

Distributed Computation of Stochastic GNE With Partial Information: An Augmented Best-Response Approach

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Abstract—In this article, we focus on the stochastic generalized Nash equilibrium problem (SGNEP) which is an important and widely used model in many different fields. In this model, subject to certain global resource constraints, a set of self-interested players aims to optimize their local objectives that depend on their own decisions and the decisions of others and are influenced by some random factors. We propose a distributed stochastic generalized Nash equilibrium seeking algorithm in a partial-decision information setting based on the Douglas–Rachford operator splitting scheme, which relaxes assumptions in the existing literature. The proposed algorithm updates players’ local decisions through augmented best-response schemes and subsequent projections onto the local feasible sets, which comprise most of the computational workload. The projected stochastic subgradient method is applied to provide approximate solutions to the augmented best-response subproblems for each player. The Robbins–Siegmund theorem is leveraged to establish the main convergence results to a true Nash equilibrium using the proposed inexact solver. Finally, we illustrate the validity of the proposed algorithm via two numerical examples: 1) a stochastic Nash–Cournot distribution game and 2) a multiproduct assembly problem with the two-stage model.

Index Terms—Distributed algorithm, generalized Nash equilibrium, operator splitting, stochastic optimization, variational inequality.

I. INTRODUCTION

IN THE Nash equilibrium problem (NEP), a set of self-interested players aims to optimize their individual payoffs which depend not only on their own decisions but also on the decisions of others [2]. The generalized Nash equilibrium problem (GNEP) extends the NEP by considering additional global resource constraints that these players should collectively satisfy [3], [4]. In numerous practical applications, such as

strategic behaviors in power markets [5], [6]; engagement of multiple humanitarian organizations in disaster relief [7]; and the traffic assignment of strategic risk-averse users [8]; in addition to couplings in objectives and global constraints, there is also uncertainty involved in the objective functions. For example, in the target-rate power-management problem for wireless networks, the power of battery-driven devices should be regulated in a real-time manner in the presence of inherent stochastic fluctuations of the underlying network [9]. These applications motivate us to consider an extension to the GNEP, called the stochastic generalized Nash equilibrium problem (SGNEP). In the SGNEP, instead of deterministic objective functions, players optimize the expected values of uncertain objective functions which are dependent on some random variables. Although the SGNEP can capture a wider range of applications, computing its Nash equilibria becomes a much more challenging problem compared to the GNEP, due to the absence of closed-form expressions of the objective functions. Fortunately, as has been shown in [10, Sec. 1.4], many SGNEPs can be formulated as stochastic variational inequalities (SVIs) and solved by leveraging existing results from this field, e.g., [11], [12], [13], [14], [15].

Our aim in this article is to develop a distributed algorithm under the partial-decision information setting for solving SGNEPs over a network of players. In the context of noncooperative games on networks, participants are self-interested and make rational decisions that maximize their own payoffs. It is unrealistic that these participants will adopt any centralized methods that require cooperation among them. Because of this, there is an enduring research interest in distributing the computation of Nash equilibria [16], [17], especially through the avenue of the operator splitting technique [18], [19]. In addition to the distributed computation, under most circumstances, participants can only have access to local information and decisions of their neighbors, which constitutes a partial-decision information setting [20], [21], [22]. One reason is that these players are reluctant to send their local information and decisions to the general public out of privacy concerns, while being willing to share these with their trusted neighbors on the whole network. Although there may exist some central node that collects and distributes the necessary information to each player, this central node is subject to performance limitations, such as single point of failure, and limited flexibility and scalability [23]. The limited capacity of communication channels also constrains information sharing over the network.

Manuscript received 30 January 2022; revised 4 July 2022 and 13 August 2022; accepted 22 September 2022. Date of publication 10 October 2022; date of current version 5 June 2023. An earlier version of this paper was presented in part at the American Control Conference 2022 [DOI: 10.23919/ACC53348.2022.9867201]. This work was supported by the National Science Foundation under Grant 2014816 and Grant 2038410. Recommended by Associate Editor B. Ghahesifard.

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Digital Object Identifier 10.1109/TCNS.2022.3212629

Significant efforts have been devoted to designing algorithms to solve SGNEPs distributedly under the full-decision information setting where each player has access to all other players' decisions. Koshal et al. [11] considered a Cartesian stochastic variational inequality problem with a monotone map. They propose a stochastic iterative Tikhonov regularization method and a stochastic iterative proximal-point method to solve it, which let players update their regularization parameters and centering terms properly after each iteration. Franci and Grammatico [24] proposed a solution based on the preconditioned forward-backward (FB) operator splitting with the expected-value pseudogradient assumed to be restricted cocoercive and approximated via the stochastic approximation (SA) scheme. To accelerate game dynamics and relax the cocoercivity assumption, Cui et al. [25] adopted a forward-backward-forward framework. These works are mostly gradient based which enjoy low complexity in implementation and updating per player step. Nevertheless, rational players would take best-response actions given others' strategies and deviate from gradient-based schemes unless forced by some external authority. The work in [26] provides an inexact generalization of the proximal best-response (BR) schemes to the SNEP, whose corresponding proximal BR map admits a contractive property. The authors in [27] further consider SNEPs with composite objectives and design a variable sample-size proximal BR scheme, under a contractive property on the proximal BR map. Yet, far less has been studied when it comes to the distributed solution to SGNEPs with merely partial information. The only existing work to our best knowledge is [28], which also relies on the FB framework along with the SA method. The convergence of the proposed algorithms has been analyzed under the assumption that the preconditioned forward operator is restricted cocoercive, which only allows comparatively small step sizes.

Our contributions can be summarized in the following aspects. First, we propose a distributed algorithm to solve the SGNEP under the partial-decision information setting based on the Douglas-Rachford splitting and the proximal mapping. In the proposed algorithm, the involved players are asked to update their decision vectors in two separate steps: 1) solving the augmented best-response subproblems and 2) projecting onto the local feasible sets after some linear transformations. The updates of their local estimates and dual variables only require some trivial linear transformations. This algorithm can deal with cases where the scenario-based objectives of players are nonsmooth, and relaxes some commonly made assumptions, such as the α -cocoercivity with $\alpha > 1/2$ in the FB splitting. Second, we establish the convergence of the proposed algorithm under assumptions concerning the properties of the pseudogradient, the extended pseudogradient, and the stochastic subgradients of the objectives. Without relying on the contractive property, the proof in this article is based on the Robbins-Siegmund theorem and extends the convergence results discussed in [26]. Drawing tools and techniques from stochastic approximation and convex analysis, we then construct a feasible inexact solver for the augmented best-response subproblems based on the projected stochastic subgradient method and discuss the prescribed accuracy within which the inexact solver should achieve such that the algorithm convergence is ensured. The proposed inexact solver

based on the projected stochastic subgradient method requires the projection onto some bounded box sets rather than the (potentially complicated) local feasible sets of the original problem, which considerably improves the computational efficiency.

The rest of this article is organized as follows. In Section II, we formally formulate the SGNEP on networks and provide some basic definitions as well as assumptions. In this section, we recast the SGNEP as the zero-finding problem of a certain operator and prove that the solution of the latter is a variational equilibrium of the former. In Section III, a distributed algorithm in a partial-decision information setting is proposed. Section IV focuses on the convergence analysis of the proposed algorithm. In this section, we also construct an inexact solver based on the projected stochastic subgradient method. In Section V, to demonstrate the theoretical findings and the proposed algorithm in practical applications, we include two numerical examples: 1) a stochastic Nash-Cournot distribution game and 2) a multi-product assembly problem with the two-stage model. Finally, Section VI concludes this article.

Basic Notations: For a set of matrices $\{V_i\}_{i \in S}$, we let $\text{blkd}(V_1, \dots, V_{|S|})$ or $\text{blkd}(V_i)_{i \in S}$ denote the diagonal concatenation of these matrices, $[V_1, \dots, V_{|S|}]$ as their horizontal stack, and $[V_1; \dots; V_{|S|}]$ as their vertical stack. For a set of vectors $\{v_i\}_{i \in S}$, $[v_i]_{i \in S}$, or $[v_1; \dots; v_{|S|}]$ denotes their vertical stack. For a matrix V and a pair of positive integers (i, j) , $[V]_{(i,j)}$ denotes the entry on the i th row and the j th column of V . For a vector v and a positive integer i , $[v]_i$ denotes the i th entry of v . Denote $\mathbb{R} := \mathbb{R} \cup \{+\infty\}$, $\mathbb{R}_+ := [0, +\infty)$, and $\mathbb{R}_{++} := (0, +\infty)$. $\mathbb{S}_+^n$ (respectively, $\mathbb{S}_{++}^n$) represents the set of all $n \times n$ symmetric positive semidefinite (respectively, definite) matrices. $\iota_S(x)$ is defined to be the indicator function of a set S , i.e., if $x \in S$, then $\iota_S(x) = 0$; otherwise, $\iota_S(x) = +\infty$. $N_S(x)$ denotes the normal cone to the set $S \subseteq \mathbb{R}^n$ at the point x : if $x \in S$, then $N_S(x) := \{u \in \mathbb{R}^n \mid \sup_{z \in S} \langle u, z - x \rangle \leq 0\}$; otherwise, $N_S(x) := \emptyset$. If $S \subseteq \mathbb{R}^n$ is a closed and convex set, the map $\text{Pj}_S : \mathbb{R}^n \rightarrow S$ denotes the projection onto S , i.e., $\text{Pj}_S(x) := \arg\min_{v \in S} \|v - x\|_2$. We use $\rightrightarrows$ to indicate a point-to-set map. For an operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $\text{Zer}(T) := \{x \in \mathbb{R}^n \mid Tx \ni 0\}$ and $\text{Fix}(T) := \{x \in \mathbb{R}^n \mid Tx \ni x\}$ denote its zero set and fixed point set, respectively. We denote $\text{dom}(T)$ the domain of the operator T and $\text{gra}(T)$ the graph of it. The resolvent and reflected resolvent of T are defined as $J_T := (I + T)^{-1}$ and $R_T := 2J_T - I$, respectively.

II. PROBLEM FORMULATION

A. Stochastic Game Formulation and SGNE

In this section, we formulate the SGNEP on networks. There are N players participating in the game, indexed by $\mathcal{N} := \{1, \dots, N\}$. Each player $i \in \mathcal{N}$ needs to determine its local decision vector $x_i \in \mathcal{X}_i$ to optimize its objective, where $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ is the local feasible set/action space of player i . This Nash equilibrium seeking problem is generalized because, besides the local constraints $\{\mathcal{X}_i\}_{i \in \mathcal{N}}$, the decision vectors of all the players should satisfy some global resource constraints, i.e., $\sum_{i \in \mathcal{N}} A_i x_i \leq c$. Here, we have the matrix $A_i \in \mathbb{R}^{m \times n_i}$ with m denoting the number of the (global) affine coupling constraints, and the constant vector $c \in \mathbb{R}^m$ representing the quantities of available resources. Altogether, for each player i , the feasible

set of the decision vector x_i is given by

$$\tilde{\mathcal{X}}_i(x_{-i}) := \mathcal{X}_i \cap \{x_i \in \mathbb{R}^{n_i} \mid A_i x_i + \sum_{j \in \mathcal{N}_{-i}} A_j x_j \leq c\} \quad (1)$$

where $\mathcal{N}_{-i} := \mathcal{N} \setminus \{i\}$, and x_{-i} denotes the stack of decision vectors except that of player i . Accordingly, the feasible set of the collective decision vector $x := [x_1; \dots; x_N]$ is given by

$$\tilde{\mathcal{X}} := \mathcal{X} \cap \{x \in \mathbb{R}^n \mid Ax - c \leq 0\} \quad (2)$$

where $\mathcal{X} := \prod_{i \in \mathcal{N}} \mathcal{X}_i$, $n := \sum_{i \in \mathcal{N}} n_i$, and $A := [A_1, A_2, \dots, A_N]$.

To capture uncertainty in practical settings, we consider stochastic games, where the objective function $\mathbb{J}_i(x_i; x_{-i})$ of each player i is the expected value of certain function J_i . Formally, given the decisions x_{-i} of the other players, each player i aims to solve the following local problem:

$$\begin{cases} \text{minimize} & x_i \in \mathcal{X}_i \mathbb{J}_i(x_i; x_{-i}) = \mathbb{E}_{\xi_i} [J_i(x_i; x_{-i}, \xi_i)] \\ \text{subject to} & A_i x_i \leq c - \sum_{j \in \mathcal{N}_{-i}} A_j x_j \end{cases} \quad (3)$$

where $\xi_i : \Omega_i \rightarrow \mathbb{R}^{n_{\xi_i}}$ is a random variable in a well-defined probability space.

Given the above formulation of the SGNEP, we have the following standing assumptions that hold throughout this article.

Assumption 1: (Scenario-Based Objectives) For each player $i \in \mathcal{N}$, given any fixed sample $\omega_i \in \Omega_i$, the scenario-based objective $J_i(\cdot; \cdot, \xi_i(\omega_i))$ is proper and continuous. In addition, $J_i(x_i; x_{-i}, \xi_i(\omega_i))$ is a convex function w.r.t. x_i given any fixed x_{-i} and $\omega_i \in \Omega_i$.

Assumption 2: (Feasible Sets) Each local feasible set $\mathcal{X}_i$ is nonempty, compact, and convex. The collective feasible set $\tilde{\mathcal{X}}$ is nonempty, and the Mangasarian–Fromovitz constraint qualification (MFCQ) holds [10, Ch. 3.2] and [29, Sec. 16.2.3].

Collectively solving the problems in (3) gives rise to the stochastic generalized Nash equilibrium (SGNE), the formal definition of which is given as follows [24], [30].

Definition 1: The collective decision $x^* \in \tilde{\mathcal{X}}$ is an SGNE if no player can benefit by unilaterally deviating from x^* . Specifically, for all $i \in \mathcal{N}$, $\mathbb{J}_i(x_i^*; x_{-i}^*) \leq \mathbb{J}_i(x_i; x_{-i}^*)$ for any $x_i \in \tilde{\mathcal{X}}_i(x_{-i}^*)$.

Under Assumption 1, the SGNE seeking problems can be transformed to the corresponding generalized quasi-variational inequality (GQVI) [29, Sec. 12.2]. As shown in [29, Prop. 12.3], Definition 1 of SGNE coincides with the following definition from the perspective of GQVI.

Definition 2: The collective decision $x^* \in \tilde{\mathcal{X}}$ is an SGNE if x^* along with a suitable $g^* \in \prod_{i \in \mathcal{N}} \partial_{x_i} \mathbb{J}_i(x_i^*; x_{-i}^*)$ is a solution of the problem GQVI($\prod_{i \in \mathcal{N}} \tilde{\mathcal{X}}_i, \prod_{i \in \mathcal{N}} \partial_{x_i} \mathbb{J}_i$), i.e.,

$$(x - x^*)^T g^* \geq 0, \forall x \in \prod_{i \in \mathcal{N}} \tilde{\mathcal{X}}_i(x_{-i}^*). \quad (4)$$

As suggested in [29, Sec. 12.2.3], under Assumptions 1 and 2, we can equivalently recast the problem in (3) into a set of inclusions by considering the Karush–Kuhn–Tucker (KKT) conditions of the above GQVI such that $\forall i \in \mathcal{N}$

$$\begin{aligned} 0 &\in \partial_{x_i} \mathbb{J}_i(x_i^*; x_{-i}^*) + A_i^T \lambda_i + N_{\mathcal{X}_i}(x_i^*) \\ 0 &\in -(Ax^* - c) + N_{\mathbb{R}_+^m}(\lambda_i) \end{aligned} \quad (5)$$

where λ_i is the Lagrangian multiplier for the global resource constraints $A_i x_i \leq c - \sum_{j \in \mathcal{N}_{-i}} A_j x_j$ for each player i .

In this article, we restrict our attention to a subset of these SGNEs where the players share the same coupled constraints and, hence, all the Lagrangian multipliers are in consensus, i.e., $\lambda_1 = \dots = \lambda_N$. This gives rise to a generalized variational inequality (GVI) problem. This subclass of the SGNEs, known as the variational stochastic generalized Nash equilibria (v-SGNEs) [3], [4], enforces the idea of economic fairness and enjoys better social stability/sensitivity [31]. We will focus on this subclass since we can leverage a variety of tools that have been developed for solving (G)VIs [10, Ch. 10–12] and design the modified best-response dynamics based on it.

Definition 3: The collective decision $x^* \in \tilde{\mathcal{X}}$ is a variational SGNE (v-SGNE) if x^* along with a suitable $g^* \in \prod_{i \in \mathcal{N}} \partial_{x_i} \mathbb{J}_i(x_i^*; x_{-i}^*)$ is a solution of the GVI($\tilde{\mathcal{X}}, \prod_{i \in \mathcal{N}} \partial_{x_i} \mathbb{J}_i$), i.e.,

$$(x - x^*)^T g^* \geq 0, \forall x \in \tilde{\mathcal{X}}. \quad (6)$$

Similarly, the KKT system of the above GVI is given by

$$\begin{aligned} 0 &\in \partial_{x_i} \mathbb{J}_i(x_i^*; x_{-i}^*) + A_i^T \lambda + N_{\mathcal{X}_i}(x_i^*) \\ 0 &\in -(Ax^* - c) + N_{\mathbb{R}_+^m}(\lambda) \end{aligned} \quad (7)$$

where λ is the Lagrangian multiplier for the global constraints in (3). Notice that the GVI in (6) is not completely equivalent to the initial SGNEP in (3) as the game may admit an SGNE, while the GVI has no solution. We make the following assumption concerning the existence of v-SGNEs.

Assumption 3: (Existence of v-SGNE) The SGNEP considered admits a nonempty set of v-SGNEs.

Remark 1: The existence and multiplicity of solutions of GNEPs with continuously differentiable objectives have been extensively studied, and the related theories can be found in [10, Ch. 2 and 3]. For the GNEPs with nonsmooth objectives, we can check the existence of v-GNEs of these GNEPs by [29, Prop. 12.11]. If the closed-form expressions of the objectives $\mathbb{J}_i(x_i; x_{-i})$ for any $i \in \mathcal{N}$ are unavailable and we cannot apply the above results, [30, Sec. 4] provides sufficient conditions to guarantee the existence of v-SGNEs based on the properties of scenario-based objectives.

B. Network Game Formulation

In network games, there exists an underlying communication graph $\mathcal{G} = (\mathcal{N}_g, \mathcal{E}_g)$, where players can communicate with their neighbors through arbitrators on the edges. The node set $\mathcal{N}_g$ denotes the set of all players, and $\mathcal{E}_g \subseteq \mathcal{N}_g \times \mathcal{N}_g$ is the set of directed edges. The cardinalities $|\mathcal{N}_g|$ and $|\mathcal{E}_g|$ are denoted by N_g and E_g . In this case, $\mathcal{N}_g = \mathcal{N}$ and $N_g = N$. We use (i, j) to denote a directed edge having node/player i as its tail and node/player j as its head. For notational brevity, let $\mathcal{N}_i$ denote the set of immediate neighbors of player i who can directly communicate with it, $\mathcal{N}_i^+ := \{j \in \mathcal{N} \mid (j, i) \in \mathcal{E}_g\}$ the set of in-neighbors of player i , and $\mathcal{N}_i^- := \{j \in \mathcal{N} \mid (i, j) \in \mathcal{E}_g\}$ the set of out-neighbors of player i . Note that although the multipliers we are going to introduce are defined in a directed fashion, we assume each node can send messages to both its in- and out-neighbors, and $\mathcal{G}$ should satisfy the following assumption.

Assumption 4: (Communicability) The underlying communication graph $\mathcal{G} = (\mathcal{N}_g, \mathcal{E}_g)$ is undirected and connected. Besides, it has no self-loops.

We next recast the SGNEP in (3) as the zero-finding problem of a certain operator that can be carried out distributedly over the communication graph $\mathcal{G}$ via the network Lagrangian of this game; see [32] for more details. Now for each player $i \in \mathcal{N}$, besides its local decision vector $y_i^i \in \mathcal{X}_i$, it keeps a local estimate $y_i^j \in \mathbb{R}^{n_j}$ of the player j 's decision for all $j \in \mathcal{N}_{-i}$, which together constitutes its augmented decision vector y_i . Here, we use y_i^i to denote the local decision of each player i to distinguish from the case where only local decision x_i are maintained and considered. We denote $y_i^{-i} := [y_i^j]_{j \in \mathcal{N}_{-i}}$ the vertical stack of $\{y_i^j\}_{j \in \mathcal{N}_{-i}}$ and $y_i := [y_i^j]_{j \in \mathcal{N}}$ the vertical stack of $\{y_i^j\}_{j \in \mathcal{N}}$, both in prespecified orders. Denote $n_{<i} = \sum_{j \in \mathcal{N}, j < i} n_j$ and $n_{>i} = \sum_{j \in \mathcal{N}, j > i} n_j$. The extended feasible set of $\mathbf{y} := [y_i]_{i \in \mathcal{N}}$ is defined as $\hat{\mathcal{X}} := \hat{\mathcal{X}}_1 \times \hat{\mathcal{X}}_2 \times \cdots \times \hat{\mathcal{X}}_N$ with each one defined as $\hat{\mathcal{X}}_i := \mathbb{R}^{n_{<i}} \times \mathcal{X}_i \times \mathbb{R}^{n_{>i}}$. For brevity, we shall write $\{y_i\}$ in replacement of the more cumbersome notation $\{y_i\}_{i \in \mathcal{N}}$ and similarly for other variables on nodes and edges (e.g., the dual variables $\{\mu_{ji}\}_{(j,i) \in \mathcal{E}_g}$ to be introduced below will be denoted simply by $\{\mu_{ji}\}$), unless otherwise specified. In the reformulated zero-finding problem, we introduced a set of dual variables $\{\lambda_i\}$ to enforce the global resource constraints. Moreover, another two sets of dual variables $\{\mu_{ji}\}$ and $\{z_{ji}\}$ are introduced to guarantee the consensus of $\{y_i\}$ and $\{\lambda_i\}$. It is worth mentioning that $\{y_i\}$ and $\{\lambda_i\}$ are maintained by players, while $\{\mu_{ji}\}$ and $\{z_{ji}\}$ are maintained by arbitrators on the edges.

We next give a brief introduction to two commonly used operators in the distributed solution of GNEP: 1) the pseudogradient $\mathbb{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and 2) the extended pseudogradient $\tilde{\mathbb{F}} : \mathbb{R}^{nN} \rightrightarrows \mathbb{R}^n$. The pseudogradient $\mathbb{F}$ is the Cartesian product of the partial subgradients of the objective functions of all players, which is given as follows:

$$\mathbb{F} : x \mapsto \prod_{i \in \mathcal{N}} [\partial_{x_i} \mathbb{J}_i(x_i; x_{-i})]. \quad (8)$$

In contrast, the extended pseudogradient $\tilde{\mathbb{F}}$ defined in (9) is a commonly used operator under the partial-decision information setting, where each player keeps the local estimates of others' decisions and then uses these estimates as the parametric inputs

$$\tilde{\mathbb{F}} : [y_i]_{i \in \mathcal{N}} \mapsto \prod_{i \in \mathcal{N}} [\partial_{y_i^i} \mathbb{J}_i(y_i^i; y_i^{-i})]. \quad (9)$$

To incorporate the extended pseudogradient $\tilde{\mathbb{F}}$ into a fixed-point iteration, we then introduce the individual selection matrices $\{\mathcal{R}_i\}_{i \in \mathcal{N}}$ and their diagonal concatenation $\mathcal{R} \in \mathbb{R}^{n \times nN}$

$$\mathcal{R}_i = [\mathbf{0}_{n_i \times n_{<i}}, \mathbf{I}_{n_i}, \mathbf{0}_{n_i \times n_{>i}}], \quad \mathcal{R} = \text{blkd}(\mathcal{R}_1, \dots, \mathcal{R}_N). \quad (10)$$

Notice that $y_i^i = \mathcal{R}_i y_i$ and $\mathcal{R}_i \mathcal{R}_i^T = \mathbf{I}_{n_i}$. Finally, the set-valued operator $\mathbb{T}$ we are going to study is given as follows:

$$\mathbb{T} : \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \mathbf{z} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{R}^T(\tilde{\mathbb{F}}(\mathbf{y}) + \Lambda^T \boldsymbol{\lambda}) + B_n \boldsymbol{\mu} + \rho_\mu L_n \mathbf{y} + N_{\hat{\mathcal{X}}}(\mathbf{y}) \\ N_{\mathbb{R}^{mN}}(\boldsymbol{\lambda}) - \Lambda \mathcal{R} \mathbf{y} + \mathbf{c} + B_m \mathbf{z} + \rho_z L_m \boldsymbol{\lambda} \\ -B_n^T \cdot \mathbf{y} \\ -B_m^T \cdot \boldsymbol{\lambda} \end{bmatrix} \quad (11)$$

where Λ is the diagonal concatenation of $\{\mathcal{A}_i\}_{i \in \mathcal{N}}$, i.e., $\Lambda := \text{blkd}(A_1, \dots, A_N)$; $\mathbf{c}$ is the vertical stack of $\{c_i\}_{i \in \mathcal{N}}$

with $\sum_{i \in \mathcal{N}} c_i = c$; $B_n := (B \otimes \mathbf{I}_n)$, $L_n := (L \otimes \mathbf{I}_n)$, $B_m := (B \otimes \mathbf{I}_m)$, $L_m := (L \otimes \mathbf{I}_m)$, B and L are the incidence matrix and Laplacian matrix of the underlying communication graph, respectively, with $L = B \cdot B^T$; and $\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$, and $\mathbf{z}$ are the stack vectors of $\{y_i\}$, $\{\lambda_i\}$, $\{\mu_{ji}\}$, and $\{z_{ji}\}$, respectively; ψ denotes the stack of the primal and dual variables, i.e., $\psi := [\mathbf{y}; \boldsymbol{\lambda}; \boldsymbol{\mu}; \mathbf{z}]$.

Theorem 1: Suppose Assumptions 1 to 4 hold, and there exists $\psi^* := [\mathbf{y}^*; \boldsymbol{\lambda}^*; \boldsymbol{\mu}^*; \mathbf{z}^*] \in \text{Zer}(\mathbb{T})$. Then, $\mathbf{y}^* = \mathbf{1}_N \otimes y^*$, $\boldsymbol{\lambda}^* = \mathbf{1}_N \otimes \lambda^*$, and (y^*, λ^*) satisfies the KKT conditions (7) for v-GNE with x^* replaced with y^* . Conversely, for a solution $(y^\dagger, \lambda^\dagger)$ of the KKT problem in (7), there exist $\boldsymbol{\mu}^\dagger$ and $\mathbf{z}^\dagger$ such that $\psi^\dagger := [\mathbf{1}_N \otimes y^\dagger; \mathbf{1}_N \otimes \lambda^\dagger; \boldsymbol{\mu}^\dagger; \mathbf{z}^\dagger] \in \text{Zer}(\mathbb{T})$.

Proof: See the proof of [32, Th. 1]. ■

Thus, finding a v-SGNE of the game in (3) is equivalent to solving for a zero point of the operator $\mathbb{T}$. To facilitate the convergence analysis of the algorithm to be proposed for the latter task, we make two parallel assumptions, either of which is instrumental for the convergence proof in Section IV.

Assumption 5: (Convergence Condition) At least one of the following statements holds.

- 1) The operator $\mathcal{R}^T \tilde{\mathbb{F}} + \frac{\rho_\mu}{2} L_n$ is maximally monotone.
- 2) The pseudogradient $\mathbb{F}$ is strongly monotone and Lipschitz continuous, i.e., there exist $\eta > 0$ and $\theta_1 > 0$, such that $\forall x, x' \in \mathbb{R}^n$, $\langle x - x', \mathbb{F}(x) - \mathbb{F}(x') \rangle \geq \eta \|x - x'\|^2$ and $\|\mathbb{F}(x) - \mathbb{F}(x')\| \leq \theta_1 \|x - x'\|$. The operator $\mathcal{R}^T \tilde{\mathbb{F}}$ is Lipschitz continuous, i.e., there exists $\theta_2 > 0$, such that $\forall \mathbf{y}, \mathbf{y}' \in \mathbb{R}^{nN}$, $\|\tilde{\mathbb{F}}(\mathbf{y}) - \tilde{\mathbb{F}}(\mathbf{y}')\| \leq \theta_2 \|\mathbf{y} - \mathbf{y}'\|$. Moreover, $\rho_\mu \geq \frac{2}{\sigma_1} (\frac{(\theta_1 + \theta_2)^2}{4\eta} + \theta_2)$, where σ_1 is the smallest positive eigenvalue of L .

III. AUGMENTED BEST-RESPONSE SCHEME

To compute the zeros of the operator $\mathbb{T}$ given in the previous section, we leverage the Douglas–Rachford (DR) splitting method which combines operator splitting and the Krasnoselskii–Mann (K–M) schemes. Given a nonexpansive operator Q with a nonempty fixed point set $\text{Fix}(Q)$, the K–M scheme [33, Sec. 5.2] suggests the following iteration:

$$\psi^{(k+1)} := \psi^{(k)} + \gamma^{(k)} (Q\psi^{(k)} - \psi^{(k)}) \quad (12)$$

where $(\gamma^{(k)})_{k \in \mathbb{N}}$ is a sequence such that $\gamma^{(k)} \in [0, 1]$ for all $k \in \mathbb{N}$ and $\sum_{k \in \mathbb{N}} \gamma^{(k)} (1 - \gamma^{(k)}) = \infty$. Here, we introduce a set of local bounded box constraints $\{\mathcal{X}_i^B\}$ which can be chosen manually as long as it satisfies $\mathcal{X}_i \subseteq \mathcal{X}_i^B$ for all $i \in \mathcal{N}$. We similarly define the extended box set $\hat{\mathcal{X}}^B := \hat{\mathcal{X}}_1^B \times \hat{\mathcal{X}}_2^B \times \cdots \times \hat{\mathcal{X}}_N^B$, where the extended box set of each player i is defined as $\hat{\mathcal{X}}_i^B := \mathbb{R}^{n_{<i}} \times \mathcal{X}_i^B \times \mathbb{R}^{n_{>i}}$. It is easy to see that the normal cones of $\hat{\mathcal{X}}^B$ and $\hat{\mathcal{X}}$ satisfy $N_{\hat{\mathcal{X}}^B} + N_{\hat{\mathcal{X}}} = N_{\hat{\mathcal{X}}}$. The motivation for introducing these box sets is to simplify the computation while maintaining boundedness for the convergence analysis as we will show later in this article. We split the operator $\mathbb{T}$ into the following operators $\mathbb{A}$ and $\mathbb{B}$:

$$\mathbb{A} : \psi \mapsto (D + \mathbb{A}_y)\psi \text{ and } \mathbb{B} : \psi \mapsto (D + \mathbb{B}_y)\psi \quad (13)$$

with D , $\mathbb{A}_y$, and $\mathcal{B}_y$ defined by

$$D = \begin{bmatrix} \frac{\rho_\mu}{2} L_n & \frac{1}{2}(\Lambda\mathcal{R})^T & \frac{1}{2}B_n & 0 \\ -\frac{1}{2}\Lambda\mathcal{R} & \frac{\rho_z}{2}L_m & 0 & \frac{1}{2}B_m \\ -\frac{1}{2}B_n^T & 0 & 0 & 0 \\ 0 & -\frac{1}{2}B_m^T & 0 & 0 \end{bmatrix} \quad (14)$$

$$\mathbb{A}_y : \psi \mapsto \begin{bmatrix} \mathcal{R}^T \tilde{\mathbb{F}}(\mathbf{y}) + N_{\hat{\mathcal{X}}^B}(\mathbf{y}) \\ \mathbf{c} \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{B}_y : \psi \mapsto \begin{bmatrix} N_{\hat{\mathcal{X}}}(\mathbf{y}) \\ N_{\mathbb{R}^{mN}}(\boldsymbol{\lambda}) \\ 0 \\ 0 \end{bmatrix}. \quad (15)$$

Furthermore, we introduce the following design matrix Φ for distributedly computing the resolvents $J_{\Phi^{-1}\mathbb{A}}$ and $J_{\Phi^{-1}\mathcal{B}}$:

$$\Phi = \begin{bmatrix} \tau_1^{-1} - \frac{\rho_\mu}{2} L_n & -\frac{1}{2}(\Lambda\mathcal{R})^T & -\frac{1}{2}B_n & 0 \\ -\frac{1}{2}\Lambda\mathcal{R} & \tau_2^{-1} - \frac{\rho_z}{2}L_m & 0 & -\frac{1}{2}B_m \\ -\frac{1}{2}B_n^T & 0 & \tau_3^{-1} & 0 \\ 0 & -\frac{1}{2}B_m^T & 0 & \tau_4^{-1} \end{bmatrix} \quad (16)$$

where $\tau_1 := \text{blkd}(\tau_{11}I_n, \dots, \tau_{1N}I_n)$ with $\tau_{11} \in \mathbb{R}_{++}, \dots, \tau_{1N} \in \mathbb{R}_{++}$; similarly for τ_2, τ_3 , and τ_4 . Notice that these step sizes $\tau_1, \dots, \tau_4$ should be small enough to guarantee that Φ is positive definite. Conservative upper bounds for these step sizes can be derived using the Gershgorin circle theorem [34].

Assumption 6: The step sizes $\tau_1, \dots, \tau_4$ are chosen properly such that the design matrix Φ in (16) is positive definite. Specifically, it suffices to choose $\tau_{1i}^{-1} > \frac{1}{2}\|A_i\|_1 + (\frac{1}{2} + \rho_\mu)d_i$, $\tau_{2i}^{-1} > \frac{1}{2}\|A_i\|_\infty + (\frac{1}{2} + \rho_z)d_i \forall i \in \mathcal{N}$, and $\tau_{3j}^{-1} > 1$, $\tau_{4j}^{-1} > 1 \forall j \in \mathcal{E}_g$.

Here, d_i denotes the degree of node/player i . In general, determining the above step sizes requires some global information acquired through coordination among players, such as a proper ρ_μ . After the incorporation of the design matrix Φ , we now work in the inner product space $\mathcal{K}$, which is a real vector space endowed with the inner product $\langle \psi_1, \psi_2 \rangle_{\mathcal{K}} = \psi_1^T \Phi \psi_2$. For brevity, let $\bar{\mathbb{A}} := \Phi^{-1}\mathbb{A}$ and $\bar{\mathcal{B}} := \Phi^{-1}\mathcal{B}$. In the DR splitting scheme, the general operator Q in (12) is given by $\mathcal{R}_* := R_{\bar{\mathcal{B}}} \circ R_{\bar{\mathbb{A}}}$ and it suggests the following exact iteration:

$$\tilde{\psi}^{(k+1)} := \mathcal{P}_*(\tilde{\psi}^{(k)}), \text{ with } \mathcal{P}_* = \text{Id} + \gamma^{(k)}(\mathcal{R}_* - \text{Id}). \quad (17)$$

Given a generic single-valued operator Q , we say that Q is restricted nonexpansive w.r.t. a set S if, for all $\psi \in \text{dom } Q$ and $\psi^* \in S$, $\|Q\psi - Q\psi^*\| \leq \|\psi - \psi^*\|$ [20]; if, in addition, $S = \text{Fix}(Q)$, then Q is quasinonexpansive [33, Def. 4.1(v)]. From the main convergence results in [32, Th. 2 and 3], if Assumptions 1 to 6 hold, even though $\mathcal{R}_*$ is not nonexpansive in a general sense, it possesses quasinonexpansiveness in the inner-product space $\mathcal{K}$, and, hence, the sequence $(y_i^{(k)})_{k \in \mathbb{N}}$ generated by the exact iteration above (see [32, Algorithm 1] for detailed implementations) will converge to a v-SGNE of the original problem defined in (3).

However, unlike the problem setting in [32], where each player has a closed-form objective function, here, the objective function is expected-value, and all too often its closed-form expression may be unavailable. Consequently, the argmin operation in the first player loop of [32, Algorithm 1] cannot be carried

Algorithm 1: Distributed v-SGNE Seeking under the Partial-Decision Information Setting.

1 Initialize: $\{\tilde{y}_i^{(0)}\}, \{\tilde{\lambda}_i^{(0)}\}, \{\tilde{\mu}_{ji}^{(0)}\}, \{\tilde{z}_{ji}^{(0)}\}$;
2 Iterate until convergence:
3 for player $i \in \mathcal{N}$ **do**
4 Communicate with neighboring players and edges
 to obtain $\tilde{y}_{iL}^{(k)}, \tilde{\mu}_{iB}^{(k)}, \tilde{\lambda}_{iL}^{(k)}, \tilde{z}_{iB}^{(k)}$;
5 $\tilde{y}_i^{-i(k+1)} = \tilde{y}_i^{(k)} - \frac{\tau_{1i}}{2}(\rho_\mu \tilde{y}_{iL}^{(k)} + \tilde{\mu}_{iB}^{(k)})$;
6 Obtain $\tilde{y}_i^{i(k+1)}$ via Subroutine 2 that approximately
 solves: $\arg\min_{v_i \in \mathcal{X}^B} [\mathbb{J}_i(v_i; \tilde{y}_i^{-i(k+1)}) + \frac{1}{2}(\tilde{\lambda}_{iL}^{(k)})^T A_i v_i$
 $+ \frac{1}{2}(\rho_\mu \tilde{y}_{iL}^{(k)} + \tilde{\mu}_{iB}^{(k)})^T v_i + \frac{1}{2\tau_{1i}} \|v_i - \tilde{y}_i^{(k)}\|^2]$;
7 $\tilde{\lambda}_{iL}^{(k+1)} = \tilde{\lambda}_{iL}^{(k)} + \tau_{2i}(A_i(\tilde{y}_i^{i(k+1)} - \frac{1}{2}\tilde{y}_i^{(k)}) - \frac{\rho_z}{2}\tilde{\lambda}_{iL}^{(k)} - \frac{1}{2}\tilde{z}_{iB}^{(k)} - c_i)$;
8 $\tilde{y}_i^{(k+1)} = 2\tilde{y}_i^{i(k+1)} - \tilde{y}_i^{(k)}, \tilde{\lambda}_i^{(k+1)} = 2\tilde{\lambda}_{iL}^{(k+1)} - \tilde{\lambda}_i^{(k)}$;
9 $\tilde{y}_i^{(k+1)} = 2\tilde{y}_i^{i(k+1)} - \tilde{y}_i^{(k)}, \tilde{\lambda}_i^{(k+1)} = 2\tilde{\lambda}_{iL}^{(k+1)} - \tilde{\lambda}_i^{(k)}$;
10 end
11 for edge $(j, i) \in \mathcal{E}_g$ **do**
12 Comm. with player i & j to obtain $\hat{y}_{ji}^{(k+1)}, \hat{\lambda}_{ji}^{(k+1)}$;
13 $\mu_{ji}^{(k+1)} = \tilde{\mu}_{ji}^{(k)} + \frac{\tau_{3i}}{2}\hat{y}_{ji}^{(k+1)}, \hat{\mu}_{ji}^{(k+1)} = 2\mu_{ji}^{(k+1)} - \tilde{\mu}_{ji}^{(k)}$;
14 $\tilde{z}_{ji}^{(k+1)} = \tilde{z}_{ji}^{(k)} + \frac{\tau_{4i}}{2}\hat{\lambda}_{ji}^{(k+1)}, \hat{z}_{ji}^{(k+1)} = 2\tilde{z}_{ji}^{(k+1)} - \tilde{z}_{ji}^{(k)}$;
15 end
16 for player $i \in \mathcal{N}$ **do**
17 Comm. & obtain $\hat{y}_{iL}^{(k+1)}, \hat{\mu}_{iB}^{(k+1)}, \hat{\lambda}_{iL}^{(k+1)}, \hat{z}_{iB}^{(k+1)}$;
18 $\tilde{y}_i^{(k+1)} = \text{Pj}_{\hat{\mathcal{Y}}_i}[\hat{y}_i^{(k+1)} - \frac{\tau_{1i}}{2}(\mathcal{R}_i^T A_i^T \hat{\lambda}_i^{(k+1)} + \rho_\mu \hat{y}_{iL}^{(k+1)} + \hat{\mu}_{iB}^{(k+1)})]$;
19 $\tilde{\lambda}_i^{(k+1)} = \text{Pj}_{\hat{\mathcal{X}}_i}[\hat{\lambda}_i^{(k+1)} + \tau_{2i}(A_i(\tilde{y}_i^{(k+1)} - \frac{1}{2}\tilde{y}_i^{(k)})$
 $- \frac{\rho_z}{2}\hat{\lambda}_{iL}^{(k+1)} - \frac{1}{2}\hat{z}_{iB}^{(k+1)})]$;
20 $\tilde{y}_i^{(k+1)} = 2\tilde{y}_i^{(k+1)} - \tilde{y}_i^{(k)}, \tilde{\lambda}_i^{(k+1)} = 2\tilde{\lambda}_i^{(k+1)} - \tilde{\lambda}_i^{(k)}$;
21 end
22 for edge $(j, i) \in \mathcal{E}_g$ **do**
23 Comm. & obtain $\tilde{y}_{ji}^{(k+1)}, \tilde{\lambda}_{ji}^{(k+1)}$;
24 $\tilde{\mu}_{ji}^{(k+1)} = \hat{\mu}_{ji}^{(k+1)} + \tau_{3i}(\tilde{y}_{ji}^{(k+1)} - \frac{1}{2}\tilde{y}_{ji}^{(k)})$;
25 $\tilde{z}_{ji}^{(k+1)} = \hat{z}_{ji}^{(k+1)} + \tau_{4i}(\tilde{\lambda}_{ji}^{(k+1)} - \frac{1}{2}\tilde{\lambda}_{ji}^{(k)})$;
26 end
27 K-M updates: $\tilde{\psi}^{(k+1)} = \tilde{\psi}^{(k)} + 2\gamma^{(k)}(\tilde{\psi}^{(k+1)} - \psi^{(k+1)})$;
28 Return: $\{\tilde{y}_i^{(k)}\}$.

out exactly. In this case, we need a desirable inexact solver such that although at each iteration step it can only get an approximate solution, the computed sequence can still eventually converge to a v-SGNE. A similar approximation strategy to resolvent has been discussed in [35]. We let $R_{\bar{\mathcal{A}}}$ denote the (scenario-based) approximate operator to the exact reflected resolvent $R_{\bar{\mathbb{A}}}$, and $\mathcal{R}$ denote the corresponding composite $R_{\bar{\mathcal{B}}} \circ R_{\bar{\mathcal{A}}}$. Substituting the operator $\mathcal{R}_*$ with $\mathcal{R}$ in [32, Algorithm 1] gives rise to the following approximate iteration:

$$\tilde{\psi}^{(k+1)} := \mathcal{P}(\tilde{\psi}^{(k)}), \text{ with } \mathcal{P} = \text{Id} + \gamma^{(k)}(\mathcal{R} - \text{Id}). \quad (18)$$

The updating steps of (18) are presented in Algorithm 1. For brevity, let $\tilde{y}_{iL}^{-i(k)} := \sum_{j \in \mathcal{N}_i^+} (\tilde{y}_i^{-i(k)} - \tilde{y}_j^{-i(k)})$, and similarly for $\tilde{y}_{iL}^{(k)}, \tilde{\lambda}_{iL}^{(k)}, \hat{y}_{iL}^{(k+1)}$, and $\hat{\lambda}_{iL}^{(k+1)}$; let $\tilde{\mu}_{iB}^{-i(k)} := \sum_{j \in \mathcal{N}_i^+} \tilde{\mu}_{ji}^{-i(k)} - \sum_{j \in \mathcal{N}_i^-} \tilde{\mu}_{ij}^{-i(k)}$, and similarly for $\tilde{\mu}_{iB}^{(k)}, \hat{\mu}_{iB}^{(k+1)}$, and $\hat{z}_{iB}^{(k+1)}$; let $\tilde{y}_{ji}^{(k+1)} := \hat{y}_i^{(k+1)} - \hat{y}_j^{(k+1)}$, and similarly for $\hat{\lambda}_{ji}^{(k+1)}, \tilde{y}_{ji}^{(k+1)}$, and $\hat{\lambda}_{ji}^{(k+1)}$.

Depending on the inexact solver adopted, $R_{\bar{\mathcal{A}}}$ usually admits no explicit formulas. Yet, as will be shown later in the next

section, we can still establish the convergence of Algorithm 1 based on some specific properties of $R_{\mathcal{A}}$.

IV. CONVERGENCE ANALYSIS AND CONSTRUCTION OF INEXACT SOLVER

A. General Convergence Results Using Approximate Solution

In this subsection, we study the sufficient conditions from a generic perspective to guarantee the convergence of Algorithm 1 to a v-SGNE of the problem (3) based on the Robbins–Siegmund theorem [36, Th. 1]. We first define the approximate error and its norm for each iteration as

$$\epsilon^{(k)} := \mathcal{R}(\tilde{\psi}^{(k)}) - \mathcal{R}_*(\tilde{\psi}^{(k)}) \text{ and } \varepsilon^{(k)} := \|\epsilon^{(k)}\|_{\mathcal{K}} \quad (19)$$

where $\tilde{\psi}^{(k)} := [\tilde{\mathbf{y}}^{(k)}; \tilde{\boldsymbol{\lambda}}^{(k)}; \tilde{\boldsymbol{\mu}}^{(k)}; \tilde{\mathbf{z}}^{(k)}]$. We next introduce the residual function $\text{res}(\tilde{\psi}) := \|\tilde{\psi} - \mathcal{R}_*(\tilde{\psi})\|_{\mathcal{K}}$ such that $\text{res}(\tilde{\psi}^*) = 0$ is a necessary condition for $\tilde{\psi}^*$ to belong to the fixed-point set of $\mathcal{R}_*$. This relation can be easily checked by using [33, Prop. 26.1(iii)]. Let $\mathcal{F}_k$ denote the σ -field comprised of $\{\tilde{\psi}^{(0)}, \{\xi_i^{(0)}\}_{i \in \mathcal{N}}, \dots, \{\xi_i^{(k-1)}\}_{i \in \mathcal{N}}\}$, where for each major iteration $k \in \mathbb{N}$, $\xi_i^{(k)} = \{\xi_{i,0}^{(k)}, \dots, \xi_{i,T_i^{(k)}-1}^{(k)}\}$, and $T_i^{(k)}$ denotes the number of noise realizations that player i observes at the k th iteration.

Theorem 2: Consider the SGNEP given in (3), and suppose Assumptions 1 to 6 hold. Moreover, $(\gamma^{(k)})_{k \in \mathbb{N}}$ is a sequence such that $\gamma^{(k)} \in [0, 1]$ and $\sum_{k \in \mathbb{N}} \gamma^{(k)}(1 - \gamma^{(k)}) = +\infty$. If the sequence $(\tilde{\psi}^{(k)})$ generated by the inexact solver satisfies

- i) $(\|\tilde{\psi}^{(k)}\|_{\mathcal{K}})_{k \in \mathbb{N}}$ is bounded a.s.;
- ii) $\sum_{k \in \mathbb{N}} \gamma^{(k)} \mathbb{E}[\varepsilon^{(k)} | \mathcal{F}_k] < \infty$ a.s.,

then $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ converges to a fixed point of $\mathcal{R}_*$ a.s., and $\lim_{k \rightarrow \infty} J_{\mathcal{A}}(\tilde{\psi}^{(k)}) = \psi^*$ a.s. Also, the corresponding entries of ψ^* satisfy $\mathbf{y}^* = (\mathbf{1}_N \otimes \mathbf{y}^*)$ and $\boldsymbol{\lambda}^* = (\mathbf{1} \otimes \boldsymbol{\lambda}^*)$. Here, $\mathbf{y}^*$ is a v-SGNE of the original SGNEP (3) and $(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ together is a solution to the KKT conditions (6) of the SGNEP.

Proof: See Appendix A. ■

Before proceeding, it is worth highlighting why we need to keep both the conditions (i) and (ii) to hold in Theorem 2. Although the condition (i), i.e., $(\|\tilde{\psi}^{(k)}\|_{\mathcal{K}})_{k \in \mathbb{N}}$ is bounded a.s., is a necessary condition for the summability statement in (ii), as has been showed in [33, Prop. 5.34] for deterministic cases, under the partial-information setting, a natural strategy is to prove the condition (i) first using a more primitive condition, and then establish the condition (ii) based on (i). The conditions (i) and (ii) to ensure the convergence in Theorem 2 will be later verified explicitly in Theorem 3 for our specific approximate strategy outlined in Section IV-B.

Remark 2: When proving Theorem 2, the inequalities invoked follow from the quasi-nonexpansiveness of the exact operator $\mathcal{R}_*$ and the Cauchy–Schwarz inequality. The proof and conclusion in Theorem 2 thus can be applied to the analysis of a general continuous operator Q in (12) and its approximation other than the operators $\mathcal{R}_*$ and $\mathcal{R}$ in this article, as long as the operator Q is quasinonexpansive and the conditions regarding $(\gamma^{(k)})_{k \in \mathbb{N}}$, $(\varepsilon^{(k)})_{k \in \mathbb{N}}$, and $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ in Theorem 2 are satisfied.

B. Construction of a Desirable Inexact Solver

As we discussed at the end of Section III, it is challenging to solve the augmented best-response subproblems that involve the exact expected-value objectives (the argmin problems in the first player for-loop of Algorithm 1). Theorem 2 suggests that we can still obtain a v-SGNE by solving these augmented best-response subproblems not precisely but up to some prescribed accuracy. In this subsection, we consider a specific scenario-based solver using the projected stochastic subgradient method [37, Ch. 2]. As has been shown in the existing literature [38], the weighted average of the projected stochastic subgradient method possesses an $O(1/t)$ convergence rate if the subgradient is unbiased and the variance of the subgradient is finite. Here, we study the explicit conditions that the projected stochastic subgradient solver should satisfy to serve as a feasible inexact solver in the context of distributed SGNEP with only partial-decision information, as suggested in Theorem 2.

We first assume the unbiasedness and finite-variance properties of a general projected stochastic subgradient method. Throughout this subsection, we use k to index the major iterations (the iteration of the v-SGNE seeking Algorithm 1) and t to index the minor iterations (the iteration of the inexact solver in the first player for-loop of Algorithm 1). Furthermore, at each major iteration k , for each player i , let the augmented scenario-based objective function be denoted by $\hat{J}_i^{(k)}(v_i; \xi_{i,t}^{(k)}) := J_i(v_i; y_i^{-i(k+1)}, \xi_{i,t}^{(k)}) + (\tilde{\varphi}_i^{(k)})^T y_i + \frac{1}{2\tau_{1i}} \|v_i - \tilde{y}_i^{i(k)}\|_2^2$, and the augmented expected-value objective function be denoted by $\hat{\mathbb{J}}_i^{(k)}(v_i) := \mathbb{J}_i(v_i; y_i^{-i(k+1)}) + (\tilde{\varphi}_i^{(k)})^T v_i + \frac{1}{2\tau_{1i}} \|v_i - \tilde{y}_i^{i(k)}\|_2^2$, where $\tilde{\varphi}_i^{(k)} := \frac{1}{2}(A_i^T \tilde{\boldsymbol{\lambda}}_i^{(k)} + \tilde{\boldsymbol{\mu}}_{iB}^{i(k)} + \rho_{\mu} \tilde{y}_{iL}^{i(k)})$. Note that $\hat{\mathbb{J}}_i^{(k)}(\cdot)$ is the objective in the first player-loop of Algorithm 1 that needs to be inexactly solved. Here, the vector $\tilde{\varphi}_i^{(k)}$ represents some augmented terms that enforce the consensus constraints and the global resource constraints. For brevity, the local estimates of the other players' decisions $y_i^{-i(k+1)}$ are omitted from the arguments of the augmented functions defined above. Let $T_i^{(k)}$ denote the total number of the projected stochastic subgradient steps taken in the k th major iteration by player i . The subgradient of the scenario-based objective function at the k th major iteration and the t th minor iteration is denoted by $g_{i,t}^{(k)} \in \partial_{y_i} \hat{J}_i^{(k)}(y_{i,t}^{i(k+1)}; \xi_{i,t}^{(k)})$, where $t = 0, 1, \dots, T_i^{(k)} - 1$.

Assumption 7: For each player $i \in \mathcal{N}$, at each major iteration k of Algorithm 1, the following statements hold.

- i) (Unbiasedness) At each minor iteration t , there exists a $g_{i,t}^{(k)} \in \partial_{y_i} \hat{J}_i^{(k)}(y_{i,t}^{i(k+1)}; \xi_{i,t}^{(k)})$ such that $\mathbb{E}[g_{i,t}^{(k)} | \sigma(\mathcal{F}_k, \xi_{i,[t]}^{(k)})]$ is a.s. a subgradient of the expected-value augmented objective $\hat{\mathbb{J}}_i^{(k)}(\cdot)$ at $y_{i,t}^{i(k+1)}$, where $\xi_{i,[t]}^{(k)} := \{\xi_{i,0}^{(k)}, \dots, \xi_{i,t-1}^{(k)}\}$ with $\xi_{i,0}^{(k)} := \emptyset$.
- ii) (Upper-bounded variance) For any $y_i^i \in \mathcal{X}_i^B$, there exists a $g_{i,i}^{(k)} \in \partial_{y_i} \hat{J}_i^{(k)}(y_i^i; \xi_i)$ such that $\mathbb{E}[\|g_{i,i}^{(k)}\|_2^2 | \mathcal{F}_k] \leq \alpha_{g,i}^2 \|\tilde{\psi}^{(k)}\|_2^2 + \beta_{g,i}^2$ a.s. for some positive constants $\alpha_{g,i}$ and $\beta_{g,i}$.

We refer the reader to the paragraph before Theorem 2 for the definitions of the stack vector $\tilde{\psi}^{(k)}$ and the filtration $(\mathcal{F}_k)_{k \in \mathbb{N}}$ as a reminder. The proposed inexact solver for the first player

Subroutine 2: Projected Stochastic Subgradient Solver.

1 **For each** $i \in \mathcal{N}$, **at the** k -th major iteration **of Alg. 1:**
2 **Initialize:** $y_{i,0}^{i(k+1)} := \tilde{y}_i^{i(k)}$;
3 **for** $t = 0$ **to** $T_i^{(k)} - 1$ **do**
4 $y_{i,t+1}^{i(k+1)} := \text{Pj}_{\mathcal{X}_i^B} [y_{i,t}^{i(k+1)} - \kappa_{i,t} \cdot g_{i,t}^{(k)}]$, with $\kappa_{i,t} := \frac{2\tau_{1i}}{t+2}$;
5 **end**
6 **Return:** $y_i^{i(k+1)} := y_{i,T_i^{(k)}}^{i(k+1)}$.

for-loop of Algorithm 1 is given in Subroutine 2. Note that the DR splitting scheme ensures local feasibility with single projection onto local feasible sets, and requires multiple projections onto relaxed bounded box sets, which considerably reduces the computational complexity compared with other methods, such as proximal-point scheme [21].

The following lemma discusses the convergence rate of Subroutine 2 as a minor updating routine inside Algorithm 1. We use $y_{i,*}^{i(k+1)}$ to denote the accurate minimizer of the expected-value augmented function $\hat{\mathbb{J}}_i^{(k)}(\cdot)$.

Lemma 1: Suppose Assumptions 1 to 7 hold. Then, for any $T = 1, \dots, T_i^{(k)}$, the distance between the approximate solution by Subroutine 2 and the accurate solution satisfies $\mathbb{E}[\|y_{i,T}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 \mid \mathcal{F}_k] \leq 4\tau_{1i}^2 T^{-1} (\alpha_{g,i}^2 \|\tilde{\psi}^{(k)}\|_2^2 + \beta_{g,i}^2)$ a.s.

Proof: See Appendix B. ■

From Lemma 1, we can conclude that for each player $i \in \mathcal{N}$, after the k th major iteration of Algorithm 1 where player i implements $T_i^{(k)}$ projected stochastic subgradient steps in Subroutine 2, $\mathbb{E}[\|y_i^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 \mid \mathcal{F}_k] \leq \frac{(2\tau_{1i})^2}{T_i^{(k)}} (\alpha_{g,i}^2 \|\tilde{\psi}^{(k)}\|_2^2 + \beta_{g,i}^2)$. Based on this result, it is straightforward to derive an upper bound for the approximate error $\varepsilon^{(k)} := \|\mathcal{R}(\tilde{\psi}^{(k)}) - \mathcal{R}_*(\tilde{\psi}^{(k)})\|_{\mathcal{K}}$. As will be shown later, this upper bound can be treated as a function of $\underline{T}^{(k)} := \min\{T_i^{(k)} : i \in \mathcal{N}\}$ which we can tune to provide a desirable sequence of approximation accuracies.

Lemma 2: Consider $(\varepsilon^{(k)})_{k \in \mathbb{N}}$ generated by Algorithm 1 using Subroutine 2 as the inexact solver. Suppose Assumptions 1 to 7 hold. Then there exist some positive constants α_ψ and β_ψ such that the following relation holds a.s.:

$$\mathbb{E}[\varepsilon^{(k)} \mid \mathcal{F}_k] \leq \left(\underline{T}^{(k)}\right)^{-1/2} \left(\alpha_\psi \|\tilde{\psi}^{(k)}\|_{\mathcal{K}} + \beta_\psi\right). \quad (20)$$

Proof: See Appendix C. ■

Lemma 2 establishes the relationship between the approximate error $\varepsilon^{(k)}$ and the stack vector $\tilde{\psi}^{(k)}$ at each major iteration k . Let $\gamma_T^{(k)} := \gamma^{(k)}(\underline{T}^{(k)})^{-1/2}$. From Theorem 2, it suffices to have $(\gamma_T^{(k)})_{k \in \mathbb{N}}$ summable and $(\|\tilde{\psi}^{(k)}\|)_{k \in \mathbb{N}}$ bounded. To this end, we next show how the abstract conditions listed in Theorem 2 can be guaranteed by some explicit choices of the parameters and finally prove the a.s. convergence of Algorithm 1.

Theorem 3: Consider the sequence $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ generated by Algorithm 1 using Subroutine 2 as an inexact solver. Suppose Assumptions 1 to 7 hold. In addition, the sequence $(\gamma^{(k)})_{k \in \mathbb{N}}$ satisfies $0 \leq \gamma^{(k)} \leq 1$ and $\sum_{k \in \mathbb{N}} \gamma^{(k)}(1 - \gamma^{(k)}) = +\infty$, and the sequence $(\gamma_T^{(k)})_{k \in \mathbb{N}}$ is absolutely summable. Then

$(\|\tilde{\psi}^{(k)}\|_{\mathcal{K}})_{k \in \mathbb{N}}$ is bounded a.s., and $\sum_{k \in \mathbb{N}} \gamma^{(k)} \mathbb{E}[\varepsilon^{(k)} \mid \mathcal{F}_k] < \infty$ a.s.

Proof: See Appendix D. ■

Combining the results of Theorems 2 and 3, we can obtain the following main result of this work.

Corollary 1: Under the same assumptions of Theorem 3, the sequence $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ will converge to a fixed point of $\mathcal{R}_*$ and the associated sequence $([g_i^{i(k)}]_{i \in \mathcal{N}})_{k \in \mathbb{N}}$ generated by $J_{\tilde{\mathcal{A}}}(\tilde{\psi}^{(k)})$ will converge to a v-SGNE of the problem (3).

Based on Corollary 1, to get a solution arbitrarily closed to a v-SGNE, we will run Algorithm 1 for a sufficiently large number of major iterations. Then, we use the obtained last-iterate $\tilde{\psi}^{(k)}$ to run Subroutine 2 for another sufficiently large number of minor iterations $\underline{T}^{(k)}$ [39, Lemma 3]. To ensure the convergence of Algorithm 1, it suffices to properly choose $(\gamma^{(k)})_{k \in \mathbb{N}}$ and $(\underline{T}^{(k)})_{k \in \mathbb{N}}$ such that $(\gamma_T^{(k)})_{k \in \mathbb{N}}$ is a summable sequence. As an example for admissible parameters, we can choose $\gamma^{(k)} = 1/k^a$ and $\underline{T}^{(k)} = k^b$, with $0 < a \leq 1$ and $a + b/2 > 1$. This can be manipulated to make the proposed algorithm work under different practical settings. For instance, if these players are working in a feedback-parsimonious setting, i.e., the available realizations of noisy first-order/gradient information per iteration are scarce, one can choose a faster decaying rate for $(\gamma^{(k)})_{k \in \mathbb{N}}$ as long as $\sum_{k \in \mathbb{N}} \gamma^{(k)} = +\infty$ and let $(\underline{T}^{(k)})_{k \in \mathbb{N}}$ grow linearly or even sublinearly. In contrast, if the available realizations are abundant, one can let $(\underline{T}^{(k)})_{k \in \mathbb{N}}$ grow superlinearly while fixing $\gamma^{(k)}$ to be some constant such that the proposed algorithm can enjoy a faster convergence rate.

V. CASE STUDY AND NUMERICAL SIMULATIONS

A. Stochastic Nash–Cournot Distribution Game

We evaluate the performance of the proposed algorithm with a Nash–Cournot distribution problem [10, Sec. 1.4.3] [40] over a transport network. Several firms (indexed by $\mathcal{N} := \{1, \dots, N\}$), who produce a common homogeneous commodity, participate in this game. These firms try to optimize their own payoffs by deciding the quantity of the commodity to produce at each factory and the quantities to distribute to different markets. A transport network is provided, with markets as the nodes and roads as the edges. Let $\mathcal{N}_T$ denote the node set of this network and $\mathcal{E}_T$ the edge set, distinguished from $\mathcal{N}_g$ and $\mathcal{E}_g$ of the underlying communication network $\mathcal{G}$. Denote cardinalities of $\mathcal{N}_T$ and $\mathcal{E}_T$ by N_T and E_T , and the incident matrix of this transport network by $B_T \in \mathbb{R}^{N_T \times E_T}$.

Each firm has N_{T_i} factories at certain nodes on this transport network, given by the set $\mathcal{N}_{T_i}$. Its decision vector $x_i \in \mathbb{R}^{E_T + N_{T_i}}$ is comprised of two parts ($x_i := [u_i; v_i]$): each entry of $u_i \in \mathbb{R}_+^{E_T}$ represents the quantity of the commodity delivered through a road in $\mathcal{E}_T$; each entry of $v_i \in \mathbb{R}_+^{N_{T_i}}$ represents the quantity of the commodity produced by one of its factories in $\mathcal{N}_{T_i}$. The indicator matrix which maps from each entry of v_i to the corresponding node on the transport network is denoted by $E_i \in \mathbb{R}^{N_T \times N_{T_i}}$, and we let $A_i := [B_T, E_i]$. These two parts (u_i and v_i) together uniquely determine the distribution of commodity $A_i x_i$ over the markets. If we assume that the factories owned by firm i have maximum production

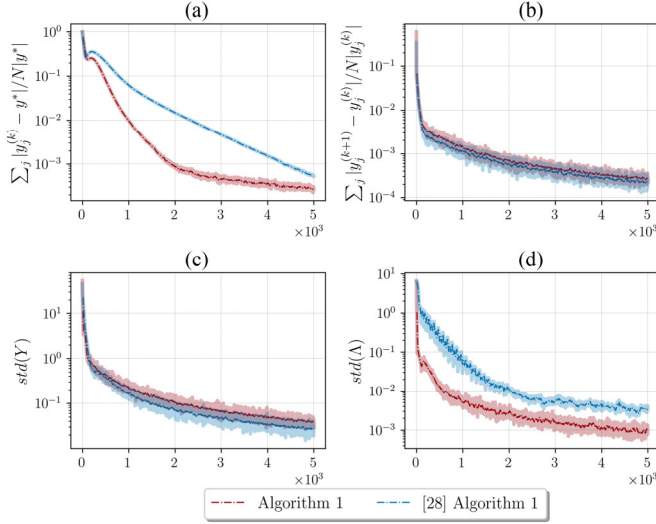


Fig. 1. Performances of Algorithm 1 in a Nash-Cournot Game.

capacities $b_i \in \mathbb{R}_{++}^{N_{T_i}}$, then each entry of the vector $u_i \in \mathbb{R}^{E_T}$ is upper-bounded by $\|b_i\|_1$, and the local feasible set $\mathcal{X}_i$ is a polytope which can be written as: $\mathcal{X}_i := \{x_i \in \mathbb{R}^{E_T + N_{T_i}} \mid 0 \leq v_i \leq b_i, 0 \leq u_i \leq \|b_i\|_1 \otimes \mathbf{1}_{E_T}, A_i x_i \geq 0\}$. The objective function of each firm i is given by: $J_i(x_i; x_{-i}, \xi_i) = x_i^T Q_i x_i + C_t(u_i) + C_p^i(v_i) - (P(Ax) + \xi_i)^T A_i x_i$, where $Q_i \in \mathbb{S}_{++}^{E_T + N_{T_i}}$, $A := [A_1, \dots, A_N]$, $x := [x_1; \dots; x_N] \in \mathbb{R}^n$ with $n := NE_T + \sum_{i \in \mathcal{N}} N_{T_i}$, and $P(Ax) := w - \Sigma Ax$ maps from the total quantities Ax of the commodity at markets to their unit prices with $w \in \mathbb{R}_{++}^{N_T}$ and $\Sigma \in \mathbb{S}_{++}^{N_T}$. The transport cost C_t is defined as the sum of the costs at all roads, i.e., $C_t(u_i) := \sum_{k \in \mathcal{E}_T} C_t^k([u_i]_k)$, where each road $k \in \mathcal{E}_T$ has $C_t^k([u_i]_k) := \eta_k([u_i]_k - (1 - \frac{1}{1+[u_i]_k}))$. The production cost C_p^i is also defined as the sum of the costs at all factories, i.e., $C_p^i(v_i) := \sum_{k \in \mathcal{N}_{T_i}} C_p^{i,k}(v_i)$, where each factory $k \in \mathcal{N}_{T_i}$ has $C_p^{i,k}([v_i]_k) := \kappa_{i,k}([v_i]_k - (1 - \frac{1}{1+[v_i]_k}))$. The total income $(P(Ax) + \xi_i)^T A_i x_i$ captures uncertainty in the unit prices through the random vector ξ_i , which has its entries independently identically distributed with mean zero.

Furthermore, we assume that each market has a maximum capacity for the commodity, and the decision vectors of the players should collectively satisfy the global resource constraints $\sum_{i \in \mathcal{N}} A_i x_i \leq c$, where $c \in \mathbb{R}_{++}^{N_T}$. Building on the discussed setups, each firm $i \in \mathcal{N}$, given the production and distribution strategies of the other players (x_{-i}), aims to solve the following optimization problem: minimize $x_i \in \mathcal{X}_i \mathbb{E}_{\xi_i}[J_i(x_i; x_{-i}, \xi_i)]$, subject to $A_i x_i \leq c - \sum_{j \in \mathcal{N}_{-i}} A_j x_j$.

1) Assumptions Verification: We use the transport network of the city of Oldenburg [41] (Fig. 2 top): it consists of $N_T = 29$ nodes (markets) and $E_T = 2 \times 34$ directed edges (roads). Five firms ($N = 5$) participates in this game, and each firm has a single factory at a given location/node $\{8, 14, 21, 10, 29\}$. Each factory has its maximum production capacity uniformly sampled from the interval $[10, 14]$, and Q_i is a diagonal matrix with the diagonal entries uniformly sampled

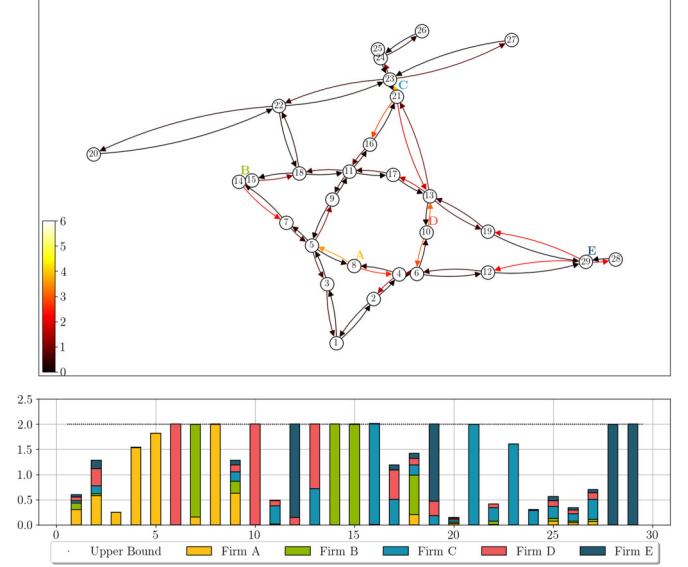


Fig. 2. v-SGNE obtained by Algorithm 1.

from $[2, 3]$. In the transport costs, we have $\frac{1}{8}\eta_k \in (0, 1]$ being the ratio between the length of road k and the maximum length of the roads in $\mathcal{E}_T$. In the production costs, we fix the coefficients $\kappa_{i,k} = 2$. In the price function $P(\cdot)$, we draw each entry of the vector w uniformly from the interval $[26, 30]$ and set $[\Sigma]_{ii} := 1$ for all $i \in \mathcal{N}_T$ and $[\Sigma]_{ji} := 0.3 \cdot (1 - \frac{1}{8}\eta_{(j,i)})$ for all $(j, i) \in \mathcal{E}_T$.

For each player $i \in \mathcal{N}$, it is easy to check by definition that $J_i(x_i; x_{-i}, \xi_i)$ and $\mathbb{J}_i(x_i; x_{-i})$ are smooth and proper, and they are convex in x_i . Moreover, the pseudogradient $\mathbb{F}$ is strongly monotone on the local compact feasible sets $\prod_{i \in \mathcal{N}} \mathcal{X}_i$ (detailed verifications are omitted due to space limit). Then by [10, Th. 2.3.3], this problem admits a unique v-SGNE. We set the communication graph of the players to be composed of an undirected circle plus two randomly selected edges. Therefore, Assumptions 1 to 4 are fulfilled. We choose $\rho_\mu = 8$ and then appropriately set the step sizes to be $\tau_1 = 0.0285 \otimes I_{N_n}$, $\tau_2 = 0.09 \otimes I_{N_m}$, $\tau_3 = 0.5 \otimes I_{E_n}$, and $\tau_4 = 0.5 \otimes I_{E_m}$. It can be checked numerically that the conditions in Assumptions 5 and 6 are satisfied. We further set $\xi_i \sim U[-2, 2]$ and can easily verify the conditions in Assumption 7.

2) Simulation Results: The sequence $(\gamma^{(k)})_{k \in \mathbb{N}}$ is fixed to be $1/2$, and the subgradient steps taken are chosen as $T^{(k)} = \lceil 10^{-4} k^{2.1} \rceil + 20$. We compare the performance of Algorithm 1 with that of [28], with $c = 4$ and the relaxed step sizes chosen as 0.04 . These step sizes are empirically pushed to near the upper limit of convergence; otherwise, [20, Lemma 6] suggests a set of minuscule and conservative step sizes ($\approx 3 \times 10^{-5}$). The performances of the proposed algorithm are shown in Fig. 1. We use the thick and semitransparent lines to illustrate the real fluctuations of the metrics throughout the iterations, while using the thin lines to exhibit the simple moving averages of the metrics with a window size of 30. The averages of the normalized distances to the v-SGNE are presented in Fig. 1(a), where the unique v-SGNE is calculated using the centralized method from [42]. Note that $y_j^{(k)}$ denotes the stack of player j 's local decision and local estimates at the k th iteration, and y^* the v-SGNE of the

game. Fig. 1(b) shows the relative lengths of the updating step at each iteration. Let $\bar{y}^{(k)} := \frac{1}{N} \sum_{j \in \mathcal{N}} y_j^{(k)}$. Fig. 1(c) exhibits how the sums of the standard deviations of the local estimates $\{y_j\}$, i.e., $\sum_{\ell=1}^n (\frac{1}{N} \sum_{j \in \mathcal{N}} ([y_j^{(k)}]_\ell - [\bar{y}^{(k)}]_\ell)^2)^{\frac{1}{2}}$, evolve over the iterations. It measures the level of consensus among different local estimates y_j . Fig. 1(d) is almost the same as Fig. 1(c) except that we are now investigating the consensus of local dual variables $\{\lambda_j\}$. The computed v-SGNE of this problem is illustrated in Fig. 2, where we use five different colors to represent the different players/firms. The top panel includes a geographic illustration, with the locations of the factories denoted by the colored letters and the total quantities transported on the roads illustrated by the brightness of the edges. The bottom panel shows the commodity contributions from the players at each market on this transport network.

B. Multiproduct Assembly Game With the Two-Stage Model

The two-stage stochastic programming problem originated from [43] and found its applications in fields, such as financial planning and control [44, Sec. 1.2], investment in power plants [44, Sec. 1.3], transportation planning during emergency response [45], etc. We now consider a multiproduct assembly problem using the two-stage model [46, Sec. 1.3.1]. In a game network with N manufacturers/players indexed by $\mathcal{N} := \{1, \dots, N\}$, each player i produces ℓ_i types of commodities. There are in total m different subassemblies that have to be ordered from a third-party vendor. Each player i needs n_i different types of subassemblies in total, and a unit of commodity j requires $h_{i,(j,v)}$ units of subassembly v , where $j = 1, \dots, \ell_i$ and $v = 1, \dots, n_i$. The demands for player i 's commodities are modeled as a random vector $D_i := [D_{i,1}; \dots; D_{i,\ell_i}]$, which has its range $\mathcal{D}_i$ inside a bounded set in the positive orthant.

We start by formulating the second-stage problem. Let the numbers of subassemblies ordered by player i be denoted by $x_i \in \mathbb{R}_+^{n_i}$, which is treated as a parameter in the second-stage problem. In this stage, player i makes a production plan about the quantity of each commodity to produce based on the realized demand vector $d_i \in \mathbb{R}_+^{\ell_i}$. This production plan should maximize the profit and at the same time not exceed the quantities of available subassemblies. The income of player i is comprised of the unit selling prices of the commodities $p_i \in \mathbb{R}^{\ell_i}$ and the unit salvage values of subassemblies that are not used $s \in \mathbb{R}^m$. Denote the numbers of produced units by $z_i \in \mathbb{R}_+^{\ell_i}$, and the numbers of subassemblies left in inventory by $y_i \in \mathbb{R}_+^{n_i}$. We introduce the matrix $H_i \in \mathbb{R}^{\ell_i \times n_i}$ with each entry $[H_i]_{(j,v)} = h_{i,(j,v)}$ and a binary matrix $A_i \in \mathbb{R}^{m \times n_i}$ mapping each entry of y_i to one among the m subassemblies. In addition, assume the full-row-rank matrix H_i has $\ell_i \leq n_i$ and no column sums to zero. Then we can define the nonsmooth function $Q_i(x_i; d_i) = \min\{-p_i^T z_i - s^T A_i y_i \mid y_i = x_i - H_i^T z_i, 0 \leq z_i \leq d_i, y_i \geq 0\}$, the minimizer of which is the best production plan.

With $Q_i(x_i; d_i)$ defined, we can then formulate the first-stage problem. The price of subassembly v per unit consists of the base cost C_v , which is a random variable and the additional cost with the increasing ratio $[\Sigma]_{(\nu,\nu)}$ per ordered unit. At this stage, when making decisions about the preorder quantities x_i

to maximize the profit, each player i is uncertain about the base prices of subassemblies and the demands for its commodities. Each player i has an expected-value objective w.r.t. the random vectors $C := [C_\nu]_{\nu=1,\dots,m}$ and D_i . Moreover, their decisions should collectively satisfy the global constraints concerning the available subassemblies. Altogether, the first-stage problem for each player i can be expressed as

$$\begin{cases} \text{minimize} & \mathbb{E} \left[\frac{1}{2} x_i^T Q_i x_i + (C + \Sigma A x)^T A_i x_i + Q_i(x_i; D_i) \right] \\ \text{subject to} & A_i x_i \leq c - \sum_{j \in \mathcal{N}_{-i}} A_j x_j \end{cases} \quad (21)$$

where $A := [A_1, \dots, A_N]$, $x := [x_1; \dots; x_N]$, $\mathcal{X}_i$ is a compact and convex local feasible set for x_i , Q_i , and Σ are diagonal matrices with each diagonal entry positive, and the constant vector $c \in \mathbb{R}^m$ denotes the quantities of available subassemblies.

Suppose $N = 5$ players participate in this game to compete for $m = 10$ types of subassemblies. The decision vector of each player i has dimension n_i chosen uniformly at random from $\{7, 8, 9, 10\}$. The local feasible set $\mathcal{X}_i$ is the direct product of n_i connected compact intervals. The communication graph consists of a directed circle and two randomly selected edges.

1) Assumptions Verification: We claim that the function $Q_i(x_i; d_i)$ is a piecewise linear function in $x_i \in \mathcal{X}_i$ given any fixed $d_i \in \mathcal{D}_i$, where $\mathcal{D}_i$ and $\mathcal{X}_i$ are both bounded. We first introduce the residual variable $r_i = d_i - z_i$ and convert the inequality constraints in $Q_i(x_i; d_i)$ to equality ones as follows:

$$\begin{cases} \text{minimize}_{z_i \geq 0, h_i \geq 0, r_i \geq 0} & -p_i^T z_i - s^T A_i h_i \\ \text{subject to} & h_i = x_i - H_i^T z_i, z_i + r_i = d_i. \end{cases} \quad (22)$$

By letting $B_i = \begin{bmatrix} H_i^T & I_{n_i} & 0_{n_i \times \ell_i} \\ I_{\ell_i} & 0_{\ell_i \times n_i} & I_{\ell_i} \end{bmatrix}$, $u_i := [z_i; h_i; r_i]$, $q_i := [-p_i; -A_i^T s; 0_{\ell_i}]$, $\tilde{I}_i = [I_{n_i}; 0_{\ell_i \times n_i}]$, and $\tilde{d}_i = [0_{n_i}; d_i]$, the above constrained linear programming can be presented as: minimize $q_i^T u_i$, while subject to $B_i u_i = \tilde{I}_i x_i + \tilde{d}_i$ and $u_i \geq 0$. Its dual problem can then be derived as

$$\text{maximize}_{v_i} \left(\tilde{I}_i x_i + \tilde{d}_i \right)^T v_i, \text{ subject to } B_i^T v_i \leq q_i. \quad (23)$$

We progress with the dual problem which only has x_i as the coefficients of the objective function. Since the feasible set $\mathcal{X}_i$ is compact inside the nonnegative orthant, the simplex method will identify a vertex solution to the problem (23), even though the problem may admit unbounded solutions. Note that the polyhedral $\mathcal{P}_i := \{v_i \in \mathbb{R}^{n_i + \ell_i} \mid B_i^T v_i \leq q_i\}$ only admits a finite number of vertices $\mathcal{V}_i := \{V_1, V_2, \dots, V_M\}$ ($-\infty$ excluded). Thus, $Q_i(x_i; d_i) := \max_{V_j \in \mathcal{V}_i} V_j^T \cdot [x_i; d_i]$, which completes the proof that $Q_i(x_i; d_i)$ is a piecewise linear function in x_i . It follows that the expected value function $\mathbb{E}_{D_i}[Q_i(x_i; D_i)]$ is a convex function in x_i [47, Sec. 3.2.1]. Applying the arguments in [32, Sec. V] to the remaining parts of $\mathbb{J}_i(x_i; x_{-i})$, we can show that the pseudogradient $\mathbb{F}$ is strongly monotone. By [29, Prop. 12.11], this multiproduct assembly problem admits a unique Nash equilibrium. It can also be checked numerically that there exists a $\rho_\mu > 0$ such that the operator $\mathcal{R}^T \tilde{\mathbb{F}} + \frac{\rho_\mu}{2} L_n$ is maximally monotone. These arguments guarantee that Assumptions 1, 3, and 5 hold for this SGNEP.

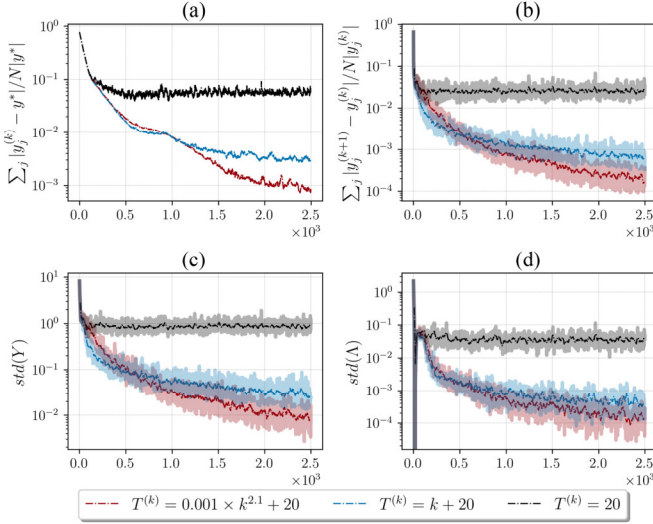


Fig. 3. Performances of Algorithm 1 for the two-stage model.

To guarantee that Assumption 7 holds, it suffices to verify that the nonsmooth parts of the objectives fulfill these conditions. We can establish the interchangeability of subdifferential and integral using [46, Th. 7.52]. We then consider the function $\phi_i(\chi) := \max_{v_i \in \mathcal{P}_i} (v_i^T \cdot \chi)$, where $\mathcal{P}_i := \{v_i \in \mathbb{R}^{n_i + \ell_i} \mid B_i^T v_i \leq q_i\}$. Since the set $\mathcal{P}_i$ is nonempty, $\phi_i(\chi)$ is the support function of $\mathcal{P}_i$. By definition, the support function $\phi_i(\chi)$ is the conjugate function of the indicator function $\iota_{\mathcal{P}_i}(\chi)$, i.e., $\phi_i(\chi) = \max_{v_i \in \mathcal{P}_i} (v_i^T \cdot \chi) = \max_{v_i \in \mathcal{P}_i} (v_i^T \cdot \chi - \iota_{\mathcal{P}_i}(v_i))$. Since the set $\mathcal{P}_i$ is convex and closed, the function $\iota_{\mathcal{P}_i}(\chi)$ is convex, lower semicontinuous and proper. By [46, Th. 7.5 and (7.24)], we obtain $\partial \phi_i(\chi) = \operatorname{argmax}_{v_i \in \mathcal{P}_i} \{v_i^T \cdot \chi - \iota_{\mathcal{P}_i}(v_i)\} = \operatorname{argmax}_{v_i \in \mathcal{P}_i} \{v_i^T \cdot \chi\}$. Moreover, by the chain rule, the subdifferential should be $\partial Q_i(x_i; d_i) = \tilde{I}_i^T \cdot \operatorname{argmax}_{v_i \in \mathcal{P}_i} \{(\tilde{I}_i x_i + \tilde{d}_i)^T \cdot v_i\}$. As we discussed in the verification of Assumption 1, the solution set of $\operatorname{argmax}_{v_i \in \mathcal{P}_i} \{(\tilde{I}_i x_i + \tilde{d}_i)^T \cdot v_i\}$ must contain at least one of $\mathcal{P}_i$'s vertices. Hence, we can always find a bounded subgradient of Q_i such that Assumption 7 (ii) holds.

2) Simulation Results: We restrict each random variable D_i to having a finite range $\{d_1, \dots, d_L\}$ with the probability distribution $\{P_1, \dots, P_L\}$. Under this restriction, the objective function of each player i can be explicitly written as: $\mathbb{J}_i(x_i; x_{-i}) = \frac{1}{2} x_i^T Q_i x_i + (\mathbb{E}[C] + \Sigma A x)^T A_i x_i + \sum_{l=1}^L P_l Q_i(x_i; d_l)$. The method proposed in [32] can then be applied to compute the unique v-SGNE for reference. The performance of Algorithm 1 when solving this multiproduct assembly problem is illustrated in Fig. 3. The thin lines reflect the simple moving averages of these metrics with a window size of 20. The curves of $T^{(k)} \propto k^{2.1}$ illustrate a steady convergence toward the v-GNE, as suggested in Theorem 3, while the trajectories of $T^{(k)} = 20$ stop decreasing after some iterations. The curves of $T^{(k)} \propto k$ also keep descending yet with a gentler trend compared with those of $T^{(k)} \propto k^{2.1}$, which suggests the possibility of some relaxations to the current conditions in Theorems 2 and 3. For the detailed figure descriptions, see Section V-A2.

VI. CONCLUSION

In this article, we study the SGNEP and propose a distributed stochastic algorithm under the partial-information setting based on solving augmented best-response subproblems induced by the Douglas–Rachford scheme. The proposed algorithm is proved to converge to a true v-SGNE if the sequence of inertial step sizes and the inverse of the number of realizations per major iteration decrease altogether at a proper rate. This raises the question if there exists a less conservative bound for this decreasing rate such that the proposed algorithm can still converge yet with a faster convergence rate and fewer observations needed per major iteration. Another interesting work remains concerning the convergence rate analysis of the proposed algorithm. As we have previously mentioned, the fixed point iteration discussed in this article engages two reflected resolvent operators, which merely admit quasicontractiveness rather than contractiveness. The convergence rate analysis under this setting remains an underexplored yet increasingly active direction [48], [49], [50]. Finally, although we only analyze the projected stochastic subgradient method, the main convergence result in Theorem 2 actually allows a lot of possibilities. It would be interesting to develop inexact solvers based on different stochastic optimization schemes that are more sample-efficient or further relax the assumptions made.

APPENDIX

A Proof of Theorem 2

Proof: The following proof is largely inspired from that of [33, Prop. 5.34] for deterministic sequences and nonexpansive operators case with suitable modifications. Given an arbitrary initial point $\tilde{\psi}^{(0)} = \tilde{\psi}_*^{(0)}$, we let $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ denote the sequence generated by (18) and $(\tilde{\psi}_*^{(k)})_{k \in \mathbb{N}}$ the auxiliary sequence constructed by letting $\tilde{\psi}_*^{(k+1)} := \mathcal{R}_*(\tilde{\psi}^{(k)})$. We next try to extract a recursive relationship w.r.t. $\|\tilde{\psi}^{(k+1)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2$ to establish that $\sum_{k \in \mathbb{N}} \operatorname{res}(\tilde{\psi}^{(k)}) < \infty$ a.s., where $\tilde{\psi}_*$ is a fixed point of $\mathcal{R}_*$. Writing the explicit formula of $\|\tilde{\psi}_*^{(k+1)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2$ yields

$$\begin{aligned} \|\tilde{\psi}_*^{(k+1)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 &= \|(1 - \gamma^{(k)})\tilde{\psi}^{(k)} + \gamma^{(k)}\mathcal{R}_*(\tilde{\psi}^{(k)}) - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 \\ &= (1 - \gamma^{(k)})\|\tilde{\psi}^{(k)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 + \gamma^{(k)}\|\mathcal{R}_*(\tilde{\psi}^{(k)}) - \mathcal{R}_*(\tilde{\psi}_*^{(k)})\|_{\mathcal{K}}^2 \\ &\quad - \gamma^{(k)}(1 - \gamma^{(k)})\|\mathcal{R}_*(\tilde{\psi}^{(k)}) - \tilde{\psi}^{(k)}\|_{\mathcal{K}}^2 \\ &\stackrel{(a)}{\leq} \|\tilde{\psi}^{(k)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 - \gamma^{(k)}(1 - \gamma^{(k)})\left(\operatorname{res}(\tilde{\psi}^{(k)})\right)^2, \end{aligned}$$

where (a) follows from the fact that $\mathcal{R}_*$ is quasi-nonexpansive. Next, we derive a recursive relationship for $\|\tilde{\psi}^{(k+1)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2$ as

$$\begin{aligned} \|\tilde{\psi}^{(k+1)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 &= \|\gamma^{(k)}\epsilon^{(k)} + \tilde{\psi}_*^{(k+1)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 \\ &= \|\tilde{\psi}_*^{(k+1)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 + \left(\gamma^{(k)}\epsilon^{(k)}\right)^2 + 2\left\langle \gamma^{(k)}\epsilon^{(k)}, \tilde{\psi}_*^{(k+1)} - \tilde{\psi}_*^{(k)} \right\rangle_{\mathcal{K}} \\ &\stackrel{(a)}{\leq} \|\tilde{\psi}^{(k)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}^2 - \gamma^{(k)}(1 - \gamma^{(k)})\left(\operatorname{res}(\tilde{\psi}^{(k)})\right)^2 \\ &\quad + \left(\gamma^{(k)}\epsilon^{(k)}\right)^2 + 2\gamma^{(k)}\epsilon^{(k)}\|\tilde{\psi}^{(k)} - \tilde{\psi}_*^{(k)}\|_{\mathcal{K}}, \end{aligned}$$

where (a) follows from the relation derived above and the Cauchy–Schwarz inequality. Taking conditional expectation $\mathbb{E}[\cdot | \mathcal{F}^{(k)}]$ on both sides yields

$$\begin{aligned} & \mathbb{E} \left[\|\tilde{\psi}^{(k+1)} - \tilde{\psi}^*\|_{\mathcal{K}}^2 | \mathcal{F}^{(k)} \right] \\ & \leq \|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}}^2 - \gamma^{(k)} \left(1 - \gamma^{(k)} \right) \left(\text{res} \left(\tilde{\psi}^{(k)} \right) \right)^2 \\ & \quad + \mathbb{E} \left[2\gamma^{(k)} \varepsilon^{(k)} \|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}} + \left(\gamma^{(k)} \varepsilon^{(k)} \right)^2 | \mathcal{F}^{(k)} \right]. \end{aligned} \quad (24)$$

Based on the fact that $\sum_{k \in \mathbb{N}} \gamma^{(k)} \mathbb{E}[\varepsilon^{(k)} | \mathcal{F}^{(k)}] < +\infty$ a.s. and $(\|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}})_{k \in \mathbb{N}}$ is bounded a.s., we can obtain that $\sum_{k \in \mathbb{N}} \mathbb{E}[2\gamma^{(k)} \varepsilon^{(k)} \|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}} + (\gamma^{(k)} \varepsilon^{(k)})^2 | \mathcal{F}^{(k)}] < \infty$ a.s. By applying the Robbins–Siegmund (R–S) Theorem [36, Th. 1], we can then conclude that on a set $\hat{\Omega}$ which has probability one, $\sum_{k \in \mathbb{N}} \gamma^{(k)} (1 - \gamma^{(k)}) \text{res}(\tilde{\psi}^{(k)})^2 < \infty$ with $\gamma^{(k)} \in [0, 1]$ and $\sum_{k \in \mathbb{N}} \gamma^{(k)} (1 - \gamma^{(k)}) = +\infty$. Now we fix an arbitrary sample path $\hat{\omega} \in \hat{\Omega}$ for subsequent analysis, while omitting $\hat{\omega}$ for brevity. In the following we will prove by contradiction that $\liminf_{k \rightarrow \infty} \text{res}(\tilde{\psi}^{(k)})^2 = 0$. Suppose otherwise that $\liminf_{k \rightarrow \infty} \text{res}(\tilde{\psi}^{(k)})^2 = \delta$, where $\delta > 0$ is some positive constant. Then there exists a sufficiently large K_δ such that for any $k > K_\delta$, $\text{res}(\tilde{\psi}^{(k)})^2 > \delta/2$. By this lower bound, we have $\sum_{k > K_\delta} \gamma^{(k)} (1 - \gamma^{(k)}) \text{res}(\tilde{\psi}^{(k)})^2 > \delta/2 \sum_{k > K_\delta} \gamma^{(k)} (1 - \gamma^{(k)}) = +\infty$, which contradicts the previous statement that $\sum_{k \in \mathbb{N}} \gamma^{(k)} (1 - \gamma^{(k)}) \text{res}(\tilde{\psi}^{(k)})^2 < \infty$. This shows that $\liminf_{k \rightarrow \infty} \text{res}(\tilde{\psi}^{(k)})^2 = 0$. As a result, there exists a subsequence, denoted by $(\tilde{\psi}^{(k_i)})_{i \in \mathbb{N}}$, such that $\lim_{i \rightarrow \infty} \text{res}(\tilde{\psi}^{(k_i)})^2 = 0$.

Moreover, the subsequence $(\tilde{\psi}^{(k_i)})_{k_i \in \mathbb{N}}$ is bounded, and thus, has a convergent subsequence $(\tilde{\psi}^{(l_i)})_{i \in \mathbb{N}}$, where $(l_i)_{i \in \mathbb{N}} \subseteq (k_i)_{i \in \mathbb{N}}$ such that $\lim_{i \rightarrow \infty} \tilde{\psi}^{(l_i)} = \tilde{\psi}^\dagger$. If Assumption 5(i) holds, by definition, $\mathcal{R}_*$ is a nonexpansive mapping. It follows from [33, Cor. 4.28] that $\tilde{\psi}^\dagger \in \text{Fix}(\mathcal{R}_*)$. If Assumption 5(ii) holds instead, from [32, Lemma 6], $\mathcal{R}_*$ is a continuous mapping, i.e., $\lim_{i \rightarrow \infty} \text{res}(\tilde{\psi}^{(l_i)}) = 0$ implies $\mathcal{R}_*(\tilde{\psi}^\dagger) = \tilde{\psi}^\dagger$ and hence $\tilde{\psi}^\dagger \in \text{Fix}(\mathcal{R}_*)$. Therefore, we can substitute $\tilde{\psi}^*$ in (24) with $\tilde{\psi}^\dagger$. By [36, Th. 1], $\lim_{k \rightarrow \infty} \|\tilde{\psi}^{(k)} - \tilde{\psi}^\dagger\|_{\mathcal{K}}^2$ exists. Since $(\tilde{\psi}^{(l_i)})_{i \in \mathbb{N}}$ is a subsubsequence of $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ converging to the fixed point $\tilde{\psi}^\dagger$, we can conclude that $\lim_{k \rightarrow \infty} \|\tilde{\psi}^{(k)} - \tilde{\psi}^\dagger\|_{\mathcal{K}}^2 = 0$, and hence $\lim_{k \rightarrow \infty} \tilde{\psi}^{(k)} = \tilde{\psi}^\dagger$. Altogether, $\tilde{\psi}^\dagger := J_{\bar{\mathcal{A}}}(\tilde{\psi}^\dagger)$ belongs to the zero set of $\mathbb{T}$ in (11). Combining this with the conclusions of Theorem 1, the proof is complete. ■

B Proof of Lemma 1

Proof: For each player $i \in \mathcal{N}$, at an arbitrary major iteration k and its minor iteration $t = 0, \dots, T_i^{(k)} - 1$, by applying the update inside the for-loop of Algorithm 2 and using the nonexpansiveness of the projection operator onto a convex set, we can obtain the following inequality of the distance between the approximate minimizer after the t th minor iteration $y_{i,t+1}^{(k+1)}$ and

the accurate minimizer $y_{i,*}^{i(k+1)}$

$$\|y_{i,t+1}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 \leq \|y_{i,t}^{i(k+1)} - \kappa_{i,t} \cdot g_{i,t}^{(k)} - y_{i,*}^{i(k+1)}\|_2^2. \quad (25)$$

Expanding the ℓ^2 norm and taking conditional expectation $\mathbb{E}[\cdot | \sigma\{\mathcal{F}_k, \xi_{i,[t]}^{(k)}\}]$ on both sides of (25) yields

$$\begin{aligned} & \mathbb{E} \left[\|y_{i,t+1}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 | \sigma\{\mathcal{F}_k, \xi_{i,[t]}^{(k)}\} \right] \\ & \leq \kappa_{i,t}^2 \mathbb{E} \left[\|g_{i,t}^{(k)}\|_2^2 | \sigma\{\mathcal{F}_k, \xi_{i,[t]}^{(k)}\} \right] + \|y_{i,t}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 \\ & \quad - 2\kappa_{i,t} \langle y_{i,t}^{i(k+1)} - y_{i,*}^{i(k+1)}, g_{i,t}^{(k)} \rangle \end{aligned} \quad (26)$$

where $g_{i,t}^{(k)} := \mathbb{E}[g_{i,t}^{(k)} | \sigma\{\mathcal{F}_k, \xi_{i,[t]}^{(k+1)}\}] \in \partial_{y_i} \hat{\mathbb{J}}_i^{(k)}(y_{i,t}^{i(k+1)})$ by Assumption 7. Using the $1/\tau_{1i}$ -strong convexity of $\hat{\mathbb{J}}_i^{(k)}$, the inner product inside (26) satisfies $\langle y_{i,t}^{i(k+1)} - y_{i,*}^{i(k+1)}, g_{i,t}^{(k)} \rangle \geq \hat{\mathbb{J}}_i^{(k)}(y_{i,t}^{i(k+1)}) - \hat{\mathbb{J}}_i^{(k)}(y_{i,*}^{i(k+1)}) + \frac{1}{2\tau_{1i}} \|y_{i,t}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2$. We then take conditional expectations $\mathbb{E}[\cdot | \mathcal{F}_k]$ on both sides of the above inequality. By the rule of successive conditioning and the fact that $y_{i,*}^{i(k+1)}$ minimizes $\hat{\mathbb{J}}_i^{(k)}(\cdot)$, the following inequality holds:

$$\begin{aligned} & \mathbb{E} \left[\|y_{i,t+1}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 | \mathcal{F}_k \right] \leq \kappa_{i,t}^2 \mathbb{E} \left[\|g_{i,t}^{(k)}\|_2^2 | \mathcal{F}_k \right] \\ & \quad + \left(1 - \frac{\kappa_{i,t}}{\tau_{1i}} \right) \mathbb{E} \left[\|y_{i,t}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 | \mathcal{F}_k \right], \text{ a.s.} \end{aligned} \quad (27)$$

Rearranging (27) and applying Assumption 7(ii) yields

$$\begin{aligned} & \frac{1}{\kappa_{i,t}} \mathbb{E} \left[\|y_{i,t+1}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 | \mathcal{F}_k \right] - \left(\frac{1}{\kappa_{i,t}} - \frac{1}{\tau_{1i}} \right) \\ & \mathbb{E} \left[\|y_{i,t}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 | \mathcal{F}_k \right] \leq \kappa_{i,t} \left(\alpha_{g,i}^2 \|\tilde{\psi}^{(k)}\|_2^2 + \beta_{g,i}^2 \right), \text{ a.s.} \end{aligned} \quad (28)$$

By setting $\kappa_{i,t} := \frac{2\tau_{1i}}{t+2}$, multiplying both sides of (28) by $(t+1)/2$, and telescoping (28) for $t = 0, \dots, T-1$, for an arbitrary $T \in \{1, \dots, T_i^{(k)}\}$, we have that the following holds a.s.:

$$\begin{aligned} & \frac{(T+1)T}{4\tau_{1i}} \mathbb{E} \left[\|y_{i,T}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 | \mathcal{F}_k \right] \\ & \leq \sum_{t=0}^{T-1} \frac{t+1}{2} \cdot \frac{2\tau_{1i}}{t+2} \left(\alpha_{g,i}^2 \|\tilde{\psi}^{(k)}\|_2^2 + \beta_{g,i}^2 \right). \end{aligned}$$

Simplifying the inequality above, we deduce that $\mathbb{E}[\|y_{i,T}^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 | \mathcal{F}_k] \leq 4\tau_{1i}^2 T^{-1} (\alpha_{g,i}^2 \|\tilde{\psi}^{(k)}\|_2^2 + \beta_{g,i}^2)$ a.s.

C Proof of Lemma 2

Proof: By the nonexpansiveness of the reflected resolvent $R_{\bar{\mathcal{B}}}$, the approximate error $\varepsilon^{(k)}$ should satisfy $\mathbb{E}[\varepsilon^{(k)} | \mathcal{F}_k] \leq \mathbb{E}[\|R_{\bar{\mathcal{A}}}(\tilde{\psi}^{(k)}) - R_{\bar{\mathcal{A}}}(\tilde{\psi}^{(k)})\|_{\mathcal{K}} | \mathcal{F}_k] = 2\mathbb{E}[\|\psi^{(k+1)} - \psi_*^{(k+1)}\|_{\mathcal{K}} | \mathcal{F}_k]$, where $\psi^{(k+1)} = [\mathbf{y}^{(k+1)}; \boldsymbol{\lambda}^{(k+1)}; \boldsymbol{\mu}^{(k+1)}; \mathbf{z}^{(k+1)}] := J_{\bar{\mathcal{A}}}(\tilde{\psi}^{(k)})$ is the stack vector obtained by using the inexact solver suggested in Algorithm 2 and $\psi_*^{(k+1)} = [\mathbf{y}_*^{(k+1)}; \boldsymbol{\lambda}_*^{(k+1)}; \boldsymbol{\mu}_*^{(k+1)}; \mathbf{z}_*^{(k+1)}] := J_{\bar{\mathcal{A}}}(\tilde{\psi}^{(k)})$ is the one using the accurate solver. Given Lemma 1 and the first two for-loops

in Algorithm 1, the approximate error of the dual variables λ has the following upper bound:

$$\begin{aligned} & \mathbb{E} \left[\|\lambda^{(k+1)} - \lambda_*^{(k+1)}\|_2 \mid \mathcal{F}_k \right] \\ & \leq \mathbb{E} \left[\|\tau_2 \Lambda \mathcal{R}(\mathbf{y}^{(k+1)} - \mathbf{y}_*^{(k+1)})\|_2 \mid \mathcal{F}_k \right] \\ & \leq \|\tau_2 \Lambda \mathcal{R}\|_2 \cdot \mathbb{E} \left[\|\mathbf{y}^{(k+1)} - \mathbf{y}_*^{(k+1)}\|_2 \mid \mathcal{F}_k \right]. \end{aligned}$$

Similar results can be trivially derived for μ and $\mathbf{z}$, the details of which are omitted for brevity. Altogether, we have that the following relation $\mathbb{E}[\|\psi^{(k+1)} - \psi_*^{(k+1)}\|_2 \mid \mathcal{F}_k] \leq C_1 \cdot \mathbb{E}[\|\mathbf{y}^{(k+1)} - \mathbf{y}_*^{(k+1)}\|_2 \mid \mathcal{F}_k]$ holds for some constant C_1 . For each $i \in \mathcal{N}$, the local estimates of others' decisions are the same in $\mathbf{y}^{i(k+1)}$ and $\mathbf{y}_*^{i(k+1)}$, while the local decisions, by Lemma 1, satisfy $\mathbb{E}[\|y_i^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 \mid \mathcal{F}_k] \leq 4\tau_{1i}^2(T_i^{(k)})^{-1}(\alpha_{g,i}^2\|\tilde{\psi}^{(k)}\|_2^2 + \beta_{g,i}^2)$ a.s. Pick the maximum coefficients $\bar{\alpha}_g := \max\{\alpha_{g,i} : i \in \mathcal{N}\}$, $\bar{\beta}_g := \max\{\beta_{g,i} : i \in \mathcal{N}\}$, $\bar{\tau}_1 := \max\{\tau_{1,i} : i \in \mathcal{N}\}$ and the minimum minor steps taken $\underline{T}^{(k)} := \min\{T_i^{(k)} : i \in \mathcal{N}\}$ over all players. By Jensen's inequality and the nonnegativity of $\alpha_{g,i}$, $\beta_{g,i}$, and $\|\tilde{\psi}^{(k)}\|$, an upper bound for the stacked local decisions and estimates is given by

$$\begin{aligned} & \mathbb{E} \left[\|\mathbf{y}^{(k+1)} - \mathbf{y}_*^{(k+1)}\|_2 \mid \mathcal{F}_k \right] \\ & \leq \left(\mathbb{E} \left[\sum_{i \in \mathcal{N}} \|y_i^{i(k+1)} - y_{i,*}^{i(k+1)}\|_2^2 \mid \mathcal{F}_k \right] \right)^{1/2} \\ & \leq 2\sqrt{N}\bar{\tau}_1 \left(\underline{T}^{(k)} \right)^{-1/2} \left(\bar{\alpha}_g \|\tilde{\psi}^{(k)}\|_2 + \bar{\beta}_g \right), \text{ a.s.} \end{aligned}$$

Combining the above inequalities, we derive the following a.s. upper bound in the Euclidean space:

$$\begin{aligned} & \mathbb{E} \left[\|\psi^{(k+1)} - \psi_*^{(k+1)}\|_2 \mid \mathcal{F}_k \right] \\ & \leq C_2 \left(\underline{T}^{(k)} \right)^{-1/2} \left(\bar{\alpha}_g \|\tilde{\psi}^{(k)}\|_2 + \bar{\beta}_g \right) \end{aligned} \quad (29)$$

where $C_2 := 2C_1\bar{\tau}_1\sqrt{N}$. We convert the above conclusion from the Euclidean space to the inner product space $\mathcal{K}$ defined by the positive definite design matrix Φ . The maximum (resp. minimum) eigenvalue of Φ is denoted by $\bar{\sigma}_\Phi$ (resp. $\underline{\sigma}_\Phi$). Then (29) implies the following relation holds a.s. in $\mathcal{K}$:

$$\begin{aligned} & \mathbb{E} \left[\|\psi^{(k+1)} - \psi_*^{(k+1)}\|_{\mathcal{K}} \mid \mathcal{F}_k \right] \\ & \leq \frac{C_2\sqrt{\bar{\sigma}_\Phi}}{\left(\underline{T}^{(k)} \right)^{1/2}} \left(\frac{\bar{\alpha}_g}{\sqrt{\underline{\sigma}_\Phi}} \|\tilde{\psi}^{(k)}\|_{\mathcal{K}} + \bar{\beta}_g \right). \end{aligned} \quad (30)$$

Hence, there exist positive constants α_ψ and β_ψ independent of k such that $\mathbb{E}[\varepsilon^{(k)} \mid \mathcal{F}_k] \leq (\underline{T}^{(k)})^{-1/2}(\alpha_\psi \|\tilde{\psi}^{(k)}\|_{\mathcal{K}} + \beta_\psi)$ a.s. ■

D Proof of Theorem 3

Proof: Consider $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ generated by the approximate iteration $\mathcal{P} = \text{Id} + \gamma^{(k)}(\mathcal{R} - \text{Id})$ and $(\tilde{\psi}_*^{(k)})_{k \in \mathbb{N}}$ generated by $\tilde{\psi}_*^{(k+1)} := \mathcal{P}_*(\tilde{\psi}^{(k)})$. Let $\tilde{\psi}^*$ denote one of the fixed points of

$\mathcal{R}_*$. To prove that $(\tilde{\psi}^{(k)})_{k \in \mathbb{N}}$ is bounded a.s., note that

$$\begin{aligned} & \mathbb{E} \left[\|\tilde{\psi}^{(k+1)} - \tilde{\psi}^*\|_{\mathcal{K}} \mid \mathcal{F}_k \right] \\ & = \mathbb{E} \left[\|\tilde{\psi}^{(k+1)} - \tilde{\psi}_*^{(k+1)} + \tilde{\psi}_*^{(k+1)} - \tilde{\psi}^*\|_{\mathcal{K}} \mid \mathcal{F}_k \right] \\ & \leq \gamma^{(k)} \mathbb{E} \left[\varepsilon^{(k)} \mid \mathcal{F}_k \right] + \mathbb{E} \left[\|\mathcal{P}_*(\tilde{\psi}^{(k)}) - \mathcal{P}_*(\tilde{\psi}^*)\|_{\mathcal{K}} \mid \mathcal{F}_k \right]. \end{aligned}$$

Let $\gamma_T^{(k)} := \gamma^{(k)}(\underline{T}^{(k)})^{-1/2}$. By applying Lemma 2 and using the fact that $\mathcal{P}_*$ is (quasi)nonexpansive, we have

$$\begin{aligned} & \mathbb{E} \left[\|\tilde{\psi}^{(k+1)} - \tilde{\psi}^*\|_{\mathcal{K}} \mid \mathcal{F}_k \right] \\ & \leq \gamma_T^{(k)} \left(\alpha_\psi \|\tilde{\psi}^{(k)}\|_{\mathcal{K}} + \beta_\psi \right) + \mathbb{E} \left[\|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}} \mid \mathcal{F}_k \right] \\ & = \gamma_T^{(k)} \left(\alpha_\psi \|\tilde{\psi}^{(k)} - \tilde{\psi}^* + \tilde{\psi}^*\|_{\mathcal{K}} + \beta_\psi \right) + \|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}} \\ & \leq \left(1 + \alpha_\psi \gamma_T^{(k)} \right) \|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}} + \gamma_T^{(k)} \left(\alpha_\psi \|\tilde{\psi}^*\|_{\mathcal{K}} + \beta_\psi \right), \text{ a.s.} \end{aligned}$$

Since $\|\tilde{\psi}^*\|_{\mathcal{K}} < \infty$ and we assume that $(\gamma_T^{(k)})_{k \in \mathbb{N}}$ is summable, by the R-S Theorem [36, Th. 1], we have that $\lim_{k \rightarrow \infty} \|\tilde{\psi}^{(k)} - \tilde{\psi}^*\|_{\mathcal{K}}$ exists and is finite a.s. Consequently, there exists a set $\hat{\Omega}$ with probability one, such that for any $\hat{\omega} \in \hat{\Omega}$, the sequence $(\|\tilde{\psi}^{(k)}(\hat{\omega}) - \tilde{\psi}^*\|_{\mathcal{K}})_{k \in \mathbb{N}}$ is bounded. Therefore, we can find some constant $B(\hat{\omega})$ which satisfies, for all $k \in \mathbb{N}$, $\|\tilde{\psi}^{(k)}(\hat{\omega})\|_{\mathcal{K}} = \|\tilde{\psi}^{(k)}(\hat{\omega}) - \tilde{\psi}^* + \tilde{\psi}^*\|_{\mathcal{K}} \leq \|\tilde{\psi}^{(k)}(\hat{\omega}) - \tilde{\psi}^*\|_{\mathcal{K}} + \|\tilde{\psi}^*\|_{\mathcal{K}} \leq B(\hat{\omega})$.

Since the deterministic sequence $(\|\tilde{\psi}^{(k)}(\hat{\omega})\|_{\mathcal{K}})_{k \in \mathbb{N}}$ is upper bounded by a constant $B(\hat{\omega})$ for any $\hat{\omega} \in \hat{\Omega}$, combining Lemma 2 and the summability of $(\gamma_T^{(k)})_{k \in \mathbb{N}}$, we finally can conclude that $\sum_{k \in \mathbb{N}} \gamma^{(k)} \mathbb{E}[\varepsilon^{(k)} \mid \mathcal{F}_k](\hat{\omega}) \leq \sum_{k \in \mathbb{N}} \gamma_T^{(k)} (\alpha_\psi \|\tilde{\psi}^{(k)}(\hat{\omega})\|_{\mathcal{K}} + \beta_\psi) \leq \sum_{k \in \mathbb{N}} \gamma_T^{(k)} (\alpha_\psi B(\hat{\omega}) + \beta_\psi) < \infty$. ■

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