)\|_{L^{2p}} + \|\phi\|_{L^\infty}\|\nabla N(\rho\eta)\|_{L^{2p}})\|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-1} \lesssim \|\rho\eta\|_{L^{2p}}\|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-1} \lesssim \|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-1}.
\end{aligned}$$ (EQ106)

For the fourth term on the right-hand side of (4.14), we observe that

$$\begin{aligned}
-\sum_k \int \partial_k(\phi R(\rho(u\cdot\nabla\eta)))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} &\lesssim \|\nabla(\phi R(\rho(u\cdot\nabla\eta)))\|_{L^{2p}}\||\nabla\tilde{\zeta}|^{2p-1}\|_{L^{2p/(2p-1)}} \\
&\lesssim (\|\phi\|_{L^\infty}\|\nabla R(\rho(u\cdot\nabla\eta))\|_{L^{2p}} + \|\nabla\phi\|_{L^\infty}\|R(\rho(u\cdot\nabla\eta))\|_{L^{2p}})\|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-1} \\
&\lesssim \|\rho\|_{L^{2p}}\|u\|_{L^\infty}\|\nabla\eta\|_{L^\infty}\|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-1} \lesssim \|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-1},
\end{aligned}$$ (EQ107)

where we used (4.20). For the fifth term on the right-hand side of (4.14), we determine that

$$\begin{aligned}
-2\sum_k \int \partial_k(\partial_j\zeta\partial_j\phi)|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} &= \frac{2(2p-1)}{p}\sum_k \int \partial_j\zeta\partial_j\phi|\partial_k\tilde{\zeta}|^{p-2}\partial_k\tilde{\zeta}\partial_k(|\partial_k\tilde{\zeta}|^p) \\
&\lesssim \|\partial_j\zeta\partial_j\phi\|_{L^4}\sum_k \||\partial_k\tilde{\zeta}|^{p-1}\|_{L^4}\|\nabla(|\partial_k\tilde{\zeta}|^p)\|_{L^2} \lesssim \bar{D}^{1/2}\|\nabla\zeta\|_{L^4}\||\nabla\tilde{\zeta}|^p\|_{L^{4(p-1)/p}}^{(p-1)/p}.
\end{aligned}$$ (EQ108)

By the Gagliardo-Nirenberg inequality, we have for the last factor

$$\begin{aligned}
\||\partial_k\tilde{\zeta}|^p\|_{L^{4(p-1)/p}}^{(p-1)/p} &\lesssim \left(\||\partial_k\tilde{\zeta}|^p\|_{L^2}^{p/(2p-2)}\|\nabla(|\partial_k\tilde{\zeta}|^p)\|_{L^2}^{(p-2)/(2p-2)}\right)^{(p-1)/p} \\
&\lesssim \||\partial_k\tilde{\zeta}|^p\|_{L^2}^{1/2}\|\nabla(|\partial_k\tilde{\zeta}|^p)\|_{L^2}^{(p-2)/2p} \lesssim \bar{D}^{(p-2)/4p}\|\partial_k\tilde{\zeta}\|_{L^{2p}}^{p/2},
\end{aligned}$$ (EQ109)

for $k = 1, 2$. Therefore, by Young's inequality, we conclude that

$$-2\sum_k \int \partial_k(\partial_j\zeta\partial_j\phi)|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} \lesssim \bar{D}^{(3p-2)/4p}|\nabla\zeta|_{L^4}|\nabla\tilde{\zeta}|_{L^{2p}}^{p/2} \le \frac{\bar{D}}{8} + C|\nabla\zeta|_{L^4}^{4p/(p+2)}|\nabla_k\tilde{\zeta}|_{L^{2p}}^{2p^2/(p+2)}.$$ (EQ110)

For the final term of (4.14), we integrate by parts and obtain

$$\begin{aligned}
&\sum_k \int \partial_k(\zeta(\phi_t - \Delta\phi + u\cdot\nabla\phi))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} \\
&\quad = -\frac{2p-1}{p}\sum_k \int \zeta(\phi_t - \Delta\phi + u\cdot\nabla\phi)|\partial_k\tilde{\zeta}|^{p-2}\partial_k\tilde{\zeta}\partial_k(|\partial_k\tilde{\zeta}|^p) \\
&\quad \lesssim \|\zeta\|_{L^{2p}}\|\phi_t - \Delta\phi + u\cdot\nabla\phi\|_{L^\infty}\sum_k \||\partial_k\tilde{\zeta}|^{p-1}\|_{L^{2p/(p-1)}}\|\nabla(|\partial_k\tilde{\zeta}|^p)\|_{L^2} \lesssim \bar{D}^{1/2}\|\nabla\tilde{\zeta}\|_{L^{2p}}^{p-1},
\end{aligned}$$ (EQ111)

using (2.3) and (4.10). Therefore, we have

$$\sum_k \int \partial_k(\zeta(\phi_t - \Delta\phi + u\cdot\nabla\phi))|\partial_k\tilde{\zeta}|^{2p-2}\partial_k\tilde{\zeta} \le \frac{\bar{D}}{8} + C\|\nabla\tilde{\zeta}\|_{L^{2p}}^{2p-2}. \tag{4.22}$$ EQ112

Introducing

$$\psi(t) = \sum_k \int |\partial_k \tilde{\zeta}|^{2p},$$

we may rewrite (4.14) by applying the above bounds as

$$(1+\psi)' + \frac{\bar{D}}{2} \lesssim o(1)(1+\psi) + (1+\psi)^{(p-1)/p} + (1+\psi)^{(2p-1)/2p} + \|\nabla\zeta\|_{L^4}^{4p/(p+2)}(1+\psi)^{p/(p+2)}. \tag{4.23}$$

It may seem that the first term in (4.23) causes an exponential increase of ψ, but importantly we have the property

$$\int_0^t (1+\psi) \lesssim t + \|\nabla\zeta\|_{L^{2p}([0,t];L^{2p})}^{2p} \lesssim 1+t, \qquad t \geq 0, \tag{4.24}$$

where we used (4.11) in the second step. Now we show that the inequality (4.24) implies that the growth of $1+\psi$ is algebraic. We divide the inequality (4.23) by $(1+\psi)^{p/(p+2)}$, obtaining

$$((1+\psi)^{2/(p+2)})' \lesssim o(1)(1+\psi)^{2/(p+2)} + (1+\psi)^{(p^2-2)/p(p+2)} + (1+\psi)^{(3p-2)/2p(p+2)} + \|\nabla\zeta\|_{L^4}^{4p/(p+2)},$$

which upon integration and applying Jensen's (or Hölder's) inequality yields for $t \geq 0$,

$$\begin{aligned}
&(1+\psi)^{2/(p+2)} \\
&\quad\lesssim 1 + o(1)\int_0^t (1+\psi)^{2/(p+2)} + \int_0^t (1+\psi)^{(p^2-2)/p(p+2)} + \int_0^t (1+\psi)^{(3p-2)/2p(p+2)} \\
&\qquad + \int_0^t \|\nabla\zeta\|_{L^4}^{4p/(p+2)} \\
&\quad\lesssim 1 + o(1)t^{1-2/(p+2)}\left(\int_0^t (1+\psi)\right)^{2/(p+2)} + t^{1-(p^2-2)/p(p+2)}\left(\int_0^t (1+\psi)\right)^{(p^2-2)/p(p+2)} \\
&\qquad + t^{1-(3p-2)/2p(p+2)}\left(\int_0^t (1+\psi)\right)^{(3p-2)/2p(p+2)} \\
&\qquad + \left(\int_0^t \|\nabla\zeta\|_{L^4}^4\right)^{p/(p+2)} t^{1-p/(p+2)},
\end{aligned} \tag{4.25}$$

where we also used $\psi(0) = 0$ since ϕ vanishes in a neighborhood of $\{t = 0\}$. Therefore, recalling (4.24), we have for $t \geq t_0$ the inequality

$$(1+\psi)^{2/(p+2)} \lesssim t,$$

where we used $\int_\delta^t \|\nabla\zeta\|_{L^4}^4 \lesssim_\delta t$ for $\delta > 0$ on the last term in (4.25). Raising the resulting inequality to $(p+2)/2$, we obtain

$$1 + \psi \leq C_p t^{(p+2)/2}.$$

By the support properties of ϕ and η, we get for $p \geq 1$

$$\|\nabla\omega\|_{L^{2p}(\Omega')} \lesssim \|\nabla\zeta\|_{L^{2p}(\Omega')} + p^{3/2} \lesssim \|\nabla\tilde{\zeta}\|_{L^{2p}} + 1 \lesssim \psi^{1/2p} + 1 \lesssim t^{(p+2)/4p}, \tag{4.26}$$

for $t \geq t_0$ concluding the proof. $\square$

Appendix A. Uniform Gronwall inequalities

In the appendix, we state and prove two Gronwall inequalities needed in the proof of Theorem 2.1. The following lemma is used to show (3.8).

Lemma A.1. *Assume that $x, y\colon [0,\infty) \to [0,\infty)$ are measurable functions with x differentiable, which satisfy*

$$\dot{x} + y \le C_0(x^2 + 1) \tag{A.1}$$

and

$$x \le C_0 y, \tag{A.2}$$

for some positive constant C_0. If

$$\int_0^\infty x(s)\,ds < \infty, \tag{A.3}$$

then

$$\lim_{t\to\infty} x(t) = 0. \tag{A.4}$$

Moreover,

$$\int_t^{t+a} y(s)\,ds \le C, \qquad t \ge 0 \tag{A.5}$$

for every a, where the constant depends on a and C_0 and

$$\limsup_{t\to\infty} \int_t^{t+a} y(s)\,ds \le Ca \tag{A.6}$$

for every $a > 0$, where the constant in depends on C_0.

Proof of Lemma A.1. In the proof, we allow all constants to depend on C_0. Let $\epsilon \in (0,1]$, and denote $b = \sqrt{\epsilon}$. Based on (A.3), there exists $t_0 > 0$ such that

$$\int_t^{t+2b} x(s)\,ds \le \epsilon, \qquad t \ge t_0. \tag{A.7}$$

Integrating the inequality $\dot{x} \le C_0(x^2+1)$ and using (A.7), we obtain

$$x(t_2) \le e^{C_0\epsilon}(x(t_1) + Cb) \lesssim x(t_1) + b \tag{A.8}$$

for all t_1 and t_2 such that $0 \le t_1 \le t_2 \le t_1 + 2b$. By (A.7), for every $t \ge t_0$, there exists $\tilde{t} \in [t, t+b]$ such that

$$x(\tilde{t}) \lesssim \frac{\epsilon}{b},$$

and thus applying (A.8) with $t_1 = \tilde{t}$ leads to

$$x(t_2) \lesssim \frac{\epsilon}{b} + b \lesssim \sqrt{\epsilon}, \qquad \tilde{t} \le t_2 \le t_1 + 2b, \tag{A.9}$$

where we used $b = \sqrt{\epsilon}$ in the last step. The inequality (A.9) holds for all $t_2 \ge t_0 + b$, and since $\epsilon > 0$ is arbitrarily small, (A.4) follows. The inequalities (A.5) and (A.6) are obtained by integrating $y \lesssim x^2 + 1$ and, for (A.6), we also use (A.4). $\square$

The next Gronwall-type lemma is needed to establish (3.19) and (3.20), which are necessary for the proofs of (2.3) and (2.7).

L02 **Lemma A.2.** *Assume that $x, y\colon [0,\infty) \to [0,\infty)$ are measurable functions with x differentiable, which satisfy*

$$\dot{x} + y \le \phi(t)(x+1) \tag{A.10}$$ EQ143

and

$$x \le C_0 y, \tag{A.11}$$ EQ142

where $\phi\colon [0,\infty) \to [0,\infty)$ is such that $\phi(t) \le C_0$ for $t \in [0,\infty)$ and $\phi(t) \to 0$, as $t \to \infty$. If also $x(0) \le C_0$, then

$$x(t) \lesssim 1, \qquad t \in [0,\infty), \tag{A.12}$$ EQ53

where the constant in (A.12) depends on C_0, and

$$\lim_{t\to\infty} x(t) = 0. \tag{A.13}$$ EQ145

Moreover,

$$\int_t^{t+1} y(s)\,ds \le C, \qquad t \ge 0, \tag{A.14}$$ EQ132

where C depends on C_0 and

$$\limsup_{t\to\infty} \int_t^{t+a} y(s)\,ds = 0, \tag{A.15}$$ EQ146

for every $a > 0$.

Proof of Lemma A.2. First, by the boundedness of ϕ, we have

$$x(t) \lesssim 1, \qquad t \in [0,T]$$ (EQ161)

and then also

$$\int_0^T y(s)\,ds \lesssim 1 \tag{A.16}$$ EQ136

for every $T > 0$, where the constant depends on T. Next, there exists $t_0 > 0$ such that

$$\dot{x} + \frac{1}{2} y \le \phi(t), \qquad t \ge t_0,$$ (EQ144)

which is obtained by choosing t_0 so large that the term containing x on the right-hand side of (A.10) is absorbed in the half of the second term on the left-hand side; see (A.11).

Let $\epsilon > 0$. Then there exists $t_1 \ge t_0$ such that

$$\dot{x} + \frac{1}{C_1} x \le \frac{\epsilon}{2}, \qquad t \ge t_1,$$ (EQ149)

where C_1 depends on C_0. This shows that as long as $x \ge C_1\epsilon$, we have $\dot{x} + (1/2C_1)x \le 0$, implying an exponential decay of x. Therefore, by increasing t_1, we can assume that

$$x(t) \le C_1 \epsilon, \qquad t \ge t_1. \tag{A.17}$$ EQ147

Since $\epsilon > 0$ was arbitrary, we obtain (A.13). To prove (A.15), note that we may assume

$$\dot{x} + \frac{1}{2} y \le \epsilon, \qquad t \ge t_1, \tag{A.18}$$ EQ150

by increasing t_1 if necessary. Integrating (A.18) between t and $t+a$, where $a>0$ is fixed as in the statement, we get

$$\int_t^{t+a} y(s)\,ds \lesssim x(t) + \epsilon a \lesssim \epsilon(1+a), \qquad t \geq t_1, \tag{EQ151}$$

where we used (A.17) in the last step. Since $\epsilon > 0$ was arbitrary, we obtain (A.14) and (A.15). □

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