

COMPACT QUANTUM METRIC SPACES FROM FREE GRAPH ALGEBRAS

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ABSTRACT. Starting with a vertex-weighted pointed graph (Γ, μ, v_0) , we form the free loop algebra $\mathcal{S}_0$ defined in Hartglass-Penneys' article on canonical C^* -algebras associated to a planar algebra. Under mild conditions, $\mathcal{S}_0$ is a non-nuclear simple C^* -algebra with unique tracial state. There is a canonical polynomial subalgebra $A \subset \mathcal{S}_0$ together with a Dirac number operator N such that $(A, L^2 A, N)$ is a spectral triple. We prove the Haagerup-type bound of Ozawa-Rieffel to verify $(\mathcal{S}_0, A, N)$ yields a compact quantum metric space in the sense of Rieffel.

We give a weighted analog of Benjamini-Schramm convergence for vertex-weighted pointed graphs. As our C^* -algebras are non-nuclear, we adjust the Lip-norm coming from N to utilize the finite dimensional filtration of A . We then prove that convergence of vertex-weighted pointed graphs leads to quantum Gromov-Hausdorff convergence of the associated adjusted compact quantum metric spaces.

As an application, we apply our construction to the Guionnet-Jones-Shlyakhtenko (GJS) C^* -algebra associated to a planar algebra. We conclude that the compact quantum metric spaces coming from the GJS C^* -algebras of many infinite families of planar algebras converge in quantum Gromov-Hausdorff distance.

1. INTRODUCTION

In Connes' noncommutative geometry [Con89, Con94], the notion of a spectral triple is an analog of a space of smooth functions on a non-commutative manifold. In [Rie98, Rie99], Rieffel initiated the study of noncommutative *metric* geometry via the notion of a compact quantum metric space. He then introduced *quantum Gromov-Hausdorff distance* as a noncommutative analogue of Gromov-Hausdorff distance to provide a framework for establishing convergence of certain spaces arising in the operator algebra and high-energy physics literature [Rie04a, Rie04b].

To the best of our knowledge, all results proving convergence in quantum Gromov-Hausdorff distance do so for sequences of *nuclear* C^* -algebras, where finite-dimensional approximations are crucial in demonstrating convergence [Rie04a, KL09, Agu19, Lat17, JRZ18, KK21]. In this article, we prove a result about quantum Gromov-Hausdorff convergence for compact quantum metric spaces associated to *non-nuclear* free graph algebras produced from vertex-weighted pointed graphs.

Given an unoriented connected graph $\Gamma = (V, E)$ with an arbitrary vertex weighting $\mu : V \rightarrow (0, \infty)$, we replace each edge $\varepsilon \in E$ between two distinct vertices by two oriented edges $\epsilon, \epsilon^{\text{op}} \in \vec{E}$ in opposite directions, and we replace each loop by a single oriented loop to obtain a strongly connected directed graph $\vec{\Gamma} = (V, \vec{E})$ which inherits the same weighting μ . One forms the Toeplitz-Cuntz-Krieger graph algebra $\mathcal{T}(\vec{\Gamma})$ [FR99] with generators $\ell(\epsilon)$ for $\epsilon \in \vec{E}$, and the *free graph algebra* [HP17] is given by

$$\mathcal{S}(\Gamma, \mu) = C^* \left(C_0(V) \cup \left\{ a_\epsilon \ell(\epsilon) + a_\epsilon^{-1} \ell(\epsilon^{\text{op}}) \mid \epsilon \in \vec{E} \right\} \right) \subset \mathcal{T}(\vec{\Gamma}).$$

Here, each $a_\epsilon \in (0, \infty)$ depends on the weighting of the source and target of ϵ , which is chosen so that $\mathcal{S}(\Gamma, \mu)$ has a semifinite trace Tr . By [Har17], $\mathcal{S}(\Gamma, \mu)$ is simple exactly when

$$\mu(v) < \sum_{\substack{\epsilon \in \vec{E} \\ s(\epsilon)=v}} \mu(t(\epsilon)); \tag{1}$$

we assume this condition in the sequel.

Now there are canonical projections $p_v \in \mathcal{S}(\Gamma, \mu)$ for the vertices $v \in V$, and by simplicity [HP17, HP14, Har17], each compression $p_v \mathcal{S}(\Gamma, \mu) p_v$ is Morita equivalent to $\mathcal{S}(\Gamma, \mu)$. We thus consider *pointed* weighted graphs, which are equipped with a basepoint $v_0 \in V$ such that $\mu(v_0) = 1$. We consider the *free loop algebra* $\mathcal{S}_0 = \mathcal{S}_0(\Gamma, \mu) := p_{v_0} \mathcal{S}(\Gamma, \mu) p_{v_0}$, which can be described as generated by loops on Γ based at v_0 . Under condition (1), $\mathcal{S}_0(\Gamma, \mu)$ also has unique trace [Har17].

The loop algebra $\mathcal{S}_0$ has a dense $*$ -subalgebra A of finite linear combinations of loops, which acts by bounded operators on $L^2(A, \text{tr}_0) \cong L^2(\mathcal{S}_0, \text{tr}_0)$. Moreover, A is *filtered* by finite dimensional $*$ -closed subspaces A_n of linear combinations of loops of length at most n , which satisfy $A_m \cdot A_n \subset A_{m+n}$ and $A_0 = \mathbb{C}1_A$. In this situation, by [OR05, Lemma 1.1], the formula

$$N = \sum_{n \geq 0} n \text{Proj}_{A_n \ominus A_{n-1}},$$

defines a *Dirac number operator* which has bounded commutator with elements of A . Thus $(A, L^2 A, N)$ is a *spectral triple* in the sense of Connes [Con94]. We prove the Haagerup-type inequality of [OR05, Theorem 1.2], which gives the following theorem.

Theorem A. *The Dirac number operator N induces a Lip-norm L on A , making $(\mathcal{S}_0, A, L)$ a compact quantum metric space in the sense of Rieffel [Rie04a].*

Thus given a connected, vertex-weighted pointed graph (Γ, μ, v_0) , we get a canonical compact quantum metric space. Given a sequence of connected vertex-weighted graphs (Γ_n, μ_n) , we say it converges *locally uniformly* to a limit (Γ, ν) if essentially on every ball of radius R about v , the graphs Γ_n eventually coincide with Γ , and the weights converge pointwise. This is a weighted analog of Benjamini-Schramm convergence [BS01]. (See Definition 3.9 for the precise definition.) With this definition in hand, we can ask whether the associated compact quantum metric spaces $(\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), L_n)$ converge in quantum Gromov-Hausdorff distance to $(\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), L)$.

Unfortunately, we were unable to solve this question due to two main problems. First, projecting an element in A_n onto A_{n-1} can increase the operator norm, similar to how truncating a Fourier series can increase the sup norm. Second, these algebras are non-nuclear, so we are missing the finite dimensional approximations which were essential to the results [Rie04a, KL09, Agu19, Lat17, JRZ18, KK21].

In analogous situations [Rie99, Agu16], one replaces the Lip-norm L with another Lip-norm $\mathcal{L}$ produced by a Minkowski functional. In our setup, we choose $\mathcal{L}$ so that it agrees with L on the spaces $A_n \ominus A_{n-1}$ of homogeneous loops, i.e., spans of loops of the same length n . While this produces a less canonical compact quantum metric space, these adjusted quantum metrics take advantage of the intrinsic finite-dimensional spaces $A_n \ominus A_{n-1}$ of homogeneous loops, which replace the finite dimensional approximations in the nuclear setting. In §3.3 below, we are able to prove that these compact quantum metric spaces converge in quantum Gromov-Hausdorff distance to the desired limit.

Theorem B. *If the sequence of vertex-weighted pointed graphs (Γ_k, μ_k, v_0^k) converges locally uniformly to (Γ, μ, v_0) , then the induced compact quantum metric spaces $(\mathcal{S}_0(\Gamma_k, \mu_k), A(\Gamma_k, \mu_k), \mathcal{L}_k)$ converge in quantum Gromov-Hausdorff distance to $(\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L})$.*

Application to subfactor theory. The original motivation in our two articles [HP17, HP14] was to develop a connection between subfactor theory and C^* -algebras with a view toward connections to Connes' non-commutative geometry [Con94]. The standard invariant of a finite index subfactor forms a shaded subfactor planar algebra [Jon21]. Here, we work with unshaded unitary factor planar algebras, which correspond to symmetrically self-dual bifinite bimodules over some factor [BHP12, Pen20]. The more categorically minded reader may choose to work directly with a unitary tensor category as in [HHP20].

A special application of the setup in this article is when:

- Γ is the principal graph of an unshaded unitary factor planar algebra $\mathcal{P}_\bullet$,
- μ is a quantum dimension vertex-weighting which satisfies the Frobenius-Perron condition, and
- $v_0 = \star$, the distinguished vertex corresponding to the empty diagram/monoidal unit object.

In this case, the cutdown $\mathcal{S}_0$ of $\mathcal{S}(\Gamma, \mu)$ at $v_0 = \star$ is isomorphic to the Guionnet-Jones-Shlyakhtenko (GJS) C^* -algebra [HP17, HP14]. This algebra is the C^* -completion of the graded algebra Gr_0 arising from their diagrammatic reproof [GJS10] of Popa's celebrated subfactor reconstruction theorem [Pop95].

When $\mathcal{P}_\bullet = \mathcal{NC}_\bullet$, the factor planar algebra of non-commuting polynomials on self-adjoint variables $X_1, \dots, X_n$, Gr_0 is exactly the algebra of non-commutative polynomials, and $\mathcal{S}_0$ is Voiculescu's reduced C^* -algebra generated by n free semi-circular elements. Thus we may view $\text{Gr}_0 \subset \mathcal{S}_0$ as a smooth subalgebra of polynomials inside the algebra of non-commutative continuous functions. We are thus in a position to study Connes' non-commutative geometry via Dirac operators and spectral triples [Con94].

In subfactor theory, there are many examples of local uniform graph convergence. For instance, we have examples coming from quantum groups at roots of unity [Jon83, Wen88, GdlHJ89, Wen90, Xu98], continuous families of subfactors [BNP07], and from composites at a fixed index [BH96, Liu15]. A particular application to subfactor theory is Corollary C below, which holds in much more generality than stated. In the corollary below, (A_n, μ_n) denotes the pointed vertex-weighted Coxeter-Dynkin diagram

$$\begin{array}{c} [1] \quad [2] \quad [3] \quad \dots \quad [n-1] \quad [n] \\ \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ \star \end{array} \quad \text{where} \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad q = \exp\left(\frac{2\pi i}{2(n+1)}\right).$$

This is the principal graph of the Temperley-Lieb-Jones (TLJ) subfactor with index $4 \cos^2(\pi/(n+1)) = (q + q^{-1})^2$ [Jon83].

Corollary C. *For a fixed n , the GJS C^* -algebra $\mathcal{S}_0(A_n, \mu_n)$ of the TLJ (sub)factor planar algebra gives a compact quantum metric space when equipped with the Lip-norm L_n from the Dirac number operator. Adjusting our Lip-norm to $\mathcal{L}_n$ as in Theorem B, the associated compact quantum metric spaces converge in quantum Gromov-Hausdorff distance to the adjusted compact quantum metric space of the GJS C^* -algebra of TLJ at $q = 1$.*

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2. BACKGROUND

2.1. Compact quantum metric spaces. We rapidly recall the notions of Gromov-Hausdorff distance, order unit space, compact quantum metric space, and quantum Gromov-Hausdorff distance from [Rie04a].

Definition 2.1. Suppose we have two compact subsets X, Y of a metric space (Z, ρ) . The Hausdorff distance between X and Y is given by

$$\text{dist}_H(X, Y) := \inf \{r > 0 \mid X \subset N_r(Y) \text{ and } Y \subset N_r(X)\},$$

where for $A \subset Z$, $N_r(A)$ is the r -neighborhood of A :

$$N_r(A) := \{z \in Z \mid \text{there is an } a \in A \text{ with } \rho(z, a) < r\}.$$

Definition 2.2. Now suppose (X, ρ_X) and (Y, ρ_Y) are independent compact metric spaces. Let $X \amalg Y$ be the disjoint union of X and Y , and let $\mathcal{M}(\rho_X, \rho_Y)$ be the set of all metrics ρ on $X \amalg Y$ such that

- ρ induces the disjoint union topology on $X \amalg Y$, and
- $\rho|_X = \rho_X$ and $\rho|_Y = \rho_Y$.

The Gromov-Hausdorff distance between (X, ρ_X) and (Y, ρ_Y) is given by

$$\text{dist}_{\text{GH}}(X, Y) = \inf \{ \text{dist}_H^\rho(X, Y) \mid X, Y \subset (X \amalg Y, \rho) \text{ and } \rho \in \mathcal{M}(\rho_X, \rho_Y) \}.$$

Definition 2.3. An order unit space (V, e) is a real vector space V together with a partial order $\leq$ with an element e called the order unit which satisfies

- (order unit) For every $v \in V$, there is an $r > 0$ such that $v \leq re$.
- (Archimedean property) If $v \leq re$ for all $r > 0$, then $v \leq 0$.

An order unit space has a norm, which is given by $\|v\| = \inf \{r > 0 \mid -re \leq v \leq re\}$.

Example 2.4. Suppose $\mathcal{A}$ is a unital C^* -algebra. Then the self-adjoint elements $\mathcal{A}_{\text{sa}}$ of $\mathcal{A}$ form an order unit space with order unit $1_{\mathcal{A}}$.

Definition 2.5. Suppose (V, e) is an order unit space.

- A state of (V, e) is a continuous linear functional $\mu \in V^*$ such that $\mu(e) = 1 = \|\mu\|$. The space of states on (V, e) is denoted $S(V)$. Given a seminorm L on (V, e) , it induces a $[0, \infty]$ -valued metric on $S(V)$ by

$$\rho_L(\mu, \nu) = \sup \{|\mu(v) - \nu(v)| \mid L(v) \leq 1\}.$$

- A Lip-norm on (V, e) is a seminorm L on V such that
 - (1) $L(v) = 0$ if and only if $v \in \mathbb{R}e$.
 - (2) The topology on $S(V)$ induced by ρ_L is the weak* topology.

Note that (2) above implies ρ_L is a genuine metric on $S(V)$ which takes only finite values.

Definition 2.6. A compact quantum metric space is a triple (V, e, L) where (V, e) is an order unit space and L is a Lip-norm on (V, e) .

The most important example for this article is the case when V is a norm-dense subspaces of $\mathcal{A}_{\text{sa}}$ for a unital C^* -algebra $\mathcal{A}$. In particular, we care about the case that $\mathcal{A}$ is obtained from a *unital pre C^* -algebra*, which consists of a pair (A, ϕ) where A is a unital complex $*$ -algebra and $\phi : A \rightarrow \mathbb{C}$ is a positive linear functional ($\phi(a^*a) \geq 0$) with $\phi(1_A) = 1_{\mathbb{C}}$ such that the left action of A on $L^2(A, \phi)$ is by bounded operators. Given a Lip-norm L on A_{sa} which induces the weak* topology on $S(\mathcal{A})$, we denote the corresponding compact quantum metric space by $(\mathcal{A}, A, L)$.

The following criterion will be useful in determining whether L is a Lip-norm on a unital C^* -algebra $\mathcal{A}$ coming from a pre C^* -algebra (A, ϕ) .

Proposition 2.7 ([OR05, Prop. 1.3]). *Let (A, ϕ) be a unital pre C^* -algebra, and let L be a seminorm on A . Then L is a Lip-norm if and only if*

$$\{a \in A \mid L(a) \leq 1 \text{ and } \phi(a) = 0\}$$

is a norm totally bounded subset of A .

Definition 2.8 ([Rie04a, Sections 3 and 4]). Suppose we have compact quantum metric spaces (V, e_V, L_V) and (W, e_W, L_W) . Then $(V \oplus W, (e_V, e_W))$ is an order unit space. Let $\mathcal{M}(V, W)$ be the set of all Lip-norms L on $V \oplus W$ which induce L_V on (V, e_V) and L_W on (W, e_W) , and let $\pi_V : V \oplus W \rightarrow V$ and $\pi_W : V \oplus W \rightarrow W$ denote the canonical projections whose dual maps $\pi_V^* : (S(V), \rho_{L_V}) \ni \varphi \mapsto \varphi \circ \pi_V \in (S(V \oplus W), \rho_L)$ and $\pi_W^* : (S(W), \rho_{L_W}) \ni \varphi \mapsto \varphi \circ \pi_W \in (S(V \oplus W), \rho_L)$ are isometries (see [Rie04a, Proposition 3.1]). The quantum Gromov-Hausdorff distance between (V, e_V, L_V) and (W, e_W, L_W) is

$$\text{dist}_q(V, W) := \inf \{ \text{dist}_H^{\rho_L}(\pi_V^*(S(V)), \pi_W^*(S(W))) \mid L \in \mathcal{M}(L_V, L_W) \}.$$

Remark 2.9. This quantity actually dominates the Gromov-Hausdorff distance between the state spaces. Indeed, the Gromov-Hausdorff distance dist_{GH} takes into account all possible isometric embeddings into any metric space, not just the particular embeddings into the metric space $S(V \oplus W)$ that define dist_q . As dist_{GH} is defined as an infimum, dist_q must dominate dist_{GH} .

The following lemma will help in providing important estimates later.

Lemma 2.10. *Let (V, e, L) be a compact quantum metric space, and let $W \subseteq V$ be a unital subspace ($e \in W$) such that $(W, e, L|_W)$ is a compact quantum metric space. If $\phi \in S(V)$, then*

$$\text{dist}_q(V, W) \leq 2 \text{dist}_H^{\|\cdot\|}(\{a \in V \mid L(a) \leq 1 \text{ and } \phi(a) = 0\}, \{a \in W \mid L|_W(a) \leq 1 \text{ and } \phi(a) = 0\}).$$

Here, $\text{dist}_H^{\|\cdot\|}$ is the Hausdorff distance in (V, e) with respect to the norm from Definition 2.3.

Proof. By [Lat16, Thm. 6.3], we have that

$$\text{dist}_q(V, W) \leq 2 \text{dist}_H^{\|\cdot\|}(\{a \in V \mid L(a) \leq 1\}, \{a \in W \mid L|_W(a) \leq 1\}).$$

However, by the discussion preceding [Lat16, Def. 3.14], we have that

$$\begin{aligned} & \text{dist}_H^{\|\cdot\|}(\{a \in V \mid L(a) \leq 1\}, \{a \in W \mid L|_W(a) \leq 1\}) \\ & \leq \text{dist}_H^{\|\cdot\|}(\{a \in V \mid L(a) \leq 1 \text{ and } \phi(a) = 0\}, \{a \in W \mid L|_W(a) \leq 1 \text{ and } \phi(a) = 0\}), \end{aligned}$$

which completes the proof. $\square$

2.2. The Ozawa-Rieffel criterion. In [OR05], Ozawa and Rieffel give criteria to determine when a filtered $*$ -algebra with a tracial state gives a compact quantum metric space. We now recall their setup and theorem.

Assumptions 2.11. For this section, A is a unital complex $*$ -algebra equipped with a trace $\text{tr} : A \rightarrow \mathbb{C}$ such that (A, tr) is a pre C^* -algebra. We further assume:

- A is filtered by $*$ -closed finite dimensional subspaces. That is, there are finite dimensional $*$ -closed subspaces $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ whose union is A which satisfy $A_m \cdot A_n \subseteq A_{m+n}$.
- The ground algebra A_0 is trivial, i.e., $A_0 = \mathbb{C}1_A$.
- The left (and right) action(s) of A on (A, tr) is bounded in $\|\cdot\|_2$, and thus extends to an action on $\mathcal{H} = L^2(A, \text{tr})$ by bounded operators.

Under these assumptions, we set $W_n = A_n \ominus A_{n-1}$ which is finite dimensional, and we let P_n be the orthogonal projection from $\mathcal{H}$ onto W_n .

Definition 2.12. The *Dirac number operator* is defined by $N := \sum_{n \geq 0} nP_n$, which is closable with dense domain.

One has the following lemma due to [OR05].

Lemma 2.13 ([OR05, Lemma 1.1]). *For every $a \in A$, $[N, a]$ is densely defined and extends to a bounded operator on $\mathcal{H}$.*

Using this lemma, we define a seminorm L on A by $L(a) = \|[N, a]\|$. Observe that L vanishes exactly on $A_0 = \mathbb{C}1_A$. We set $\mathcal{A} = \overline{A}^{\|\cdot\|}$. A main result of [OR05] is the following theorem:

Theorem 2.14 ([OR05, Theorem 1.2]). *If there exists a $C > 0$ such that for all $m, n, k \in \mathbb{N}$ and $a_k \in W_k$*

$$\|P_m a_k P_n\| \leq C \|a_k\|_2,$$

then $(\mathcal{A}, A, L)$ is a compact quantum metric space.

2.3. Free graph algebras. Let $\Gamma = (V, E)$ be a countable, connected, locally finite, undirected graph, and let $\mu : V \rightarrow \mathbb{R}_{>0}$ be a weighting on the vertices. (The examples in the later part of this article will be principal graphs of (sub)factor planar algebras with a quantum dimension weighting which satisfies the Frobenius-Perron condition.)

We now follow the construction from [HP17], summarized in [Har17, Section 2.1].

Definition 2.15. From our undirected graph Γ , we form a directed graph $\vec{\Gamma} = (V, \vec{E}, s, t)$ as follows.

- (1) For each $e \in E$ which has endpoints $\alpha \neq \beta$ in V , we get two directed edges ϵ and ϵ^{op} in $\vec{E}$ such that

$$s(\epsilon) = t(\epsilon^{\text{op}}) = \alpha \quad \text{and} \quad s(\epsilon^{\text{op}}) = t(\epsilon) = \beta.$$

- (2) For each $e \in E$ which is a loop at the vertex γ , we get one directed edge $\epsilon \in \vec{E}$ with $s(\epsilon) = t(\epsilon) = \gamma$.

Note that $\vec{\Gamma}$ inherits the weighting μ from Γ .

The algebra $C_0(V)$ is the C^* -algebra generated by the indicator functions p_v for $v \in V$ acting on $\ell^2(V)$. The $C_0(V)$ Hilbert bimodule $\mathcal{X}$ is the completion of the space of formal finite $\mathbb{C}$ -linear combinations of edges of $\vec{E}$, under the $C_0(V)$ -valued inner product given by $\langle \epsilon, \epsilon' \rangle = \delta_{\epsilon, \epsilon'} p_{t(\epsilon)}$. The action of $C_0(V)$ on $\mathcal{X}$ is given by $p_\alpha \epsilon = \delta_{\alpha, s(\epsilon)} \epsilon$ and $\epsilon p_\beta = \delta_{\alpha, t(\epsilon)} \epsilon$.

We then form the Pimsner-Fock space

$$\mathcal{F}(\vec{\Gamma}) = C_0(V) \oplus \bigoplus_{n \geq 1} \bigotimes_{C_0(V)}^n \mathcal{X}.$$

The spaces $\bigotimes_{C_0(V)}^n \mathcal{X}$ are spanned by elements of the form $\epsilon_1 \otimes \cdots \otimes \epsilon_n$ such that $\epsilon_1 \cdots \epsilon_n$ is a path in $\vec{\Gamma}$. For each edge $\epsilon \in \vec{E}$, we get bounded creation and annihilation operators on $\mathcal{F}(\vec{\Gamma})$ given by

$$\begin{aligned} \ell(\epsilon) (\epsilon_1 \otimes \cdots \otimes \epsilon_n) &:= \epsilon \otimes \epsilon_1 \otimes \cdots \otimes \epsilon_n \\ \ell(\epsilon)^* (\epsilon_1 \otimes \cdots \otimes \epsilon_n) &:= \langle \epsilon | \epsilon_1 \rangle_{C_0(V)} \epsilon_2 \otimes \cdots \otimes \epsilon_n. \end{aligned}$$

The Pimsner-Toeplitz algebra $\mathcal{T}(\vec{\Gamma})$ is the C^* -algebra generated by the $\ell(\epsilon), \ell(\epsilon)^*$. It should be noted that since the tensor products are balanced over $C_0(V)$, it follows that $\ell(\epsilon) (\epsilon_1 \otimes \cdots \otimes \epsilon_n) = 0$ whenever $t(\epsilon) \neq s(\epsilon_1)$.

Definition 2.16. The free graph algebra $\mathcal{S} = \mathcal{S}(\Gamma, \mu)$ is generated by the edge elements

$$X_\epsilon = a_\epsilon \ell(\epsilon) + a_\epsilon^{-1} \ell(\epsilon^{\text{op}}) \quad \text{where} \quad a_\epsilon = \sqrt[4]{\frac{\mu(s(\epsilon))}{\mu(t(\epsilon))}}.$$

Note that $a_\epsilon^{-1} = a_{\epsilon^{\text{op}}}$. We set $X_e := X_\epsilon + X_{\epsilon^{\text{op}}}$.

We now give the structure of the free graph algebra. There is a conditional expectation $E : \mathcal{S}(\Gamma, \mu) \rightarrow C_0(V)$ given by $E(x) = \sum_{v \in V(\Gamma)} \langle v | x v \rangle_{C_0(V)}$. We have the following lemma.

Lemma 2.17 ([GJS11, HP17]). *The algebras $\mathcal{S}_{e, \mu} = C^*(C_0(V), X_e)$ are free with amalgamation over $C_0(V)$ with respect to the conditional expectation E , i.e.*

$$\mathcal{S}(\Gamma, \mu) = \bigstar_{C_0(V)} (\mathcal{S}_{e, \mu}, E).$$

Furthermore, $\mu \circ E$ defines a (semifinite) trace Tr on $\mathcal{S}$.

One can check that $\text{Tr}(p_v) = \mu(v)$ for all $v \in V$.

3. FREE LOOP ALGEBRAS AND COMPACT QUANTUM METRIC SPACES

3.1. Free loop algebras give compact quantum metric spaces. Let $\mathcal{H}_{\Gamma, \mu} = \mathcal{F}(\vec{\Gamma}) \otimes_{C_0(V)} \ell^2(V(\Gamma), \mu)$. Here, $\ell^2(V(\Gamma), \mu)$ is the Hilbert space spanned by V , whose inner product is given by $\langle v, w \rangle = \delta_{v=w} \mu(v)^2$. Note that paths in $\vec{\Gamma}$ give an orthogonal basis for $\mathcal{H}_{\Gamma, \mu}$, and $\|\epsilon_1 \otimes \cdots \otimes \epsilon_n\|_{\mathcal{H}_{\Gamma, \mu}} = \sqrt{\mu(t(\epsilon_n))}$.

We introduce the following notation.

Notation 3.1. Let Π denote the set of all paths in $\vec{\Gamma}$, and $\sigma = \epsilon_1 \cdots \epsilon_n \in \Pi$. Let $|\sigma| = n$ denote the length of σ . We set:

- $\sigma^{\text{op}} := \epsilon_n^{\text{op}} \cdots \epsilon_1^{\text{op}}$
- $\ell(\sigma) := \ell(\epsilon_1) \cdots \ell(\epsilon_n)$.
- $a_\sigma := \sqrt[4]{\frac{\mu(s(\sigma))}{\mu(t(\sigma))}} = \sqrt[4]{\frac{\mu(s(\epsilon_1))}{\mu(t(\epsilon_n))}}$
- $X_\sigma := X_{\epsilon_1} \cdots X_{\epsilon_n}$.
- $Y_\sigma := \sum_{\sigma = \rho\tau} a_\rho a_\tau^{-1} \ell(\rho) \ell(\tau^{\text{op}})^*$.

Once it is shown in Proposition 3.2 below that $Y_\sigma \in \mathcal{S}(\Gamma, \mu)$, it will follow from faithfulness of the trace that Y_σ is the unique element in $\mathcal{S}(\Gamma, \mu)$ whose right support is under $p_{t(\sigma)}$ (i.e., precomposing with $p_{t(\sigma)}$ does nothing) and satisfies $Y_\sigma \cdot p_{t(\sigma)} = a_\sigma \sigma$. The element Y_σ is known as the *Wick word* of $a_\sigma \sigma$. Observe that $Y_\sigma^* = Y_{\sigma^{\text{op}}}$.

We now perform a change of basis from the X 's to Y 's. These Y 's will be useful later on as they are eigenvectors of the number operator.

Proposition 3.2 (Change of basis). *Suppose σ is a path in $\vec{\Gamma}$ of length n .*

- (1) $Y_\sigma \in \mathcal{S}(\Gamma, \mu)$
- (2) $Y_\sigma = X_\sigma + Q$ where Q is a linear combination of the $X_{\sigma'}$ with $|\sigma'| < n$.
- (3) $X_\sigma = Y_\sigma + P$ where P is a linear combination of the $Y_{\sigma'}$ with $|\sigma'| < n$.

Proof. We will prove this by induction on $|\sigma|$, the length of σ . If $|\sigma| = 1$, then $\sigma = \epsilon$ for some $\epsilon \in \vec{E}$ and it is apparent that $Y_\epsilon = X_\epsilon$.

Given, σ with $|\sigma| > 1$, write $\sigma = \epsilon\tau$ for $\epsilon \in \vec{E}$, and write $\tau = \epsilon'\tau'$ for $\epsilon' \in \vec{E}$. We see that

$$\begin{aligned} X_\epsilon Y_\tau &= (a_\epsilon \ell(\epsilon) + a_{\epsilon^{\text{op}}} \ell(\epsilon^{\text{op}})^*) \sum_{\tau = \tau_1 \tau_2} a_{\tau_1} a_{\tau_2}^{-1} \ell(\tau_1) \ell(\tau_2^{\text{op}})^* \\ &= \sum_{\tau = \tau_1 \tau_2} a_{\epsilon \tau_1} a_{\tau_2} \ell(\epsilon \tau_1) \ell(\tau_2^{\text{op}})^* + \delta_{\epsilon^{\text{op}}, \epsilon'} \sum_{\tau' = \tau'_1 \tau'_2} a_{\tau'_1} a_{(\tau'_2)^{\text{op}}}^{-1} \ell(\tau'_1) \ell((\tau'_2)^{\text{op}}) + a_{\epsilon^{\text{op}}} a_\tau^{-1} \ell(\epsilon^{\text{op}})^* \ell(\tau^{\text{op}})^* \\ &= Y_\sigma + \delta_{\epsilon^{\text{op}}, \epsilon'} Y_{\tau'} \end{aligned}$$

By induction, this proves (1) and (2). (3) follows directly from (2). $\square$

We now work with the following assumption:

Assumption 3.3. Our vertex-weighted graph (Γ, μ) comes with a pointing $v_0 \in V$ with $\mu(v_0) = 1$ which is minimal amongst vertex weights, i.e., $\mu(v_0) \leq \mu(v)$ for all $v \in V$.

Definition 3.4. From the free graph algebra $\mathcal{S} = \mathcal{S}(\Gamma, \mu)$, we define the free loop algebra at v_0 , denoted $\mathcal{S}_0$, as the cutdown at p_{v_0} , i.e., $\mathcal{S}_0 = p_{v_0} \mathcal{S} p_{v_0}$.

Remark 3.5. In [HP17, Har17] the K-theory of $\mathcal{S}(\Gamma, \mu)$ was shown to be given by $K_0(\mathcal{S}(\Gamma, \mu)) = \mathbb{Z}\{[p_v] | v \in V\}$, the free abelian group generated by the equivalence classes of the projections p_v , and $K_1(\mathcal{S}(\Gamma, \mu)) \cong \{0\}$. Under the mild assumption (1) from the introduction, $\mathcal{S}(\Gamma, \mu)$ is simple, so $\mathcal{S}_0(\Gamma, \mu)$ is Morita equivalent to $\mathcal{S}(\Gamma, \mu)$, and $\mathcal{S}_0(\Gamma, \mu)$ has unique trace. It follows that if (Γ, μ, v_0) and (Γ', μ', v'_0) are two pointed weighted graphs, and the additive groups generated by $\{\mu(v) | v \in V\}$ and $\{\mu(v') | v' \in V'\}$ do not agree, then $\mathcal{S}_0(\Gamma, \mu) \not\cong \mathcal{S}_0(\Gamma', \mu')$.

Let $\mathcal{H}_0$ be the Gelfand-Naimark-Segal (GNS) Hilbert space of $\mathcal{S}_0$ under the finite trace $\text{tr}_0 = \text{Tr}|_{\mathcal{S}_0}$. Let Π_0 denote the set of loops based at v_0 . Note that Π_0 is an orthonormal basis for $\mathcal{H}_0 = L^2(\mathcal{S}_0, \text{tr}_0)$. Furthermore, if $\sigma \in \Pi_0$, then Y_σ is the unique element in $\mathcal{S}_0$ satisfying $Y_\sigma v_0 = \sigma$ (note that a_σ is necessarily 1 if σ is a loop). This means that we have the following important fact:

Fact 3.6. The set $\{Y_\sigma | \sigma \in \Pi_0\}$ is an orthonormal basis for $L^2(\mathcal{S}_0, \text{tr}_0)$.

We define an unbounded operator N in $\mathcal{H}_0$ by the closure of the operator satisfying $N(\sigma) = n\sigma$ whenever σ is a loop of length n . Let A be the unital $*$ -algebra generated by the elements Y_σ , which by Proposition 3.2, is also generated by the elements X_σ . Notice that under the identification of $\mathcal{H}_0$ with $L^2(\mathcal{S}_0, \text{tr}_0)$, we may realize $N : A \rightarrow A$ as an unbounded operator satisfying $N(Y_\sigma) = nY_\sigma$ whenever σ is a loop of length n . Observe that due to cutting down $\mathcal{S}$ by p_{v_0} , it follows that the null space of N is precisely scalar multiples of p_0 , the identity in $\mathcal{S}_0$. Set $A_n := \text{span}\{Y_\sigma | |\sigma| \leq n\}$, which is $*$ -closed and finite dimensional. Furthermore, it is straightforward to see that $A_n \cdot A_m \subset A_{n+m}$, giving a $*$ -filtration of A by finite dimensional subspaces. We are now in position to use the Ozawa-Rieffel framework as in §2.2.

As above, we set $W_n := \text{span}\{Y_\sigma \mid |\sigma| = n\} = A_n \ominus A_{n-1}$, and we define P_n to be the orthogonal projection from $\mathcal{H}_0$ onto W_n .

Lemma 3.7. *If $x \in W_k$, then $\|P_m x P_n\| \leq \|x\|_2$.*

Proof. Write $x = \sum_{|\sigma|=k} b_\sigma Y_\sigma$. Note that $\|x\|_2^2 = \sum_{|\sigma|=k} |b_\sigma|^2$. We need to show that if $\xi \in P_n \mathcal{H}_0$ then we have

$$\|P_m x \xi\|_{\mathcal{H}_0} \leq \|x\|_2 \|\xi\|_{\mathcal{H}_0}$$

Write $\xi = \sum_{|\tau|=n} c_\tau \tau$. The term $P_m x P_n$ is zero unless $|m - n| \leq k \leq m + n$. Choose j such that $(k - j) + (n - j) = m$, and write

$$x = \sum_{\substack{|\rho_1|=k-j \\ |\tau_1|=j}} b_{\rho_1 \tau_1} Y_{\rho_1 \tau_1} \quad \text{and} \quad \xi = \sum_{\substack{|\rho_2|=j \\ |\tau_1|=n-j}} c_{\rho_2 \tau_2} \rho_2 \tau_2.$$

This means that

$$P_m x \xi = \left(\sum_{\substack{|\rho_1|=k-j \\ |\tau_1|=j}} b_{\rho_1 \tau_1} a_{\rho_1}^2 \ell(\rho_1) \ell(\tau_1^{\text{op}})^* \right) \cdot \sum_{\substack{|\rho_2|=j \\ |\tau_1|=n-j}} c_{\rho_2 \tau_2} \rho_2 \tau_2 = \sum_{\substack{|\rho|=k-j \\ |\sigma|=j \\ |\tau|=n-j}} a_\sigma^2 b_{\rho \sigma} c_{\sigma \tau} \rho \tau.$$

From this, we see that

$$\begin{aligned} \|P_m x \xi\|_2^2 &= \sum_{\substack{|\rho|=k-j \\ |\tau|=n-j}} \left| \sum_{|\sigma|=j} a_\sigma^2 b_{\rho \sigma} c_{\sigma \tau} \right|^2 \\ &\leq \sum_{\substack{|\rho|=k-j \\ |\tau|=n-j}} \left(\sum_{|\sigma|=j} |a_\sigma^2 b_{\rho \sigma} c_{\sigma \tau}| \right)^2 \\ &\leq \sum_{\substack{|\rho|=k-j \\ |\tau|=n-j}} \left(\sum_{|\sigma|=j} |b_{\rho \sigma} c_{\sigma \tau}| \right)^2 \\ &\leq \sum_{\substack{|\rho|=k-j \\ |\tau|=n-j}} \left[\left(\sum_{|\sigma_1|=j} |b_{\rho \sigma_1}|^2 \right) \left(\sum_{|\sigma_2|=j} |c_{\sigma_2 \tau}|^2 \right) \right] \\ &\leq \left(\sum_{|\sigma'|=k} |b_{\sigma'}|^2 \right) \cdot \left(\sum_{|\sigma''|=n} |c_{\sigma''}|^2 \right) \\ &= \|x\|_2^2 \cdot \|\xi\|_2^2 \end{aligned}$$

as desired. $\square$

Lemma 3.7 immediately implies Theorem A.

Proof of Theorem A. Recall that L is defined by $L(x) = \|[N, x]\|$, then Lemma 3.7 allows us to use the Ozawa-Rieffel criterion in Theorem 2.14 ([OR05, Theorem 1.2]) for $C = 1$ to conclude that $(\mathcal{S}_0, A, L)$ is a compact quantum metric space. $\square$

3.2. Convergence for weighted pointed graphs. We now discuss a type of convergence for vertex-weighted pointed graphs, which is a weighted analog of Benjamini-Schramm convergence [BS01]. As in the previous sections, the graphs we consider are countable, connected, locally finite, undirected, vertex-weighted, and pointed, where the base-point has minimal weight 1. The following notation will be handy.

Notation 3.8. Suppose $\Gamma = (V, E, v)$ is such a graph and $R \in \mathbb{N}$. We denote by $\Gamma(R)$ the truncation of Γ to the closed ball of radius R of Γ based at v .

Definition 3.9. Suppose we have a sequence of such graphs $(\Gamma_n = (V_n, E_n, v_n), \mu_n)$, and another graph $(\Gamma = (V, E, v), \mu)$. We say that Γ_n converges locally to Γ if for all $R \in \mathbb{N}$, there is an $N_R > 0$ such that for every $n > N_R$,

- there is a pointed graph isomorphism $\varphi_n^R : \Gamma_n(R) \rightarrow \Gamma(R)$, and
- these graph isomorphisms satisfy for all $n > \max\{N_R, N_{R+1}\}$, $\varphi_n^{R+1}|_{\Gamma_n(R)} = \varphi_n^R$.

Moreover, we say $\Gamma_n \rightarrow \Gamma$ locally uniformly if $\Gamma_n \rightarrow \Gamma$ locally and the isomorphisms φ_n^R satisfy

- for every vertex $w \in \Gamma$ with $\text{dist}(v, w) \leq R$, $\lim_{n \rightarrow \infty} \mu_n[(\varphi_n^R)^{-1}(w)] = \mu(w)$.

In other words, $(\Gamma_n, \mu_n) \rightarrow (\Gamma, \mu)$ locally uniformly if for all $R > 0$, the R -ball eventually stabilizes, and the weights converge uniformly on the R -ball.

Examples 3.10. We give several examples of local uniform graph convergence.

- (1) (Subgraphs converging to a limit graph) Consider the Coxeter-Dynkin diagrams A_n with their unique normalized Frobenius-Perron weighting, where the base-point is at the left:

$$\begin{array}{c} [1] \quad [2] \quad [3] \quad \dots \quad [n-1] \quad [n] \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ \star \end{array} \quad \text{where} \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad q = \exp\left(\frac{2\pi i}{2(n+1)}\right).$$

It is easy to see that the A_n converge to the Coxeter-Dynkin diagram A_∞ with its Frobenius-Perron weighting

$$\begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \\ \star \end{array}$$

Just observe that as $\theta \rightarrow 0$, we have $q = e^{i\theta} \rightarrow 1$, so $[n] \rightarrow n$.

- (2) (Weightings converging on the same graph) We fix the graph A_∞ , but we consider the continuous family of Frobenius-Perron weightings given by

$$\begin{array}{c} [1] \quad [2] \quad [3] \quad \dots \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \\ \star \end{array} \quad \text{where} \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad q \geq 1.$$

It is easily verified that any convergent sequence $q_n \rightarrow q_0$ gives a convergent sequence of graphs.

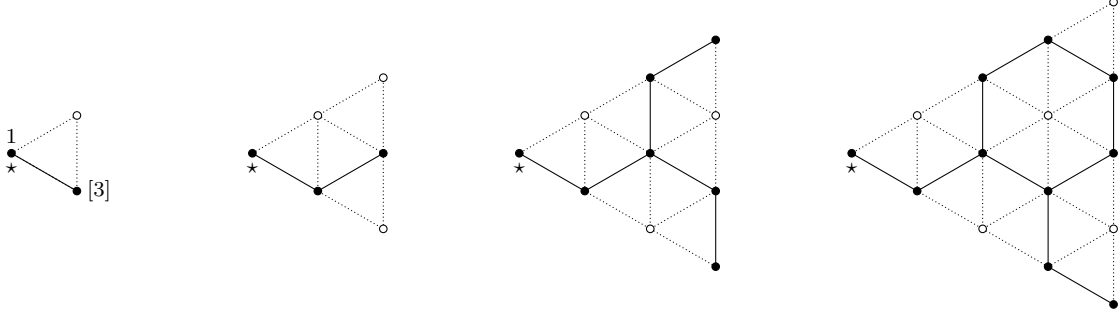
- (3) (Existence of only local isomorphisms) Consider the the affine Coxeter-Dynkin diagrams $D_n^{(1)}$ with their unique normalized Frobenius-Perron weighting:

$$\begin{array}{c} 1 \quad \quad \quad 2 \quad 2 \quad \dots \quad 2 \quad 2 \quad \quad 1 \\ \bullet \quad \quad \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \quad \quad \bullet \\ \bullet \quad \quad \bullet \quad \quad \bullet \quad \quad \bullet \quad \quad \bullet \quad \quad \bullet \\ \star \end{array}$$

It is easily verified that these graphs converge to the affine Coxeter-Dynkin diagram D_∞ with its Frobenius-Perron weighting

$$\begin{array}{c} 1 \quad \quad \quad 2 \quad 2 \quad \dots \\ \bullet \quad \quad \bullet \text{---} \bullet \text{---} \dots \\ \bullet \quad \quad \bullet \quad \quad \bullet \\ \star \end{array}$$

Remark 3.11. Examples (1) and (2) above are part of a wider family of examples coming from subfactor theory. For every simple complex Lie algebra $\mathfrak{g}$ and an appropriate root of unity q , we obtain the fusion category $\overline{\text{Rep}(\mathcal{U}_q(\mathfrak{g}))}$, the semisimplification of the category of tilting modules for the quantum group $\mathcal{U}_q(\mathfrak{g})$. The isomorphism classes of simple objects are in bijection with a truncation of the positive Weyl chamber, and the fusion rules are described by the quantum Steinberg tensor product rule, a.k.a. the quantum Racah formula [AP95, Saw06]. One way to obtain an unoriented graph is to take the *alternating part* of the fusion graph for the standard generator of $\overline{\text{Rep}(\mathcal{U}_q(\mathfrak{g}))}$, which is the principal graph of a subfactor [Wen88, Wen90]. As $k \rightarrow \infty$, we obtain a sequence of pointed bipartite weighted graphs which converge locally uniformly to the alternating part of the entire Weyl chamber. For an explicit example, we list several graphs in the sequence for $\mathfrak{su}_3$, where the shaded nodes and edges correspond to the bipartite alternating part, and the unshaded nodes and dotted lines correspond to the rest of the truncated Weyl chamber.



The generating object, which corresponds to the vertex distance 1 from $\star$, always has quantum dimension [3], and the quantum dimensions of the other vertices can be computed from this data using the adjacency matrix of the graph. See also [Ais97] for computing quantum dimensions.

3.3. Adjusting the Lip-norm. Nuclearity is often used to establish quantum Gromov-Hausdorff convergence of infinite-dimensional quantum metric spaces (see [Rie04a, KL09, Lat17, Agu19, JRZ18, KK21] where nuclearity is either implicitly or explicitly used to provide finite-dimensional approximations). Since $\mathcal{S}_0$ is exact but non-nuclear, we do not have contractive completely positive maps for finite dimensional approximations. Instead, we pass from the Lip-norm $L := \|[\cdot, N]\|$ to a new Lip-norm $\mathcal{L}$ on $\mathcal{S}_0$ defined from the finite dimensional spaces $A_n \ominus A_{n-1}$ from our filtration (A_n) of $A \subset \mathcal{S}_0$. The spaces $A_n \ominus A_{n-1}$ of homogeneous loops provide an appropriate finite dimensional approximation.

As done before and in the following definition for ease of notation, we suppress the vertex-weighted pointed graph (Γ, μ) from $\mathcal{S}_0(\Gamma, \mu)$, $A(\Gamma, \mu)$, and so on.

Definition 3.12. Let $W_n := \text{span}\{Y_\sigma \mid |\sigma| = n\}$ i.e., the span of the Wick words of length n in $\mathcal{S}_0$, and observe $A_n = \text{span} \bigcup_{k=0}^n W_k$. Set $B_n := \{x \in W_n \mid L(x) \leq 1\}$, and define

$$\mathcal{C}' := \overline{\text{conv} \bigcup_{k=0}^{\infty} B_k}, \quad \mathcal{C} := \overline{\text{conv} \bigcup_{k=1}^{\infty} B_k}, \quad \text{and} \quad \mathcal{C}_n := \text{conv} \bigcup_{k=1}^n B_k = \overline{\text{conv} \bigcup_{k=1}^n B_k},$$

where the above closures are in operator norm. We then define $\mathcal{L}$ on $\mathcal{S}_0$ to be the Minkowski functional associated to $\mathcal{C}'$, i.e.

$$\mathcal{L}(x) := \inf \{r > 0 \mid r^{-1}x \in \mathcal{C}'\}. \quad (2)$$

Observe that $\mathcal{L}(x) = \infty$ whenever $r^{-1}x \notin \mathcal{C}'$ for all $r > 0$. Clearly $\mathcal{L}$ is finite on A .

Remarks 3.13.

- (1) Since $\mathcal{L}$ is the Minkowski functional associated to $\mathcal{C}'$, and $\mathcal{C}'$ is norm-closed, then $\mathcal{C}' = \{x \in \mathcal{S}_0 \mid \mathcal{L}(x) \leq 1\}$.
- (2) By lower semi-continuity of L , we have that

$$\{x \in \mathcal{S}_0 \mid \mathcal{L}(x) \leq 1\} = \mathcal{C}' \subseteq \{x \in \mathcal{S}_0 \mid L(x) \leq 1\};$$

equivalently, $L \leq \mathcal{L}$. Thus, as L is a Lip-norm, it follows from Proposition 2.7 ([OR05, Prop. 1.3]) that $\mathcal{L}$ is a Lip-norm.

(3) Observe that if $x \in W_n$, then $L(x) = \mathcal{L}(x)$. Indeed, by lower-semicontinuity of L , we see that $L \leq 1$ on $\mathcal{C}$. So if $L(x) = 1$, then $\alpha x \notin \mathcal{C}$ for all $\alpha > 1$, and thus $\mathcal{L}(x) = 1$.

Lemma 3.14. *For every $\varepsilon > 0$, there is a $K \in \mathbb{N}$ such that $x \in A \ominus A_K$ and $L(x) \leq 1$ implies $\|x\| < 2\varepsilon/3$. Moreover, for all $n > m > K$, $\text{dist}_H^{\|\cdot\|}(\mathcal{C}_m, \mathcal{C}_n) < 4\varepsilon/3$ for all vertex-weighted pointed graphs (Γ, μ) .*

Proof. By [OR05, §3], given $a \in A$ with $L(a) \leq 1$ and $\varepsilon > 0$, there are $K > M > 0$ depending only on the constant C in Theorem 2.14 ([OR05, Theorem 1.2]) so that

- if $a^{(M)} = \sum_{|m-n| \geq M} P_m a P_n$ and $a^M = a - a^{(M)}$, then $\|a^{(M)}\| < \varepsilon/3$
- if $\hat{a}_K = \sum_{k \leq K} a_k$ and $\tilde{a}_K = a - \hat{a}_K$ (which are both in $A(t)$), $\|(\tilde{a}_K)^M\| < \varepsilon/3$.

Thus for $x \in A \ominus A_K$, we have

$$\|x\| = \|(\hat{x}_K)^M + (\tilde{x}_K)^M + x^{(M)}\| = \|(\tilde{x}_K)^M + x^{(M)}\| < 2\varepsilon/3.$$

It was shown in Lemma 3.7 that $C = 1$ regardless of the graph Γ and the vertex-weighting μ (as long as the base point v_0 has minimal weighting). Therefore, it follows that if $x \in A \ominus A_K$ and $\mathcal{L}(x) \leq 1$, then $L(x) \leq 1$ and hence $\|x\| < 2\varepsilon/3$.

Now by definition, $\mathcal{C}_m \subset \mathcal{C}_n$. Observe that when V is a vector space and $S \subset T \subset V$, then every element in $\text{conv}(T)$ is a convex combination of an element in $\text{conv}(S)$ and an element in $\text{conv}(T \setminus S)$. Hence if $x \in \mathcal{C}_n$, we have $x = \lambda y + (1 - \lambda)z$ where $y \in \mathcal{C}_m$ and $z \in \text{conv} \bigcup_{k=m+1}^n B_k$. Since $z \in A \ominus A_K$ and $\mathcal{L}(z) \leq 1$, $\|z\| < 2\varepsilon/3$ by the preceding paragraph. We conclude that $\text{dist}_H^{\|\cdot\|}(\mathcal{C}_m, \mathcal{C}_n) < 4\varepsilon/3$ as desired. $\square$

Remark 3.15. Lemma 3.14 is precisely why we introduced $\mathcal{L}$. Indeed, one of our goals in this paper is to provide natural finite-dimensional approximations of these free graph algebras, which are non-nuclear. This is accomplished in Corollary 3.16 using $\mathcal{L}$. We are not sure we can accomplish this using L since the estimates we obtained were not contractive with respect to L , which did not allow for the approximations we made in Lemma 3.14.

Corollary 3.16. *For every $\varepsilon > 0$, there is a $K \in \mathbb{N}$ such that $\text{dist}_q((\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L}), (A_k(\Gamma, \mu), \mathcal{L}|_{A_k})) < \varepsilon$ for all $k \geq K$ and for all vertex-weighted pointed graphs (Γ, μ) .*

Proof. Immediate from Lemmas 2.10 and 3.14. $\square$

Theorem 3.17. *Let (Γ_n, μ_n) be a sequence of vertex-weighted pointed graphs converging locally uniformly to (Γ, μ) , each of which have base point v_0 (here, we suppress the isomorphism data φ_n^R). Let $\mathcal{L}_n$ be the Lip-norm constructed on $\mathcal{S}_0(\Gamma_n, \mu_n)$ as in (2) above. Then $(\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), \mathcal{L}_n)$ converges in quantum Gromov-Hausdorff distance to $(\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L})$.*

Proof. We follow the strategy for convergence introduced by Rieffel in [Rie04a]:

- (1) First, uniformly approximate $(\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), \mathcal{L}_n)$ for all $n \in \mathbb{N}$ and $(\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L})$ by finite-dimensional compact quantum metric spaces.
- (2) Second, establish convergence of the finite-dimensional compact quantum metric spaces using [Rie04a, Theorem 11.2].
- (3) Finally, the argument will be complete by the triangle inequality.

Let $\varepsilon > 0$. By Corollary 3.16, there exists $K \in \mathbb{N}$ such that

$$\text{dist}_q((\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), \mathcal{L}_n), (A_K(\Gamma_n, \mu_n), \mathcal{L}_n|_{A_K})) < \frac{\varepsilon}{3}$$

for all $n \in \mathbb{N}$ and $\text{dist}_q((\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L}), (A_K(\Gamma, \mu), \mathcal{L}|_{A_K})) < \frac{\varepsilon}{3}$.

Next, $A_K(\Gamma_n, \mu_n)$ for all $n \in \mathbb{N}$ and $A_K(\Gamma, \mu)$ are finite dimensional. To prove quantum Gromov-Hausdorff convergence of $(A_K(\Gamma_n, \mu_n), \mathcal{L}_n|_{A_K})_{n \in \mathbb{N}}$ to $(A_K(\Gamma, \mu), \mathcal{L}|_{A_K})$, we use [Rie04a, Theorem 11.2], which requires finite-dimensional spaces and a continuous field of norms and Lip-norms. For sufficiently large n , all of the $\Gamma_n(K)$ coincide (again, we suppress the isomorphism data φ_n^R). It follows that for these

values of n , the vector spaces $A_K(\Gamma_n, \mu_n)$ are canonically isomorphic to the complex linear span of all loops in Γ of length at most K based at v_0 . Setting

$$V_K := \text{span}_{\mathbb{R}} \{ a\sigma + \bar{a}\sigma^{\text{op}} \mid a \in \mathbb{C} \text{ and } \sigma \in \Pi_0 \text{ such that } |\sigma| \leq K \},$$

V_K is canonically isomorphic to $A_K(\Gamma, \mu)_{\text{sa}}$ and $A_K(\Gamma_n, \mu_n)_{\text{sa}}$ for sufficiently large n .

Fix a loop σ of Γ based at v_0 , and let $Y_\sigma(n)$ be the Wick word for σ in $\mathcal{S}_0(\Gamma_n, \mu_n)$:

$$Y_\sigma(n) = \sum_{\sigma=\rho\tau} a_\rho(n) a_\tau^{-1}(n) \ell(\rho) \ell(\tau^{\text{op}})^*.$$

Since $\mu_n \rightarrow \mu$ as $n \rightarrow \infty$, it follows that $a_\rho(n) \rightarrow a_\rho$ for any path ρ in Γ . For $\xi \in V_K$, we write Y_ξ for the corresponding linear combination of Wick words in $\mathcal{S}_0(\Gamma, \mu)$, and we write $Y_\xi(n)$ for the corresponding linear combination of Wick words in $\mathcal{S}_0(\Gamma_n, \mu_n)$ for sufficiently large n . Setting $\|\xi\|_n := \|Y_\xi(n)\|_{\mathcal{S}_0(\Gamma_n, \mu_n)}$ and $\|\xi\| := \|Y_\xi\|_{\mathcal{S}_0(\Gamma, \mu)}$, we have $\|\xi\|_n \rightarrow \|\xi\|$ as $n \rightarrow \infty$. Moreover, for each k between 1 and K , note that for any $\xi \in V_K$ which is a linear combination of loops of length exactly k , $\mathcal{L}_n(Y_\xi(n))$ converges to $\mathcal{L}(Y_\xi)$. This is due to the fact that on the space W_k , L and $\mathcal{L}$ coincide. Thus by [Rie04a, Theorem 11.2], we have that

$$\lim_{n \rightarrow \infty} \text{dist}_q((A_K(\Gamma_n, \mu_n), \mathcal{L}_n|_{A_K}), (A_K(\Gamma, \mu), \mathcal{L}|_{A_K})) = 0.$$

Hence, there exists $N \in \mathbb{N}$ such that $\text{dist}_q((A_K(\Gamma_n, \mu_n), \mathcal{L}_n|_{A_K}), (A_K(\Gamma, \mu), \mathcal{L}|_{A_K})) < \frac{\varepsilon}{3}$ for all $n \geq N$. Therefore, if $n \geq N$, then

$$\begin{aligned}
& \text{dist}_q((\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), \mathcal{L}_n), (\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L})) \\
& \leq \text{dist}_q((\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), \mathcal{L}_n), (A_K(\Gamma_n, \mu_n), \mathcal{L}_n|_{A_K})) \\
& \quad + \text{dist}_q((A_K(\Gamma_n, \mu_n), \mathcal{L}_n|_{A_K})(A_K(\Gamma, \mu), \mathcal{L}|_{A_K})) \\
& \quad + \text{dist}_q((A_K(\Gamma, \mu), \mathcal{L}|_{A_K}), (\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L})) \\
& < \frac{\varepsilon}{3} + \text{dist}_q((A_K(\Gamma_n, \mu_n), \mathcal{L}_n|_{A_K})(A_K(\Gamma, \mu), \mathcal{L}|_{A_K})) + \frac{\varepsilon}{3} \\
& < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

as desired.

4. APPLICATION TO SUBFACTOR THEORY

We refer the reader to [Jon12] for the definition of a subfactor planar algebra and its principal graphs and to [BHP12] for the definition of a factor planar algebra and its principal graph.

4.1. The Guionnet-Jones-Shlyakhtenko C^* -algebras. Let $\mathcal{P}_\bullet$ be a (sub)factor planar algebra. We now give the construction from [HP17, HP14] of the Guionnet-Jones-Shlyakhtenko (GJS) C^* -algebras based on the constructions [GJS10, JSW10, GJS11]. A similar construction starting from a unitary tensor category and chosed symmetrically self-dual generator was given in [HHP20].

First, we form the graded algebra $\text{Gr}_0 = \bigoplus_{n \geq 0} \mathcal{P}_n$ with the Bacher-Walker product¹

$$x \star y = \sum_{j=0}^{\min\{m,n\}} \text{diagram} \quad \text{for} \quad x \in \mathcal{P}_m, y \in \mathcal{P}_n,$$

and trace given by

$$\mathrm{tr}(x) = \delta_{m=0}x \quad \text{for} \quad x \in \mathcal{P}_m.$$

(Since $\mathcal{P}_0 = \mathbb{C}$, the above expression gives us a number.) We note that under the GNS inner product $\langle x, y \rangle = \text{tr}_0(y^* \star x)$, the spaces $\mathcal{P}_n$ are orthogonal for distinct n .

¹This product was discovered by Bacher and Walker independently; see [BW16] and [JSW10].

Observe that since each $\mathcal{P}_n$ is $*$ -closed and finite dimensional, the subspaces $A_n = \bigoplus_{k=0}^n \mathcal{P}_k$ give Gr_0 the structure of a $*$ -filtration by finite dimensional subspaces. Moreover, since $\mathcal{P}_\bullet$ is connected, $\mathcal{P}_0 = \mathbb{C}1_{\text{Gr}_0}$. Finally, by [GJS10, JSW10], the action of Gr_0 on $(\text{Gr}_0, \text{tr}_0)$ is bounded in $\|\cdot\|_2$. Hence Assumptions 2.11 hold, and we are in the position to use the Ozawa-Rieffel criterion from Theorem 2.14 ([OR05, Theorem 1.2]).

Definition 4.1. The C^* -algebra $\mathcal{A}_0 = \overline{\text{Gr}_0}^{\|\cdot\|}$ acting on $L^2(\text{Gr}_0, \text{tr}_0)$ is called the GJS C^* -algebra of $\mathcal{P}_\bullet$.

Let (Γ, μ) be the principal graph of $\mathcal{P}_\bullet$ with its quantum dimension weighting, which satisfies the Frobenius-Perron condition. Note that the distinguished vertex $\star$ has minimal weight 1, so Assumption 3.3 holds. We have the following lemma from [HP14] which connects the GJS C^* -algebra to the free loop algebras discussed in Section 3.1.

Lemma 4.2 ([HP14, Cor. 3.4]). *The C^* -algebra $\mathcal{A}_0$ is isomorphic to the free loop algebra $\mathcal{S}_0(\Gamma, \mu)$.*

Remark 4.3. The examples of local uniform graph convergence in Examples 3.10 are all examples of principal graphs of subfactors with weightings given by quantum dimension functions which satisfy the Frobenius-Perron condition. We may thus interpret Theorem B as giving quantum Gromov-Hausdorff convergence of the compact quantum metric spaces associated to GJS C^* -algebras, with the adjusted Lip-norms.

4.2. The number operator. As in §2.2, we have the number operator $N = \sum_{n \geq 0} n P_n$ on Gr_0 , where P_n is the projection with range $B_n = A_n \ominus A_{n-1} = \mathcal{P}_n$. We end our article with some further observations about the properties of the number operator in our setup.

To begin, we give a supplementary diagrammatic proof that the number operator has bounded commutator with Gr_0 , although it follows directly from Lemma 2.13 ([OR05, Lemma 1.1]).

Lemma 4.4. *The number operator N has bounded commutator with every $x \in \text{Gr}_0$.*

Proof. To show $\|[N, x]\|_\infty$ is bounded for an arbitrary $x \in \text{Gr}_0$, it suffices to consider a fixed $x \in \mathcal{P}_m$. Suppose $y \in \mathcal{P}_n$, and we write $\hat{y}$ for its image in $L^2(\text{Gr}_0, \text{tr}_0)$. We need only treat the case $m < n$, since we may ignore the behavior of $[N, x]$ on a finite dimensional subspace. We have

$$\begin{aligned} [N, x]\hat{y} &= N(x \star y)^\wedge - x \star (Ny)^\wedge \\ &= \sum_{j=0}^m (m+n-2j) \begin{array}{c} \text{---} j \text{---} \\ \text{---} m-j \text{---} \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline y \\ \hline \end{array} \begin{array}{c} \text{---} n-j \text{---} \\ \text{---} \end{array} - \sum_{j=0}^m n \begin{array}{c} \text{---} j \text{---} \\ \text{---} m-j \text{---} \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline y \\ \hline \end{array} \begin{array}{c} \text{---} n-j \text{---} \\ \text{---} \end{array} \\ &= \sum_{j=0}^m (m-2j) \begin{array}{c} \text{---} j \text{---} \\ \text{---} m-j \text{---} \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline y \\ \hline \end{array} \begin{array}{c} \text{---} n-j \text{---} \\ \text{---} \end{array}. \end{aligned}$$

We now see we can write this sum at the end as

$$\left(\sum_{j=0}^m (m-2j) \begin{array}{c} \text{---} j \text{---} \\ \text{---} m-j \text{---} \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{c} \text{---} n-j \text{---} \\ \text{---} \end{array} \right) \hat{y}$$

where the sum in parentheses is a finite sum of bounded operators in the $\mathcal{P}_\bullet$ -Toeplitz algebra $\mathcal{T}_0(\mathcal{P}_\bullet)$ [HP17], which is independent of y . We are finished. $\square$

Proposition 4.5. *The number operator N has compact resolvent and is θ -summable, i.e., e^{-tN^2} is trace class for all $t > 0$.*

Proof. Since Γ is the principal graph of $\mathcal{P}_\bullet$, $\dim(\mathcal{P}_n)$ is the number of loops of length $2n$ on Γ based at $\star$. Letting A_Γ be the adjacency matrix of Γ , we have that A_Γ acts on $\ell^2(V)$, and $\|A_\Gamma\| \leq \delta$ by [Pop94,

§1.3.5]. Define $e_* \in \ell^2(V)$ by $e_*(v) = \delta_{v=*,}$ and note that the number of loops based at $*$ of length $2n$ is $\langle A_\Gamma^{2n} e_*, e_* \rangle$. Hence, we see

$$\dim(\mathcal{P}_n) = \langle A_\Gamma^{2n} e_*, e_* \rangle = |\langle A_\Gamma e_*, e_* \rangle| \leq \|A_\Gamma\|^{2n} \leq \delta^{2n}.$$

Thus, on $\mathcal{P}_n$, e^{-tN^2} has trace bounded above by

$$\dim(\mathcal{P}_n) e^{-tn^2} \leq \delta^{2n} e^{-tn^2}.$$

It is now clear by the root test that

$$\sum_{n \geq 0} \frac{\delta^{2n}}{e^{tn^2}} < \infty. \quad \square$$

We now use techniques from [CJS14] to show that the number operator arises as $\partial^* \partial$ where ∂ is a derivation from Gr_0 into a Hilbert space.

Definition 4.6. We define $\mathcal{K}_0 = \bigoplus_{i,j \geq 0} \mathcal{P}_{i,j,1}$, where $\mathcal{P}_{i,j,1} = \mathcal{P}_{i+j+1}$, but we think of elements as having i strings out the top, j strings out the bottom, and one string to the right:

$$\mathcal{P}_{i,j,1} \ni x \longleftrightarrow \begin{array}{c} | \\ i \\ \boxed{x} \\ | \\ j \end{array}.$$

We define $\mathcal{K}$ to be the completion of $\mathcal{K}_0$ using the inner product

$$\left\langle \begin{array}{c} | \\ i \\ \boxed{x} \\ | \\ j \end{array}, \begin{array}{c} | \\ k \\ \boxed{y} \\ | \\ \ell \end{array} \right\rangle = \delta_{i=k} \delta_{j=\ell} \star \begin{array}{c} i \\ \boxed{x} \end{array} \begin{array}{c} \boxed{y^*} \\ j \end{array} \star.$$

We have an action of $\text{Gr}_0 \otimes \text{Gr}_0^{\text{op}}$ on $\mathcal{K}$ by bounded operators. Given $x \in \mathcal{P}_m$ and $y \in \mathcal{P}_n$, we think of $x \otimes y^{\text{op}}$ as the following diagram:

$$x \otimes y^{\text{op}} = \begin{array}{c} m \\ | \\ \boxed{x} \\ | \\ \boxed{\bar{y}} \\ | \\ n \end{array}$$

which acts on $\mathcal{K}$ by left multiplication in a variation of the Bacher-Walker product. If $z \in \mathcal{P}_{k,\ell,1}$, we have

$$(x \otimes y^{\text{op}}) \cdot z = \sum_{i=0}^{\min\{m,k\}} \sum_{j=0}^{\min\{n,\ell\}} \begin{array}{c} m-i \\ | \\ \boxed{x} \\ | \\ \boxed{\bar{y}} \\ | \\ n-j \end{array} \begin{array}{c} i \\ \boxed{x} \end{array} \begin{array}{c} \boxed{\bar{y}} \\ j \end{array} \begin{array}{c} k-i \\ | \\ \boxed{z} \\ | \\ \ell-j \end{array}.$$

It is easy to see that this action is bounded using the Fock space argument of [HP17]. This means that $\mathcal{K}$ has the natural structure of a $\text{Gr}_0 - \text{Gr}_0$ bimodule. We use the notation $x \star \xi \star y = (x \otimes y^{\text{op}}) \cdot \xi$.

Definition 4.7. We define a map $\partial : \text{Gr}_0 \rightarrow \mathcal{K}$ by the linear extension of

$$\mathcal{P}_m \ni \begin{array}{c} | \\ m \\ \boxed{x} \\ | \end{array} \mapsto \sum_{j=0}^{m-1} \begin{array}{c} | \\ j \\ \boxed{x} \\ | \\ m-j-1 \end{array} = \sum_{j=0}^{m-1} \begin{array}{c} m-j-1 \\ | \\ \boxed{x} \\ | \\ j \end{array}.$$

Lemma 4.8. *The map ∂ is a closable derivation in the Bacher-Walker product.*

Proof. First, we show that ∂ is a derivation. We need to show that $\partial(x \star y) = x \star \partial(y) + \partial(x) \star y$. It is straightforward to compute

$$\partial \left(\sum_{i=0}^{\min\{m,n\}} \text{diagram}(x, y, i) \right) = \sum_{i=0}^{\min\{m,n\}} \sum_{j=0}^{n-i} \text{diagram}(x, y, i, j) + \sum_{i=0}^{\min\{m,n\}} \sum_{j=0}^{m-j} \text{diagram}(x, y, i, j)$$

The diagrams represent planar graphs with nodes x and y and various boundary labels. The first term shows x and y connected by a line with labels $m-i$ and $n-i$. The second term shows x and y connected by a line with labels $m-i$ and j . The third term shows x and y connected by a line with labels $m-j-i-1$ and j .

The right hand side is easily seen to be equal to $x \star \partial(y) + \partial(x) \star y$ after switching the order of summation. To show ∂ is closable, it is easy to calculate that in the Bacher-Walker product,

$$\partial^* \left(\text{diagram}(x, m-j-1, j) \right) = \text{diagram}(x, m)$$

Hence ∂^* is densely defined, and we are finished. $\square$

The following corollary is now immediate.

Corollary 4.9. *The number operator N is equal to $\partial^* \partial$.*

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