

Planar algebras in braided tensor categories

André Henriques, David Penneys, and James Tener

August 2, 2021

Abstract

We generalize Jones' planar algebras by internalising the notion to a pivotal braided tensor category $\mathcal{C}$. To formulate the notion, the planar tangles are now equipped with additional 'anchor lines' which connect the inner circles to the outer circle. We call the resulting notion an *anchored planar algebra*. If we restrict to the case when $\mathcal{C}$ is the category of vector spaces, then we recover the usual notion of a planar algebra.

Building on our previous work on categorified traces, we prove that there is an equivalence of categories between anchored planar algebras in $\mathcal{C}$ and pivotal module tensor categories over $\mathcal{C}$ equipped with a chosen self-dual generator. Even in the case of usual planar algebras, the precise formulation of this theorem, as an equivalence of categories, has not appeared in the literature. Using our theorem, we describe many examples of anchored planar algebras.

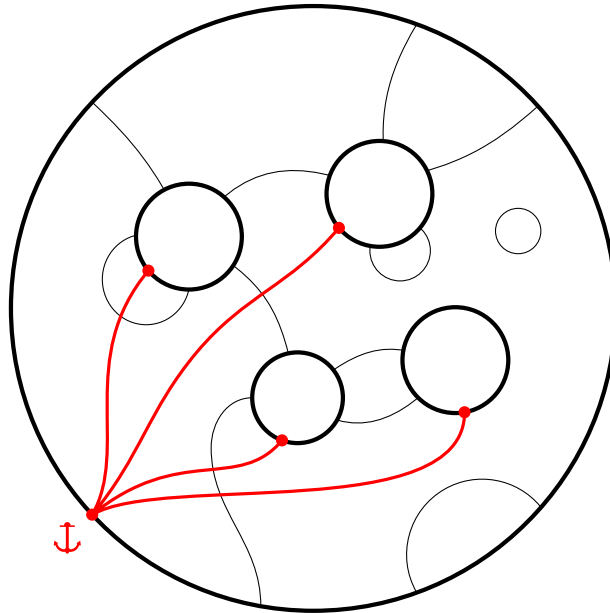


Figure 1: An anchored planar tangle

Contents

1	Introduction	3
2	Anchored planar algebras	6
2.1	Planar algebras	6
2.2	Anchored planar algebras	9
2.3	Generators for the anchored planar operad	12
3	The main theorem and examples	13
3.1	Module tensor categories	13
3.2	The main theorem	16
3.3	Examples	18
4	Constructing anchored planar algebras	24
4.1	Generic tangles and tangles in standard form	25
4.2	The ribbon braid group	29
4.3	Assigning maps to generic tangles	31
4.4	Proof of Theorem 4.1	34
4.5	Induction along a lax functor	38
5	Anchored planar algebras from module tensor categories	39
5.1	Diagram cheat sheet from [HPT16]	40
5.2	Constructing the anchored planar algebra	43
5.3	Functoriality	45
6	Module tensor categories from anchored planar algebras	50
6.1	Reconstructing the category $\mathcal{M}$	50
6.2	The adjoint pair $\Phi \dashv \mathrm{Tr}_{\mathcal{C}}$	52
6.3	The tensor structure on $\mathcal{M}$	54
6.4	The pivotal structure	59
6.5	The central functor	61
6.6	Functoriality	64
7	Equivalence of categories	66
7.1	Module tensor categories to planar algebras and back	66
7.2	Naturality of Ψ	73
7.3	Planar algebras to module tensor categories and back	76
A	An associativity type relation	80
B	Anchored planar tangles with coupons	81
C	The tube string calculus for the categorified trace	85

1 Introduction

In [Jon99], Jones introduced the notion of a *planar algebra* as an axiomatization of the standard invariant of a finite index subfactor. A planar algebra (in vector spaces) is a sequence of vector spaces $\mathcal{P}[0], \mathcal{P}[1], \mathcal{P}[2], \dots$ called the *box spaces* of the planar algebra, along with an action of planar tangles¹, i.e., for every planar tangle T we have a linear map $Z(T) : \mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_r] \rightarrow \mathcal{P}[k_0]$. For example:

$$Z \left(\begin{array}{c} \text{Diagram: A large circle containing two smaller circles labeled 1 and 2. Circle 1 is on the left, circle 2 is on the right. They are connected by a line. There are red dots on the boundaries of the circles and on the connecting line.} \end{array} \right) : \mathcal{P}[3] \otimes \mathcal{P}[5] \rightarrow \mathcal{P}[6]$$

The operation of sticking a tangle inside another is required, by the axioms of a planar algebra, to correspond to the composition of multilinear maps.

In this paper, we internalize the notion of planar algebra to a braided pivotal tensor category $\mathcal{C}$. The resulting notion is called an *anchored planar algebra*. The box spaces $\mathcal{P}[k]$ are now objects of $\mathcal{C}$, and the planar tangles are replaced by *anchored planar tangles* (see Figure 1 for an example). An anchored planar algebra associates to each anchored planar tangle T a morphism in $\mathcal{C}$:

$$Z \left(\begin{array}{c} \text{Diagram: A large circle containing two smaller circles labeled 1 and 2. Red lines (anchors) connect the boundaries of the circles to the outer boundary of the large circle.} \end{array} \right) : \mathcal{P}[3] \otimes \mathcal{P}[5] \rightarrow \mathcal{P}[6].$$

In general, this is a morphism $Z(T) \in \text{Hom}_{\mathcal{C}}(\mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_r], \mathcal{P}[k_0])$, where the order of the tensor factors $\mathcal{P}[k_1], \dots, \mathcal{P}[k_r]$ is determined by the anchor lines (the red lines in the picture).

We recall that a tensor category is called *pivotal* if every object is equipped with an isomorphism to its double dual, satisfying certain axioms. There is a well-known algebraic classification of planar algebras [MPS10, Yam12, BHP12] which goes as follows:

$$\left\{ \text{Planar algebras} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Pairs } (\mathcal{D}, X), \text{ where } \mathcal{D} \text{ is a pivotal category and} \\ X \in \mathcal{D} \text{ is a symmetrically self-dual generator} \end{array} \right\}. \quad (1)$$

Here, an object X is called symmetrically self-dual if it is equipped with an isomorphism to its dual, subject to a certain symmetry condition. For a pair $(\mathcal{D}, X)$ as above, the k -th box space of the associated planar algebra $\mathcal{P}$ is given by the invariants in $X^{\otimes k}$:

$$\mathcal{P}[k] := \text{Hom}_{\mathcal{D}}(1_{\mathcal{D}}, X^{\otimes k}).$$

¹While Jones worked with shaded planar tangles in his study of subfactors, in this article we work with unshaded planar tangles.

Conversely, the action of planar tangles is given by the graphical calculus of $\mathcal{D}$.

The main goal of our paper is to generalise the correspondence (1) to the case of planar algebras internal to $\mathcal{C}$, i.e., anchored planar algebras. Our classification result is formulated as an equivalence of categories. It establishes a correspondence between anchored planar algebras in $\mathcal{C}$, and a certain type of *module tensor categories* for $\mathcal{C}$ (Definition 3.1).

Here, a module tensor category is a simultaneous generalisation of the notion of $\mathcal{C}$ -module category and of the notion of tensor category (a monoidal category which is linear over some field). First and foremost, a module tensor category $\mathcal{M}$ is a tensor category. In addition to being a tensor category, it comes equipped with a tensor functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ which gives it with the structure of a $\mathcal{C}$ -module category: $c \cdot m = \Phi(c) \otimes m$. Now, because $\mathcal{C}$ is braided, there is another left action of $\mathcal{C}$ on $\mathcal{M}$, given by $c \cdot m = m \otimes \Phi(c)$. In a module tensor category, those two actions are isomorphic, i.e., we are provided with isomorphisms $e_{\Phi(c),m} : \Phi(c) \otimes m \rightarrow m \otimes \Phi(c)$. The latter are subject to the following three axioms:

$$\begin{aligned} e_{\Phi(c),x \otimes y} &= (\text{id}_x \otimes e_{\Phi(c),y}) \circ (e_{\Phi(c),x} \otimes \text{id}_y) \\ e_{\Phi(c \otimes d),x} &= (e_{\Phi(c),x} \otimes \text{id}_{\Phi(d)}) \circ (\text{id}_{\Phi(c)} \otimes e_{\Phi(d),x}) \\ e_{\Phi(c),\Phi(d)} &= \Phi(\beta_{c,d}) \end{aligned}$$

where β is the braiding in $\mathcal{C}$. The above conditions can be re-packaged into the single data of a braided functor $\Phi^z : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M})$ from the category $\mathcal{C}$ into the Drinfel'd center of $\mathcal{M}$.

We say that a module tensor category is *pivotal* if both $\mathcal{M}$ and Φ^z are pivotal, and *pointed* if it comes equipped with a symmetrically self-dual object $m \in \mathcal{M}$ which generates it as a module tensor category. Our main theorem is:

Theorem A. *There is an equivalence of categories*

$$\left\{ \text{Anchored planar algebras in } \mathcal{C} \right\} \cong \left\{ \begin{array}{l} \text{Pointed pivotal module} \\ \text{tensor categories over } \mathcal{C} \end{array} \right\}$$

We warn the reader that equipping the collection of all pointed pivotal module tensor categories with the structure of a category is not totally obvious (they are more naturally a 2-category). The precise version of our theorem is stated as Theorem 3.7 in the body of this paper.

Given a pointed pivotal module tensor category $(\mathcal{M}, m)$, the n -th box object of the associated anchored planar algebra is given by the formula

$$\mathcal{P}[n] = \text{Tr}_{\mathcal{C}}(m^{\otimes n}),$$

where $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$ is the right adjoint of Φ . If one removes the condition that the generator $m \in \mathcal{M}$ is symmetrically self-dual, then one obtains a classification of *oriented* planar algebras (i.e., planar algebras where the strands are oriented); for simplicity we only consider the unoriented case. We note that, even when $\mathcal{C} = \text{Vec}$, our theorem yields a version of the equivalence (1) which is more precise than has previously appeared in the literature.

In our previous article [HPT16], we showed that the functor $\text{Tr}_{\mathcal{C}}$ admits a ‘calculus of strings on tubes’ (see Section 5.1 for an overview). As a corollary of our main theorem, we can now prove that this calculus of strings on tubes is invariant under all 3-dimensional isotopies (Appendix C).

Examples of anchored planar algebras have already appeared in the literature, in the work of Jaffe-Liu on parafermions and reflection positivity [JL17, JLW16]. In their work, a notion of ‘planar para algebra’ is presented, which is equivalent to that of an anchored planar algebra in the braided tensor category $\mathbf{Vec}(\mathbb{Z}/N\mathbb{Z})$ of $\mathbb{Z}/N\mathbb{Z}$ -graded vector spaces (Example 3.9). By our main theorem (Theorem A), the notion of a planar para algebra is equivalent to that of a module tensor category over $\mathbf{Vec}(\mathbb{Z}/N\mathbb{Z})$. The parafermion planar para algebras constructed in [JL17] then correspond, under the equivalence, to Tambara-Yamagami categories associated to $\mathbb{Z}/N\mathbb{Z}$. They lie in the larger family of Tambara-Yamagami module tensor categories over $\mathbf{Vec}(A)$, where A is an abelian group (Example 3.8).

In the other direction, the algebraic classification given in Theorem A allows us to construct many examples of anchored planar algebras, including examples from near group categories (Section 3.3.1), and from ADE module tensor categories over Temperley-Lieb-Jones categories (Section 3.3.2). We explicitly compute the box objects $\mathcal{P}[k]$ in all our examples.

Our paper is structured as follows. In Section 2, we review material on planar algebras and introduce the notion of anchored planar algebra. In Section 3, we review the notion of module tensor category, and state our main theorem (Theorem 3.7). Using our theorem, we then provide a number of examples of anchored planar algebras. In Section 4, we explain how to construct anchored planar algebras via generators and relations. In Section 5, we use the categorified trace associated to a module tensor category [HPT16] to construct a functor $\Lambda : \mathbf{ModTens}_* \rightarrow \mathbf{APA}$ from the category of pointed $\mathcal{C}$ -module tensor categories to the category of anchored planar algebras in $\mathcal{C}$. In Section 6, we construct a functor $\Delta : \mathbf{APA} \rightarrow \mathbf{ModTens}_*$ going the other way. Finally, in Section 7 we complete the proof of our main theorem and show that the two functors Λ and Δ witness an equivalence of categories.

Outlook. Planar algebras were introduced by Vaughan Jones [Jon99] as a tool to study the standard invariants of subfactors, and they have played a crucial role in recent progress on classification of subfactors [Pet10, BMPS12, JMS14, AMP15]. In particular, planar algebras provide a framework in which one can apply categorical methods to the study of subfactors. To any (unshaded) planar algebra one may associate a tensor category $\mathcal{M}$ equipped with a generating object m , and planar algebras for which $\mathcal{M}$ is a fusion category are called *finite depth*. By celebrated theorems of Popa [Pop90, Pop95], for any unitary finite depth planar algebra, there exists a bimodule ${}_RX_R$ over the hyperfinite factor R of either type II_1 or III_1 such that the map $m \mapsto X$ extends to a tensor functor $\mathcal{M} \rightarrow \mathbf{Bim}(R)$ which is fully faithful. Furthermore, such an R - R -bimodule X is unique up to conjugation by an invertible bimodule (see also [Izu93] and [Izu17, Thm. 2.2] in the type III_1 setting).

The algebraic characterization of anchored planar algebras given in this article enables the study of analogous statements for *anchored* planar algebras. In the recent paper [HP17], the first two authors introduced the notion of the *commutant* $\mathcal{C}'$ of a unitary fusion category $\mathcal{C}$. This is the sub-tensor category $\mathcal{C}' \subset \mathbf{Bim}(R)$ consisting of all bimodules that commute with the elements of $\mathcal{C}$. Commutants of unitary fusion categories provide examples of *bicommutant categories*, which are tensor categories equivalent to their double commutant in $\mathbf{Bim}(R)$. Bicommutant categories are higher categorical analogs of von Neumann algebra, and they play an important role in the study of fully extended topological quantum field theories [Hen17]. We conjecture that anchored planar algebras in $\mathcal{Z}(\mathcal{C})$ play the same role for the

bicommutant category $\mathcal{C}'$ as do usual planar algebras for $\text{Bim}(R)$:

Conjecture 1.1. *For every finite depth unitary anchored planar algebra in $\mathcal{Z}(\mathcal{C})$, with associated pointed module tensor category $(\mathcal{M}, m)$, there exists an object $X \in \mathcal{C}'$ such that the map $m \mapsto X$ extends to a fully faithful functor of module tensor categories $\mathcal{M} \rightarrow \mathcal{C}'$. Moreover, such an object $X \in \mathcal{C}'$ is unique up to conjugation by an invertible object of $\mathcal{C}'$.*

These conjectures are the subject of ongoing research by the authors.

Acknowledgements. The authors would like to thank Bruce Bartlett, Vaughan Jones, Zhengwei Liu, Scott Morrison, Mathew Pugh, Noah Snyder, and Kevin Walker for helpful discussions.

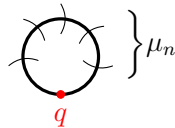
André Henriques gratefully acknowledges the Leverhulme trust and the EPSRC grant “Quantum Mathematics and Computation” for financing his visiting position in Oxford. André Henriques has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 674978). David Penneys was supported by the NSF DMS grants 1500387/1655912 and 1654159, and David Penneys and James Tener were partially supported by NSF DMS grant 0856316. James Tener was partially supported by an AMS-Simons travel grant. James Tener would also like to thank the Max Planck Institute for Mathematics for support during the course of this work.

2 Anchored planar algebras

2.1 Planar algebras

Planar algebras were first introduced by Jones [Jon99] in the context of his program to understand and classify subfactors [Pet10, BMPS12, JMS14, AMP15]. More generally, in the spirit of the tangle hypothesis [Lur09], one can expect objects similar to planar algebras to be relevant whenever dealing with dualizable 1-morphisms in a 2-category. We start with a brief definition, following [MPS10, BHP12].

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disc. We fix a base point $q \in \partial\mathbb{D}$ on its boundary. For each $n \in \mathbb{N}$, we also fix a set $\mu_n = (\mu_n^1, \dots, \mu_n^n)$ consisting of n germs of arcs that intersect $\partial\mathbb{D}$ transversely, not at q :



Definition 2.1. A planar tangle $T = (T, X, Q)$ consists of:

- A subset $T \subseteq \mathbb{D}$ of the form

$$T = \mathbb{D} \setminus (\mathring{D}_1 \cup \dots \cup \mathring{D}_r), \quad r \geq 0,$$

where $D_i = \{z \in \mathbb{D} : |z - a_i| \leq r_i\} \subset \mathring{\mathbb{D}}$ are disjoint closed discs contained in the interior of $\mathbb{D}$. We let $D_0 := \mathbb{D}$ and write $\partial^i T := \partial D_i$, so that

$$\partial T = \bigcup_{i=0}^r \partial^i T.$$

The boundary components $\partial^1 T, \dots, \partial^r T$ are called *input circles* (the input circles are ordered), and $\partial^0 T = \partial \mathbb{D}$ is called the *output circle* of T .

- A closed 1-dimensional smooth submanifold $X \subset T$ satisfying $\partial X = X \cap \partial T$. The components of X are called *strands* and the points of ∂X are called *boundary points*.
- A set $Q = \{q_0, \dots, q_r\}$, $q_0 = q$, $q_i \in \partial^i T \setminus X$, called the *anchor points*.

If $X \cap \partial^i T$ has cardinality k and $f : \mathbb{C} \rightarrow \mathbb{C}$ is the unique affine linear map that sends $\partial \mathbb{D}$ to $\partial^i T$ and q to q_i (f is the identity if $i = 0$), we demand that the strands X be tangent to $f(\mu_k)$ to infinite order.

A planar tangle T is of type $(k_1, \dots, k_r; k_0)$ if it has r input circles, and for each $i \in \{0, \dots, r\}$ we have $|X \cap \partial^i T| = k_i$. The collection of all tangles (for fixed q and μ_n) admits the structure of a *topological coloured operad* [BV73]. This amounts to the following structure. If $S = (S, Y, P)$ and $T = (T, X, Q)$ are tangles of types $(k_1, \dots, k_r; k_0)$ and $(\ell_1, \dots, \ell_s; \ell_0)$, and if $\ell_0 = k_i$ for some $i \geq 1$, then we can form a new tangle

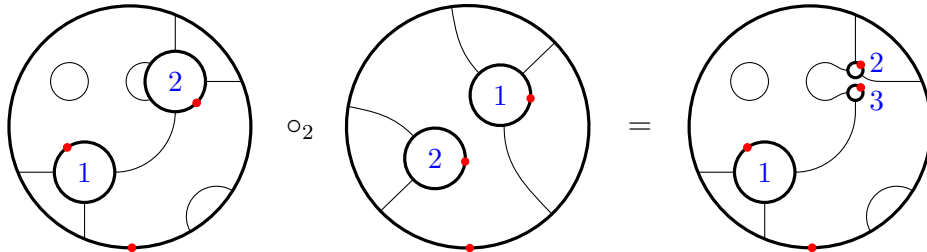
$$T \circ_i S \text{ of type } (k_1, \dots, k_{i-1}, \ell_1, \dots, \ell_s, k_{i+1}, \dots, k_r; k_0),$$

called the *i-th operadic composite* of T and S . This tangle is defined as follows. Letting $f : \mathbb{C} \rightarrow \mathbb{C}$ be the unique affine linear map that sends $\partial^0 S$ to $\partial^i T$ and q to q_i , it is given by

$$T \circ_i S := (T \cup f(S), X \cup f(Y), Q \cup f(P) \setminus \{q_i\}). \quad (2)$$

Our conventions about the behaviour of strands near their boundary ensure that this operation is well defined, and in particular that $X \cup f(Y)$ is a smooth manifold.

Example 2.2. Here is an instance of operadic composition of tangles:



The *planar operad* is the coloured operad with set of colours given by the natural numbers, and operations of type $(k_1, \dots, k_r; k_0)$ given by isotopy classes of planar tangles of type $(k_1, \dots, k_r; k_0)$.

We note that the planar operad is independent, up to canonical isomorphism, of the choices of basepoint $q \in \partial \mathbb{D}$ and germs of arcs μ_n .

A *planar algebra* is an algebra over the planar operad [Jon99]:

Definition 2.3. A planar algebra in $\mathbf{Vec}$ is a sequence of vector spaces $\mathcal{P} = (\mathcal{P}[n])_{n \geq 0}$ together with an action of the planar operad, i.e., for every isotopy class of planar tangle T of type $(k_1, \dots, k_r; k_0)$, a linear map $Z(T) : \mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_r] \rightarrow \mathcal{P}[k_0]$. For example,

$$Z \left(\begin{array}{c} \text{Diagram of a tangle } T \text{ with two input discs labeled 1 and 2, and a single output disc.} \end{array} \right) : \mathcal{P}[3] \otimes \mathcal{P}[5] \rightarrow \mathcal{P}[6].$$

This data must satisfy the following axioms:

- (identity) the identity tangle (which only has radial strings and no rotation between marked points) acts as the identity transformation.

$$Z \left(\begin{array}{c} \text{Diagram of an identity tangle with } n \text{ parallel strings.} \end{array} \right) = \text{id}_{\mathcal{P}[n]}$$

Here, we draw an n next to a string to indicate n parallel strings.

- (composition) operadic composition of tangles corresponds to composition of maps. Specifically, if S is of type $(k_1, \dots, k_r; k_0)$, T is of type $(\ell_1, \dots, \ell_s; \ell_0)$, and $\ell_0 = k_i$, then

$$Z(T \circ_i S) = Z(T) \circ (\text{id}_{\mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_{i-1}]} \otimes Z(S) \otimes \text{id}_{\mathcal{P}[k_{i+1}] \otimes \dots \otimes \mathcal{P}[k_r]}).$$

For example, the composite tangle $T \circ_2 S$ from Example 2.2 yields the equation

$$Z(T \circ_2 S) = Z(T) \circ (\text{id}_{\mathcal{P}[3]} \otimes Z(S)) : \mathcal{P}[3] \otimes \mathcal{P}[3] \otimes \mathcal{P}[2] \rightarrow \mathcal{P}[6]$$

- (symmetry) renumbering the input discs amounts to precomposing by a permutation. Namely, if T is of type $(k_1, \dots, k_r; k_0)$, $\sigma \in \mathfrak{S}_r$ is a permutation, and T^σ is the tangle of type $(k_{\sigma(1)}, \dots, k_{\sigma(r)}; k_0)$ obtained by renumbering the input circles of T , then

$$Z(T^\sigma) = Z(T) \circ \sigma : \mathcal{P}[k_{\sigma(1)}] \otimes \dots \otimes \mathcal{P}[k_{\sigma(r)}] \rightarrow \mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_r] \rightarrow \mathcal{P}[k_0].$$

The vector space $\mathcal{P}[n]$ is called the n^{th} *box space* of the planar algebra $\mathcal{P}$. A planar algebra is called *connected* if its zeroth box space $\mathcal{P}[0]$ is one dimensional. In that case, the map

$$Z \left(\begin{array}{c} \text{Diagram of a single circle with a marked point.} \end{array} \right) : \mathbb{C} \rightarrow \mathcal{P}[0]$$

is necessarily an isomorphism. If $\mathcal{P}$ is a connected planar algebra, then closed loops count for a multiplicative factor δ , for some $\delta \in \mathbb{C}$.

Even though it has never been phrased in this precise form, the following theorem is essentially well known [MPS10, Yam12, BHP12]:

Theorem 2.4. *There is an equivalence of categories²*

$$\left\{ \text{Planar algebras} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Pairs } (\mathcal{C}, X), \text{ where } \mathcal{C} \text{ is a pivotal category and} \\ X \in \mathcal{C} \text{ is a symmetrically self-dual generator} \end{array} \right\},$$

where for an object X to generate a tensor category means that every other object is isomorphic to a direct summand of a direct sum of powers of X .

The main result of our paper (Theorem A) is a generalisation of the above result.

2.2 Anchored planar algebras

Anchored planar tangles are a variant of planar tangles where we add *anchor lines* connecting the anchor points. Anchored planar algebras are very similar to planar algebras, with the notable difference that the symmetry axiom gets replaced by a braiding axiom.

Definition 2.5. An *anchored planar tangle* is a quadruple $T = (T, X, Q, A)$, where (T, X, Q) is a planar tangle (Definition 2.1), and A is a system of anchor lines. Here, a system of anchor lines is a collection of paths in T that connect the output anchor point q_0 to the input anchor points q_i and which, unlike the strings, are only considered up to isotopy. The anchor lines (always drawn in red in our pictures) may intersect the strings, but should not intersect each other. We also insist that the numbering of the input circles corresponds to the clockwise numbering of the anchor lines as they approach q_0 (indicated by the little blue arrow in the picture):



This means that, when considering anchored planar tangles, we may drop the numbering of the input circles, as it is entirely determined by the topology of the anchor lines.

Let S and T be anchored planar tangles of types $(k_1, \dots, k_r; k_0)$ and $(\ell_1, \dots, \ell_s; \ell_0)$, with systems of anchor lines A_S and A_T . If $\ell_0 = k_i$ for some $i \geq 1$, then their i -th operadic composition $T \circ_i S$ is defined as follows. The underlying planar tangle is as in (2). It is then equipped with the system of anchor lines obtained from $A_T \cup f(A_S)$ (f as in (2)) by replacing the line that connects q_0 to q_i with s parallel lines. We illustrate this process by

²Strictly speaking, such pairs $(\mathcal{C}, X)$ form a 2-category. However, this 2-category is equivalent to a usual category. See Lemma 3.6 and the paragraph thereafter for a discussion of this subtle point.

Note that, unlike the planar operad, the anchored planar operad does not admit an action of the symmetric group $\mathfrak{S}_n$ on its set of n -ary operations. It is therefore not an operad, but rather a non- Σ operad [May72, Def.3.12]. It carries however actions of the ribbon braid groups (see Section 4.2), and so it is what one might call a colored *ribbon braided operad*. To our knowledge, this particular variant of the notion of operad has not appeared in the literature. It is a straightforward modification of the notion of a braided operad [Fie, Def.3.2].

Recall that a pivotal category is a rigid tensor category (= tensor category with left and right duals) equipped with a monoidal natural isomorphism $\varphi_a : a \rightarrow a^{\vee\vee}$ between the identity functor and the double dual functor.

Given an object a of a braided pivotal category, we define its twist $\theta_a : a \rightarrow a$ by the formula

10

where $\beta_{a,b} : a \otimes b \rightarrow b \otimes a$ denotes the braiding. It satisfies $\theta_{a \otimes b} = \beta_{b,a} \circ \beta_{a,b} \circ (\theta_a \otimes \theta_b)$. We refer the reader to [HPT16, §2 and §A.2] for an extended discussion of braided pivotal categories.

An anchored planar algebra is an algebra over the anchored planar operad:

Definition 2.7. Let $\mathcal{C}$ be a braided pivotal category. An *anchored planar algebra* in $\mathcal{C}$ is a sequence $\mathcal{P} = (\mathcal{P}[n])_{n \geq 0}$ of objects of $\mathcal{C}$, along with operations

$$Z(T) : \mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_r] \rightarrow \mathcal{P}[k_0]$$

for every isotopy class of anchored planar tangle T of type $(k_1, \dots, k_r; k_0)$, subject to the following axioms:

- (identity) the identity anchored tangle acts as the identity morphism
- (composition) if S and T are anchored planar tangles of type $(k_1, \dots, k_r; k_0)$ and $(\ell_1, \dots, \ell_s; \ell_0)$, and if $\ell_0 = k_i$, then

$$Z(T \circ_i S) = Z(T) \circ (\text{id}_{\mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_{i-1}]} \otimes Z(S) \otimes \text{id}_{\mathcal{P}[k_{i+1}] \otimes \dots \otimes \mathcal{P}[k_r]}) \quad (5)$$

- (anchor dependence) the following relations hold:

$$\begin{aligned} \text{-- (braiding)} \quad & Z \left(\begin{array}{c} i \quad j \quad k \\ \text{[Diagram: Two strands, one from } i \text{ and one from } j \text{, cross each other. The } i \text{ strand goes to } k \text{ and the } j \text{ strand goes to } i \text{ after crossing.}] \end{array} \right) = Z \left(\begin{array}{c} i \quad j \quad k \\ \text{[Diagram: Two strands, one from } i \text{ and one from } j \text{, cross each other. The } i \text{ strand goes to } j \text{ and the } j \text{ strand goes to } k \text{ after crossing.}] \end{array} \right) \circ \beta_{\mathcal{P}[j], \mathcal{P}[i+k]} \\ \text{-- (twist)} \quad & Z \left(\begin{array}{c} n \\ \text{[Diagram: A single strand with } n \text{ parallel components, twisted.}] \end{array} \right) = \theta_{\mathcal{P}[n]} \end{aligned}$$

(Recall that a little number n next to a string to indicates n parallel strings.)

$\mathcal{P}[n]$ is called the n^{th} *box object* of the anchored planar algebra $\mathcal{P}$.

Remark 2.8. For any anchored planar algebra in $\mathcal{C}$, the anchor dependence axiom implies that $\mathcal{P}[0]$ has trivial twist, since the following tangles are isotopic:

$$\theta_{\mathcal{P}[0]} = Z \left(\begin{array}{c} \text{[Diagram: A circle with a red strand entering from the bottom, looping around, and exiting from the bottom.]} \end{array} \right) = Z \left(\begin{array}{c} \text{[Diagram: A circle with a red strand entering from the bottom and exiting from the bottom.]} \end{array} \right) = \text{id}_{\mathcal{P}[0]}.$$

Remark 2.9. If we write

$$f \circ_i g := f \circ (\text{id}_{b_1 \otimes \dots \otimes b_{i-1}} \otimes g \otimes \text{id}_{b_{i+1} \otimes \dots \otimes b_n}) : b_1 \otimes \dots \otimes b_{i-1} \otimes a_1 \otimes \dots \otimes a_m \otimes b_{i+1} \otimes \dots \otimes b_n \rightarrow c$$

for the operadic composition of morphisms $f : b_1 \otimes \dots \otimes b_n \rightarrow c$ and $g : a_1 \otimes \dots \otimes a_m \rightarrow b_i$ in $\mathcal{C}$, then axiom (5) can be elegantly rephrased as

$$Z(T \circ_i S) = Z(T) \circ_i Z(S).$$

2.3 Generators for the anchored planar operad

In this section, we present a collection of elements of the anchored planar operad, along with a number of relations that they satisfy. The tangles that we present below generate the anchored planar algebra operad as a non- Σ operad, but we will not need this result. What is more relevant is that these tangles generate the anchored planar algebra operad as a ribbon braided operad, a fact which will become evident from the role that they will play further down, in Algorithm 4.9. However, as we have decided not to discuss ribbon braided operads, we will not attempt to formalize this statement.

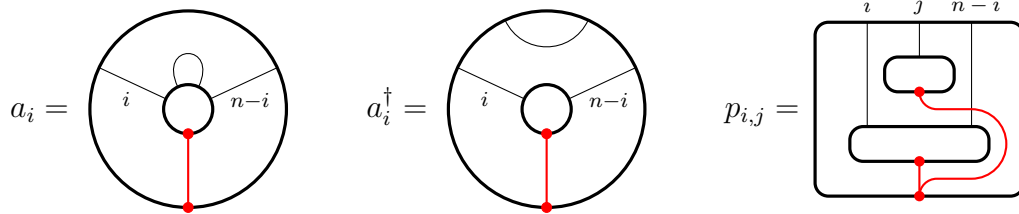
Throughout this section, the word ‘tangle’ means ‘isotopy class of anchored planar tangle’:

Definition 2.10 (generating tangles). (1) The unit tangle u has no input disks and no strings: $u = \bigcirc$.

(2) For every $n \in \mathbb{N}_{\geq 0}$, the identity tangle id_n has one input disk, n radial strings, and a radial anchor line: $\text{id}_n = \bigcirc_n$.

(3) For every $n \in \mathbb{N}_{\geq 0}$ and every $0 \leq i \leq n$, we let a_i be the tangle of type $(n+2; n)$ which caps the $(i+1)$ -st and $(i+2)$ -nd strings of the input circle, and we let $a_i^\dagger$ be tangle of type $(n; n+2)$ which caps the $(i+1)$ -st and $(i+2)$ -nd strings of the output circle.

(4) Finally, for every $n \in \mathbb{N}_{\geq 0}$ and every $0 \leq i \leq n$ and $j \geq 0$, we let $p_{i,j}$ be the tangle of type $(n, j; n+j)$ drawn here below:



We now list a number of relations satisfied by the above tangles, similar to the ones which appear in [Jon99, Pen12]:

Proposition 2.11. *The following relations hold amongst the tangles defined above. Here, we write $\circ$ in place of $\circ_1$ when there is no possible confusion.*

$$(A1) \quad p_{0,j} \circ_1 u = \text{id}_j, \quad p_{i,0} \circ_2 u = \text{id}_n$$

$$(A2) \quad a_i \circ a_j = a_{j-2} \circ a_i \text{ for } i+1 < j$$

$$(A3) \quad a_i^\dagger \circ a_j^\dagger = a_{j+2}^\dagger \circ a_i^\dagger \text{ for } i \leq j$$

$$(A4) \quad a_i \circ a_j^\dagger = \begin{cases} a_{j-2}^\dagger \circ a_i & \text{for } i < j-1 \\ \text{id}_n & \text{for } i = j \pm 1 \\ \delta \text{id}_n & \text{for } i = j \\ a_j^\dagger \circ a_{i-2} & \text{for } i > j+1 \end{cases} \quad \text{where } \delta \text{ refers to a free-floating circular strand}$$

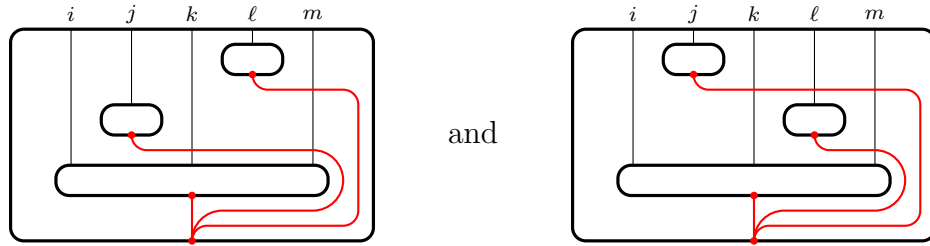
$$(A5) \quad a_i \circ p_{j,k} = \begin{cases} p_{j-2,k} \circ_1 a_i & \text{for } i+1 < j \\ p_{j,k-2} \circ_2 a_{i-j} & \text{for } j < i+1 < j+k \\ p_{j,k} \circ_1 a_{i-k} & \text{for } i+1 > j+k \end{cases}$$

$$(A6) \quad a_i^\dagger \circ p_{j,k} = \begin{cases} p_{j+2,k} \circ_1 a_i^\dagger & \text{for } i \leq j \\ p_{j,k+2} \circ_2 a_{i-j}^\dagger & \text{for } j \leq i \leq j+k \\ p_{j,k} \circ_1 a_{i-k}^\dagger & \text{for } i \geq j+k \end{cases}$$

$$(A7) \quad p_{i+j,k} \circ_1 p_{i,j+\ell} = p_{i,j+k+\ell} \circ_2 p_{j,k}$$

Proof. Immediate by drawing pictures. □

Remark 2.12. The relation $p_{i+j+k,\ell} \circ_1 p_{i,j} = p_{i,j} \circ_1 p_{i+k,\ell}$ holds in the absence of anchor lines ([Jon99, Lemma 4.1.20, (i)]), but not in their presence. These tangles are given by:



The underlying planar tangles are isotopic, but the anchor lines are not the same.

3 The main theorem and examples

Given a braided pivotal category $\mathcal{C}$, there is the notion of a *module tensor category* over $\mathcal{C}$. This is a tensor category $\mathcal{M}$ equipped with a multiplicative version of an action of $\mathcal{C}$ (see Definition 3.1). A module tensor category is called *pointed* if it is equipped with a chosen self-dual object m that generates $\mathcal{M}$ as a $\mathcal{C}$ -module tensor category (see Definition 3.4 for a precise definition).

Our main theorem (Theorem 3.7) says that there is an equivalence between the notions of anchored planar algebra in $\mathcal{C}$ and of pointed module tensor category over $\mathcal{C}$. This will be used later, in Section 3.3, to construct many examples of anchored planar algebras.

3.1 Module tensor categories

The notion of module tensor category over a braided pivotal category was studied in [HPT16]:

Definition 3.1. Let $\mathcal{C}$ be a braided tensor category. A *module tensor category* over $\mathcal{C}$ is a tensor category $\mathcal{M}$ together with a braided tensor functor $\Phi^z : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M})$ from $\mathcal{C}$ to the Drinfel'd center of $\mathcal{M}$. If $\mathcal{C}$ and $\mathcal{M}$ are moreover pivotal and if Φ^z is a pivotal functor, then we call $\mathcal{M}$ a *pivotal module tensor category*.

From now on, we shall use the term “module tensor category” to mean “pivotal module tensor category”. The composite

$$\Phi := F \circ \Phi^{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{M}$$

of $\Phi^{\mathcal{Z}}$ with the forgetful functor $F : \mathcal{Z}(\mathcal{M}) \rightarrow \mathcal{M}$ is called the *action functor*. We write $e_{\Phi(c)}$ for the half-braiding associated to $\Phi^{\mathcal{Z}}(c)$, so that $\Phi^{\mathcal{Z}}(c) = (\Phi(c), e_{\Phi(c)})$. If the action functor Φ admits a right adjoint, then we denote it by

$$\mathrm{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C} \tag{6}$$

and call it a *categorified trace*. It comes equipped with *traciator* isomorphisms

$$\tau_{x,y} : \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \rightarrow \mathrm{Tr}_{\mathcal{C}}(y \otimes x)$$

satisfying $\tau_{x,y \otimes z} = \tau_{z \otimes x, y} \circ \tau_{x \otimes y, z}$. For more details, we refer the reader to [HPT16].

Definition 3.2. Suppose $(\mathcal{M}_1, \Phi_1^{\mathcal{Z}})$ and $(\mathcal{M}_2, \Phi_2^{\mathcal{Z}})$ are module tensor categories over $\mathcal{C}$. A functor of module tensor categories $(G, \gamma) : (\mathcal{M}_1, \Phi_1^{\mathcal{Z}}) \rightarrow (\mathcal{M}_2, \Phi_2^{\mathcal{Z}})$ consists of the following data:

- a pivotal tensor functor $G = (G, \nu, i) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$. Here, the structure maps $\nu_{x,y} : G(x) \otimes G(y) \rightarrow G(x \otimes y)$ and $i : 1_{\mathcal{M}_2} \rightarrow G(1_{\mathcal{M}_1})$ satisfy the obvious naturality, associativity and unitality axioms, along with the following compatibility with the pivotal structure:

$$G(\varphi_x) = \delta_x^{-1} \circ \delta_{x^\vee} \circ \varphi_{G(x)}$$

where $\delta_x := ((G(\mathrm{ev}_x) \circ \nu_{x^\vee, x}) \otimes \mathrm{id}_{G(x)^\vee}) \circ (\mathrm{id}_{G(x)^\vee} \otimes \mathrm{coev}_{G(x)}) : G(x^\vee) \rightarrow G(x)^\vee$.

- an *action coherence* monoidal natural isomorphism $\gamma : \Phi_2 \Rightarrow G\Phi_1$. This consists of isomorphisms $\gamma_c : \Phi_2(c) \rightarrow G(\Phi_1(c))$, natural in c , making the following diagram commute:

$$\begin{array}{ccc} \Phi_2(c) \otimes \Phi_2(d) & \xrightarrow{\gamma_c \otimes \gamma_d} & G(\Phi_1(c)) \otimes G(\Phi_1(d)) \\ \downarrow \cong & & \downarrow \cong \\ \Phi_2(c \otimes d) & \xrightarrow{\gamma_{c \otimes d}} G(\Phi_1(c \otimes d)) \xrightarrow{\cong} G(\Phi_1(c) \otimes \Phi_1(d)) \end{array}$$

and such that γ_{1c} is equal to the obvious composite of unit coherences of Φ_i and G .

Moreover, for every $x \in \mathcal{M}_1$ and $c \in \mathcal{C}$, the above pieces of data should satisfy the following compatibility axiom with half-braidings:

$$\begin{array}{ccccc} \Phi_2(c) \otimes G(x) & \xrightarrow{\gamma_c \otimes \mathrm{id}} & G(\Phi_1(c)) \otimes G(x) & \xrightarrow{\cong} & G(\Phi_1(c) \otimes x) \\ \downarrow e_{\Phi_2(c), G(x)} & & & & \downarrow G(e_{\Phi_1(c), x}) \\ G(x) \otimes \Phi_2(c) & \xrightarrow{\mathrm{id} \otimes \gamma_c} & G(x) \otimes G(\Phi_1(c)) & \xrightarrow{\cong} & G(x \otimes \Phi_1(c)) \end{array} \tag{7}$$

Definition 3.3. Given two functors of module tensor categories (G, γ^G) and (H, γ^H) from $(\mathcal{M}_1, \Phi_1^Z)$ to $(\mathcal{M}_2, \Phi_2^Z)$, a natural transformation $\kappa : (G, \gamma^G) \Rightarrow (H, \gamma^H)$ is a family of isomorphisms $\kappa_x : G(x) \rightarrow H(x)$ for $x \in \mathcal{M}_1$, natural in x . They are monoidal in the sense that $\kappa_{1_{\mathcal{M}_1}} \circ i_G = i_H$ and that the following diagram commutes:

$$\begin{array}{ccc} F(x \otimes y) & \xrightarrow{\cong} & F(x) \otimes F(y) \\ \downarrow \kappa_{x \otimes y} & & \downarrow \kappa_x \otimes \kappa_y \\ G(x \otimes y) & \xrightarrow{\cong} & G(x) \otimes G(y), \end{array} \quad (8)$$

and they are subject to the following compatibility axiom with the action coherence isomorphisms:

$$\begin{array}{ccc} \Phi_2(c) & \xrightarrow{\gamma_c^G} & G(\Phi_1(c)) \\ & \searrow \gamma_c^H & \swarrow \kappa_{\Phi_1(c)} \\ & H(\Phi_1(c)) & \end{array} \quad (9)$$

An object m in a pivotal category is called *symmetrically self-dual* if it comes equipped with an isomorphism $\psi : m \rightarrow m^\sim$ satisfying $\psi^\sim \circ \varphi_m = \psi$. Here, $\varphi_m : m \rightarrow m^{\sim\sim}$ is the isomorphism provided by the pivotal structure and

$$\psi^\sim = (\text{ev}_{m^\sim} \otimes \text{id}_m) \circ (\text{id}_{m^{\sim\sim}} \otimes \psi \otimes \text{id}_{m^\sim}) \circ (\text{id}_{m^{\sim\sim}} \otimes \text{coev}_m) : m^{\sim\sim} \rightarrow m^\sim \quad (10)$$

is the dual of ψ . Upon identifying m with $m^\sim$ via ψ , we write

$$\begin{aligned} \tilde{\text{ev}}_m &:= \text{ev}_m \circ (\psi \otimes \text{id}_m) : m \otimes m \rightarrow 1 \quad \text{and} \\ \tilde{\text{coev}}_m &:= (\text{id}_m \otimes \psi^{-1}) \circ \text{coev}_m : 1 \rightarrow m \otimes m \end{aligned} \quad (11)$$

for the evaluation and coevaluation maps. Symmetrically self-dual objects admit a graphical calculus of unoriented strands where all isotopies are allowed, and where the caps and cups are given by the above modified evaluation and coevaluation maps.

Definition 3.4. A module tensor category $\mathcal{M}$ is called *pointed* if it comes equipped with a chosen symmetrically self-dual object $m \in \mathcal{M}$ that generates it as a module tensor category over $\mathcal{C}$. Here, we say that m generates $\mathcal{M}$ as a module tensor category if every object of $\mathcal{M}$ is a direct summand of a sum of objects of the form $\Phi(c) \otimes m^{\otimes n}$, for some $c \in \mathcal{C}$.

A functor of pointed module tensor categories is a functor $G : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ of module tensor categories that sends the distinguished object $m_1 \in \mathcal{M}_1$ to the distinguished object $m_2 \in \mathcal{M}_2$ and satisfies $\psi_2 = \delta_{m_1} \circ G(\psi_1)$.

A natural transformation between two functors $G, H : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ of pointed module tensor category is a natural transformation $\kappa : G \Rightarrow H$ that satisfies $\kappa_{m_1} = \text{id}_{m_2}$.

Definition 3.5. For a fixed braided pivotal category $\mathcal{C}$, we denote by **ModTens** the 2-category of (pivotal) module tensor categories which admit direct sums, are idempotent complete, and whose action functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ admits a right adjoint. Note that we omit the dependence on $\mathcal{C}$ from the notation. Similarly, the 2-category of pointed module categories with these properties is denoted **ModTens**_{*}.

From now on, we restrict our attention to (pivotal) module tensor categories which are objects of $\mathbf{ModTens}$.

Lemma 3.6. *Let $G, H : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be two 1-morphisms in $\mathbf{ModTens}_*$. Then there exists at most one 2-morphism from G to H . Moreover, any 2-morphism in $\mathbf{ModTens}_*$ is invertible.*

Proof. By definition, a natural transformation $\kappa : G \Rightarrow H$ of functors between pointed module tensor categories satisfies $\kappa_{m_1} = \text{id}_{m_2}$. By the commutativity of the diagram (8), this completely determines κ on objects of the form $m_1^{\otimes n}$. By the same diagram, we also see that $\kappa_{m_1^{\otimes n}}$ is invertible. Similarly, the commutativity of (9) completely determines κ on objects of the form $\Phi_1(c)$ for $c \in \mathcal{C}$, and $\kappa_{\Phi_1(c)}$ is invertible. By one more application of (8), we see that κ is completely determined and invertible on objects of the form $\Phi_1(c) \otimes m_1^{\otimes n}$. At last, κ is completely determined and invertible on any direct summand of a direct sum of objects of the form $\Phi_1(c) \otimes m_1^{\otimes n}$, which is all of $\mathcal{M}_1$ by our assumption that m_1 generates $\mathcal{M}_1$. $\square$

Consider the equivalence relation on 1-morphisms of $\mathbf{ModTens}_*$ according to which two functors $G, H : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ are equivalent if there exists a (necessarily invertible) natural transformation $\kappa : G \Rightarrow H$. We denote $\tau_{\leq 1}(\mathbf{ModTens}_*)$ the 1-category with same objects as $\mathbf{ModTens}_*$, and with 1-morphisms given by equivalence classes of 1-morphisms of $\mathbf{ModTens}_*$ with respect to the above equivalence relation. By the previous lemma, the projection functor

$$\mathbf{ModTens}_* \rightarrow \tau_{\leq 1}(\mathbf{ModTens}_*)$$

is an equivalence of 2-categories (where $\mathbf{ModTens}_*$ is treated as a 2-category with only identity 2-morphisms). The moral of the above discussion is that, even though $\mathbf{ModTens}_*$ is not a 1-category, it is harmless to treat it as if it were one.

3.2 The main theorem

Fix a braided pivotal tensor category $\mathcal{C}$ which admits finite direct sums and is idempotent complete. Let $\mathbf{APA}$ be the category of anchored planar algebras in $\mathcal{C}$ (we omit $\mathcal{C}$ from the notation), and let $\mathbf{ModTens}_*$ and $\tau_{\leq 1}(\mathbf{ModTens}_*)$ be as in the previous section. The following result generalizes the well known result according to which the data of a planar algebra (in vector spaces) is equivalent to that of a pair $(\mathcal{D}, X)$, where $\mathcal{D}$ is a pivotal tensor category and $X \in \mathcal{D}$ is a generating symmetrically self-dual object [MPS10, BHP12, Yam12]:

Main Theorem 3.7 (Theorem A). *There exists an equivalence of categories*

$$\left\{ \text{Anchored planar algebras in } \mathcal{C} \right\} \cong \left\{ \begin{array}{l} \text{Pointed pivotal module} \\ \text{tensor categories over } \mathcal{C} \end{array} \right\}$$

between the category $\mathbf{APA}$ of anchored planar algebras in $\mathcal{C}$, and the category $\tau_{\leq 1}(\mathbf{ModTens}_)$ of pointed pivotal module tensor categories over $\mathcal{C}$ whose action functor admits a right adjoint.*

Proof. We construct functors $\Lambda : \tau_{\leq 1}(\mathbf{ModTens}_*) \rightarrow \mathbf{APA}$ and $\Delta : \mathbf{APA} \rightarrow \tau_{\leq 1}(\mathbf{ModTens}_*)$ in Sections 5 and 6, respectively. In Sections 7.1 and 7.2, we construct a natural isomorphism $\Delta\Lambda \Rightarrow \text{id}_{\tau_{\leq 1}(\mathbf{ModTens}_*)}$. Finally, in Section 7.3, we show that the equality $\Lambda\Delta = \text{id}_{\mathbf{APA}}$ holds on the nose. $\square$

We quickly sketch the construction of the above two functors. The functor Λ sends a pointed module tensor category $(\mathcal{M}, m)$ to the anchored planar algebra $\mathcal{P}$ with n th box object

$$\mathcal{P}[n] := \text{Tr}_{\mathcal{C}}(m^{\otimes n}), \quad (12)$$

and action of the generating tangles (Definition 2.10) given by Theorem 5.1.

The functor Δ sends an anchored planar algebra $\mathcal{P}$ to the category $\mathcal{M}$ constructed as follows. We first construct $\mathcal{M}_0$, whose objects are formal expressions “ $\Phi(c) \otimes m^{\otimes n}$ ” and whose morphisms are given by

$$\mathcal{M}_0(\Phi(c) \otimes m^{\otimes n_1}, \Phi(d) \otimes m^{\otimes n_2}) := \mathcal{C}(c, d \otimes \mathcal{P}[n_2 + n_1]).$$

$\mathcal{M}$ is then obtained from $\mathcal{M}_0$ by formally adding direct sums, and idempotent completing. The composition of morphisms, tensor structure, and structure of module tensor category are described in Section 6.

3.2.1 Generalisations

There exist a number of variations of Theorem 3.7, which we list below without proof. We believe that these more general statements can be proved in a way essentially identical to our main theorem:

- If the generating object $m \in \mathcal{M}$ is not required to be self-dual, then the notion of anchored planar algebra needs to be modified to include orientations on the strands. We call the resulting notion an *oriented anchored planar algebra*. In an oriented anchored planar algebra, the box objects are no longer indexed by numbers. They are indexed by sequences of $+$ and $-$, encoding the orientations of the strands. For example, in an oriented anchored planar algebra, we have:

$$Z \left(\text{Diagram} \right) : \mathcal{P}[+, -, +] \otimes \mathcal{P}[-, +, -, +, +] \rightarrow \mathcal{P}[+, -, +, -, +, +].$$

There is an equivalence of categories:

$$\left\{ \begin{array}{l} \text{Oriented anchored} \\ \text{planar algebras in } \mathcal{C} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Pivotal module tensor categories over } \mathcal{C} \\ \text{equipped with a chosen generator} \end{array} \right\}$$

Here, the condition that m generates $\mathcal{M}$ means that every object of the category is isomorphic to a direct summand of a direct sum of objects of the form $\Phi(c) \otimes m^{\otimes n_1} \otimes (m^\vee)^{\otimes n_2} \otimes m^{\otimes n_3} \otimes (m^\vee)^{\otimes n_4} \otimes \dots$

- If instead of requiring that the generator $m \in \mathcal{M}$ is symmetrically self-dual, we take it anti-symmetrically self-dual, i.e., if one replaces the condition $\psi^\vee \circ \varphi_m = \psi$ in the paragraph above (10) by $\psi^\vee \circ \varphi_m = -\psi$, then the strands need to be equipped with a

“disorientation” in the sense of [CMW09, Fig. 3]. Here, a disorientation is the data of an orientation, along with a finite collection of orientation-reversing points, drawn as $\rightarrow \rightarrow \leftarrow$ and $\leftarrow \rightarrow \leftarrow$. These are subject to the local relations $Z(\rightarrow \rightarrow \leftarrow \leftarrow) = Z(\rightarrow \rightarrow)$ and $Z(\rightarrow \rightarrow \leftarrow) = -Z(\rightarrow \leftarrow \leftarrow)$. We call the resulting notion a *disoriented anchored planar algebra*. It is convenient (but not necessary) to insist that the strands only meet the boundary circles in the positive orientation: in that way, the box objects are indexed by $\mathbb{N}_{\geq 0}$.

We have an equivalence of categories:

$$\left\{ \begin{array}{l} \text{Disoriented anchored} \\ \text{planar algebras in } \mathcal{C} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Pivotal module tensor categories over } \mathcal{C} \text{ equipped} \\ \text{with a chosen anti-symmetrically self-dual generator} \end{array} \right\}$$

The crucial step which gets affected by the presence of disorientations is the last equality in (42), which now acquires a minus sign.

- Instead of a single generator, we may consider a collection of symmetrically self-dual objects $\{m_a\}_{a \in \Gamma}$ that jointly generate $\mathcal{M}$. The appropriate generalisation of anchored planar algebras then includes multiple colours of strings, one for each element of Γ . We call the resulting notion a *colored anchored planar algebra*. The box objects of a colored anchored planar algebra are now indexed by words in the alphabet Γ . For example:

$$Z \left(\begin{array}{c} \text{Diagram: A circle with two smaller circles inside. A red strand enters from the bottom, goes up, loops around the left inner circle, and exits. A blue strand enters from the top, goes down, loops around the right inner circle, and exits. A green strand enters from the top, goes down, loops around the left inner circle, and exits. A red strand enters from the bottom, goes up, loops around the right inner circle, and exits. The strands are labeled with letters a, b, c. The diagram is enclosed in large parentheses.} \end{array} \right) : \mathcal{P}[aab] \otimes \mathcal{P}[aabc] \rightarrow \mathcal{P}[aacccb], \quad a, b, c \in \Gamma.$$

For every set Γ , we then get an equivalence of categories:

$$\left\{ \begin{array}{l} \Gamma\text{-colored anchored} \\ \text{planar algebras in } \mathcal{C} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Pivotal module tensor categories over } \mathcal{C} \\ \text{equipped with a collection of symmetrically} \\ \text{self-dual generators indexed by } \Gamma \end{array} \right\}$$

- Dropping the self-dual condition, we may also consider collections of arbitrary objects that jointly generate $\mathcal{M}$. In that case, the appropriate generalisation of planar algebra is a *colored oriented anchored planar algebra*.

There is an equivalence of categories:

$$\left\{ \begin{array}{l} \Gamma\text{-colored oriented anchored} \\ \text{planar algebras in } \mathcal{C} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Pivotal module tensor categories over } \mathcal{C} \\ \text{with a collection of generators indexed by } \Gamma \end{array} \right\}$$

We note that in a Γ -colored oriented anchored planar algebra, the box objects are now indexed by words in $\Gamma \cup \Gamma^\sim$.

3.3 Examples

Using Theorem 3.7, we now provide a list of examples of anchored planar algebras. For each anchored planar algebra, we compute the box objects by means of the formula (12).

3.3.1 Near group categories

Let A be an abelian group and let $\mathcal{C} := \text{Vec}(A)$ be the category of A -graded vector spaces. Given a symmetric bicharacter $\langle \cdot, \cdot \rangle : A \times A \rightarrow U(1)$ (which we will later take to be non-degenerate), there is a braided structure on $\mathcal{C}$ with braiding $\beta_{a,b} : a \otimes b \rightarrow b \otimes a$ given by $\langle a, b \rangle \text{id}_{a+b}$. Note that two such categories are braided equivalent iff the associated quadratic forms $a \mapsto \langle a, a \rangle$ are equal [FRS04, Prop. 2.14]. A braided category as above has a preferred pivotal structure, characterised by the fact that all the quantum dimensions of the simple objects are 1. Its twist (4) are given by $\theta_a = \langle a, a \rangle \text{id}_a$.

Example 3.8 (Tambara-Yamagami). Let A be a finite abelian group, and let $\langle \cdot, \cdot \rangle$ be a non-degenerate symmetric bicharacter. Let $N := |A|$, and let τ be a square root of N^{-1} . The Tambara-Yamagami fusion category $\mathcal{M} := \mathcal{TY}(A, \langle \cdot, \cdot \rangle, \tau)$ [TY98] has distinguished representatives of simple objects indexed by the set $A \cup \{m\}$. We follow the conventions in [Shi11, §3]. The monoidal structure on $A \subset \mathcal{M}$ is given by group multiplication, with trivial associator. The remaining fusion rules are:

$$a \otimes m := m, \quad m \otimes a := m, \quad m \otimes m := \bigoplus_{a \in A} a.$$

The associator $\alpha_{a,m,b} : (a \otimes m) \otimes b \rightarrow a \otimes (m \otimes b)$ is $\langle a, b \rangle \text{id}_m$ (we refer to [Shi11, §3] for the remaining associators). We work with the pivotal structure on $\mathcal{M}$ given by $\varphi_a = \text{id}_a$ for $a \in A$ (so that $\dim(a) = 1$) and $\varphi_m = \text{id}_m$ (so that $\dim(m) = \tau^{-1}$). Note that, with respect to this pivotal structure, the object m is symmetrically self-dual.

The center $\mathcal{Z}(\mathcal{M})$ contains $2N$ invertible objects, labeled by pairs (a, ε) where $a \in A$ and $\varepsilon^2 = \langle a, a \rangle$ [Izu01]. The half-braiding $e_{(a,\varepsilon)}$ is given by $e_{(a,\varepsilon),b} = \langle a, b \rangle \text{id}_{a+b}$ and $e_{(a,\varepsilon),m} = \varepsilon \text{id}_m$, from which it follows that $\theta_{(a,\varepsilon)} = \langle a, a \rangle \text{id}_{(a,\varepsilon)}$. Using the formula for $\alpha_{a,m,b}$, one easily computes that $(a, \varepsilon) \otimes (b, \eta) = (a + b, \varepsilon \eta \langle a, b \rangle)$ in $\mathcal{Z}(\mathcal{M})$.

We now specialize to $A = \mathbb{Z}/N\mathbb{Z}$, with bicharacter $\langle i, j \rangle = q^{ij}$, $q = \exp(2\pi i/N)$. There is a distinguished copy of A inside $\mathcal{Z}(\mathcal{M})$, given by $\varepsilon_a = q^{a^2(N+1)/2}$. The thus obtained inclusion functor $\Phi^{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M})$ endows $\mathcal{M}$ with the structure of a module tensor category over $\mathcal{C} = \text{Vec}(\mathbb{Z}/N\mathbb{Z})$ (with braiding given by the above bicharacter, and equipped with its canonical pivotal structure). The action functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ is fully faithful, and its adjoint $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$ is the projection functor from $\mathcal{M}$ onto its full $\text{Vec}(\mathbb{Z}/N\mathbb{Z})$ subcategory. In other words, $\text{Tr}_{\mathcal{C}}(m) = 0$, and $\text{Tr}_{\mathcal{C}}(a) = a$ for all $a \in \mathbb{Z}/N\mathbb{Z}$. Applying formula (12) to this situation, we see that the anchored planar algebra $\mathcal{P}$ associated to $\mathcal{M}$ and m is given by:

$$\mathcal{P}[2k] = \text{Tr}_{\mathcal{C}}(m^{\otimes 2k}) \cong \mathbb{C}[\mathbb{Z}/N\mathbb{Z}]^{\otimes k} \quad \text{and} \quad \mathcal{P}[2k+1] = 0.$$

Example 3.9 (Planar para algebras). Let $\mathcal{C} = \text{Vec}(\mathbb{Z}/N\mathbb{Z})$ be as in the previous example. The recently introduced planar para algebras [JL17, Def. 2.3] are the same thing as anchored planar algebras in $\text{Vec}(\mathbb{Z}/N\mathbb{Z})$: the para isotopy axiom (iv) is our braid anchor dependence, and the rotation axiom (v) is our twist anchor dependence.

Applying the functor $\Delta : \text{APA} \rightarrow \text{ModTens}_*$ (see Section 6 below) to their main example of the parafermion planar para algebra, one recovers the Tambara-Yamagami category $\mathcal{TY}(A, \langle \cdot, \cdot \rangle, \tau)$ associated to the positive square root of N^{-1} . Indeed, if m is the strand in the parafermion planar para algebra, then $m \otimes m \cong \mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$. Thus, the parafermion

planar para algebra is given, as in Example 3.8, by a braided functor $\text{Vec}(\mathbb{Z}/N\mathbb{Z}) \rightarrow \mathcal{Z}(\mathcal{M})$ to the center of a Tambara-Yamagami fusion category.

Example 3.10 (Near groups). These examples were pointed out to us by Zhengwei Liu. Let A be an abelian group of order N , and let $\mathcal{M}$ be a near group category with fusion rules

$$a \otimes m = m, \quad m \otimes a = m, \quad m \otimes m = Nm \oplus \bigoplus_{a \in A} a.$$

Such categories are known to exist for many groups (up to order 13) [EG14, Table 2], and are conjectured to exist for all finite cyclic groups.

By [Izu01], [EG14, Thm. 4], there is a non-degenerate symmetric bicharacter $\langle \cdot, \cdot \rangle : A \times A \rightarrow U(1)$ which is an invariant of $\mathcal{M}$. Now, by [EG14, §4.3], each invertible object $a \in A$ lifts uniquely to the center $\mathcal{Z}(\mathcal{M})$, and the braidings and twists on $A \subset \mathcal{Z}(\mathcal{M})$ are given by the bicharacter: $\beta_{a,b} = \langle a, b \rangle$ and $\theta_a = \langle a, a \rangle$. Hence, $\mathcal{Z}(\mathcal{M})$ admits a braided pivotal functor from the corresponding ribbon category $\text{Vec}(A)$.

Let $\mathcal{P}$ be the anchored planar algebra associated to $(\mathcal{M}, m)$. The action functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ is full, and its adjoint $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$ is the projection onto the copy of $\text{Vec}(A)$ inside $\mathcal{M}$. For small k , the box objects $\mathcal{P}[k] = \text{Tr}_{\mathcal{C}}(m^{\otimes k})$ are given by $\mathcal{P}[0] = 1$, $\mathcal{P}[1] = 0$, and $\mathcal{P}[2] = \bigoplus_{a \in A} a$. For larger $k \geq 1$, we have $\mathcal{P}[k] = f(k) \bigoplus_{a \in A} a$, where the function f is given by the recursive formula

$$f(1) = 0 \quad f(2) = 1 \quad f(k+1) = N(f(k) + f(k-1)).$$

We note that the quantity $f(k)$ depends polynomially on N , with coefficients given by diagonals in Pascal's triangle:

$$f(k) = \sum_{i=2}^{\lfloor k/2 \rfloor + 1} \binom{k-i}{i-2} N^{k-i}.$$

3.3.2 Temperley-Lieb-Jones module categories

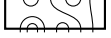
Let $\mathcal{C}$ be non-degenerately braided pivotal fusion category, let $A \in \mathcal{C}$ be a connected étale algebra (an algebra is called *étale* if it is both commutative and separable), and let $\mathcal{M} = \text{Mod}_{\mathcal{C}}(A)$ be the category of right A -modules. By [DMNO13, Cor. 3.30], we then have

$$\mathcal{Z}(\mathcal{M}) \cong \mathcal{C} \boxtimes \mathcal{D},$$

where $\mathcal{D}$ is the full subcategory of $\mathcal{M}$ consisting of dyslectic A -modules, equipped with the braiding opposite to the one inherited from $\mathcal{C}$. Thus, we see that $\mathcal{M}$ becomes a module tensor category for both $\mathcal{C}$ and $\mathcal{D}$:

$$\Phi_{\mathcal{C}}^{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M}) \quad \text{and} \quad \Phi_{\mathcal{D}}^{\mathcal{Z}} : \mathcal{D} \rightarrow \mathcal{Z}(\mathcal{M}).$$

The functor $\Phi_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{M}$ is dominant, while the functor $\Phi_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{M}$ is fully faithful (so that its adjoint $\text{Tr}_{\mathcal{D}} : \mathcal{M} \rightarrow \mathcal{D}$ is the projection onto a full subcategory of $\mathcal{M}$ equivalent to $\mathcal{D}$).

We now specialize to the case where $\mathcal{C}$ is the Temperley-Lieb-Jones (TLJ) A_n category. Let $q := \exp(\pi i/(n+1))$ and $\delta := q + q^{-1}$. The TLJ A_n category is the quotient of the free pivotal idempotent complete category on a single symmetrically self-dual object m of dimension δ by the ideal of negligible morphisms (morphisms $f : x \rightarrow y$ such that $\text{tr}(fg) = 0$ for all $g : y \rightarrow x$). More concretely, it is obtained by taking the category whose objects are the natural numbers and whose morphisms are linear combinations of Kauffman diagrams (things like this: ) , equating free floating loops with δ , taking the idempotent completion, and then modding out by the negligible morphisms. This category has n simple objects, denoted $\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}$, called the Jones-Wenzl idempotents. The object $\mathbf{1}$ is the unit, and the object $\mathbf{2}$ is the generator m (the strand). We also write $\mathbf{g}$ for the object $\mathbf{n}$. We equip $\mathcal{C}$ with the braiding given by

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} := iq^{1/2} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} - iq^{-1/2} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \quad (13)$$

on the generator. Note that this is not the only possible braiding, and that the classification of étale algebras depends on the choice of braiding [HPT16, Rem. 3.14].

Example 3.11. Let $\mathcal{C}$ be the TLJ A_n category, and let $(\mathcal{M}, m) = (A_n, \mathbf{2})$ with its obvious structure of module tensor category over $\mathcal{C} = A_n$. The functor $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$ is the identity functor, and the associated anchored planar algebra $\mathcal{P}$ is given by $\mathcal{P}[k] = \underline{\text{Hom}}_{\mathcal{C}}(1, m^{\otimes k}) = m^{\otimes k}$ (the object represented by k parallel strands).

Example 3.12 (D_{2n}). When $\mathcal{C}$ is the TLJ A_{4n-3} category, $A := \mathbf{1} \oplus \mathbf{g} \in \mathcal{C}$ is a commutative algebra object, and its category of modules $\mathcal{M}$ is a D_{2n} tensor category. We write $\underline{1}, \underline{2}, \dots, \underline{2n-2}, \underline{2n-1}, \underline{2n-1}'$ for the simples of $\mathcal{M}$.

If we view $(\mathcal{M}, m) = (D_{2n}, \underline{2})$ as a pointed module tensor category over $\mathcal{C}$, then the box objects of the associated anchored planar algebra can be computed as follows:

$$\mathcal{P}[k] = \text{Tr}_{\mathcal{C}}(\underline{2}^{\otimes k}) = \text{Tr}_{\mathcal{C}}(\Phi(\mathbf{2}^{\otimes k})) = (\mathbf{1} \oplus \mathbf{g}) \otimes \mathbf{2}^{\otimes k}.$$

Here, $\Phi : A_{4n-3} \rightarrow D_{2n} = \text{Mod}_{A_{4n-3}}(A)$ is the free A -module functor, and its adjoint $\text{Tr}_{\mathcal{C}} : D_{2n} \rightarrow A_{4n-3}$ is the functor that forgets the module structure (see [HPT16, Ex. 3.12]).

Example 3.13 (D_{2n}). Let $\mathcal{M} = D_{2n}$ be as in the previous example. By [BEK01, Cor. 4.9], the center of $\mathcal{M}$ is given by $\mathcal{Z}(D_{2n}) \cong \mathcal{C} \boxtimes \mathcal{D} = A_{4n-3} \boxtimes \frac{1}{2} \overline{D}_{2n}$, where the bar on the second factor indicates that it is equipped with the opposite of the natural braiding.

If we view $(\mathcal{M}, m) = (D_{2n}, \underline{2})$ as a pointed module tensor category over $\mathcal{D} := \frac{1}{2} \overline{D}_{2n}$, then $\text{Tr}_{\mathcal{D}} : \mathcal{M} \rightarrow \mathcal{D}$ is the projection functor onto the copy of $\mathcal{D}$ inside D_{2n} . The associated anchored planar algebra in $\mathcal{D}$ is given by $\mathcal{P}[k] = \underline{2}^{\otimes k}$ when k is even, and $\mathcal{P}[k] = 0$ when k is odd.

Example 3.14 (D_4). Let $(\mathcal{M}, m) = (D_4, \underline{2})$ be as in the previous examples (note that it is also a Tambara-Yamagami category), and let

$$\mathcal{C} := \mathcal{Z}(\mathcal{M}) = A_5 \boxtimes \frac{1}{2} \overline{D}_4 = A_5 \boxtimes \text{Vec}(\mathbb{Z}/3\mathbb{Z}).$$

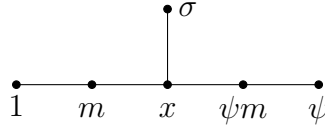
As before, we write $\mathbf{1}, \mathbf{2}, \dots, \mathbf{5}$ for the objects of A_5 , and $\underline{1}, \underline{2}, \underline{3}, \underline{3}'$ for the object of D_4 . The functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ is given on pairs $(p, q) \in \mathcal{Z}(D_4)$ by $\Phi(p, q) = p \otimes q$:

$$\begin{array}{ccccc} (\mathbf{1}, \underline{1}) \mapsto \underline{1} & (\mathbf{2}, \underline{1}) \mapsto \underline{2} & (\mathbf{3}, \underline{1}) \mapsto \underline{3} \oplus \underline{3}' & (\mathbf{4}, \underline{1}) \mapsto \underline{2} & (\mathbf{5}, \underline{1}) \mapsto \underline{1} \\ (\mathbf{1}, \underline{3}) \mapsto \underline{3} & (\mathbf{2}, \underline{3}) \mapsto \underline{2} & (\mathbf{3}, \underline{3}) \mapsto \underline{1} \oplus \underline{3}' & (\mathbf{4}, \underline{3}) \mapsto \underline{2} & (\mathbf{5}, \underline{3}) \mapsto \underline{3} \\ (\mathbf{1}, \underline{3}') \mapsto \underline{3}' & (\mathbf{2}, \underline{3}') \mapsto \underline{2} & (\mathbf{3}, \underline{3}') \mapsto \underline{1} \oplus \underline{3} & (\mathbf{4}, \underline{3}') \mapsto \underline{2} & (\mathbf{5}, \underline{3}') \mapsto \underline{3}' \end{array}$$

The box objects $\mathcal{P}[k] = \text{Tr}_{\mathcal{C}}(\underline{2}^{\otimes k})$ of the associated anchored planar algebra are then given by

$$\begin{aligned} \mathcal{P}[0] &= (\mathbf{1}, \underline{1}) + (\mathbf{3}, \underline{3}) + (\mathbf{3}, \underline{3}') + (\mathbf{5}, \underline{1}) \\ \mathcal{P}[1] &= (\mathbf{2}, \underline{1}) + (\mathbf{2}, \underline{3}) + (\mathbf{2}, \underline{3}') + (\mathbf{4}, \underline{1}) + (\mathbf{4}, \underline{3}) + (\mathbf{4}, \underline{3}') \\ \mathcal{P}[2] &= (\mathbf{1}, \underline{1}) + (\mathbf{1}, \underline{3}) + (\mathbf{1}, \underline{3}') + 2(\mathbf{3}, \underline{1}) + 2(\mathbf{3}, \underline{3}) + 2(\mathbf{3}, \underline{3}') + (\mathbf{5}, \underline{1}) + (\mathbf{5}, \underline{3}) + (\mathbf{5}, \underline{3}') \\ \mathcal{P}[k] &= 3\mathcal{P}[k-2] \text{ for } k \geq 3. \end{aligned}$$

Example 3.15 (E_6). Let $\mathcal{C}$ be the TLJ A_{11} category, with simple objects $\mathbf{1}, \mathbf{2}, \dots, \mathbf{11}$, and let $\mathcal{M}$ be its E_6 module tensor category. We write



for the simple objects of $\mathcal{M}$. By [BEK01, Cor.4.9], $\mathcal{Z}(E_6) \cong \mathcal{C} \boxtimes \mathcal{D} = A_{11} \boxtimes \overline{A}_3$, where $\mathcal{D} = \overline{A}_3 = \langle 1, \sigma, \psi \rangle \subset \mathcal{M}$. Using the E_6 fusion rules $m \otimes y = \bigoplus_{z: \text{neighbour of } y \text{ in } \overline{A}_3} z$, one easily computes

$$\begin{aligned} m^{\otimes 2k} &= a_k 1 + b_k x + (a_k - 1)\psi \\ m^{\otimes 2k+1} &= a_{k+1} m + b_k \sigma + (a_{k+1} - 1)\psi m \end{aligned} \tag{14}$$

where a_k and b_k are given by the recursive formula $a_k = a_{k-1} + b_{k-1}$, $b_k = b_{k-1} + 2a_k - 1$, $a_0 = 1$, $b_0 = 0$. The functor $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$ is given by

$$\begin{aligned} \text{Tr}_{\mathcal{C}}(1) &= \mathbf{1} + \mathbf{7} & \text{Tr}_{\mathcal{C}}(m) &= \mathbf{2} + \mathbf{6} + \mathbf{8} & \text{Tr}_{\mathcal{C}}(x) &= \mathbf{3} + \mathbf{5} + \mathbf{7} + \mathbf{9} \\ \text{Tr}_{\mathcal{C}}(\psi m) &= \mathbf{4} + \mathbf{6} + \mathbf{10} & \text{Tr}_{\mathcal{C}}(\psi) &= \mathbf{5} + \mathbf{11} & \text{Tr}_{\mathcal{C}}(\sigma) &= \mathbf{4} + \mathbf{8} \end{aligned} \tag{15}$$

(see [HPT16, Sub-Ex. 3.13 & Rem. 3.14]). Combining (14) and (15), we obtain the following explicit formulas for the box objects of the associated anchored planar algebra:

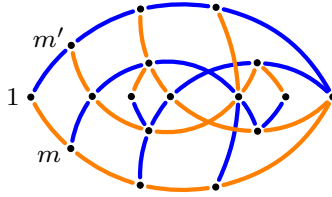
$$\begin{aligned} \mathcal{P}[2k] &= \text{Tr}_{\mathcal{C}}(a_k 1 + b_k x + (a_k - 1)\psi) \\ &= a_k(\mathbf{1} + \mathbf{7}) + b_k(\mathbf{3} + \mathbf{5} + \mathbf{7} + \mathbf{9}) + (a_k - 1)(\mathbf{5} + \mathbf{11}) \\ &= a_k \mathbf{1} + b_k \mathbf{3} + (a_k + b_k - 1)\mathbf{5} + (a_k + b_k)\mathbf{7} + b_k \mathbf{9} + (a_k - 1)\mathbf{11} \\ \mathcal{P}[2k+1] &= \text{Tr}_{\mathcal{C}}(a_{k+1} m + b_k \sigma + (a_{k+1} - 1)\psi m) \\ &= a_{k+1}(\mathbf{2} + \mathbf{6} + \mathbf{8}) + b_k(\mathbf{4} + \mathbf{8}) + (a_{k+1} - 1)(\mathbf{4} + \mathbf{6} + \mathbf{10}) \\ &= a_{k+1} \mathbf{2} + (a_{k+1} + b_k - 1)\mathbf{4} + (2a_{k+1} - 1)\mathbf{6} + (a_{k+1} + b_k)\mathbf{8} + (a_{k+1} - 1)\mathbf{10} \end{aligned}$$

Example 3.16 (E_6). Let us now view $\mathcal{M} = E_6$ as a module tensor category over $\mathcal{D} = \overline{A}_3 \subset \mathcal{Z}(\mathcal{M})$. The functor $\Phi_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{M}$ is fully faithful, so its adjoint $\text{Tr}_{\mathcal{D}} : \mathcal{M} \rightarrow \mathcal{D}$ is the projection functor onto the subcategory $\langle 1, \sigma, \psi \rangle \subset \mathcal{M}$. Using formula (14) for the powers of m , the box objects of the associated anchored planar algebra are easily computed:

$$\mathcal{P}[2k] = \text{Tr}_{\mathcal{D}}(m^{\otimes 2k}) = a_k 1 + (a_k - 1)\psi, \quad \mathcal{P}[2k+1] = \text{Tr}_{\mathcal{D}}(m^{\otimes 2k+1}) = b_k \sigma.$$

Example 3.17 ($D_{10}-E_7$). Let $\mathcal{C}$ be the TLJ A_{17} category, and let $A = \mathbf{1} + \mathbf{9} + \mathbf{17}$ be the algebra object whose category of modules is E_7 . Let $\mathcal{M}$ be the category of A - A -bimodules. This category is Morita equivalent to $\mathcal{C}$, with $\text{Mod}_{\mathcal{C}}(A)$ as the equivalence bimodule. Its center is thus given by $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{C}) = A_{17} \boxtimes \overline{A}_{17}$, where the second A_{17} is equipped with the opposite braiding [Müg03, Thm. 7.10].

By identifying A - A -bimodules with $A \otimes A^{op}$ -modules (braided tensor product), and computing $A \otimes A^{op} \in \mathcal{C}$ (this object has a unique algebra structure), one can show that $\mathcal{M}$ is equivalent to $D_{10} \oplus E_7$ as a $\mathcal{C}$ -module category (and also as a $\overline{\mathcal{C}}$ -module category). Finally, by playing around with quantum dimensions and using the existence of a dominant functor $A_{17} \boxtimes \overline{A}_{17} \rightarrow \mathcal{M}$, one obtains the following structure for $\mathcal{M}$:



This category has 17 simple objects, represented by the vertices of the above graph. It is generated by two symmetrically self-dual objects $m, m' \in \mathcal{M}$, and the operations of fusion with m and with m' are encoded by the orange and blue edges, respectively. This fusion category was studied in [Got10, §5.3.5], where it was realised as the category of K - K bi-unitary connections for K the graph E_7 .

The inclusion $\mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C} = \mathcal{Z}(\mathcal{M})$ of the first copy of $\mathcal{C}$ (the one corresponding to the blue edges) equips $\mathcal{M}$ with the structure of a $\mathcal{C}$ -module tensor category. Let $\mathcal{P}$ be the anchored planar algebra in $\mathcal{C}$ associated to the pair $(\mathcal{M}, m)$. Its odd box objects $\mathcal{P}[2k+1]$ are all zero, and its even box objects start as follows:

$$\mathcal{P}[0] = \mathbf{1} + \mathbf{17}$$

$$\mathcal{P}[2] = \mathbf{1} + \mathbf{9} + \mathbf{17}$$

$$\mathcal{P}[4] = 2 \cdot \mathbf{1} + \mathbf{5} + 3 \cdot \mathbf{9} + \mathbf{13} + 2 \cdot \mathbf{17}$$

$$\mathcal{P}[6] = 5 \cdot \mathbf{1} + 5 \cdot \mathbf{5} + \mathbf{7} + 9 \cdot \mathbf{9} + \mathbf{11} + 5 \cdot \mathbf{13} + 5 \cdot \mathbf{17}$$

$$\mathcal{P}[8] = 14 \cdot \mathbf{1} + \mathbf{3} + 20 \cdot \mathbf{5} + 7 \cdot \mathbf{7} + 29 \cdot \mathbf{9} + 7 \cdot \mathbf{11} + 20 \cdot \mathbf{13} + \mathbf{15} + 14 \cdot \mathbf{17}$$

$$\mathcal{P}[10] = 42 \cdot \mathbf{1} + 9 \cdot \mathbf{3} + 75 \cdot \mathbf{5} + 36 \cdot \mathbf{7} + 99 \cdot \mathbf{9} + 36 \cdot \mathbf{11} + 75 \cdot \mathbf{13} + 9 \cdot \mathbf{15} + 42 \cdot \mathbf{17}$$

The latter are computed by taking the various powers of m :

$$\begin{array}{ccc}
m^{\otimes 2} = 1 & m^{\otimes 4} = 2 & m^{\otimes 6} = 5 \\
m^{\otimes 8} = 14 & m^{\otimes 10} = 42 &
\end{array}$$

and applying the functor $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$.

4 Constructing anchored planar algebras

In this section we show that, in order to construct an anchored planar algebra, it is enough to provide the data associated to a certain small set of tangles (the ones which appear in Definition 2.10), and to verify a certain list of relations.

Theorem 4.1. *Let $\mathcal{C}$ be a braided pivotal category and let $\mathcal{P} = (\mathcal{P}[n])_{n \geq 0}$ be a sequence of objects of $\mathcal{C}$. Let*

$$\eta : 1 \rightarrow \mathcal{P}[0] \quad \alpha_i : \mathcal{P}[n+2] \rightarrow \mathcal{P}[n] \quad \alpha_i^\dagger : \mathcal{P}[n] \rightarrow \mathcal{P}[n+2] \quad \varpi_{i,j} : \mathcal{P}[n] \otimes \mathcal{P}[j] \rightarrow \mathcal{P}[n+j]$$

be morphisms (numbering and indices as in Definition 2.10) satisfying the following relations (the first seven are similar to (A1) – (A7); the last two are related to the two anchor dependence axioms, and are discussed in Remark 4.2 below):

$$(C1) \quad \varpi_{0,j} \circ (\eta \otimes \text{id}_j) = \text{id}_j, \quad \varpi_{i,0} \circ (\text{id}_n \otimes \eta) = \text{id}_n$$

$$(C2) \quad \alpha_i \circ \alpha_j = \alpha_{j-2} \circ \alpha_i \text{ for } i+1 < j$$

$$(C3) \quad \alpha_i^\dagger \circ \alpha_j^\dagger = \alpha_{j+2}^\dagger \circ \alpha_i^\dagger \text{ for } i \leq j$$

$$(C4) \quad \alpha_i \circ \alpha_j^\dagger = \begin{cases} \alpha_{j-2}^\dagger \circ \alpha_i & \text{for } i < j-1 \\ \text{id}_n & \text{for } i = j \pm 1 \\ \alpha_j^\dagger \circ \alpha_{i-2} & \text{for } i > j+1 \end{cases}$$

$$(C5) \quad \alpha_i \circ \varpi_{j,k} = \begin{cases} \varpi_{j-2,k} \circ (\alpha_i \otimes \text{id}_j) & \text{for } i+1 < j \\ \varpi_{j,k-2} \circ (\text{id}_n \otimes \alpha_{i-j}) & \text{for } j < i+1 < j+k \\ \varpi_{j,k} \circ (\alpha_{i-k} \otimes \text{id}_j) & \text{for } i+1 > j+k \end{cases}$$

$$(C6) \quad \alpha_i^\dagger \circ \varpi_{j,k} = \begin{cases} \varpi_{j+2,k} \circ (\alpha_i^\dagger \otimes \text{id}_j) & \text{for } i \leq j \\ \varpi_{j,k+2} \circ (\text{id}_n \otimes \alpha_{i-j}^\dagger) & \text{for } j \leq i \leq j+k \\ \varpi_{j,k} \circ (\alpha_{i-k}^\dagger \otimes \text{id}_j) & \text{for } i \geq j+k \end{cases}$$

$$(C9) \quad \alpha_n \circ \cdots \circ \alpha_{2n-1} \circ \varpi_{n,n} \circ [(\alpha_{n-1}^\dagger \circ \cdots \circ \alpha_0^\dagger \circ \eta) \otimes \text{id}_n] = \theta_{\mathcal{P}[n]},$$

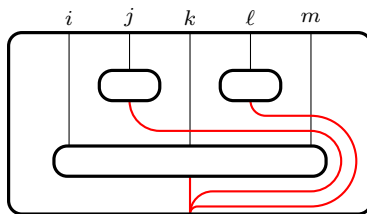
where the twist θ is as in (4), id_n stands for $\text{id}_{\mathcal{P}[n]}$, and $\beta_{j,\ell}$ stands for $\beta_{\mathcal{P}[j], \mathcal{P}[\ell]}$.

Then the assignment $T \mapsto Z(T)$ given in Algorithm 4.9 below endows $\mathcal{P}$ with the structure of an anchored planar algebra in $\mathcal{C}$.

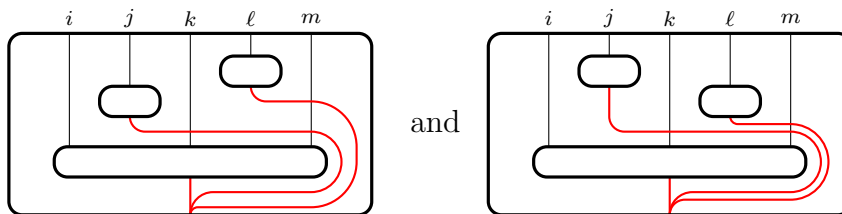
The proof of Theorem 4.1 will be presented at the end of Section 4.4.

Remarks 4.2.

- (1) The relation (C8) above is a variant of the first anchor dependence relation in Definition 2.7. Consider the following non-generic tangle:



which we can make generic in the following two ways:



The left and right tangles corresponds to the left and right hand sides of (C8).

- (2) The relation (C9) above is a variant of the twist anchor dependence axiom:

$$Z \left(\text{Diagram with nested circles and a red line} \right) = \theta_{\mathcal{P}[n]}$$

4.1 Generic tangles and tangles in standard form

The moduli space of anchored planar tangles is an infinite dimensional manifold $\mathbb{M}$ modelled on a Fréchet space. In this section, we define a stratification of $\mathbb{M}$ and study the associated notion of *generic* anchored planar tangle. We then prove that any two generic anchored planar tangles can be connected by a finite sequence of certain moves. Later on, we define

a notion of anchored planar tangle with *underlying tangle in standard form*, and describe a procedure which takes a generic anchored planar tangle and brings it into standard form.

Recall that, before stating Definitions 2.1 and 2.5 (the definitions of planar tangles and of anchored planar tangles), we had assumed the data of a point $q \in \partial\mathbb{D}$, and of certain collections of germs of arcs $\mu_n = (\mu_n^1, \dots, \mu_n^n)$. It will be convenient to make a number of technical restrictions to the above data: we assume that q is at the bottom of $\mathbb{D}$ (angle $-\pi/2$), that the arcs μ_n^i cross the boundary of $\mathbb{D}$ orthogonally, that they have everywhere non-zero curvature, and that they intersect $\partial\mathbb{D}$ only in the top hemisphere (angles strictly between 0 and π).

Definition 4.3. An anchored planar tangle is called *generic* if it satisfies:

- (G1) the y -coordinates of the local maxima/minima of strings as well as those of the centers of the input disks are all distinct.
- (G2) if a string has a horizontal tangent at some point, then the second derivative at that point does not vanish.
- (G3) no boundary point $p \in \partial X$ is on the equator of a circle (angle 0 or π).
- (G4) no anchor point q_i is at the top of a circle (angle $\pi/2$).

We let $\mathbb{M}_0 \subset \mathbb{M}$ denote the open dense subset consisting of anchored planar tangles that satisfy (G1) – (G4). The simplest way in which a tangle can fail to be generic is if:

- (S1) the y -coordinates of two centers of circles and/or critical points of strands coincide; the tangle otherwise satisfies (G2), (G3), (G4).
- (S2) a strand has a horizontal inflection point (and the third derivative at that point does not vanish); the tangle otherwise satisfies (G1), (G3), (G4).
- (S3) a strand is attached on the equator of a circle; the tangle otherwise satisfies (G1), (G2), (G4).
- (S4) an anchor point is on the top of its circle; the tangle otherwise satisfies (G1), (G2), (G3).

Each of the above conditions (S1) – (S4) determines a submanifold of $\mathbb{M}$ of codimension one (it is locally determined by one equation). Let $\mathbb{M}_1$ denote their union. Its boundary $\mathbb{M}_{\geq 2} := \partial\mathbb{M}_1 \subset \mathbb{M}$ has codimension ≥ 2 , leading to a stratification

$$\mathbb{M} = \mathbb{M}_0 \cup \mathbb{M}_1 \cup \mathbb{M}_{\geq 2}$$

of the moduli space of anchored planar tangles.

Let $T_1, T_2 \in \mathbb{M}_0$ be two generic tangles which are isotopic. An isotopy between T_1 and T_2 is a path $[0, 1] \rightarrow \mathbb{M}$. By transversality, such a path can be isotoped to one which avoids $\mathbb{M}_{\geq 2}$ and which intersects $\mathbb{M}_1$ transversely. Each transversal intersection correspond to one of the following moves:

- (M1) exchange the relative order of the y -coordinates of the centers of two input discs.
- (M2) exchange the order of the y -coordinates of the center of a disc and of a critical point of a string (two cases, depending on whether the critical point is a cup or a cap).
- (M3) exchange the relative order of the heights of two critical points of strings (four sub-cases: cup/cup, cup/cap, cap/cup, cap/cap).
- (M4) replace a cup followed by a cap by a string that goes straight down : $\nearrow \rightarrow \mid$ (this move comes in two flavors which are each other's mirror images).
- (M5) move a strand's boundary point past the equator of an input circle (two variants, depending on whether the boundary point is on the left or on the right of the circle).
- (M6) swing an anchor point from slightly to the right to slightly to the left of the north pole.

The moves (M1) – (M3) correspond to crossing a component of the stratum (S1), and the moves (M4) – (M6) correspond to crossing the strata (S2) – (S4), respectively.

The condition about the arcs μ_n having non-zero curvature is needed in the following technical lemma:

Lemma 4.4. *Let $\gamma : [0, 1] \rightarrow \mathbb{M}$ be a smooth path which avoids $\mathbb{M}_{\geq 2}$ and intersects $\mathbb{M}_1$ transversely. Then γ intersects $\mathbb{M}_1$ in finitely many points.*

Proof. First of all, there can be at most finitely many (M1) events, as any accumulation point of such events would be a point where γ fails to be transverse to the submanifold $\mathbb{M}_1$. Similarly, there can be at most finitely many occurrences of the moves (M5) and (M6).

The situation for the moves (M2) – (M4) is different, and the lemma would be false in the absence of the curvature condition on the arcs μ_n .

Write (T_t, X_t, Q_t, A_t) for the tangle $\gamma(t)$. Let $\underline{X} \subset [0, 1] \times \mathbb{D}$ be the 2-manifold which is the union of all the $\{t\} \times X_t$, and let

$$\partial \underline{X} := \bigcup_{t \in [0, 1]} \{t\} \times \partial X_t \quad \quad \dot{\underline{X}} := \underline{X} \setminus \partial \underline{X} = \bigcup_{t \in [0, 1]} \{t\} \times \dot{X}_t.$$

By Morse theory, the cups and caps form a 1-dimensional submanifold $\dot{C} \subset \dot{\underline{X}}$. The projection map $\dot{C} \rightarrow [0, 1]$ is then itself a Morse function, whose critical points are the (M4) events. Let C be the closure of $\dot{C}$ in $\underline{X}$. By our condition that the μ_n cross $\partial \mathbb{D}$ orthogonally, one can identify the points of $C \cap \partial \underline{X}$ with the (M5) events. Moreover, because of our condition on the curvature of the μ_n , C is a submanifold of $\underline{X}$ (in the sense of manifolds with boundary), and $C \cap \partial \underline{X}$ consists of regular points for the projection to $[0, 1]$. The projection $C \rightarrow [0, 1]$ is therefore Morse on all of C .

Since C is a manifold and since it is compact, the projection map $C \rightarrow [0, 1]$ has at most finitely critical points. Our path γ therefore sees at most finitely many (M4) events.

Finally, there can be at most finitely many (M2) – (M3) events. An accumulation point of such events would either be a point of $\dot{C}$ where the transversality condition with $\mathbb{M}_1$ fails, or a point of ∂C where the condition (G1) fails in the definition of (S3) $\subset \mathbb{M}_1$. $\square$

We have proved:

Proposition 4.5. *Any two generic anchored planar tangles can be connected by a finite sequence of the moves (M1) – (M6).* $\square$

Generic tangles can be brought to a certain standard form by an algorithm that we now describe.

Definition 4.6. A generic anchored planar tangle is said to have *underlying tangle in standard form* if:

- all the anchor points are on the bottom of the disks (by our conditions on the μ_n , this implies that all the strings are attached to the top halves of the input disks).
- the projections onto the y -axis of all the input disks, and of all the local maxima/minima of the strings are disjoint.

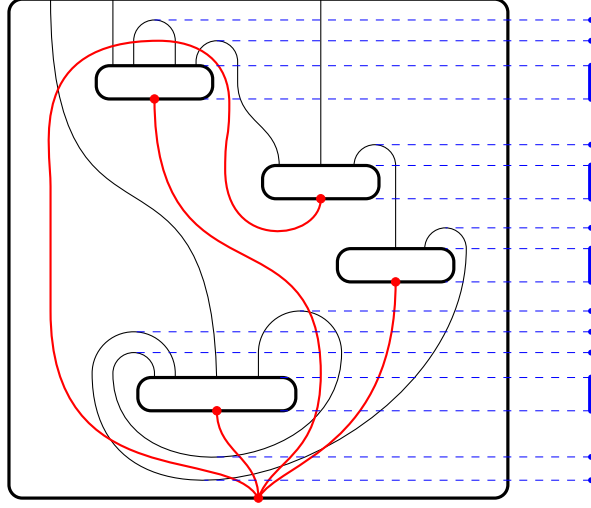
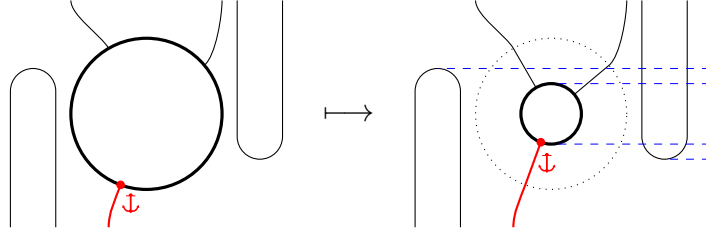


Figure 2: An example of an anchored planar tangle with underlying tangle in standard form (except that the circles are drawn more like rectangles). The y -projections of the input circles and of the maxima/minima of strings are indicated in blue. They are all disjoint, as required by the definition.

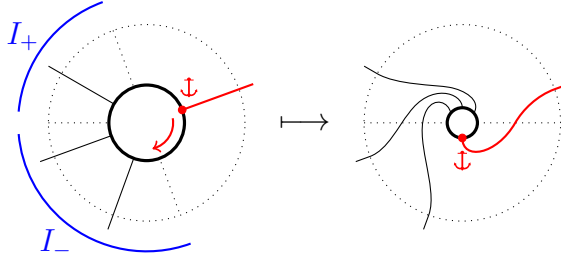
The following algorithm takes as input a generic anchored planar tangle (Definition 4.3) and produces as output an anchored planar tangle whose underlying tangle is in standard form (Definition 4.6). The output of the algorithm is well defined up to isotopy within the space of anchored planar tangles whose underlying tangle is in standard form.

Algorithm 4.7 (Standard form). Let T be a generic anchored planar tangle.

- (1) First, we shrink all input circles so that their y -projections do not overlap with those of the other input disks, nor with those of the local maxima/minima of the strings. The strings attached to these circles are extended by straight lines:



- (2) Next, we rotate the input circles so as to let the anchor points sink to the bottom. The direction of rotation depends on whether the anchor point is in the left half of right half of the input circle. The boundary points rotate by the same amount (by our choice of μ_n , this implies that they end up above the equator). At the same time, we further shrink the input circles, so as to make sure that the new critical y -values satisfy the conditions of a tangle in standard form (Definition 4.6), and that no unnecessary new maxima/minima are introduced.



No new local maxima/minima are created on the strings attached above the equator (attached in the blue interval I_+). The strings attached below the equator (in the blue interval I_-) acquire exactly one new local maximum.

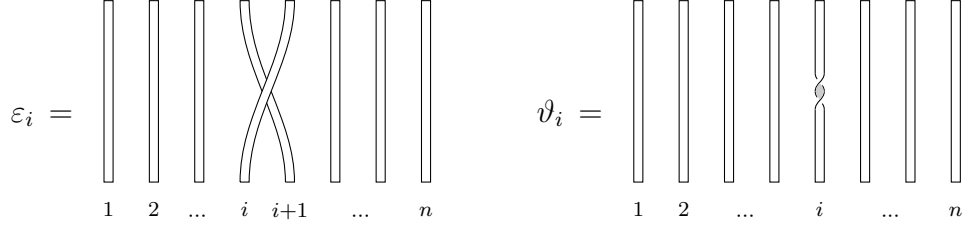
4.2 The ribbon braid group

The ribbon braid group $\mathcal{RB}_n = \mathcal{B}_n \ltimes \mathbb{Z}^n$ is the semi-direct product of the braid group $\mathcal{B}_n$ with the free abelian group of rank n with respect to the action that permutes the coordinates.

Definition 4.8. The ribbon braid group is the group $\mathcal{RB}_n$ with generators $\varepsilon_1, \dots, \varepsilon_{n-1}$, $\vartheta_1, \dots, \vartheta_n$ and relations:

- $\varepsilon_i \varepsilon_{i+1} \varepsilon_i = \varepsilon_{i+1} \varepsilon_i \varepsilon_{i+1}$ for all $i = 1, \dots, n-1$,
- $\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i$ if $|i - j| > 1$,
- $\varepsilon_i \vartheta_i = \vartheta_{i+1} \varepsilon_i$, $\varepsilon_i \vartheta_{i+1} = \vartheta_i \varepsilon_i$,
- $\varepsilon_i \vartheta_j = \vartheta_j \varepsilon_i$ for $j \notin \{i, i+1\}$.

The elements of $\mathcal{RB}_n$ are best visualized as ribbons that braid among each other, and twirl on themselves:



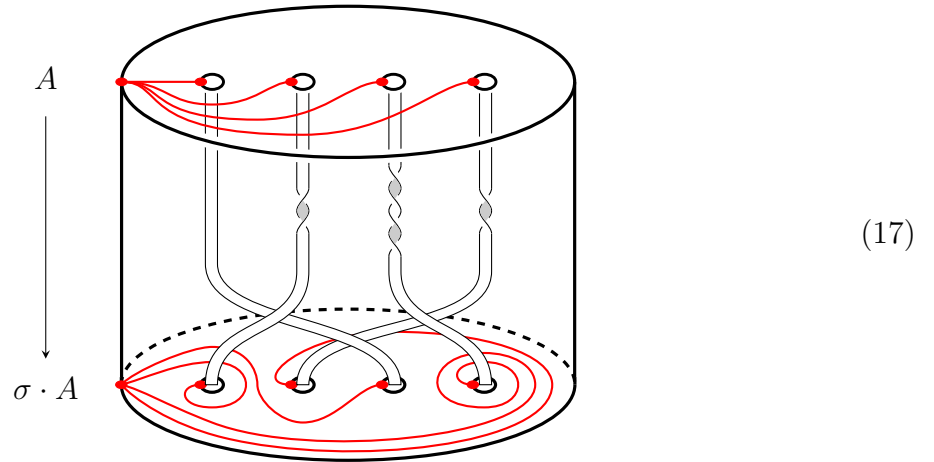
Reading left to right, we compose from bottom to top so that, for example, $\varepsilon_1 \vartheta_1 =$ .

Let $\mathfrak{S}_n$ be the symmetric group, and let $\sigma \mapsto \pi_\sigma : \mathcal{RB}_n \rightarrow \mathfrak{S}_n$ be the obvious homomorphism. Given objects $a_1, \dots, a_n$ in a braided pivotal category, an element $\sigma \in \mathcal{RB}_n$ induces a map

$$P(\sigma) : a_1 \otimes \dots \otimes a_n \rightarrow a_{\pi_\sigma^{-1}(1)} \otimes \dots \otimes a_{\pi_\sigma^{-1}(n)} \quad (16)$$

by the rules $P(\varepsilon_i) := \text{id}_{a_1 \otimes \dots \otimes a_{i-1}} \otimes \beta_{a_i, a_{i+1}} \otimes \text{id}_{a_{i+2} \otimes \dots \otimes a_n}$, $P(\vartheta_i) := \text{id}_{a_1 \otimes \dots \otimes a_{i-1}} \otimes \theta_{a_i} \otimes \text{id}_{a_{i+1} \otimes \dots \otimes a_n}$, and $P(\sigma\tau) = P(\sigma) \circ P(\tau)$, where β is the braiding in $\mathcal{C}$ and the twist θ is defined in (4).

Let T be a planar tangle whose input circles have disjoint y -projections. Then there is an action of $\mathcal{RB}_n$ on the set of systems of anchor lines on T . We assume without loss of generality that T has no strands (as the anchor lines don't interact with the strands). Let $\sigma \in \mathcal{RB}_n$ be a ribbon braid, and let A be a systems of anchor lines. To compute $\sigma \cdot A$, we place the ribbon braid vertically under the tangle and connect the i th ribbon to the i th input circle in order of increasing y -projection. Then we let the input circles flow down following the ribbons. The anchor lines follow by continuity while the circles braid and twist around each other. Here is an example of this process



with $\sigma = \varepsilon_2^{-1} \varepsilon_3^{-1} \varepsilon_1 \vartheta_2 \vartheta_3^{-2} \vartheta_4^{-1}$.

Let $A_{\text{st}} :=$  denote the 'standard' system of anchor lines. We may identify the elements of $\mathcal{RB}_n$ with systems of anchor lines via the map $\sigma \mapsto \sigma \cdot A_{\text{st}}$. Using that cor-

respondence, we can then transport the operad structure from anchored tangles to ribbon braids. We illustrate the resulting operad structure on $(\mathcal{RB}_n)_{n \geq 0}$ with an example:

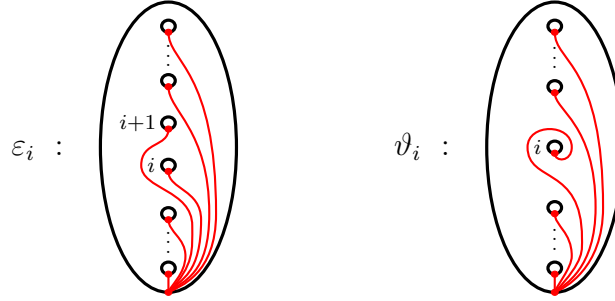
$$\begin{array}{c} | \quad \text{X} \end{array} \circ_3 \begin{array}{c} \text{U} \end{array} = \begin{array}{c} | \quad \text{U} \end{array} \quad (18)$$

The operadic composition of ribbon braids is compatible with their action (16) on tensor products of objects of $\mathcal{C}$ in the sense that

$$P(\sigma \circ_i \tau) = P(\sigma) \circ_i P(\tau). \quad (19)$$

Here, the $\circ_i$ in the left hand side is as in (18), and the $\circ_i$ in the right hand side is the one mentioned in Remark 2.9.

For the reader's convenience, we finish this section by including the pictures of the systems of anchor lines which correspond to the generators ε_i and ϑ_i of the ribbon braid group:



4.3 Assigning maps to generic tangles

Given objects $\mathcal{P}[n] \in \mathcal{C}$ and morphisms $\eta, \alpha_i, \alpha_i^\dagger, \varpi_{i,j}$ as in Theorem 4.1, we now present an algorithm which assigns to a generic anchored planar tangle T of type $(k_1, \dots, k_r; k_0)$ a morphism $Z(T) : \mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_r] \rightarrow \mathcal{P}[k_0]$ in $\mathcal{C}$.

Algorithm 4.9. Let $T = (T, X, Q, A)$ be a generic anchored planar tangle.

- (1) First, apply Algorithm 4.7 so as to bring the underlying planar tangle of T into standard form. The resulting tangle is determined canonically up to isotopy within the space of anchored planar tangles whose underlying tangle is in standard form.
- (2) At this point, it becomes technically convenient to replace all circles by rectangles. We do this while taking care not to change the minimal and maximal y -coordinates of the input boxes, and not to change the y -coordinates of the local maxima/minima of the strings. We call the resulting rectangular anchored planar tangle $T' = (T', X', Q', A')$ (this is no longer an anchored planar tangle in the sense of Definition 2.5). It is well defined up to isotopy within the space of rectangular anchored planar tangles whose underlying rectangular tangle is in standard form (obvious analog of Definition 4.6).

- (3) Endow T' with a new system A_{st} of *standard anchor lines*, and call the result $T_{\text{st}} = (T', X', Q', A_{\text{st}})$. Here, standard anchor lines are anchor lines which travel from the anchor points on the input circles horizontally to the right towards the external boundary, then down along that boundary, and finally come together at the very bottom (see Figure 3 for an example of this procedure). Finally, let $\sigma_T \in \mathcal{RB}_r$ be the ribbon braid group element uniquely determined by the equation

$$\sigma_T \cdot A_{\text{st}} = A'.$$

Here, the dot denotes the action of ribbon braids on anchor lines described in Section 4.2.

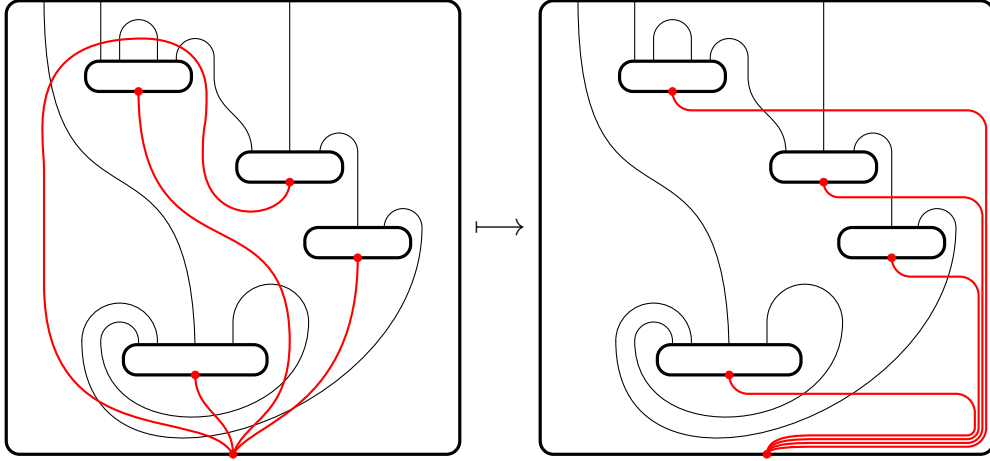
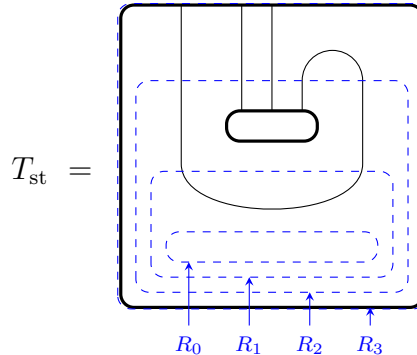


Figure 3: Endowing a tangle in standard form with standard anchor lines

- (4) We now write T_{st} as a composite of generating tangles, in a way similar to the algorithm described in [Jon99, JP11]. Let C be the union of all the input rectangles and all the critical points of strings. Consider a nested sequence of closed rectangles

$$R_0 \subset R_1 \subset \dots \subset R_N$$

with the following properties: R_N is the outer rectangle of T_{st} ; the y -projection of R_0 lies below that of C ; the x -projection of R_0 contains that of C ; the strands don't intersect the vertical sides of the rectangles; R_i contains R_{i-1} in its interior; finally and most importantly, the y -projection of R_i minus that of R_{i-1} contains exactly one component of the y -projection of C . We illustrate this with an example:



Let $T_0 := R_0$, and let $T_i := R_i \setminus \mathring{R}_{i-1}$ for $i \geq 1$. By construction, each T_i is isotopic to a generating tangle (Definition 2.10), and we have $T_{\text{st}} = T_N \circ_1 \dots \circ_1 T_2 \circ_1 T_1 \circ_1 T_0$ (see Figure 4 for an example). Letting $\mathcal{T}_0 = \eta$, and letting $\mathcal{T}_i = \alpha_j$ if $T_i = a_j$, $\mathcal{T}_i = \alpha_j^\dagger$ if $T_i = a_j^\dagger$, and $\mathcal{T}_i = \varpi_{j,k}$ if $T_i = p_{j,k}$ (η , α_j , $\alpha_j^\dagger$, $\varpi_{j,k}$ as in Theorem 4.1), we set

$$Z(T_{\text{st}}) := \mathcal{T}_N \circ_1 \dots \circ_1 \mathcal{T}_2 \circ_1 \mathcal{T}_1 \circ_1 \mathcal{T}_0. \quad (20)$$

Note that the operation $\circ_1$ is associative, and that the prescription (20) is therefore well defined.

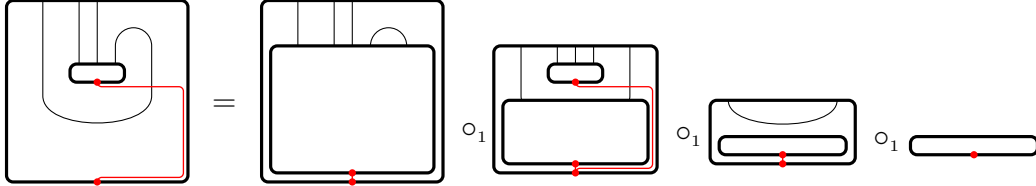


Figure 4: Decomposing the tangle T_{st} into generating tangles

(5) Finally, we let

$$Z(T) := Z(T_{\text{st}}) \circ P(\sigma_T), \quad (21)$$

where $P(\sigma_T) : \mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_r] \rightarrow \mathcal{P}[k_{\pi_T^{-1}(1)}] \otimes \dots \otimes \mathcal{P}[k_{\pi_T^{-1}(r)}]$ is the action (described in Section 4.2) of the ribbon braid σ_T on a tensor product of objects of $\mathcal{C}$, and π_T is the image of σ_T in the symmetric group.

We illustrate the above algorithm by computing the value of certain simple tangles:

Example 4.10. $Z(\bigcirc) = \eta$.

Example 4.11. We have $Z\left(\bigcirc \begin{smallmatrix} n \\ \circ \end{smallmatrix}\right) = \text{id}_{\mathcal{P}[n]}$.

Proof. Following Algorithm 4.9, $Z(\text{id}_n) := \varpi_{0,n} \circ_1 \eta$, which equals $\text{id}_{\mathcal{P}[n]}$ by (C1). $\square$

Example 4.12. We have $Z\left(\begin{smallmatrix} i & j \\ \text{loop} \end{smallmatrix}\right) = \alpha_i$, and $Z\left(\begin{smallmatrix} i & j \\ \text{loop}^\dagger \end{smallmatrix}\right) = \alpha_i^\dagger$.

Proof. The first tangle evaluates to $\alpha_i \circ_1 \varpi_{0,i+j+2} \circ_1 \eta$, which simplifies to $\alpha_i \circ_1 \text{id}_{\mathcal{P}[i+j+2]} = \alpha_i$ by (C1). By the same argument, the second tangle evaluates to $\alpha_i^\dagger \circ_1 \varpi_{0,i+j} \circ_1 \eta = \alpha_i^\dagger$. $\square$

Example 4.13. We have $Z\left(\begin{smallmatrix} i & j & k \\ \text{loop} \end{smallmatrix}\right) = \varpi_{i,j}$, and $Z\left(\begin{smallmatrix} i & j & k \\ \text{loop}^\dagger \end{smallmatrix}\right) = \varpi_{i,j} \circ \beta_{\mathcal{P}[j], \mathcal{P}[i+k]}.$

Proof. $Z(p_{i,j}) := \varpi_{i,j} \circ_1 \varpi_{0,i+k} \circ_1 \eta$ simplifies to $\varpi_{i,j} \circ_1 \text{id}_{\mathcal{P}[i+k]} = \varpi_{i,j}$. The second tangle differs from the first one by its anchor lines. By (21), it evaluates to $Z(p_{i,j}) \circ \beta_{\mathcal{P}[j], \mathcal{P}[i+k]}$. $\square$

Example 4.14. We have $Z \left(\text{tangle diagram} \right) = \theta_{\mathcal{P}[n]}$ and $Z \left(\text{tangle diagram} \right) = \theta_{\mathcal{P}[n]}^{-1}$.

Proof. These tangles are identical to the one in Example 4.11, except for the position of their anchor lines. By (21), they evaluate to $Z(\text{id}_n) \circ \theta_{\mathcal{P}[n]}$ and $Z(\text{id}_n) \circ \theta_{\mathcal{P}[n]}^{-1}$, respectively. $\square$

Example 4.15. We have $Z \left(\text{tangle diagram} \right) = \theta_{\mathcal{P}[n]}$ and $Z \left(\text{tangle diagram} \right) = \theta_{\mathcal{P}[n]}^{-1}$.

Proof. Following Algorithm 4.9, the first tangle evaluates to $\alpha_n \circ_1 \cdots \circ_1 \alpha_{2n-1} \circ_1 \varpi_{n,n} \circ_1 \alpha_{n-1}^\dagger \circ_1 \cdots \circ_1 \alpha_0^\dagger \circ_1 \eta$, which is equal to $\theta_{\mathcal{P}[n]}$ by (C9). The second tangle is the inverse up to isotopy of the first one. By Propositions 4.17 and 4.18 below, it evaluates to the inverse morphism $\theta_{\mathcal{P}[n]}^{-1}$. $\square$

Example 4.16. We have $Z \left(\text{tangle diagram} \right) = \text{id}_{\mathcal{P}[n]}$.

Proof. This follows from the prescription (21), and from the first part of Example 4.15 (the part which does not rely on Propositions 4.17 and 4.18). $\square$

4.4 Proof of Theorem 4.1

As a first step towards Theorem 4.1, we verify that the assignment $T \mapsto Z(T)$ is compatible with composition of tangles:

Proposition 4.17. *Let T and S be generic anchored planar tangles with underlying planar tangles in standard form. Assume that the composition $T \circ_a S$ is allowed, and let $Z(T)$, $Z(S)$, $Z(T \circ_a S)$ be as provided by Algorithm 4.9. Then we have $Z(T \circ_a S) = Z(T) \circ_a Z(S)$.*

Proof. We first check the special case when S is a generating tangle (Definition 2.10), $T = p_{i,j}$, and $a = 2$. If $S = u$, we check using (C1):

$$Z(T \circ_2 S) = Z(p_{i,0} \circ_2 u) = Z(\text{id}_n) = \text{id}_{\mathcal{P}[n]} = \varpi_{i,0} \circ_2 \eta = Z(T) \circ_2 Z(S)$$

If $S = a_k$, we check using (C5):

$$Z(T \circ_2 S) = Z(p_{i,j} \circ_2 a_k) = Z(a_{i+k} \circ_1 p_{i,j+2}) = \alpha_{i+k} \circ_1 \varpi_{i,j+2} = \varpi_{i,j} \circ_2 \alpha_k = Z(T) \circ_2 Z(S).$$

If $S = a_k^\dagger$, we check using (C6):

$$Z(T \circ_2 S) = Z(p_{i,j} \circ_2 a_k^\dagger) = Z(a_{i+k}^\dagger \circ_1 p_{i,j-2}) = \alpha_{i+k}^\dagger \circ_1 \varpi_{i,j-2} = \varpi_{i,j} \circ_2 \alpha_k^\dagger = Z(T) \circ_2 Z(S).$$

Finally, if $S = p_{k,l}$, we check using (C7):

$$Z(T \circ_2 S) = Z(p_{i,j} \circ_2 p_{k,l}) = Z(p_{i+k,l} \circ_1 p_{i,j-l}) = \varpi_{i+k,l} \circ_1 \varpi_{i,j-l} = \varpi_{i,j} \circ_2 \varpi_{k,l} = Z(T) \circ_2 Z(S).$$

Let us now assume that S is a generating tangle (Definition 2.10) and that T has standard anchor lines. Decompose T into generating tangles $T = T_N \circ_1 \dots \circ_1 T_0$ as in Figure 4. By (20) and by the computations performed in Examples 4.10–4.13, we get that $Z(T) = Z(T_N) \circ_1 \dots \circ_1 Z(T_0)$. We then have

$$T \circ_a S = [[T_N \circ_1 \dots \circ_1 T_k] \circ_1 [T_{k-1} \circ_1 \dots \circ_1 T_0]] \circ_a S = [[T_N \circ_1 \dots \circ_1 T_k] \circ_2 S] \circ_1 [T_{k-1} \circ_1 \dots \circ_1 T_0]$$

for some index k with $T_k = p_{i,j}$. It follows that

$$\begin{aligned} Z(T \circ_a S) &= Z([[T_N \circ_1 \dots \circ_1 T_k] \circ_2 S] \circ_1 [T_{k-1} \circ_1 \dots \circ_1 T_0]) \\ &= Z(T_N \circ_1 \dots \circ_1 T_{k+1} \circ_1 [T_k \circ_2 S] \circ_1 T_{k-1} \circ_1 \dots \circ_1 T_0) \\ &= Z(T_N \circ_1 \dots \circ_1 T_{k+1} \circ_1 T'_k \circ_1 T''_k \circ_1 T_{k-1} \circ_1 \dots \circ_1 T_0) \\ &= Z(T_N) \circ_1 \dots \circ_1 Z(T_{k+1}) \circ_1 Z(T'_k) \circ_1 Z(T''_k) \circ_1 Z(T_{k-1}) \circ_1 \dots \circ_1 Z(T_0) \\ &= Z(T_N) \circ_1 \dots \circ_1 Z(T_{k+1}) \circ_1 Z(T'_k \circ_1 T''_k) \circ_1 Z(T_{k-1}) \circ_1 \dots \circ_1 Z(T_0) \\ &= Z(T_N) \circ_1 \dots \circ_1 Z(T_{k+1}) \circ_1 Z(T_k \circ_2 S) \circ_1 Z(T_{k-1}) \circ_1 \dots \circ_1 Z(T_0) \\ &= Z(T_N) \circ_1 \dots \circ_1 Z(T_{k+1}) \circ_1 [Z(T_k) \circ_2 Z(S)] \circ_1 Z(T_{k-1}) \circ_1 \dots \circ_1 Z(T_0) \\ &= [[Z(T_N) \circ_1 \dots \circ_1 Z(T_k)] \circ_2 Z(S)] \circ_1 [Z(T_{k-1}) \circ_1 \dots \circ_1 Z(T_0)] \\ &= [[Z(T_N) \circ_1 \dots \circ_1 Z(T_k)] \circ_1 [Z(T_{k-1}) \circ_1 \dots \circ_1 Z(T_0)]] \circ_a Z(S) \\ &= Z(T) \circ_a Z(S), \end{aligned}$$

where $T'_k \circ_1 T''_k$ is the standard way (as in Figure 4) of decomposing $T_k \circ_2 S$ into generating tangles (except when $S = u$, in which case the terms involving T'_k and T''_k should be omitted), and we have used the previous computations to pass from the 6th to the 7th line.

We now take T and S to be arbitrary planar tangles (in standard form), equipped with standard anchor lines. Write $S = S_N \circ_1 \dots \circ_1 S_0$ as products of generating tangles, so that $Z(S) = Z(S_N) \circ_1 \dots \circ_1 Z(S_0)$. By repeated use of the previous step, we get

$$\begin{aligned} Z(T \circ_a S) &= Z(T \circ_a [S_N \circ_1 \dots \circ_1 S_2 \circ_1 S_1 \circ_1 S_0]) \\ &= Z([[\dots [T \circ_a S_N] \circ_a \dots] \circ_a S_2] \circ_a S_1] \circ_a S_0) \\ &= Z([[\dots [T \circ_a S_N] \circ_a \dots] \circ_a S_2] \circ_a S_1) \circ_a Z(S_0) \\ &= [Z([\dots [T \circ_a S_N] \circ_a \dots] \circ_a S_2) \circ_a Z(S_1)] \circ_a Z(S_0) \\ &= \dots \\ &= [[[\dots [Z(T) \circ_a Z(S_N)] \circ_a \dots] \circ_a Z(S_2)] \circ_a Z(S_1)] \circ_a Z(S_0) \\ &= Z(T) \circ_a [Z(S_N) \circ_1 \dots \circ_1 Z(S_2) \circ_1 Z(S_1) \circ_1 Z(S_0)] \\ &= Z(T) \circ_a Z(S) \end{aligned}$$

as desired.

Finally, we treat the general case. Let T and S to be anchored planar tangles with underlying planar tangles in standard form, and let $\sigma_S, \sigma_T, \sigma_{T \circ_a S}$ be the ribbon braid elements which enter in the definition (21) of Z , so that $Z(T) = Z(T_{\text{st}}) \circ P(\sigma_T)$, $Z(S) = Z(S_{\text{st}}) \circ P(\sigma_S)$, and $Z(T \circ_a S) = Z((T \circ_a S)_{\text{st}}) \circ P(\sigma_{T \circ_a S})$. Letting π_T be the image of σ_T in the symmetric group, we have $(T \circ_a S)_{\text{st}} = T_{\text{st}} \circ_{\pi_T(a)} S_{\text{st}}$. It follows from our previous computations that

$$Z((T \circ_a S)_{\text{st}}) = Z(T_{\text{st}} \circ_{\pi_T(a)} S_{\text{st}}) = Z(T_{\text{st}}) \circ_{\pi_T(a)} Z(S_{\text{st}}).$$

We also have

$$P(\sigma_{T \circ_a S}) = P(\sigma_T \circ_a \sigma_S) = P(\sigma_T) \circ_a P(\sigma_S),$$

where the first equality holds by the definition of the operadic composition of ribbon braids (Section 4.2), and the second equality is equation (19). Putting all this together, we get

$$\begin{aligned} Z(T \circ_a S) &= Z((T \circ_a S)_{\text{st}}) \circ P(\sigma_{T \circ_a S}) \\ &= [Z(T_{\text{st}}) \circ_{\pi_T(a)} Z(S_{\text{st}})] \circ [P(\sigma_T) \circ_a P(\sigma_S)] \\ &= [Z(T_{\text{st}}) \circ P(\sigma_T)] \circ_a [Z(S_{\text{st}}) \circ P(\sigma_S)] \\ &= Z(T) \circ_a Z(S). \end{aligned} \quad \square$$

We now prove that the assignment $T \mapsto Z(T)$ is invariant under all isotopies of anchored planar tangles:

Proposition 4.18. *Suppose T and S are generic anchored planar tangles which are isotopic. Then $Z(T) = Z(S)$.*

Proof. In Section 4.1, we saw that any two generic anchored planar tangles which are isotopic can be connected by a finite sequence of the moves (M1) – (M6). It is therefore enough to show that $Z(T) = Z(S)$ whenever T and S are related by one of these moves. We proceed case by case:

(M1) T is obtained from S by exchanging the relative order of the y -coordinates of the centers of two input circles. Then, up to permuting T and S , the decompositions of T_{st} and S_{st} into generating tangles (as in Figure 4) look like

$$\begin{aligned} T_{\text{st}} &= T_N \circ_1 \dots \circ_1 T_{s+2} \circ_1 p_{i+j+k,\ell} \circ_1 p_{i,j} \circ_1 T_{s-1} \circ_1 \dots \circ_1 T_0 \\ S_{\text{st}} &= T_N \circ_1 \dots \circ_1 T_{s+2} \circ_1 p_{i,j} \circ_1 p_{i+k,\ell} \circ_1 T_{s-1} \circ_1 \dots \circ_1 T_0 \end{aligned}$$

(see part (1) of Remark 4.2). We also have $\sigma_S = \varepsilon_a \sigma_T$ for some appropriate index a . It follows that

$$\begin{aligned} Z(T) &= Z(T_{\text{st}}) \circ P(\sigma_T) \\ &= [Z(T_N) \circ_1 \dots \circ_1 Z(p_{i+j+k,\ell}) \circ_1 Z(p_{i,j}) \circ_1 \dots \circ_1 Z(T_0)] \circ P(\sigma_T) \\ &= [Z(T_N) \circ_1 \dots \circ_1 \varpi_{i+j+k,\ell} \circ_1 \varpi_{i,j} \circ_1 \dots \circ_1 Z(T_0)] \circ P(\sigma_T) \\ &= [Z(T_N) \circ_1 \dots \circ_1 [[\varpi_{i,j} \circ_1 \varpi_{i+k,\ell}] \circ (\text{id} \otimes \beta_{\mathcal{P}[j], \mathcal{P}[\ell]})] \circ_1 \dots \circ_1 Z(T_0)] \circ P(\sigma_T) \\ &= [Z(T_N) \circ_1 \dots \circ_1 \varpi_{i,j} \circ_1 \varpi_{i+k,\ell} \circ_1 \dots \circ_1 Z(T_0)] \circ P(\varepsilon_a) \circ P(\sigma_T) \\ &= [Z(T_N) \circ_1 \dots \circ_1 Z(p_{i,j}) \circ_1 Z(p_{i+k,\ell}) \circ_1 \dots \circ_1 Z(T_0)] \circ P(\varepsilon_a \sigma_T) \\ &= Z(S_{\text{st}}) \circ P(\sigma_S) = Z(S), \end{aligned}$$

where we have used (C8) in the fourth equality.

(M2) T is obtained from S by exchanging the relative order of the y -coordinates of the center of and input circle and of a critical point of a string. Then, up to permuting T and S , the decompositions of T_{st} and S_{st} into generating tangles are of the form

$$\begin{aligned} T_{\text{st}} &= T_N \circ_1 \dots \circ_1 T_{s+2} \circ_1 T_{s+1} \circ_1 T_s \circ_1 T_{s-1} \circ_1 \dots \circ_1 T_0 \\ S_{\text{st}} &= T_N \circ_1 \dots \circ_1 T_{s+2} \circ_1 S_{s+1} \circ_1 S_s \circ_1 T_{s-1} \circ_1 \dots \circ_1 T_0 \end{aligned}$$

with

$$\begin{array}{ccccccc}
T_{s+1} = a_i & & T_{s+1} = a_i & & T_{s+1} = a_i^\dagger & & T_{s+1} = a_i^\dagger \\
T_s = p_{j,k} & & T_s = p_{j,k} & & T_s = p_{j,k} & & T_s = p_{j,k} \\
S_{s+1} = p_{j-2,k} & \text{or} & S_{s+1} = p_{j,k} & \text{or} & S_{s+1} = p_{j+2,k} & \text{or} & S_{s+1} = p_{j,k} \\
S_s = a_i & & S_s = a_{i-k} & & S_s = a_i^\dagger & & S_s = a_{i-k}^\dagger \\
\text{for } i+1 < j & & \text{for } i+1 > j+k & & \text{for } i \leq j & & \text{for } i \geq j+k.
\end{array} \tag{22}$$

By (C5) and (C6) we note that, in every case, the equation

$$Z(T_{s+1}) \circ_1 Z(T_s) = Z(S_{s+1}) \circ_1 Z(S_s)$$

holds. We also have $\sigma_S = \sigma_T$. It follows that

$$\begin{aligned}
Z(T) &= Z(T_{\text{st}}) \circ P(\sigma_T) = [Z(T_N) \circ_1 \dots \circ_1 Z(T_{s+1}) \circ_1 Z(T_s) \circ_1 \dots \circ_1 Z(T_0)] \circ P(\sigma_T) \\
&= [Z(T_N) \circ_1 \dots \circ_1 Z(S_{s+1}) \circ_1 Z(S_s) \circ_1 \dots \circ_1 Z(T_0)] \circ P(\sigma_S) \\
&= Z(S_{\text{st}}) \circ P(\sigma_S) = Z(S).
\end{aligned}$$

(M3) T is obtained from S by exchanging the relative order of the y -coordinates of two critical points of strings. This case is entirely similar to the previous one, with

$$\begin{array}{ccccccc}
T_{s+1} = a_i & & T_{s+1} = a_i^\dagger & & T_{s+1} = a_i & & T_{s+1} = a_i \\
T_s = a_j & & T_s = a_j^\dagger & & T_s = a_j^\dagger & & T_s = a_j^\dagger \\
S_{s+1} = a_{j-2} & \text{or} & S_{s+1} = a_{j+2}^\dagger & \text{or} & S_{s+1} = a_{j-2}^\dagger & \text{or} & S_{s+1} = a_j^\dagger \\
S_s = a_i & & S_s = a_i^\dagger & & S_s = a_i & & S_s = a_{i-2} \\
\text{for } i+1 < j & & \text{for } i \leq j & & \text{for } i < j-1 & & \text{for } i > j+1.
\end{array}$$

instead of (22). The relevant relations are (C2), (C3), and (C4).

(M4) T is obtained from S by a Morse cancellation: $\nearrow \rightarrow \mid$. The argument to show $Z(T) = Z(S)$ uses (C4), and is similar to the previous ones.

(M5) T is obtained from S by moving the attaching point of a strand past the equator of some input circle. The invariance of Z under that move is a direct consequence of the corresponding statement for the moves (M2) and (M4).

(M6) T is obtained from S by swinging an anchor point from slightly to the right of the north pole to slightly to the left of the north pole of an input circle. By applying Algorithm 4.7 (which is step (1) of Algorithm 4.9) to T and to S , we obtain tangles T' and S' that differ from each other by a 2π -rotation around some input circle:

$$T' = \text{diagram} \longleftarrow \text{diagram} \longrightarrow \text{diagram} = S'$$

The diagrams show three circular tangles. The first diagram (T') has a red strand entering from the top, looping around the circle, and exiting at the bottom. The second diagram is a circle with a red dot at the top and a vertical line segment at the bottom labeled 'n'. The third diagram (S') has a red strand entering from the top, looping around the circle, and exiting at the bottom. The first and third diagrams are identical.

By the already established moves (M2) – (M4) and by Proposition 4.17, checking the

$$\text{equality } Z\left(\text{diagram}\right) = Z\left(\text{diagram}\right) \text{ is equivalent to checking that}$$

The diagrams are the same as in the previous block, showing the equivalence of the two tangles T' and S'.

holds. The latter follows from the computations performed in Examples 4.11 and 4.16.

This concludes the proof of isotopy invariance of the assignment $T \mapsto Z(T)$. $\square$

Proof of Theorem 4.1. Given an anchored planar tangle T , pick a generic anchored planar tangle $\tilde{T}$ which is isotopic to T and set

$$Z(T) := Z(\tilde{T}),$$

where the meaning of the right hand side is provided by Algorithm 4.9. We note that, by Proposition 4.18, the value $Z(\tilde{T})$ is independent of the choice of perturbation $\tilde{T}$ of T , and that the map $T \mapsto Z(T)$ is well-defined on isotopy classes of tangles.

It remains to show that $((\mathcal{P}[n])_{n \geq 0}, Z)$ satisfies the axioms of an anchored planar algebra. The action of the identity tangle was computed in Example 4.11. The gluing axiom was proved in Proposition 4.17. Finally, the two anchor dependence axioms were established in Examples 4.13 and 4.14, respectively. $\square$

4.5 Induction along a lax functor

Let $\mathcal{C}$ and $\mathcal{D}$ be braided pivotal categories, and let $\Psi : \mathcal{C} \rightarrow \mathcal{D}$ be a lax monoidal functor which is braided and balanced. In this section, we show that if $\mathcal{P} = (\mathcal{P}[n])_{n \geq 0}$ is an anchored planar algebra in $\mathcal{C}$, then its image $\Psi\mathcal{P} = (\Psi(\mathcal{P}[n]))_{n \geq 0}$ is naturally equipped with the structure of an anchored planar algebra in $\mathcal{D}$.

Recall that a functor $\Psi : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories is called *lax monoidal* if it comes equipped with a morphism $i : 1_{\mathcal{D}} \rightarrow \Psi(1_{\mathcal{C}})$ and a family of morphisms $\mu_{a,b} : \Psi(a) \otimes_{\mathcal{D}} \Psi(b) \rightarrow \Psi(a \otimes_{\mathcal{C}} b)$ which are natural in a and b , and satisfy the following two axioms (we suppress the associators and the unitors for better readability):

$$\mu_{1,a} \circ (i \otimes \text{id}_{\Psi(a)}) = \text{id}_{\Psi(a)} = \mu_{a,1} \circ (\text{id}_{\Psi(a)} \otimes i), \quad \mu_{a,b \otimes c} \circ (\text{id}_{\Psi(a)} \otimes \mu_{b,c}) = \mu_{a \otimes b, c} \circ (\mu_{a,b} \otimes \text{id}_{\Psi(c)}).$$

Given objects $c_1, \dots, c_n$ in $\mathcal{C}$, we write $\mu_{c_1, \dots, c_n} : \Psi(c_1) \otimes \dots \otimes \Psi(c_n) \rightarrow \Psi(c_1 \otimes \dots \otimes c_n)$ for the obvious composite of μ 's (which is independent of the way one performs the composite, as a consequence of the second axiom listed above). By convention, we also let $\mu_{\emptyset} = i$ when $n = 0$, and $\mu_c = \text{id}_c$ when $n = 1$.

When $\mathcal{C}$ and $\mathcal{D}$ are braided, we say that Ψ is *braided* if the equation $\Psi(\beta_{a,b}) \circ \mu_{a,b} = \mu_{b,a} \circ \beta_{\Psi(a), \Psi(b)}$ holds for every $a, b \in \mathcal{C}$. When the two categories are furthermore pivotal, with twists θ as in (4) equipping them with the structure of balanced tensor categories [HPT16, §A.2], we say that Ψ is *balanced* if $\Psi(\theta_c) = \theta_{\Psi(c)}$ for all $c \in \mathcal{C}$.

Theorem 4.19. *Let $\Psi : \mathcal{C} \rightarrow \mathcal{D}$ be a braided balanced lax monoidal functor between braided pivotal categories. Then for every anchored planar algebra $\mathcal{P} = (\mathcal{P}[n])_{n \geq 0}$ in $\mathcal{C}$, the morphisms*

$$Z_{\Psi\mathcal{P}}(T) := \Psi(Z_{\mathcal{P}}(T)) \circ \mu_{\mathcal{P}[k_1], \dots, \mathcal{P}[k_n]} \quad \begin{array}{l} (T \text{ an anchored planar tangle} \\ \text{of type } (k_1, \dots, k_r; k_0)) \end{array}$$

equip $\Psi\mathcal{P} := (\Psi(\mathcal{P}[n]))_{n \geq 0}$ with the structure of an anchored planar algebra in $\mathcal{D}$.

Proof. We check the axioms in Definition 2.7. The identity axiom holds because $\Psi : \mathcal{C} \rightarrow \mathcal{D}$ is a functor. The composition axiom follows from the associativity and naturality of μ . Indeed, if S and T are anchored planar tangles of type $(k_1, \dots, k_r; k_0)$ and $(\ell_1, \dots, \ell_s, \ell_0)$ with $\ell_0 = k_i$, we have

$$\begin{aligned}
Z_{\Psi\mathcal{P}}(T \circ_i S) &= \Psi(Z_{\mathcal{P}}(T \circ_i S)) \circ \mu_{\mathcal{P}[k_1], \dots, \mathcal{P}[k_{i-1}], \mathcal{P}[\ell_1], \dots, \mathcal{P}[\ell_s], \mathcal{P}[k_{i+1}], \dots, \mathcal{P}[k_n]} \\
&= \Psi(Z_{\mathcal{P}}(T) \otimes (\text{id}_{\mathcal{P}[k_1]} \otimes \dots \otimes \text{id}_{\mathcal{P}[k_{i-1}]} \otimes Z_{\mathcal{P}}(S) \otimes \text{id}_{\mathcal{P}[k_{i+1}]} \otimes \dots \otimes \text{id}_{\mathcal{P}[k_r]})) \\
&\quad \circ \mu_{\mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_{i-1}], \mathcal{P}[\ell_1] \otimes \dots \otimes \mathcal{P}[\ell_s], \mathcal{P}[k_{i+1}] \otimes \dots \otimes \mathcal{P}[k_n]} \\
&\quad \circ (\mu_{\mathcal{P}[k_1], \dots, \mathcal{P}[k_{i-1}]} \otimes \mu_{\mathcal{P}[\ell_1], \dots, \mathcal{P}[\ell_s]} \otimes \mu_{\mathcal{P}[k_{i+1}], \dots, \mathcal{P}[k_n]}) \\
&= \Psi(Z_{\mathcal{P}}(T)) \circ \mu_{\mathcal{P}[k_1] \otimes \dots \otimes \mathcal{P}[k_{i-1}], \mathcal{P}[k_i], \mathcal{P}[k_{i+1}] \otimes \dots \otimes \mathcal{P}[k_n]} \\
&\quad \circ (\text{id}_{\Psi\mathcal{P}[k_1] \otimes \dots \otimes \Psi\mathcal{P}[k_{i-1}]} \otimes \Psi(Z_{\mathcal{P}}(S)) \otimes \text{id}_{\Psi\mathcal{P}[k_{i+1}] \otimes \dots \otimes \Psi\mathcal{P}[k_r]}) \\
&\quad \circ (\mu_{\mathcal{P}[k_1], \dots, \mathcal{P}[k_{i-1}]} \otimes \mu_{\mathcal{P}[\ell_1], \dots, \mathcal{P}[\ell_s]} \otimes \mu_{\mathcal{P}[k_{i+1}], \dots, \mathcal{P}[k_n]}) \\
&= \Psi(Z_{\mathcal{P}}(T)) \circ \mu_{\mathcal{P}[k_1], \dots, \mathcal{P}[k_r]} \\
&\quad \circ (\text{id}_{\Psi\mathcal{P}[k_1]} \otimes \dots \otimes \text{id}_{\Psi\mathcal{P}[k_{i-1}]} \otimes (\Psi(Z_{\mathcal{P}}(S)) \circ \mu_{\mathcal{P}[\ell_1], \dots, \mathcal{P}[\ell_s]}) \\
&\quad \quad \otimes \text{id}_{\Psi\mathcal{P}[k_{i+1}]} \otimes \dots \otimes \text{id}_{\Psi\mathcal{P}[k_r]}) \\
&= Z_{\Psi\mathcal{P}}(T) \circ (\text{id}_{\Psi\mathcal{P}[k_1]} \otimes \dots \otimes \text{id}_{\Psi\mathcal{P}[k_{i-1}]} \otimes Z_{\Psi\mathcal{P}}(S) \otimes \text{id}_{\Psi\mathcal{P}[k_{i+1}]} \otimes \dots \otimes \text{id}_{\Psi\mathcal{P}[k_r]}).
\end{aligned}$$

Finally, the two anchor dependence axioms follow from the fact that Ψ is braided and balanced, respectively. Letting

$$p_{i,j,k} = \text{diagram} \quad p'_{i,j,k} = \text{diagram} \quad \text{and} \quad t_n = \text{diagram},$$

we have

$$\begin{aligned}
Z_{\Psi\mathcal{P}}(p'_{i,j,k}) &= \Psi(Z_{\mathcal{P}}(p'_{i,j,k})) \circ \mu_{\mathcal{P}[j], \mathcal{P}[i+k]} \\
&= \Psi(Z_{\mathcal{P}}(p_{i,j,k})) \circ \Psi(\beta_{\mathcal{P}[j], \mathcal{P}[i+k]}) \circ \mu_{\mathcal{P}[j], \mathcal{P}[i+k]} \\
&= \Psi(Z_{\mathcal{P}}(p_{i,j,k})) \circ \mu_{\mathcal{P}[i+k], \mathcal{P}[j]} \circ \beta_{\Psi\mathcal{P}[j], \Psi\mathcal{P}[i+k]} \\
&= Z_{\Psi\mathcal{P}}(p_{i,j,k}) \circ \beta_{\Psi\mathcal{P}[j], \Psi\mathcal{P}[i+k]}
\end{aligned}$$

and

$$Z_{\Psi\mathcal{P}}(t_n) = \Psi(Z_{\mathcal{P}}(t_n)) = \Psi(\theta_{\mathcal{P}[n]}) = \theta_{\Psi\mathcal{P}[n]}.$$

□

5 Anchored planar algebras from module tensor categories

In this section we will use Theorem 4.1 in combination with the results of our previous paper [HPT16] to construct an anchored planar algebra $\mathcal{P}$ from a pair $(\mathcal{M}, m)$ consisting of a module tensor category $\mathcal{M}$ and a symmetrically self-dual object $m \in \mathcal{M}$. The anchored planar algebra only depends on the sub-module tensor category generated by m , so we may assume without loss of generality that m generates $\mathcal{M}$ as a $\mathcal{C}$ -module tensor category.

5.1 Diagram cheat sheet from [HPT16]

Let $\mathcal{C}$ be a braided pivotal category, and let $\mathcal{M}$ be a module tensor category whose action functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ admits a right adjoint $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$. Following [HPT16], we have a diagrammatic calculus of strings on cylinders for the categorified trace $\text{Tr}_{\mathcal{C}}$, where the tubes are allowed to branch and braid. We recall the main results of our earlier paper, which we will use to construct anchored planar algebras (the objects of $\mathcal{M}$ which appear below are not assumed to be self-dual, and so the strings should really be oriented).

(T1) The *categorified trace* $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$ can be thought of as lifting strings and coupons in the plane (corresponding to $\mathcal{M}$) up into tubes in 3-space (corresponding to $\mathcal{C}$):

$$f : x \rightarrow y \quad \begin{array}{c} \text{plane} \\ \text{---} x \text{---} \text{box } f \text{---} y \end{array} \quad \mapsto \quad \text{Tr}(f) = \begin{array}{c} \text{tube} \\ \text{---} y \text{---} \text{box } f \text{---} x \end{array}$$

The left adjoint $\Phi : \mathcal{M} \rightarrow \mathcal{C}$ of $\text{Tr}_{\mathcal{C}}$ is represented graphically by placing strings and coupons from 3-space close to the 2-dimensional plane (corresponding to $\mathcal{M}$):

$$g : c \rightarrow d \quad \begin{array}{c} d \\ \text{---} \text{box } g \text{---} \\ c \end{array} \quad \mapsto \quad \Phi(g) = \begin{array}{c} \text{plane} \\ \text{---} c \text{---} \text{box } g \text{---} d \end{array}$$

(The box labelled g in the above display does not touch the plane.) Taking mates under the adjunction $\mathcal{C}(c, \text{Tr}_{\mathcal{C}}(x)) \cong \mathcal{M}(\Phi(c), x)$ is represented graphically by [HPT16, (6)] which we reproduce here for the reader's convenience:

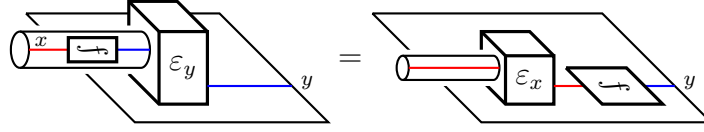
$$\begin{array}{c} \text{tube} \\ \text{---} x \text{---} \text{box } h \text{---} \\ c \end{array} \in \mathcal{C}(c, \text{Tr}_{\mathcal{C}}(x)) \quad \longleftrightarrow \quad \begin{array}{c} \text{plane} \\ \text{---} c \text{---} \text{box } \tilde{h} \text{---} x \end{array} \in \mathcal{M}(\Phi(c), x)$$

Visually, we think of the above adjunction as the operation of opening the tube, like an umbrella, and flattening it to a plane. (This time, the box labelled $\tilde{h}$ does touch the plane.)

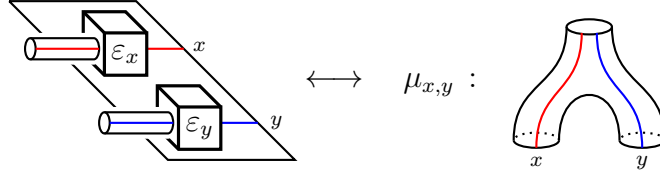
(T2) The *attaching map* ε is the counit of the adjunction $\Phi \dashv \text{Tr}_{\mathcal{C}}$. We denote it as follows:

$$\varepsilon_x : \Phi(\text{Tr}_{\mathcal{C}}(x)) \rightarrow x \quad \begin{array}{c} \text{plane} \\ \text{---} \text{box } \varepsilon_x \text{---} x \end{array}$$

(T3) The attaching map ε is a natural transformation: $\varepsilon_y \circ \Phi(\text{Tr}_C(f)) = f \circ \varepsilon_x$

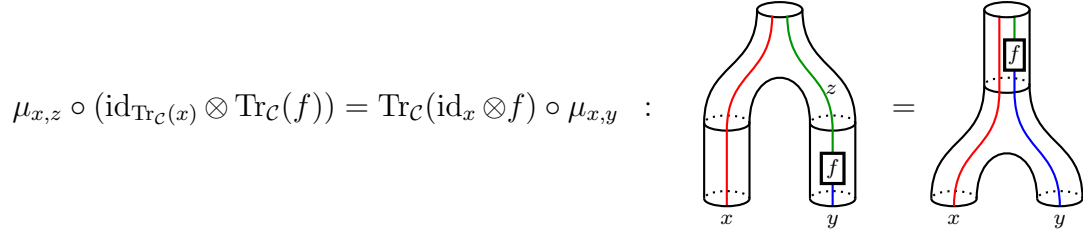


(T4) The *multiplication map* $\mu_{x,y} : \text{Tr}_C(x) \otimes \text{Tr}_C(x) \rightarrow \text{Tr}_C(x \otimes y)$ is the mate of the map $\varepsilon_x \otimes \varepsilon_y : \Phi(\text{Tr}_C(x) \otimes \text{Tr}_C(x)) \rightarrow x \otimes y$. We represent it by a pair of pants with strings:

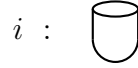


The multiplication map is associative (see (T15) below for a stronger statement).

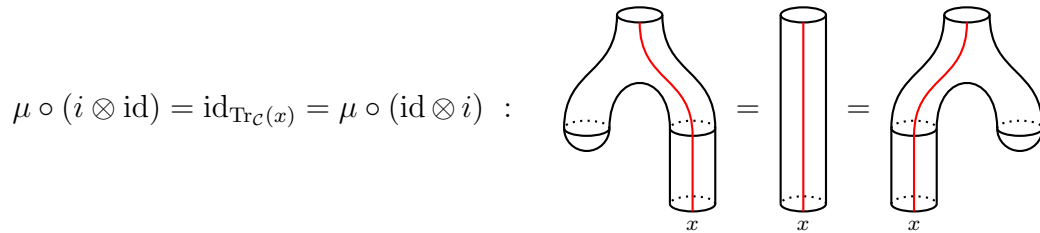
(T5) The multiplication map is natural in x and y . For example, we have



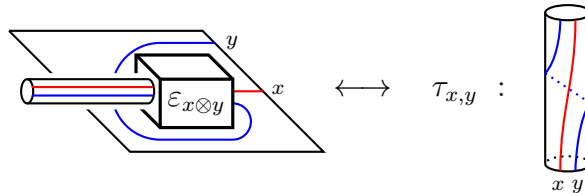
(T6) The *unit map* $i : 1_C \rightarrow \text{Tr}_C(\Phi(1_C)) = \text{Tr}_C(1_{\mathcal{M}})$ is the unit of the adjunction $\Phi \dashv \text{Tr}_C$ evaluated on the unit object 1_C . We denote it by a cup:



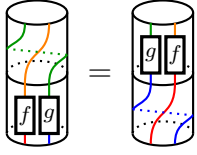
(T7) The unit map i is compatible with the multiplication map μ :

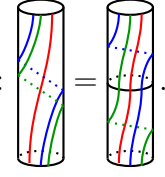


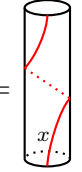
(T8) The *traciator* $\tau_{x,y} : \text{Tr}_C(x \otimes y) \rightarrow \text{Tr}_C(y \otimes x)$ is the mate of the following morphism under the adjunction $\Phi \dashv \text{Tr}_C$:



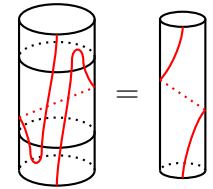
We sometimes write $\tau_{x,y}^+$ for $\tau_{x,y}$, and $\tau_{x,y}^-$ for $\tau_{y,x}^{-1}$.

(T9) The traciator is natural: .

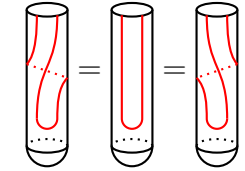
(T10) The traciator satisfies the cyclic invariance property $\tau_{x,y \otimes z} = \tau_{z \otimes x,y} \circ \tau_{x \otimes y,z}$: .

(T11) The traciator with $1_{\mathcal{M}} \in \mathcal{M}$ agrees with the twist in $\mathcal{C}$: $\tau_{1,x} = \theta_{\text{Tr}_{\mathcal{C}}(x)}$. .

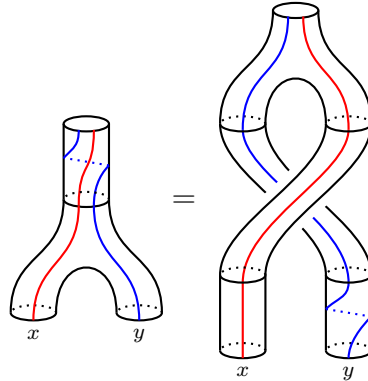
(T12) The zig-zag relation holds around the back of the tube:

$$\text{Tr}_{\mathcal{C}}(\text{id}_x \otimes \text{ev}_{x^{\vee}}) \circ \tau_{x,x^{\vee} \otimes x}^- \circ \text{Tr}_{\mathcal{C}}(\text{coev}_{x^{\vee}} \otimes \text{id}_x) = \tau_{1,x} :$$


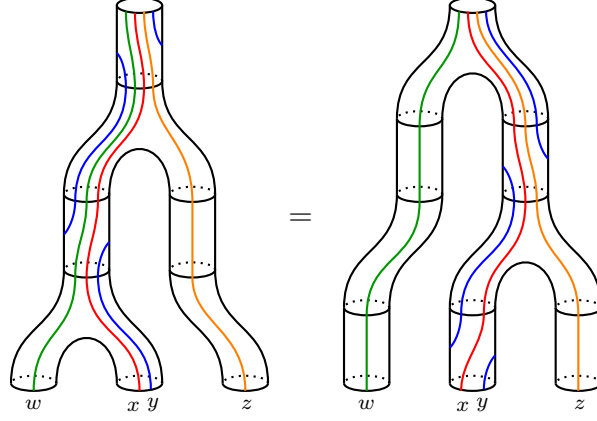
(T13) The traciator is compatible with the unit map:

$$\tau_{x^{\vee},x} \circ \text{Tr}_{\mathcal{C}}(\text{coev}_{x^{\vee}}) \circ i = \text{Tr}_{\mathcal{C}}(\text{coev}_x) \circ i = \tau_{x^{\vee},x}^- \circ \text{Tr}_{\mathcal{C}}(\text{coev}_{x^{\vee}}) \circ i :$$


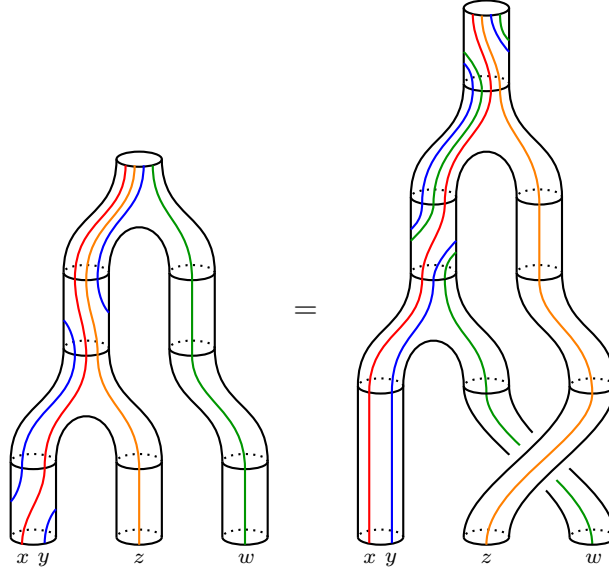
(T14) For every $x, y \in \mathcal{M}$, we have $\tau_{x,y} \circ \mu_{x,y} = \mu_{y,x} \circ \beta_{\text{Tr}_{\mathcal{C}}(x), \text{Tr}_{\mathcal{C}}(y)} \circ (\text{id}_{\text{Tr}_{\mathcal{C}}(x)} \otimes \theta_{\text{Tr}_{\mathcal{C}}(y)})$:



(T15) Finally, the following strong version of associativity holds:



along with the following variant (see Appendix A for a proof):



5.2 Constructing the anchored planar algebra

Let $\mathcal{C}$ be a braided pivotal category. For the remainder of this section, we fix a module tensor category $\mathcal{M}$ and a symmetrically self-dual object $m \in \mathcal{M}$. We assume that $\Phi := F \circ \Phi^z : \mathcal{C} \rightarrow \mathcal{M}$ has a right adjoint, denoted

$$\mathrm{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}.$$

Recall the morphisms $\tilde{\mathrm{ev}}_m : m \otimes m \rightarrow 1$ and $\tilde{\mathrm{coev}}_m : 1 \rightarrow m \otimes m$ from (11).

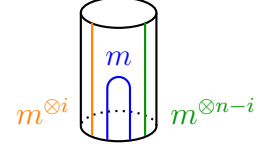
Theorem 5.1. *Let $\mathcal{C}$ be a braided pivotal category. Let $\mathcal{M}$ and m be as above, and let*

$$\mathcal{P}[n] := \mathrm{Tr}_{\mathcal{C}}(m^{\otimes n}). \quad (23)$$

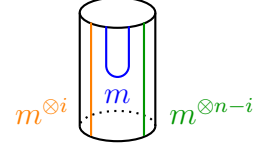
Then the maps η , α_i , $\alpha_i^\dagger$, $\varpi_{i,j}$ defined below satisfy the assumptions of Theorem 4.1, and thus endow $\mathcal{P} = (\mathcal{P}[n])_{n \geq 0}$ with the structure of an anchored planar algebra in $\mathcal{C}$:

$$\eta := i : 1 \rightarrow \mathrm{Tr}_{\mathcal{C}}(1_{\mathcal{M}}) : \quad \text{cup}$$

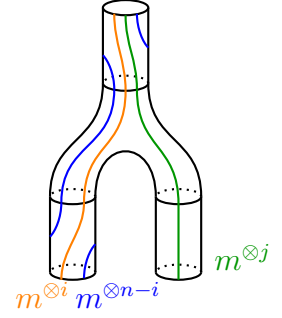
$$\alpha_i := \mathrm{Tr}_{\mathcal{C}}(\mathrm{id}_{m^{\otimes i}} \otimes \tilde{\mathrm{ev}}_m \otimes \mathrm{id}_{m^{\otimes n-i}}) : \mathrm{Tr}_{\mathcal{C}}(m^{\otimes n+2}) \rightarrow \mathrm{Tr}_{\mathcal{C}}(m^{\otimes n}) :$$



$$\alpha_i^{\dagger} := \mathrm{Tr}_{\mathcal{C}}(\mathrm{id}_{m^{\otimes i}} \otimes \mathrm{co}\tilde{\mathrm{ev}}_m \otimes \mathrm{id}_{m^{\otimes n-i}}) : \mathrm{Tr}_{\mathcal{C}}(m^{\otimes n}) \rightarrow \mathrm{Tr}_{\mathcal{C}}(m^{\otimes n+2}) :$$



$$\varpi_{i,j} := \tau_{m^{\otimes n-i}, m^{\otimes i+j}}^- \circ \mu_{m^{\otimes n}, m^{\otimes j}} \circ (\tau_{m^{\otimes i}, m^{\otimes n-i}}^+ \otimes 1_{\mathrm{Tr}_{\mathcal{C}}(m^{\otimes j})}) : \mathrm{Tr}_{\mathcal{C}}(m^{\otimes n}) \otimes \mathrm{Tr}_{\mathcal{C}}(m^{\otimes j}) \rightarrow \mathrm{Tr}_{\mathcal{C}}(m^{\otimes n+j}) :$$



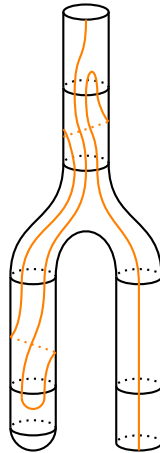
Proof. We show that the maps η , α_i , $\alpha_i^{\dagger}$, $\varpi_{i,j}$ satisfy the relations (C1) – (C9) in Theorem 4.1.

The relation (C1) holds by (T7). The three relations (C2) – (C4) hold because the maps $\mathrm{id}_{m^{\otimes i}} \otimes \tilde{\mathrm{ev}}_m \otimes \mathrm{id}_{m^{\otimes n-i}}$ and $\mathrm{id}_{m^{\otimes i}} \otimes \mathrm{co}\tilde{\mathrm{ev}}_m \otimes \mathrm{id}_{m^{\otimes n-i}}$ satisfy corresponding relations in $\mathcal{M}$, and $\mathrm{Tr}_{\mathcal{C}}$ is a functor. The relations (C5) – (C6) holds by (T5) and (T9).

Using (T10), the relation (C7) can be shown to be equivalent to the first equation in (T15) with $x = m^{\otimes m+i}$, $y = m^{\otimes j}$, $z = m^{\otimes \ell}$, $w = m^{\otimes k}$, upon precomposing both sides by $\tau_{m^{\otimes i}, m^{\otimes m}} \otimes \mathrm{id}_{\mathrm{Tr}_{\mathcal{C}}(m^{\otimes j+\ell}) \otimes \mathrm{Tr}_{\mathcal{C}}(m^{\otimes k})}$ and postcomposing both sides by $\tau_{m^{\otimes m}, m^{\otimes i+j+k+\ell}}^-$.

Again using (T10), the relation (C8) can be shown to be equivalent to the second equation in (T15) with $x = m^{\otimes m+i}$, $y = m^{\otimes k}$, $z = m^{\otimes j}$, $w = m^{\otimes \ell}$, upon precomposing both sides by $\tau_{m^{\otimes i+k}, m^{\otimes m}} \otimes \mathrm{id}_{\mathrm{Tr}_{\mathcal{C}}(m^{\otimes j}) \otimes \mathrm{Tr}_{\mathcal{C}}(m^{\otimes \ell})}$ and postcomposing both sides by $\tau_{m^{\otimes m}, m^{\otimes i+j+k+\ell}}^-$.

Finally, to show (C9), we note that the left hand side of that relation can be represented diagrammatically as



with the orange strand standing for $m^{\otimes n}$. By applying in order (T13), (T5), (T7), (T12), and (T11), we see that this map is equal to $\theta_{\text{Tr}_{\mathcal{C}}(m^{\otimes n})}$, as desired. $\square$

We denote by $\Lambda(\mathcal{M}, m)$ the anchored planar algebra associated to the pointed module tensor category $(\mathcal{M}, m)$ by means of the construction described in Theorem 5.1.

5.3 Functoriality

In the previous section, given a pivotal module tensor category $\mathcal{M}$ and a symmetrically self-dual object $m \in \mathcal{M}$, we constructed an anchored planar algebra $\mathcal{P} = \Lambda(\mathcal{M}, m)$. Our next goal is to upgrade this to a functor

$$\Lambda : \text{ModTens}_* \rightarrow \text{APA}$$

between the category of pointed pivotal module tensor categories over $\mathcal{C}$ and the category of anchored planar algebras in $\mathcal{C}$. We recall that all module tensor categories are assumed pivotal, even if do not mention it explicitly.

Let $(\mathcal{M}_1, m_1)$ and $(\mathcal{M}_2, m_2)$ be pointed module tensor categories, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the associated anchored planar algebras. Given a morphism $G : (\mathcal{M}_1, m_1) \rightarrow (\mathcal{M}_2, m_2)$ in ModTens_* (Definition 3.4), we wish to describe the corresponding morphism of anchored planar algebras

$$\Lambda(G) : \mathcal{P}_1 = \Lambda(\mathcal{M}_1, m_1) \longrightarrow \mathcal{P}_2 = \Lambda(\mathcal{M}_2, m_2).$$

Recall from Definition 3.2 that a morphism of module tensor categories is a pair $G = (G, \gamma)$, with $\gamma : \Phi_2 \Rightarrow G\Phi_1$. Let $\text{Tr}_{\mathcal{C}}^i : \mathcal{M}_i \rightarrow \mathcal{C}$, $i = 1, 2$, be the categorified traces associated to $\mathcal{M}_1$ and $\mathcal{M}_2$.

Definition 5.2. For each $x \in \mathcal{M}_1$, we define $\zeta_x : \text{Tr}_{\mathcal{C}}^1(x) \rightarrow \text{Tr}_{\mathcal{C}}^2(G(x))$ to be the mate of

$$\Phi_2(\text{Tr}_{\mathcal{C}}^1(x)) \xrightarrow{\gamma_{\text{Tr}_{\mathcal{C}}^1(x)}} G(\Phi_1(\text{Tr}_{\mathcal{C}}^1(x))) \xrightarrow{G(\varepsilon_x^1)} G(x)$$

under the adjunction $\Phi_2 \dashv \text{Tr}_{\mathcal{C}}^2$.

Definition 5.3. The map $\Lambda(G) : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ associated to $G : (\mathcal{M}_1, m_1) \rightarrow (\mathcal{M}_2, m_2)$ is the sequence of morphisms

$$(\Lambda(G)[n] : \mathcal{P}_1[n] \rightarrow \mathcal{P}_2[n])_{n \geq 0} \tag{24}$$

given by

$$\Lambda(G)[n] : \mathcal{P}_1[n] = \text{Tr}_{\mathcal{C}}^1(m_1^{\otimes n}) \xrightarrow{\zeta_{m_1^{\otimes n}}} \text{Tr}_{\mathcal{C}}^2(G(m_1^{\otimes n})) \xrightarrow{\cong} \text{Tr}_{\mathcal{C}}^2(m_2^{\otimes n}) = \mathcal{P}_2[n].$$

Our next goal is to show that this is a morphism of anchored planar algebras. With this purpose in mind, we first list some properties of the natural transformation ζ :

Lemma 5.4. *The following diagram commutes:*

$$\begin{array}{ccc} \Phi_2(\text{Tr}_{\mathcal{C}}^1(x)) & \xrightarrow{\gamma_{\text{Tr}_{\mathcal{C}}^1(x)}} & G(\Phi_1(\text{Tr}_{\mathcal{C}}^1(x))) \\ \downarrow \Phi_2(\zeta_x) & & \downarrow G(\varepsilon_x^1) \\ \Phi_2(\text{Tr}_{\mathcal{C}}^2(G(x))) & \xrightarrow{\varepsilon_{G(x)}^2} & G(x) \end{array}$$

Proof. The morphism ζ_x is the mate of $G(\varepsilon_x^1) \circ \gamma_{\mathrm{Tr}_C^1(x)}$. By [HPT16, Lem. 4.3], it is also the mate of $\varepsilon_{G(x)}^2 \circ \Phi_2(\zeta_x)$. The two composites therefore agree. $\square$

Lemma 5.5. *For all $x, y \in \mathcal{M}_1$, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Tr}_C^1(x) \otimes \mathrm{Tr}_C^1(y) & \xrightarrow{\mu_{x,y}^1} & \mathrm{Tr}_C^1(x \otimes y) \\ \downarrow \zeta_x \otimes \zeta_y & & \downarrow \zeta_{x \otimes y} \\ \mathrm{Tr}_C^2(G(x)) \otimes \mathrm{Tr}_C^2(G(y)) & \xrightarrow{\mu_{G(x), G(y)}^2} & \mathrm{Tr}_C^2(G(x) \otimes G(y)) \xrightarrow{\cong} \mathrm{Tr}_C^2(G(x \otimes y)) \end{array}$$

Proof. The mate of $\zeta_{x \otimes y} \circ \mu_{x,y}^1$ is $G(\varepsilon_{x \otimes y}^1) \circ \gamma_{\mathrm{Tr}_C^1(x \otimes y)} \circ \Phi_2(\mu_{x,y}^1)$, and the mate of $\mu_{G(x), G(y)}^2 \circ \zeta_x \otimes \zeta_y$ is $\varepsilon_{G(x) \otimes G(y)}^2 \circ \Phi_2(\mu_{G(x), G(y)}^2 \circ (\zeta_x \otimes \zeta_y))$. It is therefore enough to show that

$$G(\varepsilon_{x \otimes y}^1) \circ \gamma_{\mathrm{Tr}_C^1(x \otimes y)} \circ \Phi_2(\mu_{x,y}^1) = \varepsilon_{G(x) \otimes G(y)}^2 \circ \Phi_2(\mu_{G(x), G(y)}^2 \circ (\zeta_x \otimes \zeta_y))$$

(note that we have suppressed the isomorphism $\nu_{x,y} : G(x) \otimes G(y) \rightarrow G(x \otimes y)$). This is verified as follows:

$$\begin{aligned} & G(\varepsilon_{x \otimes y}^1) \circ \gamma_{\mathrm{Tr}_C^1(x \otimes y)} \circ \Phi_2(\mu_{x,y}^1) \\ &= G(\varepsilon_{x \otimes y}^1) \circ G(\Phi_1(\mu_{x,y}^1)) \circ \gamma_{\mathrm{Tr}^1(x) \otimes \mathrm{Tr}^1(y)} && \text{(naturality of } \gamma) \\ &= G(\varepsilon_{x \otimes y}^1 \circ \Phi_1(\mu_{x,y}^1)) \circ \gamma_{\mathrm{Tr}^1(x) \otimes \mathrm{Tr}^1(y)} \\ &= G(\varepsilon_x^1 \otimes \varepsilon_y^1) \circ \gamma_{\mathrm{Tr}^1(x) \otimes \mathrm{Tr}^1(y)} && \text{([HPT16, Lem. 4.6])} \\ &= (G(\varepsilon_x^1) \otimes G(\varepsilon_y^1)) \circ (\gamma_{\mathrm{Tr}^1(x)} \otimes \gamma_{\mathrm{Tr}^1(y)}) \\ &= (G(\varepsilon_x^1) \circ \gamma_{\mathrm{Tr}^1(x)}) \otimes (G(\varepsilon_y^1) \circ \gamma_{\mathrm{Tr}^1(y)}) \\ &= (\varepsilon_{G(x)}^2 \circ \Phi_2(\zeta_x)) \otimes (\varepsilon_{G(y)}^2 \circ \Phi_2(\zeta_y)) && \text{(Lemma 5.4)} \\ &= (\varepsilon_{G(x)}^2 \otimes \varepsilon_{G(y)}^2) \circ \Phi_2(\zeta_x \otimes \zeta_y) \\ &= \varepsilon_{G(x) \otimes G(y)}^2 \circ \Phi_2(\mu_{G(x), G(y)}^2 \circ \Phi_2(\zeta_x \otimes \zeta_y)) && \text{([HPT16, Lem. 4.6])} \\ &= \varepsilon_{G(x) \otimes G(y)}^2 \circ \Phi_2(\mu_{G(x), G(y)}^2 \circ (\zeta_x \otimes \zeta_y)) \end{aligned} \quad \square$$

Let us write $\tau_{x,y}^i : \mathrm{Tr}_C^i(x \otimes y) \rightarrow \mathrm{Tr}_C^i(y \otimes x)$, $i = 1, 2$, for the traciator associated to $\mathrm{Tr}^i : \mathcal{M}_i \rightarrow \mathcal{C}$. Recall from Definition 3.2 that for every $c \in \mathcal{C}$ and $x \in \mathcal{M}_1$, we have

$$G(e_{\Phi_1(c), x}) \circ (\gamma_c \otimes \mathrm{id}_{G(x)}) = (\mathrm{id}_{G(x)} \otimes \gamma_c) \circ e_{\Phi_2(c), G(x)}. \quad (25)$$

Lemma 5.6. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Tr}_C^1(x \otimes y) & \xrightarrow{\tau_{x,y}^1} & \mathrm{Tr}_C^1(y \otimes x) \\ \downarrow \zeta_{x \otimes y} & & \downarrow \zeta_{y \otimes x} \\ \mathrm{Tr}_C^2(G(x \otimes y)) & & \mathrm{Tr}_C^2(G(x \otimes y)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Tr}_C^2(G(x) \otimes G(y)) & \xrightarrow{\tau_{G(x), G(y)}^2} & \mathrm{Tr}_C^2(G(y) \otimes G(x)) \end{array}$$

Proof. We verify $\zeta_{y \otimes x} \circ \tau_{x,y}^1 = \tau_{G(x), G(y)}^2 \circ \zeta_{x \otimes y}$ upon taking mates. Let $c := \text{Tr}_C^1(x \otimes y)$ and $d := \text{Tr}_C^1(y \otimes x)$. The mate of $\zeta_{y \otimes x} \circ \tau_{x,y}^1$ is:

$$\begin{aligned} G(\varepsilon_{y \otimes x}^1) \circ \gamma_d \circ \Phi_2(\tau_{x,y}^1) &= G(\varepsilon_{y \otimes x}^1) \circ G\Phi_1(\tau_{x,y}^1) \circ \gamma_c \\ &= G(\varepsilon_{y \otimes x}^1 \circ \Phi_1(\tau_{x,y}^1)) \circ \gamma_c = G(\text{mate of } \tau_{x,y}^1) \circ \gamma_c. \end{aligned}$$

By definition (see (T8) or [HPT16, Def. 4.13]), the mate of $\tau_{x,y}^1$ is

$$(\text{id}_{y \otimes x} \otimes \tilde{\text{ev}}_y) \circ (\text{id}_y \otimes \varepsilon_{x \otimes y}^1 \otimes \text{id}_{y^\sim}) \circ (e_{\Phi_1(c), y} \otimes \text{id}_{y^\sim}) \circ (\text{id}_{\Phi_1(c)} \otimes \text{coev}_y) \quad (26)$$

where $\tilde{\text{ev}}_y = \text{ev}_{y^\sim} \circ (\varphi_y \otimes \text{id}_{y^\sim})$ and $\varphi_y : y \rightarrow y^\sim$ is the pivotal structure. Let $\dot{x} := G(x)$ and $\dot{y} := G(y)$. Applying G to (26) and precomposing with $\gamma_{\text{Tr}_C^1(x \otimes y)}$, we obtain:

$$\begin{aligned} &G(\text{mate of } \tau_{x,y}^1) \circ \gamma_c \\ &= (\text{id}_{\dot{y} \otimes \dot{x}} \otimes \tilde{\text{ev}}_{\dot{y}}) \circ (\text{id}_{\dot{y}} \otimes G(\varepsilon_{x \otimes y}^1) \otimes \text{id}_{\dot{y}^\sim}) \circ (G(e_{\Phi_1(c), y}) \otimes \text{id}_{\dot{y}^\sim}) \circ (\text{id}_{G\Phi_1(c)} \otimes \text{coev}_{\dot{y}}) \circ \gamma_c \\ &= (\text{id}_{\dot{y} \otimes \dot{x}} \otimes \tilde{\text{ev}}_{\dot{y}}) \circ (\text{id}_{\dot{y}} \otimes G(\varepsilon_{x \otimes y}^1) \otimes \text{id}_{\dot{y}^\sim}) \circ ([G(e_{\Phi_1(c), y}) \circ (\gamma_c \otimes \text{id}_{\dot{y}})] \otimes \text{id}_{\dot{y}^\sim}) \circ (\text{id}_{\Phi_2(c)} \otimes \text{coev}_{\dot{y}}) \\ &= (\text{id}_{\dot{y} \otimes \dot{x}} \otimes \tilde{\text{ev}}_{\dot{y}}) \circ (\text{id}_{\dot{y}} \otimes G(\varepsilon_{x \otimes y}^1) \otimes \text{id}_{\dot{y}^\sim}) \circ ([(\text{id}_{\dot{y}} \otimes \gamma_c) \circ e_{\Phi_2(c), \dot{y}}] \otimes \text{id}_{\dot{y}^\sim}) \circ (\text{id}_{\Phi_2(c)} \otimes \text{coev}_{\dot{y}}) \\ &= (\text{id}_{\dot{y} \otimes \dot{x}} \otimes \tilde{\text{ev}}_{\dot{y}}) \circ (\text{id}_{\dot{y}} \otimes [G(\varepsilon_{x \otimes y}^1) \circ \gamma_c] \otimes \text{id}_{\dot{y}^\sim}) \circ (e_{\Phi_2(c), \dot{y}} \otimes \text{id}_{\dot{y}^\sim}) \circ (\text{id}_{\Phi_2(c)} \otimes \text{coev}_{\dot{y}}) \\ &= (\text{id}_{\dot{y} \otimes \dot{x}} \otimes \tilde{\text{ev}}_{\dot{y}}) \circ (\text{id}_{\dot{y}} \otimes [\varepsilon_{\dot{x} \otimes \dot{y}}^2 \circ \Phi_2(\zeta_{x \otimes y})] \otimes \text{id}_{\dot{y}^\sim}) \circ (e_{\Phi_2(c), \dot{y}} \otimes \text{id}_{\dot{y}^\sim}) \circ (\text{id}_{\Phi_2(c)} \otimes \text{coev}_{\dot{y}}) \\ &= (\text{id}_{\dot{y} \otimes \dot{x}} \otimes \tilde{\text{ev}}_{\dot{y}}) \circ (\text{id}_{\dot{y}} \otimes \varepsilon_{\dot{x} \otimes \dot{y}}^2 \otimes \text{id}_{\dot{y}^\sim}) \circ (e_{\Phi_2(d), \dot{y}} \otimes \text{id}_{\dot{y}^\sim}) \circ (\text{id}_{\Phi_2(d)} \otimes \text{coev}_{\dot{y}}) \circ \Phi_2(\zeta_{x \otimes y}) \\ &= \text{mate of } (\tau_{\dot{x}, \dot{y}}^2 \circ \zeta_{x \otimes y}). \end{aligned}$$

The first equality holds because G is a pivotal functor, the third one holds by equation (25), the fifth one holds by Lemma 5.4, and the sixth one holds by the naturality of $e_{\Phi_2(-), \dot{y}}$ (see [HPT16, Def. 3.5]). $\square$

Lemma 5.7. *The following diagram commutes:*

$$\begin{array}{ccc} 1_C & \xrightarrow{i^1} & \text{Tr}_C^1(1_{\mathcal{M}_1}) \\ \downarrow i^2 & & \downarrow \zeta_1 \\ \text{Tr}_C^2(1_{\mathcal{M}_2}) & \xrightarrow{\cong} & \text{Tr}_C^1(G(1_{\mathcal{M}_1})) \end{array}$$

Proof. We check the equality $\zeta_1 \circ i^1 = i^2$ upon taking mates. Recall that i^2 is the unit of the adjunction $\Phi_2 \dashv \text{Tr}_C^2$, evaluated on the object $1_{\mathcal{M}_2}$. Its mate is the identity on $1_{\mathcal{M}_2}$. We now compute the mate of $\zeta_1 \circ i^1$. To lessen the notational confusion between indices and unit objects, we write $\underline{1}$ for $1_{\mathcal{M}_1}$. We have:

$$\begin{aligned} \text{mate of } (\zeta_1 \circ i^1) &= \varepsilon_{G(\underline{1})}^2 \circ \Phi_2(\zeta_1 \circ i^1) \\ &= \varepsilon_{G(\underline{1})}^2 \circ \Phi_2(\zeta_1) \circ \Phi_2(i^1) \\ &= G(\varepsilon_{\underline{1}}^1) \circ \gamma_{\text{Tr}_C^2(\underline{1})} \circ \Phi_2(i^1) \\ &= G(\varepsilon_{\underline{1}}^1) \circ G\Phi_1(i^1) \circ \gamma_{1_C} \\ &= G(\varepsilon_{\underline{1}}^1 \circ \Phi_1(i^1)) = G(\text{id}_{\underline{1}}) = \text{id}_{1_{\mathcal{M}_2}} \end{aligned}$$

where the third equality holds by Lemma 5.4, the fourth equality is the naturality of γ , and the sixth one is [HPT16, (12)]. Towards the end of the calculation, we have suppressed the unit coherences of Φ_i and of G . $\square$

We are now in position to show that the sequence $(\Lambda(G)[n] : \mathcal{P}_1[n] \rightarrow \mathcal{P}_2[n])_{n \geq 0}$ defines a morphism of anchored planar algebras.

Proposition 5.8. *The map $\Lambda(G) : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ given in Definition 5.3 is a morphism of anchored planar algebras.*

Proof. In view of Theorem 4.1, a morphism of anchored planar algebras $\mathcal{P}_1 \rightarrow \mathcal{P}_2$ is equivalent to a sequence of morphisms $(\mathcal{P}_1[n] \rightarrow \mathcal{P}_2[n])_{n \geq 0}$ making the following diagrams commute:

$$\begin{array}{ccccccc} 1 & \xrightarrow{\eta} & \mathcal{P}_1[0] & \mathcal{P}_1[n+2] & \xrightarrow{\alpha_i} & \mathcal{P}_1[n] & \mathcal{P}_1[n] & \xrightarrow{\alpha_i^\dagger} & \mathcal{P}_1[n+2] & \mathcal{P}_1[n] \otimes \mathcal{P}_1[j] & \xrightarrow{\varpi_{i,j}} & \mathcal{P}_1[n+j] \\ \parallel & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ 1 & \xrightarrow{\eta} & \mathcal{P}_2[0] & \mathcal{P}_2[n+2] & \xrightarrow{\alpha_i} & \mathcal{P}_2[n] & \mathcal{P}_2[n] & \xrightarrow{\alpha_i^\dagger} & \mathcal{P}_2[n+2] & \mathcal{P}_2[n] \otimes \mathcal{P}_2[j] & \xrightarrow{\varpi_{i,j}} & \mathcal{P}_2[n+j] \end{array}$$

We show that the morphisms (24) satisfy the above conditions.

The commutativity of the first diagram is the content of Lemma 5.7. The commutativity of the middle diagrams

$$\begin{array}{ccc} \mathcal{P}_1[n+2] & \xrightarrow{\alpha_i} & \mathcal{P}_1[n] \\ \Lambda(G)[n+2] = \zeta_{m_1^{\otimes n+2}} \downarrow & & \downarrow \zeta_{m_1^{\otimes n}} \\ \mathcal{P}_2[n+2] & \xrightarrow{\alpha_i} & \mathcal{P}_2[n] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{P}_1[n] & \xrightarrow{\alpha_i^\dagger} & \mathcal{P}_1[n+2] \\ \zeta_{m_1^{\otimes n}} \downarrow & & \downarrow \zeta_{m_1^{\otimes n+2}} \\ \mathcal{P}_2[n] & \xrightarrow{\alpha_i^\dagger} & \mathcal{P}_2[n+2] \end{array}$$

follows from the naturality of ζ , and the relations $G(\tilde{\text{ev}}_{m_1}) = \tilde{\text{ev}}_{m_2}$ and $G(\text{co}\tilde{\text{ev}}_{m_1}) = \text{co}\tilde{\text{ev}}_{m_2}$, which we explain below. We only show the first equation, as the other one is similar. For clarity, we reintroduce the coherence ν , which we had often suppressed. The computation goes as follows:

$$\begin{aligned} G(\tilde{\text{ev}}_{m_1}) \circ \nu_{m_1, m_1} &= G(\text{ev}_{m_1} \circ (\psi_1 \otimes \text{id}_{m_1})) \circ \nu_{m_1, m_1} \\ &= G(\text{ev}_{m_1}) \circ G(\psi_1 \otimes \text{id}_{m_1}) \circ \nu_{m_1, m_1} \\ &= G(\text{ev}_{m_1}) \circ \nu_{m_1, m_1} \circ (G(\psi_1) \otimes \text{id}_{m_2}) \\ &= \text{ev}_{G(m_1)} \circ ([\delta_{m_1} \circ G(\psi_1)] \otimes \text{id}_{m_2}) = \text{ev}_{m_2} \circ (\psi_2 \otimes \text{id}_{m_2}) = \tilde{\text{ev}}_{m_2}, \end{aligned}$$

where the fourth equality uses the identity $G(\text{ev}_{m_1}) \circ \nu_{m_1, m_1} = \text{ev}_{G(m_1)} \circ (\delta_{m_1} \otimes \text{id})$ (which holds by the definition of δ ; see Definition 3.2), and the fifth equality follows from the relation $\delta_{m_1} \circ G(\psi_1) = \psi_2$ (see Definition 3.4).

Finally, the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{P}_1[n] \otimes \mathcal{P}_1[j] & \xrightarrow{\varpi_{i,j}} & \mathcal{P}_1[n+j] \\ \Lambda(G)[n] \otimes \Lambda(G)[j] \downarrow & & \downarrow \Lambda(G)[n+j] \\ \mathcal{P}_2[n] \otimes \mathcal{P}_2[j] & \xrightarrow{\varpi_{i,j}} & \mathcal{P}_2[n+j] \end{array}$$

follows from Lemmas 5.5 and 5.6:

$$\begin{array}{ccccccc}
\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n}) \otimes \mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes j}) & \xrightarrow{\tau^+ \otimes \mathrm{id}} & \mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n}) \otimes \mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes j}) & \xrightarrow{\mu} & \mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n+j}) & \xrightarrow{\tau^-} & \mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n+j}) \\
\downarrow \Lambda(G)[n] \otimes \Lambda(G)[j] & \text{5.6} & \downarrow \Lambda(G)[n] \otimes \Lambda(G)[j] & \text{5.5} & \downarrow \Lambda(G)[n+j] & \text{5.6} & \downarrow \Lambda(G)[n+j] \\
\mathrm{Tr}_{\mathcal{C}}^2(m_2^{\otimes n}) \otimes \mathrm{Tr}_{\mathcal{C}}^2(m_2^{\otimes j}) & \xrightarrow{\tau^+ \otimes \mathrm{id}} & \mathrm{Tr}_{\mathcal{C}}^2(m_2^{\otimes n}) \otimes \mathrm{Tr}_{\mathcal{C}}^2(m_2^{\otimes j}) & \xrightarrow{\mu} & \mathrm{Tr}_{\mathcal{C}}^2(m_2^{\otimes n+j}) & \xrightarrow{\tau^-} & \mathrm{Tr}_{\mathcal{C}}^2(m_2^{\otimes n+j})
\end{array}$$

□

In the previous proposition, we have done all the hard work for:

Theorem 5.9. *Let $\mathcal{C}$ be a braided pivotal category. Then the map*

$$\Lambda : \mathrm{ModTens}_* \rightarrow \mathrm{APA}$$

given by Theorem 5.1 at the level of objects and by Definition 5.3 at the level of morphisms is a functor from the (2-)category of pointed pivotal module tensor categories over $\mathcal{C}$ to the category of anchored planar algebras in $\mathcal{C}$.

Proof. Given two functors

$$(\mathcal{M}_1, m_1) \xrightarrow{(G_1, \gamma^1)} (\mathcal{M}_2, m_2) \xrightarrow{(G_2, \gamma^2)} (\mathcal{M}_3, m_3),$$

with composite $(G_3, \gamma^3) := (G_2, \gamma^2) \circ (G_1, \gamma^1)$, we still need to show that $\Lambda(G_3) = \Lambda(G_2) \circ \Lambda(G_1)$. Thus, for every $n \in \mathbb{N}$, we need to show $\Lambda(G_3)[n] = \Lambda(G_2)[n] \circ \Lambda(G_1)[n]$, i.e.,

$$\zeta_{m_1^{\otimes n}}^3 = \zeta_{m_2^{\otimes n}}^2 \circ \zeta_{m_1^{\otimes n}}^1. \quad (27)$$

Note that, by definition, $\gamma_c^3 = G_2(\gamma_c^1) \circ \gamma_c^2$. We verify (27) upon taking mates:

$$\begin{aligned}
\text{mate of } \zeta_{m_1^{\otimes n}}^3 &= G_3(\varepsilon_{m_1^{\otimes n}}^1) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^3 \\
&= G_2 G_1(\varepsilon_{m_1^{\otimes n}}^1) \circ G_2(\gamma_{\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^1) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^2 \\
&= G_2(G_1(\varepsilon_{m_1^{\otimes n}}^1) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^1) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^2 \\
&= G_2(\varepsilon_{G_1(m_1^{\otimes n})}^2 \circ \Phi_2(\zeta_{m_1^{\otimes n}}^1)) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^2 && \text{(Lemma 5.4)} \\
&= G_2(\varepsilon_{m_2^{\otimes n}}^2) \circ G_2 \Phi_2(\zeta_{m_1^{\otimes n}}^1) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^2 \\
&= G_2(\varepsilon_{m_2^{\otimes n}}^2) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}^2(m_2^{\otimes n})}^2 \circ \Phi_3(\zeta_{m_1^{\otimes n}}^1) && \text{(naturality of } \gamma^2) \\
&= \text{mate of } (\zeta_{m_2^{\otimes n}}^2 \circ \zeta_{m_1^{\otimes n}}^1)
\end{aligned}$$

We also need to check that $\Lambda(\mathrm{id}_{(\mathcal{M}, m)}) = \mathrm{id}_{\Lambda(\mathcal{M}, m)}$. By definition, the map $\Lambda(\mathcal{M}, m)[n] \rightarrow \Lambda(\mathcal{M}, m)[n]$ associated to $(G, \gamma) := \mathrm{id}_{(\mathcal{M}, m)}$ is the mate of $G(\varepsilon_{m^{\otimes n}}) \circ \gamma_{\mathrm{Tr}_{\mathcal{C}}(m^{\otimes n})} = \varepsilon_{m^{\otimes n}}$ (Definition 5.2). But ε is the counit of the adjunction, and so its mate is an identity morphism. □

We finish the section by noting that, since APA is a 1-category, the functor Λ descends to a functor $\tau_{\leq 1}(\mathrm{ModTens}_*) \rightarrow \mathrm{APA}$. We will not distinguish between these two meanings of Λ .

6 Module tensor categories from anchored planar algebras

In the previous section, we saw how to construct an anchored planar algebra $\mathcal{P}$ from a pointed module tensor category $(\mathcal{M}, \Phi^z, m)$. Our next goal is to provide a construction that goes the other way.

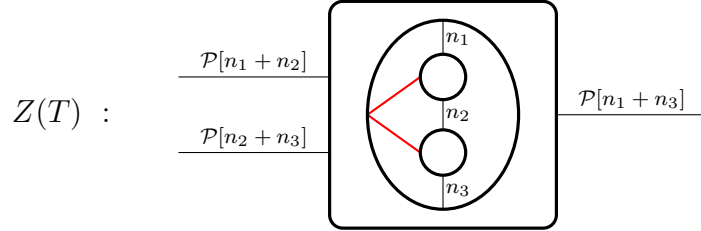
Given braided pivotal category $\mathcal{C}$ and an anchored planar algebra $\mathcal{P}$ in $\mathcal{C}$, we will construct a pivotal tensor category $\mathcal{M}$, a braided pivotal functor $\Phi^z : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M})$, and the symmetrically self-dual object m that generates $\mathcal{M}$ as a module tensor category (Definition 3.4). In Section 7, we will show that these two constructions are each other's inverses.

6.1 Reconstructing the category $\mathcal{M}$

In this section, given an anchored planar algebra, we construct a category $\mathcal{M}$ with a distinguished object $m \in \mathcal{M}$.

Our construction makes heavy use of the graphical calculus for morphisms in $\mathcal{C}$. Given an anchored planar algebra $(\mathcal{P}, Z)$, an anchored planar tangle T yields a morphism $Z(T)$ in $\mathcal{C}$. We represent this by a coupon labeled by the tangle T . For example, the multiplication

tangle $T = \text{multiplication tangle}$ yields a map $Z(T) : \mathcal{P}[n_1 + n_2] \otimes \mathcal{P}[n_2 + n_3] \rightarrow \mathcal{P}[n_1 + n_3]$, denoted

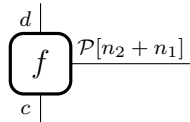


The above diagram reads from left to right; later on, we will use some more complicated diagrams with strands coming in and out from multiple directions. We adopt the convention that the strands that come in from the left and from the bottom are input strands, and that the strands that come out from the top and from the right are output strands.

As a way towards constructing $\mathcal{M}$, we first construct a certain full subcategory $\mathcal{M}_0$ of $\mathcal{M}$. The latter is obtained from $\mathcal{M}_0$ by formally adding finite direct sums, and passing to the idempotent completion. The objects of $\mathcal{M}_0$ are formal symbols “ $\Phi(c) \otimes m^{\otimes n}$ ”, where c is an object of $\mathcal{C}$, and $n \geq 0$. The hom spaces are given by

$$\mathcal{M}_0(\Phi(c) \otimes m^{\otimes n_1}, \Phi(d) \otimes m^{\otimes n_2}) := \mathcal{C}(c, d \otimes \mathcal{P}[n_2 + n_1]). \quad (28)$$

We represent an element f of the above hom space as follows:



We then let

$$m := \Phi(1_{\mathcal{C}}) \otimes m^{\otimes 1} \in \mathcal{M}_0 \subset \mathcal{M}. \quad (29)$$

The composition in $\mathcal{M}_0$ of two morphisms $f : \text{"}\Phi(a) \otimes m^{\otimes n_1}\text{"} \rightarrow \text{"}\Phi(b) \otimes m^{\otimes n_2}\text{"}$ and $g : \text{"}\Phi(b) \otimes m^{\otimes n_2}\text{"} \rightarrow \text{"}\Phi(c) \otimes m^{\otimes n_3}\text{"}$ is given by

$$g \circ f := \begin{array}{c} c \\ | \\ \boxed{g} \\ | \\ b \\ | \\ \boxed{f} \\ | \\ a \end{array} \begin{array}{l} \mathcal{P}[n_3 + n_2] \\ \mathcal{P}[n_2 + n_1] \end{array} \begin{array}{c} \boxed{\begin{array}{c} \text{---} n_3 \text{---} \\ | \\ \text{---} n_2 \text{---} \\ | \\ \text{---} n_1 \text{---} \end{array}} \\ \mathcal{P}[n_3 + n_1] \end{array} \quad (30)$$

and the identity morphism on an object $\text{"}\Phi(c) \otimes m^{\otimes n}\text{"}$ is given by

$$c \begin{array}{c} | \\ \boxed{\begin{array}{c} \text{---} n \text{---} \\ | \\ \text{---} n \text{---} \end{array}} \\ | \end{array} \mathcal{P}[2n] \quad (31)$$

The following computations shows that the composition of morphisms is associative:

$$\begin{aligned} (h \circ g) \circ f &= \begin{array}{c} d \\ | \\ \boxed{h} \\ | \\ c \\ | \\ \boxed{g} \\ | \\ b \\ | \\ \boxed{f} \\ | \\ a \end{array} \begin{array}{l} \mathcal{P}[n_4 + n_3] \\ \mathcal{P}[n_3 + n_2] \\ \mathcal{P}[n_2 + n_1] \end{array} \begin{array}{c} \boxed{\begin{array}{c} \text{---} n_4 \text{---} \\ | \\ \text{---} n_3 \text{---} \\ | \\ \text{---} n_2 \text{---} \end{array}} \\ \mathcal{P}[n_4 + n_2] \\ \mathcal{P}[n_4 + n_1] \end{array} \\ &= \begin{array}{c} d \\ | \\ \boxed{h} \\ | \\ c \\ | \\ \boxed{g} \\ | \\ b \\ | \\ \boxed{f} \\ | \\ a \end{array} \begin{array}{l} \mathcal{P}[n_4 + n_3] \\ \mathcal{P}[n_3 + n_2] \\ \mathcal{P}[n_2 + n_1] \end{array} \begin{array}{c} \boxed{\begin{array}{c} \text{---} n_4 \text{---} \\ | \\ \text{---} n_3 \text{---} \\ | \\ \text{---} n_2 \text{---} \end{array}} \\ \mathcal{P}[n_4 + n_1] \end{array} \\ &= \begin{array}{c} d \\ | \\ \boxed{h} \\ | \\ c \\ | \\ \boxed{g} \\ | \\ b \\ | \\ \boxed{f} \\ | \\ a \end{array} \begin{array}{l} \mathcal{P}[n_4 + n_3] \\ \mathcal{P}[n_3 + n_2] \\ \mathcal{P}[n_2 + n_1] \end{array} \begin{array}{c} \boxed{\begin{array}{c} \text{---} n_4 \text{---} \\ | \\ \text{---} n_3 \text{---} \\ | \\ \text{---} n_2 \text{---} \end{array}} \\ \mathcal{P}[n_3 + n_1] \\ \mathcal{P}[n_4 + n_1] \end{array} \\ &= h \circ (g \circ f) \end{aligned}$$

and unital:

$$\begin{aligned}
\text{id} \circ f &= \text{Diagram 1} \\
&= \text{Diagram 2} = f \\
&= \text{Diagram 3} \\
&= f \circ \text{id}.
\end{aligned}$$

The diagrams are as follows:

- Diagram 1:** A box labeled f with input a and output b . A wire labeled $\mathcal{P}[n_2 + n_1]$ connects the output of f to a larger box. This larger box contains three circles labeled n_1, n_2, n_3 from bottom to top. A red line connects the output of f to the n_2 circle. A dashed box encloses the n_2 circle and a red dot on the input wire a . A wire labeled $\mathcal{P}[2n_2]$ connects the n_2 circle to the n_3 circle. The output of the large box is labeled $\mathcal{P}[n_2 + n_1]$.
- Diagram 2:** A box labeled f with input a and output b . A wire labeled $\mathcal{P}[n_2 + n_1]$ connects the output of f to a box containing two circles labeled n_1, n_2 . A red line connects the output of f to the n_2 circle. The output of this box is labeled $\mathcal{P}[n_2 + n_1]$ and is equal to f .
- Diagram 3:** A box labeled f with input a and output b . A wire labeled $\mathcal{P}[n_2 + n_1]$ connects the output of f to a larger box. This larger box contains two circles labeled n_1, n_2 from bottom to top. A red line connects the output of f to the n_1 circle. A dashed box encloses the n_1 circle and a red dot on the input wire a . A wire labeled $\mathcal{P}[2n_1]$ connects the n_1 circle to the n_2 circle. The output of the large box is labeled $\mathcal{P}[n_2 + n_1]$.

6.2 The adjoint pair $\Phi \dashv \text{Tr}_{\mathcal{C}}$

Our next goal is to construct a pair of adjoint functors

$$\Phi : \mathcal{C} \xrightarrow{\sim} \mathcal{M} : \text{Tr}_{\mathcal{C}}. \quad (32)$$

The functor Φ lands in the full subcategory $\mathcal{M}_0$ of $\mathcal{M}$. It is given by

$$\begin{aligned}
\Phi(c) &:= \text{“}\Phi(c) \otimes m^{\otimes 0}\text{”} \quad \text{and} \\
\Phi(f : a \rightarrow b) &:= \text{Diagram} \in \mathcal{M}(\Phi(a), \Phi(b)) = \mathcal{C}(a, b \otimes \mathcal{P}[0]).
\end{aligned} \quad (33)$$

The diagram for $\Phi(f)$ is a box labeled f with input a and output b . A wire labeled $\mathcal{P}[0]$ connects the output of f to a box containing a single circle labeled 0 . A red dot is on the input wire a .

The functor $\text{Tr}_{\mathcal{C}}$ is first defined on $\mathcal{M}_0$, and then formally extended to $\mathcal{M}$ using the fact that $\mathcal{C}$ admits direct sums and is idempotent complete. It is given on objects by

$$\text{Tr}_{\mathcal{C}}(\text{“}\Phi(c) \otimes m^{\otimes n}\text{”}) := c \otimes \mathcal{P}[n], \quad (34)$$

and on morphisms by

$$\text{Tr}_{\mathcal{C}}(f) := \text{Diagram} \in \mathcal{C}(a \otimes \mathcal{P}[n], b \otimes \mathcal{P}[p]). \quad (35)$$

The diagram for $\text{Tr}_{\mathcal{C}}(f)$ is a box labeled f with input a and output b . A wire labeled $\mathcal{P}[p + n]$ connects the output of f to a larger box. This larger box contains three circles labeled $0, n, p$ from bottom to top. A red line connects the output of f to the n circle. A wire labeled $\mathcal{P}[n]$ connects the 0 circle to the n circle. The output of the large box is labeled $\mathcal{P}[p]$.

for $f \in \mathcal{M}_0(\Phi(a) \otimes m^{\otimes n}, \Phi(b) \otimes m^{\otimes p}) = \mathcal{C}(a, b \otimes \mathcal{P}[p+n])$. To see that this defines a functor, we check:

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{C}}(g) \circ \mathrm{Tr}_{\mathcal{C}}(f) &= \text{Diagram 1} \\
&= \text{Diagram 2} \\
&= \text{Diagram 3} \\
&= \mathrm{Tr}_{\mathcal{C}}(g \circ f)
\end{aligned}$$

The diagrams are as follows:

- Diagram 1:** A vertical line with inputs a and b at the bottom and output c at the top. Box f (input a , output b) is connected to box g (input b , output c). Box f is connected to a box with two circles (top p , bottom n) via a line labeled $\mathcal{P}[p+n]$. Box g is connected to a box with two circles (top q , bottom p) via a line labeled $\mathcal{P}[q+p]$. The box with circles p, n is connected to a box with circles q, p via a line labeled $\mathcal{P}[p]$. The box with circles q, p is connected to a box with two circles (top q , bottom 0) via a line labeled $\mathcal{P}[q]$. Red lines connect the top circle to the bottom circle in each of the three box pairs.
- Diagram 2:** Similar to Diagram 1, but the box with circles q, p is merged into the box with circles q, n via a line labeled $\mathcal{P}[q]$. The box with circles q, n is connected to the box with circles $q, 0$ via a line labeled $\mathcal{P}[q+n]$.
- Diagram 3:** Similar to Diagram 2, but the box with circles q, n is merged into the box with circles $q, 0$ via a line labeled $\mathcal{P}[n]$.

and

$$\mathrm{Tr}_{\mathcal{C}}(\mathrm{id}_{\Phi(a) \otimes m^{\otimes n}}) = \text{Diagram 4} = \text{Diagram 5} = \mathrm{id}_{a \otimes \mathcal{P}[n]}.$$

The diagrams are as follows:

- Diagram 4:** A vertical line with input a at the bottom and output a at the top. A box with one circle (top n) is connected to a box with two circles (top n , bottom 0) via a line labeled $\mathcal{P}[2n]$. The box with circles $n, 0$ is connected to a box with two circles (top n , bottom 0) via a line labeled $\mathcal{P}[n]$. Red lines connect the top circle to the bottom circle in each of the two box pairs.
- Diagram 5:** A vertical line with input a at the bottom and output a at the top. A box with one circle (top n) is connected to a box with one circle (top n) via a line labeled $\mathcal{P}[n]$. Red lines connect the top circle to the bottom circle in each of the two box pairs.

Lemma 6.1. *The identity map*

$$\mathcal{M}(\Phi(a), \Phi(c) \otimes m^{\otimes n}) = \mathcal{C}(a, c \otimes \mathcal{P}[n]) \longrightarrow \mathcal{C}(a, c \otimes \mathcal{P}[n]) = \mathcal{C}(a, \mathrm{Tr}_{\mathcal{C}}(\Phi(c) \otimes m^{\otimes n}))$$

is natural in $a \in \mathcal{C}$ and in $\Phi(c) \otimes m^{\otimes n} \in \mathcal{M}$. It exhibits $\mathrm{Tr}_{\mathcal{C}} : \mathcal{M}_0 \rightarrow \mathcal{C}$ as the right adjoint of $\Phi : \mathcal{C} \rightarrow \mathcal{M}_0$.

Proof. First, given $f \in \mathcal{C}(a, b)$, we must show that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{M}_0(\Phi(b), \Phi(c) \otimes m^{\otimes n}) = \mathcal{C}(b, c \otimes \mathcal{P}[n]) & \xrightarrow{\mathrm{id}} & \mathcal{C}(b, c \otimes \mathcal{P}[n]) \\
\downarrow - \circ \Phi(f) & & \downarrow - \circ f \\
\mathcal{M}_0(\Phi(a), \Phi(c) \otimes m^{\otimes n}) = \mathcal{C}(a, c \otimes \mathcal{P}[n]) & \xrightarrow{\mathrm{id}} & \mathcal{C}(a, c \otimes \mathcal{P}[n])
\end{array}$$

Starting with g in the upper left corner, we compute:

$$g \circ \Phi(f) = \text{diagram} = g \circ f.$$

Second, given $g \in \mathcal{M}_0(\text{“}\Phi(c) \otimes m^{\otimes n}\text{”}, \text{“}\Phi(d) \otimes m^{\otimes p}\text{”})$, we must show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_0(\Phi(a), {}^{\text{“}}\Phi(c) \otimes m^{\otimes n}\text{”}) := \mathcal{C}(a, c \otimes \mathcal{P}[n]) & \xrightarrow{\text{id}} & \mathcal{C}(a, c \otimes \mathcal{P}[n]) \\ \downarrow g \circ - & & \downarrow \text{Tr}_{\mathcal{C}}(g) \circ - \\ \mathcal{M}_0(\Phi(a), {}^{\text{“}}\Phi(d) \otimes m^{\otimes p}\text{”}) := \mathcal{C}(a, d \otimes \mathcal{P}[p]) & \xrightarrow{\text{id}} & \mathcal{C}(a, d \otimes \mathcal{P}[p]) \end{array}$$

Starting with f in the upper left corner, we compute:

Diagrammatic representation of the composition of two functions g and f . The diagram shows two input wires, a and c , entering boxes f and g respectively. The output of f is connected to the input of g via a wire labeled $\mathcal{P}[p+n]$. The output of g is connected to a larger box labeled $\mathcal{P}[p]$ via a wire labeled $\mathcal{P}[p+n]$. The output of this box is connected to a final box labeled $\mathcal{P}[p]$ via a wire labeled $\mathcal{P}[p]$. The final box is connected to the output wire d via a wire labeled $\mathcal{P}[p+n]$. The diagram is labeled $g \circ f =$ on the left and $= \text{Tr}_{\mathcal{C}}(g) \circ f$ on the right.

1

By the above lemma, we get an adjunction

$$\Phi : \mathcal{C} \xrightarrow{\leftarrow} \mathcal{M}_0 : \mathrm{Tr}_{\mathcal{C}}. \quad (36)$$

The adjunction (32) follows formally. It is obtained from (36) by applying the 2-functor

$$\{\text{linear categories}\} \rightarrow \{\text{additive idempotent complete linear categories}\}, \quad (37)$$

which adjoins finite direct sums and idempotent completes.

6.3 The tensor structure on $\mathcal{M}$

We now endow $\mathcal{M}$ with a monoidal structure. To do so, we first define such a structure on $\mathcal{M}_0$ and then extend it formally to $\mathcal{M}$ by applying of the 2-functor (37). At the level of objects, the monoidal structure is given by

$$\text{“}\Phi(c) \otimes m^{\otimes n_1}\text{”} \otimes \text{“}\Phi(d) \otimes m^{\otimes n_2}\text{”} := \text{“}\Phi(c \otimes d) \otimes m^{\otimes n_2+n_1}\text{”} \quad 1_{\mathcal{M}_0} := \Phi(1_c). \quad (38)$$

The tensor product of morphisms $f_i \in \mathcal{M}_0(\Phi(a_i) \otimes m^{\otimes n_i}, \Phi(b_i) \otimes m^{\otimes p_i})$, $i = 1, 2$, is given by:

$$f_1 \otimes f_2 := \begin{array}{c} \begin{array}{|c|} \hline b_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline b_2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline f_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline f_2 \\ \hline \end{array} \end{array} \begin{array}{c} \mathcal{P}[p_2+n_2] \\ \mathcal{P}[p_1+n_1] \end{array} \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \xrightarrow{\mathcal{P}[\sum p_i + \sum n_i]} \quad (39)$$

Note that tensoring a morphism $f \in \mathcal{M}_0(\Phi(a) \otimes m^{\otimes n}, \Phi(b) \otimes m^{\otimes p})$ with an identity morphism $\text{id}_{\Phi(c) \otimes m^{\otimes q}}$ is computed as follows:

$$\begin{aligned} f \otimes \text{id} &= \begin{array}{c} \begin{array}{|c|} \hline b \\ \hline \end{array} \quad \begin{array}{|c|} \hline c \\ \hline \end{array} \\ \begin{array}{|c|} \hline f \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} \end{array} \xrightarrow{\mathcal{P}[p+2q+n]} = \begin{array}{c} \begin{array}{|c|} \hline b \\ \hline \end{array} \quad \begin{array}{|c|} \hline c \\ \hline \end{array} \\ \begin{array}{|c|} \hline f \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} \end{array} \xrightarrow{\mathcal{P}[p+2q+n]} \\ \text{id} \otimes f &= \begin{array}{c} \begin{array}{|c|} \hline c \\ \hline \end{array} \quad \begin{array}{|c|} \hline b \\ \hline \end{array} \\ \begin{array}{|c|} \hline \text{Diagram 4} \\ \hline \end{array} \quad \begin{array}{|c|} \hline f \\ \hline \end{array} \end{array} \xrightarrow{\mathcal{P}[p+2q+n]} = \begin{array}{c} \begin{array}{|c|} \hline b \\ \hline \end{array} \quad \begin{array}{|c|} \hline c \\ \hline \end{array} \\ \begin{array}{|c|} \hline f \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{Diagram 5} \\ \hline \end{array} \end{array} \xrightarrow{\mathcal{P}[p+2q+n]} \end{aligned}$$

We now show that the above operation defines a functor $\otimes : \mathcal{M}_0 \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$. Given morphisms $f_i \in \mathcal{M}_0(\Phi(a_i) \otimes m^{\otimes n_i}, \Phi(b_i) \otimes m^{\otimes p_i})$ and $g_i \in \mathcal{M}_0(\Phi(b_i) \otimes m^{\otimes p_i}, \Phi(c_i) \otimes m^{\otimes q_i})$, for $i = 1, 2$, we compute:

$$(g_1 \circ f_1) \otimes (g_2 \circ f_2) = \begin{array}{c} \begin{array}{|c|} \hline g_2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline f_2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline g_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline f_1 \\ \hline \end{array} \end{array} \begin{array}{c} \mathcal{P}[q_2+p_2] \\ \mathcal{P}[p_2+n_2] \\ \mathcal{P}[q_1+p_1] \\ \mathcal{P}[p_1+n_1] \end{array} \begin{array}{|c|} \hline \text{Diagram 6} \\ \hline \end{array} \xrightarrow{\mathcal{P}[\sum q_i + \sum n_i]} \quad (39)$$

$$\begin{aligned}
&= \text{Diagram 1} \\
&= \text{Diagram 2} \\
&= \text{Diagram 3} \\
&= (g_1 \otimes g_2) \circ (f_1 \otimes f_2)
\end{aligned}$$

The diagrams are as follows:

- Diagram 1:** A vertical stack of four boxes labeled g_2, f_2, g_1, f_1 from top to bottom. To the right of each box is a label: $\mathcal{P}[q_2+p_2]$, $\mathcal{P}[p_2+n_2]$, $\mathcal{P}[q_1+p_1]$, and $\mathcal{P}[p_1+n_1]$ respectively. These four boxes are connected to a large circle containing four smaller circles labeled q_1, q_2, p_1, p_2 at the top and n_1, n_2 at the bottom. Red lines connect the boxes to the circles: g_2 to q_1, q_2 ; f_2 to p_1, p_2 ; g_1 to q_1, p_1 ; and f_1 to p_1, n_1 . A single line exits the right of the large circle labeled $\mathcal{P}[\sum q_i + \sum n_i]$.
- Diagram 2:** Similar to Diagram 1, but the boxes are rearranged: g_2 and g_1 are on the left, and f_2 and f_1 are on the right. The labels and connections are adjusted accordingly.
- Diagram 3:** The diagram is split into two horizontal sections. The top section has boxes g_2 and g_1 connected to a circle with q_1, q_2, p_1, p_2 . The bottom section has boxes f_2 and f_1 connected to a circle with p_1, p_2, n_1, n_2 . Red lines connect the boxes to the circles. Lines from the two circles connect to a final circle on the right containing n_i, p_i, q_i . Labels $\mathcal{P}[\sum q_i + \sum p_i]$ and $\mathcal{P}[\sum p_i + \sum n_i]$ are on the lines entering the final circle. The final circle has a line exiting labeled $\mathcal{P}[\sum q_i + \sum n_i]$.

where we used the anchor dependence axiom (Definition 2.7) in the third equality. Finally, we verify that $\text{id}_{\Phi(a) \otimes m^{\otimes n}} \otimes \text{id}_{\Phi(b) \otimes m^{\otimes p}} = \text{id}_{\Phi(a) \otimes m^{\otimes n} \otimes \Phi(b) \otimes m^{\otimes p}}$ holds:

$$\left| \begin{array}{c} a \\ \text{Diagram 4} \end{array} \right| = \left| \begin{array}{c} a \otimes b \\ \text{Diagram 5} \end{array} \right|$$

The diagrams are as follows:

- Diagram 4:** A circle with a red dot and label n is connected to a circle with a red dot and label p . These are connected to a large circle containing two smaller circles labeled n and p . Red lines connect the boxes to the circles. A line exits the right of the large circle labeled $\mathcal{P}[n + 2p + n]$.
- Diagram 5:** A circle with a red dot and label $n+p$ is connected to a line labeled $\mathcal{P}[2(n+p)]$.

The associator and unitor isomorphisms of $\mathcal{M}_0$ are inherited from those of $\mathcal{C}$. The associator $\alpha : (\Phi(a) \otimes m^{\otimes n} \otimes \Phi(b) \otimes m^{\otimes p}) \otimes \Phi(c) \otimes m^{\otimes q} \rightarrow \Phi(a) \otimes m^{\otimes n} \otimes (\Phi(b) \otimes$

$$m^{\otimes p''} \otimes \left(\Phi(c) \otimes m^{\otimes q''} \right) \text{ is given by}$$

$$\begin{array}{c}
 a \otimes (b \otimes c) \\
 \hline
 \boxed{\alpha_{a,b,c}} \\
 \hline
 (a \otimes b) \otimes c
 \end{array}
 \quad
 \begin{array}{c}
 \boxed{\boxed{r}} \\
 \hline
 \mathcal{P}[2r]
 \end{array}
 \quad
 r = n + p + q,$$

and the unitors $\lambda: \Phi(1_C) \otimes \text{“}\Phi(a) \otimes m^{\otimes n}\text{”} \rightarrow \text{“}\Phi(a) \otimes m^{\otimes n}\text{”}$ and $\rho: \text{“}\Phi(a) \otimes m^{\otimes n}\text{”} \otimes \Phi(1_C) \rightarrow \text{“}\Phi(a) \otimes m^{\otimes n}\text{”}$ are given by

[illegible]

The associator is natural in its three inputs:

$\alpha \circ ((f_1 \otimes f_2) \otimes f_3)$

$=$

$=$

$=$

$= (f_1 \otimes (f_2 \otimes f_3)) \circ \alpha.$

Similarly, the two unitors λ and ρ are natural (we only show this for λ):

The diagram illustrates the proof of the identity $\lambda \circ (\text{id} \otimes f) = f \circ \lambda$ in the rewriting theory of the lambda-calculus. The proof is presented as a sequence of equalities between string diagrams.

Top Diagram: The expression $\lambda \circ (\text{id} \otimes f)$ is represented by a string diagram. It features a vertical line for the input a (labeled 1_C) and a vertical line for the input b . The input b passes through a box labeled λ . The input a passes through a box labeled f . The output of f is labeled $\mathcal{P}[p+n]$. This output then enters a large box containing a smaller box labeled λ and a circle labeled p . The output of the large box is labeled $\mathcal{P}[p+n]$.

Middle Diagram: The expression is rewritten as $f \circ \lambda$. This is shown by a string diagram where the input a passes through a box labeled f , and the input b passes through a box labeled λ . The output of f is labeled $\mathcal{P}[p+n]$. The output of λ is labeled $\mathcal{P}[p+n]$.

Bottom Diagram: The expression is further rewritten as $f \circ \lambda$. This is shown by a string diagram where the input a passes through a box labeled f , and the input b passes through a box labeled λ . The output of f is labeled $\mathcal{P}[p+n]$. The output of λ is labeled $\mathcal{P}[p+n]$.

Note that the fourth equality follows from the relation $\lambda_{a \otimes c} \circ \alpha_{1_c, a, c} = \lambda_a \otimes \text{id}_c$, which holds true in any tensor category.

Finally, we verify that the pentagon and triangle axioms are satisfied:

The diagram illustrates the proof of the associativity of composition for comonoids. It consists of two parts, each showing an equality between two string diagrams.

Top part: The left side represents the expression $(\text{id} \otimes \alpha) \circ \alpha \circ (\alpha \otimes \text{id})$. The string diagram shows three input wires. The top wire passes through an identity box. The middle and bottom wires pass through a box labeled α . The top and middle wires then pass through another box labeled α . The bottom wire passes through a box labeled α . The output is a single wire. The right side represents the expression $\alpha \circ \alpha$, which is a single box labeled α with two input wires.

Bottom part: The left side represents the expression $(\text{id} \otimes \lambda) \circ \alpha$. The string diagram shows two input wires. The top wire passes through an identity box. The bottom wire passes through a box labeled α . The top and bottom wires then pass through a box labeled λ . The output is a single wire. The right side represents the expression $\rho \otimes \text{id}$, which is a box labeled ρ with two input wires.

This finishes the proof that $\mathcal{M}_0$, and thus $\mathcal{M}$, is a tensor category.

Note that, by (29), (33) and (38), there is a canonical isomorphism

$$“\Phi(c) \otimes m^{\otimes n}” \cong \Phi(c) \otimes m^{\otimes n}.$$

We may therefore blur the distinction between the formal symbol “ $\Phi(c) \otimes m^{\otimes n}$ ” and the tensor product $\Phi(c) \otimes m^{\otimes n}$.

We also note that $\Phi : \mathcal{C} \rightarrow \mathcal{M}_0$ (and therefore $\Phi : \mathcal{C} \rightarrow \mathcal{M}$) is visibly a tensor functor.

6.4 The pivotal structure

The goal of this section is to show that the category $\mathcal{M}$ is pivotal, and that the object $m \in \mathcal{M}$ is symmetrically self-dual. Recall that the balanced structure on $\mathcal{C}$ was defined in (4). We denote the twist and its inverse graphically by:

$$\theta_a = \left. \vphantom{\int} \right\}_{a^+}^a \quad \text{and} \quad \theta_a^{-1} = \left. \vphantom{\int} \right\}_a^{a^+}.$$

The rigidity of $\mathcal{C}$ implies the corresponding property for $\mathcal{M}_0$ (which then implies that $\mathcal{M}$ is rigid):

Lemma 6.2. *For any $c \in \mathcal{C}$, $n \geq 0$, the morphisms*

$$\text{ev}_{\Phi(c) \otimes m \otimes n} := \text{ev}_c \text{ (diagram)} \quad \text{and} \quad \text{coev}_{\Phi(c) \otimes m \otimes n} := \text{coev}_c \text{ (diagram)}$$

exhibit $\Phi(c^\vee) \otimes m^{\otimes n}$ as the dual object of $\Phi(c) \otimes m^{\otimes n}$ in $\mathcal{M}_0$.

Proof. We need to show the two zig-zag equations. The first one is as follows:

$$(\mathrm{id} \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes \mathrm{id}) = c \left| \begin{array}{c} \text{diagram} \end{array} \right. = c \left| \begin{array}{c} \text{diagram} \end{array} \right. = \mathrm{id}.$$

The other one is similar.

From now on, we identify $(\Phi(c) \otimes m^{\otimes n})^\sim$ with $\Phi(c^\sim) \otimes m^{\otimes n}$ by the above lemma. We now equip $\mathcal{M}_0$ (and hence $\mathcal{M}$) with a pivotal structure:

$$\varphi_{\Phi(a) \otimes m^{\otimes n}} : \Phi(a) \otimes m^{\otimes n} \rightarrow (\Phi(a) \otimes m^{\otimes n})^{\vee\vee} = \Phi(a^{\vee\vee}) \otimes m^{\otimes n}$$

$$\varphi_{\Phi(a) \otimes m^{\otimes n}} := \text{diagram} = \text{diagram} \quad (40)$$

Lemma 6.3. *The isomorphism φ is natural.*

Proof. Let $f : \Phi(a) \otimes m^n \rightarrow \Phi(b) \otimes m^p$ be a morphism in $\mathcal{M}_0$. We must show that

$$\varphi \circ f =$$

is equal to $f^{\sim\sim} \circ \varphi$, where $f^{\sim\sim} : \Phi(a^{\sim\sim}) \otimes m^n \rightarrow \Phi(b^{\sim\sim}) \otimes m^p$ is the double dual of f :

$$f^{\sim\sim} = (\text{ev} \otimes \text{id}) \circ (\text{id} \otimes \text{ev} \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes f \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \text{coev} \otimes \text{id}) \circ (\text{id} \otimes \text{coev}).$$

We compute:

$$f^{\sim\sim} \circ \varphi =$$

where the last equality follows from the compatibility between the full braiding and the twist, valid in any balanced tensor category. $\square$

Lemma 6.4. *The isomorphism φ is monoidal: $\varphi_a \otimes \varphi_b = \varphi_{a \otimes b}$.*

Proof. We compute:

$$\begin{array}{c} a^{\sim\sim} \\ | \\ \boxed{\varphi_a} \quad \boxed{n} \end{array} \quad \begin{array}{c} b^{\sim\sim} \\ | \\ \boxed{\varphi_b} \quad \boxed{p} \end{array} \quad \boxed{\begin{array}{c} n \\ p \end{array}} = \begin{array}{c} (a \otimes b)^{\sim\sim} \\ | \\ \boxed{\varphi_{a \otimes b}} \quad \boxed{n+p} \end{array} \quad \begin{array}{c} a \otimes b \\ | \\ \boxed{n+p} \end{array}.$$

$\square$

The object m is visibly self-dual. Its evaluation of coevaluations maps are given by:

$$\text{ev}_m = \text{1}_c \quad \boxed{\begin{array}{c} n \\ n \end{array}} \quad \text{P}[2] \quad \text{and} \quad \text{coev}_m = \text{1}_c \quad \boxed{\begin{array}{c} n \\ n \end{array}} \quad \text{P}[2]. \quad (41)$$

We now show that m is symmetrically self-dual (Section 3.1). Upon identifying $m^\sim$ with m by means of (41), we let $\psi : m \rightarrow m^\sim$ be the identity map. In the same way, we identify $m^{\sim\sim}$ with $m^\sim$ and with m . The isomorphism $\varphi : m \rightarrow m^{\sim\sim}$ described in (40) is then also the identity map, as $\varphi_1 = 1$ in $\mathcal{C}$. Finally, letting $\psi^\sim$ be as in (10), we verify that the desired equation $\psi^\sim \circ \varphi = \psi$ holds true:

$$\psi^\sim = \text{1}_c \quad \boxed{\begin{array}{c} n \\ n \end{array}} \quad \text{P}[2] = \boxed{\begin{array}{c} n \\ n \end{array}} \quad \text{P}[2] = \boxed{\begin{array}{c} n \\ n \end{array}} \quad \text{P}[2]. \quad (42)$$

6.5 The central functor

In this section, we equip the functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ with the structure of a central functor, i.e., we provide a factorization

$$\begin{array}{ccc} & \mathcal{Z}(\mathcal{M}) & \\ \Phi^{\mathcal{Z}} \nearrow & & \searrow F \\ \mathcal{C} & \xrightarrow{\Phi} & \mathcal{M}. \end{array}$$

To do so, we associate to each object $c \in \mathcal{C}$ a half-braiding $e_{\Phi(c)}$, so that $(\Phi(c), e_{\Phi(c)})$ is an object of $\mathcal{Z}(\mathcal{M})$. Once again, it is enough to do this at the level of the category $\mathcal{M}_0$. For $c \in \mathcal{C}$ and $\Phi(a) \otimes m^{\otimes n} \in \mathcal{M}_0$, we define:

$$e_{\Phi(c), \Phi(a) \otimes m^{\otimes n}} := \Phi(\beta_{c,a}) \otimes \text{id}_{m^{\otimes n}} = \begin{array}{c} a \\ \diagdown \\ c \end{array} \quad \boxed{\begin{array}{c} n \\ n \end{array}} \quad \text{P}[2n]. \quad (43)$$

Lemma 6.5. *The isomorphisms $e_{\Phi(c)}$ are natural.*

Proof. Given a morphism $f : \Phi(a) \otimes m^{\otimes n} \rightarrow \Phi(b) \otimes m^{\otimes p}$ in $\mathcal{M}_0$, we compute:

$$\begin{aligned}
 (f \otimes \text{id}) \circ e_{\Phi(c), \Phi(a) \otimes m^{\otimes n}} &= \text{Diagram 1} \\
 &= \text{Diagram 2} \\
 &= e_{\Phi(c), \Phi(b) \otimes m^{\otimes p}} \circ (\text{id} \otimes f). \quad \square
 \end{aligned}$$

The diagrams represent string diagrams in a monoidal category. Diagram 1 shows a box labeled f with inputs a and c and output b , followed by a box with two circles labeled n and p , and a larger box with two circles labeled n and p . Diagram 2 shows the same components but with the boxes rearranged to show the naturality of the isomorphism $e_{\Phi(c)}$.

Lemma 6.6. *The isomorphisms $e_{\Phi(a)}$ satisfy the hexagon axiom*

$$(\text{id}_{\Phi(b) \otimes m^{\otimes n}} \otimes e_{\Phi(a), \Phi(c) \otimes m^{\otimes p}}) \circ (e_{\Phi(a), \Phi(b) \otimes m^{\otimes n}} \otimes \text{id}_{\Phi(c) \otimes m^{\otimes p}}) = e_{\Phi(a), \Phi(b \otimes c) \otimes m^{\otimes n+p}}.$$

Proof. We compute:

$$\text{Diagram 1} = \text{Diagram 2}$$

The diagrams represent string diagrams in a monoidal category. Diagram 1 shows a box labeled f with inputs a and c and output b , followed by a box with two circles labeled n and p , and a larger box with two circles labeled $n+p$ and $n+p$. Diagram 2 shows the same components but with the boxes rearranged to show the hexagon axiom.

We can now define the functor $\Phi^{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M})$. It is given by $\Phi^{\mathcal{Z}}(c) := (\Phi(c), e_{\Phi(c)})$ on objects, and

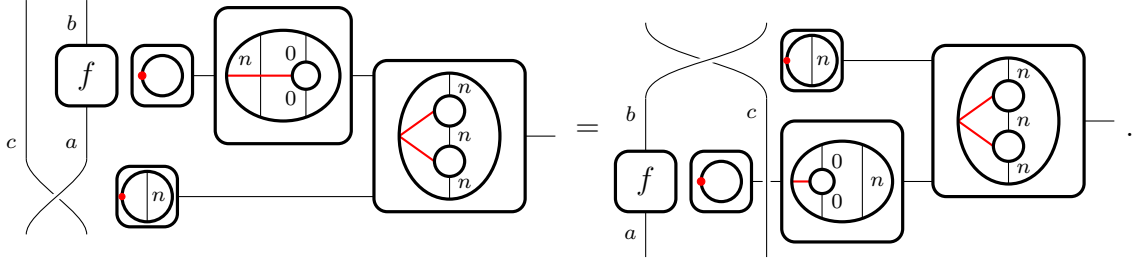
$$\Phi^{\mathcal{Z}}(f) := \Phi(f) = \text{Diagram 1}$$

The diagram shows a box labeled f with inputs a and b and output c , followed by a box with two circles labeled n and p , and a larger box with two circles labeled $n+p$ and $n+p$.

on morphisms. To see that $\Phi^{\mathcal{Z}}(f) : \Phi^{\mathcal{Z}}(a) \rightarrow \Phi^{\mathcal{Z}}(b)$ is a morphism in $\mathcal{Z}(\mathcal{M})$, i.e. that the equation

$$(\text{id} \otimes f) \circ e_{\Phi(a), \Phi(c) \otimes m^{\otimes n}} = e_{\Phi(b), \Phi(c) \otimes m^{\otimes n}} \circ (f \otimes \text{id})$$

holds for any $\Phi(c) \otimes m^{\otimes n} \in \mathcal{M}_0$, we verify:

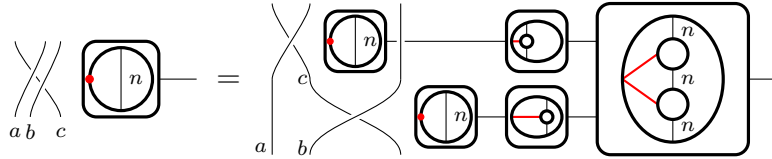


The equations $\Phi^{\mathcal{Z}}(g \circ f) = \Phi^{\mathcal{Z}}(g) \circ \Phi^{\mathcal{Z}}(f)$ and $\Phi^{\mathcal{Z}}(\text{id}) = \text{id}$ are trivial to check, and so we get a functor $\Phi^{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M})$.

Our next task is to show that $\Phi^{\mathcal{Z}}$ is a tensor functor. Note that, by (38), the functor $\Phi : \mathcal{C} \rightarrow \mathcal{M}_0$ is a strict tensor functor:

$$\Phi(a \otimes b) = \Phi(a) \otimes \Phi(b), \quad \Phi(1) = 1.$$

The half-braiding of $\Phi^{\mathcal{Z}}(a \otimes b)$ is identical to that of $\Phi^{\mathcal{Z}}(a) \otimes \Phi^{\mathcal{Z}}(b)$:



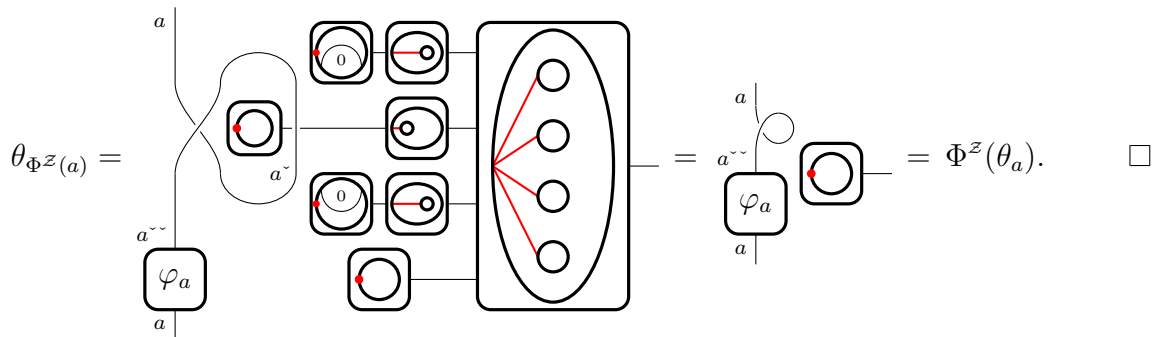
and the half-braiding on $\Phi^{\mathcal{Z}}(1)$ is trivial. It follows that $\Phi^{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M}_0)$, and hence $\Phi^{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{M})$, are strict tensor functors.

Lemma 6.7. *The functor $\Phi^{\mathcal{Z}}$ is braided.*

Proof. For all $a, b \in \mathcal{C}$, we have $\Phi^{\mathcal{Z}}(\beta_{a,b}) = \begin{array}{c} b \\ \diagdown \\ a \end{array} \begin{array}{c} \diagup \\ b \end{array} \begin{array}{c} \diagdown \\ a \end{array} \begin{array}{c} \diagup \\ b \end{array} = e_{\Phi(a), \Phi(b)}.$ □

Lemma 6.8. *The functor $\Phi^{\mathcal{Z}}$ is compatible with the twists: $\theta_{\Phi^{\mathcal{Z}}(a)} = \Phi^{\mathcal{Z}}(\theta_a)$.*

Proof. We have



This finishes the construction of the module tensor category $\mathcal{M}$. We sum up the results of this section in the following theorem:

Theorem 6.9. *Given an anchored planar algebra $\mathcal{P}$ in a braided pivotal category $\mathcal{C}$, we can construct a pair $(\mathcal{M}, m)$ where $\mathcal{M}$ is a pivotal module tensor category, and $m \in \mathcal{M}$ is a symmetrically self-dual object. Moreover, the functor $\Phi = F \circ \Phi^z : \mathcal{C} \rightarrow \mathcal{M}$ admits a right adjoint $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$.*

We denote by $\Delta(\mathcal{P})$ the pointed module tensor category associated to the anchored planar algebra $\mathcal{P}$ by means of the above construction.

6.6 Functoriality

In Theorem 6.9, given an anchored planar algebra, we constructed a pointed module tensor category $\Delta(\mathcal{P})$. Our next goal is to upgrade this to a functor

$$\Delta : \text{APA} \rightarrow \text{ModTens}_*$$

between the category of anchored planar algebras in $\mathcal{C}$ and that of pointed pivotal module tensor categories over $\mathcal{C}$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be anchored planar algebras, and let $\mathcal{M}_1 := \Delta(\mathcal{P}_1)$ and $\mathcal{M}_2 := \Delta(\mathcal{P}_2)$ be the associated pointed module tensor categories. Given a morphism $H : \mathcal{P}_1 \rightarrow \mathcal{P}_2$, we describe the associated functor $\Delta(H) = (\Delta(H), \gamma^{\Delta(H)}) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$. It is enough to give $\Delta(H)$ on the full subcategory of $\mathcal{M}_1$ with objects of the form $\Phi_1(c) \otimes m_1^{\otimes n}$. We let

$$\Delta(H)(\Phi_1(c) \otimes m_1^{\otimes n}) := \Phi_2(c) \otimes m_2^{\otimes n} \quad (44)$$

and $(\gamma^{\Delta(H)})_c := \text{id} : \Phi_2(c) \rightarrow G(\Phi_1(c))$. The image of a morphism $f \in \mathcal{M}_1(\Phi(a) \otimes m_1^{\otimes k}, \Phi(b) \otimes m_1^{\otimes n}) := \mathcal{C}(a, b \otimes \mathcal{P}[n+k])$ is then given by $\Delta(H)(f) := (\text{id}_b \otimes H[n+k]) \circ f$:

$$\Delta(H)(f) = \Delta(H) \left(\begin{array}{c} b \\ | \\ \boxed{f} \\ | \\ a \end{array} \begin{array}{c} \xrightarrow{\mathcal{P}_1[n+k]} \end{array} \right) := \begin{array}{c} b \\ | \\ \boxed{f} \\ | \\ a \end{array} \begin{array}{c} \xrightarrow{\mathcal{P}_1[n+k]} \end{array} \boxed{H} \begin{array}{c} \xrightarrow{\mathcal{P}_2[n+k]} \end{array}. \quad (45)$$

To see that $\Delta(H)$ is a functor, we use that H is a map of anchored planar algebras, and thus commutes with the action of tangles:

$$\begin{aligned} \Delta(H)(g \circ f) &= \begin{array}{c} \boxed{g} \\ \boxed{f} \end{array} \begin{array}{c} \text{tangle} \end{array} \boxed{H} = \begin{array}{c} \boxed{g} \boxed{H} \\ \boxed{f} \boxed{H} \end{array} \begin{array}{c} \text{tangle} \end{array} = \Delta(H)(g) \circ \Delta(H)(f) \\ \Delta(H)(\text{id}_{\Phi_1(c) \otimes m_1^{\otimes n}}) &= c \left| \begin{array}{c} \boxed{n} \\ \text{red dot} \end{array} \begin{array}{c} \xrightarrow{\mathcal{P}_1[2n]} \end{array} \boxed{H} \xrightarrow{\mathcal{P}_2[2n]} \right| = c \left| \begin{array}{c} \boxed{n} \\ \text{red dot} \end{array} \begin{array}{c} \xrightarrow{\mathcal{P}_2[2n]} \end{array} \right| = \text{id}_{\Phi_2(c) \otimes m_2^{\otimes n}}. \end{aligned}$$

The functor $\Delta(H)$ is a tensor functor because $\Delta(H)(1_{\mathcal{M}_1}) = \Delta(H)(\Phi_1(1_{\mathcal{C}})) = \Phi_2(1_{\mathcal{C}}) = 1_{\mathcal{M}_2}$ and

$$\begin{aligned} \Delta(H)((\Phi_1(c) \otimes m_1^{\otimes n}) \otimes (\Phi_1(d) \otimes m_1^{\otimes n'})) &= \Delta(H)(\Phi_1(c \otimes d) \otimes m_1^{\otimes n+n'}) \\ &= \Phi_2(c \otimes d) \otimes m_2^{\otimes n+n'} = (\Phi_2(c) \otimes m_2^{\otimes n}) \otimes (\Phi_2(d) \otimes m_2^{\otimes n'}) \\ &= \Delta(H)(\Phi_1(c) \otimes m_1^{\otimes n}) \otimes \Delta(H)(\Phi_1(d) \otimes m_1^{\otimes n'}), \end{aligned}$$

and because:

$$\begin{aligned}
\Delta(H)(f \otimes g) &= \text{Diagram 1} \\
&= \text{Diagram 2} \\
&= \text{Diagram 3} = \Delta(H)(f) \otimes \Delta(H)(g).
\end{aligned}$$

It is pivotal because

$$\Delta(H)(\varphi_{\Phi_1(c) \otimes m_1^{\otimes n}}) = \text{Diagram 1} = \text{Diagram 2} = \varphi_{\Phi_2(c) \otimes m_2^{\otimes n}}.$$

$\Delta(H)$ is compatible with the half-braidings, and thus a functor of module tensor categories (Definition 3.2):

$$\Delta(H)(e_{\Phi_1(c), \Phi_1(a) \otimes m_1^{\otimes n}}) = \text{Diagram 1} = \text{Diagram 2} = e_{\Phi_2(c), \Phi_2(a) \otimes m_2^{\otimes n}}.$$

Finally, it is a functor of pointed module tensor categories (Definition 3.4) as it sends m_1 to m_2 and the condition $\psi_2 = \delta_{m_1} \circ \Delta(H)(\psi_1)$ holds true (both sides are identity morphisms).

It remains to show that the assignment $H \mapsto \Delta(H)$ is a functor, i.e., that the relations $\Delta(H_1 \circ H_2) = \Delta(H_1) \circ \Delta(H_2)$ and $\Delta(\text{id}) = \text{id}$ are satisfied. Both are obvious from the definition (45) of Δ . We have shown:

Theorem 6.10. *Let $\mathcal{C}$ be a braided pivotal category which admits finite direct sums and is idempotent complete. Then the map*

$$\Delta : \text{APA} \rightarrow \text{ModTens}_*$$

given by Theorem 6.9 at the level of objects and by (44) and (45) at the level of morphisms is a functor from the category of anchored planar algebras in $\mathcal{C}$ to the (2-)category of pointed pivotal module tensor categories over $\mathcal{C}$.

Note that we may compose Δ with the projection $\text{ModTens}_* \xrightarrow{\cong} \tau_{\leq 1}(\text{ModTens}_*)$ to get a functor $\text{APA} \rightarrow \tau_{\leq 1}(\text{ModTens}_*)$.

7 Equivalence of categories

In this section, we complete the proof of Theorem 3.7 and show that the functors

$$\Lambda : \text{ModTens}_* \rightleftarrows \text{APA} : \Delta$$

witness an equivalence of (2-)categories. In view of Lemma 3.6, this is equivalent to the functors $\Lambda : \tau_{\leq 1}(\text{ModTens}_*) \rightleftarrows \text{APA} : \Delta$ witnessing an equivalence of ordinary categories.

The proof is in two steps. In Sections 7.1 and 7.2 we construct a natural isomorphism $\Delta \circ \Lambda \Rightarrow \text{id}$, and in Section 7.3 we show that $\Lambda \circ \Delta = \text{id}$.

7.1 Module tensor categories to planar algebras and back

Let $(\mathcal{M}, m)$ be a pointed module tensor category, and let $(\mathcal{M}', m')$ be its image under $\Delta \circ \Lambda$. We wish to construct an equivalence

$$\Psi = (\Psi, \gamma^\Psi) : (\mathcal{M}', m') \rightarrow (\mathcal{M}, m)$$

(an isomorphism in $\tau_{\leq 1}(\text{ModTens}_*)$).

Let $\Phi : \mathcal{C} \rightarrow \mathcal{M}$ and $\Phi' : \mathcal{C} \rightarrow \mathcal{M}'$ be the action functors of $\mathcal{M}$ and of $\mathcal{M}'$. Since $\mathcal{M}$ admits direct sums and is idempotent complete, it is enough to define Ψ on the full subcategory $\mathcal{M}'_0 \subset \mathcal{M}'$ with objects $\Phi'(c) \otimes m'^{\otimes n}$. At the level of objects, the functor $\Psi : \mathcal{M}' \rightarrow \mathcal{M}$ is given by:

$$\Psi(\Phi'(c) \otimes m'^{\otimes n}) := \Phi(c) \otimes m^{\otimes n}. \quad (46)$$

At the level of morphisms, the image of

$$f \in \mathcal{M}'(\Phi'(a) \otimes m'^{\otimes i}, \Phi'(b) \otimes m'^{\otimes j}) = \mathcal{C}(a, b \otimes \mathcal{P}[j+i]) = \mathcal{C}(a, b \otimes \text{Tr}_{\mathcal{C}}(m^{\otimes j+i}))$$

under Ψ is given by:

$$\begin{aligned} \Psi(f) &:= \text{Diagram} \\ &= (\text{id}_{\Phi(b)} \otimes \text{id}_{m^{\otimes j}} \otimes \tilde{\text{ev}}_{m^{\otimes i}}) \circ (\text{id}_{\Phi(b)} \otimes \varepsilon_{m^{\otimes i+j}} \otimes \text{id}_{m^{\otimes i}}) \circ (\Phi(f) \otimes \text{id}_{m^{\otimes i}}), \end{aligned} \quad (47)$$

where we have written $\mathcal{P}$ for $\Lambda(\mathcal{M})$. The action coherence isomorphisms $(\gamma^\Psi)_c : \Phi(c) \rightarrow \Psi\Phi'(c)$ are all identity morphisms.

Note that, at this point, it is not obvious that Ψ is a functor. Nevertheless, it is easy to see that, if Ψ is a functor, then it is actually an equivalence of categories:

Lemma 7.1. *If $\Psi : \mathcal{M}' \rightarrow \mathcal{M}$ is a functor, then it is an equivalence of categories.*

Proof. We first show that $\Psi|_{\mathcal{M}'_0}$ is fully faithful. This holds true because the image $\Psi(f)$ of a morphism f is also its image under the following sequence of isomorphisms:

$$\begin{aligned}
\mathcal{M}'(\Phi'(a) \otimes m'^{\otimes k}, \Phi'(b) \otimes m'^{\otimes n}) &= \mathcal{C}(a, b \otimes \mathcal{P}[n+k]) \\
&= \mathcal{C}(a, b \otimes \text{Tr}_{\mathcal{C}}(m^{\otimes n+k})) \\
&\cong \mathcal{C}(b^\vee \otimes a, \text{Tr}_{\mathcal{C}}(m^{\otimes n+k})) \\
&\cong \mathcal{M}(\Phi(b^\vee \otimes a), m^{\otimes n+k}) \\
&\cong \mathcal{M}(\Phi(b)^\vee \otimes \Phi(a), m^{\otimes n} \otimes m^{\otimes k}) \\
&\cong \mathcal{M}(\Phi(a) \otimes m^{\otimes k}, \Phi(b) \otimes m^{\otimes n}).
\end{aligned}$$

This implies that Ψ is also fully faithful. Finally, the functor Ψ is essentially surjective because m generates $\mathcal{M}$ (Definition 3.4). $\square$

We now show that Ψ is a functor:

Lemma 7.2. Ψ preserves composition of morphisms: $\Psi(g) \circ \Psi(f) = \Psi(g \circ f)$.

Proof. It is enough to check this condition when f and g are morphisms in $\mathcal{M}'_0$. Let $f : \Phi'(a) \otimes m'^{\otimes i} \rightarrow \Phi'(b) \otimes m'^{\otimes j}$ and $g : \Phi'(b) \otimes m'^{\otimes j} \rightarrow \Phi'(c) \otimes m'^{\otimes k}$ be composable morphisms. Then

$$\begin{aligned}
\Psi(g) \circ \Psi(f) &= \text{Diagram 1} \\
&= \text{Diagram 2} \\
&= \text{Diagram 3} \quad [\text{HPT16, Lem. 4.6}] \\
&= \text{Diagram 4} \quad (\text{naturality of } \varepsilon).
\end{aligned}$$

We need to show that this is equal to the image of

$$g \circ f \in \mathcal{M}'(\Phi'(a) \otimes m'^{\otimes i}, \Phi'(c) \otimes m'^{\otimes k}) = \mathcal{C}(a, c \otimes \mathcal{P}[i+k])$$

under Ψ , where $\mathcal{P} = \Lambda(\mathcal{M}, m)$ and $g \circ f$ is as in (30). By chasing through the various definitions (Theorem 5.1 and Algorithm 4.9) one verifies that the value of $\mathcal{P} = (\mathcal{P}, Z)$ on the multiplication tangle is given by

$$Z\left(\begin{array}{c} \text{Diagram: A circle with two internal circles. The top circle has a red line from its left side to the left side of the bottom circle. The bottom circle has a red line from its left side to the left side of the top circle. The top circle is labeled with 'k' and the bottom circle with 'i'. The left side of the top circle is labeled 'j'.$$

It follows that $\Psi(g \circ f) = \Psi(g) \circ \Psi(f)$, as desired. $\square$

Lemma 7.3. Ψ preserves identity morphisms: $\Psi(\text{id}_X) = \text{id}_{\Psi(X)}$.

Proof. It is enough to check this condition when X is an object of $\mathcal{M}'_0$. The identity morphism $\text{id}_{\Phi'(a) \otimes m'^{\otimes n}}$ is equal to the morphism in $\mathcal{C}$ given by

$$a \quad \left| \quad \begin{array}{c} \text{Diagram: A circle with a vertical line through its center. The left side of the circle is labeled 'a' and the right side is labeled 'n'. The circle is labeled 'P[2n]'. \right| \in \mathcal{C}(a, a \otimes \mathcal{P}[2n]).$$

By definition, $\Psi(\text{id}_{\Phi'(a) \otimes m'^{\otimes n}})$ is then given by:

$$\begin{array}{c} \text{Diagram: A horizontal line labeled 'a' above a parallelogram. Inside the parallelogram is a box labeled 'ε' with a vertical line through its center. The left side of the box is labeled 'n' and the right side is labeled 'n'. The box is labeled 'ε'.$$

$\square$

We now wish to endow Ψ with the structure of a tensor functor. It is enough to do this on $\Psi|_{\mathcal{M}'_0} : \mathcal{M}'_0 \rightarrow \mathcal{M}$. Recall that, by definition, $(\Phi'(a) \otimes m'^{\otimes k}) \otimes (\Phi'(b) \otimes m'^{\otimes n}) = \Phi'(a \otimes b) \otimes m'^{\otimes k+n}$, and $1_{\mathcal{M}'} = \Phi'(1_{\mathcal{C}})$. So we must provide natural isomorphisms

$$\Psi(\Phi'(a) \otimes m'^{\otimes k}) \otimes \Psi(\Phi'(b) \otimes m'^{\otimes n}) \rightarrow \Psi(\Phi'(a \otimes b) \otimes m'^{\otimes k+n}), \quad 1_{\mathcal{M}} \rightarrow \Psi(\Phi'(1_{\mathcal{C}})),$$

subject to the usual associativity and unitality conditions. We let the former be the composite

$$\begin{aligned} (\Phi(a) \otimes m^{\otimes k}) \otimes (\Phi(b) \otimes m^{\otimes n}) &\rightarrow \Phi(a) \otimes (m^{\otimes k} \otimes \Phi(b)) \otimes m^{\otimes n} \\ &\xrightarrow{e^{-1}_{\Phi(b), m^{\otimes k}}} \Phi(a) \otimes (\Phi(b) \otimes m^{\otimes k}) \otimes m^{\otimes n} \rightarrow \Phi(a \otimes b) \otimes m^{\otimes k+n}, \end{aligned}$$

and the latter be the canonical isomorphism $1_{\mathcal{M}} \rightarrow \Phi(1_{\mathcal{C}})$.

Proposition 7.4. *The above isomorphisms equip $\Psi : \mathcal{M}' \rightarrow \mathcal{M}$ with the structure of a tensor functor.*

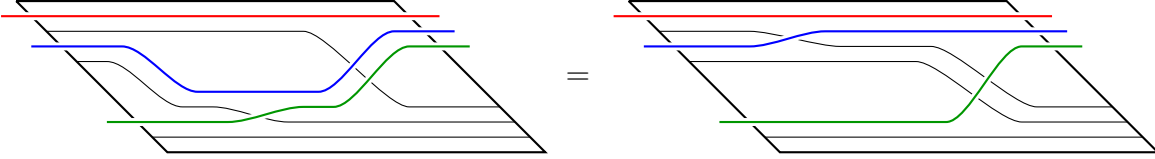
Proof. It is enough to check that $\Psi|_{\mathcal{M}'_0}$ is a tensor functor. Let $X = \Phi'(a) \otimes m'^{\otimes n}$, $Y = \Phi'(b) \otimes m'^{\otimes p}$, $Z = \Phi'(c) \otimes m'^{\otimes q}$ be objects. The associativity condition

$$\begin{array}{ccccc} (\Psi(X) \otimes \Psi(Y)) \otimes \Psi(Z) & \longrightarrow & \Psi(X \otimes Y) \otimes \Psi(Z) & \longrightarrow & \Psi((X \otimes Y) \otimes Z) \\ \alpha \downarrow & & & & \downarrow \alpha \\ \Psi(X) \otimes (\Psi(Y) \otimes \Psi(Z)) & \longrightarrow & \Psi(X) \otimes \Psi(Y \otimes Z) & \longrightarrow & \Psi(X \otimes (Y \otimes Z)) \end{array}$$

can be rewritten as:

$$\begin{array}{ccccc} ((\Phi(a) \otimes m^{\otimes n}) \otimes (\Phi(b) \otimes m^{\otimes p})) \otimes (\Phi(c) \otimes m^{\otimes q}) & \longrightarrow & \Phi((a \otimes b) \otimes m'^{\otimes n+p}) \otimes (\Phi(c) \otimes m^{\otimes q}) & \longrightarrow & \Phi((a \otimes b) \otimes c) \otimes m^{\otimes n+p+q} \\ \alpha \downarrow & & & & \downarrow \alpha \\ (\Phi(a) \otimes m^{\otimes n}) \otimes ((\Phi(b) \otimes m^{\otimes p}) \otimes (\Phi(c) \otimes m^{\otimes q})) & \longrightarrow & (\Phi(a) \otimes m^{\otimes n}) \otimes (\Phi(b \otimes c) \otimes m^{\otimes p+q}) & \longrightarrow & \Phi(a \otimes (b \otimes c)) \otimes m^{\otimes n+p+q} \end{array}$$

This diagram is commutative because of the following equality of morphisms in $\mathcal{M}$:



The unitality conditions

$$\begin{array}{ccc} \Psi(1_{\mathcal{M}'} \otimes \Psi(X)) \longleftarrow 1_{\mathcal{M}} \otimes \Psi(X) & & \Psi(X) \otimes 1_{\mathcal{M}} \longrightarrow \Psi(X) \otimes \Psi(1_{\mathcal{M}'}) \\ \downarrow & \text{and} & \downarrow \\ \Psi(1_{\mathcal{M}'} \otimes X) \longrightarrow \Psi(X) & & \Psi(X) \longleftarrow \Psi(X \otimes 1_{\mathcal{M}'}) \end{array}$$

are left as an exercise to the reader.

It remains to show that the map $\Psi(X) \otimes \Psi(Y) \rightarrow \Psi(X \otimes Y)$ is natural in X and Y . Given two morphisms in $\mathcal{M}'$ (which we may take with out loss of generality in $\mathcal{M}'_0$)

$$f : \Phi'(a) \otimes m'^{\otimes i} \rightarrow \Phi'(b) \otimes m'^{\otimes j}, \quad g : \Phi'(c) \otimes m'^{\otimes k} \rightarrow \Phi'(d) \otimes m'^{\otimes \ell},$$

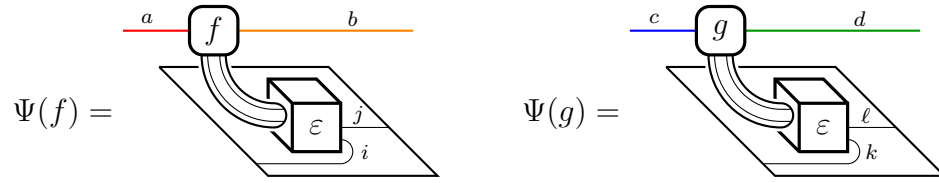
we need to argue that the following diagram commutes:

$$\begin{array}{ccc} \Psi(\Phi'(a) \otimes m'^{\otimes i}) \otimes \Psi(\Phi'(c) \otimes m'^{\otimes k}) & \longrightarrow & \Psi((\Phi'(a) \otimes m'^{\otimes i}) \otimes (\Phi'(c) \otimes m'^{\otimes k})) \\ \Psi(f) \otimes \Psi(g) \downarrow & & \downarrow \Psi(f \otimes g) \\ \Psi(\Phi'(b) \otimes m'^{\otimes j}) \otimes \Psi(\Phi'(d) \otimes m'^{\otimes \ell}) & \longrightarrow & \Psi((\Phi'(b) \otimes m'^{\otimes j}) \otimes (\Phi'(d) \otimes m'^{\otimes \ell})) \end{array}$$

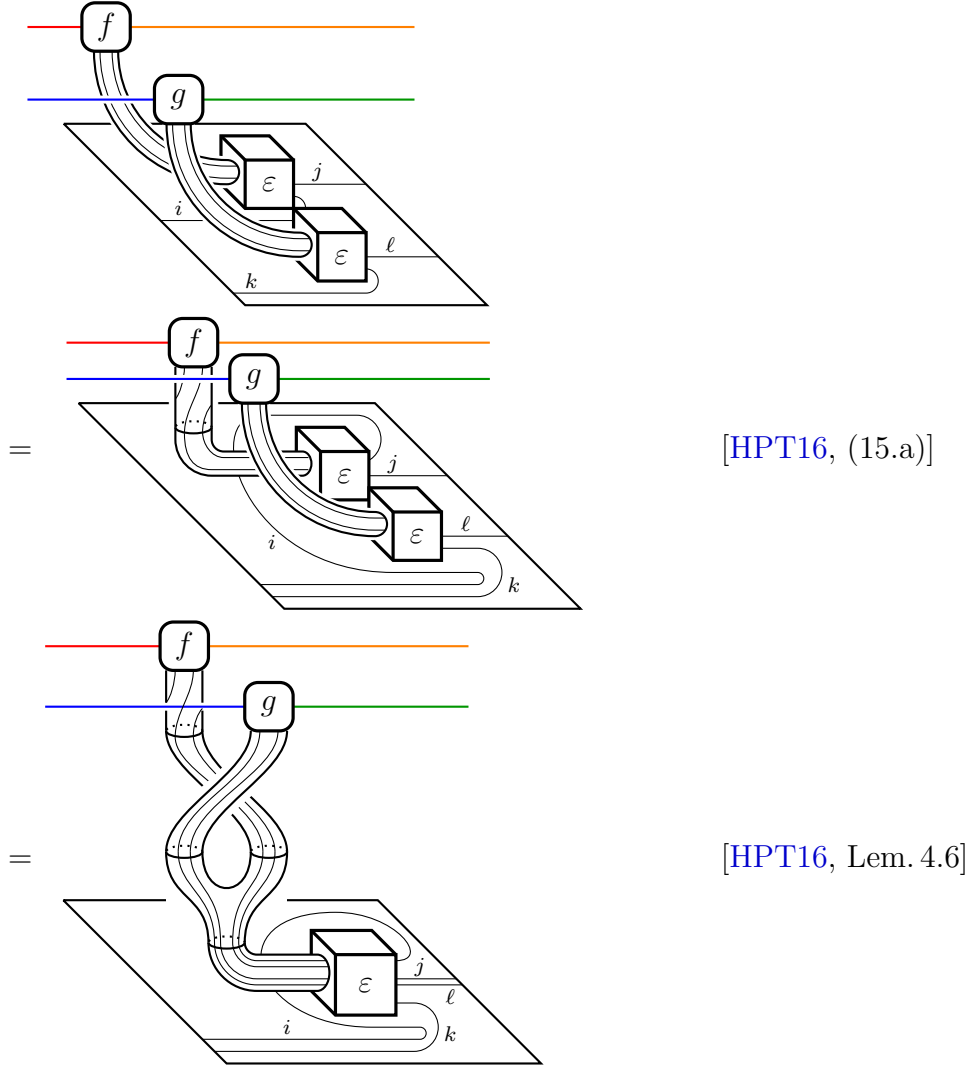
Equivalently:

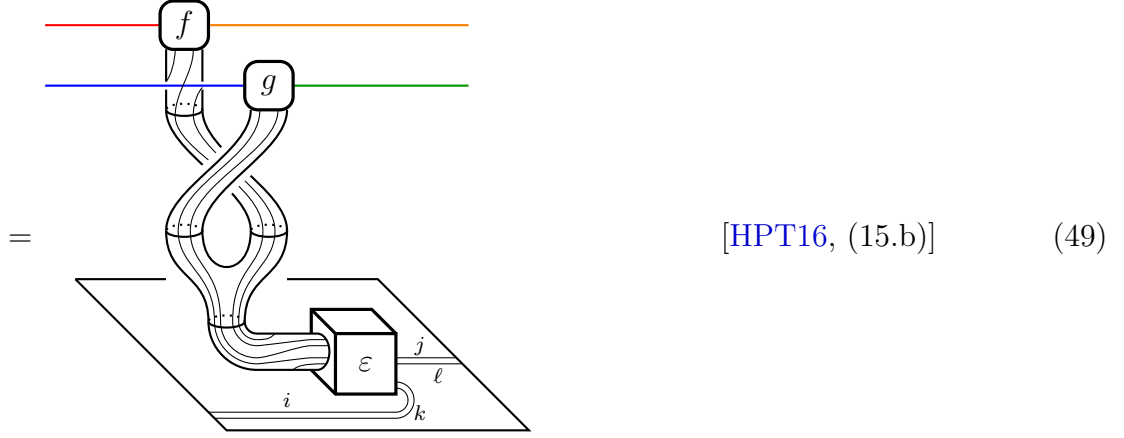
$$\begin{array}{ccc} (\Phi(a) \otimes m^{\otimes i}) \otimes (\Phi(c) \otimes m^{\otimes k}) & \longrightarrow & \Phi(a \otimes b) \otimes m'^{\otimes i+k} \\ \Psi(f) \otimes \Psi(g) \downarrow & & \downarrow \Psi(f \otimes g) \\ (\Phi(b) \otimes m^{\otimes j}) \otimes (\Phi(d) \otimes m^{\otimes \ell}) & \longrightarrow & \Phi(b \otimes d) \otimes m^{\otimes j+\ell} \end{array} \quad (48)$$

Recall the definition



of $\Psi(f)$ and $\Psi(g)$. The composite $\begin{array}{c} \square \\ \rightarrow \end{array}$ in (48) is given by





We need to show that this is equal to the image of

$$\begin{aligned}
f \otimes g &\in \mathcal{M}'((\Phi'(a) \otimes m'^{\otimes i}) \otimes (\Phi'(c) \otimes m'^{\otimes k}), (\Phi'(b) \otimes m'^{\otimes j}) \otimes (\Phi'(d) \otimes m'^{\otimes \ell})) \\
&= \mathcal{M}'(\Phi'(a \otimes c) \otimes m'^{\otimes i+k}, \Phi'(b \otimes d) \otimes m'^{\otimes j+\ell}) \\
&= \mathcal{C}(a \otimes c, b \otimes d \otimes \mathcal{P}[j + \ell + k + i])
\end{aligned}$$

under Ψ . Recall that $f \otimes g$ was defined in (39). By the anchor dependence relation (Definition 2.7) and the definition of $\varpi_{i,j}$ (Theorem 5.1) the value of $\mathcal{P} = (\mathcal{P}, Z)$ on the tensor product tangle is given by

$$\begin{aligned}
Z\left(\begin{array}{c} \text{Diagram with two circles and strands } i, j, k, \ell \end{array}\right) &= Z\left(\begin{array}{c} \text{Diagram with two circles and strands } i, j, k, \ell \end{array}\right) \circ \beta_{\mathcal{P}[\ell+k], \mathcal{P}[j+i]} = Z(p_{j, \ell+k}) \circ \beta_{\mathcal{P}[\ell+k], \mathcal{P}[j+i]} \\
&= \varpi_{j, \ell+k} \circ \beta_{\mathcal{P}[\ell+k], \mathcal{P}[j+i]} \\
&= \tau_{m^{\otimes i}, m^{\otimes j+\ell+k}}^- \circ \mu_{m^{\otimes j+i}, m^{\otimes \ell+k}} \circ (\tau_{m^{\otimes j}, m^{\otimes i}}^+ \otimes \text{id}) \circ \beta_{\mathcal{P}[\ell+k], \mathcal{P}[j+i]} : \\
&\quad \text{Tr}_{\mathcal{C}}(m^{\otimes \ell+k}) \otimes \text{Tr}_{\mathcal{C}}(m^{\otimes j+i}) \rightarrow \text{Tr}_{\mathcal{C}}(m^{\otimes j+\ell+k+i}).
\end{aligned}$$

This composite is exactly the one which appears in (49), and so the latter is equal $\Psi(f \otimes g)$. The commutativity of (48) follows. $\square$

We also need to check that Ψ is a pivotal functor (first item in Definition 3.2):

Lemma 7.5. *The functor $\Psi : \mathcal{M}' \rightarrow \mathcal{M}$ is compatible with the pivotal structures:*

$$\Psi(\varphi_X) = \delta_X^{-1} \circ \delta_X^\sim \circ \varphi_{\Psi(X)}.$$

Proof. Once again, it is enough to check this compatibility on objects of the form $X = \Phi'(a) \otimes m'^{\otimes n}$. Using Lemma 6.2 to identify $(\Phi'(a) \otimes m'^{\otimes n})^\sim$ with $\Phi'(a^\sim) \otimes m'^{\otimes n}$, the desired equation becomes:

$$\Psi(\varphi_{\Phi'(a) \otimes m'^{\otimes n}}) = \delta_{\Phi'(a^\sim) \otimes m'^{\otimes n}}^{-1} \circ \delta_{\Phi'(a) \otimes m'^{\otimes n}}^\sim \circ \varphi_{\Phi(a) \otimes m^{\otimes n}}. \quad (50)$$

Applying the definition (47) of Ψ to the morphism $\varphi_{\Phi'(a) \otimes m'^{\otimes n}}$ (described in (40)) yields the formula:

$$\Psi(\varphi_{\Phi'(a) \otimes m'^{\otimes n}}) = \Phi(\varphi_a) \otimes \text{id}_{m^{\otimes n}}.$$

Let $\psi : m \rightarrow m^\sim$ be the isomorphism which equips m with the structure of a symmetrically self-dual object (paragraph before Definition 3.4), and let $\dot{\delta}_a : \Phi(a^\sim) \rightarrow \Phi(a)^\sim$ be the natural isomorphism coming from the fact that Φ is a tensor functor. Using the monoidal property of φ , the fact that Φ is a pivotal functor, and the axiom satisfied by ψ , we get:

$$\varphi_{\Phi(a) \otimes m^{\otimes n}} = \varphi_{\Phi(a)} \otimes \varphi_m^{\otimes n} = [(\dot{\delta}_a^\sim)^{-1} \circ \dot{\delta}_a^\sim \circ \Phi(\varphi_a)] \otimes [(\psi^\sim)^{-1} \circ \psi]^{\otimes n}$$

Finally, using the definition of δ (Definition 3.2), we compute:

$$\begin{aligned} \delta_{\Phi'(a) \otimes m'^{\otimes n}} &= (\Psi(\text{ev}_{\Phi'(a) \otimes m'^{\otimes n}}) \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_{\Psi(\Phi'(a) \otimes m'^{\otimes n})}) \\ &= ((\Phi(\text{ev}_a) \otimes \tilde{\text{ev}}_{m^{\otimes n}}) \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_{\Phi(a) \otimes m^{\otimes n}}) \quad [(47) \text{ and Lem. 6.2}] \\ &= ((\Phi(\text{ev}_a) \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_{\Phi(a)})) \otimes ((\tilde{\text{ev}}_m \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_m))^{\otimes n} \\ &= \dot{\delta}_a \otimes \psi^{\otimes n} \end{aligned}$$

It follows that $\delta_{\Phi'(a^\sim) \otimes m'^{\otimes n}} = \dot{\delta}_a^{-1} \otimes (\psi^{-1})^{\otimes n}$ and $\delta_{\Phi'(a) \otimes m'^{\otimes n}^\sim} = \dot{\delta}_a^\sim \otimes (\psi^\sim)^{\otimes n}$. The right hand side of (50) is therefore given by:

$$[\dot{\delta}_a^{-1} \otimes (\psi^{-1})^{\otimes n}] \circ [\dot{\delta}_a^\sim \otimes (\psi^\sim)^{\otimes n}] \circ [[(\dot{\delta}_a^\sim)^{-1} \circ \dot{\delta}_a^\sim \circ \Phi(\varphi_a)] \otimes [(\psi^\sim)^{-1} \circ \psi]^{\otimes n}] = \Phi(\varphi_a) \otimes \text{id}_{m^{\otimes n}}.$$

This finishes the proof. $\square$

Recall that $(\mathcal{M}', m')$ is the image of $(\mathcal{M}, m)$ under the functor $\Delta \circ \Lambda$. By Lemmas 7.1, 7.2, 7.3, 7.5 and Proposition 7.4, we know that $\Psi : \mathcal{M}' \rightarrow \mathcal{M}$ is an equivalence of pivotal tensor categories. To verify that Ψ is an equivalence of *pointed* pivotal tensor categories (Definition 3.4), we still need to check that $\Psi(m') = m$ and that

$$\psi = \delta_{m'} \circ \Psi(\psi'). \quad (51)$$

Here, $\psi : m \rightarrow m^\sim$ and $\psi' : m' \rightarrow (m')^\sim$ are the isomorphisms which make m and m' into symmetrically self-dual objects, and $\delta_{m'} : \Psi((m')^\sim) \rightarrow \Psi(m')^\sim$ is as described in Definition 3.2. The first condition $\Psi(m') = m$ is clear. The second condition (51) reduces to $\psi = \delta_{m'}$ because $\psi' = \text{id}_{m'}$ (see the end of Section 6.4). Here, we have identified m' with its dual object by means of the pairing (41).

By definition, $\delta_{m'}$ is given by

$$\delta_{m'} : \Psi((m')^\sim) = \Psi(m') = m \xrightarrow{\text{id}_m \otimes \text{coev}_m} m \otimes m \otimes m^\sim \xrightarrow{\Psi(\tilde{\text{ev}}_{m'}) \otimes \text{id}_{m^\sim}} m^\sim = \Psi(m')^\sim.$$

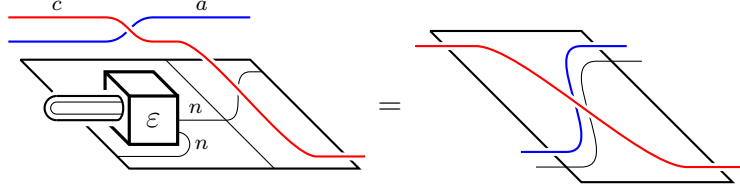
Now, applying the definition (47) of Ψ to the morphism $\tilde{\text{ev}}_{m'}$ (which is equal to (41) because $\psi' = \text{id}_{m'}$), we learn that $\Psi(\tilde{\text{ev}}_{m'}) = \tilde{\text{ev}}_m$. Finally, going back to the definition (11) of $\tilde{\text{ev}}_m$, we get $\psi = \delta_{m'}$, as desired. We conclude that Ψ is an equivalence of pointed pivotal tensor categories.

In order to show that $\Psi : (\mathcal{M}', m') \rightarrow (\mathcal{M}, m)$ is an isomorphism in $\mathbf{ModTens}_*$, it remain to show that it is a functor of module tensor categories:

Lemma 7.6. $\Psi : \mathcal{M}' \rightarrow \mathcal{M}$ satisfies the compatibility (7) in the definition of functor between module tensor categories:

$$\begin{array}{ccc} \Phi(c) \otimes \Psi(X) & \xrightarrow{\cong} & \Psi(\Phi'(c) \otimes X) \\ \downarrow e_{\Phi(c), \Psi(X)} & & \downarrow \Psi(e_{\Phi'(c), X}) \\ \Psi(X) \otimes \Phi(c) & \xrightarrow{\cong} & \Psi(X \otimes \Phi'(c)) \end{array} \quad (52)$$

Proof. Without loss of generality, we take X of the form $\Phi'(a) \otimes m'^{\otimes n}$. Recall that $e_{\Phi'(c), \Phi'(a) \otimes m'^{\otimes n}}$ was defined in (43). The composite $\overleftarrow{\square}$ in (52) is then given by:



The latter is visibly equal to $e_{\Phi(c), \Psi(\Phi'(a) \otimes m'^{\otimes n})} = e_{\Phi(c), \Phi(a) \otimes m^{\otimes n}}$. $\square$

Recall that the functors $\Lambda : \mathbf{ModTens}_* \rightarrow \mathbf{APA}$ and $\Delta : \mathbf{APA} \rightarrow \mathbf{ModTens}_*$ were described in Theorems 5.9 and 6.10, respectively. We summarize the results of this section:

Proposition 7.7. *For any pointed module tensor category $(\mathcal{M}, m) \in \mathbf{ModTens}_*$, there exists an isomorphism in $\mathbf{ModTens}_*$*

$$\Psi_{(\mathcal{M}, m)} : \Delta \circ \Lambda(\mathcal{M}, m) \rightarrow (\mathcal{M}, m).$$

Our next task is to investigate the question of whether these isomorphisms assemble to a natural transformation $\Psi : \Delta \circ \Lambda \Rightarrow \text{id}_{\mathbf{ModTens}_*}$.

7.2 Naturality of Ψ

Let $\mathcal{M}_1 = (\mathcal{M}_1, m_1)$ and $\mathcal{M}_2 = (\mathcal{M}_2, m_2)$ be pointed module tensor categories. Let $G : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a functor of pointed module tensor categories (a morphism in $\mathbf{ModTens}_*$), and let $G' : \mathcal{M}'_1 \rightarrow \mathcal{M}'_2$ be its image under $\Delta \circ \Lambda$. Let also $\Psi_1 := \Psi_{\mathcal{M}_1} : \mathcal{M}'_1 \rightarrow \mathcal{M}_1$ and $\Psi_2 := \Psi_{\mathcal{M}_2} : \mathcal{M}'_2 \rightarrow \mathcal{M}_2$ be as in Proposition 7.7.

We consider the following diagram:

$$\begin{array}{ccc} \mathcal{M}'_1 & \xrightarrow{G'} & \mathcal{M}'_2 \\ \downarrow \Psi_1 & & \downarrow \Psi_2 \\ \mathcal{M}_1 & \xrightarrow{G} & \mathcal{M}_2. \end{array} \tag{53}$$

Recall from Section 3.1 that $\tau_{\leq 1}(\mathbf{ModTens}_*)$ is obtained from $\mathbf{ModTens}_*$ by identifying all 1-morphisms which are connected by a (necessarily invertible) 2-morphism. The diagram (53) does not commute in $\mathbf{ModTens}_*$, but its image down in $\tau_{\leq 1}(\mathbf{ModTens}_*)$ does commute:

Proposition 7.8. *There exists an (invertible) natural transformation*

$$\kappa : \Psi_2 \circ G' \Rightarrow G \circ \Psi_1$$

of pointed module tensor category functors (a 2-morphism in $\mathbf{ModTens}_$) between the composites $\Psi_2 \circ G'$ and $G \circ \Psi_1$.*

Proof. As usual, it is enough to define κ on those objects of $\mathcal{M}'_1$ of the form $\Phi'_1(c) \otimes m_1^{\otimes n}$. Let $\gamma_c : \Phi_2(c) \rightarrow G\Phi_1(c)$ be the action coherence isomorphism associated to the functor G . By definition ((44) and (46)), we have:

$$\begin{aligned} \Psi_2 \circ G'(\Phi'_1(c) \otimes m_1'^{\otimes n}) &= \Psi_2(\Phi'_2(c) \otimes m_2'^{\otimes n}) = \Phi_2(c) \otimes m_2^{\otimes n} \\ \text{and } G \circ \Psi_1(\Phi'_1(c) \otimes m_1'^{\otimes n}) &= G(\Phi_1(c) \otimes m_1^{\otimes n}) \cong G\Phi_1(c) \otimes m_2^{\otimes n}. \end{aligned}$$

Using the above identifications, we define the components of κ as follows:

$$\begin{aligned} \kappa_{\Phi'_1(c) \otimes m_1'^{\otimes n}} : \Psi_2 G'(\Phi'_1(c) \otimes m_1'^{\otimes n}) &= \Phi_2(c) \otimes m_2^{\otimes n} \\ &\xrightarrow{\gamma_c \text{id}_{m_2^{\otimes n}}} G\Phi_1(c) \otimes m_2^{\otimes n} \cong G\Psi_1(\Phi'_1(c) \otimes m_1'^{\otimes n}). \end{aligned} \quad (54)$$

Clearly, this satisfies the condition $\kappa_{m'_1} = \text{id}_{m_2}$ of being pointed (Definition 3.4). We must show that κ is natural with respect to morphisms in $\mathcal{M}'_1$, monoidal (Definition 3.3), and compatible with the action coherence isomorphisms (Definition 3.3, Eq. (9)).

We start by the naturality of κ . Given a morphism $f \in \mathcal{M}'_1(\Phi'_1(a) \otimes m_1^{\otimes n}, \Phi'_1(b) \otimes m_1^{\otimes k}) = \mathcal{C}(a, b \otimes \mathrm{Tr}_{\mathcal{C}}^1(m_1^{\otimes k+n}))$, we need to show that the following diagram commutes:

$$\begin{array}{ccc} \Psi_2 G'(\Phi'_1(a) \otimes m_1'^n) & \xrightarrow{\kappa_{\Phi'_1(a)} \otimes m_1'^n} & G\Psi_1(\Phi'_1(a) \otimes m_1'^n) \\ \downarrow \Psi_2 G'(f) & & \downarrow G\Psi_1(f) \\ \Psi_2 G'(\Phi'_1(b) \otimes m_1'^k) & \xrightarrow{\kappa_{\Phi'_1(b)} \otimes m_1'^k} & G\Psi_1(\Phi'_1(b) \otimes m_1'^k) \end{array}$$

Recall that $G' = \Delta\Lambda(G)$. The composite $\sqcup_{\rightarrow}$ is given by:

$$\begin{aligned}
& (\gamma_b \otimes \text{id}_{m_2^{\otimes k}}) \circ (\text{id}_{\Phi_2(b)} \otimes \text{id}_{m_2^{\otimes k}} \otimes \tilde{\text{ev}}_{m_2^{\otimes n}}) \circ (\text{id}_{\Phi_2(b)} \otimes \varepsilon_{m_2^{\otimes k+n}}^2 \otimes \text{id}_{m_2^n}) \circ (\Phi_2(G'(f)) \otimes \text{id}_{m_2^{\otimes n}}) \\
& \quad \quad \quad [\text{Def. (47) of } \Psi_2] \\
&= (\gamma_b \otimes \text{id}_{m_2^{\otimes k}} \otimes \tilde{\text{ev}}_{m_2^{\otimes n}}) \circ (\text{id}_{\Phi_2(b)} \otimes \varepsilon_{m_2^{\otimes k+n}}^2 \otimes \text{id}_{m_2^{\otimes n}}) \circ (\Phi_2[(\text{id}_b \otimes \Lambda G[k+n]) \circ f] \otimes \text{id}_{m_2^{\otimes n}}) \\
& \quad \quad \quad [\text{Def. (45) of } \Delta \Lambda G] \\
&= (\gamma_b \otimes \text{id}_{m_2^{\otimes k}} \otimes \tilde{\text{ev}}_{m_2^{\otimes n}}) \circ (\text{id}_{\Phi_2(b)} \otimes [\varepsilon_{m_2^{\otimes k+n}}^2 \circ \Phi_2(\zeta_{m_1^{\otimes k+n}})]) \otimes \text{id}_{m_2^{\otimes n}} \circ (\text{id}_{\Phi_2(b)} \otimes \Phi_2(f) \otimes \text{id}_{m_2^{\otimes n}}) \\
& \quad \quad \quad [\text{Def. 5.3 of } \Lambda G] \\
&= (\gamma_b \otimes \text{id}_{m_2^{\otimes k}} \otimes \tilde{\text{ev}}_{m_2^{\otimes n}}) \circ (\text{id}_{\Phi_2(b)} \otimes [G(\varepsilon_{m_1^{\otimes k+n}}^1) \circ \gamma_{\text{Tr}_C^1(m_1^{\otimes k+n})}]) \otimes \text{id}_{m_2^{\otimes n}} \circ (\text{id}_{\Phi_2(b)} \otimes \Phi_2(f) \otimes \text{id}_{m_2^{\otimes n}}) \\
& \quad \quad \quad [\text{Lemma 5.4}] \\
&= (\text{id}_{G\Phi_1(b)} \otimes \text{id}_{m_2^{\otimes k}} \otimes \tilde{\text{ev}}_{m_2^{\otimes n}}) \circ (\text{id}_{G\Phi_1(b)} \otimes G(\varepsilon_{m_1^{\otimes k+n}}^1) \otimes \text{id}_{m_2^{\otimes n}}) \circ ([\gamma_{b \otimes \text{Tr}_C^1(m_1^{\otimes k+n})} \circ \Phi_2(f)] \otimes \text{id}_{m_2^{\otimes n}}) \\
& \quad \quad \quad [\gamma \text{ is monoidal}] \\
&= (\text{id}_{G\Phi_1(b)} \otimes \text{id}_{m_2^{\otimes k}} \otimes \tilde{\text{ev}}_{m_2^{\otimes n}}) \circ (\text{id}_{G\Phi_1(b)} \otimes G(\varepsilon_{m_1^{\otimes k+n}}^1) \otimes \text{id}_{m_2^{\otimes n}}) \circ ([G\Phi_1(f) \circ \gamma_a] \otimes \text{id}_{m_2^{\otimes n}}) \\
& \quad \quad \quad [\gamma \text{ is natural}] \\
&= (\text{id}_{G\Phi_1(b)} \otimes \text{id}_{m_2^{\otimes k}} \otimes \tilde{\text{ev}}_{m_2^{\otimes n}}) \circ (\text{id}_{G\Phi_1(b)} \otimes G(\varepsilon_{m_1^{\otimes k+n}}^1) \otimes \text{id}_{m_2^{\otimes n}}) \circ (G\Phi_1(f) \otimes \text{id}_{m_2^{\otimes n}}) \circ (\gamma_a \otimes \text{id}_{m_2^{\otimes n}}).
\end{aligned}$$

Using the definition (47) of Ψ_1 and the fact that $G(\tilde{e}v_{m_1}^{\otimes n}) = \tilde{e}v_{m_2}^{\otimes n}$ (coming from the fact that G is a pivotal functor), one recognizes the last line as the composite $\overline{}\downarrow$ in the above diagram.

To show that κ is monoidal, we argue that for every pair of objects $X, Y \in \mathcal{M}'_1$, the following diagram is commutative (Definition 3.3, Eq. (8)):

$$\begin{array}{ccccc}
\Psi_2 G'(X \otimes Y) & \xrightarrow{\cong} & \Psi_2(G'(X) \otimes G'(Y)) & \xrightarrow{\cong} & \Psi_2 G'(X) \otimes \Psi_2 G'(Y) \\
\downarrow \kappa_{X \otimes Y} & & & & \downarrow \kappa_X \otimes \kappa_Y \\
G\Psi_1(X \otimes Y) & \xrightarrow{\cong} & G(\Psi_1(X) \otimes \Psi_1(Y)) & \xrightarrow{\cong} & G\Psi_1(X) \otimes G\Psi_1(Y).
\end{array} \tag{55}$$

Without loss of generality, we take $X = \Phi'_1(a) \otimes m_1'^{\otimes n}$ and $Y = \Phi'_1(b) \otimes m_1'^{\otimes k}$. We rewrite (55) as the boundary of the following pasting diagram:

$$\begin{array}{ccccc}
\Phi_2(a \otimes b) \otimes m_2^{n+k} & \xrightarrow{\cong} & \Phi_2(a) \otimes \Phi_2(b) \otimes m_2^n \otimes m_2^k & \xrightarrow{\text{id} \otimes e_{\Phi_2(b), m_2^n} \otimes \text{id}} & \Phi_2(a) \otimes m_2^n \otimes \Phi_2(b) \otimes m_2^k \\
\downarrow \gamma_{a \otimes b} \otimes \text{id} & & \downarrow \gamma_a \otimes \gamma_b \otimes \text{id} \otimes \text{id} & & \downarrow \gamma_a \otimes \text{id} \otimes \gamma_b \otimes \text{id} \\
G\Phi_1(a \otimes b) \otimes m_2^{n+k} & \xrightarrow{\cong} & G\Phi_1(a) \otimes G\Phi_1(b) \otimes G(m_1^n) \otimes G(m_1^k) & & \\
\downarrow \cong & \swarrow \cong & \downarrow \text{id} \otimes G(e_{\Phi_1(b), m_1^n}) \otimes \text{id} & \searrow \cong & \\
& G\Phi_1(a) \otimes G(\Phi_1(b) \otimes m_1^n) \otimes G(m_1^k) & \xrightarrow{\text{id} \otimes G(e_{\Phi_1(b), m_1^n}) \otimes \text{id}} & G\Phi_1(a) \otimes G(m_1^n \otimes \Phi_1(b)) \otimes G(m_1^k) & \\
\downarrow \cong & \swarrow \cong & \downarrow G(\text{id} \otimes e_{\Phi_1(b), m_1^n} \otimes \text{id}) & \searrow \cong & \\
G(\Phi_1(a) \otimes \Phi_1(b) \otimes m_1^n \otimes m_1^k) & \xrightarrow{\text{id} \otimes e_{\Phi_1(b), m_1^n} \otimes \text{id}} & G(\Phi_1(a) \otimes m_1^n \otimes \Phi_1(b) \otimes m_1^k) & \xrightarrow{\cong} & G\Phi_1(a) \otimes m_2^n \otimes G\Phi_1(b) \otimes m_2^k
\end{array}$$

The upper left rectangle commutes because γ is monoidal. The irregular hexagon commutes because γ is compatible with the half-braidings (7). The other three regions are visibly commutative. The commutativity of (55) follows.

The unitality condition $\kappa_{1_{\mathcal{M}'_1}} \circ i_{\Psi_2 \circ G'} = i_{G \circ \Psi_1}$ for κ (Definition 3.3) is a consequence of the corresponding unitality condition $\gamma_{1_c} = G(i_{\Psi_1}) \circ i_G \circ i_{\Phi_2}^{-1}$ of γ (Definition 3.2).

At last, we show that κ is compatible with the action coherence isomorphisms (Definition 3.3, Eq. (9)):

$$\begin{array}{ccccccc}
\Phi_2(c) & \xrightarrow{\gamma_c^{\Psi_2}} & \Psi_2 \Phi'_2(c) & \xrightarrow{\Psi_2(\gamma_c^{G'})} & \Psi_2 G'(\Phi'_1(c)) & & \\
\downarrow = & & & & \downarrow \kappa_{\Phi'_1(c)} & & \\
\Phi_2(c) & \xrightarrow{\gamma_c^G} & G\Phi_1(c) & \xrightarrow{G(\gamma_c^{\Psi_1})} & G\Psi_1(\Phi'_1(c)). & &
\end{array} \tag{56}$$

Here, γ^F refers to the action coherence isomorphism associated to the functor F . By definition, $\gamma_c^{G'} = \text{id}$ (line below (44)) and $\gamma_c^{\Psi_i} = \text{id}$ (line below (47)). The diagram (56) therefore reduces to the equation $\gamma_c^G = \kappa_{\Phi'_1(c)}$, which holds by the definition (54) of κ . $\square$

Combining all the results of Sections 7.1 and 7.2, we have proven:

Theorem 7.9. *The composite*

$$\Delta \circ \Lambda : \tau_{\leq 1}(\text{ModTens}_*) \rightarrow \text{APA} \rightarrow \tau_{\leq 1}(\text{ModTens}_*)$$

is naturally isomorphic to the identity functor $\text{id}_{\tau_{\leq 1}(\text{ModTens}_)}$.* $\square$

7.3 Planar algebras to module tensor categories and back

In this section, we finish the proof that $\Lambda : \text{ModTens}_* \rightleftarrows \text{APA} : \Delta$ form an equivalence of categories by arguing that $\Lambda \circ \Delta$ is naturally isomorphic to id_{APA} . If one takes (34) as our definition of $\text{Tr}_{\mathcal{C}}$, then we will prove that there is actually an equality $\Lambda \circ \Delta = \text{id}_{\text{APA}}$ on the nose.

Let $\mathcal{P}$ be an anchored planar algebra and let $\mathcal{P}'$ be its image under $\Lambda \circ \Delta$. Let $(\mathcal{M}, m) := \Delta(\mathcal{P})$ so that, following (23), $\mathcal{P}'[n] = \text{Tr}_{\mathcal{C}}(m^{\otimes n})$. If one takes (34) as our definition of $\text{Tr}_{\mathcal{C}}$ then, indeed, $\text{Tr}_{\mathcal{C}}(m^{\otimes n}) = \mathcal{P}[n]$, and so we have $\mathcal{P}[n] = \mathcal{P}'[n]$. We still need to check that the action of the generating tangles u , a_i , $a_i^\dagger$, p_{ij} (Definition 2.10) given by $\mathcal{P}$ and by $\mathcal{P}'$ agree. Specifically, we need to check:

$$\eta = Z(u), \quad \alpha_i = Z(a_i), \quad \alpha_i^\dagger = Z(a_i^\dagger) \quad \text{and} \quad \varpi_{i,j} = Z(p_{i,j}),$$

where $Z(-)$ denotes the action of $\mathcal{P}$, and the maps η , α_i , $\alpha_i^\dagger$, $\varpi_{i,j}$ are as described in Theorem 5.1. The value of any tangle being determined by the values of the generating tangles (Algorithm 4.9), the above conditions will then imply that $\mathcal{P}$ and $\mathcal{P}'$ are equal as anchored planar algebras.

Recall that a morphism $f \in \mathcal{M}(\Phi(c) \otimes m^{\otimes n_1}, \Phi(d) \otimes m^{\otimes n_2}) := \mathcal{C}(c, d \otimes \mathcal{P}[n_2 + n_1])$ is represented diagrammatically by

$$\begin{array}{c} d \\ | \\ \boxed{f} \\ | \\ c \end{array} \xrightarrow{\mathcal{P}[n_2 + n_1]}. \quad (57)$$

Lemma 7.10. *For all $i = 1, \dots, n-1$*

- (1) *the map $\alpha_i : \mathcal{P}[n+2] \rightarrow \mathcal{P}[n]$ is equal to $Z(a_i) : \mathcal{P}[n+2] \rightarrow \mathcal{P}[n]$, and*
- (2) *the map $\alpha_i^\dagger : \mathcal{P}[n] \rightarrow \mathcal{P}[n+2]$ is equal to $Z(a_i^\dagger) : \mathcal{P}[n] \rightarrow \mathcal{P}[n+2]$.*

Proof. We only prove the first statement, the other one is similar. Recall from Theorem 5.1 that α_i is the image of $\text{id}_i \otimes \tilde{\text{ev}}_m \otimes \text{id}_{n-i} : m^{\otimes n+2} \rightarrow m^{\otimes n}$ under $\text{Tr}_{\mathcal{C}}$.

By the definition (41) of ev_m , the fact that $\text{ev}_m = \tilde{\text{ev}}_m$ (paragraph after (41)), and the definition (39) of the tensor product in $\mathcal{M}$, we have

$$\text{id}_i \otimes \tilde{\text{ev}}_m \otimes \text{id}_{n-i} = \begin{array}{c} \vdots \\ \boxed{\text{tangle}} \\ \vdots \end{array} \xrightarrow{\mathcal{P}[2n+2]}.$$

1_c

Therefore, by (35), $\alpha_i := \text{Tr}_{\mathcal{C}}(\text{id}_i \otimes \tilde{\text{ev}}_m \otimes \text{id}_{n-i})$ satisfies:

$$\alpha_i = \begin{array}{c} \vdots \\ \boxed{\text{tangle}} \\ \vdots \end{array} \xrightarrow{\mathcal{P}[n+2]} \begin{array}{c} \boxed{\text{tangle}} \\ \text{---} \end{array} \xrightarrow{\mathcal{P}[n]} = Z \left(\begin{array}{c} \boxed{\text{tangle}} \\ \text{---} \end{array} \right) \circ_1 Z \left(\begin{array}{c} \boxed{\text{tangle}} \\ \text{---} \end{array} \right) = Z(a_i).$$

□

Lemma 7.11. *The map $\eta : 1_{\mathcal{C}} \rightarrow \mathcal{P}[0]$ is equal to $Z(u)$.*

Proof. By definition, η is the unit of the adjunction $\Phi \dashv \text{Tr}_{\mathcal{C}}$ evaluated on $1_{\mathcal{C}} \in \mathcal{C}$ (Theorem 5.1 and (T6)). Equivalently, it is the image of $\text{id}_{\Phi(1_{\mathcal{C}})}$ under the bijection $\mathcal{M}(\Phi(1_{\mathcal{C}}), \Phi(1_{\mathcal{C}})) \cong \mathcal{C}(1_{\mathcal{C}}, \text{Tr}_{\mathcal{C}}(\Phi(1_{\mathcal{C}})))$ provided by the adjunction. The latter is described in Lemma 6.1. It is the identity map

$$\mathcal{M}(\Phi(1_{\mathcal{C}}), \Phi(1_{\mathcal{C}})) := \mathcal{C}(1_{\mathcal{C}}, \mathcal{P}[0]) \rightarrow \mathcal{C}(1_{\mathcal{C}}, \mathcal{P}[0]) = \mathcal{C}(1_{\mathcal{C}}, \text{Tr}_{\mathcal{C}}(\Phi(1_{\mathcal{C}}))).$$

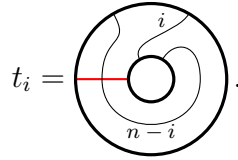
Therefore $\eta = \text{id}_{\Phi(1_{\mathcal{C}})}$. Combining this with the definition (31) of identities in $\mathcal{M}$, it follows that $\eta = \text{id}_{\Phi(1_{\mathcal{C}})} = Z(u)$. $\square$

Before checking the relation $\varpi_{i,j} = Z(p_{i,j})$, we investigate the traciator (T8) and multiplication map (T4) in the category $\mathcal{M} = \Delta(\mathcal{P})$.

Lemma 7.12. *The traciator*

$$\tau_{m^{\otimes i}, m^{\otimes n-i}} \in \mathcal{C}(\text{Tr}_{\mathcal{C}}(m^{\otimes i} \otimes m^{\otimes n-i}), \text{Tr}_{\mathcal{C}}(m^{\otimes n-i} \otimes m^{\otimes i})) = \mathcal{C}(\mathcal{P}[n], \mathcal{P}[n])$$

is equal to $Z(t_i)$, where t_i is the rotation tangle



As similar result holds true for $\tau_{m^{\otimes i}, m^{\otimes n-i}}^{-1}$ and the inverse rotation tangle.

Proof. Set $j = n - i$. Recall from (T8) that $\tau_{m^{\otimes i}, m^{\otimes j}}$ is the mate of

$$(\text{id}_{m^{\otimes n}} \otimes \tilde{\text{ev}}_{m^{\otimes j}}) \circ (\text{id}_{m^{\otimes j}} \otimes \varepsilon_{m^{\otimes n}} \otimes \text{id}_{m^{\otimes j}}) \circ (e_{\Phi(\mathcal{P}[n]), m^{\otimes j}} \otimes \text{id}_{m^{\otimes j}}) \circ (\text{id}_{\Phi(\mathcal{P}[n])} \otimes \text{co}\tilde{\text{ev}}_{m^{\otimes j}})$$

under the bijection $\mathcal{C}(\mathcal{P}[n], \mathcal{P}[n]) \cong \mathcal{M}(\Phi(\mathcal{P}[n]), m^{\otimes n})$. The latter is the identity map, and may therefore be suppressed (Lemma 6.1). Using the definition (43) of the half-braiding, we can then compute:

$$\tau_{m^{\otimes i}, m^{\otimes j}} = \begin{array}{c} \begin{array}{|c|} \hline \begin{array}{c} \text{Diagram of } \tau_{m^{\otimes i}, m^{\otimes j}} \text{ as a composition of boxes and wires} \end{array} \\ \hline \end{array} \end{array} \xrightarrow{\mathcal{P}[n]} \begin{array}{|c|} \hline \begin{array}{c} \text{Diagram of } Z(t_i) \end{array} \\ \hline \end{array} = Z(t_i).$$

$\square$

Lemma 7.13. *The multiplication map*

$$\mu_{m^{\otimes i}, m^{\otimes j}} \in \mathcal{C}(\text{Tr}_{\mathcal{C}}(m^{\otimes i}) \otimes \text{Tr}_{\mathcal{C}}(m^{\otimes j}), \text{Tr}_{\mathcal{C}}(m^{\otimes i+j})) = \mathcal{C}(\mathcal{P}[i] \otimes \mathcal{P}[j], \mathcal{P}[i+j])$$

is equal to $Z\left(\begin{array}{c} \text{Diagram of } \mu_{m^{\otimes i}, m^{\otimes j}} \end{array}\right)$.

Proof. Recall from (T4) that $\mu_{m^{\otimes i}, m^{\otimes j}}$ is the mate of $\varepsilon_{m^{\otimes i}} \otimes \varepsilon_{m^{\otimes j}}$ under the adjunction $\mathcal{C}(\mathrm{Tr}_{\mathcal{C}}(m^{\otimes i}) \otimes \mathrm{Tr}_{\mathcal{C}}(m^{\otimes j}), \mathrm{Tr}_{\mathcal{C}}(m^{\otimes i+j})) \cong \mathcal{M}(\Phi(\mathrm{Tr}_{\mathcal{C}}(m^{\otimes i}) \otimes \mathrm{Tr}_{\mathcal{C}}(m^{\otimes j})), m^{\otimes i+j})$. As in the previous proofs, we suppress that bijection and just write $\mu_{m^{\otimes i}, m^{\otimes j}} = \varepsilon_{m^{\otimes i}} \otimes \varepsilon_{m^{\otimes j}}$.

Under the adjunction $\mathcal{C}(\mathcal{P}[i], \mathcal{P}[i]) \cong \mathcal{M}(\Phi(\mathcal{P}[i]), m^{\otimes i})$, the counit $\varepsilon_{m^{\otimes i}}$ corresponds to the identity $\mathrm{id}_{\mathcal{P}[i]}$. In the notation (57), we represented it by a through-strand:

$$\varepsilon_{m^{\otimes i}} = \int_{\mathcal{P}[i]} \in \mathcal{M}(\Phi(\mathcal{P}[i]), m^{\otimes i}). \quad (58)$$

We now use the definition (39) of the tensor product in $\mathcal{M}$ to compute $\mu_{m^{\otimes i}, m^{\otimes j}} = \varepsilon_{m^{\otimes i}} \otimes \varepsilon_{m^{\otimes j}}$:

$$\begin{array}{c} \text{P}[i] \quad \text{P}[j] \end{array} \rightarrow \text{P}[i+j] = Z \left(\begin{array}{c} i \quad j \\ \circ \quad \circ \end{array} \right) \circ \beta_{\mathcal{P}[j], \mathcal{P}[i]}^{-1} = Z \left(\begin{array}{c} i \quad j \\ \circ \quad \circ \end{array} \right).$$

□

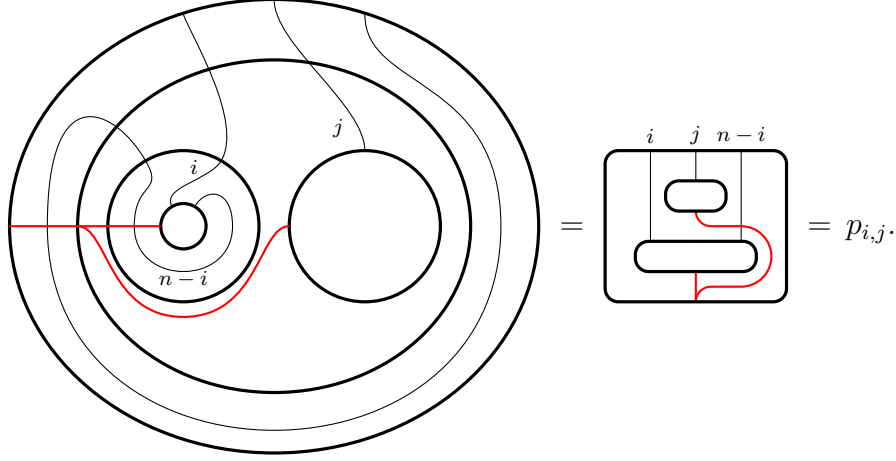
As a consequence of the Lemmas 7.12 and 7.13, we get:

Lemma 7.14. *The relation $\varpi_{i,j} = Z(p_{i,j})$ holds.*

Proof. By the definition of $\varpi_{i,j}$ (Theorem 5.1) and by the Lemmas 7.12 and 7.13, we have:

$$\begin{aligned} \varpi_{i,j} &= \tau_{m^{\otimes i+j}, m^{\otimes n-i}}^{-1} \circ \mu_{m^{\otimes n}, m^{\otimes j}} \circ (\tau_{m^{\otimes i}, m^{\otimes n-i}} \otimes \mathrm{id}_{\mathcal{P}[j]}) \\ &= \tau_{m^{\otimes i+j}, m^{\otimes n-i}}^{-1} \circ_1 \mu_{m^{\otimes n}, m^{\otimes j}} \circ_1 \tau_{m^{\otimes i}, m^{\otimes n-i}} \\ &= Z \left(\begin{array}{c} i+j \\ \circ \\ n-i \end{array} \right) \circ_1 Z \left(\begin{array}{c} n \quad j \\ \circ \quad \circ \end{array} \right) \circ_1 Z \left(\begin{array}{c} i \\ \circ \\ n-i \end{array} \right) \\ &= Z \left(\begin{array}{c} i+j \\ \circ \\ n-i \end{array} \right) \circ_1 \left(\begin{array}{c} n \quad j \\ \circ \quad \circ \end{array} \right) \circ_1 \left(\begin{array}{c} i \\ \circ \\ n-i \end{array} \right) = Z(p_{i,j}). \end{aligned}$$

Here, the last equality is best checked by drawing:



□

So far, we have checked that the functor $\Lambda \circ \Delta : \text{APA} \rightarrow \text{APA}$ is the identity on objects. We now check that it is the identity on morphisms. Let $H : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ be a map of anchored planar algebras, and let $H' : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ be its image under $\Lambda \circ \Delta$:

Lemma 7.15. *For every n , the two maps $H[n] : \mathcal{P}_1[n] \rightarrow \mathcal{P}_2[n]$ and $H'[n] : \mathcal{P}_1[n] \rightarrow \mathcal{P}_2[n]$ are equal.*

Proof. Let $(\mathcal{M}_i, m_i) := \Delta(\mathcal{P}_i)$ and $G := \Delta(H) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$. Recall from (24) that

$$H'[n] = \Lambda(G)[n] = \zeta_{m_1^{\otimes n}} : \mathcal{P}_1[n] = \text{Tr}_{\mathcal{C}}^1(m_1^{\otimes n}) \rightarrow \text{Tr}_{\mathcal{C}}^2(m_2^{\otimes n}) = \mathcal{P}_2[n]$$

is the mate of $G(\varepsilon_{m_1^{\otimes n}}^1) \circ \gamma_{\text{Tr}_{\mathcal{C}}^1(m_1^{\otimes n})}^G = G(\varepsilon_{m_1^{\otimes n}}^1)$, where the last equality holds because $\gamma^G = \gamma^{\Delta(H)} = \text{id}$ (see the line below (44)). As before, the adjunction $\mathcal{C}(\mathcal{P}_1[n], \mathcal{P}_2[n]) \cong \mathcal{M}_2(\Phi_2(\mathcal{P}_1[n]), m_2^{\otimes n})$ being an identity map, we simply write $H'[n] = G(\varepsilon_{m_1^{\otimes n}}^1)$. Recall from (58) that

$$\varepsilon_{m_1^{\otimes n}}^1 = \left. \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|_{\mathcal{P}_1[n]} \in \mathcal{M}_1(\Phi_1(\mathcal{P}_1[n]), m_1^{\otimes n}).$$

By (45), it follows that

$$H'[n] = \Delta(H)(\varepsilon_{m_1^{\otimes n}}^1) = \left. \begin{array}{c} \boxed{H[n]} \\ \text{---} \end{array} \right|_{\mathcal{P}_1[n]}^{\mathcal{P}_2[n]} = H[n].$$

□

Combining all the results of this section, we have proven:

Theorem 7.16. *The composite*

$$\Lambda \circ \Delta : \text{APA} \rightarrow \text{ModTens}_* \rightarrow \text{APA}$$

is the identity functor.

□

This concludes the proof of Theorem 3.7.

□

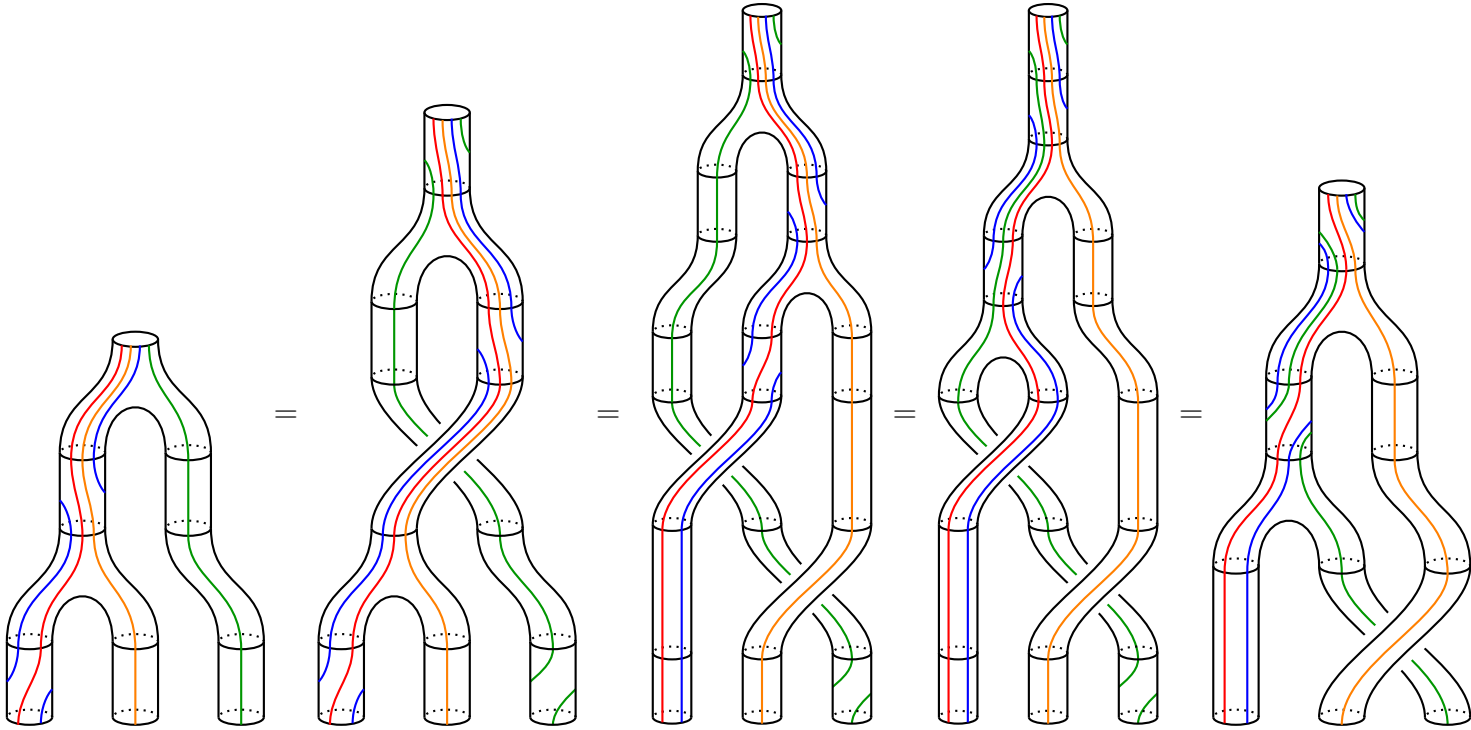
A An associativity type relation

Let $\mathcal{C}$ be a braided pivotal category and let $\text{Tr}_{\mathcal{C}} : \mathcal{M} \rightarrow \mathcal{C}$ be as in Section 3.1.

Lemma A.1. *For any $x, y, z, w \in \mathcal{M}$, the following maps are equal in $\mathcal{C}(\text{Tr}_{\mathcal{C}}(x \otimes y) \otimes \text{Tr}_{\mathcal{C}}(z) \otimes \text{Tr}_{\mathcal{C}}(w), \text{Tr}_{\mathcal{C}}(x \otimes z \otimes y \otimes w))$:*

$$\begin{aligned} & \mu_{x \otimes z \otimes y, w} \circ (\tau_{y, x \otimes z}^- \otimes \text{id}_{\text{Tr}_{\mathcal{C}}(w)}) \circ (\mu_{y \otimes x, z} \otimes \text{id}_{\text{Tr}_{\mathcal{C}}(w)}) \circ (\tau_{x, y}^+ \otimes \text{id}_{\text{Tr}_{\mathcal{C}}(z)} \otimes \text{id}_{\text{Tr}_{\mathcal{C}}(w)}) \\ &= \tau_{y \otimes w, x \otimes z}^- \circ \mu_{y \otimes w \otimes x, z} \circ (\tau_{x, y \otimes w}^+ \otimes \text{id}_{\text{Tr}_{\mathcal{C}}(z)}) \circ (\mu_{x \otimes y, w} \otimes \text{id}_{\text{Tr}_{\mathcal{C}}(z)}) \circ (\text{id}_{\text{Tr}_{\mathcal{C}}(x \otimes y)} \otimes \beta_{\text{Tr}_{\mathcal{C}}(z), \text{Tr}_{\mathcal{C}}(w)}). \end{aligned}$$

Proof. Consider the following five morphisms, named (a)–(e) from left to right.



We get the following equalities: (a) = (b) by relation (T14), (b) = (c) by the braiding in $\mathcal{C}$, (c) = (d) by the first relation in (T15), and (d) = (e) by the relations (T10), (T14), and the braiding in $\mathcal{C}$. $\square$

B Anchored planar tangles with coupons

Let $\mathcal{C}$ be a braided tensor category, and let $\mathcal{M}$ be a pivotal module tensor category over $\mathcal{C}$. The constructions in Sections 4 and 5 generalize to assign a morphism

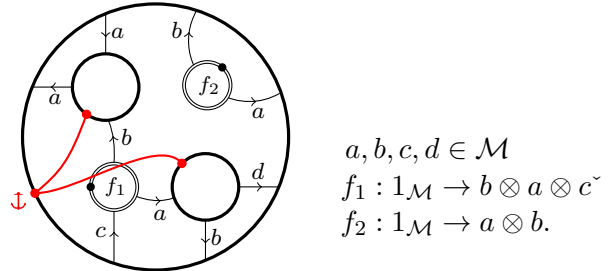
$$Z \left(\begin{array}{c} \text{Diagram of a disc with two holes and a red strand} \end{array} \right) : \begin{array}{l} \text{Tr}_{\mathcal{C}}(a \otimes a^\sim \otimes b) \otimes \text{Tr}_{\mathcal{C}}(a^\sim \otimes a \otimes b^\sim \otimes c \otimes b) \\ \rightarrow \text{Tr}_{\mathcal{C}}(a \otimes a^\sim \otimes c \otimes c^\sim \otimes c \otimes b) \end{array} \quad (59)$$

to every oriented anchored planar tangle T with strands labeled by objects of $\mathcal{M}$. In this appendix, we further generalize the construction to assign a morphism $Z(T)$ to every anchored planar tangle T with strands labeled by objects and coupons labeled by morphisms in $\mathcal{M}$.

Definition B.1. Let $\mathcal{M}$ be a pivotal tensor category. A *labeled oriented anchored planar tangle with coupons* consists of the following data:

- A disc with holes $T = \mathbb{D} \setminus (\mathring{D}_1 \cup \dots \cup \mathring{D}_r)$, as in Definition 2.1.
- Anchor points $Q = \{q_0, \dots, q_r\}$ and anchor lines as in Definition 2.5.
- Disjoint closed discs $C_1, \dots, C_s \subset T$ (the coupons) with marked points $c_i \in \partial C_i$. The coupons are not ordered. The anchor lines are allowed to intersect the coupons.
- A closed 1-dimensional oriented submanifold $X \subset T \setminus (\mathring{C}_1 \cup \dots \cup \mathring{C}_s)$ (the strands). The boundary of X lies on $\partial T \cup \bigcup \partial C_i$ and it does not touch the q_i and the c_i .
- An object of $\mathcal{M}$ for each connected component of X .
- A morphism $f_i \in \mathcal{M}(1, x_1 \otimes \dots \otimes x_k)$ for each coupon C_i , where the $x_1, \dots, x_k \in \mathcal{M}$ are as follows. Let m_j be the label of the j th strand that one encounters as one walks clockwise on ∂C_i , starting at c_i . We set $x_j = m_j$ if that j th strand is oriented outwards, and $x_j = (m_j)^\sim$ otherwise.

We illustrate this notion with an example:



Note that tangles with coupons have already appeared in the literature: the A_2 -planar tangle [EP10, Fig. 14] is an example of a tangle with coupons as in Definition B.1 (in which the trivalent junctures are the coupons).

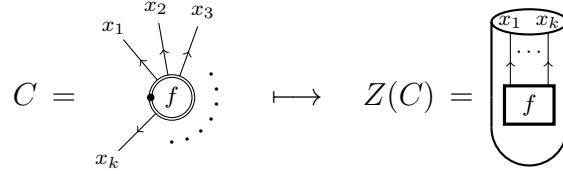
To an isotopy class of such tangles T , we wish to associate a morphism $Z(T)$. For the above example, this will be a map

$$Z(T) : \text{Tr}_C(a \otimes a^\sim \otimes b^\sim) \otimes \text{Tr}_C(d \otimes b \otimes a^\sim) \rightarrow \text{Tr}_C(a \otimes a^\sim \otimes b \otimes a \otimes d \otimes b \otimes c^\sim).$$

We proceed as follows. First, for each coupon C , labeled by some $f \in \mathcal{M}(1, x_1 \otimes \cdots \otimes x_k)$, we associate the element

$$Z(C) := \text{Tr}_C(f) \circ i : 1_C \rightarrow \text{Tr}_C(x_1 \otimes \cdots \otimes x_k).$$

In the graphical calculus of Section 5.1, this is represented by:



Let $\check{T}$ be tangle obtained from T by sliding the anchor lines off of coupons, removing the interiors of the coupons, and adding new anchor lines that connect the marked points c_i to anchor point q_0 . This is a tangle without coupons, and so it has an associated morphism $Z(\check{T})$ as in (59). We define $Z(T)$ by:

$$Z(T) := [[Z(\check{T}) \circ_{j_1} Z(C_1)] \circ_{j_2} Z(C_2)] \cdots \circ_{j_s} Z(C_s), \quad (60)$$

where we write j_i for the order of arrival at q_0 of the anchor line which connects it to C_i , and the coupons have been renumbered so as to have $j_1 > j_2 > \cdots > j_s$.

Theorem B.2. *Let T be a labeled oriented anchored planar tangle with coupons. Then the morphism $Z(T)$ defined above only depends on the isotopy class of T .*

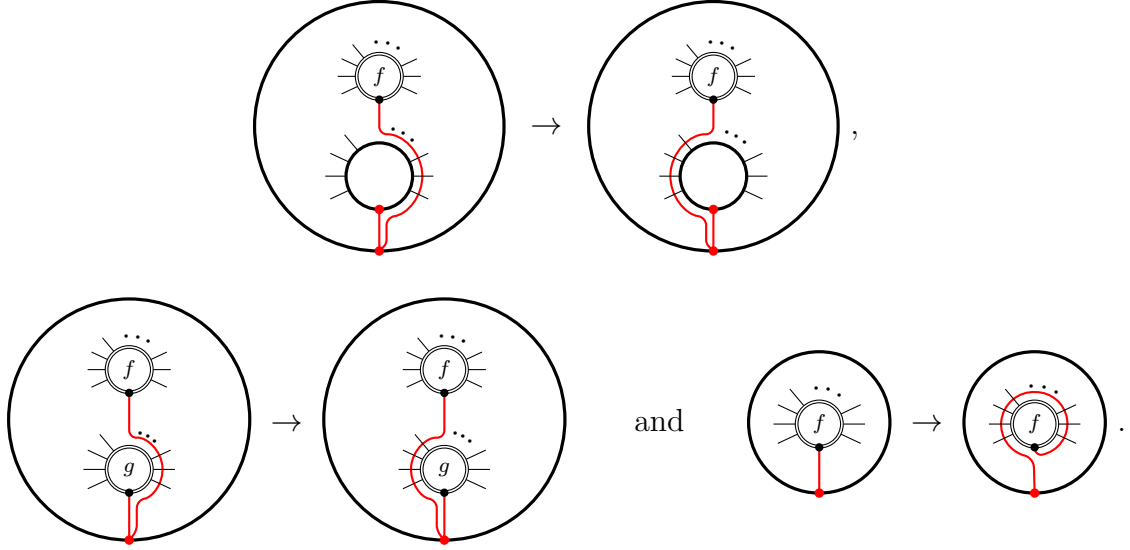
Let S and T be labeled oriented anchored planar tangles with coupons such that the labels on the i th input circle of T agree with those on the outer circle of S . Then the familiar composition formula holds:

$$Z(T \circ_i S) = Z(T) \circ_i Z(S).$$

Proof. In the construction of $\check{T}$, we had to move the anchor lines off the coupons, and add new anchor lines from q_0 to c_i . We need to show that $Z(T)$ does not depend on these choices. The isotopy invariance of $Z(T)$ will then follow from the corresponding property of $Z(\check{T})$ (proven in Sections 4 and 5 modulo the fact that, in those sections, we were not working with orientations and labelings).

We first argue that $Z(T)$ does not depend on the choice of anchor lines from q_0 to c_i . Fix a tangle T whose anchor lines do not intersect the coupons, and let us assume that $\check{T}$ and $\check{T}'$ are obtained from $T \setminus (\check{C}_1 \cup \cdots \cup \check{C}_s)$ by adding the required new anchor lines in two different ways. We write $Z(T)$ and $Z(T)'$ for the quantity (60) computed using $\check{T}$ and $\check{T}'$, respectively. Our goal is to show that $Z(T) = Z(T)'$.

The systems of anchor lines on $\check{T}$ and on $\check{T}'$ are related by a sequence of the following moves:



(In the language of Section 4.2, these correspond to right-composition of a ribbon braid by one of the generators ε_i or ϑ_i .) It is enough to prove $Z(T) = Z(T)'$ when $\check{T}$ differs from $\check{T}'$ by a single one of the above moves. If $\check{T}$ and $\check{T}'$ differ by the first type of move, then we have:

$$\begin{aligned}
Z(T)' &:= [[Z(\check{T}') \circ_{j'_1} \dots] \circ_{j'_\ell} Z(C'_\ell)] \circ_{j'_{\ell+1}} \dots \\
&= [[Z(\check{T}') \circ_{j_1} \dots] \circ_{j_{\ell-1}} Z(C_\ell)] \circ_{j_{\ell+1}} \dots \\
&= [[(Z(\check{T}) \circ_{j_{\ell-1}} \beta) \circ_{j_1} \dots] \circ_{j_{\ell-1}} (Z(C_\ell) \otimes \text{id})] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_{\ell-1}} (\beta \circ (Z(C_\ell) \otimes \text{id}))] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_{\ell-1}} (\text{id} \otimes Z(C_\ell))] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} Z(C_\ell)] \circ_{j_{\ell+1}} \dots = Z(T),
\end{aligned}$$

where $C'_\ell = C_\ell$ is the coupon labelled by f . If they differ by the second type of move, the computation is similar:

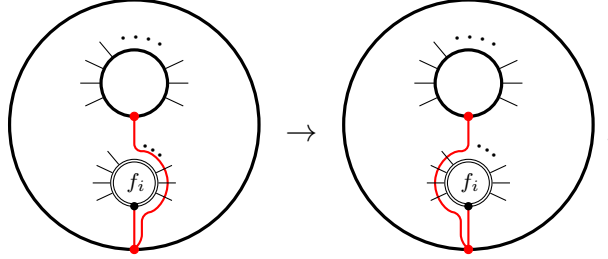
$$\begin{aligned}
Z(T)' &= [[Z(\check{T}') \circ_{j'_1} \dots] \circ_{j'_{\ell-1}} Z(C'_{\ell-1})] \circ_{j'_\ell} Z(C'_\ell)] \circ_{j'_{\ell+2}} \dots \\
&= [[(Z(\check{T}) \circ_{j_\ell} \beta) \circ_{j_1} \dots] \circ_{j_\ell} (Z(C'_\ell) \otimes Z(C'_{\ell-1}))] \circ_{j_{\ell+2}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} (\beta \circ (Z(C_{\ell-1}) \otimes Z(C_\ell)))] \circ_{j_{\ell+2}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} (Z(C_\ell) \otimes Z(C_{\ell-1}))] \circ_{j_{\ell+2}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_{\ell-1}} Z(C_{\ell-1})] \circ_{j_\ell} Z(C_\ell)] \circ_{j_{\ell+2}} \dots = Z(T),
\end{aligned}$$

where $C'_\ell = C_{\ell-1}$ is the coupon labelled by f and $C'_{\ell-1} = C_\ell$ is the coupon labelled by g . Finally, if $\check{T}$ and $\check{T}'$ differ by the third type of move, then:

$$\begin{aligned}
Z(T)' &= [[Z(\check{T}') \circ_{j'_1} \dots] \circ_{j'_\ell} Z(C'_\ell)] \circ_{j'_{\ell+1}} \dots \\
&= [[(Z(\check{T}) \circ_{j_\ell} \theta) \circ_{j_1} \dots] \circ_{j_\ell} Z(C_\ell)] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} (\theta \circ Z(C_\ell))] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} Z(C_\ell)] \circ_{j_{\ell+1}} \dots = Z(T).
\end{aligned}$$

In the above three computations, we have used in an essential way that the source of $Z(C'_\ell)$ (the unit object 1_C) has trivial braiding and trivial twist.

We now show that $Z(T)$ does not depend on the way we slide the anchor lines off of the coupons. Given the freedom that we have just established of choosing the anchor lines connecting q_0 and c_i , any two ways of sliding the initial anchor lines off of the coupons differ by (a composition of) the following move:



If $\check{T}$ and $\check{T}'$ are related by the above move, then we have:

$$\begin{aligned}
Z(T)' &= [[Z(\check{T}') \circ_{j'_1} \dots] \circ_{j'_\ell} Z(C'_\ell)] \circ_{j'_{\ell+1}} \dots \\
&= [[Z(\check{T}') \circ_{j_1} \dots] \circ_{j_{\ell+1}} Z(C_\ell)] \circ_{j_{\ell+1}} \dots \\
&= [[(Z(\check{T}) \circ_{j_\ell} \beta) \circ_{j_1} \dots] \circ_{j_\ell} (\text{id} \otimes Z(C_\ell))] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} (\beta \circ (\text{id} \otimes Z(C_\ell)))] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} (Z(C_\ell) \otimes \text{id})] \circ_{j_{\ell+1}} \dots \\
&= [[Z(\check{T}) \circ_{j_1} \dots] \circ_{j_\ell} Z(C_\ell)] \circ_{j_{\ell+1}} \dots = Z(T),
\end{aligned}$$

where $C'_\ell = C_\ell$ is the coupon labelled by f . This finishes the proof that $Z(T)$ only depends on the isotopy class of T .

Let now S and T be tangles with coupons whose i th operadic composition is defined (the labels on $\partial^i T$ agree with those on $\partial^0 S$). By (60), $Z(T) = [[Z(\check{T}) \circ_{j_1} Z(C_1)] \dots \circ_{j_s} Z(C_s)]$ and $Z(S) = [[Z(\check{S}) \circ_{k_1} Z(D_1)] \dots \circ_{k_t} Z(D_t)]$, where the C 's are the coupons of T and the D 's are the coupons of S . If choose the new anchor lines on $\check{T}$ so as to arrive at q_0 after those of T (with respect to the usual clockwise order (3)), then we compute:

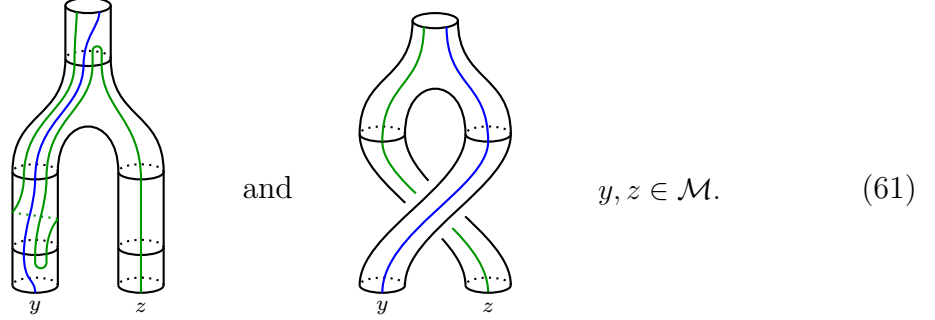
$$\begin{aligned}
Z(T) \circ_i Z(S) &= [[Z(\check{T}) \circ_{j_1} Z(C_1)] \dots \circ_{j_s} Z(C_s)] \circ_i Z(S) \\
&= [[[Z(\check{T}) \circ_i Z(S)] \circ_{j'_1} Z(C_1)] \dots \circ_{j'_s} Z(C_s)] \\
&= [[[Z(\check{T}) \circ_i [[Z(\check{S}) \circ_{k_1} Z(D_1)] \dots \circ_{k_t} Z(D_t)]] \circ_{j'_1} Z(C_1)] \dots \circ_{j'_s} Z(C_s)] \\
&= [[[Z(\check{T}) \circ_i Z(\check{S})] \circ_{k'_1} Z(D_1)] \dots \circ_{k'_t} Z(D_t)] \circ_{j'_1} Z(C_1)] \dots \circ_{j'_s} Z(C_s)] \\
&= [[[Z(\check{T}) \circ_i Z(\check{S})] \circ_{j''_1} Z(C_1)] \dots \circ_{j''_s} Z(C_s)] \circ_{k'_1} Z(D_1)] \dots \circ_{k'_t} Z(D_t)] \\
&= [[Z(\widetilde{T \circ_i S}) \circ_{j''_1} Z(C_1)] \dots \circ_{j''_s} Z(C_s)] \circ_{k'_1} Z(D_1)] \dots \circ_{k'_t} Z(D_t)] \\
&= Z(T \circ_i S),
\end{aligned}$$

where $k'_n = k_n + i - 1$, $j'_n = j_n + a - 1$, $j''_n = j_n + b - 1$, a is the number of inputs of $Z(S)$, and b is the number of inputs of $\check{Z}(S)$. $\square$

C The tube string calculus for the categorified trace

In this appendix, we show that the calculus of tubes with strings introduced in our earlier paper [HPT16] and reviewed in Section 5.1 is invariant under all 3-dimensional isotopies.

We illustrate the non-triviality of the problem with an example. Consider the following two ‘tube string diagrams’:



Here, as usual, $\mathcal{M}$ is a module tensor category over some braided pivotal category $\mathcal{C}$. Following [HPT16], we can associate morphisms $\mathrm{Tr}_{\mathcal{C}}(y) \otimes \mathrm{Tr}_{\mathcal{C}}(z) \rightarrow \mathrm{Tr}_{\mathcal{C}}(z \otimes y)$ to the above two pictures. These are:

$$\mathrm{Tr}_{\mathcal{C}}(\mathrm{id}_{z \otimes y} \otimes \mathrm{ev}_z) \circ \mu_{z \otimes y \otimes z^-, z} \circ (\tau_{y \otimes z^-, z} \otimes \mathrm{id}_{\mathrm{Tr}_{\mathcal{C}}(z)}) \circ (\mathrm{Tr}_{\mathcal{C}}(\mathrm{id}_y \otimes \mathrm{coev}_{z^-}) \otimes \mathrm{id}_{\mathrm{Tr}_{\mathcal{C}}(z)})$$

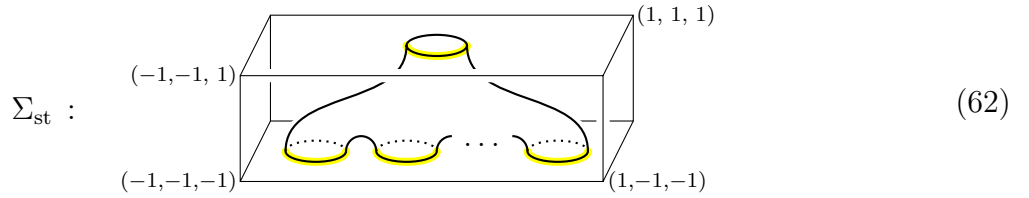
and $\mu_{z, y} \circ \beta_{\mathrm{Tr}_{\mathcal{C}}(y), \mathrm{Tr}_{\mathcal{C}}(z)}.$

Now, as it happens, the two pictures in (61) are isotopic, which raises the expectation that those two morphisms should be equal. That is indeed true, but a direct proof using the identities in Section 5.1 as axioms is a rather non-trivial exercise³.

The goal of this appendix is to show that whenever two string tube string diagrams are isotopic, the corresponding morphisms in $\mathcal{C}$ are equal. We prove this by establishing a dictionary between tube string diagrams and anchored planar tangles.

Definition C.1. A *tube string diagram* consists of:

- An embedded 2-manifold $\Sigma \subset [-1, 1]^3$ with no local maxima in its interior, which is isotopic to the following standard surface



via an isotopy that fixes the boundary. Here, $\partial\Sigma = \partial\Sigma_{\mathrm{st}}$ consists of $k \geq 0$ ‘input circles’ $\partial^1\Sigma, \dots, \partial^k\Sigma$ (on the $z = -1$ plane) whose centers have zero y -coordinate, and of one ‘output circle’ $\partial^0\Sigma$ (on the $z = 1$ plane) centered at $(0, 0, 1)$.

We write $\partial_{\mathrm{vis}}\Sigma$ for the part of $\partial\Sigma$ whose y -coordinate is negative: the ‘visible’ part of the boundary, highlighted in yellow in the above picture.

³*Hint:* the proof uses (T13), (T14) and (T15).

- Disjoint closed discs $C_1, \dots, C_s \subset \mathring{\Sigma}$ (the coupons) with marked points $c_i \in \partial C_i$.
- A closed 1-dimensional oriented submanifold $X \subset \Sigma \setminus (\mathring{C}_1 \cup \dots \cup \mathring{C}_s)$ (the strands); the boundary of X lies in $\partial_{\text{vis}} \Sigma \cup \bigcup \partial C_i$ and does not touch the marked points c_i .
- An object of $\mathcal{M}$ for each connected component of X .
- A morphism $f_i \in \mathcal{M}(1, x_1 \otimes \dots \otimes x_k)$ for each coupon C_i , as in Definition B.1.

We now explain how to associate an oriented anchored planar tangle with coupons, well defined up to isotopy, to a tube string diagram. This assignment will be denoted

$$\Sigma \mapsto T(\Sigma).$$

We refer the reader to Figure 5 for an example.

Given a tube string diagram $\Sigma \subset [0, 1]^3$, let us write $D_i \subset \partial[0, 1]^3$ for the disc bounded by $\partial^i \Sigma$ (so that $\partial^i \Sigma = \partial D_i$), and let $\Sigma^+ := \Sigma \cup (D_1 \cup \dots \cup D_k)$. Pick an orientation preserving diffeomorphism $\varphi : \Sigma^+ \rightarrow \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ with the property that $\varphi(C_i)$ and $\varphi(D_i)$ are round disks, and let

$$T := \mathbb{D} \setminus (\varphi(\mathring{D}_1) \cup \dots \cup \varphi(\mathring{D}_k)).$$

We can then use the diffeomorphism $\varphi : \Sigma \rightarrow T$ to transfer the strands and coupons from Σ to T . Our next task is to equip T with anchor lines. We will first define the anchor lines on Σ , and then transfer them to T using φ .

Let $\Gamma := \partial[0, 1]^3 \setminus (\mathring{D}_0 \cup \dots \cup \mathring{D}_k)$, and let W be the 3-manifold bounded by $\Gamma \cup \Sigma$ (the solid cube minus the stuff inside Σ). Then

$$\Gamma \hookrightarrow W \hookleftarrow \Sigma$$

is a trivial cobordism⁴. It therefore induces a diffeomorphism $\psi : \Gamma \rightarrow \Sigma$, well defined up to isotopy (because surface diffeomorphisms are isotopic if and only if they are homotopic [FM12, §1.4]).

Let $q_i \in \partial^i \Sigma = \partial D_i$ be the points on the back of the tubes (the points with maximal y -coordinate), and consider the following system of standard lines, $A_\Gamma \subset \Gamma$, which connect q_0 to the other points q_i . The i th line starts at q_i , travels straight to $[-1, 1] \times \{1\} \times \{-1\}$, then straight to $[-1, 1] \times \{1\} \times \{1\}$, and then straight to q_0 . The anchor lines on Σ are the image of those lines under the diffeomorphism $\psi : \Gamma \rightarrow \Sigma$

$$A_\Sigma := \psi(A_\Gamma)$$

and are characterised by the fact that $A_\Gamma \cup A_\Sigma$ bound k disjoint disks in W . At last, we transfer the anchor lines A_Σ from Σ to T by means of the diffeomorphism $\varphi : \Sigma \rightarrow T$. This construction establishes a bijection between isotopy classes of tube string diagrams and isotopy classes of labeled oriented anchored planar tangle with coupons.

If Σ and Σ' are two tube string diagrams and if the labels along the i th input of Σ agree with those along the output of Σ' , then we can glue them together in the obvious way to form a new tube string diagram $\Sigma \circ_i \Sigma'$, well defined up to isotopy.

⁴A cobordism isomorphic to $\Sigma \times \{0\} \hookrightarrow \Sigma \times [0, 1] \cup_{\partial \Sigma \times [0, 1]} \partial \Sigma \hookleftarrow \Sigma \times \{1\}$.

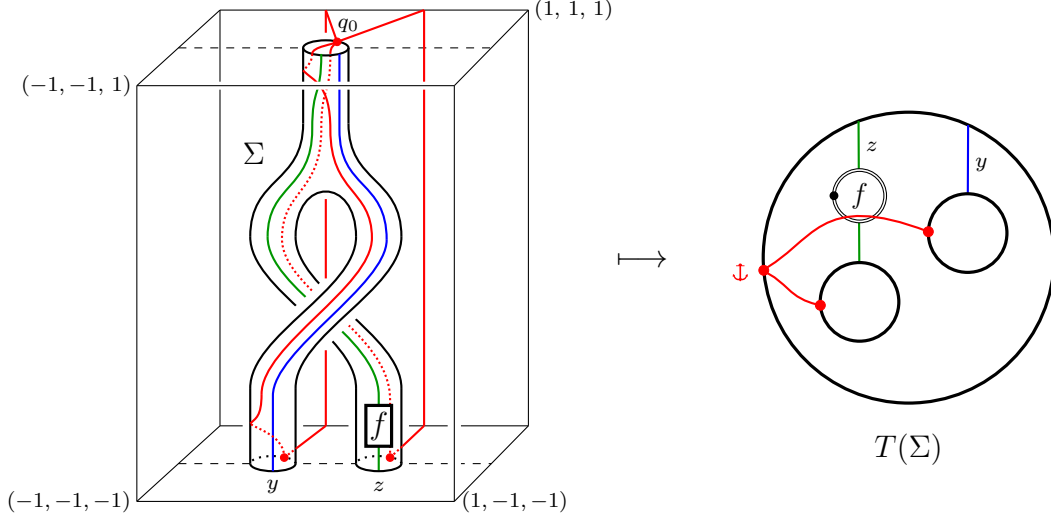
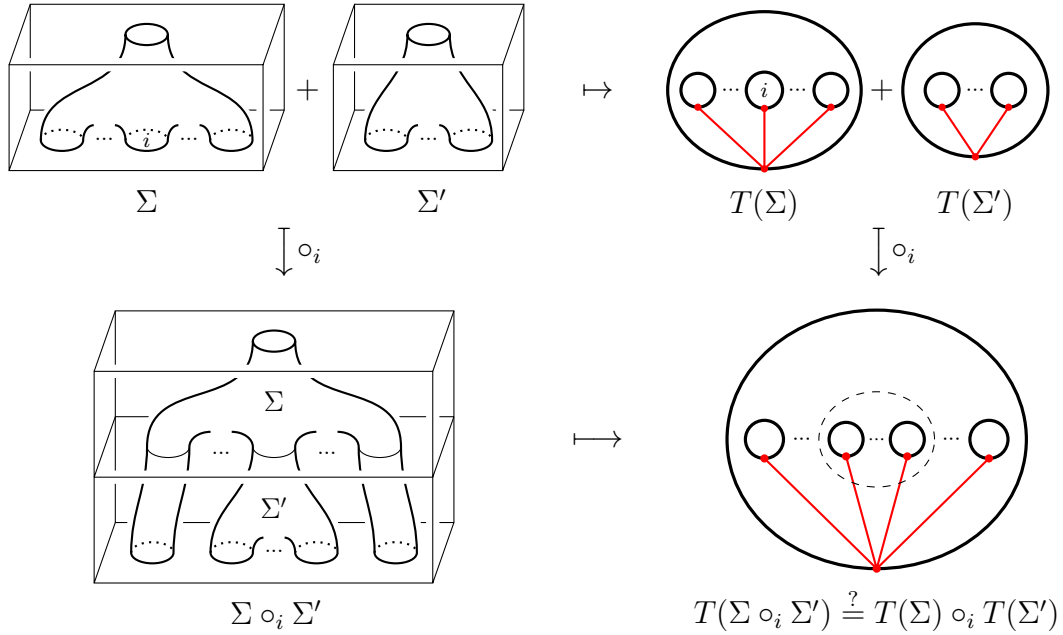


Figure 5: The construction of an oriented anchored planar tangle with coupons $T(\Sigma)$ from a tube string diagram Σ (the green and blue strands are oriented, even though this is not indicated in the picture).

Lemma C.2. *The map $\Sigma \mapsto T(\Sigma)$ is compatible with the operadic composition of tangles and of tube string diagrams: $T(\Sigma \circ_i \Sigma') = T(\Sigma) \circ_i T(\Sigma')$.*

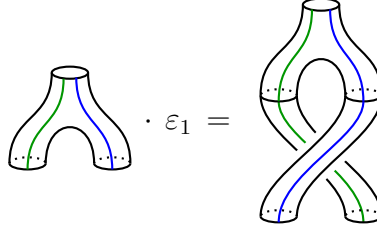
Proof. Let Σ and Σ' be two tube string diagrams whose i th composition is well defined. The operations $\Sigma \mapsto T(\Sigma)$ and $\circ_i$ being isotopy invariant, we may assume without loss of generality that the underlying surfaces of Σ and Σ' are in standard form (62). In that case, the construction of $T(\Sigma)$ from Σ is very simple and we have:



Let T be the manifold depicted on the bottom right (a disc with holes), along with the anchor lines as drawn. The tangle $T(\Sigma \circ_i \Sigma')$ is the result of taking T and transferring onto

it the pattern of strands and coupons from Σ and Σ' using the diffeomorphism $\Sigma \cup_i \Sigma' \rightarrow \Sigma \circ_i \Sigma' \rightarrow T$. Similarly, the tangle $T(\Sigma) \circ_i T(\Sigma')$ is obtained by transferring that same pattern of strands and coupons via the diffeomorphisms $\Sigma \cup_i \Sigma' \rightarrow T(\Sigma) \cup_i T(\Sigma') \rightarrow T$. Those diffeomorphisms are clearly isotopic, so $T(\Sigma \circ_i \Sigma') = T(\Sigma) \circ_i T(\Sigma')$. $\square$

Given a ribbon braid $\sigma \in \mathcal{RB}_n$ and a tube string diagram Σ with n inputs, one can form a new tube string diagram $\Sigma \cdot \sigma$ by gluing σ on the bottom of Σ . For example:



(we write ε_i and ϑ_i for the generators of $\mathcal{RB}_n$, as in Section 4.2). This defines a right⁵ action of the ribbon braid group on the set of isotopy classes of tube string diagrams with n inputs.

For a fixed planar tangle T with n inputs, there is also a right action of $\mathcal{RB}_n$ on the set of systems of anchor lines on T . We now describe that action. Let T_{st} be the ‘standard tangle’, with input circles along the y -axis, and standard system of anchor lines A_{st} , as in Section 4.2. Given a system of anchor lines A on T , pick a diffeomorphism $\varphi : T_{\text{st}} \rightarrow T$ (disregarding strands and coupons) that sends A_{st} to A . We define:

$$A \cdot \sigma := \varphi(\sigma \cdot A_{\text{st}}),$$

where $\sigma \cdot A_{\text{st}}$ is as described in Section 4.2. Equivalently, if ψ_σ is the diffeomorphism of T_{st} defined in the picture (17), so that $\sigma \cdot A_{\text{st}} = \psi_\sigma(A_{\text{st}})$, then the right action of σ on $A = \varphi(A_{\text{st}})$ is given by:

$$\varphi(A_{\text{st}}) \cdot \sigma := \varphi \circ \psi_\sigma(A_{\text{st}}).$$

Note that, by taking $\varphi = \psi_{\sigma'}$ in the above formula for some $\sigma' \in \mathcal{RB}_n$, one gets the relation $(\sigma' \cdot A_{\text{st}}) \cdot \sigma = \sigma' \sigma \cdot A_{\text{st}}$.

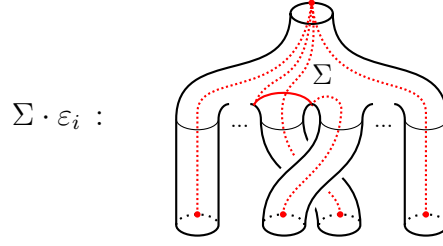
Lemma C.3. *The map $\Sigma \mapsto T(\Sigma)$ is equivariant with respect to the above right actions of the ribbon braid group: $T(\Sigma \cdot \sigma) = T(\Sigma) \cdot \sigma$.*

Proof. It is enough to prove the lemma for the generators ε_i and ϑ_i . If $\sigma = \vartheta_i$, then $T(\Sigma \cdot \vartheta_i)$ differs from $T(\Sigma)$ by the fact that its strands make an extra twist around the i th input disc, whereas $T(\Sigma) \cdot \vartheta_i$ differs from $T(\Sigma)$ by the fact that the i th anchor line makes an extra twist. Those two anchored planar tangles are isotopic by means of a Dehn twist.

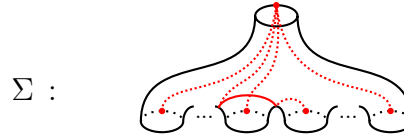
We now treat the case $\sigma = \varepsilon_i$. Let Σ be a tube string diagram. The operation $\Sigma \mapsto T(\Sigma)$ and the right actions of σ being isotopy invariant, we may assume, as in Lemma C.2, that the underlying surface of Σ is in standard form. To compute $T(\Sigma \cdot \varepsilon_i)$, one starts with T ,

⁵Here, we read braids from top to bottom (e.g., $\varepsilon_1 \varepsilon_2 = \text{diagram}$). This is the opposite of the convention used in Section 4.2.

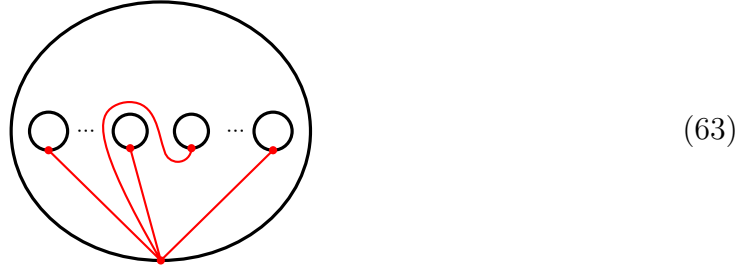
adds a braid between the i th and $(i+1)$ st legs, and computes the anchor lines, as in Figure 5. This yields the following picture:



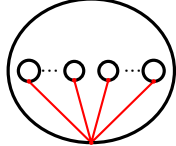
Now, by ‘shortening the legs’, one can map the above picture, via a diffeomorphism, back to Σ . Note that the anchor lines are now twisted:



One can then map these anchor lines onto the standard tangle T_{st} , using the standard diffeomorphism $\varphi_{\text{st}} : \Sigma \rightarrow T_{\text{st}}$ which ‘flattens things out’:



The tangle $T(\Sigma \cdot \varepsilon_i)$ is the above picture, along with the pattern of stands and coupons transferred from Σ via the diffeomorphism φ_{st} .

The anchor lines (63) are equal to  $\cdot \varepsilon_i$, and so $T(\Sigma \cdot \varepsilon_i) = T(\Sigma) \cdot \varepsilon_i$. $\square$

Let $P(\sigma)$ is as in (16).

Lemma C.4. *The assignment $T \mapsto Z(T)$ defined in (60) is compatible with the right action of the braid group: $Z(T \cdot \sigma) = Z(T) \circ P(\sigma)$.*

Proof. We first prove the lemma when T has no coupons. Let T be such a tangle. Put T in standard form (Section 4.1) and write its anchor lines as $A = \sigma_T \cdot A_{\text{st}}$, as in Algorithm 4.9. By (21), we then have $Z(T) = Z(T, A_{\text{st}}) \circ P(\sigma_T)$, and so:

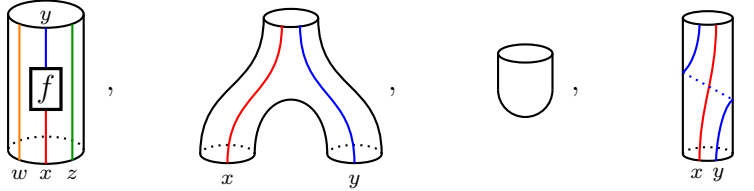
$$\begin{aligned}
 Z(T \cdot \sigma) &= Z((T, \sigma_T \cdot A_{\text{st}}) \cdot \sigma) \\
 &= Z(T, \sigma_T \sigma \cdot A_{\text{st}}) \\
 &= Z(T, A_{\text{st}}) \circ P(\sigma_T \sigma) \\
 &= Z(T, A_{\text{st}}) \circ P(\sigma_T) \circ P(\sigma) \\
 &= Z(T) \circ P(\sigma).
 \end{aligned}$$

If T has coupons, then we can write it as a composite $T = T' \circ_1 T''$ of an annular tangle T' (with coupons), and a tangle without coupons T'' . We then have:

$$\begin{aligned}
Z(T \cdot \sigma) &= Z((T' \circ_1 T'') \cdot \sigma) \\
&= Z(T' \circ_1 (T'' \cdot \sigma)) \\
&= Z(T') \circ Z(T'' \cdot \sigma) \\
&= Z(T') \circ Z(T'') \circ P(\sigma) \\
&= Z(T' \circ_1 T'') \circ P(\sigma) \\
&= Z(T) \circ P(\sigma).
\end{aligned}$$

□

Given a tube string diagram labelled by objects and morphisms of $\mathcal{M}$, we now define its evaluation in $\mathcal{C}$. Consider the following ‘elementary tube string diagrams’ (all the strands are oriented, even though this is not visible in our pictures)



possibly modified by replacing each stand by many parallel strands (or no strands), changing the orientations of the strands, and replacing the coupon by a cup or cap. (To conform to the definition, we should really have replaced the morphism $f : x \rightarrow y$ by its mate $\tilde{f} : 1 \rightarrow y \otimes x^\vee$, and equipped the coupon with a marked point on its left side, but we will ignore these details.) Suppose that Σ is a tube string diagram which is given to us as a composition of elementary tube string diagrams Σ_i and a braid $\sigma \in \mathcal{B}_n$:

$$\Sigma = [[[\Sigma_1 \circ_{j_1} \Sigma_2] \circ_{j_2} \Sigma_3 \dots] \circ_{j_{k-1}} \Sigma_k] \cdot \sigma. \quad (64)$$

Then, following [HPT16], we define

$$Z(\Sigma) := [[[Z(\Sigma_1) \circ_{j_1} Z(\Sigma_2)] \circ_{j_2} \dots] \circ_{j_{k-1}} Z(\Sigma_k)] \circ P(\sigma), \quad (65)$$

where the value of $Z(-)$ on the elementary tube string diagrams is given by

$$Z\left(\text{Cylinder with } f\right) := \text{Tr}_{\mathcal{C}}(\text{id}_w \otimes f \otimes \text{id}_z), \quad Z\left(\text{Y-junction}\right) := \mu_{x,y}, \quad Z\left(\text{Cup}\right) := i, \quad Z\left(\text{Cylinder with strands}\right) := \tau_{x,y},$$

and $P(\sigma)$ is as in (16). We can now state the main theorem of this section:

Theorem C.5. *Let Σ and Σ' be two tube string diagrams which are presented as composites of elementary tube string diagrams and a braid (as in (64)), and let $Z(\Sigma)$ and $Z(\Sigma')$ be as in (65). Then*

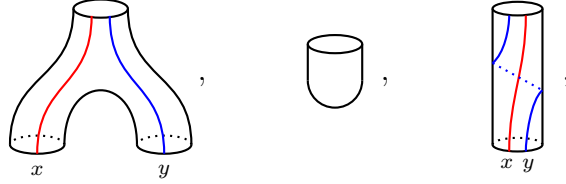
$$\Sigma \text{ and } \Sigma' \text{ are isotopic} \implies Z(\Sigma) = Z(\Sigma').$$

Proof. By Theorem B.2, it is enough to argue that for every tube string diagram as in (64), we have

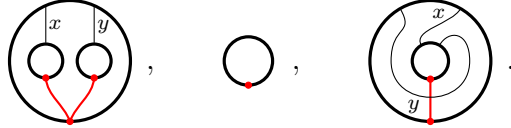
$$Z(\Sigma) = Z(T(\Sigma)) \quad (66)$$

(where the two sides are defined in (65) and (60), respectively). Indeed, assuming that fact, if Σ and Σ' are isotopic tube string diagrams, then $Z(\Sigma) = Z(T(\Sigma)) = Z(T(\Sigma')) = Z(\Sigma')$, where the middle equality holds by Theorem B.2.

If Σ is equal to one of the following elementary tube string diagrams (potentially modified by replacing strands with some number of parallel strands, and changing orientations)



then $T(\Sigma)$ is given by:



By the analog of Lemmas 7.13, 7.11, and 7.12 for labeled oriented tangles, we then have

$$Z\left(\text{circle with two small circles inside, labeled } x \text{ and } y\right) = \mu_{x,y}, \quad Z(\text{circle with a red dot}) = i, \quad \text{and} \quad Z\left(\text{circle with a red strand labeled } y \text{ and a blue strand labeled } x\right) = \tau_{x,y}, \quad (67)$$

and so equation (66) holds for those tangles. The same argument goes through when the strands labelled x and y are replaced by an arbitrary numbers of parallel strands.

If

$$\Sigma = \text{cylinder with a box labeled } f \text{ and strands } w, x, z \text{ at the bottom and } y \text{ at the top}, \quad \text{then} \quad T(\Sigma) = \text{rectangle with a box labeled } \tilde{f} \text{ and strands } w, y, z \text{ at the top and } x \text{ at the bottom}, \quad (68)$$

where $\tilde{f} : 1 \rightarrow y \otimes x^\sim$ is the mate of $f : x \rightarrow y$. Applying the definition (60) of $Z(-)$ to the right hand side of (68), we get:

$$Z(T(\Sigma)) = Z\left(\text{rectangle with a box labeled } \tilde{f} \text{ and strands } w, y, z \text{ at the top and } x \text{ at the bottom}\right) \circ_2 \left(\text{Tr}_C(\tilde{f}) \circ i\right).$$

By the second part of Theorem B.2, this is then equal to:

$$Z\left(\text{rectangle with a box labeled } \tilde{f} \text{ and strands } w, y, z \text{ at the top and } x^\sim, x \text{ at the bottom}\right) \circ_1 Z\left(\text{rectangle with a box labeled } \tilde{f} \text{ and strands } w, y, x^\sim, x, z \text{ at the top and } x \text{ at the bottom}\right) \circ_2 \left(\text{Tr}_C(\tilde{f}) \circ i\right).$$

By the analog of Examples 4.12 and 4.13 for labeled oriented tangles (recall that α_i and $\varpi_{i,j}$ are defined in Theorem 5.1), we then get:

$$Z(T(\Sigma)) = Z\left(\begin{array}{c} \text{Diagram of a tube string diagram with a coupon labeled } f \text{ and labels } w, x, z \text{ at the bottom} \end{array}\right).$$

It is now an easy exercise, using the identities in Section 5.1, to check that the right hand side simplifies to $\text{Tr}_C(\text{id}_w \otimes f \otimes \text{id}_z)$. This finishes the proof of equation (66) for the tube string diagram in (68). The variants of (68) where the coupon is replaced by a cup or a cap can be treated by similar methods, and are left to the reader as an exercise.

So far, we have shown that equation (66) holds true for all the elementary tube string diagrams. We now consider the general case. If $\Sigma = [[\Sigma_1 \circ_{j_1} \Sigma_2] \circ_{j_2} \Sigma_3 \dots] \cdot \sigma$ is a composite of elementary tube string diagrams Σ_i and a braid σ , then we have:

$$\begin{aligned} Z(T(\Sigma)) &= Z(T([\Sigma_1 \circ_{j_1} \Sigma_2] \circ_{j_2} \Sigma_3 \dots] \cdot \sigma)) \\ &= Z(T([\Sigma_1 \circ_{j_1} \Sigma_2] \circ_{j_2} \Sigma_3 \dots] \cdot \sigma)) \\ &= Z(T([\Sigma_1 \circ_{j_1} \Sigma_2] \circ_{j_2} \Sigma_3 \dots]) \circ P(\sigma)) \\ &= Z([T(\Sigma_1) \circ_{j_1} T(\Sigma_2)] \circ_{j_2} T(\Sigma_3) \dots] \circ P(\sigma)) \\ &= [[Z(T(\Sigma_1)) \circ_{j_1} Z(T(\Sigma_2))] \circ_{j_2} Z(T(\Sigma_3)) \dots] \circ P(\sigma)) \\ &= [[Z(\Sigma_1) \circ_{j_1} Z(\Sigma_2)] \circ_{j_2} Z(\Sigma_3) \dots] \circ P(\sigma) \\ &= Z(\Sigma), \end{aligned}$$

where the second equality holds by Lemma C.3, the third equality holds by Lemma C.4, the fourth equality holds by Lemma C.2, the fifth equality holds by the second half of Theorem B.2, the sixth equality holds because the Σ_i are elementary tube string diagrams, and the last equality is the definition (65) of $Z(\Sigma)$. $\square$

References

- [AMP15] Narjess Afzaly, Scott Morrison, and David Penneys, *The classification of subfactors with index at most $5\frac{1}{4}$* , 2015, [arXiv:1509.00038](#), to appear Mem. Amer. Math. Soc.
- [BEK01] Jens Böckenhauer, David E. Evans, and Yasuyuki Kawahigashi, *Longo-Rehren subfactors arising from α -induction*, Publ. Res. Inst. Math. Sci. **37** (2001), no. 1, 1–35, [MR1815993](#) [arXiv:math/0002154v1](#).
- [BHP12] Arnaud Brothier, Michael Hartglass, and David Penneys, *Rigid C^* -tensor categories of bimodules over interpolated free group factors*, J. Math. Phys. **53** (2012), no. 12, 123525, 43, [MR3405915](#) [DOI:10.1063/1.4769178](#) [arXiv:1208.5505](#).

- [BMPS12] Stephen Bigelow, Scott Morrison, Emily Peters, and Noah Snyder, *Constructing the extended Haagerup planar algebra*, Acta Math. **209** (2012), no. 1, 29–82, [MR2979509](#), [arXiv:0909.4099](#), [DOI:10.1007/s11511-012-0081-7](#).
- [BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin, 1973, [MR0420609](#).
- [CMW09] David Clark, Scott Morrison, and Kevin Walker, *Fixing the functoriality of Khovanov homology*, Geom. Topol. **13** (2009), no. 3, 1499–1582, [MR2496052](#) [DOI:10.2140/gt.2009.13.1499](#) [arXiv:math.GT/0701339](#).
- [DMNO13] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik, *The Witt group of non-degenerate braided fusion categories*, J. Reine Angew. Math. **677** (2013), 135–177, [MR3039775](#) [arXiv:1009.2117](#).
- [EG14] David E. Evans and Terry Gannon, *Near-group fusion categories and their doubles*, Adv. Math. **255** (2014), 586–640, [MR3167494](#) [arXiv:1208.1500](#) [DOI:10.1016/j.aim.2013.12.014](#).
- [EP10] David E. Evans and Mathew Pugh, *A_2 -planar algebras I*, Quantum Topol. **1** (2010), no. 4, 321–377, [MR2733244](#) [DOI:10.4171/QT/8](#) [arXiv:0906.4225](#).
- [Fie] Zbigniew Fiedorowicz, *Symmetric bar construction*, Available at <https://people.math.osu.edu/fiedorowicz.1/>.
- [FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012, [MR2850125](#).
- [FRS04] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert, *TFT construction of RCFT correlators. III. Simple currents*, Nuclear Phys. B **694** (2004), no. 3, 277–353, [MR2076134](#) [DOI:10.1016/j.nuclphysb.2004.05.014](#) [arXiv:hep-th/0403157](#).
- [Got10] Satoshi Goto, *On Ocneanu’s theory of double triangle algebras for subfactors and classification of irreducible connections on the Dynkin diagrams*, Expo. Math. **28** (2010), no. 3, 218–253, [MR2670999](#) [DOI:10.1016/j.exmath.2009.11.001](#).
- [Hen17] André G. Henriques, *What Chern-Simons theory assigns to a point*, Proc. Natl. Acad. Sci. USA **114** (2017), no. 51, 13418–13423, [MR3747830](#) [DOI:10.1073/pnas.1711591114](#) [arXiv:1503.06254](#).
- [HP17] André Henriques and David Penneys, *Bicommutant categories from fusion categories*, Selecta Math. (N.S.) **23** (2017), no. 3, 1669–1708, [MR3663592](#) [DOI:10.1007/s00029-016-0251-0](#) [arXiv:1511.05226](#).
- [HPT16] André Henriques, David Penneys, and James Tener, *Categorified trace for module tensor categories over braided tensor categories*, Doc. Math. **21** (2016), 1089–1149, [MR3578212](#) [arXiv:1509.02937](#).
- [Izu93] Masaki Izumi, *On type II and type III principal graphs of subfactors*, Math. Scand. **73** (1993), no. 2, 307–319.
- [Izu01] Masaki Izumi, *The structure of sectors associated with Longo-Rehren inclusions. II. Examples*, Rev. Math. Phys. **13** (2001), no. 5, 603–674, [MR1832764](#) [DOI:10.1142/S0129055X01000818](#).
- [Izu17] Masaki Izumi, *A Cuntz algebra approach to the classification of near-group categories*, Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones’ 60th birthday, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 46, Austral. Nat. Univ., Canberra, 2017, [MR3635673](#) [arXiv:1512.04288](#), pp. 222–343.
- [JL17] Arthur Jaffe and Zhengwei Liu, *Planar Para Algebras, Reflection Positivity*, Comm. Math. Phys. **352** (2017), no. 1, 95–133, [MR3623255](#) [DOI:10.1007/s00220-016-2779-4](#) [arXiv:1602.02662](#).
- [JLW16] Arthur Jaffe, Zhengwei Liu, and Alex Wozniakowski, *Qudit isotopy*, 2016, [arXiv:1602.02671](#).

- [JMS14] Vaughan F. R. Jones, Scott Morrison, and Noah Snyder, *The classification of subfactors of index at most 5*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), no. 2, 277–327, [MR3166042](#), [arXiv:1304.6141](#), [DOI:10.1090/S0273-0979-2013-01442-3](#).
- [Jon99] Vaughan F. R. Jones, *Planar algebras I*, 1999, [arXiv:math.QA/9909027](#).
- [JP11] Vaughan F. R. Jones and David Penneys, *The embedding theorem for finite depth subfactor planar algebras*, Quantum Topol. **2** (2011), no. 3, 301–337, [arXiv:1007.3173](#), [MR2812459](#), [DOI:10.4171/QT/23](#).
- [Lur09] Jacob Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, [MR2555928](#) [arXiv:0905.0465](#), pp. 129–280.
- [May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. 271 [MR0420610](#), Available at <http://www.math.uchicago.edu/~may/BOOKS/gils.pdf>.
- [MPS10] Scott Morrison, Emily Peters, and Noah Snyder, *Skein theory for the D_{2n} planar algebras*, J. Pure Appl. Algebra **214** (2010), no. 2, 117–139, [arXiv:0808.0764](#) [MR2559686](#) [DOI:10.1016/j.jpaa.2009.04.010](#).
- [Müg03] Michael Müger, *From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors*, J. Pure Appl. Algebra **180** (2003), no. 1-2, 159–219, [MR1966525](#) [DOI:10.1016/S0022-4049\(02\)00248-7](#) [arXiv:math.CT/0111205](#).
- [Pen12] David Penneys, *A cyclic approach to the annular Temperley-Lieb category*, J. Knot Theory Ramifications **21** (2012), no. 6, 1250049, 40, [MR2903179](#) [arXiv:0912.1320](#) [DOI:10.1142/S0218216511010012](#).
- [Pet10] Emily Peters, *A planar algebra construction of the Haagerup subfactor*, Internat. J. Math. **21** (2010), no. 8, 987–1045, [MR2679382](#), [DOI:10.1142/S0129167X10006380](#), [arXiv:0902.1294](#).
- [Pop90] Sorin Popa, *Classification of subfactors: the reduction to commuting squares*, Invent. Math. **101** (1990), no. 1, 19–43, [MR1055708](#), [DOI:10.1007/BF01231494](#).
- [Pop95] ———, *Classification of subfactors and their endomorphisms*, CBMS Regional Conference Series in Mathematics, vol. 86, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1995, [MR1339767](#).
- [Shi11] Kenichi Shimizu, *Frobenius-Schur indicators in Tambara-Yamagami categories*, J. Algebra **332** (2011), 543–564, [MR2774703](#) [arXiv:1005.4500](#) [DOI:10.1016/j.jalgebra.2011.02.002](#).
- [TY98] Daisuke Tambara and Shigeru Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra **209** (1998), no. 2, 692–707, [MR1659954](#).
- [Yam12] Shigeru Yamagami, *Representations of multicategories of planar diagrams and tensor categories*, 2012, [arXiv:1207.1923](#).