

Representations of fusion categories and their commutants

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Abstract

A bicommutant category is a higher categorical analog of a von Neumann algebra. We study the bicommutant categories which arise as the commutant $\mathcal{C}'$ of a fully faithful representation $\mathcal{C} \rightarrow \text{Bim}(R)$ of a unitary fusion category $\mathcal{C}$. Using results of Izumi, Popa, and Tomatsu about existence and uniqueness of representations of unitary (multi)fusion categories, we prove that if $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent unitary fusion categories, then their commutant categories $\mathcal{C}'$ and $\mathcal{D}'$ are equivalent as bicommutant categories. In particular, they are equivalent as tensor categories:

$$\left(\mathcal{C} \simeq_{\text{Morita}} \mathcal{D} \right) \implies \left(\mathcal{C}' \simeq_{\text{tensor}} \mathcal{D}' \right).$$

This categorifies the well-known result according to which the commutants (in some representations) of Morita equivalent finite dimensional C^* -algebras are isomorphic von Neumann algebras, provided the representations are ‘big enough’.

We also introduce a notion of positivity for bi-involutive tensor categories. For dagger categories, positivity is a property (the property of being a C^* -category). But for bi-involutive tensor categories, positivity is extra structure. We show that unitary fusion categories and $\text{Bim}(R)$ admit distinguished positive structures, and that fully faithful representations $\mathcal{C} \rightarrow \text{Bim}(R)$ automatically respect these positive structures.

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1 Introduction

Given a von Neumann algebra R , we write $\text{Bim}(R)$ for its category of separable bimodules. We view

$$\text{Bim}(R) = \text{End}(R\text{-Mod})$$

as a categorical analog of $B(H)$, the $*$ -algebra of bounded operators on a separable Hilbert space (and $R\text{-Mod}$ as a categorical analog of a Hilbert space). The category of R - R -bimodules is a *bi-involutive tensor category* [HP17, Def. 4.1]; the two involutions are given by the dagger structure $f \mapsto f^*$, and by the functor which sends a bimodule to its complex conjugate (with left and right actions given by $a\bar{\xi}b = \bar{b}^*\xi a^*$).

If $\mathcal{C}$ is a semisimple rigid C^* -tensor category or if $\mathcal{C} = \text{Bim}(R)$ for some von Neumann algebra R , then it comes naturally equipped with cones $\mathcal{P}_{a,b} \subset \text{Hom}(a \otimes \bar{a}, b \otimes \bar{b})$ for every $a, b \in \mathcal{C}$. We call such a collection of cones $\mathcal{P}_{a,b}$ (subject to various axioms) a *positive structure* on $\mathcal{C}$ (Def. 5.1).

A *representation* of a C^* -tensor category $\mathcal{C}$ is a C^* -tensor functor

$$\alpha : \mathcal{C} \rightarrow \text{Bim}(R).$$

We call the representation *fully faithful* if the underlying functor α is fully faithful. If $\mathcal{C}$ is equipped with bi-involutive and positive structures, and if α is compatible with those structures, then we call α a *positive representation*.

One of our results is that when $\mathcal{C}$ is a rigid C^* -tensor category, and when we restrict to fully faithful representations, then the notions of representation and of positive representation are equivalent:

Theorem A (Thm. 5.18). *Let $\mathcal{C}$ be a semisimple rigid C^* -tensor category. Then every fully faithful representation $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ extends uniquely to a positive representation. Moreover, every isomorphism of representations is an isomorphism of positive representations.*

Monoidal categories categorify monoids, and fusion categories categorify finite dimensional semisimple algebras (indeed, if $\mathcal{C}$ is a fusion category, then its Grothendieck ring $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a multimatrix algebra [Lus87, 1.2(a)]). Similarly, *bicommutant categories*, a notion due to the first author [Hen17], are categorical analogs of von Neumann algebras.¹

Given a category $\mathcal{C}$ equipped with a positive representation $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$, its *commutant category*

$$\mathcal{C}' = \{(X, e_X) \mid X \in \text{Bim}(R), e_X \text{ exhibits } X \text{ as commuting with the image of } \mathcal{C}\}$$

is the category whose objects are pairs (X, e_X) with $X \in \text{Bim}(R)$, and

$$e_X = \{e_{X,c} : X \boxtimes_R \alpha(c) \xrightarrow{\cong} \alpha(c) \boxtimes_R X\}_{c \in \mathcal{C}}$$

is a unitary half-braiding. The commutant category is again a bi-involutive tensor category with positive structure, and the forgetful functor $(X, e_X) \mapsto X$ provides a positive representation $\mathcal{C}' \rightarrow \text{Bim}(R)$. Iterating, we can form the double commutant $\mathcal{C}'' := (\mathcal{C}')'$. We call $(\mathcal{C}, \alpha)$ a *bicommutant category* if the canonical inclusion $\iota : \mathcal{C} \rightarrow \mathcal{C}''$ is an equivalence (Def. 6.1).

In [HP17], given a fully faithful representation $\mathcal{C} \rightarrow \text{Bim}(R)$ of a unitary fusion category, we proved that the double commutant $\mathcal{C}''$ is isomorphic to $\mathcal{C} \otimes_{\text{Vec}} \text{Hilb}$, and that it is a bicommutant category. We also proved [HP17, Lem 6.1 and Thm A] that $\mathcal{C}'$ is a bicommutant category. Note that, unlike with the usual von Neumann bicommutant theorem, given $\mathcal{C}$ and α as above there is no formal argument guaranteeing that $\mathcal{C}''$ is a bicommutant category.

In the present article, we further study these examples of bicommutant categories. Our main theorem is a categorical analog of the following well known result about finite dimensional Neumann algebras: if H is a separable infinite dimensional Hilbert space, and $A, B \subset B(H)$ are finite dimensional subalgebras which are Morita equivalent, and whose central projections are infinite, then $A' \cong B'$.

Theorem B (Thm. 6.8). *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be Morita equivalent unitary fusion categories. Let $\alpha_0 : \mathcal{C}_0 \rightarrow \text{Bim}(R_0)$ and $\alpha_1 : \mathcal{C}_1 \rightarrow \text{Bim}(R_1)$ be fully faithfully representations, where R_0 and R_1 are hyperfinite factors which are either both of type II or both of type III₁. And let $\mathcal{C}'_i$ be the commutant category of $\mathcal{C}_i$ inside $\text{Bim}(R_i)$.*

Then $\mathcal{C}'_0$ and $\mathcal{C}'_1$ are equivalent as bicommutant categories. In particular, they are equivalent as tensor categories:

$$\left(\mathcal{C}_0 \simeq_{\text{Morita}} \mathcal{C}_1 \right) \implies \left(\mathcal{C}'_0 \simeq_{\text{tensor}} \mathcal{C}'_1 \right).$$

Here, two unitary fusion categories $\mathcal{C}_0$ and $\mathcal{C}_1$ are said to be *Morita equivalent* if they can be realised as corners inside a unitary 2×2 multifusion category

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_0 & \mathcal{M} \\ \mathcal{M}^* & \mathcal{C}_1 \end{pmatrix}.$$

¹In the original definition of bicommutant categories [Hen17], positive structures were not mentioned. We believe that this was a mistake. We fix this by slightly altering the definition.

The proof of Theorem B goes along the following lines. We first use Popa’s reconstruction theorem [Pop95a, Thm. 3.1] to construct a fully faithful representation $\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus 2})$, where R is a factor isomorphic to R_1 . We then use Izumi–Popa–Tomatsu’s uniqueness theorem [Pop95b, Pop90, Izu17, Tom21] to show that $\alpha|_{\mathcal{C}_i}$ and α_i are isomorphic representations of $\mathcal{C}_i$. In particular, the commutant category of $\mathcal{C}_i$ inside $\text{Bim}(R)$ is equivalent to the commutant category of $\mathcal{C}_i$ inside $\text{Bim}(R_i)$. To finish, we invoke the following general theorem about the commutant categories of unitary $k \times k$ multifusion categories:

Theorem C (Thm. 6.6). *Let $\mathcal{C}$ be a unitary $k \times k$ multifusion category, and let*

$$\mathcal{C} \longrightarrow \text{Bim}(R_1 \oplus \dots \oplus R_k)$$

be a fully faithful representation, where the R_i are factors. Let $\mathcal{C}'$ be the commutant of $\mathcal{C}$ inside $\text{Bim}(R_1 \oplus \dots \oplus R_k)$, and $\mathcal{C}'_i$ the commutant of the i th corner $\mathcal{C}_i \subset \mathcal{C}$ inside $\text{Bim}(R_i)$. Then the obvious functor $\mathcal{C}' \rightarrow \mathcal{C}'_i$ is an equivalence of categories.

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2 Unitary multifusion categories

2.1 Tensor categories

We briefly discuss the various types of categories that appear in our article. For more details, we refer the reader to [GLR85, Sel11, EGNO15] and [HP17, §2-3]. All our categories are assumed to be linear over $\mathbb{C}$ (i.e., enriched over complex vector spaces), admit direct sums, and be idempotent complete, and all our functors are assumed to be linear.

A category $\mathcal{C}$ is called:

- a *dagger category* if for every $x, y \in \mathcal{C}$ we are given an antilinear map $*$: $\mathcal{C}(x, y) \rightarrow \mathcal{C}(y, x)$, called the adjoint, satisfying $f^{**} = f$ and $(f \circ g)^* = g^* \circ f^*$. A morphism f is called *unitary* if $f^* = f^{-1}$.
- a *C*-category* if it is a dagger category and admits a faithful dagger functor $\mathcal{C} \rightarrow \text{Hilb}$ whose image is norm closed at the level of hom-spaces.
- a *tensor category* if we are given a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ together with a unit object $1 \in \mathcal{C}$, an associator and left and right unitors isomorphisms which are natural and satisfy the pentagon and triangle axioms. We will suppress these coherence isomorphisms whenever possible.

- a *dagger tensor category* if $\mathcal{C}$ has both the structures of a dagger category and of a tensor category, the associators and unitors are unitary, and $(f \otimes g)^* = f^* \otimes g^*$.
- a C^* -*tensor category* if $\mathcal{C}$ is a dagger tensor category whose underlying dagger category is a C^* -category.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called:

- a *dagger functor* between dagger categories if $F(f^*) = F(f)^*$ for all $f \in \mathcal{C}(x, y)$.
- a *tensor functor* between tensor categories if F is equipped with natural isomorphisms $\mu_{x,y} : F(x) \otimes F(y) \rightarrow F(x \otimes y)$ and $i : 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}})$ satisfying

$$\begin{aligned} \mu_{x,y \otimes z} \circ (\text{id}_{F(x)} \otimes \mu_{y,z}) &= \mu_{x \otimes y, z} \circ (\mu_{x,y} \otimes \text{id}_{F(z)}) \quad \text{and} \\ \mu_{1,x} \circ (i \otimes \text{id}_{F(x)}) &= \text{id}_{F(x)} = \mu_{x,1} \circ (\text{id}_{F(x)} \otimes i). \end{aligned}$$

- an *anti-tensor functor* if it is equipped with natural isomorphisms $\nu_{x,y} : F(x) \otimes F(y) \rightarrow F(y \otimes x)$ and $j : 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}})$ satisfying similar conditions. Equivalently, this is a tensor functor $\mathcal{C}^{\text{mp}} \rightarrow \mathcal{D}$, where $\mathcal{C}^{\text{mp}}$ denotes the tensor category with opposite monoidal structure.
- a *dagger tensor functor* between dagger tensor categories if F is both a dagger functor and a tensor functor, and the natural isomorphisms μ and i are unitary. (There is an analogous definition for a dagger anti-tensor functor.)

A tensor category $\mathcal{C}$ is called *rigid* if every object $x \in \mathcal{C}$ admits both a left and a right dual, that is, if there exist objects $x^\sim$ and ${}^\sim x$, and morphisms $\text{ev}_x : x^\sim \otimes x \rightarrow 1$, $\text{coev}_x : 1 \rightarrow x \otimes x^\sim$, $\text{ev}_{{}^\sim x} : x \otimes {}^\sim x \rightarrow 1$, $\text{coev}_{{}^\sim x} : 1 \rightarrow {}^\sim x \otimes x$ satisfying the duality equations $(\text{id} \otimes \text{ev})(\text{coev} \otimes \text{id}) = \text{id}$ and $(\text{ev} \otimes \text{id})(\text{id} \otimes \text{coev}) = \text{id}$.

If $\mathcal{C}$ is a rigid C^* -tensor category, then the above equations only determine $x^\sim$ and ${}^\sim x$ up to canonical isomorphism (as opposed to canonical unitary isomorphism). However, when the unit object of $\mathcal{C}$ admits a direct sum decomposition into simple objects, the above issue can be remedied.

Let $\mathcal{C}$ be a rigid C^* -tensor category whose unit object is a sum of simple objects $1 = \bigoplus 1_i$ (also known as a semisimple rigid C^* -tensor category), and let $p_i : 1 \rightarrow 1$ be the corresponding orthogonal projections. Given a morphism $f : x \rightarrow y$ in $\mathcal{C}$, we write $f_{ij} : x \rightarrow y$ for $p_i \otimes f \otimes p_j$.

Lemma 2.1 ([BDH14, Thms. 4.12 and 4.22]). *Let $\mathcal{C}$ be as above. Then, for every object $x \in \mathcal{C}$, there exists an object $\bar{x}$ along with morphisms $\text{ev}_x : \bar{x} \otimes x \rightarrow 1$ and $\text{coev}_x : 1 \rightarrow x \otimes \bar{x}$ satisfying the duality equations²*

$$(\text{id} \otimes \text{ev}_x)(\text{coev}_x \otimes \text{id}) = \text{id} \quad (\text{ev}_x \otimes \text{id})(\text{id} \otimes \text{coev}_x) = \text{id} \quad (1)$$

and the balancing condition (which is a version of sphericality): for all $f : x \rightarrow x$, if

$$\text{ev}_x(\text{id} \otimes f_{ij}) \text{ev}_x^* = \lambda p_j \quad \text{and} \quad \text{coev}_x^*(f_{ij} \otimes \text{id}) \text{coev}_x = \lambda' p_i \quad \text{then} \quad \lambda = \lambda'. \quad (2)$$

The object $\bar{x}$ is determined up to unique unitary isomorphism by ev_x and coev_x subject to the above equations.

²We omit various unitors and associators to keep the equations compact.

The object $\bar{x}$ is called the *conjugate* of x . The conjugate $\bar{x}$ is canonically isomorphic to $x^\sim$ and to $\check{x}$ (and it is meaningless to ask whether the isomorphisms $\bar{x} \rightarrow x^\sim$ and $\bar{x} \rightarrow \check{x}$ are unitary given that $x^\sim$ and $\check{x}$ are only well defined up to canonical isomorphism).

Let us now consider the situation of a dagger tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between rigid tensor C^* -categories, and let us assume as before that the unit objects of $\mathcal{C}$ and of $\mathcal{D}$ are direct sums of simples. In general, for x an object of $\mathcal{C}$, the morphisms³ $F(\text{ev}_x)$ and $F(\text{coev}_x)$ do not exhibit $F(\bar{x})$ as the conjugate of $F(x)$, because they might fail the balancing condition. A typical example where this problem occurs is the functor $\mathcal{C} \rightarrow \text{End}(\mathcal{C})$ given by the left action of a unitary fusion category $\mathcal{C}$ on itself.

However, if F is full (i.e. surjective at the level of morphisms)⁴, then F sends balanced solutions to balanced solutions, and the canonical isomorphism $\chi_x : F(\bar{x}) \rightarrow \overline{F(x)}$ is unitary:

Lemma 2.2. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a full dagger tensor functor between semisimple rigid C^* -tensor category. Then:*

- For every $x \in \mathcal{C}$, with $\text{ev}_x : \bar{x} \otimes x \rightarrow 1_{\mathcal{C}}$ and $\text{coev}_x : 1_{\mathcal{C}} \rightarrow x \otimes \bar{x}$ as in Lemma 2.1,

$$\begin{aligned} \tilde{\text{ev}}_{F(x)} &:= F(\text{ev}_x) \circ \mu_{\bar{x}, x} : F(\bar{x}) \otimes F(x) \rightarrow 1_{\mathcal{D}} \\ \text{and } \tilde{\text{coev}}_{F(x)} &:= \mu_{x, \bar{x}}^{-1} \circ F(\text{coev}_x) : 1_{\mathcal{D}} \rightarrow F(x) \otimes F(\bar{x}) \end{aligned}$$

form balanced solutions of the duality equations (are solutions of (1) and (2)).

- The isomorphism

$$\chi_x := (\tilde{\text{ev}}_{F(x)} \otimes \text{id}_{\overline{F(x)}}) \circ (\text{id}_{F(\bar{x})} \otimes \text{coev}_{F(x)}) : F(\bar{x}) \rightarrow \overline{F(x)}$$

is unitary, and is also given by $\chi_x = (\text{id}_{\overline{F(x)}} \otimes \tilde{\text{coev}}_{F(x)}^*) \circ (\text{ev}_{F(x)}^* \otimes \text{id}_{F(\bar{x})})$.

Proof. It is easy to see that $\tilde{\text{ev}}_{F(x)}$ and $\tilde{\text{coev}}_{F(x)}$ satisfy condition (1). To check (2), let $g : F(x) \rightarrow F(x)$ be an endomorphism and let $g_{ij} := q_i \otimes g \otimes q_j$, where $q_i, q_j \in \text{End}(1_{\mathcal{D}})$ are minimal projections. Let $p_i, p_j \in \text{End}(1_{\mathcal{C}})$ be the corresponding minimal projections in $\mathcal{C}$, satisfying $F(p_i) = q_i$ and $F(p_j) = q_j$. Since F is full, $g = F(f)$ for some $f : x \rightarrow x$. By assumption, we have $\text{ev}_x(\text{id}_{\bar{x}} \otimes f_{ij}) \text{ev}_x^* = \lambda p_j$ and $\text{coev}_x(f_{ij} \otimes \text{id}_{\bar{x}}) \text{coev}_x^* = \lambda' p_i$ with $\lambda = \lambda'$. It follows that

$$\begin{aligned} \tilde{\text{ev}}_{F(x)} \circ (\text{id}_{F(\bar{x})} \otimes g_{ij}) \circ \tilde{\text{ev}}_{F(x)}^* &= (F(\text{ev}_x) \circ \mu_{\bar{x}, x}) \circ (\text{id}_{F(\bar{x})} \otimes F(f_{ij})) \circ ((F(\text{ev}_x) \circ \mu_{\bar{x}, x})^*) \\ &= F(\text{ev}_x \circ (\text{id}_{\bar{x}} \otimes f_{ij}) \circ \text{ev}_x^*) = F(\lambda p_j) = \lambda q_j \end{aligned}$$

and

$$\begin{aligned} \tilde{\text{coev}}_{F(x)}^* \circ (g_{ij} \otimes \text{id}_{F(\bar{x})}) \circ \tilde{\text{coev}}_{F(x)} &= F(\mu_{x, \bar{x}}^{-1} \circ \text{coev}_x)^* \circ (F(f_{ij}) \otimes \text{id}_{F(\bar{x})}) \circ (\mu_{x, \bar{x}}^{-1} \circ \text{coev}_x) \\ &= F(\text{coev}_x^* \circ (f \otimes \text{id}_{\bar{x}}) \circ \text{coev}_x) = F(\lambda' p_i) = \lambda' q_i \end{aligned}$$

with $\lambda = \lambda'$, as desired.

The isomorphism $\chi_x = (\tilde{\text{ev}}_{F(x)} \otimes \text{id}_{\overline{F(x)}}) \circ (\text{id}_{F(\bar{x})} \otimes \text{coev}_{F(x)})$ is unitary by [BDH14, Thm. 4.22]. It is equal to $(\text{id}_{\overline{F(x)}} \otimes \tilde{\text{coev}}_{F(x)}^*) \circ (\text{ev}_{F(x)}^* \otimes \text{id}_{F(\bar{x})})$ since the latter is visibly the inverse of χ_x . $\square$

³We omit the structure isomorphisms μ and i for better readability.

⁴Note that, by Lemma 3.4, such functors are almost always fully faithful.

2.2 Unitary multifusion categories

A category is called *semisimple* if every object is a finite direct sum of simple objects.

Definition 2.3. A semisimple rigid tensor category with finitely many isomorphism classes of simple objects, and whose unit object is simple is called a *fusion category*. If we drop the condition that the unit object is simple, then we get the notion of a multifusion category: a *multifusion category* is a tensor category which is rigid, semisimple, and which has finitely many isomorphism classes of simple objects.

A *unitary (multi)fusion category* is a dagger tensor category whose underlying dagger category is a C^* -category, and whose underlying tensor category is a (multi)fusion category.

A multifusion category is called *indecomposable* if it is not the direct sum of two non-trivial multifusion categories.

Let $\mathcal{C}$ be an indecomposable multifusion category and let $1 = \bigoplus_{i=1}^k 1_i$ be the decomposition of its unit object into simple objects. Then the subcategories $\mathcal{C}_{ij} := 1_i \otimes \mathcal{C} \otimes 1_j$ are all non-zero, and we may write $\mathcal{C} = \bigoplus_{ij} \mathcal{C}_{ij}$. The tensor product $\otimes : \mathcal{C}_{ij} \times \mathcal{C}_{j'k} \rightarrow \mathcal{C}$ takes values in $\mathcal{C}_{ik}$, and is non-zero if and only if $j = j'$. Each $\mathcal{C}_i := \mathcal{C}_{ii}$ is a fusion category, and each $\mathcal{C}_{ij}$ is an invertible bimodule (a Morita equivalence) between $\mathcal{C}_i$ and $\mathcal{C}_j$ [Müg03, Def. 4.2] [ENO10, Prop. 4.2]. We call $\mathcal{C}_i$ a *corner* of $\mathcal{C}$.

There is an analogy between indecomposable multifusion categories and matrix algebras. Inspired by that analogy, if $\mathcal{C}$ is an indecomposable multifusion category whose unit object is a sum of k irreducible objects, then we call $\mathcal{C}$ a $k \times k$ *multifusion category*,

The datum of a 2×2 multifusion category is equivalent to the data of a fusion category $\mathcal{C}_0$, a fusion category $\mathcal{C}_1$, and a Morita equivalence $\mathcal{M}$ between $\mathcal{C}_0$ and $\mathcal{C}_1$. Letting $\mathcal{M}^*$ be the inverse bimodule of $\mathcal{M}$ (which is unique up to contractible choice), the associated multifusion category is given by

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_0 & \mathcal{M} \\ \mathcal{M}^* & \mathcal{C}_1 \end{pmatrix}. \quad (3)$$

Its unit object $1_{\mathcal{C}} = 1_0 \oplus 1_1$ is the sum of the unit object of $\mathcal{C}_0$ and that of $\mathcal{C}_1$.

When $\mathcal{C}_0$ and $\mathcal{C}_1$ are unitary fusion categories, we define a *unitary Morita equivalence* to be a unitary 2×2 multifusion category as in (3). This is the analog of [Müg03, Def. 5.15] within the context of tensor C^* -categories (Müger only discusses dagger and pivotal tensor categories).

Let $\mathcal{C}$ be a unitary multifusion category. Then the *dimension* of an object $x \in \mathcal{C}_{ij}$ is the unique number d_x that satisfies

$$\mathrm{ev}_x \circ \mathrm{ev}_x^* = d_x p_j \quad \text{and} \quad \mathrm{coev}_x^* \circ \mathrm{coev}_x = d_x p_i. \quad (4)$$

It is such that $d_x = 0$ iff $x = 0$, $d_x \geq 1$ for $x \neq 0$, $d_x = 1$ iff x is invertible, $d_{x \oplus y} = d_x + d_y$ for $x, y \in \mathcal{C}_{ij}$, and $d_{x \otimes y} = d_x \cdot d_y$ for $x \in \mathcal{C}_{ij}$ and $y \in \mathcal{C}_{jk}$ [BDH14, Prop. 5.2] (see also [LR97]). Extending the assignments $x \mapsto d_x$ by additivity to all the objects of $\mathcal{C}$, we get a ring homomorphism $K_0(\mathcal{C}) \rightarrow M_n(\mathbb{R})$.

The *global dimension* of a fusion category is the sum of the squares of the dimensions of its simple objects. By [Müg03, Prop. 5.17], Morita equivalent unitary fusion categories always have same global dimension. As a consequence, if $\mathcal{C}$ is an indecomposable unitary

multifusion category, then all its corners $\mathcal{C}_i$ have same global dimension. We denote this common quantity by $D = D(\mathcal{C})$ and call it the global dimension of $\mathcal{C}$.

The following result generalises [Müg03, Prop. 5.17] and appears as [ENO05, Prop. 2.17]. For every i and j , let $\text{Irr}(\mathcal{C}_{ij})$ be a set of representatives of the irreducible objects of $\mathcal{C}_{ij}$.

Lemma 2.4. *Let $\mathcal{C}$ be an indecomposable unitary multifusion category. Then, for every i and j , we have*

$$\sum_{x \in \text{Irr}(\mathcal{C}_{ij})} d_x^2 = D(\mathcal{C}).$$

We will provide a proof of this lemma in the next section for completeness, and for the convenience of the reader.

2.3 Graphical calculus for unitary multifusion categories

Recall that, in the familiar string diagram formalism for tensor categories, objects are denoted by strands and morphisms are denoted by coupons [JS91, Sel11]. There is an analogous string diagram formalism for 2-categories, where objects are denoted by shaded regions, 1-morphisms by strands, and 2-morphisms by coupons [Sel11, §8][BDH14, §2]. A multifusion category can be thought of as a 2-category whose objects are the irreducible summands of 1, and whose morphisms are given by $\text{Hom}(1_i, 1_j) = \mathcal{C}_{ij}$. As such, we can apply the graphical calculus for 2-categories to depict objects and morphisms in a multifusion category. The shadings of the regions denote the various irreducible summands of the unit object:

$$\square = 1_i \quad \text{light gray} \square = 1_j \quad \text{medium gray} \square = 1_k \quad \text{dark gray} \square = 1_\ell$$

A line between shading i and shading j indicates an object of $\mathcal{C}_{ij}$, and a coupon  is a morphism in that category.

A coupon  with multiple lines on the bottom and on the top denotes a morphism from a tensor product of objects of $\mathcal{C}$ (each of them living in some $\mathcal{C}_{ij}$) to another such tensor product.

Let now $\mathcal{C}$ be a unitary multifusion category. If $x \in \mathcal{C}_{ij}$, $y \in \mathcal{C}_{jk}$, and $z \in \mathcal{C}_{ki}$ are irreducible objects, we let $e_\alpha \in \text{Hom}(1, x \otimes y \otimes z)$ and $e^\alpha \in \text{Hom}(1, \bar{z} \otimes \bar{y} \otimes \bar{x})$ denote dual bases, and consider the canonical element

$$\sqrt{d_x d_y d_z} \cdot \sum_{\alpha} e_\alpha \otimes e^\alpha.$$

As in [HP17, §2.5], we use the convention that a pair of colored nodes denotes a labelling by the above canonical element:

$$\begin{array}{c} x \ y \\ \diagdown \diagup \\ \bullet \\ \diagup \diagdown \\ z \end{array} \otimes \begin{array}{c} z \\ \diagdown \diagup \\ \bullet \\ \diagup \diagdown \\ x \ y \end{array} := \sqrt{d_x d_y d_z} \cdot \sum_{\alpha} \begin{array}{c} x \ y \\ \diagdown \diagup \\ \bullet \\ \diagup \diagdown \\ z \end{array} e_\alpha \otimes \begin{array}{c} z \\ \diagdown \diagup \\ \bullet \\ \diagup \diagdown \\ x \ y \end{array} e^\alpha \quad (5)$$

The following lemma is identical to [HP17, Lemma 2.16]. We include it without proof. Let $N_{x,y}^z := \dim(\text{Hom}(x \otimes y, z))$.

Lemma 2.5. Let $\bigcirc = 1_i$, $\bigcirc = 1_j$, $\bigcirc = 1_k$, $\bigcirc = 1_\ell$. Then the following relations hold:

$$\begin{array}{c} z \\ | \\ x \text{---} y \\ | \\ z \end{array} = \sqrt{d_x d_y d_z^{-1}} \cdot N_{x,y}^z \begin{array}{c} | \\ | \\ | \end{array} \quad (\text{Bigon 1})$$

$$\begin{array}{c} z \\ | \\ x \text{---} y \\ | \\ z \end{array} \otimes \begin{array}{c} x \text{---} y \\ | \\ z \end{array} \otimes \begin{array}{c} z \\ | \\ x \text{---} y \\ | \\ x \text{---} y \end{array} = \sqrt{d_x d_y d_z^{-1}} \cdot \begin{array}{c} | \\ | \\ | \end{array} \otimes \begin{array}{c} x \text{---} y \\ | \\ z \end{array} \otimes \begin{array}{c} z \\ | \\ x \text{---} y \\ | \\ x \text{---} y \end{array} \quad (\text{Bigon 2})$$

$$\sum_{z \in \text{Irr}(\mathcal{C}_{ik})} \sqrt{d_z} \begin{array}{c} x \text{---} y \\ | \\ z \\ | \\ x \text{---} y \end{array} = \sqrt{d_x d_y} \cdot \begin{array}{c} | \\ | \\ | \end{array} \quad (\text{Fusion})$$

$$\sum_{v \in \text{Irr}(\mathcal{C}_{i\ell})} \begin{array}{c} y \text{---} z \\ | \\ v \\ | \\ w \text{---} x \end{array} \otimes \begin{array}{c} \bar{z} \text{---} \bar{y} \\ | \\ \bar{v} \\ | \\ \bar{x} \text{---} \bar{w} \end{array} = \sum_{u \in \text{Irr}(\mathcal{C}_{jk})} \begin{array}{c} y \text{---} z \\ | \\ u \\ | \\ w \text{---} x \end{array} \otimes \begin{array}{c} \bar{z} \text{---} \bar{y} \\ | \\ \bar{u} \\ | \\ \bar{x} \text{---} \bar{w} \end{array} \quad (\text{I=H})$$

The next lemma is a version of [HP17, Lemma 2.17].

Lemma 2.6. Let $\bigcirc = 1_i$, $\bigcirc = 1_j$, $\bigcirc = 1_k$. Then the following relation holds:

$$\sum_{\substack{a \in \text{Irr}(\mathcal{C}_{ij}) \\ b \in \text{Irr}(\mathcal{C}_{jk})}} \begin{array}{c} y \\ | \\ a \text{---} b \\ | \\ x \end{array} \otimes \begin{array}{c} \bar{y} \\ | \\ \bar{b} \text{---} \bar{a} \\ | \\ \bar{x} \end{array} = D \cdot \delta_{x,y} \begin{array}{c} | \\ | \\ | \end{array} \otimes \begin{array}{c} | \\ | \\ | \end{array} \quad (6)$$

Proof of Lemma 2.4 and of Lemma 2.6. We use the I=H relation to rewrite the left hand side of (6) as

$$\sum_{\substack{a \in \text{Irr}(\mathcal{C}_{ij}) \\ b \in \text{Irr}(\mathcal{C}_i)}} \begin{array}{c} y \\ | \\ a \text{---} b \\ | \\ x \end{array} \otimes \begin{array}{c} \bar{y} \\ | \\ \bar{b} \text{---} \bar{a} \\ | \\ \bar{x} \end{array}$$

The only terms which contribute to the sum are the ones with $b = 1_i$. We are left with

$$\delta_{x,y} \sum_{a \in \text{Irr}(\mathcal{C}_{ij})} \begin{array}{c} \bigcirc \\ | \\ a \end{array} \cdot \begin{array}{c} \bigcirc \\ | \\ \bar{a} \end{array} \cdot \begin{array}{c} | \\ | \\ | \end{array} \otimes \begin{array}{c} | \\ | \\ | \end{array} = \delta_{x,y} \left(\sum_{a \in \text{Irr}(\mathcal{C}_{ij})} d_a^2 \right) \cdot \begin{array}{c} | \\ | \\ | \end{array} \otimes \begin{array}{c} | \\ | \\ | \end{array}.$$

By symmetry, the left hand side of (6) is also equal to

$$\delta_{x,y} \left(\sum_{b \in \text{Irr}(\mathcal{C}_{jk})} d_b^2 \right) \cdot \begin{array}{c} | \\ | \\ | \end{array} \otimes \begin{array}{c} | \\ | \\ | \end{array}.$$

It follows that $\sum_{x \in \text{Irr}(\mathcal{C}_{ij})} d_x^2 = \sum_{x \in \text{Irr}(\mathcal{C}_{jk})} d_x^2$.

For every i, j, k, ℓ , we conclude (using that $\mathcal{C}$ is indecomposable, i.e., that none of the $\mathcal{C}_{ij}$ is zero) that

$$\sum_{x \in \text{Irr}(\mathcal{C}_{ij})} d_x^2 = \sum_{x \in \text{Irr}(\mathcal{C}_{jk})} d_x^2 = \sum_{x \in \text{Irr}(\mathcal{C}_{k\ell})} d_x^2. \quad \square$$

3 Representations of unitary multifusion categories

A representation of a C^* -algebra A is a $*$ -homomorphism $A \rightarrow B(H)$, for H some Hilbert space. We take the perspective that a C^* -tensor category $\mathcal{C}$ is a higher categorical analog of C^* -algebra, and that a good higher categorical analog of a Hilbert space is a W^* -category, i.e., a category of the form $R\text{-Mod}$, for R some von Neumann algebra. A representation of $\mathcal{C}$ on $R\text{-Mod}$ is then a dagger tensor functor $\mathcal{C} \rightarrow \text{End}(R\text{-Mod}) = \text{Bim}(R)$ into the category of R - R -bimodules.

3.1 Representations of C^* -tensor categories

Given a von Neumann algebra R with separable predual, let us write $\text{Bim}(R)$ for the category of bimodules whose underlying Hilbert space is separable. It is a C^* -tensor category when equipped with the *Connes fusion product* [Sau83], [Con94, Appendix B.δ], [SY17]

$$\boxtimes_R : \text{Bim}(R) \times \text{Bim}(R) \rightarrow \text{Bim}(R).$$

Here, given a right module X and a left module Y , their fusion $X \boxtimes_R Y$ is the completion of the vector space $\text{Hom}_R(L^2 R, X) \otimes_R Y$ with respect to the inner product

$$\langle a_1 \otimes y_1, a_2 \otimes y_2 \rangle := \langle \langle a_2 | a_1 \rangle_R y_1, y_2 \rangle,$$

where $\langle a_2 | a_1 \rangle_R := a_2^* \circ a_1 \in \text{Hom}(L^2 R_R, L^2 R_R) = R$. The fusion can be equivalently described as a completion of $X \otimes_R \text{Hom}_R(L^2 R, Y)$, and as a completion of $\text{Hom}_R(L^2 R, X) \otimes_R L^2 R \otimes_R \text{Hom}_R(L^2 R, Y)$.

The unit object of the above monoidal structure is provided by the standard form of the von Neumann algebra $L^2 R \in \text{Bim}(R)$ [Haa75, Tak03]. It is an R - R -bimodule which, for every faithful state φ , is canonically isomorphic to the GNS Hilbert space $L^2(R, \varphi)$. The image of $1 \in R \hookrightarrow L^2(R, \varphi) \cong L^2(R)$ is denoted $\sqrt{\varphi} \in L^2(R)$.

Definition 3.1. Let $\mathcal{C}$ be a C^* -tensor category. A C^* -representation of $\mathcal{C}$ is a dagger tensor functor

$$\mathcal{C} \rightarrow \text{Bim}(R)$$

for some von Neumann algebra R .

Given two C^* -representations $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(S)$, a *morphism* from α to β consists of a bimodule ${}_S \Phi_R$ along with unitary isomorphisms $\phi_c : \Phi \boxtimes_R \alpha(c) \rightarrow \beta(c) \boxtimes_S \Phi$, natural in $c \in \mathcal{C}$, which satisfy the following half-braiding condition:

$$\begin{array}{ccc} \Phi \boxtimes_R \alpha(c) \boxtimes_R \alpha(d) & \xrightarrow{\phi_c \boxtimes \text{id}} & \beta(c) \boxtimes_S \Phi \boxtimes_R \alpha(d) \xrightarrow{\text{id} \boxtimes \phi_d} \beta(c) \boxtimes_S \beta(d) \boxtimes_S \Phi \\ \downarrow \text{id} \boxtimes \mu^\alpha & & \downarrow \mu^\beta \boxtimes \text{id} \\ \Phi \boxtimes_R \alpha(c \otimes d) & \xrightarrow{\phi_{c \otimes d}} & \beta(c \otimes d) \boxtimes_S \Phi. \end{array} \quad (7)$$

An *isomorphism* between α and β is a morphism $(\Phi, \phi) : \alpha \rightarrow \beta$, where Φ is an invertible bimodule.

Example 3.2. Let R be a factor and g an outer automorphism that squares to the identity. Let $\alpha : \text{Hilb}_{\text{fd}}[\mathbb{Z}/2] \rightarrow \text{Bim}(R)$ be the trivial representation, and let $\beta : \text{Hilb}_{\text{fd}}[\mathbb{Z}/2] \rightarrow \text{Bim}(R)$ be the representation induced by g . Then $\Phi = L^2 R \oplus L^2 R_g$, together with the obvious isomorphisms $\phi_x : \Phi \boxtimes_R \alpha(x) \rightarrow \beta(x) \boxtimes_R \Phi$ is a morphism from α to β .

Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *faithful* if the maps $\text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$ are injective, and *fully faithful* if these maps are isomorphisms. A C^* -representation $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ is called *fully faithful* if the functor α is fully faithful. When $\mathcal{C}$ is semisimple, this is equivalent to the condition that the simple objects of $\mathcal{C}$ remain simple in $\text{Bim}(R)$, and that non-isomorphic objects of $\mathcal{C}$ remain non-isomorphic in $\text{Bim}(R)$.

Lemma 3.3. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are semisimple rigid tensor categories with simple units, and $F : \mathcal{C} \rightarrow \mathcal{D}$ is a tensor functor. Then F is fully faithful if and only if*

$$\nexists c \in \text{Irr}(\mathcal{C}) \setminus \{1_{\mathcal{C}}\} \text{ such that } F(c) \text{ contains } 1_{\mathcal{D}} \text{ as a summand.}$$

Proof. If F is fully faithful, then the conclusion of the lemma is clearly satisfied. If F is not fully faithful, then either a simple object in $\mathcal{C}$ has non-simple image in $\mathcal{D}$, or two non-isomorphic simples in $\mathcal{C}$ have isomorphic images in $\mathcal{D}$. If $c \in \mathcal{C}$ is simple but $F(c)$ is not simple, then $c \otimes c^\sim \oplus 1_{\mathcal{C}} \mapsto F(c) \otimes F(c)^\sim \oplus 1_{\mathcal{D}}$, and the latter contains $1_{\mathcal{D}}$ since $F(c)$ is not simple. If $c, d \in \mathcal{C}$ are non-isomorphic simples with $F(c) \cong F(d)$, then $1_{\mathcal{C}}$ is not a summand of $c \otimes d^\sim$, but $F(c \otimes d^\sim) \cong F(c) \otimes F(c)^\sim$ contains $1_{\mathcal{D}}$. $\square$

Lemma 3.4. *Let $\mathcal{C}$ be an indecomposable multifusion category (or, more generally, an indecomposable semisimple rigid C^* -tensor category) and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor, where $\mathcal{D}$ is not the zero category. Then F is a faithful functor.*

Proof. Decomposing each object of $\mathcal{C}$ as a direct sum of simples, it is easy to check that the map $\text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$ is injective for every $x, y \in \mathcal{C}$ if and only if it is injective for every $x, y \in \text{Irr}(\mathcal{C})$. Let $x, y \in \text{Irr}(\mathcal{C})$ be simple objects. If $x \neq y$, there is nothing to show. If $x = y$, we need to show that $F(\text{id}_x) \neq 0$. Equivalently, we need to show that $F(x) \not\cong 0$ in $\mathcal{D}$.

Let $x \in \text{Irr}(\mathcal{C}_{ij})$ be a simple object. We assume by contradiction that $F(x) \cong 0$, and show that $\mathcal{D}$ is the zero category. The coevaluation map $1_i \rightarrow x \otimes x^\sim$ exhibits 1_i as a direct summand of $x \otimes x^\sim$. Therefore $F(1_i)$ is a direct summand of $F(x \otimes x^\sim)$. The latter is isomorphic to $F(x) \otimes F(x^\sim) \cong 0 \otimes F(x^\sim) \cong 0$, so $F(1_i) \cong 0$. For every object $y \in \mathcal{C}_{ki}$, we therefore have

$$F(y) \cong F(y \otimes 1_i) \cong F(y) \otimes F(1_i) \cong F(y) \otimes 0 \cong 0.$$

Now, 1_k is a direct summand of $y \otimes y^\sim$, so $F(1_k)$ is a direct summand of $F(y \otimes y^\sim)$. The latter is zero because $F(y) \cong 0$, so $F(1_k)$ is zero. All the 1_k 's go to zero. It follows that $1_{\mathcal{D}} \cong F(\bigoplus_k 1_k) \cong \bigoplus_k F(1_k) \cong 0$, so $\mathcal{D}$ is the zero category. $\square$

3.2 Uniqueness of representations of unitary fusion categories

Given a von Neumann algebra M and a normal $*$ -endomorphism $\rho : M \rightarrow M$, let ${}_{\rho}L^2M \in \text{Bim}(M)$ denote the bimodule obtained from L^2M by twisting its left action by ρ . It is well known that when M is a factor of type III or II_{∞} , the map $\rho \mapsto {}_{\rho}L^2M$ induces an equivalence of C^* -tensor categories

$$\text{End}(M) \cup \{0\} \xrightarrow{\simeq} \text{Bim}(M). \quad (8)$$

Here, $\text{End}(M)$ is the category whose objects are normal $*$ -endomorphisms, whose morphisms are given by

$$\text{Hom}(\rho, \sigma) := \{a \in M \mid a\rho(x) = \sigma(x)a \ \forall x \in M\},$$

with $*$ -operation $a \mapsto a^*$, and tensor product operation given by the composition of endomorphisms: $\sigma \otimes \rho := \rho \circ \sigma$. More generally, for any properly infinite von Neumann algebra M , $\text{Bim}(M)$ is equivalent to the idempotent completion of $\text{End}(M)$.

In [Izu17, Thm. 2.2], Izumi uses Popa's subfactor uniqueness theorem [Pop95b] to prove that representations of unitary fusion categories as endomorphisms of a hyperfinite type III_1 factor with separable predual are unique up to isomorphism (for the same notion of isomorphism as the one in Definition 3.1). We could use Popa's uniqueness theorem for hyperfinite finite depth II_1 subfactors [Pop90] to prove the analogous result for hyperfinite II_1 and II_{∞} factors. We will instead give a unified proof for uniqueness of representations into hyperfinite II_1 , II_{∞} , and III_1 factors based on the following powerful theorem recently proven by Tomatsu (we phrase it in the special case when M is a factor):

Theorem 3.5 ([Tom21, Thm. D]). *Let $\mathcal{C}$ be an amenable rigid C^* -tensor category. Let α and β be centrally free cocycle actions of $\mathcal{C}$ on a properly infinite factor M with separable predual. Suppose that $\alpha(c)$ and $\beta(c)$ are approximately unitarily equivalent for all $c \in \mathcal{C}$. Then α and β are strongly cocycle conjugate.*

Let us first explain the terms which appear in the above theorem.

Here, *cocycle actions* are exactly dagger tensor functors $\mathcal{C} \rightarrow \text{End}(M) \cup \{0\}$, and two such being *cocycle conjugate* [Tom21, Def. 5.6] means that there exists an automorphism Φ of M and a unitary monoidal equivalence $\phi : \text{Ad}(\Phi) \circ \alpha \Rightarrow \beta$

$$\begin{array}{ccc} & \alpha \nearrow & \text{End}(M) \cup \{0\} \\ \mathcal{C} & & \downarrow \text{Ad}(\Phi) \\ & \beta \searrow & \text{End}(M) \cup \{0\} \end{array} \quad \phi \not\Downarrow \quad (9)$$

(The adverb *strongly* in *strongly cocycle conjugate* means we can take Φ to be approximately inner; this condition is not necessary for our purposes here.) The definition of an *amenable* rigid C^* -tensor category appears in [HI98] as a certain Følner type condition, and unitary fusion categories are visibly amenable.

An endomorphism $\rho \in \text{End}(M)$ is called *centrally trivial* if the induced endomorphism ρ^{ω} on the ultrapower M^{ω} restricts to the identity endomorphism on the central sequence algebra M_{ω} [MT09, §2.4 and Def. 4.1]. It is called *properly centrally non-trivial* if no direct

summand of ρ is centrally trivial.⁵ Finally, a dagger tensor functor $\alpha : \mathcal{C} \rightarrow \text{End}(M) \cup \{0\}$ is *centrally free* if $\alpha(c)$ is properly centrally non-trivial for every $c \in \text{Irr}(\mathcal{C}) \setminus \{1_{\mathcal{C}}\}$. The following result is an immediate corollary of Lemma 3.3:

Lemma 3.6. *Let $\mathcal{C}$ be a rigid C^* -tensor category, M an infinite factor, and $\alpha : \mathcal{C} \rightarrow \text{End}(M) \cup \{0\}$ a C^* -representation. Then α is centrally free if and only if it is fully faithful and for every $c \in \text{Irr}(\mathcal{C}) \setminus \{1_{\mathcal{C}}\}$, its image $\alpha(c) \in \text{End}(M)$ is not centrally trivial.* $\square$

The following result of Masuda and Tomatsu characterises centrally trivial endomorphisms of hyperfinite factors.

Theorem 3.7 ([MT09, Thm. 4.12, Lem. 4.10]). *Let R be a hyperfinite factor of type II_{∞} or III_{λ} for $\lambda \in (0, 1]$. Then the set of irreducible centrally trivial endomorphisms of R is exactly the set of automorphisms of the form $\text{Ad}(u) \circ \sigma_t^{\phi}$, where $\phi \in R_*$ is a normal state, and $(\sigma_t^{\phi})_{t \in \mathbb{R}}$ denotes the modular automorphism group.*

As a corollary of Lemma 3.6 and Theorem 3.7, we get:

Corollary 3.8. *Let $\mathcal{C}$ be a unitary fusion category, R a hyperfinite factor of type II_{∞} or III_1 , and $\alpha : \mathcal{C} \rightarrow \text{End}(R) \cup \{0\}$ a C^* -representation. Then α is centrally free if and only if it is fully faithful.*

Proof. When R is of type II_{∞} , the automorphisms σ_t^{ϕ} are all inner, so the conditions in Lemma 3.6 are trivially satisfied. When R is of type III_1 , the automorphisms σ_t^{ϕ} have infinite order in $\text{Out}(M)$. Since $\mathcal{C}$ has only finitely many types of simple objects, no non-trivial σ_t^{ϕ} can be in the image of α . $\square$

For the definition of *approximately unitarily equivalent*, we refer the reader to [Tom21, §2.4]. We will only use this concept in the following lemma:

Lemma 3.9. *Let R be a hyperfinite factor with separable predual, and let $\sigma, \rho \in \text{End}(R)$ be dualizable finite depth⁶ endomorphisms.*

- *If R is of type III_1 , then σ and ρ are always approximately unitarily equivalent.*
- *If R is of type II_{∞} , σ and ρ are irreducible and their dimensions agree $d_{\sigma} = d_{\rho}$, then σ and ρ are approximately unitarily equivalent.*

Proof. Let $\overline{\text{Int}}_r(R) \subset \text{End}(R)$ denote the set of approximately inner endomorphisms of rank r [Tom21, Def. 2.2]. If we can show that ρ and σ are approximately inner of rank r for the same number $r \in \mathbb{R}_{>0}$, then we are finished by [Tom21, Prop. 2.7], which states any two approximately inner endomorphisms of rank r are approximately unitarily equivalent.

If R is of type III_1 , then by [MT09, Cor. 3.16(4)] the set $\overline{\text{Int}}_r(R)$ is independent of r , and equal to the set of dualizable endomorphisms R . The result follows.

If R is of type II_{∞} , fix a faithful normal semifinite trace Tr on R . By [Izu03, Cor. 4.4 and Rem. 4.6], both ρ and σ admit Connes-Takesaki modules $\text{Mod}(\rho), \text{Mod}(\sigma) \in \mathbb{R}_{>0}$. The

⁵The equivalence between this definition and the one presented in [Tom21, Def. 2.16] follows along the same lines as the proof of [MT07, Lem. 8.3].

⁶An endomorphism $\rho \in \text{End}(R)$ has *finite depth* if the C^* -tensor category generated by ρ is fusion.

Connes-Takesaki module is multiplicative [Izu03, Prop. 4.2(1)] so, by the finite depth assumption, $\text{Mod}(\rho)$ and $\text{Mod}(\sigma)$ generate finite subgroups of $\mathbb{R}_{>0}$, hence trivial. Thus, ρ and σ both have trivial Connes-Takesaki modules. Finally, by [MT09, Lem 2.15], we have $\rho, \sigma \in \overline{\text{Int}}_r(R)$ for $r = d_\sigma = d_\rho$. $\square$

With the above preliminaries in place, the following result is an immediate corollary of Tomatsu's Theorem (Thm. 3.5):

Theorem 3.10. *Let $\mathcal{C}$ be a unitary fusion category, and let*

$$\alpha : \mathcal{C} \rightarrow \text{Bim}(R), \quad \beta : \mathcal{C} \rightarrow \text{Bim}(S)$$

be fully faithful C^ -representations, where R and S are hyperfinite factors with separable preduals that are either both of type II (either II_1 or II_∞), or both of type III_1 . Then there is an isomorphism $(\Phi, \phi) : \alpha \rightarrow \beta$.*

Proof. The hyperfinite II_1 and II_∞ factors are Morita equivalent, so we may assume that R and S are either both of type II_∞ , or both of type III_1 . By the uniqueness of the hyperfinite II_∞ and III_1 factors ([MvN43, Con76, Haa87, Haa16]), we may furthermore assume without loss of generality that $R = S$.

View α and β as representations $\mathcal{C} \rightarrow \text{End}(R) \cup \{0\}$ under the equivalence (8). By Corollary 3.8, α and β are centrally free. By Lemma 3.9, for every $c \in \text{Irr}(\mathcal{C})$, $\alpha(c)$ and $\beta(c)$ are approximately unitarily equivalent. By Tomatsu's Theorem (Thm. 3.5), α and β are therefore strongly cocycle conjugate. In particular, there exists $\Phi \in \text{Aut}(R)$ and a unitary monoidal equivalence $\phi : \text{Ad}(\Phi) \circ \alpha \Rightarrow \beta$, as in (9). Applying the equivalence (8) once again, we are finished. $\square$

3.3 Existence of representations of 2×2 unitary fusion categories

In [HP17, §3.1], we explained how results of Popa [Pop95a, Thm. 3.1] (see also [FR13, Thm. 4.1]) can be used to construct a fully faithful representation $\mathcal{C} \rightarrow \text{Bim}(R)$ of a unitary fusion category $\mathcal{C}$ on a hyperfinite factor R which is not type I (or, more generally, any factor which tensorially absorbs the hyperfinite II_1 factor). In this section, we extend this to 2×2 unitary multifusion categories.

Let $A \subset B$ be an inclusion of factors. Recall from [BDH14, Def. 5.1 and 5.10] that its *index* $[B : A]$ is the square of the dimension of the bimodule ${}_A L^2(B)_B$. By definition, the latter is specified by the conditions (4) when ${}_A L^2(B)_B$ is dualizable, and is otherwise infinite. By [BDH14, Cor. 7.14], unless A and B are finite dimensional, the above index is equal to the index of Longo's *minimal conditional expectation* $E_0 : B \rightarrow A$ (defined in [Lon89, Thm. 5.5]).

Let us now assume that $A \subset B$ has finite index. In the diagrammatic calculus, denoting A and B by the regions

$$\square = A \quad \blacksquare = B,$$

the standard solutions of the duality equations (1, 2) are denoted by the shaded/unshaded cups and caps:

$$\begin{aligned} R &:= \text{unshaded cup} : L^2(A) \rightarrow {}_A L^2(B) \boxtimes_B L^2(B)_A & R^* &:= \text{shaded cap} \\ S &:= \text{shaded cup} : L^2(B) \rightarrow {}_B L^2(B) \boxtimes_A L^2(B)_B & S^* &:= \text{unshaded cap} \end{aligned}$$

The dimension $d = [B : A]^{1/2}$ is then specified by the balancing equations $R^* \circ R = d \operatorname{id}_{L^2(A)}$ and $S^* \circ S = d \operatorname{id}_{L^2(B)}$.

It was shown in [DGG14, §5] that the collection $\mathcal{P}_\bullet = \{\mathcal{P}_{n,\pm}\}$ of finite dimensional C^* -algebras

$$\begin{aligned} \mathcal{P}_{2n,+} &:= \operatorname{End}_{A-A} (L^2(B)^{\boxtimes_{A^n}}) & \mathcal{P}_{2n+1,+} &:= \operatorname{End}_{A-B} (L^2(B)^{\boxtimes_{A^{n+1}}}) \\ \mathcal{P}_{2n,-} &:= \operatorname{End}_{B-B} (L^2(B)^{\boxtimes_{A^{n+1}}}) & \mathcal{P}_{2n+1,-} &:= \operatorname{End}_{B-A} (L^2(B)^{\boxtimes_{A^{n+1}}}) \end{aligned}$$

has the structure of a C^* -planar algebra [Jon21, Def. 1.37]. Note that the planar algebra structure on $\mathcal{P}_\bullet$ depends on the choice of R and S above. Since they were chosen to satisfy the balancing condition (2), the planar algebra $\mathcal{P}_\bullet$ is *spherical*, i.e., it satisfies

$$\begin{array}{c} \text{cup} \end{array} f = \begin{array}{c} \text{cap} \end{array} f \quad \forall f \in \operatorname{End}_{A-B}(L^2(B)).$$

Remark 3.11. When $A \subset B$ is a finite index II_1 subfactor which is *extremal* (i.e., the traces $\operatorname{tr}_{A'}$ and tr_B agree on the relative commutant $A' \cap B$), Jones [Jon21, Thm. 4.2.1] gave another construction of a spherical C^* -planar algebra from $A \subset B$, commonly referred to as the *standard invariant* of the subfactor. It was shown in [DGG14, Proof of Thm. 5.4 and Rem. 5.5] that, in this setting, the C^* -planar algebras defined in [Jon21, Thm. 4.2.1] and in [DGG14, §5] are isomorphic.

The construction in [DGG14, §5] can be applied in greater generality. Let $\mathcal{C}$ be a rigid C^* -tensor category whose unit decomposes as a direct sum of two simple objects $1_{\mathcal{C}} = 1_+ \oplus 1_-$, and let $X = 1_+ \otimes X \otimes 1_-$ be an object that generates $\mathcal{C}$ (i.e., such that every object of $\mathcal{C}$ is isomorphic to a direct sum of direct summands of alternating tensor powers of X and $\bar{X}$). We may then set

$$\mathcal{P}_{n,+} := \operatorname{Hom}_{\mathcal{C}}(1_+, (X \otimes \bar{X})^{\otimes n}) \quad \text{and} \quad \mathcal{P}_{n,-} := \operatorname{Hom}_{\mathcal{C}}(1_-, (\bar{X} \otimes X)^{\otimes n}).$$

Using the balanced solutions of the duality equations (1, 2), the graphical calculus for $\mathcal{C}$ equips $\mathcal{P}_\bullet = \{\mathcal{P}_{n,\pm}\}$ with the structure of a spherical C^* -planar algebra.

One can recover $\mathcal{C}$ from the C^* -planar algebra $\mathcal{P}_\bullet$ in the following way. Consider the non-idempotent complete C^* tensor category $\mathcal{C}^\circ$, whose objects are the symbols $X \otimes \bar{X} \otimes X \otimes \dots \otimes X$, $X \otimes \bar{X} \otimes X \otimes \dots \otimes \bar{X}$, $\bar{X} \otimes X \otimes \bar{X} \otimes \dots \otimes X$, $\bar{X} \otimes X \otimes \bar{X} \otimes \dots \otimes \bar{X}$, and whose hom-spaces are given by the $\mathcal{P}_{n,\pm}$. Then the idempotent completion of $\mathcal{C}^\circ$ is $\mathcal{C}$ (the obvious inclusion functor $\mathcal{C}^\circ \rightarrow \mathcal{C}$ exhibits $\mathcal{C}$ as the idempotent completion of $\mathcal{C}^\circ$). This construction is known as the *category of projections* of $\mathcal{P}_\bullet$. The generating object X is the strand of the planar algebra in $\mathcal{P}_{1,+}$, and the action of cups and caps from the planar algebra give balanced solutions of the duality equations.

The above two constructions are each other's inverses [Pen20, Thm. C and §4] (see also [HPT16]):

Theorem 3.12. *There is an equivalence of categories⁷*

$$\left\{ \begin{array}{l} \text{Spherical } C^*\text{-planar algebras } \mathcal{P}_\bullet \text{ with each } \mathcal{P}_{n,\pm} \text{ finite dimensional and } \mathcal{P}_{0,\pm} \cong \mathbb{C} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Pairs } (\mathcal{C}, X) \text{ with } \mathcal{C} \text{ a rigid } C^* \text{ tensor category} \\ \text{with } 1_{\mathcal{C}} = 1_+ \oplus 1_- \text{ a simple decomposition and} \\ \text{a generator } X \in \mathcal{C} \text{ such that } X = 1_+ \otimes X \otimes 1_- \end{array} \right\}.$$

⁷Similarly to [HPT16, Lem. 3.5], the collection of pairs $(\mathcal{C}, X)$ forms a 2-category which is equivalent to a 1-category.

To summarize, in [Jon21], Jones constructed a map from extremal finite index II_1 subfactors $A \subset B$ to C^* -planar algebras. In [DGG14] (see also [Pen20]) a map was constructed from pairs $(\mathcal{C}, X)$ as above to C^* -planar algebras. As explained in Remark 3.11, Jones' construction factors through this second map, by taking $\mathcal{C} \subset \text{Bim}(A \oplus B)$ to be the subcategory generated by ${}_A L^2 B_B$, and $X = {}_A L^2 B_B$. So we have a commutative diagram.

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \text{Finite index sub-} \\ \text{factors } A \subset B \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{l} \text{Spherical } C^*\text{-planar} \\ \text{algebras } \mathcal{P}_\bullet \end{array} \right\} \\
 & \searrow & \updownarrow \cong \\
 & & \left\{ \begin{array}{l} \text{Rigid } C^* \text{ tensor categories} \\ \mathcal{C} \text{ with generator } X \end{array} \right\}
 \end{array} \tag{10}$$

Definition 3.13. A C^* -planar algebra is called *finite depth* if the corresponding rigid C^* tensor category $\mathcal{C}$ is multifusion (has finitely many isomorphism classes of simple objects).

A finite index subfactor $A \subset B$ is called *finite depth* if its standard invariant is finite depth, equivalently, if only finitely many isomorphism classes of A - A (equivalently A - B , B - A , or B - B) bimodules occur as direct summands of $\bigoplus_{n \in \mathbb{N}} L^2(B)^{\boxtimes n}$.

Remark 3.14. A finite index, finite depth II_1 subfactor is automatically extremal [Pop90, 3.7.1].

By [Pop95a, Thm. 3.1], as explained in [Jon21, Proof of Thm. 4.3.1], the horizontal map in (10) is surjective. Specifically, given a spherical C^* -planar algebra $\mathcal{P}_\bullet$, there exists an extremal II_1 subfactor $A \subset B$ whose standard invariant is isomorphic to $\mathcal{P}_\bullet$. Moreover, when $\mathcal{P}_\bullet$ has finite depth, then A and B can be taken to be hyperfinite.

Theorem 3.15. *Let R be any hyperfinite non type I factor. Then every 2×2 unitary multifusion category $\mathcal{C}$ admits a fully faithful representation into $\text{Bim}(R^{\oplus 2})$.*

Proof. Let $X \in \mathcal{C}_{01}$ be any object which generates $\mathcal{C}$ as an involutive category (i.e., such that every simple object of $\mathcal{C}$ appears as a direct summand of either $X \otimes \bar{X} \otimes X \otimes \dots \otimes X$, $X \otimes \bar{X} \otimes X \otimes \dots \otimes \bar{X}$, $\bar{X} \otimes X \otimes \bar{X} \otimes \dots \otimes X$, or $\bar{X} \otimes X \otimes \bar{X} \otimes \dots \otimes \bar{X}$). For example, we may take $X = \bigoplus_{c \in \text{Irr}(\mathcal{C}_{01})} c$. Let $\mathcal{P}_\bullet$ be the finite depth spherical C^* -planar algebra associated to the pair $(\mathcal{C}, X)$. By Popa's Theorem ([Pop95a, Thm. 3.1], [Jon21, Proof of Thm. 4.3.1]), there exists a hyperfinite II_1 subfactor $A \subset B$ whose standard invariant is isomorphic to $\mathcal{P}_\bullet$.

By the commutativity of (10), the map $X \mapsto {}_A L^2 B_B$ extends to a fully faithful representation $\mathcal{C} \rightarrow \text{Bim}(A \oplus B) \cong \text{Bim}(R_{\text{II}_1}^{\oplus 2})$, where R_{II_1} denotes the hyperfinite II_1 factor. By [HP17, Lem. 3.4], since R is hyperfinite and not of type I, $R \bar{\otimes} R_{\text{II}_1} \cong R$. We get the desired fully faithful representation by tensoring with R :

$$\mathcal{C} \longrightarrow \text{Bim}(A \oplus B) \cong \text{Bim}(R_{\text{II}_1}^{\oplus 2}) \xrightarrow{- \bar{\otimes} R} \text{Bim}(R^{\oplus 2}). \quad \square$$

Corollary 3.16. *Let $\mathcal{C}$ be a unitary fusion category, and let R be a hyperfinite factor which is not of type I. Then there exists a fully faithful representation $\mathcal{C} \rightarrow \text{Bim}(R)$.*

Proof. Apply the previous theorem to the 2×2 multifusion category $\begin{pmatrix} \mathcal{C} & \mathcal{C} \\ \mathcal{C} & \mathcal{C} \end{pmatrix}$. $\square$

It should be possible to extend the above result to the case of $k \times k$ unitary multifusion categories, but the argument is a bit tricky so, for now, we are content to make the following conjecture, which we leave to a future joint article.

Conjecture 3.17. *Let $\mathcal{C}$ be a $k \times k$ unitary multifusion category, and let R be a hyperfinite factor which is not of type I. Then there exists a fully faithful representation $\mathcal{C} \rightarrow \text{Bim}(R^{\oplus k})$. Moreover, if $\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus k})$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(S^{\oplus k})$ are fully faithful representations, where R and S are either both of type II or both of type III₁, then α and β are isomorphic in the sense of Definition 3.1.*

4 Bi-involutive tensor categories

4.1 Bi-involutive tensor categories

An *involutive* tensor category [Egg11] is a tensor category $\mathcal{C}$ equipped with an anti-linear anti-tensor functor

$$\bar{\cdot} : \mathcal{C} \rightarrow \mathcal{C}, \quad \nu_{x,y} : \bar{x} \otimes \bar{y} \rightarrow \overline{y \otimes x}, \quad r : 1 \rightarrow \bar{1}$$

which squares to the identity in the sense that we are given isomorphisms $\varphi_x : x \rightarrow \bar{\bar{x}}$, natural in x , satisfying $\overline{\varphi_x} = \varphi_{\bar{x}}$, $\varphi_1 = \bar{r} \circ r$, and $\varphi_{x \otimes y} = \overline{\nu_{y,x}} \circ \nu_{\bar{x},\bar{y}} \circ (\varphi_x \otimes \varphi_y)$. The object $\bar{x}$ is called the *conjugate* of x .

Definition 4.1. A *bi-involutive* tensor category [HP17, §2.1] is a dagger tensor category equipped with an involutive structure such that the functor $\bar{\cdot}$ is a dagger functor, and the structure isomorphisms ν, r, φ are unitary.

A bi-involutive functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between bi-involutive tensor categories is a dagger tensor functor with unitary isomorphisms $\chi_x : F(\bar{x}) \rightarrow \overline{F(x)}$, natural in x , satisfying

$$\begin{aligned} \chi_{\bar{x}} &= \overline{\chi_x}^{-1} \circ \varphi_{F(x)} \circ F(\varphi_x)^{-1}, \\ \chi_{1_{\mathcal{C}}} &= \bar{i} \circ r_{\mathcal{D}} \circ i^{-1} \circ F(r_{\mathcal{C}})^{-1}, \\ \text{and} \quad \chi_{x \otimes y} &= \overline{\mu_{x,y}} \circ \nu_{F(y), F(x)} \circ (\chi_y \otimes \chi_x) \circ \mu_{\bar{y}, \bar{x}}^{-1} \circ F(\nu_{y,x})^{-1}. \end{aligned}$$

Example 4.2. The tensor category of Hilbert spaces and bounded linear maps is a bi-involutive tensor category, where the involution $\bar{\cdot}$ sends a Hilbert space to its complex conjugate. The subcategories of finite dimensional Hilbert spaces, and of separable Hilbert spaces are similarly bi-involutive tensor categories.

Example 4.3. Every unitary multifusion category is a bi-involutive tensor category. More generally, any semisimple rigid C*-tensor category is a bi-involutive tensor category. The conjugate is described in Lemma 2.1, and the structure map φ_x is given by

$$\varphi_x := (\text{id} \otimes \text{ev}_x) \circ (\text{ev}_{\bar{x}}^* \otimes \text{id}) : x \rightarrow \bar{\bar{x}}.$$

By [BDH14, Thm. 4.22], the isomorphism φ_x is unitary; it is therefore also given by the formula $\varphi_x = (\text{coev}_x^* \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_{\bar{x}})$.

In the graphical calculus for rigid tensor categories [Sel11], objects are commonly depicted by oriented strands which may bend up and down. The strands could equally well have been chosen *cooriented* given that, in the presence of an ambient orientation (an orientation of the plane in which the string diagrams are drawn), an orientation is equivalent to a coorientation. However, when dealing with unitary (multi)fusion categories, or, more generally, when dealing with bi-involutive tensor categories, *it is preferable to use coorientations*. The involutions $f \mapsto f^*$ and $f \mapsto \bar{f}$ are then conveniently encoded by the reflections along the coordinate axes:

$$\begin{array}{ccc} \begin{array}{c} y \\ | \\ \boxed{f} \\ | \\ x \end{array} & \mapsto & \begin{array}{c} x \\ | \\ \boxed{f^*} \\ | \\ y \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} y \\ | \\ \boxed{f} \\ | \\ x \end{array} & \mapsto & \begin{array}{c} \bar{y} \\ | \\ \boxed{\bar{f}} \\ | \\ \bar{x} \end{array} . \end{array} \quad (11)$$

When x and y are dualizable objects in a semisimple rigid C^* -tensor category and their conjugates $\bar{x}$ and $\bar{y}$ are given by (1) and (2), then, by [BDH14, (4.17)], the two involutions (11) are related by

$$\begin{array}{c} \bar{y} \\ | \\ \boxed{\bar{f}} \\ | \\ \bar{x} \end{array} = \begin{array}{c} \bar{y} \\ | \\ \boxed{f^*} \\ | \\ \bar{x} \end{array} = \begin{array}{c} \bar{y} \\ | \\ \boxed{f^*} \\ | \\ \bar{x} \end{array} \quad \text{and} \quad \begin{array}{c} x \\ | \\ \boxed{f^*} \\ | \\ y \end{array} = \begin{array}{c} x \\ | \\ \boxed{f} \\ | \\ y \end{array} = \begin{array}{c} x \\ | \\ \boxed{f} \\ | \\ y \end{array} . \quad (12)$$

4.2 The bi-involutive structure on $\text{Bim}(R)$

The tensor category of bimodules over a von Neumann algebra R is naturally a bi-involutive tensor category. The involution $\bar{\cdot} : \text{Bim}(R) \rightarrow \text{Bim}(R)$ sends a bimodule to its complex conjugate, with left and right actions given by $a\bar{\xi}b := b^*\xi a^*$.

The definition of the coherence

$$\nu : \bar{X} \boxtimes_R \bar{Y} \rightarrow \overline{Y \boxtimes_R X}$$

on $\text{Bim}(R)$, and more generally on the bicategory of von Neumann algebras, involves three instances of the modular conjugation J . It is given by

$$\begin{aligned} \nu : \text{Hom}_{-R}(L^2R, \bar{X}) \otimes L^2R \otimes \text{Hom}_{R-}(L^2R, \bar{Y}) \\ \longrightarrow \overline{\text{Hom}_{-R}(L^2R, Y) \otimes L^2R \otimes \text{Hom}_{R-}(L^2R, X)} \\ f \otimes \xi \otimes g \longmapsto \overline{g^* \otimes J\xi \otimes f^*} \end{aligned} \quad (13)$$

where $g^* := \bar{g} \circ J$.

It was shown in [BDH14, §6] that when A and B are von Neumann algebras with atomic centers, the dual of a dualizable bimodule ${}_AX_B$ is canonically identified with its complex conjugate ${}_B\bar{X}_A$.⁸ We recall the construction of standard solutions $\text{ev}_X : \bar{X} \boxtimes_B X \rightarrow L^2B$ and $\text{coev}_X : L^2A \rightarrow X \boxtimes_B \bar{X}$ of the conjugate equations (1) and (2).

⁸The results in [BDH14] are phrased in the context of von Neumann algebras with finite dimensional centers but extend verbatim to the case of von Neumann algebras with atomic centers.

Let B' denote the commutant of the right B -action on X . The bimodule $Y := {}_A L^2(B')_{B'}$ is dualizable; let $\tilde{Y}$ be its dual, and let $r : L^2 A \rightarrow Y \boxtimes_{B'} \tilde{Y}$ and $s^* : \tilde{Y} \boxtimes_A Y \rightarrow L^2(B')$ be standard solutions of the conjugate equations. By [Sau83, Prop. 3.1], there exists a canonical A - A bimodule isomorphism $X \boxtimes_B \overline{X} \cong L^2(B')$. The non-normalised minimal conditional expectation [BDH14, (6.8)] (see also [KL92, Hia88])

$$E : B' \rightarrow A$$

reads $E(b) := r^* \circ (b \boxtimes_{B'} \text{id}_{\tilde{Y}}) \circ r \in \text{Hom}_A(L^2 A, L^2 A) = A$, and the standard evaluation and coevaluation morphisms are given by

$$\text{coev}_X := \left(\begin{array}{c} L^2 A \rightarrow L^2(B') \cong X \boxtimes_B \overline{X} \\ \sqrt{\psi} \mapsto \sqrt{\psi \circ E} \end{array} \right) \quad \text{ev}_X := \text{coev}_{\overline{X}}^*, \quad (14)$$

where ψ and $\psi \circ E$ are positive elements of the preduals $L^1 A$ and $L^1 B'$, respectively.

4.3 Representations of bi-involutive categories

In the presence of bi-involutive structures, the notion of C^* -representation (Definition 3.1) can be enhanced in the following way:

Definition 4.4. Let $\mathcal{C}$ be a bi-involutive tensor category. A *bi-involutive representation* of $\mathcal{C}$ is a bi-involutive functor $\mathcal{C} \rightarrow \text{Bim}(R)$, for some von Neumann algebra R .

Given two bi-involutive representations $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(S)$, and a morphism $(\Phi, \phi) : \alpha \rightarrow \beta$ in the sense of Definition 3.1, it would be nice if we could formulate a compatibility condition with the bi-involutive structures, i.e., a compatibility between (Φ, ϕ) and the isomorphisms χ^α and χ^β . Unfortunately, we do not know how to formulate such a condition.

However, if Φ is dualizable, then there does exist an additional condition that one can impose (the diagram in Definition 4.5), and which is not a consequence of the previous requirements. The possibility/necessity of this further coherence was missed in our earlier paper [HP17]. We are thus in a position to define a notion of a dualizable morphism from α to β , which is not the same as that of a morphism which happens to be dualizable. We call it a *dualizable-morphism* for lack of a better name:

Definition 4.5. Let $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(S)$ be bi-involutive representations of a bi-involutive tensor category $\mathcal{C}$. A dualizable-morphism from α to β is a morphism

$$(\Phi, \phi) : \alpha \rightarrow \beta$$

of the underlying C^* -representations, where Φ is dualizable, such that the following diagram

commutes for all $c \in \mathcal{C}$:

$$\begin{array}{ccc}
\Phi \boxtimes_R \alpha(\bar{c}) & \xrightarrow{\phi_{\bar{c}}} & \beta(\bar{c}) \boxtimes_S \Phi \xrightarrow{\chi_c^\beta \boxtimes \text{id}_\Phi} \overline{\beta(c)} \boxtimes_S \Phi \\
\downarrow \text{id}_\Phi \boxtimes \chi_c^\alpha & & \uparrow \text{coev}_\Phi^* \boxtimes \text{id} \\
\Phi \boxtimes_R \overline{\alpha(c)} & & \Phi \boxtimes_R \overline{\Phi} \boxtimes_S \overline{\beta(c)} \boxtimes_S \Phi \\
\downarrow \text{id} \boxtimes \text{ev}_\Phi^* & & \uparrow \nu_{\Phi, \beta(c)}^{-1} \\
\Phi \boxtimes_R \overline{\alpha(c)} \boxtimes_R \overline{\Phi} \boxtimes_S \Phi & \xrightarrow{\nu_{\alpha(c), \Phi}} & \Phi \boxtimes_R \overline{\Phi} \boxtimes_S \alpha(c) \boxtimes_S \Phi \xrightarrow{\overline{\phi_c}} \Phi \boxtimes_R \overline{\beta(c)} \boxtimes_S \Phi \boxtimes_S \Phi.
\end{array} \quad (15)$$

Here, ev_Φ and coev_Φ refer to the standard evaluation and coevaluation morphisms (14).

Suppressing the ν 's (and the associators), the coherence (15) is described by the following diagrammatic equation, where the crossing on the left is $\phi_{\bar{c}}$ and the crossing on the right is ϕ_c :

Note that, when $\mathcal{C}$ is rigid, that condition is equivalent to

where the crossing on the right is ϕ_c^{-1} . The latter condition makes sense even when Φ is not dualizable.

When Φ is invertible, the maps ϕ in Definition 3.1 can be re-expressed as a unitary monoidal natural isomorphism

$$\begin{array}{ccc}
\mathcal{C} & \begin{array}{l} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} & \begin{array}{c} \text{Bim}(R) \\ \downarrow \text{Ad}(\Phi) \\ \text{Bim}(S) \end{array} \\
& \phi \not\cong &
\end{array} \quad (17)$$

in which case the coherence (15) becomes easier to display:

$$\begin{array}{ccc}
\Phi \boxtimes_R \alpha(\bar{c}) \boxtimes_R \overline{\Phi} & \xrightarrow{\phi_{\bar{c}}} & \beta(\bar{c}) \\
\downarrow \text{id} \boxtimes \chi_c^\alpha \boxtimes \text{id} & & \downarrow \chi_c^\beta \\
\Phi \boxtimes_R \overline{\alpha(c)} \boxtimes_R \overline{\Phi} & \longrightarrow & \overline{\Phi} \boxtimes_R \alpha(c) \boxtimes_R \overline{\Phi} \xrightarrow{\overline{\phi_c}} \overline{\beta(c)}.
\end{array} \quad (18)$$

5 Positive structures

As explained in Example 4.3, every rigid C*-tensor category $\mathcal{C}$ has a canonical bi-involutive structure $x \mapsto \bar{x}$; it is furthermore equipped with distinguished evaluation and coevaluation morphisms $\text{ev}_x : \bar{x} \otimes x \rightarrow 1$ and $\text{coev}_x : 1 \rightarrow x \otimes \bar{x}$. However, if we just start with $\mathcal{C}$ as a bi-involutive tensor category (i.e., if the involution $x \mapsto \bar{x}$ is provided as part of the data, as opposed to constructed from the C*-tensor structure), then there is no way of knowing which maps $\bar{x} \otimes x \rightarrow 1$ and $1 \rightarrow x \otimes \bar{x}$ to call the evaluation and coevaluation morphisms, even when $\mathcal{C}$ is unitary fusion.

A *positive structure* on $\mathcal{C}$ determines those morphisms up to positive scalar (when x is irreducible). For bi-involutive tensor C*-categories which are not rigid, such as $\text{Bim}(R)$, then the evaluation and coevaluation morphisms typically fail to exist. But the notion of positive structure is still meaningful.

5.1 Positive structures on bi-involutive tensor categories

Let $\mathcal{C}$ be a bi-involutive C*-tensor category.

Definition 5.1. A *positive structure* on $\mathcal{C}$ is a collection of subsets

$$\mathcal{P}_{a,b} \subset \text{Hom}_{\mathcal{C}}(a \otimes \bar{a}, b \otimes \bar{b}) \quad \text{for every } a, b \in \mathcal{C}$$

called *cp maps*⁹ that satisfy the following axioms:

- $0 \in \mathcal{P}_{a,b}$ and $\text{id}_{a \otimes \bar{a}} \in \mathcal{P}_{a,a}$,
- the $\mathcal{P}_{a,b}$ are closed under addition, positive scaling, composition, and adjoints,
- (cp maps are real) every cp map $\theta \in \mathcal{P}_{a,b}$ satisfies $\bar{\theta} = \theta$, i.e., the following diagram commutes

$$\begin{array}{ccc} a \otimes \bar{a} & \xrightarrow{\theta} & b \otimes \bar{b} \\ \downarrow \tilde{\nu} & & \downarrow \tilde{\nu} \\ \overline{a \otimes \bar{a}} & \xrightarrow{\bar{\theta}} & \overline{b \otimes \bar{b}} \end{array} \quad (19)$$

where $\tilde{\nu} := \nu \circ (\varphi \boxtimes \text{id})$, and

- (cp maps are closed under amplification) for every cp map θ and every morphism f the map $f \otimes \theta \otimes \bar{f}$ is also cp.

Example 5.2. The category of Hilbert spaces is equipped with a canonical positive structure, by declaring a linear map $\theta : H \otimes \bar{H} \rightarrow K \otimes \bar{K}$ to be cp if for every Hilbert space L , the map

$$\text{id}_L \otimes \theta \otimes \text{id}_{\bar{L}} : \mathcal{J}_2(L \otimes H) \cong L \otimes H \otimes \bar{H} \otimes \bar{L} \rightarrow L \otimes K \otimes \bar{K} \otimes \bar{L} \cong \mathcal{J}_2(L \otimes K)$$

sends positive Hilbert-Schmidt operators to positive Hilbert-Schmidt operators. Here, $\mathcal{J}_2(H) := H \otimes \bar{H} \subset B(H)$ denotes the ideal of Hilbert-Schmidt operators.

⁹The letters *cp* stands for “completely positive”. We warn the reader that positive maps $a \otimes \bar{a} \rightarrow a \otimes \bar{a}$ (i.e., maps which can be written as $f^* \circ f$) are typically not cp.

More generally, we have:

Example 5.3. For every von Neumann algebra M , the bi-involutive tensor category $\text{Bim}(M)$ has a canonical positive structure, explained in more generality in Section 5.2 below.

If $\mathcal{C}$ is a bi-involutive C^* -tensor category equipped with a positive structure $\mathcal{P}$, then one can form a new category whose objects are in bijection with those of $\mathcal{C}$, and whose hom-sets are the cones $\mathcal{P}_{a,b}$. If a is dualizable with dual $\bar{a}$, then the cones $\mathcal{P}_{a,b}$ satisfy a version of Frobenius reciprocity:

Lemma 5.4. *Let $\mathcal{C}$ be a bi-involutive C^* -tensor category equipped with a positive structure, and let $a, b, c \in \mathcal{C}$ be objects. If a is dualizable in $\mathcal{C}$ and satisfies $\bar{a} \cong a^\vee$, then we have an isomorphism*

$$\mathcal{P}_{a \otimes b, c} \xrightarrow{\cong} \mathcal{P}_{b, \bar{a} \otimes c} \quad (20)$$

given by $\theta \mapsto (\text{id}_{\bar{a}} \otimes \theta \otimes \text{id}_a) \circ (\text{ev}_a^* \otimes \text{id}_{b \otimes \bar{b}} \otimes \overline{\text{ev}_a^*})$.

Proof. The inverse map sends $\theta \in \mathcal{P}_{b, \bar{a} \otimes c}$ to $(\text{coev}_a^* \otimes \text{id}_{c \otimes \bar{c}} \otimes \overline{\text{coev}_a^*}) \circ (\text{id}_a \otimes \theta \otimes \text{id}_{\bar{a}})$. $\square$

Note that the isomorphism (20) does depend on the choice of identification $\bar{a} \cong a^\vee$.

5.2 Positive structure on the 2-category of von Neumann algebras

In this section, we describe the canonical positive structure on the bi-involutive 2-category $\mathbf{vN}$ of von Neumann algebras. One recovers the positive structure mentioned in Example 5.3 by restricting to the full sub 2-category whose only object is M (and, by taking $M = \mathbb{C}$, one obtains Example 5.2).

Let A and B be von Neumann algebras. For ${}_A X_B$ a bimodule, and $n \in \mathbb{N}$, we define the cone

$$\mathbf{P}_{X,n} \subset \mathbb{C}^n \otimes {}_A X \boxtimes_B \overline{X}_A \otimes \mathbb{C}^n = X^{\oplus n} \boxtimes_B \overline{X}^{\oplus n}$$

to be the closed positive span of vectors of the form

$$g \otimes \xi \otimes g^* \in \text{Hom}_{-B}(L^2 B, X^{\oplus n}) \otimes_B L^2 B \otimes_B \text{Hom}_{B-}(L^2 B, \overline{X}^{\oplus n}) \subset X^{\oplus n} \boxtimes_B \overline{X}^{\oplus n},$$

where $g \in \text{Hom}_{-B}(L^2 B, X^{\oplus n})$, $\xi \in L^2_+ B$, and $g^* = \bar{g} \circ J$.

Definition 5.5. Let A , B_1 , and B_2 be von Neumann algebras, and let ${}_A X_{B_1}$ and ${}_A Y_{B_2}$ be bimodules. A map

$$\theta : {}_A X \boxtimes_{B_1} \overline{X}_A \rightarrow {}_A Y \boxtimes_{B_2} \overline{Y}_A$$

is called cp if $\text{id}_{\mathbb{C}^n} \otimes \theta \otimes \text{id}_{\mathbb{C}^n}$ sends $\mathbf{P}_{X,n}$ to $\mathbf{P}_{Y,n}$ for every $n \in \mathbb{N}$. The collection of all cp maps θ as above is denoted $\mathcal{P}_{X,Y}$.

We claim that the cones $\mathcal{P}_{X,Y} \subset \text{Hom}_{A-A}(X \boxtimes_{B_1} \overline{X}, Y \boxtimes_{B_2} \overline{Y})$ equip $\mathbf{vN}$ with a positive structure in the sense of Definition 5.1. It is clear from the definition that $\text{id}_{X \boxtimes_{B_1} \overline{X}} \in \mathcal{P}_{X,X}$, and that cp maps are closed under addition and composition. The fact that $(\mathcal{P}_{X,Y})^* = \mathcal{P}_{Y,X}$ is a consequence of the cones $\mathbf{P}_{X,n}$ and $\mathbf{P}_{Y,n}$ being self-dual [Sau83, Prop. 3.1]. The remaining two axioms are checked in Propositions 5.6 and 5.9 below.

Proposition 5.6. *Let ${}_AX_{B_1}, {}_AY_{B_2} \in \mathbf{vN}$ be bimodules. Then for every cp map $\theta \in \mathcal{P}_{X,Y}$, we have $\bar{\theta} = \theta$ (i.e., the diagram (19) commutes).*

Proof. Since $\mathbf{P}_X$ linearly spans ${}_AX \boxtimes_{B_1} \overline{X}_A$ [Sau83, Prop. 3.1], and since all morphisms are $\mathbb{C}$ -linear, it is enough to check that this diagram commutes when applied to vectors $\eta := f \otimes \xi \otimes f^* \in \mathbf{P}_X$. By Definition (13), $\tilde{\nu}(\eta) = \bar{\eta}$. Hence

$$\bar{\theta}(\tilde{\nu}(\eta)) = \bar{\theta}(\bar{\eta}) = \overline{\theta(\eta)} = \tilde{\nu}(\theta(\eta)). \quad \square$$

Lemma 5.7. *Let ${}_AX_{B_1}, {}_AY_{B_2} \in \mathbf{vN}$ be bimodules, and let $\theta \in \text{Hom}_{A-A}(X \boxtimes_{B_1} \overline{X}, Y \boxtimes_{B_2} \overline{Y})$. Then*

$$\theta \in \mathcal{P}_{X,Y} \quad \Leftrightarrow \quad \text{id}_{\ell^2} \otimes \theta \otimes \text{id}_{\ell^2} \in \mathcal{P}_{\ell^2 \otimes X, \ell^2 \otimes Y}.$$

Proof. Let $\mathbf{P}_{X,\infty} \subset \ell^2 \otimes {}_AX \boxtimes_{B_1} \overline{X}_A \otimes \ell^2$ be the closed positive span of

$$g \otimes \xi \otimes g^* \in \text{Hom}_{-B}(L^2 B, X^{\oplus \infty}) \otimes L^2 B \otimes \text{Hom}_{B-}(L^2 B, \overline{X}^{\oplus \infty}) \quad \text{for } \xi \in L^2_+ B.$$

By definition, $\text{id}_{\ell^2} \otimes \theta \otimes \text{id}_{\ell^2}$ is cp if and only if it sends $\mathbf{P}_{X,\infty}$ to $\mathbf{P}_{Y,\infty}$. Let $p_n \in B(\ell^2)$ be the projection onto the span of the n first basis vectors of $\ell^2 = \ell^2(\mathbb{N})$. Then $\mathbf{P}_{X,\infty}$ is the closure of $\bigcup_{n \in \mathbb{N}} \mathbf{P}_{X,n}$ inside $\ell^2 \otimes {}_AX \boxtimes_{B_1} \overline{X}_A \otimes \ell^2$. If $\text{id}_{\ell^2} \otimes \theta \otimes \text{id}_{\ell^2}$ maps $\mathbf{P}_{X,\infty}$ to $\mathbf{P}_{Y,\infty}$, then $p_n \otimes \theta \otimes p_n$ maps $\mathbf{P}_{X,n}$ to $\mathbf{P}_{Y,n}$ for all $n \in \mathbb{N}$, so $\theta \in \mathcal{P}_{X,Y}$.

Conversely, assume $\theta \in \mathcal{P}_{X,Y}$. If $\eta \in \mathbf{P}_{X,\infty}$, then $(p_n \otimes \text{id}_{X \boxtimes_{B_1} \overline{X}} \otimes p_n)\eta \in \mathbf{P}_{X,n}$ for all $n \in \mathbb{N}$. Since $\text{id}_{\mathbb{C}^n} \otimes \theta \otimes \text{id}_{\mathbb{C}^n}$ maps $\mathbf{P}_{X,n}$ to $\mathbf{P}_{Y,n}$, we have $(p_n \otimes \theta \otimes p_n)\eta \in \mathbf{P}_{Y,n}$. Now $(p_n \otimes \theta \otimes p_n)\eta \rightarrow (\text{id}_{\ell^2} \otimes \theta \otimes \text{id}_{\ell^2})\eta$ as $n \rightarrow \infty$, and thus $\text{id}_{\ell^2} \otimes \theta \otimes \text{id}_{\ell^2}$ maps $\mathbf{P}_{X,\infty}$ to $\mathbf{P}_{Y,\infty}$. $\square$

Lemma 5.8. *Let ${}_AX_{B_1}, {}_AY_{B_2}$, and ${}_CZ_A$ be bimodules between von Neumann algebras. Then, for every cp map $\theta \in \mathcal{P}_{X,Y}$, we have*

$$\text{id}_Z \boxtimes \theta \boxtimes \text{id}_{\overline{Z}} \in \mathcal{P}_{Z \boxtimes_A X, Z \boxtimes_A Y}.$$

Proof. The action of C is irrelevant to the statement of the lemma, so we may treat Z as a mere right A -module. Without loss of generality, we may assume that the right A -action is faithful (otherwise, letting $q \in A$ be the central projection onto the support of Z , we may replace X and Y by the corresponding summands qX and qY , on which qA acts).

Since Z_A is a faithful module, we may identify $\ell^2 \otimes Z_A$ with $(\ell^2 \otimes L^2 A)_A$. By two applications of Lemma 5.7, we then have:

$$\begin{aligned} \theta \text{ is cp} &\iff \text{id}_{\ell^2} \otimes \theta \otimes \text{id}_{\ell^2} \text{ is cp} \\ &\iff (\text{id}_{\ell^2 \otimes L^2 A}) \boxtimes_A \theta \boxtimes_A (\text{id}_{L^2 A \otimes \ell^2}) \text{ is cp} \\ &\iff (\text{id}_{\ell^2 \otimes Z}) \boxtimes_A \theta \boxtimes_A (\text{id}_{\overline{Z} \otimes \ell^2}) \text{ is cp} \\ &\iff \text{id}_Z \boxtimes_A \theta \boxtimes_A \text{id}_{\overline{Z}} \text{ is cp.} \end{aligned} \quad \square$$

Proposition 5.9. *For every cp map $\theta \in \mathcal{P}_{X,Y}$ and $f \in \text{Hom}_{C-A}(W, Z)$, the linear map $f \boxtimes_A \theta \boxtimes_A \bar{f}$ is cp.*

Proof. The relevant map is the composite of $\text{id}_Z \boxtimes_A \theta \boxtimes_A \text{id}_{\overline{Z}}$ and $f \boxtimes_A \text{id}_{Y \boxtimes_{B_2} \overline{Y}} \boxtimes_A \bar{f}$. The former is cp by Lemma 5.8. To see that the latter is cp, simply note that the image of $g \otimes \xi \otimes g^* \in \mathbf{P}_{Z \boxtimes_A Y, n}$ under $\text{id}_{\mathbb{C}^n} \otimes f \boxtimes_A \text{id}_{Y \boxtimes_{B_2} \overline{Y}} \boxtimes_A \bar{f} \otimes \text{id}_{\mathbb{C}^n}$ is $(f \circ g) \otimes \xi \otimes (f \circ g)^*$, which is visibly in $\mathbf{P}_{W \boxtimes_A Y, n}$. $\square$

Recall from (14) that for a bimodule ${}_AX_B$ between von Neumann algebras with atomic centers, the standard solutions $\text{ev}_X : \overline{X} \boxtimes_A X \rightarrow L^2 B$ and $\text{coev}_X : L^2 A \rightarrow X \boxtimes_B \overline{X}$ of the conjugate equations are given by

$$\text{coev}_X : \sqrt{\psi} \mapsto \sqrt{\psi \circ E} \quad \text{and} \quad \text{ev}_X = \text{coev}_{\overline{X}}^*,$$

where E is the non-normalised minimal conditional expectation [BDH14, (6.8)]. Given an inclusion of von Neumann algebras $A \subset B$ and a conditional expectation $E : B \rightarrow A$, let us write $L^2 E$ for the corresponding linear map $L^2 A \rightarrow L^2 B : \sqrt{\psi} \mapsto \sqrt{\psi \circ E}$.

Proposition 5.10. *Let A, B be von Neumann algebras with atomic centers, and let ${}_AX_B$ be a dualizable bimodule. Then the standard solutions*

$$\text{ev}_X : \overline{X} \boxtimes_A X \rightarrow L^2 B \quad \text{and} \quad \text{coev}_X : L^2 A \rightarrow X \boxtimes_B \overline{X}$$

of the conjugate equations satisfy $\text{ev}_X \in \mathcal{P}_{\overline{X}, L^2 B}$ and $\text{coev}_X \in \mathcal{P}_{L^2 A, X}$.

Proof. We only treat the case of coev (the proof for ev follows by taking adjoints). We have canonical isomorphisms $L^2 A^{\oplus n} \boxtimes_A \overline{L^2 A}^{\oplus n} \cong L^2(A) \otimes M_n(\mathbb{C})$ and

$$X^{\oplus n} \boxtimes_B \overline{X}^{\oplus n} \cong (X \boxtimes_B \overline{X}) \otimes M_n(\mathbb{C}) \cong L^2 B' \otimes M_n(\mathbb{C}),$$

where the isomorphism $X \boxtimes_B \overline{X} \cong L^2 B'$ is as in [Sau83, Prop. 3.1]. Under this identification, the map $\text{id}_{\mathbb{C}^n} \otimes \text{coev}_X \otimes \text{id}_{\mathbb{C}^n}$ corresponds to $(L^2 E) \otimes \text{id}_{M_n(\mathbb{C})}$. Next, we note that the following square of A - A bimodule maps commutes:

$$\begin{array}{ccc} L^2 A \otimes M_n(\mathbb{C}) & \xrightarrow{L^2 E \otimes \text{id}_{M_n(\mathbb{C})}} & L^2 B' \otimes M_n(\mathbb{C}) \\ \downarrow \cong & & \downarrow \cong \\ L^2(A \otimes M_n(\mathbb{C})) & \xrightarrow{L^2(E \otimes \text{id}_{M_n(\mathbb{C})})} & L^2(B' \otimes M_n(\mathbb{C})). \end{array}$$

Indeed, for an element of the form $\xi \otimes e_{11} \in L^2 A \otimes M_n(\mathbb{C})$ with $\xi \in L_+^2 A$, the two maps visibly agree. The commutativity follows as all four maps are $M_n(A)$ - $M_n(A)$ -bilinear.

Finally, since the bottom arrow $L^2(E \otimes \text{id}_{M_n(\mathbb{C})})$ maps $L_+^2(A \otimes M_n(\mathbb{C}))$ to $L_+^2(B' \otimes M_n(\mathbb{C}))$, the map $\text{id}_{\mathbb{C}^n} \otimes \text{coev}_X \otimes \text{id}_{\mathbb{C}^n} = \text{id}_{\mathbb{C}^n} \otimes L^2 E \otimes \text{id}_{\mathbb{C}^n} = L^2 E \otimes \text{id}_{M_n(\mathbb{C})}$ sends $\mathbf{P}_{L^2 A, n}$ to $\mathbf{P}_{X, n}$. $\square$

5.3 Positive structure on rigid C^* -tensor categories

We show that every unitary multifusion category, indeed any semisimple rigid C^* -tensor category, admits canonical bi-involutive and positive structures. The notion of cp map presented in this section is originally due to Selinger [Sel07, Cor. 4.13 & §4.4].

Let $\mathcal{C}$ be a semisimple rigid C^* -tensor category. Recall from [Pen20] that a *unitary dual functor* is an assignment $c \mapsto (\bar{c}, \text{ev}_c, \text{coev}_c)$ of a dual object plus duality data to every object $c \in \mathcal{C}$, in such a way that the canonical isomorphisms

$$\nu_{a,b} := ((\text{ev}_a \circ (\text{id}_{\bar{a}} \otimes \text{ev}_b \otimes \text{id}_a)) \otimes \text{id}_{\overline{b \otimes a}}) \circ (\text{id}_{\bar{a} \otimes \bar{b}} \otimes \text{coev}_{b \otimes a}) : \bar{a} \otimes \bar{b} \rightarrow \overline{b \otimes a}$$

are unitary, and such that for every $f : a \rightarrow b$ the conjugate morphism

$$\bar{f} := (\text{id}_{\bar{b}} \otimes \text{coev}_a^*) \circ (\text{id}_{\bar{b}} \otimes f^* \otimes \text{id}_{\bar{a}}) \circ (\text{ev}_b^* \otimes \text{id}_{\bar{a}}) = \text{diagram}$$

satisfies $\bar{f} = (\text{ev}_a \otimes \text{id}_{\bar{b}}) \circ (\text{id}_{\bar{a}} \otimes f^* \otimes \text{id}_{\bar{b}}) \circ (\text{id}_{\bar{a}} \otimes \text{coev}_b)$ (ensuring the consistency of the graphical calculus (12)). A unitary dual functor induces a bi-involutive structure on $\mathcal{C}$ via the structure maps

$$\bar{\cdot} : \mathcal{C} \rightarrow \mathcal{C}, \quad \nu_{a,b} : \bar{b} \otimes \bar{a} \rightarrow \overline{a \otimes b}, \quad r : 1 \rightarrow \bar{1}, \quad \varphi_c : c \rightarrow \bar{\bar{c}}, \quad (21)$$

where $\nu_{a,b}$ is as above, r is coev_1 (followed by a left unitor isomorphism), and

$$\varphi_c := (\text{coev}_c^* \otimes \text{id}_{\bar{c}}) \circ (\text{id}_c \otimes \text{coev}_{\bar{c}}) = (\text{id}_{\bar{c}} \otimes \text{ev}_c) \circ (\text{ev}_{\bar{c}}^* \otimes \text{id}_c),$$

where the second equality is [Pen20, Cor. 3.10].

Remark 5.11. As explained in Example 4.3, any semisimple rigid C^* -tensor category (for instance a unitary multifusion category) admits a distinguished unitary dual functor, characterised by the conjugate equations (1) and (2). A typical unitary dual functor does not, however, satisfy the balancing condition (2).

By [Pen20, Cor. B], any two unitary dual functors yield canonically equivalent bi-involutive structures. Specifically, if $(\bar{\cdot}^1, \text{ev}^1, \text{coev}^1)$ and $(\bar{\cdot}^2, \text{ev}^2, \text{coev}^2)$ are unitary dual functors, letting

$$\chi_c : \bar{c}^1 \rightarrow \bar{c}^2 \quad (22)$$

be the unitary in the polar decomposition of $\tilde{\chi}_c := (\text{ev}_c^1 \otimes \text{id}_{\bar{c}^2}) \circ (\text{id}_{\bar{c}^1} \otimes \text{coev}_c^2)$, then $(\mu \equiv \text{id}, i \equiv \text{id}, \chi)$ is an equivalence of bi-involutive tensor categories.

Given a unitary dual functor on $\mathcal{C}$, let

$$\mathcal{P}_{a,b} := \{ \theta_f : a \otimes \bar{a} \rightarrow b \otimes \bar{b} \mid c \in \mathcal{C}, f : a \otimes c \rightarrow b \}, \quad (23)$$

where $\theta_f := (f \otimes \bar{f}) \circ (\text{id}_a \otimes \text{coev}_c \otimes \text{id}_{\bar{a}}) = \text{diagram}$ (24)

We call the elements of $\mathcal{P}_{a,b}$ cp maps. Our next goal is to prove that this is a positive structure on $\mathcal{C}$, and that it is independent of the choice of unitary dual functor.

Lemma 5.12. *A map $\theta \in \mathcal{C}(a \otimes \bar{a}, b \otimes \bar{b})$ is cp as in (23) if and only if its one-click rotation*

$$\rho(\theta) := \text{diagram} = (\text{id}_{\bar{a} \otimes b} \otimes \text{ev}_b) \circ (\text{id}_{\bar{a}} \otimes \theta \otimes \text{id}_b) \circ (\text{ev}_a^* \otimes \text{id}_{\bar{a} \otimes b})$$

is a positive operator in the C^* -algebra $\mathcal{C}(\bar{a} \otimes b, \bar{a} \otimes b)$. Similarly, θ is cp if and only if $\rho^{-1}(\theta) := (\text{ev}_b \otimes \text{id}_{\bar{b} \otimes a}) \circ (\text{id}_{\bar{b}} \otimes \theta \otimes \text{id}_a) \circ (\text{id}_{\bar{b} \otimes a} \otimes \text{ev}_a^*)$ is positive.

Proof. If $\theta = \theta_f$ for some $f : a \otimes c \rightarrow b$, then $\rho(\theta) = (\text{id}_{\bar{a}} \otimes f) \circ ((\text{ev}_a^* \circ \text{ev}_a) \otimes \text{id}_b) \circ (\text{id}_{\bar{a}} \otimes f^*) \geq 0$. Conversely, if $\theta \in \mathcal{C}(a \otimes \bar{a}, b \otimes \bar{b})$ is such that $\rho(\theta) \geq 0$, then we may write $\rho(\theta)$ as $g \circ g^*$ for some $g : c \rightarrow \bar{a} \otimes b$. Setting $f := (\text{coev}_a^* \otimes \text{id}_b) \circ (\text{id}_a \otimes g)$, we then have $\theta = \theta_f \in \mathcal{P}_{a,b}$. The second statement is similar. $\square$

The subsets $\mathcal{P}_{a,b} \subset \mathcal{C}(a \otimes \bar{a}, b \otimes \bar{b})$ defined in (23) form a positive structure in the sense of Definition 5.1. It is straightforward to verify that the $\mathcal{P}_{a,b}$ contain identities, are closed under composition, closed under adjoints, and that $f \otimes \theta \otimes \bar{f}$ is cp whenever θ is cp. To see that $\mathcal{P}_{a,b}$ is closed under addition we note that, by Lemma 5.12, if $\theta, \psi \in \mathcal{P}_{a,b}$, then $\rho(\theta)$ and $\rho(\psi)$ are positive, and thus so is $\rho(\theta) + \rho(\psi) = \rho(\theta + \psi)$. Hence $\theta + \psi \in \mathcal{P}_{a,b}$.

Our next task is to show that this positive structure is independent of the choice of unitary dual functor. Let $(\bar{\cdot}^1, \text{ev}^1, \text{coev}^1)$ and $(\bar{\cdot}^2, \text{ev}^2, \text{coev}^2)$ be two unitary dual functors on $\mathcal{C}$. Then the equivalence (22) between the corresponding bi-involutive structures sends cp maps to cp maps:

Proposition 5.13. *Let $\mathcal{C}$ be a semisimple rigid C^* -tensor category equipped with two unitary dual functors. Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be the corresponding sets of cp maps, as defined in (23). Then*

$$\theta \in \mathcal{P}_{a,b}^1 \iff (\text{id}_b \otimes \chi_b) \circ \theta \circ (\text{id}_a \otimes \chi_a^{-1}) \in \mathcal{P}_{a,b}^2,$$

where χ_a and χ_b are as in (22).

Proof. We only prove the “ $\Rightarrow$ ” implication (the other one follows by exchanging the roles of $\mathcal{P}^1$ and $\mathcal{P}^2$). Given $f : a \otimes c \rightarrow b$, let $\theta_f^\varepsilon := (f \otimes \bar{f}) \circ (\text{id}_a \otimes \text{coev}_c^\varepsilon \otimes \text{id}_{\bar{a}}) \in \mathcal{P}_{a,b}^i$, for $\varepsilon = 1, 2$. We need to show that

$$\forall f : a \otimes c \rightarrow b \quad (\text{id}_b \otimes \chi_b) \circ \theta_f^1 \circ (\text{id}_a \otimes \chi_a^{-1}) \in \mathcal{P}_{a,b}^2.$$

Pick orthogonal direct sum decompositions into simples $a = \bigoplus a_i$, $b = \bigoplus b_j$, $c = \bigoplus c_k$. Letting $f_{ijk} : a_i \otimes c_k \rightarrow b_j$ be the matrix elements of f , so that $f = \sum f_{ijk}$, we then have $\theta_f^\varepsilon = \sum_{ijk} \theta_{f_{ijk}}^\varepsilon$. Similarly, $\chi_a = \sum_i \chi_{a_i}$ and $\chi_b = \sum_j \chi_{b_j}$. It follows that

$$\begin{aligned} (\text{id}_b \otimes \chi_b) \circ \theta_f^1 \circ (\text{id}_a \otimes \chi_a^{-1}) &= \sum_{ijk} (\text{id}_{b_j} \otimes \chi_{b_j}) \circ \theta_{f_{ijk}}^1 \circ (\text{id}_{a_i} \otimes \chi_{a_i}^{-1}) \\ &= \sum_{ijk} [\text{positive number}] \cdot (\text{id}_{b_j} \otimes \tilde{\chi}_{b_j}) \circ \theta_{f_{ijk}}^1 \circ (\text{id}_{a_i} \otimes \tilde{\chi}_{a_i}^{-1}) \\ &= \sum_{ijk} [\text{positive number}] \cdot \theta_{f_{ijk}}^2 \in \mathcal{P}_{a,b}^2 \end{aligned}$$

where the last equality is most easily checked by using the definition $\theta_f = \text{diagram of } \theta_f$ of θ_f . $\square$

The cone $\mathcal{P}_{1,a} \subset \mathcal{C}(1, a \otimes \bar{a})$ is self-dual in the following sense. Let $\mathcal{C}$ be a semisimple rigid C^* -tensor category, and let φ be a faithful state on the finite dimensional abelian C^* -algebra $\text{End}_{\mathcal{C}}(1)$.

is positive. If $\theta = \theta_f \in \mathcal{P}_{a,b}$, then for every $g \geq 0$ the morphism (25) can be written as $(\text{id}_d \otimes f) \circ (\sqrt{g} \otimes \text{id}_c) \circ (\sqrt{g} \otimes \text{id}_c) \circ (\text{id}_d \otimes f^*)$, and is thus clearly positive. Conversely, if $\theta \in \mathcal{C}(a \otimes \bar{a}, b \otimes \bar{b})$ is cp in the sense of [JP17], then setting $d = \bar{a}$ and $g = \text{ev}_a^* \circ \text{ev}_a$ in (25), we get $\rho(\theta) \geq 0$, thus $\theta \in \mathcal{P}_{a,b}$ by Lemma 5.12.

5.4 Positive representations

Let $\mathcal{C}$ and $\mathcal{D}$ be bi-involutive tensor categories equipped with positive structures. A bi-involutive functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to *respect the positive structures* if for every cp map $\theta : a \otimes \bar{a} \rightarrow b \otimes \bar{b}$ in $\mathcal{C}$, the following composite is cp in $\mathcal{D}$:

$$F(a) \otimes \overline{F(a)} \xrightarrow{\text{id} \otimes \chi_a^{-1}} F(a) \otimes F(\bar{a}) \xrightarrow{\mu_{a,\bar{a}}} F(a \otimes \bar{a}) \xrightarrow{F(\theta)} F(b \otimes \bar{b}) \xrightarrow{\mu_{b,\bar{b}}^{-1}} F(b) \otimes F(\bar{b}) \xrightarrow{\text{id} \otimes \chi_b} F(b) \otimes \overline{F(b)}. \quad (26)$$

Definition 5.17. Let $\mathcal{C}$ be as above. A *positive representation* of $\mathcal{C}$ is a bi-involutive functor $\mathcal{C} \rightarrow \text{Bim}(R)$ (as in Definition 4.4) which respects the positive structures. Here, $\text{Bim}(R)$ is equipped with the positive structure described in Section 5.2.

By the results of the previous section, a semisimple rigid C^* -tensor category $\mathcal{C}$ admits canonical bi-involutive and positive structures (21) and (23) (remember that, by (22) and Proposition 5.13, these are independent of the choice of unitary dual functor on $\mathcal{C}$). As we have seen, given a tensor functor $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$, there are many layers of structure that one may or may not require this functor to preserve. Specifically, one could require α to be:

- (i) a C^* -representation (Definition 3.1),
- (ii) a bi-involutive representation (Definition 4.4),
- (iii) or, finally, a positive representation (Definition 5.17).

Surprisingly, at least when α is fully faithful, options (i) and (iii) yield equivalent notions. In contrast, option (ii) yields a non-equivalent, and less well-behaved notion. The equivalence between (i) and (iii) is formalised in the following theorem:

Theorem 5.18 (Thm. A). *Let $\mathcal{C}$ be a semisimple rigid C^* -tensor category. Then every fully faithful C^* -representation $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ extends uniquely to a positive representation.*

Given fully faithful C^ -representation $\alpha_i : \mathcal{C} \rightarrow \text{Bim}(R_i)$, $i = 1, 2$, then every isomorphism $\alpha_1 \cong \alpha_2$ of C^* -representations (an invertible bimodule ${}_{R_2}\Phi_{R_1}$ together with a unitary monoidal natural isomorphism ϕ as in (17)) is an isomorphism of positive representations. I.e., the coherence (18) is automatically satisfied.*

Proof. This is the content of Lemmata 5.20 and 5.21 below. □

It is natural to ask whether the statement of Theorem 5.18 also holds true without the requirement of α being fully faithful. We leave this as an open question (we do not know of any counterexamples).

Let $\mathcal{C}$ be a semisimple rigid C^* -tensor category. Once again, we equip $\mathcal{C}$ with its canonical bi-involutive and positive structures, described in Section 5.3.

Lemma 5.21. *Let $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(S)$ be positive fully faithful representations. And let Φ be a morphism between the underlying C^* -representations (a bimodule ${}_S\Phi_R$ together with unitary natural isomorphisms $\phi_c : \Phi \boxtimes_R \alpha(c) \rightarrow \beta(c) \boxtimes_S \Phi$ satisfying the half-braiding condition (7)). Then the coherence (16) is automatically satisfied.*

In particular, under the above assumptions, when ${}_S\Phi_R$ is dualizable, the coherence (15) is automatically satisfied.

Proof. Let $\text{ev}^\alpha := \alpha(\text{coev}_c)^* \circ \mu_{c,\bar{c}}^\alpha$, and $\text{ev}^\beta := \beta(\text{coev}_c)^* \circ \mu_{c,\bar{c}}^\beta$. Then we have

where the first and last equalities hold by the second formula for χ in Lemma 5.20, and the third equality holds by the half-braiding condition (7) followed by the naturality of ϕ . $\square$

From now on, given a semisimple rigid C^* -tensor category $\mathcal{C}$, by a *fully faithful representation* of $\mathcal{C}$, we shall always mean a fully faithful C^* -representation, equivalently a fully faithful positive representation. By Theorem 5.18, two fully faithful representations are equivalent as C^* -representations if and only if they are equivalent as positive representations.

6 Commutant categories

The present section is devoted to proof of Theorem B. The latter states that if $\mathcal{C}_0$ and $\mathcal{C}_1$ are Morita equivalent unitary fusion categories equipped with fully faithful representations $\mathcal{C}_0 \rightarrow \text{Bim}(R_0)$ and $\mathcal{C}_1 \rightarrow \text{Bim}(R_1)$, where R_0 and R_1 are hyperfinite factors which are either both of type II or both of type III_1 , then the commutant categories $\mathcal{C}'_0$ and $\mathcal{C}'_1$ are equivalent as bi-involutive tensor categories with positive structures.

6.1 Bicommutant categories

We start by recalling and correcting the notion of bicommutant category from [Hen17] and [HP17, §3] (the correction does not affect any of the previous results). The new feature, which was not present in [Hen17, HP17], is that of a positive structure (Def. 5.1).

Let $\mathcal{C}$ be a bi-involutive tensor category equipped with a positive structure, and let $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$ be a positive representation. The *commutant category* $\mathcal{C}'$ is the category whose objects are pairs (X, e_X) with $X \in \text{Bim}(R)$ and $e_X = (e_{X,c})_{c \in \mathcal{C}}$ a unitary half-braiding

$$\begin{array}{c} c \quad X \\ \searrow \quad \nearrow \\ X \quad c \end{array} \quad e_{X,c} : X \boxtimes \alpha(c) \rightarrow \alpha(c) \boxtimes X,$$

natural in c , and subject to the ‘hexagon’ axiom:

$$\begin{array}{c} c \otimes d \quad X \\ \diagup \quad \diagdown \\ X \quad c \otimes d \end{array} = \begin{array}{c} c \quad d \quad X \\ | \quad \diagdown \quad | \\ X \quad c \quad d \end{array} \quad \begin{aligned} & ((\mu_{c,d}^\alpha)^{-1} \boxtimes \text{id}_X) \circ e_{X,c \otimes d} \circ (\text{id}_X \boxtimes \mu_{c,d}^\alpha) \\ &= (\text{id}_{\alpha(c)} \boxtimes e_{X,d}) \circ (e_{X,c} \boxtimes \text{id}_{\alpha(d)}) \end{aligned}$$

Morphisms $(X, e_X) \rightarrow (Y, e_Y)$ are morphisms $f : X \rightarrow Y$ in $\text{Bim}(R)$ that satisfy

$$\begin{array}{c} c \quad Y \\ | \quad \boxed{f} \\ X \quad c \end{array} = \begin{array}{c} c \quad Y \\ \boxed{f} \quad | \\ X \quad c \end{array}.$$

The commutant $\mathcal{C}'$ is a tensor category with $(X, e_X) \otimes (Y, e_Y) = (X \boxtimes Y, e_{X \boxtimes Y})$, where the half-braiding $e_{X \boxtimes Y, c}$ is given by

$$e_{X \boxtimes Y, c} := \begin{array}{c} c \quad X \quad Y \\ | \quad \diagdown \quad | \\ X \quad Y \quad c \end{array} : X \boxtimes Y \boxtimes \alpha(c) \rightarrow \alpha(c) \boxtimes X \boxtimes Y.$$

The commutant category is naturally a bi-involutive category. The dagger structure inherited from $\text{Bim}(R)$, and the conjugate of (X, e_X) given by $\overline{X} \in \text{Bim}(R)$ along with the unitary half-braiding

$$e_{\overline{X}, c} : \overline{X} \otimes c \xrightarrow{\text{id} \otimes \varphi_c} \overline{X} \otimes \bar{c} \xrightarrow{\nu_{X, \bar{c}}} \bar{c} \otimes \overline{X} \xrightarrow{\overline{e_{X, \bar{c}}}^{-1}} \overline{X} \otimes \bar{c} \xrightarrow{\nu_{\bar{c}, X}^{-1}} \bar{c} \otimes \overline{X} \xrightarrow{\varphi_c^{-1} \otimes \text{id}} c \otimes \overline{X}$$

(which is an abbreviation for

$$\begin{aligned} \overline{X} \otimes \alpha(c) &\xrightarrow{\text{id} \otimes \alpha(\varphi_c)} \overline{X} \otimes \alpha(\bar{c}) \xrightarrow{\text{id} \otimes \chi_{\bar{c}}} \overline{X} \otimes \overline{\alpha(\bar{c})} \xrightarrow{\nu_{X, \alpha(\bar{c})}} \overline{\alpha(\bar{c})} \otimes \overline{X} \xrightarrow{\overline{e_{X, \bar{c}}}^{-1}} \\ &\longrightarrow \overline{X} \otimes \alpha(\bar{c}) \xrightarrow{\nu_{\alpha(\bar{c}), X}^{-1}} \overline{\alpha(\bar{c})} \otimes \overline{X} \xrightarrow{\overline{\chi_{\bar{c}} \otimes \text{id}}} \overline{\alpha(\bar{c})} \otimes \overline{X} \xrightarrow{\varphi_{\alpha(c)}^{-1} \otimes \text{id}} c \otimes \overline{X} \quad). \end{aligned}$$

It is moreover equipped with a positive structure, once again inherited from $\text{Bim}(R)$:

$$\mathcal{P}_{(X, e_X), (Y, e_Y)}^{\mathcal{C}'} := \mathcal{P}_{X, Y}^{\text{Bim}(R)} \cap \text{Hom}_{\mathcal{C}'}((X, e_X), (Y, e_Y)).$$

The commutant category admits an evident positive representation $\alpha' : \mathcal{C}' \rightarrow \text{Bim}(R)$ given by forgetting the half-braiding: $(X, e_X) \mapsto X$. So we may iterate the commutant construction to obtain $\mathcal{C}'' = (\mathcal{C}')'$. Note that there is a natural inclusion functor $\iota : \mathcal{C} \rightarrow \mathcal{C}''$ given by $c \mapsto (\alpha(c), e_c)$, where for $\underline{X} = (X, e_X) \in \mathcal{C}'$ the map $e_{c, \underline{X}} : \alpha(c) \boxtimes X \rightarrow X \boxtimes \alpha(c)$ is given by $e_{c, \underline{X}} := e_{X, c}^{-1}$. Naturality and the hexagon axiom are easily verified, as is the fact that morphisms in $\mathcal{C}$ give morphisms in $\mathcal{C}''$.

The following is a slight modification of [HP17, Def. 3.2]:

Definition 6.1. A *bicommutant category* is a bi-involutive tensor category with positive structure $\mathcal{C}$ such that there exists a hyperfinite von Neumann algebra R and a positive representation $\mathcal{C} \rightarrow \text{Bim}(R)$ for which the inclusion functor $\iota : \mathcal{C} \hookrightarrow \mathcal{C}''$ is an equivalence.

An equivalence of bicommutant categories is an equivalence of the underlying bi-involutive tensor categories, which respects the positive structures.

By [HP17, Lem 6.1 and Thm A], for any fully faithful representation $\mathcal{C} \rightarrow \text{Bim}(R)$ of a unitary fusion category $\mathcal{C}$, the commutant category $\mathcal{C}'$ is a bicommutant category.

Remark 6.2. In [HP17], the definition of a bicommutant category demanded R to be a hyperfinite factor. In principle, we could allow R to be any von Neumann algebra.

6.2 Description of the commutant

Let $\mathcal{C}$ be a unitary fusion category equipped with a fully faithful representation $\alpha : \mathcal{C} \rightarrow \text{Bim}(R)$, where R is a factor. We recall from [HP17, §4.1] the definition of the functor $\Delta : \text{Bim}(R) \rightarrow \mathcal{C}'$:

$$\Delta(X) := \bigoplus_{c \in \text{Irr}(\mathcal{C})} c \boxtimes X \boxtimes \bar{c}, \quad e_{\Delta(X), a} := \sum_{b, c \in \text{Irr}(\mathcal{C})} \sqrt{d_a^{-1}} \quad \begin{array}{c} a \quad b \quad X \quad \bar{b} \\ | \quad | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \quad | \\ c \quad X \quad \bar{c} \quad a \end{array} \quad (29)$$

where we have used the graphical convention (5) for pairs of colored nodes. The functor (29) is a categorical version of a (non-normalised) conditional expectation. Let $D = D(\mathcal{C})$ be the global dimension of $\mathcal{C}$.

Lemma 6.3. If $(X, e_X) \in \mathcal{C}'$, then the map $u_X : X \rightarrow \Delta(X)$ given by

$$\begin{aligned} u_X &:= \frac{1}{\sqrt{D}} \sum_{c \in \text{Irr}(\mathcal{C})} \sqrt{d_c} \quad \begin{array}{c} c \quad X \quad \bar{c} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \end{array} \\ &= \frac{1}{\sqrt{D}} \sum_{c \in \text{Irr}(\mathcal{C})} \sqrt{d_c} \cdot (e_{X, c} \otimes \text{id}_{\bar{c}}) \circ (\text{id}_X \otimes \text{coev}_c) \end{aligned}$$

is an isometry, and is a morphism in $\mathcal{C}'$. The projector $p_X := u_X u_X^* \in \text{End}_{\mathcal{C}'}(\Delta(X))$ is given by

$$p_X = \frac{1}{D} \sum_{a, x, y \in \text{Irr}(\mathcal{C})} \sqrt{d_a} \quad \begin{array}{c} y \quad X \quad \bar{y} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ x \quad X \quad \bar{x} \end{array}.$$

Proof. We first check that u_X is an isometry:

$$u_X^* u_X = \frac{1}{D} \sum_{c \in \text{Irr}(\mathcal{C})} d_c \quad \begin{array}{c} c \\ | \\ \bigoplus_X \end{array} = \frac{1}{D} \left(\sum_{c \in \text{Irr}(\mathcal{C})} d_c^2 \right) \text{id}_X = \text{id}_X.$$

To see that u_X is a morphism in $\mathcal{C}'$, i.e., that it commutes with the half-braiding, we compute

$$e_{\Delta(X),a}(u_X \otimes \text{id}) = \frac{1}{\sqrt{D}} \sum_{x,y \in \text{Irr}(\mathcal{C})} \sqrt{\frac{d_y}{d_a}} \begin{array}{c} a \quad x \quad X \quad \bar{x} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ y \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ a \end{array} = \frac{1}{\sqrt{D}} \sum_{x,y \in \text{Irr}(\mathcal{C})} \sqrt{d_x} \begin{array}{c} a \quad x \quad X \quad \bar{x} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ X \quad \text{---} \quad \text{---} \quad \text{---} \\ a \end{array} = (\text{id} \otimes u_X) e_{X,a},$$

where we have used Lemma 2.5 (Fusion) for the second equality. Finally, we check that

$$u_X u_X^* = \frac{1}{D} \sum_{x,y \in \text{Irr}(\mathcal{C})} \sqrt{d_x d_y} \begin{array}{c} y \quad X \quad \bar{y} \\ \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ x \quad X \quad \bar{x} \end{array} = \frac{1}{D} \sum_{\substack{x,y,a \\ \in \text{Irr}(\mathcal{C})}} \sqrt{d_a} \begin{array}{c} y \quad X \quad \bar{y} \\ \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ x \quad \text{---} \quad \text{---} \\ X \quad \text{---} \quad \bar{x} \end{array} = \frac{1}{D} \sum_{\substack{x,y,a \\ \in \text{Irr}(\mathcal{C})}} \sqrt{d_a} \begin{array}{c} y \quad X \quad \bar{y} \\ \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ x \quad X \quad \bar{x} \end{array} = p_X.$$

Once again, we have used Lemma 2.5 (Fusion) for the second equality.

The relation $p_X^2 = p_X$ follows from the previous computations. We present a second proof of this relation for the benefit of the reader, and for later reference:

$$\begin{aligned} p_X^2 &= \frac{1}{D^2} \sum_{\substack{x,y,z,a,b \\ \in \text{Irr}(\mathcal{C})}} \sqrt{d_a d_b} \begin{array}{c} z \quad X \quad \bar{z} \\ \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ y \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ x \quad X \quad \bar{x} \end{array} = \frac{1}{D^2} \sum_{\substack{x,y,z,a,b,c \\ \in \text{Irr}(\mathcal{C})}} \sqrt{d_c} \begin{array}{c} z \quad X \quad \bar{z} \\ \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ y \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ x \quad X \quad \bar{x} \end{array} \\ &= \frac{1}{D^2} \sum_{\substack{x,y,z,a,b,c \\ \in \text{Irr}(\mathcal{C})}} \sqrt{d_c} \begin{array}{c} z \quad X \quad \bar{z} \\ \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ y \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ x \quad X \quad \bar{x} \end{array} = \frac{1}{D^2} \sum_{\substack{x,y,z,a,b,c \\ \in \text{Irr}(\mathcal{C})}} \sqrt{d_c} \begin{array}{c} z \quad X \quad \bar{z} \\ \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ y \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \\ x \quad X \quad \bar{x} \end{array} = p_X. \end{aligned} \tag{30}$$

Here, we have used Lemma 2.5 (Fusion) for the second equality, Lemma 2.5 (I=H) for the four equality, and Lemma 2.6 for the last equality. $\square$

Recall that a functor $F : \mathcal{S} \rightarrow \mathcal{T}$ is called *dominant* if every object of $\mathcal{T}$ is isomorphic to a subobject of an object of the form $F(s)$, for some $s \in \mathcal{S}$.

Corollary 6.4. *The functor $\Delta : \text{Bim}(R) \rightarrow \mathcal{C}'$ is dominant.*

Proof. If $\underline{X} = (X, e_X) \in \mathcal{C}'$, then $\underline{X}$ is a direct summand of $\Delta(X)$. Indeed, more is true. If $\underline{X} \in \mathcal{C}'$, then $\Delta(X) \cong \underline{X} \otimes \Delta(1)$. $\square$

6.3 Commutants of multifusion categories

Let $\mathcal{C}$ be a $k \times k$ unitary multifusion category. As in Section 2.2, we write $\mathcal{C}_{ij}$ for $1_i \otimes \mathcal{C} \otimes 1_j$, and $\mathcal{C}_i$ for $\mathcal{C}_{ii}$. Let $R_1, \dots, R_k$ be factors, let $\mathcal{C} \rightarrow \text{Bim}(R_1 \oplus \dots \oplus R_k)$ be a fully faithful representation, and let $\mathcal{C}'$ be the corresponding commutant category.

Letting $z_i \in R_1 \oplus \cdots \oplus R_k$ denote the i -th central projection, any $(R_1 \oplus \cdots \oplus R_k)$ -bimodule X can be decomposed as

$$X = \bigoplus_{i,j \in \{1, \dots, k\}} X_{ij}$$

with $X_{ij} = z_i X z_j$. In that way, we may think of a bimodule $X \in \text{Bim}(R_1 \oplus \cdots \oplus R_k)$ as a matrix of Hilbert spaces $X = (X_{ij})$, where each X_{ij} is an R_i - R_j bimodule.

Lemma 6.5. *Let $\mathcal{C}$ be as above, and let $(X, e_X) \in \mathcal{C}'$ be in of commutant category. Then $X_{ij} = 0$ for all $i \neq j$.*

Proof. Let $A := R_1 \oplus \cdots \oplus R_k$, and write $1 = \bigoplus_{i=1}^k 1_i \in \text{Bim}(A)$, where $1_i = L^2 R_i$. Since $1_i \in \mathcal{C}$ commutes with X , we have $X_{ij} = 1_i \boxtimes_A X_{ij} \boxtimes_A 1_j \cong X_{ij} \boxtimes_A 1_i \boxtimes_A 1_j = 0$. $\square$

In the diagrams that follow, the shading of regions will correspond to the various R_i :

$$\text{light gray circle} = R_i \quad \text{medium gray circle} = R_j \quad \text{dark gray circle} = R_\ell \quad \text{etc.}$$

By the previous lemma, for any $(X, e_X) \in \mathcal{C}'$, we can decompose X as

$$X = \bigoplus_{i \in \{1, \dots, k\}} X_i, \tag{31}$$

with $X_i = z_i X = X z_i$. Similarly, the half-braiding e_X decomposes as a family of isomorphisms

$$e_{X,c} = \begin{array}{c} c \quad X_j \\ \text{light gray region} \\ X_i \quad c \end{array} : X_i \boxtimes_{R_i} c \rightarrow c \boxtimes_{R_j} X_j \tag{32}$$

for every $c \in \mathcal{C}_{ij}$, and $1 \leq i, j \leq k$. Here, the shadings  and  on the two sides of the strand c represent R_i and R_j , respectively.

Let $\mathcal{C}'_i$ denote the commutant of $\mathcal{C}_i$ inside $\text{Bim}(R_i)$. For every $i \in \{1, \dots, k\}$, there is an obvious forgetful/projection functor

$$\begin{aligned} \Pi_i : \mathcal{C}' &\rightarrow \mathcal{C}'_i \\ (X, e_X) &\mapsto (X_i, e_{X_i}), \end{aligned} \tag{33}$$

where X_i denotes the i -th summand in the decomposition (31) of X , and

$$e_{X_i} = \{e_{X_i,c} : X_i \boxtimes_{R_i} c \rightarrow c \boxtimes_{R_i} X_i\}_{c \in \mathcal{C}_i}$$

is as in (32). The functor Π_i is bi-involutive (in particular it is a tensor functor), and respects the positive structures.

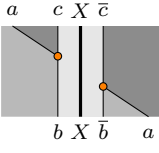
Theorem 6.6 (Thm. C). *For every $i \in \{1, \dots, k\}$, the functor $\Pi_i : \mathcal{C}' \rightarrow \mathcal{C}'_i$ is an equivalence of categories.*

Proof. We assume without loss of generality that $i = 1$, and reserve the shading $\square$ for R_1 . Consider the following functor (which is a modification of the functor Δ from the previous section), given by

$$\Delta_1 : \text{Bim}(R_1) \rightarrow \mathcal{C}'$$

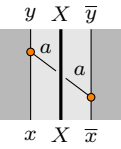
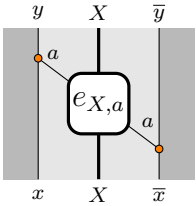
$$\Delta_1(X) := \bigoplus_{j \in \{1, \dots, k\}} \bigoplus_{c \in \text{Irr}(\mathcal{C}_{j1})} c \boxtimes X \boxtimes \bar{c}$$

with half-braiding

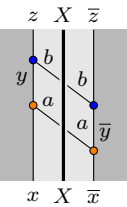
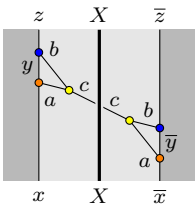
$$e_{\Delta_1(X), a} := \sum_{\substack{b \in \text{Irr}(\mathcal{C}_{i,1}) \\ c \in \text{Irr}(\mathcal{C}_{j1})}} \sqrt{d_a^{-1}} \quad \text{for } a \in \mathcal{C}_{ij}, \quad (34)$$


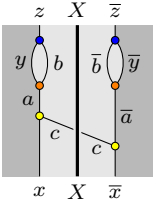
where $\square = R_i$ and $\blacksquare = R_j$.

Recall that $D := \dim(\mathcal{C}_i) = \dim(\mathcal{C}_j)$ for all $1 \leq i, j \leq k$. Let (X, e_X) be an object of $\mathcal{C}'$. We claim that, similarly to Lemma 6.3, the morphism

$$p_X := \frac{1}{D} \sum_{j \in \{1, \dots, k\}} \sum_{\substack{a \in \text{Irr}(\mathcal{C}_1) \\ \square = R_j \\ x, y \in \text{Irr}(\mathcal{C}_{j1})}} \sqrt{d_a} \quad \text{for } a \in \mathcal{C}_{ij}, \quad (35)$$



is a projector, and is an element of $\text{End}_{\mathcal{C}'}(\Delta_1(X))$. The relation $p_X^* = p_X$ follows from the unitarity of the half-braiding, and is left to the reader as an exercise. The relation $p_X^2 = p_X$ is proven along the same lines as (30), using Lemmata 2.5 and 2.6:

$$p_X^2 = \frac{1}{D^2} \sum_{\square = R_j} \sum_{\substack{a, b \in \text{Irr}(\mathcal{C}_1) \\ x, y, z \in \text{Irr}(\mathcal{C}_{j1})}} \sqrt{d_a d_b} \quad \text{for } a, b \in \mathcal{C}_{ij}, \quad (36)$$



$$= \frac{1}{D^2} \sum_{\square = R_j} \sum_{\substack{a, b, c \in \text{Irr}(\mathcal{C}_1) \\ x, y, z \in \text{Irr}(\mathcal{C}_{j1})}} \sqrt{d_c} \quad \text{for } a, b, c \in \mathcal{C}_{ij}, \quad (37)$$


$$= p_X.$$

Finally, to see that p_X is a morphism of $\mathcal{C}'$, i.e. that it commutes with the half-braiding (34),

we compute

$$\begin{aligned}
e_{\Delta_1(X),a} (p_X \boxtimes \text{id}_a) &= \frac{1}{D} \sum_{\substack{b \in \text{Irr}(\mathcal{C}_1) \\ x,y \in \text{Irr}(\mathcal{C}_{i1}) \\ z \in \text{Irr}(\mathcal{C}_{j1})}} \sqrt{\frac{d_b}{d_a}} \quad \begin{array}{c} \text{Diagram 1: A square with a diagonal line from top-left to bottom-right. The top edge is labeled $a, z, X, \bar{z}$ and the bottom edge is labeled $x, X, \bar{x}, a$. A blue dot is at (z, X) and an orange dot is at (x, X). A line segment connects (z, X) to (x, X), passing through a point labeled b on the diagonal. Another line segment connects (z, X) to $(\bar{x}, a)$, passing through a point labeled $\bar{y}$ on the diagonal. The region to the right of the diagonal is shaded gray. \end{array} \\
&= \frac{1}{D} \sum_{\substack{b \in \text{Irr}(\mathcal{C}_1) \\ x \in \text{Irr}(\mathcal{C}_{i1}) \\ y,z \in \text{Irr}(\mathcal{C}_{j1})}} \sqrt{\frac{d_b}{d_a}} \quad \begin{array}{c} \text{Diagram 2: A square with a diagonal line from top-left to bottom-right. The top edge is labeled $a, z, X, \bar{z}$ and the bottom edge is labeled $x, X, \bar{x}, a$. An orange dot is at (z, X) and a blue dot is at (x, X). A line segment connects (z, X) to (x, X), passing through a point labeled b on the diagonal. Another line segment connects (z, X) to $(\bar{x}, a)$, passing through a point labeled $\bar{y}$ on the diagonal. The region to the right of the diagonal is shaded gray. \end{array} = (\text{id}_a \boxtimes p_X) e_{\Delta_1(X),a}
\end{aligned}$$

for $\text{lightgray} = R_i$, $\text{darkgray} = R_j$ and $a \in \mathcal{C}_{ij}$, where the middle equality holds by Lemma 2.5 (I=H).

Since $\mathcal{C}'$ is idempotent complete, the above computations allow us to define, for every $\underline{X} = (X, e_X) \in \mathcal{C}'_1$, a new object $\Phi_1(\underline{X}) := p_X(\Delta_1(X)) \in \mathcal{C}'$, as the image of the projector (35). We claim that the resulting functor

$$\begin{aligned}
\Phi_1 : \mathcal{C}'_1 &\longrightarrow \mathcal{C}' \\
(X, e_X) &\mapsto p_X(\Delta_1(X))
\end{aligned}$$

is an inverse of the functor $\Pi_1 : \mathcal{C}' \rightarrow \mathcal{C}'_1$ defined in (33).

The equation $\Pi_1 \circ \Phi_1 \simeq \text{id}_{\mathcal{C}'_1}$ is a direct consequence of Lemma 6.3. More precisely, for any object $\underline{X} = (X, e_X)$ in $\mathcal{C}'_1$ the isometry u_X induces a unitary between $\underline{X}$ and $\Pi_1 \circ \Phi_1(\underline{X})$. These morphisms assemble to a unitary natural transformation $\text{id}_{\mathcal{C}'_1} \xrightarrow{\sim} \Pi_1 \circ \Phi_1$.

It remains to show that $\Phi_1 \circ \Pi_1 \simeq \text{id}_{\mathcal{C}'}$. For $(X, e_X) \in \mathcal{C}'$, let X_1 be as in (33). Then, as in Lemma 6.3, the map

$$u_X := \frac{1}{\sqrt{D}} \sum_{\text{lightgray} = R_j} \sum_{x \in \text{Irr}(\mathcal{C}_{j1})} \sqrt{d_x} \quad \begin{array}{c} \text{Diagram: A square with a diagonal line from top-left to bottom-right. The top edge is labeled $x, X_1, \bar{x}$ and the bottom edge is labeled X_j. A blue dot is at (x, X_1) and an orange dot is at (x, X_j). A line segment connects (x, X_1) to (x, X_j), passing through a point labeled x on the diagonal. The region to the right of the diagonal is shaded gray. \end{array} : X \rightarrow \Delta_1(X_1)$$

is an isometry, and an morphism in $\mathcal{C}'$. It is an isometry because

$$u_X^* u_X = \frac{1}{D} \sum_{\text{lightgray} = R_j} \sum_{x \in \text{Irr}(\mathcal{C}_{j1})} d_x \quad \begin{array}{c} \text{Diagram: A square with a diagonal line from top-left to bottom-right. The top edge is labeled X_j and the bottom edge is labeled X_j. A blue dot is at (x, X_j) and an orange dot is at (x, X_j). A line segment connects (x, X_j) to (x, X_j), passing through a point labeled x on the diagonal. The region to the right of the diagonal is shaded gray. \end{array} = \frac{1}{D} \sum_{j=1}^k \left(\sum_{x \in \text{Irr}(\mathcal{C}_{j1})} d_x^2 \right) \text{id}_{X_j} = \frac{1}{D} \sum_{j=1}^k D \cdot \text{id}_{X_j} = \text{id}_X.$$

And it is a morphism in $\mathcal{C}'$ because

$$\begin{aligned}
e_{\Delta_1(X_1),a}(u_X \otimes \text{id}) &= \frac{1}{\sqrt{D}} \sum_{\substack{x \in \text{Irr}(\mathcal{C}_{j1}) \\ y \in \text{Irr}(\mathcal{C}_{i1})}} \sqrt{\frac{d_y}{d_a}} \text{Diagram} \\
&= \frac{1}{\sqrt{D}} \sum_{x \in \text{Irr}(\mathcal{C}_{j1})} \sqrt{d_x} \text{Diagram} = (\text{id} \otimes u_X) e_{X,a}
\end{aligned}$$

The first diagram is a square with a vertical line. The top edge is labeled a , x , X_1 , $\bar{x}$. The bottom edge is labeled X_i , a . The left edge has a point y and the right edge has a point $\bar{y}$. The second diagram is a square with a vertical line. The top edge is labeled a , $\bar{x}$, X_1 , x . The bottom edge is labeled X_i , a . The left edge has a point x and the right edge has a point $\bar{x}$.

for all $a \in \mathcal{C}_{ij}$ and $1 \leq i, j \leq k$, where we have used Lemma 2.5 (Fusion) for the second equality.

Moreover, as in Lemma 6.3, one readily checks that $u_X u_X^* = p_{X_1}$:

$$\begin{aligned}
u_X u_X^* &= \frac{1}{D} \sum_{\substack{\bullet = R_j \\ x, y \in \text{Irr}(\mathcal{C}_{j1})}} \sum \sqrt{d_x d_y} \text{Diagram} = \frac{1}{D} \sum_{\substack{\bullet = R_j \\ x, y \in \text{Irr}(\mathcal{C}_{j1})}} \sum_{a \in \text{Irr}(\mathcal{C}_1)} \sqrt{d_a} \text{Diagram} \\
&= \frac{1}{D} \sum_{\substack{\bullet = R_j \\ x, y \in \text{Irr}(\mathcal{C}_{j1})}} \sum_{a \in \text{Irr}(\mathcal{C}_1)} \sqrt{d_a} \text{Diagram} = p_{X_1}.
\end{aligned}$$

The first diagram is a square with a vertical line. The top edge is labeled y , X_1 , $\bar{y}$. The bottom edge is labeled x , X_1 , $\bar{x}$. The left edge has a point x and the right edge has a point $\bar{x}$. The second diagram is a square with a vertical line. The top edge is labeled y , X_1 , $\bar{y}$. The bottom edge is labeled x , X_1 , $\bar{x}$. The left edge has a point a and the right edge has a point $\bar{a}$.

For every object $X \in \mathcal{C}'$, the isometry $u_X : X \rightarrow \Delta_1(X)$ therefore induces a unitary $X \rightarrow p_{X_1}(\Delta_1(X_1)) = \Phi_1(X_1) = \Phi_1 \circ \Pi_1(X)$. The latter assemble to a unitary natural transformation $\text{id}_{\mathcal{C}} \xrightarrow{\sim} \Phi_1 \circ \Pi_1$. $\square$

Corollary 6.7. *For any $i, j \in \{1, \dots, k\}$, there is an equivalence $\mathcal{C}'_i \simeq \mathcal{C}'_j$ (equivalence of bi-involutive categories, respecting the positive structures).*

Proof. By Theorem 6.6, the two functors

$$\mathcal{C}'_i \xleftarrow{\Pi_i} \mathcal{C}' \xrightarrow{\Pi_j} \mathcal{C}'_j$$

are equivalences of categories. These functors are bi-involutive, and respect the positive structures. $\square$

We are now in position to prove our main theorem.

Theorem 6.8 (Thm. B). *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be Morita equivalent unitary fusion categories, and let*

$$\alpha_0 : \mathcal{C}_0 \rightarrow \text{Bim}(R_0), \quad \alpha_1 : \mathcal{C}_1 \rightarrow \text{Bim}(R_1)$$

be fully faithfully representations, where R_0 and R_1 are hyperfinite factors. Assume that R_0 and R_1 are either both of type II, or both of type III₁.

Let $\mathcal{C}'_i$ be the commutant category of $\mathcal{C}_i$ inside $\text{Bim}(R_i)$. Then $\mathcal{C}'_0$ and $\mathcal{C}'_1$ are equivalent as bicommutant categories (equivalent as bi-involutive tensor categories with positive structures).

Proof. Let $\mathcal{C} = \begin{pmatrix} \mathcal{C}_0 & \mathcal{M} \\ \mathcal{M}^* & \mathcal{C}_1 \end{pmatrix}$ be a unitary 2×2 multifusion category witnessing the Morita equivalence. By Theorem 3.15, there exists a fully faithful representation

$$\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus 2}), \quad (36)$$

where R is a factor isomorphic to R_1 .

Let $\tilde{\alpha}_i := \alpha|_{\mathcal{C}_i} : \mathcal{C}_i \rightarrow \text{Bim}(R)$ denote the restriction of α to $\mathcal{C}_i \subset \mathcal{C}$. By Theorem 3.10, $\tilde{\alpha}_i$ and α_i are isomorphic as C^* -representations:

$$\begin{array}{ccc} & & \text{Bim}(R) \\ & \nearrow \tilde{\alpha}_i & \downarrow \simeq \\ \mathcal{C}_i & & \text{Bim}(R_i) \\ & \searrow \alpha_i & \end{array}$$

And by Theorem 5.18, they are isomorphic as positive representations. The commutant category of $\mathcal{C}_i$ inside $\text{Bim}(R)$ is therefore equivalent to the commutant category of $\mathcal{C}_i$ inside $\text{Bim}(R_i)$ (as as bi-involutive tensor categories with positive structures). The result follows by applying Corollary 6.7 to the representation (36). $\square$

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