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Optimal stabilizing rates of switched linear control systems under arbitrary known switchings ☆

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ABSTRACT

The problem of stabilizing discrete-time switched linear control systems using continuous control input under arbitrary mode switchings is studied. It is assumed that at each time instance the switching mode can be arbitrarily chosen but is always known by the controller designing the continuous control input; thus the continuous controller is of the general form of an ensemble of mode-dependent state feedback controllers. Under this setting, the fastest (worst-case) stabilizing rate is proposed as a quantitative metric of the systems' stabilizability. Conditions are derived on when this stabilizing rate can be exactly achieved by an admissible control policy and a counter example is given to show that the stabilizing rate may not always be attained by a mode-dependent linear state feedback control policy. Bounds on the stabilizing rate are derived using (semi)norms. When such bounds are tight, the corresponding extremal norms are characterized geometrically. Numerical algorithms based on ellipsoid and polytope norms are developed for computing bounds of the stabilizing rate and illustrated through examples.

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1. Introduction and overview

In this paper, we study the stabilization problem of switched linear control systems (SLCS):

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \in \mathbb{Z}_+ := \{0, 1, \ldots\}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the continuous control input, $\sigma(t) \in \mathcal{M} := \{1, \ldots, m\}$ is the switching mode, and $(A_i, B_i)_{i \in \mathcal{M}}$ are the subsystem matrices in different modes. The set of all switching sequences $\sigma = (\sigma(0), \sigma(1), \ldots)$ is denoted by $\mathcal{S} := \mathcal{M}^\infty$.

1.1. Mode-conscious vs. Mode-resilient stabilization

Different from the existing work (e.g., DeCarlo, Branicky, Pettersson, & Lennartson, 2000; Hu, Ma, & Lin, 2008; Lin & Antsaklis, 2008; Zhang, Abate, Hu, & Vitus, 2009) where both the control input u and the switching signal σ are utilized to stabilize the SLCS, the problem studied here assumes that only u can be controlled by the user to stabilize the SLCS, while σ can be arbitrary. Further, it is assumed that, at each time t, the user when designing the control input $u(t)$ has the knowledge of $\sigma(t)$ as well as the state $x(t)$. Under these assumptions, a valid feedback control policy for the user is given by $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1, \ldots)$ where $\mathbf{u}_t$ is the feedback control law at time t: $u(t) = \mathbf{u}_t(\sigma(t), x(t))$. Denote by $\mathcal{U}$ the set of all such feedback control policies $\mathbf{u}$, and by $x(\cdot; \sigma, \mathbf{u}, x(0))$ the solution of the SLCS under $\mathbf{u} \in \mathcal{U}$ and the switching sequence $\sigma \in \mathcal{S}$, with the initial state $x(0)$.

Definition 1.1. The SLCS is called mode-conscious (or σ_*)-asymptotically stabilizable if there exists $\mathbf{u} \in \mathcal{U}$ such that, for any $\sigma \in \mathcal{S}$ and any $x(0)$, we have $x(t; \sigma, \mathbf{u}, x(0)) \to 0$ as $t \to \infty$.

In other words, mode-conscious stabilization is the problem of stabilizing the SLCS using continuous input u under arbitrary but *known* switching modes. A related but different problem, called

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the *mode-resilient stabilization problem*, is treated in Hu, Shen, and Lee (2017) where it is assumed that, at each time t, the user decides $u(t)$ without the knowledge of $\sigma(t)$; hence the user's control law is of the form $u(t) = \mathbf{u}_t(x(t))$. If a valid user control policy exists so that $x(t) \to 0$ as $t \to \infty$ under arbitrary switching sequence σ and initial state $x(0)$, then the SLCS is called *mode-resilient stabilizable* or σ^*-stabilizable. Two examples are given below to illustrate the difference of these two problems.

Example 1.1. As an application, consider a networked control system consisting of a discrete-time LTI plant $x_r(t+1) = A_r x_r(t) + B_r u_r(t)$ and a remote controller. At each time t, the controller will first receive the state $x_r(t)$ from the plant, and then use it to calculate $u_r(t)$ and send it to the plant, both over a communication channel. Suppose such communications are subject to frequent blackouts (due to, e.g., data loss, attacks) which can each time last up to a maximum of $m-1$ times steps, and during blackouts, the plant will keep using the last received control command from the controller. By denoting $0 = t_0 < t_1 < t_2 < \cdots$ the times at which the communications are successful in both directions, the sampled state and control sequences $x(k) = x_r(t_k)$ and $u(k) = u_r(t_k)$ follows the SLCS $x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)$ where $\sigma(k) = t_{k+1} - t_k + 1 \in \mathcal{M}$ and $A_i = (A_r)^i$, $B_i = [(A_r)^{i-1} + \cdots + I]B_r$ for $i \in \mathcal{M}$. If the controller does (resp. does not) know the duration of the current communication blackout, then designing a controller to stabilize the plant becomes an instance of the σ_*- (resp. σ^*-)stabilization problem. See Hu et al. (2017) and Yu, Wang, Chu, and Xie (2004) for details on the latter case.

Example 1.2. As another example, consider a pursuer and an evader with the respective positions $x_p, x_e \in \mathbb{R}^n$. Suppose the evader has several distinctive modes of evading maneuvers of the form $x_e(t+1) = x_e(t) + F_i[x_e(t) - x_p(t)]$ for $i \in \mathcal{M}$ and $F_i \in \mathbb{R}^{n\times n}$; while the pursuer has the dynamics $x_p(t+1) = x_p(t) + B_p u(t)$. Then, their relative displacement $x := x_e - x_p$ follows the SLCS (1) with $A_i = I + F_i$ and $B_i = B_p$. Depending on if the pursuer knows the evader's current evading mode, designing control $u(t)$ so that the pursuer can capture the evader (i.e., $x_p = x_e$) is either a σ_*-stabilization problem or a σ^*-stabilization problem.

With the knowledge of switching modes, the task of mode-conscious stabilization is considerably easier than mode-resilient stabilization, as illustrated by the following example. Consider the SLCS on $\mathbb{R}$ with two modes: $x(t+1) = x(t) + u(t)$ and $x(t+1) = x(t) - u(t)$, where $B_1 = 1$ and $B_2 = -1$. Given any $x(t) \neq 0$ at time t, if the user does not know the current mode $\sigma(t)$, its best action is to choose $u(t) = 0$, for otherwise the adversary will choose whichever mode with $B_{\sigma(t)}u(t) \cdot x(t) > 0$, resulting in $|x(t+1)| > |x(t)|$. Hence, the SLCS is not mode-resilient stabilizable. On the other hand, if the user knows $\sigma(t)$, hence $B_{\sigma(t)}$, the SLCS can be steered to the origin in one step by the controller $u(t) = -x(t)/B_{\sigma(t)}$. More discussions on the connection and difference of the two problems are given in Remarks 1.1 and 3.1.

Despite this relative ease, the mode-conscious stabilization remains a difficult problem. For one, it subsumes the stability problem of autonomous switched linear systems (SLSs) under arbitrary switching as a special case with $B_i = 0$ for all i (c.f. Remark 1.2), which by itself is well known to be an NP-hard problem (Tsitsiklis & Blondel, 1997). Following is an example with $B_i \neq 0$:

$$A_1 = \begin{bmatrix} 0.5 & 2 \\ 0 & 0.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ A_2 = \begin{bmatrix} 0.5 & 0 \\ 2 & 0.5 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2}$$

Suppose at each time t, the mode $\sigma(t)$ is chosen such that $\sigma(t) = 1$ if $|x_1(t)| \le |x_2(t)|$, and $\sigma(t) = 2$ if $|x_1(t)| > |x_2(t)|$. Regardless of the user's choice on $u(t)$, $\|x(t+1)\|_\infty \ge \frac{3}{2}\|x(t)\|_\infty$ for all t. This implies that the SLCS is not σ_*-stabilizable, even though each individual subsystem (A_i, B_i) is stabilizable. The latter is a necessary but not sufficient condition for σ_*-stabilizability.

1.2. Stabilizing rate

In this paper, instead of merely deriving stabilizability conditions, we adopt a quantitative approach by studying a stabilizability metric defined as follows.

Definition 1.2. The constant $\rho \in [0, \infty)$ is called an *attainable (exponential) stabilizing rate* of the SLCS if there exist $\mathbf{u} \in \mathcal{U}$ and $K \in [0, \infty)$ such that for all $x(0) \in \mathbb{R}^n$ and all $\sigma \in \mathcal{S}$, we have

$$\|x(t; \sigma, \mathbf{u}, x(0))\| \le K\rho^t \|x(0)\|, \quad \forall\, t \in \mathbb{Z}_+. \tag{3}$$

The *σ_*-stabilizing rate* of the SLCS, denoted by $\rho_* \in [0, \infty)$, is the infimum of all attainable stabilizing rates ρ.

Loosely speaking, ρ_* is the optimal worst-case exponential decay rate of the SLCS' state trajectory using continuous input under arbitrary switchings, where "worst-case" here is w.r.t. the arbitrary choice of σ. Knowing ρ_* allows us to compare the degree of stabilizability of different SLCSs and measure the robustness of stabilizability to perturbations to system parameters. In particular, if $\rho_* < 1$, the SLCS is σ_*-asymptotically stabilizable. In Section 2 we will show that the converse is also true.

Remark 1.1. The value of ρ_* in Definition 1.2 is independent of the norm $\|\cdot\|$ since all norms are equivalent. If we restrict the user control policy $\mathbf{u}$ to be of the form $u(t) = \mathbf{u}_t(x(t)), \forall\, t \in \mathbb{Z}_+$, a similar rate $\rho^* \ge 0$ can be defined, which provides a quantitative metric of the SLCS's mode-resilient stabilizability (see Hu et al., 2017). Obviously, $\rho^* \ge \rho_*$. The gap represents the information premium of the user's knowledge of the current mode. See Example 4.1 for an example.

Remark 1.2. An autonomous SLS $x(t+1) = A_{\sigma(t)}x(t)$ is called absolutely stable or stable under arbitrary switching (Liberzon, 2012) if $x(t) \to 0$ for any switching sequence σ. By considering the SLS as a SLCS with $B_i = 0$ for all i, the σ_*-stabilizing rate ρ_* is reduced to the *joint spectral radius* (JSR) of the matrix set $\{A_i\}_{i\in\mathcal{M}}$ (Jungers, 2009; Rota & Strang, 1960) and provides a quantitative metric of the SLS's absolute stability.

1.3. Main questions

In this paper, we will address the following questions.

Question 1.1. *Can the σ_*-stabilizing rate ρ_* be achieved exactly by a user control policy $\mathbf{u} \in \mathcal{U}$, i.e., does (3) hold when $\rho = \rho_*$?*

We will deal with this question in Section 3. In particular, for an SLCS with $\rho_* = 1$, if the answer is yes, then the SLCS is marginally stabilizable as the user can keep any state trajectory bounded under arbitrary switchings.

Question 1.2. *If the σ_*-stabilizing rate ρ_* is achievable, can it always be achieved by a mode-dependent linear state feedback controller of the form $u(t) = K_{\sigma(t)}x(t)$?*

Mode-dependent linear state feedback controllers are specified by a set of feedback gain matrices $\{K_i\}_{i\in\mathcal{M}}$. If the answer to Question 1.2 is affirmative, then control synthesis can be greatly simplified. The closed-loop system under such a controller is the autonomous SLS: $x(t+1) = (A_{\sigma(t)} + B_{\sigma(t)} K_{\sigma(t)}) x(t)$. By Remark 1.2, Question 1.2 is equivalent to whether ρ_* is equal to the JSR of the matrix set $\{A_i + B_i K_i\}_{i \in \mathcal{M}}$ for some properly chosen $\{K_i\}_{i \in \mathcal{M}}$. This will be addressed in Section 5.

Question 1.3. *For a general SLCS, is there a systematic way to compute tight upper and lower bounds of ρ_*?*

In Sections 4 and 6, we will present a seminorms-based approach and various iterative algorithms that generate increasingly accurate bounds of ρ_*.

1.4. Previous work and contributions

The stabilization of SLSs and SLCSs has been extensively studied in the literature (Lin & Antsaklis, 2009; Shorten, Wirth, Mason, Wulff, & King, 2007; Sun & Ge, 2005). A large portion of the existing work (e.g., Fiacchini, Girard, & Jungers, 2016; Geromel & Colaneri, 2006; Wicks, Peleties, & DeCarlo, 1998) focuses on the SLS switching stabilization problem, namely, stabilizing the SLS using σ. For SLCSs, the stabilization using both u and σ has received considerable attention, e.g., DeCarlo et al. (2000), Hu et al. (2008), Lin and Antsaklis (2008) and Zhang et al. (2009). The stabilization of SLCSs using u under arbitrary σ has also been previously explored. For example, the case when the continuous controller for u does not know the current mode (i.e., σ^*-stabilization) is studied in Hu et al. (2017), Khargonekar, Petersen, and Zhou (1990), Kothare, Balakrishnan, and Morari (1996) and Mao (2003). When the continuous controller knows the current mode (i.e., the σ_*-stabilization problem), stabilizability conditions are derived based on parameter-dependent quadratic Lyapunov functions method (Daafouz & Bernussou, 2001; Fang, Lin, & Antsaklis, 2004; Hetel, Daafouz, & Iung, 2006; Xie, Wang, Hao, & Xie, 2003), multiple Lyapunov function method (Daafouz, Riedinger, & Iung, 2002; Philippe, Essick, Dullerud, Jungers, et al., 2015), piecewise quadratic Lyapunov function method (Molchanov & Pyatnitskiy, 1989), Lyapunov-like function method with average dwell time constraints (Zhang & Shi, 2009), time-varying quadratic Lyapunov function method for continuous-time SLCSs with dwell time constraints (Allerhand & Shaked, 2011), and for uncertain LTI sampled-data systems (Naghshtabrizi, Hespanha, & Teel, 2008), to name a few. In these papers, the stabilizing controllers are assumed to be static or mode-dependent linear state feedbacks. As shown in Section 5 of the present paper, this assumption introduces conservativeness. Another factor that makes these existing stabilizing conditions sufficient but not necessary is that they employ variants of quadratic Lyapunov functions, which are in general not "optimal" (see Section 4 for definitions of optimal ones). An exact characterization of the σ_*-stabilizability condition and more accurate methods for computing the stabilizing rate call for further research and new techniques.

Compared to our previous work on the σ^*-stabilization problem in Hu et al. (2017), this paper deals with a different problem (σ_*-stabilization) and contains substantial new contributions, which will be elaborated as follows.

- *Similarities*: To answer Question 1.1 in Section 1.3, we will introduce the notions of defectiveness, reducibility, and extremal/Barabanov norms. Similar but differently defined notions were proposed in Hu et al. (2017). Certain results in this paper concerning these notions (e.g., Proposition 3.1, Theorem 4.1, and Proposition 4.3) have their counterparts in Hu et al. (2017), even though the proofs are different due to the different problems under study. These proofs are included in the Appendix.
- *Major differences*: Substantial new results of this paper include the geometry of extremal norms, complexity of Barabanov norms, a highly nontrivial example showing that the rate ρ_* may not be attained by any mode-dependent linear state feedback (cf. Section 5.1) as well as certain families of SLCSs when this is possible (cf. Section 5.2), and algorithms for computing ρ_*.

This paper is also significantly expanded from its conference version in Hu, Shen, and Lee (2018). New contents include Theorem 2.1(iii), Proposition 4.2, Theorem 4.1, the entire Section 5, all the examples (except Example 4.1), and the proofs of all the main technical results.

The main contributions of this paper are summarized as follows: (i) conditions are developed for marginal σ_*-stabilizability using the notions of defectiveness and reducibility (c.f. Section 3); (ii) analytical bounds on the stabilizing rate are established using (semi)norms and conditions are given on when such bounds are tight (c.f. Section 4); (iii) it is shown that, except for some special cases, the optimal user control policy may not be a mode-dependent linear state feedback controller (c.f. Section 5); and (iv) numerical algorithms are developed for computing the stabilizing rate (c.f. Section 6). Finally, conclusions are given in Section 7.

2. Preliminary results

We first derive some preliminary results and useful facts. A notion related to σ_*-asymptotic stabilizability in Definition 1.1 is given as follows.

Definition 2.1. The SLCS is call σ_*-exponentially stabilizable if there exist $\mathbf{u} \in \mathcal{U}$, $\rho \in [0, 1)$, and $K \in [0, \infty)$ such that $\|x(t; \sigma, \mathbf{u}, x(0))\| \le K\rho^t \|x(0)\|$, $\forall\, t \in \mathbb{Z}_+$, $\forall\, x(0) \in \mathbb{R}^n$, $\forall\, \sigma \in \mathcal{S}$.

By Definition 1.2, the SLCS is σ_*-exponentially stabilizable if and only if $\rho_* < 1$. It is obvious that σ_*-exponential stabilizability implies σ_*-asymptotic stabilizability. The following theorem, proved in Appendix A, shows that the converse is also true.

Theorem 2.1. *The following statements are equivalent:*

(i) *The SLCS is σ_*-exponentially stabilizable;*
(ii) *The SLCS is σ_*-asymptotically stabilizable;*
(iii) *For any $z \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$, and $\varepsilon > 0$, there exist $\mathbf{u}_{z,\sigma,\varepsilon} \in \mathcal{U}$ and $T_{z,\sigma,\varepsilon} \in \mathbb{Z}_+$ such that $\|x(T_{z,\sigma,\varepsilon}; \sigma, \mathbf{u}_{z,\sigma,\varepsilon}, z)\| \le \varepsilon \cdot \|z\|$.*

Consequently, through the rest of this paper, we will refer to either notion simply as σ_*-stabilizability.

The following result states that the σ_*-stabilizing rate ρ_* is positively homogeneous of degree one with respect to the collective scale of $\{A_i\}_{i \in \mathcal{M}}$ but independent of the scale of any individual B_i. The latter is hardly surprising since ρ_* depends on each B_i only through its range space $\mathcal{R}(B_i)$, i.e., $\rho_* = \rho_*(\{A_i\}_{i \in \mathcal{M}}, \{\mathcal{R}(B_i)\}_{i \in \mathcal{M}})$.

Lemma 2.1. *Let ρ_* be the σ_*-stabilizing rate of the SLCS $\{(A_i, B_i)\}_{i \in \mathcal{M}}$. Given scalar constants α and $\beta_i \neq 0$, $i \in \mathcal{M}$, the SLCS $\{(\alpha A_i, \beta_i B_i)\}_{i \in \mathcal{M}}$ has the σ_*-stabilizing rate $|\alpha|\, \rho_*$.*

Proof. It is trivial when $\alpha = 0$. Suppose $\alpha \neq 0$. If the SLCS $\{(A_i, B_i)\}_{i \in \mathcal{M}}$ has the solution $x(t; \sigma, \mathbf{u}, x(0))$ under a control policy $\mathbf{u} \in \mathcal{U}$, then under the control policy $\hat{\mathbf{u}} \in \mathcal{U}$ such that $\hat{\mathbf{u}}_t(i, x) = \alpha^{t+1} (\beta_i)^{-1} \mathbf{u}_t(i, x)$, $\forall\, i \in \mathcal{M}$, $x \in \mathbb{R}^n$, it can be easily proved by induction that $\hat{x}(t; \sigma, \hat{\mathbf{u}}, x(0)) = \alpha^t x(t; \sigma, \mathbf{u}, x(0))$ is the solution to the SLCS $\{(\alpha A_i, \beta_i B_i)\}_{i \in \mathcal{M}}$ starting from the initial state $x(0)$. The desired result then follows. $\square$

Owing to this homogeneity property, a common technique we will employ when studying a general SLCS with $\rho_* > 0$ is to study the scaled SLCS $\{(A_i/\rho_*, B_i)\}_{i \in \mathcal{M}}$. This allows us to focus on SLCSs with $\rho_* = 1$ when studying properties of SLCSs with the same homogeneity.

3. Defectiveness and reducibility

In this section, we will present a partial answer to Question 1.1 given in Section 1.3.

Definition 3.1. The SLCS is called *nondefective* if there exist $\mathbf{u} \in \mathcal{U}$ and $K \in [0, \infty)$ such that $\|x(t; \sigma, \mathbf{u}, x(0))\| \leq K(\rho_*)^t \|x(0)\|, \forall t$, for all $x(0) \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$. Otherwise, it is called *defective*.

Recall that ρ_* is the infimum of ρ for which (3) holds. Nondefective SLCSs are those SLCSs where this infimum can be exactly achieved. The SLCS is called σ_*-*marginally stabilizable* if there exists $\mathbf{u} \in \mathcal{U}$ so that $x(t)$ is bounded for all t, σ and $x(0)$. This is the case if either (i) $\rho_* < 1$; or (ii) $\rho_* = 1$ and the SLCS is nondefective. See Guglielmi and Zennaro (2001, 2003) and Jungers (2009) for related definitions of nondefectiveness for switched linear systems.

In the particular case $\rho_* = 0$, the SLCS is nondefective if and only if it is controllable to the origin in one time step for all $\sigma(0)$ and $x(0)$. As an example, note that the LTI system (A, B) with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has $\rho_* = 0$ since it is controllable to the origin in two time steps; it is defective since it cannot be steered to the origin in one time step from $x(0) = (0, 1)$. Here, $(0, 1)$ denotes a column vector in $\mathbb{R}^2$. In the following, we will focus on the nontrivial case $\rho_* > 0$.

As an example of defective SLCSs with $\rho_* > 0$, consider the following system:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Given the state $x(t)$ at any time t, for any user input $u(t)$, $x_1(t+1)+x_3(t+1) = x_1(t)+x_3(t)$ remains unchanged, while the increase from $x_2(t)$ to $x_2(t+1)$ is either $x_1(t)$ if $\sigma(t) = 1$ or $x_3(t)$ if $\sigma(t) = 2$. Thus, for $x(0)$ with $x_1(0) + x_3(0) \neq 0$, say, $x_1(0) + x_3(0) > 0$, the growth of $\|x(t)\|$ is dictated by the growth of $x_2(t)$, since x_1 and x_3 are under the user's control and their difference can be made to zero as shown below. The slowest growth rate of $x_2(t)$ against all possible σ is achieved when the user chooses $u(0)$ so that $x_1(1) = x_3(1) = [x_1(0) + x_3(0)]/2 > 0$, and then $u(t) \equiv 0$ for $t \in \mathbb{Z}_+$. This yields $x_2(t) = c + \frac{x_1(0)+x_3(0)}{2} t$ for some constant c. The linear growth rate of $x_2(t)$, hence of $\|x(t)\|$, implies that $\rho_* = 1$. The SLCS is defective as $\|x(t)\|$ is unbounded despite the user's best effort.

For general SLCSs, nondefectiveness is difficult to verify. For a sufficient condition, we call a subspace V of $\mathbb{R}^n$ *control* σ_*-*invariant* if for each $z \in V$ and each $i \in \mathcal{M}$, there exists $v_i \in \mathbb{R}^p$ such that $A_i z + B_i v_i \in V$. Two trivial control σ_*-invariant subspaces are $\{0\}$ and $\mathbb{R}^n$.

Definition 3.2. The SLCS is called *irreducible* if it has no nontrivial control σ_*-invariant subspaces. Otherwise, it is called *reducible*.

When $B_i = 0$ for all i, the notion of an irreducible SLCS coincides with that of an irreducible SLS (Jungers, 2009). An example of irreducible SLCS with nonzero B_i is given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

The only one-dimensional subspace V control invariant under mode 2 is the eigenspace of A_2, i.e., the one spanned by $(1, 0)$. However, such a V is not control invariant under mode 1.

In Appendix B, we prove the following sufficient condition for nondefectiveness.

Proposition 3.1. *If the SLCS with $\rho_* > 0$ is irreducible, then it is nondefective.*

As a result of Proposition 3.1, the SLCS in (4) is nondefective. This will be verified in Example 4.2 later on.

Irreducibility is a sufficient but not necessary condition for nondefectiveness: one can easily find examples of SLCSs that are reducible but nondefective. For example, the SLCS given in (2) is reducible since any 1D subspace of $\mathbb{R}^2$ not aligned with either B_1 or B_2 is a control σ_*-invariant subspace; and it is nondefective as will be shown in Section 4. A necessary and sufficient condition for nondefectiveness will be presented in Theorem 4.1.

Remark 3.1. In the study of mode-resilient stabilization problem, we can similarly define the notions of nondefective and irreducible SLCSs, with the control σ_*-invariant subspace replaced by the control σ^*-invariant subspace, namely, a subspace V such that for any $x \in V$, there exists $v \in \mathbb{R}^p$ such that $A_i x + B_i v \in V$ for all $i \in \mathcal{M}$. One can also show that irreducibility implies nondefectiveness (Hu et al., 2017), which is a much stronger result than Proposition 3.1 as the requirement for a subspace to be control σ^*-invariant is far more stringent than it being control σ_*-invariant.

4. Bounds of σ_*-stabilizing rate

In this section, to answer Question 1.3 in Section 1.3, we develop a systematic approach to establish bounds of the σ_*-stabilizing rate ρ_*. Recall that a *seminorm* on $\mathbb{R}^n$ is a nonnegative function $\xi : \mathbb{R}^n \to \mathbb{R}_+$ that is subadditive and absolutely homogeneous of degree one (Rudin, 1979, Definition 1.33), i.e., $\xi(x+y) \leq \xi(x) + \xi(y)$ for all $x, y \in \mathbb{R}^n$, and $\xi(\lambda \cdot x) = |\lambda| \cdot \xi(x)$ for all $\lambda \in \mathbb{R}, x \in \mathbb{R}^n$. A seminorm is convex, and thus continuous, on $\mathbb{R}^n$. If a seminorm is further positive definite, i.e., $\xi(x) = 0$ only if $x = 0$, then it becomes a norm on $\mathbb{R}^n$.

Define an operator[1] $\mathcal{F}$ so that, for any seminorm ξ on $\mathbb{R}^n$, $\mathcal{F}(\xi)$ is the function

$$\mathcal{F}(\xi) : z \in \mathbb{R}^n \mapsto \max_{i \in \mathcal{M}} \inf_{v \in \mathbb{R}^p} \xi(A_i z + B_i v) \in \mathbb{R}_+. \tag{5}$$

In the above definition, using a similar argument as in Hu et al. (2017, Lemma IV.2), one can show that $\inf_v \xi(A_i z + B_i v)$ is attained by (possibly many) minimizers v for any z and $i \in \mathcal{M}$.

It is easily verified that $\mathcal{F}$ maps the seminorm ξ to another seminorm $\mathcal{F}(\xi)$, which we denote by $\xi_\sharp$. In particular, if $\xi = \|\cdot\|$ is a norm, then $\xi_\sharp$, which we denoted by $\|\cdot\|_\sharp$, is a seminorm but not necessarily a norm on $\mathbb{R}^n$.

Proposition 4.1 (Hu et al., 2018). *Let $\alpha, \beta \geq 0$ be constants.*

(i) If a nonzero seminorm ξ satisfies $\xi_\sharp \geq \alpha \xi$, then $\rho_* \geq \alpha$.
(ii) If a norm $\|\cdot\|$ satisfies $\alpha \|\cdot\| \leq \|\cdot\|_\sharp \leq \beta \|\cdot\|$, then $\alpha \leq \rho_* \leq \beta$.

The proof of Proposition 4.1 can be found in Hu et al. (2018); hence it is omitted.

As an example, for the SLCS in (2), the argument right after (2) shows that $\|\cdot\|_{\infty,\sharp} \geq \frac{3}{2} \|\cdot\|_\infty$. Thus, in view of Proposition 4.1, we have $\rho_* \geq \frac{3}{2}$.

Definition 4.1. A seminorm ξ on $\mathbb{R}^n$ is called a *lower extremal seminorm* of the SLCS if $\xi_\sharp \geq \rho_* \xi$. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called an *extremal norm* if $\|\cdot\|_\sharp \leq \rho_* \|\cdot\|$, and a *Barabanov norm* (Barabanov, 1988; Jungers, 2009) if $\|\cdot\|_\sharp = \rho_* \|\cdot\|$.

[1] The operator $\mathcal{F}$ is similar to the operator $\mathcal{T}$ for mode-resilient stabilization in Hu et al. (2017), with the crucial difference that the order of max and inf is exchanged due to the difference in information structures.

Proposition 4.1 implies that the task of finding tight lower and upper bounds of ρ_* can be reduced to finding lower extremal seminorms and extremal norms of the SLCS, if they exist. A Barabanov norm is both a lower seminorm and an extreme norm and can characterize ρ_* precisely by itself. These notions are generalizations of their counterparts for autonomous SLSs (Barabanov, 1988; Wirth, 2002).

Using Definition 4.1 and the definition of $\|\cdot\|_\sharp$, we obtain a useful geometric characterization of the extremal and Barabanov norms as follows. Toward this end, we first introduce some notations. For a set $\mathcal{X} \subset \mathbb{R}^n$ and $i \in \mathcal{M}$, denote by $A_i\mathcal{X}$ the image of $\mathcal{X}$ under the linear transform A_i, by $\partial\mathcal{X}$ and $\mathrm{int}(\mathcal{X})$ the boundary and interior of $\mathcal{X}$ respectively (with respect to the topology of $\mathbb{R}^n$), and by $P_{B_i^\perp}$ the orthogonal projection operator onto the subspace $\mathcal{R}(B_i)^\perp = \ker(B_i^T)$. For a closed convex set $\mathcal{U}$ in $\mathbb{R}^n$, let $\mathrm{rbd}(\mathcal{U})$ and $\mathrm{ri}(\mathcal{U})$ denote the relative boundary and relative interior of $\mathcal{U}$ respectively (Rockafellar, 1970), where $\mathrm{rbd}(\mathcal{U}) = \mathcal{U} \setminus \mathrm{ri}(\mathcal{U})$. The silhouette of a closed convex set $\mathcal{C}$ viewed along the direction of a subspace $\mathcal{V}$ is defined as $\mathrm{rbd}(P_{\mathcal{V}^\perp}\mathcal{C})$.

Proposition 4.2 (*Geometry of Extremal Norms*)**.** *Let $\mathcal{B} := \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ and $\partial\mathcal{B} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ be the unit ball and unit sphere of the norm $\|\cdot\|$, respectively.*

(1) $\|\cdot\|$ is an extremal norm if and only if $P_{B_i^\perp}(A_i\mathcal{B}) \subseteq P_{B_i^\perp}(\rho_*\mathcal{B})$ for each $i \in \mathcal{M}$;

(2) $\|\cdot\|$ is a Barabanov norm if and only if $P_{B_i^\perp}(A_i\mathcal{B}) \subseteq P_{B_i^\perp}(\rho_*\mathcal{B})$, $\forall\, i \in \mathcal{M}$ and $\partial\mathcal{B} = \cup_{i\in\mathcal{M}}\mathcal{X}_i$, where $\mathcal{X}_i := \{x \in \partial\mathcal{B} \mid P_{B_i^\perp}(A_ix) \in \mathrm{rbd}(P_{B_i^\perp}(\rho_*\mathcal{B}))\}$ consists of all $x \in \partial\mathcal{B}$ so that A_ix lies on the silhouette of $\rho_*\mathcal{B}$ when viewed along the direction of $\mathcal{R}(B_i)$.

Proof. (1) By homogeneity, $\|\cdot\|$ is an extremal norm if and only if $\|z\|_\sharp \le \rho_*$ for all $z \in \mathcal{B}$. "Only if": suppose $\|\cdot\|$ is an extremal norm. Then for any $z \in \mathcal{B}$ and each $i \in \mathcal{M}$, $\inf_{v\in\mathbb{R}^p}\|A_iz + B_iv\| \le \rho_*$. It follows from the comment after (5) that there exists $v_* \in \mathbb{R}^p$ such that $\|A_iz + B_iv_*\| = \inf_{v\in\mathbb{R}^p}\|A_iz + B_iv\| \le \rho_*$. Hence, $A_iz + B_iv_* \in \rho_*\mathcal{B}$. This implies that $P_{B_i^\perp}(A_iz) = P_{B_i^\perp}(A_iz + B_iv_*) \in P_{B_i^\perp}(\rho_*\mathcal{B})$. Therefore, $P_{B_i^\perp}(A_i\mathcal{B}) \subseteq P_{B_i^\perp}(\rho_*\mathcal{B})$, $\forall\, i \in \mathcal{M}$.
"If": suppose for any $z \in \mathcal{B}$ and any $i \in \mathcal{M}$, $P_{B_i^\perp}(A_iz) \in P_{B_i^\perp}(\rho_*\mathcal{B})$. Then there exists $\widehat{z} \in \rho_*\mathcal{B}$ such that $P_{B_i^\perp}(A_iz) = P_{B_i^\perp}(\widehat{z})$. Thus $w := \widehat{z} - A_iz \in \mathcal{R}(B_i)$, i.e., $w = B_iv_*$ for some $v_* \in \mathbb{R}^p$. Therefore, $\|z\|_\sharp = \inf_v \|A_iz + B_iv\| \le \|A_iz + B_iv_*\| = \|\widehat{z}\| \le \rho_*$, i.e., $\|\cdot\|$ is an extremal norm.

(2) "Only if": suppose $\|\cdot\|$ is a Barabanov norm. By Part (1), $P_{B_i^\perp}(A_i\mathcal{B}) \subseteq P_{B_i^\perp}(\rho_*\mathcal{B})$ for each $i \in \mathcal{M}$. It suffices to show that $\partial\mathcal{B} = \cup_{i\in\mathcal{M}}\mathcal{X}_i$, or equivalently, $\partial\mathcal{B} \subseteq \cup_{i\in\mathcal{M}}\mathcal{X}_i$. Since $\|\cdot\|_\sharp = \rho_*\|\cdot\|$, for any $z \in \partial\mathcal{B}$, there exists $i \in \mathcal{M}$ such that $\inf_v\|A_iz + B_iv\| = \rho_*$. This implies that $\|A_iz + B_iv\| \ge \rho_*$ for all v. Further, Part (1) implies that $P_{B_i^\perp}(A_iz) \in P_{B_i^\perp}(\rho_*\mathcal{B})$, where it can be shown that $P_{B_i^\perp}(\rho_*\mathcal{B})$ is a convex and compact set. We claim that $P_{B_i^\perp}(A_iz) \in \mathrm{rbd}(P_{B_i^\perp}(\rho_*\mathcal{B}))$. Suppose otherwise. Then noting that $\mathrm{ri}(P_{B_i^\perp}(\rho_*\mathcal{B})) = P_{B_i^\perp}(\mathrm{ri}(\rho_*\mathcal{B})) = P_{B_i^\perp}(\mathrm{int}(\rho_*\mathcal{B}))$ (Rockafellar, 1970, Theorem 6.6), there exists v' such that $A_iz + B_iv' \in \mathrm{int}(\rho_*\mathcal{B})$ or equivalently $\|A_iz + B_iv'\| < \rho_*$. This yields a contradiction. Therefore, $P_{B_i^\perp}(A_iz) \in \mathrm{rbd}(P_{B_i^\perp}(\rho_*\mathcal{B}))$, and thus $z \in \mathcal{X}_i$.
"If": By Part (1), the first condition implies $\|\cdot\|_\sharp \le \rho_*\|\cdot\|$. It suffices to show that for any $z \in \partial\mathcal{B}$, $\|z\|_\sharp = \max_{i\in\mathcal{M}}\inf_v\|A_iz+B_iv\| \ge \rho_*$. Since $\partial\mathcal{B} = \cup_{i\in\mathcal{M}}\mathcal{X}_i$, there exists $i \in \mathcal{M}$ such that $P_{B_i^\perp}(A_iz) \in \mathrm{rbd}(P_{B_i^\perp}(\rho_*\mathcal{B}))$. We claim that $\|A_iz + B_iv\| \ge \rho_*$ for all v. Suppose otherwise, then there exists v' such that $A_iz + B_iv' \in \mathrm{int}(\rho_*\mathcal{B})$. Hence, $P_{B_i^\perp}(A_iz) = P_{B_i^\perp}(A_iz + B_iv') \in P_{B_i^\perp}(\mathrm{int}(\rho_*\mathcal{B})) = \mathrm{ri}(P_{B_i^\perp}(\rho_*\mathcal{B}))$, contradiction. Thus $\|z\|_\sharp \ge \rho_*, \forall\, z \in \partial\mathcal{B}$. □

The following result describes exactly the class of SLCSs for which an extremal norm exists. Its proof is given in Appendix C.

Theorem 4.1. *The SLCS has an extremal norm if and only if it is nondefective.*

Stronger conditions than nondefectiveness are in general needed to ensure the existence of a Barabanov norm. For illustration, consider the following family of SLCSs:

$$A_1 = \begin{bmatrix} a_1 & \star \\ 0 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \ A_2 = \begin{bmatrix} a_2 & \star \\ \star & \star \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{6}$$

where $|a_1| < 1$, $|a_2| < 1$, and $\star$ denotes any value in $\mathbb{R}$. For such a SLCS, $\rho_* = 1$ as the adversary will always choose mode 1 for $x(0)$ with $x_2(0) \neq 0$. Moreover, ρ_* can be achieved by the user control policy $u(t) = -K_{\sigma(t)}x(t)$ with K_1 the first row of A_1 and K_2 the second row of A_2; hence the SLCS is nondefective. However, if a Barabanov norm $\|\cdot\|$ exists, then $\rho_*\|(1,0)\| = \|(1,0)\|_\sharp$ is given by

$$\max\Big(\inf_v \|(a_1 + v, 0)\|, \inf_v \|(a_2, v + \star)\|\Big) = \inf_{u\in\mathbb{R}} \|(a_2, u)\|,$$

where $\star$ in the above equation corresponds to the (2, 1)-element of A_2. However, this is impossible since $\inf_u \|(a_2, u)\| \le \|(a_2, 0)\| = |a_2| \cdot \|(1,0)\| < \|(1,0)\|$. Thus, such a SLCS does not attain Barabanov norms.

The following result, proved in Appendix D, gives a sufficient condition for the existence of Barabanov norms.

Proposition 4.3. *If the SLCS is irreducible, then it has a Barabanov norm.*

By Theorem 4.1 and Proposition 4.3, Question 1.1 raised in Section 1 can be answered affirmatively via the construction of extremal and Barabanov norms, with the latter providing a sufficient test by Proposition 3.1.

We next present several examples of SLCSs whose Barabanov norms can be explicitly constructed.

Example 4.1. Consider the following SLCS:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ \ A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let $\gamma = (\sqrt{5} + 1)/2 \approx 1.6180$, which satisfies $\gamma(\gamma - 1) = 1$. Define the norm on $\mathbb{R}^2$ as

$$\|z\| := \max\{|z_1|,\ \gamma|z_1 + z_2|\},\ \ \forall\, z = (z_1, z_2) \in \mathbb{R}^2. \tag{7}$$

We now use the conditions in the second part of Proposition 4.2 to verify geometrically that the above defined norm $\|\cdot\|$ is indeed a Barabanov norm. The unit sphere of $\|\cdot\|$ is shown in the top figure of Fig. 1. The images of the unit ball $\mathcal{B}$ after a scaling by γ^{-1} and after the linear transforms by A_1 and A_2 are shown in the bottom figure. Note that $A_1\mathcal{B}$ and $\gamma^{-1}\mathcal{B}$ have the same projection onto the x_1-axis; while $A_2\mathcal{B}$ and $\gamma^{-1}\mathcal{B}$ have the same projection onto the x_2-axis. This verifies that $P_{B_1^\perp}(A_1\mathcal{B}) \subseteq P_{B_1^\perp}(\rho_*\mathcal{B})$ and $P_{B_2^\perp}(A_2\mathcal{B}) \subseteq P_{B_2^\perp}(\rho_*\mathcal{B})$, respectively. Furthermore, among the four edges of $\mathcal{B}$, the two slanted ones after the linear transform of A_1 becomes the two vertical edges of $A_1\mathcal{B}$ which, when viewed in the B_1 direction (i.e. top down), are in the silhouette of $\gamma^{-1}\mathcal{B}$. Similarly, the images of the two vertical edges of the unit sphere under A_2 are in the silhouette of $\gamma^{-1}\mathcal{B}$ in the left–right view. This verifies the remaining conditions in Proposition 4.2 and shows that the norm $\|\cdot\|$ in (7) is indeed a Barabanov norm and $\rho_* = \gamma^{-1} \approx 0.6180$. In contrast, it is found via numerical computation in Hu et al. (2017) that the σ^*-stabilizing rate ρ^* of the SLCS satisfies $\rho^* \in [1.2183, 1.2239]$. The gap between ρ^* and ρ_* is due to information premium of the user's knowledge of the current mode. In this case, stabilization is possible with this knowledge but impossible without it.

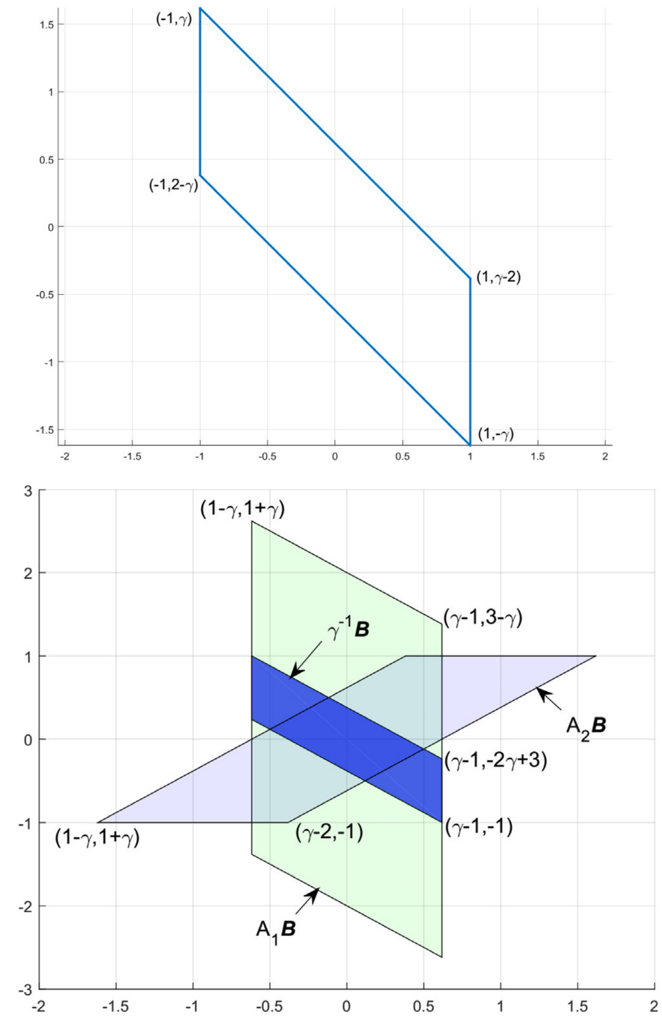

Fig. 1. Top: unit sphere of the Barabanov norm of the SLCS in Example 4.2; Bottom: unit ball after scaling by γ^{-1} and the linear transforms A_1 and A_2.

To find the optimal control policy, we first consider $x(t) = (z_1, 1)$. If $\sigma(t) = 1$, then $u^*(t) \in \arg\min_v \max\{|z_1+1|, \gamma|2z_1+v|\}$, which is the interval between the two values $-2z_1 \pm (z_1+1)/\gamma$. If $\sigma(t) = 2$, then $u^*(t) = \arg\min_v \max\{|v+1|, \gamma|v-z_1+1|\} = (\gamma-1)z_1 - 1$. We next consider $z = (1, 0)$. In this case, $u^*(t)$ can be of arbitrary values between $-2 \pm \gamma^{-1}$ if $\sigma(t) = 1$; and $u^*(t) = \gamma - 1$ if $\sigma^*(t) = 2$. By homogeneity, the above control policy can be extended to arbitrary $z \in \mathbb{R}^2$. In particular, the following mode-dependent linear state feedback controller achieves the σ_*-stabilizing rate ρ_*: $u(t) = K_{\sigma(t)}x(t)$, where

$$K_1 = \begin{bmatrix} -2 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} (\gamma-1) & 1 \end{bmatrix}. \tag{8}$$

Example 4.2. We now revisit the SLCS given in (2). Define the norm $\|z\| := \max\{|4z_1+z_2|, |z_1+4z_2|\}$ for $z = (z_1, z_2) \in \mathbb{R}^2$. Similar to the previous example, one can verify that $\|\cdot\|_\sharp = \frac{3}{2}\|\cdot\|$. Thus, $\|\cdot\|$ is a Barabanov norm of the SLCS and $\rho_* = \frac{3}{2}$. In Fig. 2, we plot the unit ball of the norm $\|\cdot\|$ after a scaling by ρ_* and after the linear transforms by A_1 and A_2. As in the previous example, ρ_* can be achieved by a mode-dependent linear state feedback controller (details are omitted). The argument in the paragraph right after (2) in Section 1 shows that $\|\cdot\|_{\infty,\sharp} \geq \frac{3}{2}\|\cdot\|_\infty$, i.e., $\|\cdot\|_\infty$ is a lower extremal (semi)norm.

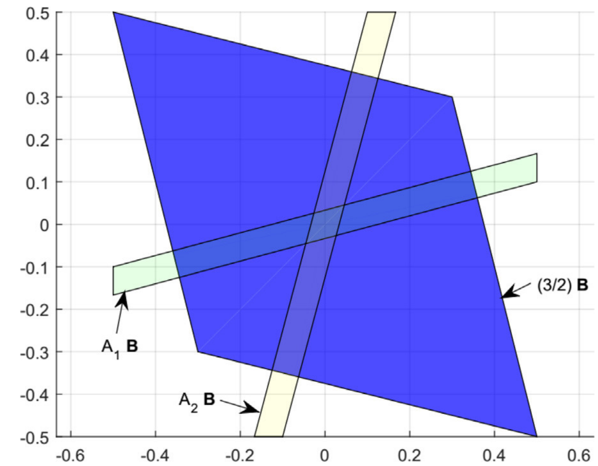

Fig. 2. Unit ball $\mathcal{B}$ of the Barabanov norm of the SLCS given by (2) after scaling by $\frac{3}{2}$ and the linear transforms by A_1, A_2.

In general, the unit ball of a Barabanov norm (if it exists) may have a high complexity, even for a bimodal (i.e., two-mode) SLCS on $\mathbb{R}^2$, as shown by the following example.

Example 4.3. Consider the bimodal SLCS in (4), which is recapped here for convenience:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

As shown in Section 3, this SLCS is irreducible; hence it has a Barabanov norm by Proposition 4.3. We construct the unit ball $\mathcal{B}$ of this norm as follows. Let $\rho \in (1, 2)$ be a constant whose value is to be determined later. Set $t := (\rho-1)^{-1} - 1 > 0$. Define a sequence $z_{t-k} := ((t-k)/\rho^{t-k}, 1/\rho^{t-k}) \in \mathbb{R}^2$ for each $k \in \mathbb{Z}_+$, which satisfies $A_2 z_{t-k-1} = \rho z_{t-k}$. Denote by $\overline{z_{t-k-1}z_{t-k}}$ the line segment between z_{t-k-1} and z_{t-k}. Then $A_2\overline{z_{t-k-1}z_{t-k}} = \rho \cdot \overline{z_{t-k}z_{t-k+1}}$ for $k \geq 1$ and $A_2\overline{z_{t-1}z_t}$ is along the direction $(1, -1)$. In the top figure of Fig. 3, we plot the line segments $\overline{z_{t-k-1}z_{t-k}}$ together with their symmetric images across the origin.

From the point $-z_t$, we plot a line L_2 along the direction of $(-1, 1)$, which intersects a certain line segment $\overline{z_{t-\ell-1}z_{t-\ell}}$ for some $\ell \in \mathbb{Z}_+$ at the point y. Then half of the boundary $\partial\mathcal{B}$ of the unit ball $\mathcal{B}$ is given by $\overline{(-z_t)y}$, the line segment between $-z_t$ and y, followed by the line segments $\overline{yz_{t-\ell}}, \overline{z_{t-\ell}z_{t-\ell+1}}, \ldots, \overline{z_{t-1}z_t}$, whose symmetric images across the origin form the other half of $\partial\mathcal{B}$ (see the top of Fig. 3). It is easy to see that $A_2\,\partial\mathcal{B} \subset \rho\,\mathcal{B}$. In fact, each of the line segments $\overline{yz_{t-\ell}}, \overline{z_{t-\ell}z_{t-\ell+1}}, \ldots, \overline{z_{t-1}z_t}$ after the transform of A_2 becomes part (or all) of the next line segment (clockwise direction) on $\partial\mathcal{B}$ scaled by ρ; while $A_2\overline{(-z_t)y} \subset \rho\,\mathcal{B}$ as $A_2(-z_t) \in \rho\,\overline{(-z_t)y}$ and $A_2 y \in \rho\,\overline{z_{t-\ell}z_{t-\ell+1}}$. Furthermore, note that the images of $\overline{(-z_t)y}$ and $\overline{z_t(-y)}$ under A_1 are two horizontal line segments. We can choose ρ so that one of them has the same height (i.e., x_2-coordinate) as that of ρy, which would imply that $A_1\mathcal{B}$ and $\rho\mathcal{B}$ has the same silhouette along the B_1 direction. Numerical solution yields the value of such a ρ as 1.2493. Thus, the norm whose unit ball is the corresponding $\mathcal{B}$ is a Barabanov norm of the SLCS and $\rho_* \approx 1.2493$. Fig. 3 plots the unit ball $\mathcal{B}$ and its images under the scaling by ρ_* and the linear transforms A_1 and A_2. For this bimodal SLCS, its Barabanov norm has a polytopic unit ball with 14 faces. Following similar steps, we can show that if the $(2, 1)$-element of A_1 is changed from 1 to an arbitrarily small $\varepsilon > 0$, then the resulting SLCS attains a Barabanov norm whose polytopic unit ball has arbitrarily many faces.

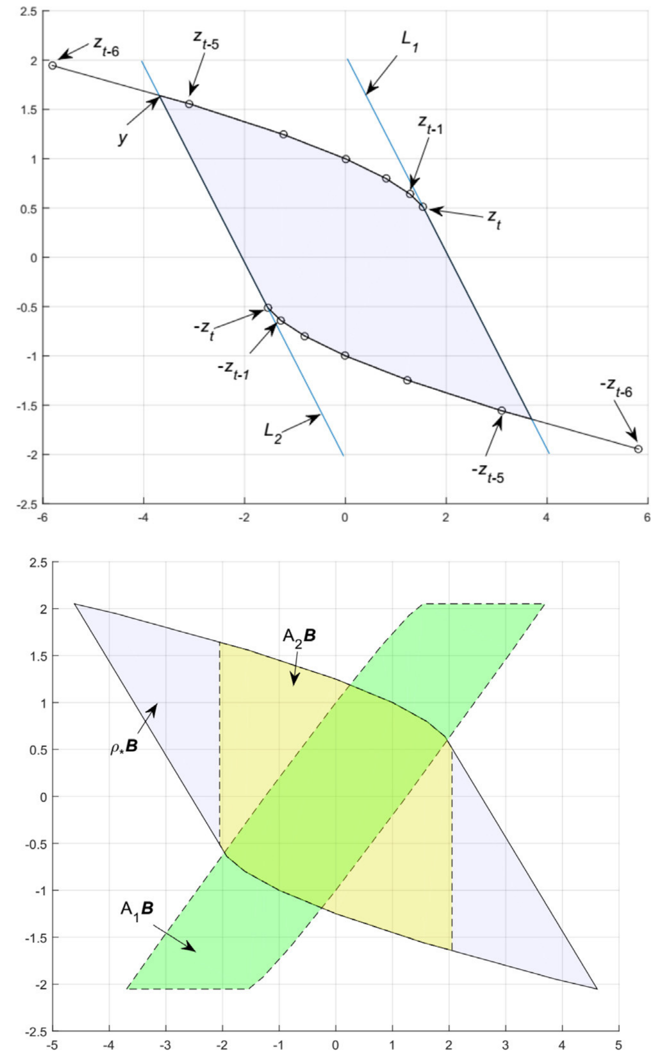

Fig. 3. Top: unit ball $\mathcal{B}$ of the Barabanov norm of the SLCS (4); Bottom: $\mathcal{B}$ after scaling by ρ_* and after the linear transforms by A_1, A_2.

Example 4.4. A Barabanov norm may not always attain a polytopic unit ball. Consider the following bimodal SLCS:

$$A_1 = \begin{bmatrix} \cos(\alpha\pi) & -\sin(\alpha\pi) & 0 \\ \sin(\alpha\pi) & \cos(\alpha\pi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = 0,$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

where $\alpha > 0$ is an irrational number. Due to subsystem (A_1, B_1), the unit ball of any Barabanov norm (if exists) must be invariant to all rotations around the x_3-axis; hence such a unit ball cannot be a polytope. It is easy to check that the Euclidean norm is a Barabanov norm. Another Barabanov norm is given by the one whose unit ball is the cylindrical set $\{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1^2 + z_2^2 \leq 1 \text{ and } |z_3| \leq 1\}$. Thus, Barabanov norms when exist may not be unique.

5. Mode-dependent linear state feedback controllers

Denote by $\mathcal{L}$ the set of all mode-dependent linear state feedback control policies for the user, i.e., the policies of the form $\mathbf{u}_t(\sigma(t), x(t)) = K_{\sigma(t)}x(t)$, $\forall t$, for a finite set of feedback gain matrices $\{K_i\}_{i\in\mathcal{M}}$. Under such a policy, the closed-loop SLCS becomes an autonomous SLS with the subsystem matrices $A_i + B_iK_i$, $i \in \mathcal{M}$, and the σ_*-stabilizing rate is reduced to the JSR of the matrix set $\{A_i+B_iK_i\}_{i\in\mathcal{M}}$ (see Remark 1.2). Denote by $\bar{\rho}_*$ the smallest possible JSR achieved by different choices of $\{K_i\}_{i\in\mathcal{M}}$. Since $\mathcal{L} \subset \mathcal{U}$, we have $\bar{\rho}_* \geq \rho_*$.

5.1. Example of suboptimality

We now present an example where the optimal σ_*-stabilizing rate ρ_* can be attained by some control policy in $\mathcal{U}$ but not by any control policy in $\mathcal{L}$.

A regular icosahedron, one of the five Platonic solids, is a convex polyhedron with 12 vertices denoted by v_i, 20 faces, and 30 edges. In Fig. 4, a regular icosahedron $\mathcal{B}_{\text{icosa}}$ with all its edges of length 2 is plotted. The cartesian coordinates of its 12 vertices are all the cyclic permutations of $(0, \pm 1, \pm\gamma)$, where $\gamma = (\sqrt{5}+1)/2$ is the golden ratio (Coxeter, 1973). Being a symmetric convex body with nonempty interior, $\mathcal{B}_{\text{icosa}}$ is the unit ball of a norm on $\mathbb{R}^3$, which we denote by $\|\cdot\|_{\text{icosa}}$. We will next construct an SLCS whose Barabanov norm is exactly $\|\cdot\|_{\text{icosa}}$. Important for our construction are the following facts. First, when viewed along a direction that passes through the origin and the center of a face (e.g., as in Fig. 4(a)), the silhouette of $\mathcal{B}_{\text{icosa}}$ is the relative boundary of a regular hexagon, each edge of which is generic, i.e., being the projection image of a single edge of $\mathcal{B}_{\text{icosa}}$. On the other hand, when viewed along a direction that passes through the origin and the center of an edge, e.g., from top down as in Fig. 4(b), the silhouette of $\mathcal{B}_{\text{icosa}}$ is the relative boundary of an irregular hexagon with four "singular" edges being the projection images of four faces of $\mathcal{B}_{\text{icosa}}$. Specifically, the four non-horizontal edges of the irregular hexagon in Fig. 4(b) are the top down projection images of the triangular faces $\overline{v_1v_6v_7}$, $\overline{v_5v_6v_7}$, $\overline{v_2v_8v_9}$, and $\overline{v_{10}v_8v_9}$. Furthermore, as shown in Fig. 4(c), a linear transformation exists that transforms the irregular hexagon silhouette in Fig. 4(b) to fit tightly inside the regular hexagonal silhouette in Fig. 4(a), with the four singular edges of the former on the boundary of the latter.

We now construct the first subsystem (A_1, B_1). Let $w \in \mathbb{R}^3$ be the unit (outward) normal of the face $\overline{v_2v_8v_9}$, i.e., $w = (w_1, w_2, w_3) = \frac{1}{\sqrt{9\gamma+6}}(2\gamma+1, -\gamma, 0)$. Define

$$A_1 = \begin{bmatrix} -w_2 & 0 & w_1 \\ w_1 & 0 & w_2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \operatorname{diag}\left(\frac{2}{\sqrt{3}}, \frac{2}{\gamma^2}, 1\right), \quad B_1 = w. \tag{9}$$

The linear transform represented by A_1 is the composition of two transforms. The first scales the x_1 and x_2 coordinates so that the scaled icosahedron has a top-down view as in Fig. 4(c). This is followed by a rotation that rotates the x_3-axis to w and the x_2-axis to the x_3-axis so that the top-down silhouette of the scaled icosahedron in Fig. 4(c) becomes the silhouette of $A_1\mathcal{B}_{\text{icosa}}$ viewed from the w direction. This is shown in Fig. 5, where the original icosahedron $\mathcal{B}_{\text{icosa}}$ (dashed lines) and the transformed icosahedron $A_1\mathcal{B}_{\text{icosa}}$ (solid lines) are shown together and viewed from the w direction in (a) and a generic direction in (b). As verified by Fig. 5(a), viewed from the w direction, the four singular edges of the silhouette of $A_1\mathcal{B}_{\text{icosa}}$ are on the silhouette of $\mathcal{B}_{\text{icosa}}$. This can also be seen in Fig. 5(b): the line segment $\overline{v_1v_3}$ (resp. $\overline{v_1v_4}$) is on the same plane as the face $\overline{v_5'v_6'v_7'}$ (resp. $\overline{v_1'v_6'v_7'}$), where $v_i' := A_1v_i$ for all vertices v_i of $\mathcal{B}_{\text{icosa}}$.

Suppose at an arbitrary t the state of the SLCS is at $x(t) \neq 0$ which, due to homogeneity, can be assumed to lie on the boundary of $\mathcal{B}_{\text{icosa}}$, and the adversary chooses $\sigma(t) = 1$, i.e., the above constructed subsystem (A_1, B_1), to evolve the system. We now find the optimal user control $u(t)$ so that $x(t+1) = A_1x(t) + B_1u(t)$ has the smallest $\|\cdot\|_{\text{icosa}}$ norm.

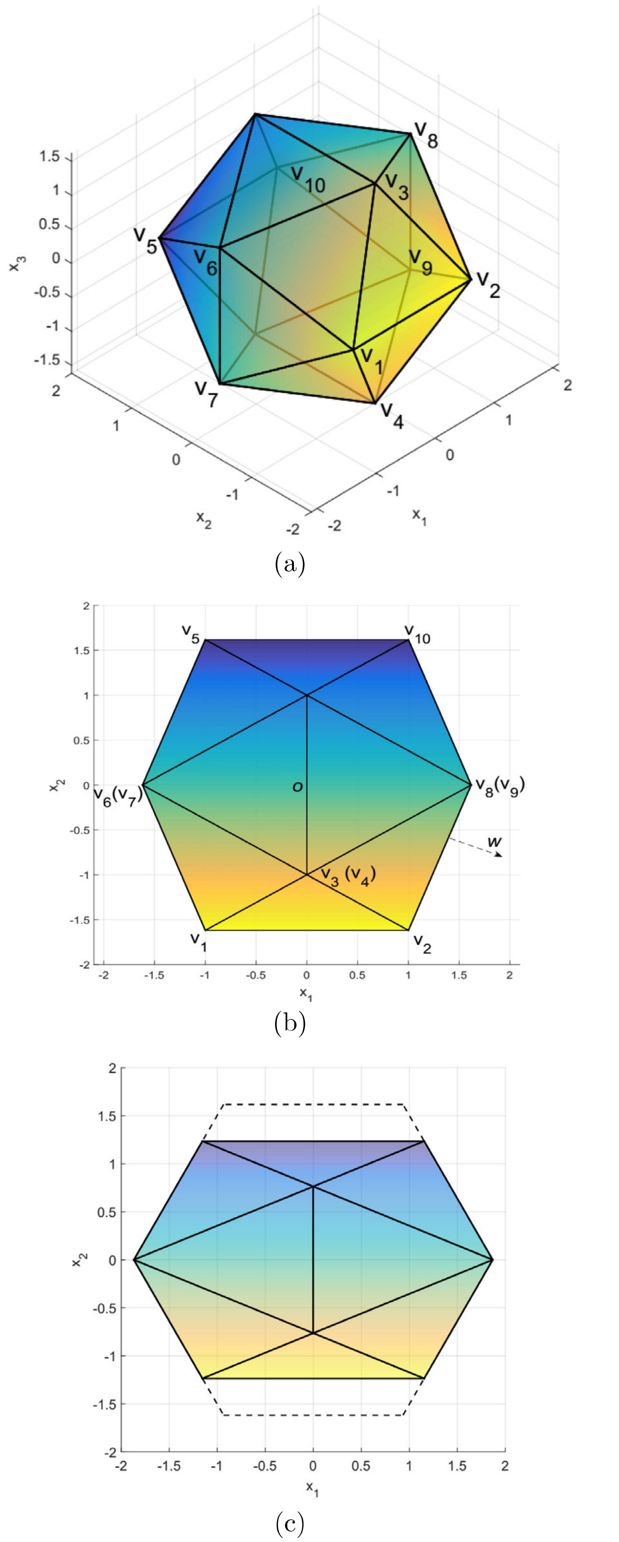

Fig. 4. (a) Icosahedron $\mathcal{B}_{\text{icosa}}$ viewed from an angle through the origin and the center of an face; (b) $\mathcal{B}_{\text{icosa}}$ viewed from the top down; (c) Top down view of $\mathcal{B}_{\text{icosa}}$ after the linear transform by $\text{diag}(\frac{2}{\sqrt{3}}, \frac{2}{\gamma^2}, 1)$, with the dashed line representing the regular hexagonal silhouette in (a) after a proper rotation.

- Case 1: If $x(t)$ is on the face $\overline{v_5v_6v_7}$, then $A_1x(t)$ is on the face $\overline{v_5'v_6'v_7'}$ of $A_1\mathcal{B}_{\text{icosa}}$. In this case, the smallest possible $\|x(t+1)\|_{\text{icosa}}$ is 1, which is achieved when $x(t+1)$ is placed on the line segment $\overline{v_1v_3}$ by the following unique choice of $u(t)$:

$$u^*(t) = \begin{bmatrix} \frac{3\sqrt{3}-\sqrt{15}}{6} & \frac{7\sqrt{3}-3\sqrt{15}}{2} & -1 \end{bmatrix} x(t). \tag{10}$$

  By symmetry, the same conclusion holds if $x(t)$ is on the opposite face $\overline{v_2v_8v_9}$.
- Case 2: If $x(t)$ is on the face $\overline{v_1v_6v_7}$, then $A_1x(t)$ is on the face $\overline{v_1'v_6'v_7'}$ of $A_1\mathcal{B}_{\text{icosa}}$. The smallest possible $\|x(t+1)\|_{\text{icosa}}$ (which is also 1) is achieved when $x(t+1)$ is placed on the line segment $\overline{v_1v_4}$ by the following unique choice of $u(t)$:

$$u^*(t) = \begin{bmatrix} \frac{3\sqrt{3}-\sqrt{15}}{6} & -\frac{7\sqrt{3}-3\sqrt{15}}{2} & -1 \end{bmatrix} x(t). \tag{11}$$

  The same conclusion holds if $x(t)$ is on the opposite face $\overline{v_8v_9v_{10}}$. Note that the gain matrices in (10) and (11) are different due to the fact that the two line segments, $\overline{v_1v_3}$ and $\overline{v_1v_4}$, and the origin are not on the same plane.
- Case 3: If $x(t)$ is not on the faces $\overline{v_5v_6v_7}$, $\overline{v_2v_8v_9}$, $\overline{v_1v_6v_7}$, or $\overline{v_8v_9v_{10}}$, then $A_1x(t)$ when viewed along the w direction is not on the silhouette in Fig. 5(a). In this case, by a proper choice of $u(t)$, we can make $\|x(t+1)\|_{\text{icosa}}$ strictly less than 1.

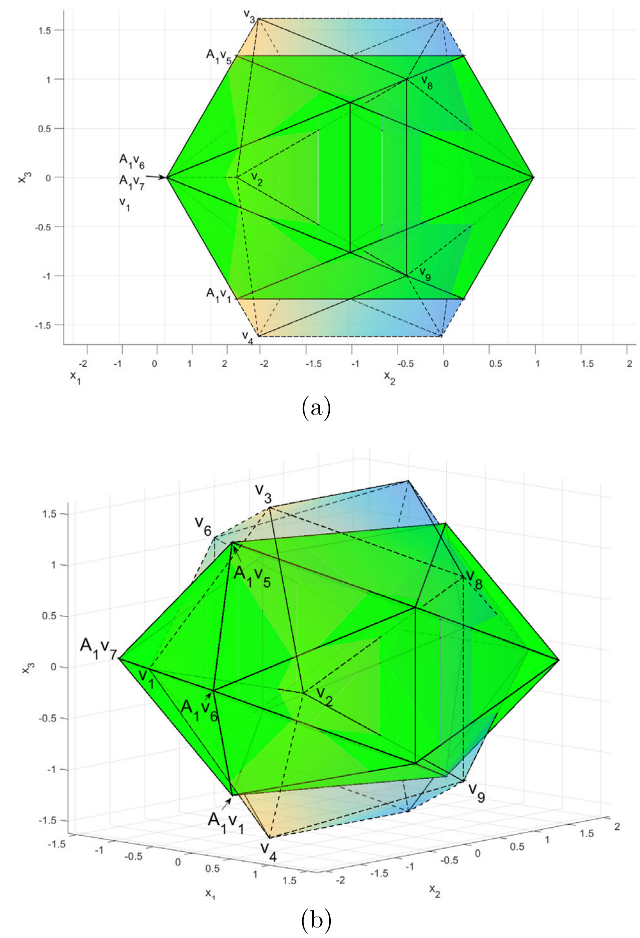

Fig. 5. The original icosahedron $\mathcal{B}_{\text{icosa}}$ (dashed lines) and the transformed $A_1\mathcal{B}_{\text{icosa}}$ (bold lines) viewed from: (a) the direction of w; (b) a generic direction.

Denote by $C_1 = \{\alpha_1v_5 + \alpha_2v_6 + \alpha_3v_7 \mid \alpha_1, \alpha_2, \alpha_3 \geq 0\}$ the convex cone spanned by the face $\overline{v_5v_6v_7}$. Similarly, denote by C_2 the convex cone spanned by the face $\overline{v_1v_6v_7}$. Then, $-C_1$ and $-C_2$ are the convex cones spanned by the faces $\overline{v_2v_8v_9}$ and $\overline{v_8v_9v_{10}}$, respectively. Define $\Omega_1 = C_1 \cup (-C_1) \cup C_2 \cup (-C_2)$. By homogeneity, the conclusions in the above three cases can be extended to arbitrary $x(t) \in \mathbb{R}^3$:

(i) If $x(t) \in C_1 \cup (-C_1)$, then $\min_{u(t)} \|x(t+1)\|_{\text{icosa}} = \|x(t)\|_{\text{icosa}}$. The minimum is achieved by $u^*(t)$ in (10), under which $x(t+1)$ is on the plane spanned by $\overline{v_1 v_3}$ and the origin;
(ii) If $x(t) \in C_2 \cup (-C_2)$, then $\min_{u(t)} \|x(t+1)\|_{\text{icosa}} = \|x(t)\|_{\text{icosa}}$. The minimum is achieved by $u^*(t)$ in (11), under which $x(t+1)$ is on the plane spanned by $\overline{v_1 v_4}$ and the origin;
(iii) If $x(t) \notin \Omega_1$, then $\min_{u(t)} \|x(t+1)\|_{\text{icosa}} < \|x(t)\|_{\text{icosa}}$.

The optimal controllers in Cases 1–2 have the form:

$$u^*(t) = \frac{3\sqrt{3}-\sqrt{15}}{6}x_1(t) + \frac{7\sqrt{3}-3\sqrt{15}}{2}|x_2(t)| - x_3(t). \tag{12}$$

Let $\Omega_2, \ldots, \Omega_5$ be such that each Ω_i is a cone spanned by two adjacent faces of $\mathcal{B}_{\text{icosa}}$ together with their opposite faces and $\cup_{i=1}^5 \Omega_i = \mathbb{R}^3$. For each Ω_i, $i = 2, \ldots, 5$, similar to (A_1, B_1), we can construct a subsystem (A_i, B_i) under which $\min_{u(t)} \|x(t+1)\|_{\text{icosa}} < \|x(t)\|_{\text{icosa}}$ if $x(t) \notin \Omega_i$; and $\min_{u(t)} \|x(t+1)\|_{\text{icosa}} = \|x(t)\|_{\text{icosa}}$ if $x(t) \in \Omega_i$ with the minimum achieved by a unique optimal controller $u^*(t)$ of a form similar to (12). Furthermore, we can choose Ω_i and the corresponding rotation matrix in A_i carefully so that the SLCS $\{(A_i, B_i)\}_{i=1,\ldots,5}$ is ergodic under the optimal controller: starting from any $x(0)$, the state trajectory $x(t)$ will visit each half of Ω_i (as spanned by one face and its opposite face) periodically with the period 10. As $\mathcal{B}_{\text{icosa}}$ satisfies the geometric conditions in Proposition 4.2, $\|\cdot\|_{\text{icosa}}$ is a Barabanov norm of the SLCS with $\rho_* = 1$. Indeed, for any $x(t) \neq 0$, we have $x(t) \in \Omega_i$ for some $i \in \{1, \ldots, 5\}$. Then the worst-case mode choice would be $\sigma(t) = i$, against which the user can at best maintain the same $\|\cdot\|_{\text{icosa}}$ norm of $x(t+1)$ as $x(t)$ using a nonlinear feedback controller.

Suppose the user adopts a control policy in $\mathcal{L}$ and the state at time t is at, e.g., $x(t) \in \Omega_1$. Then under $\sigma(t) = 1$ and the linear feedback controller $u(t) = K_1 x(t)$, the user can ensure $\|x(t+1)\|_{\text{icosa}} = \|x(t)\|_{\text{icosa}}$ for $x(t)$ in at most half of Ω_1 by choosing K_1 according to either (10) or (11). The ergodicity property of the SLCS under the optimal control policy then implies that there exists $\tau \leq 10$ such that $\|x(t+\tau)\|_{\text{icosa}} > \|x(t)\|_{\text{icosa}}$. As a result, the exponential growth rate of $\|x(t)\|_{\text{icosa}}$ under any $u \in \mathcal{L}$ is uniformly bounded from below away from 1. This implies that $\bar{\rho}_* > \rho_* = 1$. After a scaling by $\alpha \in (1/\bar{\rho}_*, 1)$, the α-scaled SLCS can be σ_*-stabilized by some $\mathbf{u}^* \in \mathcal{U}$ but not by any $\mathbf{u} \in \mathcal{L}$. Consequently, the answer to Question 1.2 in Section 1 is negative.

5.2. Cases of optimality

It is worth mentioning that for some SLCSs, their optimal control policy can indeed be found in $\mathcal{L}$. An example has been given in Example 4.1 where, among the many possible optimal controllers, one of them as given in (8) is in $\mathcal{L}$. In this section, we will study some families of SLCSs which attain optimal controllers in $\mathcal{L}$.

Proposition 5.1. *For a nondefective SLCS $\{(A_i, B_i)\}$, there exist a user control policy $\mathbf{u} \in \mathcal{L}$ and some $K \in [0, \infty)$ such that $\|x(t; \sigma, \mathbf{u}, x(0))\| \leq K(\rho_*)^t \|x(0)\|$, $\forall t$, $\forall \sigma \in \mathcal{S}$, $\forall x(0)$, if one of the following holds:*

(a) The state space is $\mathbb{R}^2$, i.e., $A_i \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times p}$, $\forall i \in \mathcal{M}$.
(b) The state space is $\mathbb{R}^n$ and, for each $i \in \mathcal{M}$, the dimension of the range space of B_i is either 0, $n-1$, or n.

Proof. We first prove for case (a). If $\rho_* = 0$, then the SLCS is stabilizable to the origin in one step, i.e, $A_i = B_i K_i$ for some K_i, $\forall i$. We can then choose $\mathbf{u} \in \mathcal{L}$ with $u(t) = -K_{\sigma(t)} x(t)$. In what follows, we assume that $\rho_* > 0$. After a proper scaling, we further assume $\rho_* = 1$.

By Theorem 4.1, the nondefectiveness assumption implies the existence of an extreme norm $\|\cdot\|$, which satisfies $\max_i \inf_v \|A_i z + B_i v\| \leq \|z\|$, $\forall z$. Thus, with the control policy $\mathbf{u}$ such that $u(t) \in \arg\min_v \|A_{\sigma(t)} x(t) + B_{\sigma(t)} v\|$, we have $\|x(t+1)\| \leq \|x(t)\|$, $\forall t$, $\forall \sigma(t)$. It remains to prove that there is one such $\mathbf{u}$ in $\mathcal{L}$, i.e., for each $\sigma(t) = i$, there exists $K_i \in \mathbb{R}^{p\times 2}$ such that $K_i z \in \arg\min_v \|A_i z + B_i v\|$, $\forall x(t) = z$. If $\mathcal{R}(B_i) = \{0\}$, i.e., $B_i = 0$, the choice of $K_i = 0$ is trivial. If $\mathcal{R}(B_i) = \mathbb{R}^2$, then there is some K_i such that $A_i + B_i K_i = 0$, e.g., $K_i = -B_i^\dagger A_i$ where $B_i^\dagger$ is the pseudo-inverse of B_i. Suppose B_i is rank one. Without loss of generality, we assume the first column of B_i is a nonzero vector $w_i \in \mathbb{R}^2$; hence $\mathcal{R}(B_i) = \text{span}\{w_i\}$. For the unit ball $\mathcal{B}$ of $\|\cdot\|$, there exists some $y_i \in \partial\mathcal{B}$ that is on the silhouette of $\mathcal{B}$ when viewed along the w_i direction, or more precisely, $P_{B_i^\perp}(y_i) \in \text{rbd}(P_{B_i^\perp}(\mathcal{B}))$. Note that w_i and y_i cannot be of the same direction as $\mathcal{B}$ has nonempty interior. Consider a control input v^* with $v_2^* = \cdots = v_p^* = 0$ and $v_1^* \in \mathbb{R}$ such that $\tilde{y}_i := A_i z + v_1^* w_i = \alpha y_i$ for some $\alpha \in \mathbb{R}$, i.e., $v_1^* = k_i z := -(y_i^\perp)^T A_i / \left((y_i^\perp)^T w_i\right) \cdot z$. A consequence of such a choice is that the line passing through $A_i z$ along the direction of w_i is a supporting plane (line) of the scaled unit ball $\alpha\mathcal{B}$ at the supporting point $\tilde{y}_i \in \partial(\alpha\mathcal{B})$. This implies that $\|A_i z + B_i v\|$ achieves its minimum value α at $v = v^*$. As $v^* = K_i z$ where the first row of K_i is k_i and the rest of the rows are zero, we have proved the statement (a). The proof of statement (b) is an immediate extension of the above proof and thus omitted. □

6. Computation algorithms

In this section, numerical algorithms for computing the σ_*-stabilizing rate ρ_* of SLCS will be developed based on the results in Section 4.

6.1. Ellipsoid norms

A positive semidefinite matrix $P \succeq 0$ defines a seminorm $\|z\|_P := (z^T P z)^{1/2}$. If $P \succ 0$ is positive definite, then $\|z\|_P$ is a norm, called an ellipsoid norm. Simple computation shows

$$\|z\|_{P\sharp} = \max_{i\in\mathcal{M}} z^T \left(A_i^T P A_i - A_i^T P B_i (B_i^T P B_i)^\dagger B_i^T P A_i\right) z.$$

The condition that $\|\cdot\|_{P\sharp} \leq \beta \|\cdot\|_P$ is equivalent to

$$A_i^T P A_i - A_i^T P B_i (B_i^T P B_i)^\dagger B_i^T P A_i \preceq \beta^2 P, \quad \forall i \in \mathcal{M}.$$

The smallest β^* for the above to hold can be obtained by solving the above (nonconvex) problem. For an easier bound, consider control policies in $\mathcal{L}$, i.e., $\mathbf{u}_t(i, x) = K_i x$, and write

$$\begin{aligned}\|z\|_{P\sharp} &= \max_i \inf_v (A_i z + B_i v)^T P (A_i z + B_i v) \\ &\leq \max_i \inf_{K_i} z^T (A_i + B_i K_i)^T P (A_i + B_i K_i) z.\end{aligned}$$

Thus, a sufficient condition for $\|\cdot\|_{P\sharp} \leq \beta\|\cdot\|_P$ is

$$\exists K_i \text{ such that } (A_i + B_i K_i)^T P (A_i + B_i K_i) \preceq \beta^2 P, \ \forall i.$$

By letting $Q = P^{-1}$, $F_i = K_i P^{-1}$, and using Schur complement, we can rewrite the above as:

$$\exists Q \text{ and } F_i \text{ s.t.} \begin{bmatrix} \beta Q & A_i Q + B_i F_i \\ Q A_i^T + F_i^T B_i^T & \beta Q \end{bmatrix} \succeq 0, \ \forall i. \tag{13}$$

By solving the LMI feasibility problem (13) with decreasing β, we obtain an upper bound of ρ_*. This bound is conservative since extremal or Barabanov norms are generally not ellipsoid norms and the optimal user control policy may not be in $\mathcal{L}$.

6.2. Polytope norms

A less conservative but more computationally intensive approach is based on polytope norms. Let $C = \{c_1, \ldots, c_\ell\} \subset \mathbb{R}^n$

be such that $c_1, \ldots, c_\ell$ span $\mathbb{R}^n$. Then $\|z\|_C := \max_{j=1,\ldots,\ell} |c_j^T z|$ defines a (polytope) norm on $\mathbb{R}^n$. Applying the operator (5), we have

$$\|z\|_{C\sharp} = \max_i \inf_v \max_j \left| c_j^T (A_i z + B_i v) \right| .$$

For each fixed $i \in \mathcal{M}$, $\inf_v \max_j \left| c_j^T (A_i z + B_i v) \right|$ is the optimal value of the linear program:

$$\min_{v,y} \;\; y \;\; \text{s.t.} \;\; \pm c_j^T (A_i z + B_i v) \le y, \;\; j = 1, \ldots, \ell. \tag{14}$$

By strong duality, the optimal value of (14) is equal to that of its dual problem, $\max_{c \in \Omega_i} c^T z$, where Ω_i is the bounded, centrally symmetric polytope in $\mathbb{R}^n$ given by

$$\Omega_i := \left\{ \sum_{j=1}^{\ell} (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j \,\middle|\, \sum_{j=1}^{\ell} (\theta_{ij}^+ + \theta_{ij}^-) = 1, \right.$$
$$\left. \sum_{j=1}^{\ell} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0, \; \theta_{ij}^+ \ge 0, \; \theta_{ij}^- \ge 0 \right\}.$$

Then, $\|z\|_{C\sharp} = \max_i \max_{c \in \Omega_i} c^T z = \max_{c \in \Omega} c^T z$, where

$$\Omega = \left\{ \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j \,\middle|\, \sum_{i,j} (\theta_{ij}^+ + \theta_{ij}^-) = 1, \right.$$
$$\left. \sum_{j} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0, \; \forall i, \; \theta_{ij}^+ \ge 0, \; \theta_{ij}^- \ge 0 \right\}$$

is the convex hull of $\cup_{i \in \mathcal{M}} \Omega_i$ (denoted by $\Omega := \mathrm{Co}\,(\cup_{i \in \mathcal{M}} \Omega_i)$). Here, $i \in \mathcal{M}$ and $j = 1, \ldots, \ell$.

The condition that $\|\cdot\|_{C\sharp} \ge \alpha \|\cdot\|_C$ is equivalent to $\mathrm{Co}\,(\alpha c_1, \ldots, \alpha c_\ell) \subset \Omega$, i.e., $\alpha c_k \in \Omega$ for each $k = 1, \ldots, \ell$. The largest α for this to hold provides a lower bound of ρ_*, which is given by $\alpha^* = \min_k \alpha_k^*$ where $\alpha_k^* := \sup\{\alpha \ge 0 \mid \alpha c_k \in \Omega\}$ can be computed by solving a linear program. In the ideal case, $\|\cdot\|_C$ resembles a Barabanov norm (if exists), i.e., $\mathrm{Co}\,(c_1, \ldots, c_\ell)$ and Ω have similar shape. This would imply that α_k^* have similar values (i.e., low eccentricity).

Based on the above discussions, we present Algorithm 1 below as an answer to Question 1.3 in Section 1. The algorithm adaptively changes the polytope norm $\|\cdot\|_C$ to achieve increasingly tight lower bounds of ρ_*. Using the polytope norm $\|\cdot\|_C$ established by Algorithm 1, one can solve a set of linear programs to find the smallest β such that $\|\cdot\|_{C\sharp} \le \beta \|\cdot\|$ holds, which yields an upper bound of ρ_*. By increasing the number ℓ of vectors in the set C, the obtained bounds can be made arbitrarily tight. However, the computational complexity will also increase exponentially fast with ℓ.

Algorithm 1

Initialize $C \in \mathbb{R}^{n \times \ell}$ with columns c_k, $k = 1, \ldots, \ell$
repeat
  for $k = 1, \ldots, \ell$ **do**
    Compute $\alpha_k^* = \sup\{\alpha \ge 0 \,|\, \alpha c_k \in \Omega\}$
  end for
  $\bar{\alpha}^* \leftarrow \sqrt[\ell]{\prod_{k=1}^{\ell} \alpha_k^*}$
  for $k = 1, \ldots, \ell$ **do**
    $c_k \leftarrow (\alpha_k^* / \bar{\alpha}^*) \cdot c_k$
  end for
until $(\max_k \alpha_k^*)/(\min_k \alpha_k^*) \le 1 + \varepsilon$ or maximum number of iterations is reached
return $\alpha^* = \min_k \alpha_k^*$

We now test the algorithms on the SLCS (4), whose σ_*-stabilizing rate has been found in Example 4.3 to be $\rho_* \approx 1.2493$. By using Algorithm 1 with C initialized to have 72 unit vectors equally dividing half of the unit circle, we obtain a polytopic norm $\|\cdot\|_C$ whose unit ball is very close to the one depicted in Fig. 3. It produces the bounds $1.2474 \le \rho_* \le 1.2638$. In comparison, we obtain the upper bound $\rho_* \le 1.4143$ by solving the LMI problem (13), and the upper bound $\rho_* \le 1.3305$ when using the algorithm in Daafouz and Bernussou (2001, Theorem 4).

7. Conclusions

The optimal stabilizing rate is proposed as a quantitative metric of the stabilizability of SLCSs using continuous input under arbitrary but known mode switchings. It is shown that the optimal stabilizing rate may not always be attainable and, even if it is attainable, it may not be achieved by a mode-dependent linear state feedback controller. Theoretical and numerical techniques based on (semi)norms are proposed to compute bounds of the optimal stabilizing rate. Future research includes, e.g., finding larger families of SLCS for which the optimal stabilizing rate can be achieved by mode-dependent linear state feedback controller.

Appendix A. Proof of Theorem 2.1

Obviously, (i) implies (ii), and (ii) implies (iii); we only need to show that (iii) implies (i). Suppose (iii) holds. Consider a fixed $z \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, and set $\varepsilon = 0.5$. Then for any σ, there exists $\mathbf{u}_{z,\sigma} \in \mathcal{U}$ such that $T_{z,\sigma} := \min\{t \mid \|x(t; \sigma, \mathbf{u}_{z,\sigma}, z)\| < 0.5\}$ is finite. We claim that $T_{z,\sigma}$ is uniformly bounded in σ, i.e.,

Claim: there exists $T_z \in \mathbb{Z}_+$ such that for any $\sigma \in \mathcal{S}$,
we can find $\mathbf{u}_{z,\sigma} \in \mathcal{U}$ so that $T_{z,\sigma} \le T_z$.

Suppose the claim fails. Then there exist a sequence of switching sequences $(\sigma^{(k)})$ and a strictly increasing time sequence (T_k) such that for each k, $\|x(t; \sigma^{(k)}, \mathbf{u}, z)\| \ge 0.5$ for all $t = 0, 1, \ldots, T_k$ under any $\mathbf{u} \in \mathcal{U}$. At each t, since $\sigma^{(k)}(t)$, $k \in \mathbb{Z}_+$, take values in the finite set $\mathcal{M}$, at least one value, denoted by $\sigma^{(\infty)}(t)$, appears infinitely often. Denote by $\sigma^{(\infty)} \in \mathcal{S}$ the switching sequence $(\sigma^{(\infty)}(0), \sigma^{(\infty)}(1), \ldots)$. By repeatedly taking subsequences of $(\sigma^{(k)})$ and induction on t, we have $\|x(t; \sigma^{(\infty)}, \mathbf{u}, z)\| \ge 0.5, \forall t$ under any $\mathbf{u} \in \mathcal{U}$, a contradiction to (iii). Hence, the claim holds.

For a fixed $z \in \mathbb{S}^{n-1}$ and a given switching sequence σ, let $\mathbf{u}_{z,\sigma}$ be a control policy such that $T_{z,\sigma}$ is the first time satisfying $\|x(t; \sigma, \mathbf{u}_{z,\sigma}, z)\| < 0.5$ with $T_{z,\sigma} \le T_z$, and $u_{z,\sigma}(t)$ be the control input value produced by this control policy at t. Define the admissible control policy $\widetilde{\mathbf{u}}_{z,\sigma} := (u_{z,\sigma}(0), u_{z,\sigma}(1), \ldots)$. Clearly, under the given σ and $\widetilde{\mathbf{u}}_{z,\sigma}$, $\|x(t; \sigma, \widetilde{\mathbf{u}}_{z,\sigma}, v)\|$ is continuous in v at each t. Therefore, there exists a neighborhood $\mathcal{U}_{z,\sigma}$ of z such that for any $v \in \mathcal{U}_{z,\sigma}$, $\|x(t; \sigma, \widetilde{\mathbf{u}}_{z,\sigma}, v)\| < 0.5$ for some $t \le T_z$. Since there are only finitely many σ's up to the time T_z, this neighborhood can be chosen uniformly with respect to σ, i.e., we can find a neighborhood $\mathcal{U}_z$ of z and a control policy $\widetilde{\mathbf{u}}_z$ (which is the ensemble of all the $\widetilde{\mathbf{u}}_{z,\sigma}$'s defined above) such that for any $\sigma \in \mathcal{S}$ and any $v \in \mathcal{U}_z$, $\|x(t; \sigma, \widetilde{\mathbf{u}}_z, v)\| < 0.5$ for some $t \le T_z$. Since $\mathbb{S}^{n-1}$ is compact, there exist finitely many $z^{(1)}, \ldots, z^{(p)} \in \mathbb{S}^{n-1}$ for some $p \in \mathbb{N}$ such that the corresponding neighborhoods $\mathcal{U}_{z^{(1)}}, \ldots, \mathcal{U}_{z^{(p)}}$ cover $\mathbb{S}^{n-1}$. Define $T_* := \max_{j=1,\ldots,p} T_{z^{(j)}}$. Let $\mathbf{u}^*$ be the control policy obtained by piecing together $\widetilde{\mathbf{u}}_{z^{(j)}}$, i.e., if $z \in \mathcal{U}_{z^{(j)}}$, then $\widetilde{\mathbf{u}}_{z^{(j)}}$ is invoked, $\forall j = 1, \ldots, p$. Therefore, for any $z \in \mathbb{S}^{n-1}$ and any $\sigma \in \mathcal{S}$, $\|x(t; \sigma, \mathbf{u}^*, z)\| \le 0.5$ for some $t \le T_*$. It can be verified that $\sup_{t \in [0, T_*], z \in \mathbb{S}^{n-1}, \sigma \in \mathcal{S}} \|x(t; \sigma, \mathbf{u}^*, z)\| < \infty$. Using this result and a standard argument for switching systems (Shen & Hu, 2012, Proposition 2.1), it can be shown that by repeating $\mathbf{u}^*$ whenever the state solution's norm is reduced by at least half for the first time (which takes no more than T_* time), we obtain an admissible control policy that exponentially stabilizes the SLCS.

Appendix B. Proof of Proposition 3.1

By scaling all of the A_i's matrices by $1/\rho_*$, we can assume without loss of generality that $\rho_* = 1$. Define the extended real valued function

$$\zeta(z) := \sup_{\sigma\in\mathcal{S}} \inf_{\mathbf{u}\in\mathcal{U}} \sup_{t\in\mathbb{Z}_+} \|x(t;\sigma,\mathbf{u},z)\|, \quad \forall z\in\mathbb{R}^n. \tag{B.1}$$

It is easily seen that ζ is subadditive, absolutely homogeneous of degree one, and positive definite (since $\zeta(\cdot)\geq\|\cdot\|$). Hence, $\mathcal{W} := \{z \mid \zeta(z)<\infty\}$ is a subspace of $\mathbb{R}^n$. We claim that it is control σ_*-invariant. In fact, for any $z\in\mathcal{W}$, there exist $K\in[0,\infty)$ and $\mathbf{u}\in\mathcal{U}$ such that for any $\sigma\in\mathcal{S}$, $\|x(t;\sigma,\mathbf{u},z)\|\leq K$, $\forall t$. Let $\sigma(0)=i$ be arbitrary, and let $v=\mathbf{u}_0(z,i)$ be the corresponding control produced by the policy $\mathbf{u}$ at time 0. Denote $\sigma_+ = (\sigma(1),\sigma(2),\ldots)$ and $\mathbf{u}_+ = (\mathbf{u}_1,\mathbf{u}_2,\ldots)$. Then $x(1) = A_iz+B_iv$ is such that for any $\sigma_+\in\mathcal{S}$, $\|x(t;\sigma_+,\mathbf{u}_+,x(1))\| = \|x(t+1;\sigma,\mathbf{u},x(0))\|\leq K$. This shows that $x(1)\in\mathcal{W}$, i.e., $\mathcal{W}$ is control σ_*-invariant.

By the irreducibility assumption, $\mathcal{W}$ is either $\{0\}$ or $\mathbb{R}^n$. In this and the next paragraph we will prove by contradiction that the former is impossible. Suppose otherwise, i.e., $\mathcal{W}=\{0\}$. Then for an arbitrary $z\in\mathbb{S}^{n-1}$, there exists σ_z such that $\inf_{\mathbf{u}\in\mathcal{U}}\sup_{t\in\mathbb{Z}_+}\|x(t;\sigma_z,\mathbf{u},z)\| > 2$, which implies that for any $\mathbf{u}\in\mathcal{U}$, there exists $s_{z,\sigma_z,\mathbf{u}}\in\mathbb{Z}_+$ such that $\|x(s_{z,\sigma_z,\mathbf{u}};\sigma_z,\mathbf{u},z)\|>2$. We claim that $s_{z,\sigma_z,\mathbf{u}}$ is uniformly bounded in $z\in\mathbb{S}^{n-1}$, σ_z, and $\mathbf{u}\in\mathcal{U}$, namely, there exists $N\in\mathbb{Z}_+$ such that for any $z\in\mathbb{S}^{n-1}$, there exists σ_z such that for any $\mathbf{u}\in\mathcal{U}$, there exists $t\leq N$ so that $\|x(t;\sigma_z,\mathbf{u},z)\|>2$. Suppose not, then there exist a strictly increasing sequence of times (s_k), a sequence (z_k) in $\mathbb{S}^{n-1}$, and a sequence of control policies $(\mathbf{u}^k)$ such that for each k, $\|x(t;\sigma,\mathbf{u}^k,z_k)\|\leq 2$ for all σ and all $t\in\{0,1,\ldots,s_k\}$. It follows from the similar argument for (Hu et al., 2017, Theorem III.1) that there exist $z_*\in\mathbb{S}^{n-1}$ and a control policy $\mathbf{u}^*$ such that $\sup_{t\in\mathbb{Z}_+}\|x(t;\sigma,\mathbf{u}^*,z_*)\|\leq 2$ for all $\sigma\in\mathcal{S}$. This implies that $\zeta(z_*)\leq 2$ such that $z_*\in\mathcal{W}$, a contradiction. Therefore, the claim holds. Hence, starting from any initial state $z\in\mathbb{S}^{n-1}$, there exists a switching sequence under which the state norm will be more than doubled at some time $t\leq N$ regardless of $\mathbf{u}\in\mathcal{U}$. When this occurs at time t, the adversary can start a new switching sequence $\sigma_{x(t)/\|x(t)\|}$. Repeated indefinitely, this process leads to a switching sequence $\sigma\in\mathcal{S}$ under which the state solution grows exponentially fast to infinity regardless of $\mathbf{u}\in\mathcal{U}$, contradicting the assumption that $\rho_*=1$. Therefore, $\mathcal{W}\neq\{0\}$.

Since the SLCS is irreducible, we have $\mathcal{W}=\mathbb{R}^n$, i.e., ζ is pointwise finite on $\mathbb{R}^n$. Together with the properties established at the beginning of the proof, we conclude that ζ is a norm. Hence, $\zeta(\cdot)\leq K\|\cdot\|$ for a constant $K\in[0,\infty)$, or equivalently, the SLCS is nondefective.

Appendix C. Proof of Theorem 4.1

Suppose the SLCS has an extremal norm $\|\cdot\|$, i.e., $\|z\|_\sharp = \max_i\inf_v\|A_iz+B_iv\| \leq \rho_*\|z\|$ for all z. Then under the user control policy $\mathbf{u}_t(\sigma(t),x(t)) := \arg\min_v\|A_{\sigma(t)}x(t)+B_{\sigma(t)}v\|$, we have $\|x(t+1)\|\leq\rho_*\|x(t)\|$, hence $\|x(t)\|\leq(\rho_*)^t\|x(0)\|$ for all t, i.e., the SLCS is nondefective.

For the other direction, we only prove for the case of $\rho_*>0$ since the case of $\rho_*=0$ is straightforward. By replacing each A_i with A_i/ρ_*, we further assume without loss of generality that $\rho_*=1$. Since the SLCS is nondefective, there exist a constant $K\in[0,\infty)$ and a user control policy $\mathbf{u}\in\mathcal{U}$ such that $\|x(t;\sigma,\mathbf{u},z)\|\leq K\|z\|$, $\forall t, \forall\sigma\in\mathcal{S},\forall z$, where $\|\cdot\|$ is a generic (but not necessarily extremal) norm. This implies that the function ζ defined in (B.1) is bounded, i.e., $\zeta(\cdot)\leq K\|\cdot\|$, and thus is pointwise finite on $\mathbb{R}^n$. Since ζ is subadditive, absolutely homogeneous of degree one, and positive definite, it is a norm. We claim that ζ is an extremal norm. Indeed, any $\sigma\in\mathcal{S}$ and $\mathbf{u}\in\mathcal{U}$ can be written as $\sigma=(\sigma(0),\sigma_+)$ and $\mathbf{u}=(\mathbf{u}_0,\mathbf{u}_+)$ where $\sigma(0)=i\in\mathcal{M}$, $\sigma_+\in\mathcal{S}$, $\mathbf{u}_0(i,z)=v\in\mathbb{R}^p$, and $\mathbf{u}_+\in\mathcal{U}$. For any $z\in\mathbb{R}^n$, we have

$$\begin{aligned}\zeta(z) &= \sup_{\sigma(0)}\sup_{\sigma_+}\inf_{\mathbf{u}_0}\inf_{\mathbf{u}_+}\max\left\{\|z\|, \sup_{t\geq 1}\|x(t;\sigma,\mathbf{u},z)\|\right\}\\ &= \max\left\{\|z\|, \sup_{i\in\mathcal{M}}\inf_{v}\sup_{\sigma_+\in\mathcal{S}}\inf_{\mathbf{u}_+\in\mathcal{U}}\sup_{t\geq 0}\|x(t;\sigma_+,\mathbf{u}_+,A_iz+B_iv)\|\right\} \qquad (C.1)\\ &= \max\left\{\|z\|, \max_{i\in\mathcal{M}}\inf_v\zeta(A_iz+B_iv)\right\}\\ &= \max\left\{\|z\|,\zeta_\sharp(z)\right\}\geq\zeta_\sharp(z). \qquad (C.2)\end{aligned}$$

Note that in deriving the second equality, we switch the order of $\sup_{\sigma_+}$ and $\inf_{\mathbf{u}_0}$ as the feedback control law $\mathbf{u}_0$ is based on z and $\sigma(0)$ but not on σ_+. This shows that ζ is an extremal norm.

Appendix D. Proof of Proposition 4.3

Suppose the SLCS is irreducible, and we consider $\rho_*>0$ first. By scaling the matrices A_i's by $1/\rho_*$, we assume without loss of generality that $\rho_*=1$. Define

$$\chi(z) := \sup_{\sigma\in\mathcal{S}}\inf_{\mathbf{u}\in\mathcal{U}}\limsup_{t\to\infty}\|x(t;\sigma,\mathbf{u},z)\|, \quad \forall z\in\mathbb{R}^n,$$

which is pointwise finite, since the irreducibility of the SLCS implies the nondefectiveness as shown in Proposition 3.1. Clearly, χ is a seminorm on $\mathbb{R}^n$. By a similar argument for the derivation of (C.2), we obtain, for any $z\in\mathbb{R}^n$, $\sigma=(\sigma(0),\sigma_+)\in\mathcal{S}$, and $\mathbf{u}=(\mathbf{u}_0,\mathbf{u}_+)\in\mathcal{U}$,

$$\begin{aligned}\chi(z) &= \sup_{\sigma(0)}\sup_{\sigma_+}\inf_{\mathbf{u}_0}\inf_{\mathbf{u}_+}\limsup_{t\to\infty}\|x(t+1;\sigma,\mathbf{u},z)\|\\ &= \sup_i\inf_v\sup_{\sigma_+}\inf_{\mathbf{u}_+}\limsup_{t\to\infty}\left\|x(t;\sigma_+,\mathbf{u}_+,A_iz+B_iv)\right\|\\ &\qquad\qquad (i=\sigma(0),\ v=\mathbf{u}_0(i,z))\\ &= \max_{i\in\mathcal{M}}\inf_{v\in\mathbb{R}^p}\chi(A_iz+B_iv)=\chi_\sharp(z). \qquad (D.1)\end{aligned}$$

We next show that χ is a norm, or equivalently, the subspace $\mathcal{N}_\chi := \{z\mid\chi(z)=0\}$ is $\{0\}$. First, we claim that $\mathcal{N}_\chi$ is a control σ_*-invariant subspace. To see this, let $z\in\mathcal{N}_\chi$ be arbitrary. Then (D.1) implies that, for each $i\in\mathcal{M}$, $\inf_v\chi(A_iz+B_iv)=0$. Since χ is a seminorm, it follows from the comment after (5) that for each $i\in\mathcal{M}$, there exists $v_i^*\in\mathbb{R}^p$ such that $\chi(A_iz+B_iv_i^*)=\inf_v\chi(A_iz+B_iv)=0$. This shows that $\mathcal{N}_\chi$ is control σ_*-invariant.

Since the SLCS is irreducible, $\mathcal{N}_\chi$ is either $\{0\}$ or $\mathbb{R}^n$. We show that $\mathcal{N}_\chi\neq\mathbb{R}^n$. Suppose not, i.e., $\chi\equiv 0$ on $\mathbb{R}^n$. Fix an arbitrary $z\in\mathbb{S}^{n-1}$. Then for any σ, $\inf_{\mathbf{u}}\limsup_{t\to\infty}\|x(t;\sigma,\mathbf{u},z)\|=0$. Hence, for any σ and $\varepsilon>0$, there exists $\mathbf{u}_{z,\sigma,\varepsilon}$ such that $\limsup_{t\to\infty}\|x(t;\sigma,\mathbf{u}_{z,\sigma,\varepsilon},z)\|<\varepsilon$. This implies that $\|x(T_{z,\sigma,\varepsilon};\sigma,\mathbf{u}_{z,\sigma,\varepsilon},z)\|\leq\varepsilon$ for some $T_{z,\sigma,\varepsilon}\in\mathbb{Z}_+$. By Theorem 2.1, the SLCS is σ_*-exponentially stabilizable, i.e., $\rho_*<1$. This contradicts the assumption that $\rho_*=1$. Hence, $\mathcal{N}_\chi=\{0\}$ so that χ is a norm on $\mathbb{R}^n$. Along with (D.1), this shows that χ is a Barabanov norm.

Finally, we consider the case of $\rho_*=0$. Since the SLCS is irreducible and hence nondefective, each subsystem (A_i,B_i) is stabilizable to the origin in one time step, i.e., $A_i=B_iK_i$ for a matrix K_i, $\forall i\in\mathcal{M}$. Any norm $\|\cdot\|$ on $\mathbb{R}^n$ satisfies $\|\cdot\|_\sharp=0$ and is thus a Barabanov norm.

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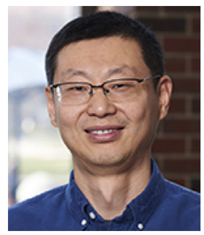

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