ON THE WELL-POSEDNESS OF THE ANISOTROPICALLY-REDUCED TWO-DIMENSIONAL KURAMOTO-SIVASHINSKY EQUATION

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Abstract. We address the global existence and uniqueness of solutions for the anisotropically reduced 2D Kuramoto-Sivashinsky equations in a periodic domain with initial data $u_{01} \in L^2$ and $u_{02} \in H^{-1+\eta}$ for $\eta > 0$.

1. Introduction

We address the well-posedness of the anisotropically-reduced Kuramoto-Sivashinsky equation (r-KSE) as introduced in [LY]

$$\begin{aligned} &\partial_t u_1 - \nu\Delta u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 = \sigma u_1 \\ &\partial_t u_2 + \Delta^2 u_2 + \lambda\Delta u_2 + u_1\partial_x u_2 + u_2\partial_y u_2 = 0 \\ &\partial_y u_1 = \partial_x u_2 \end{aligned} \tag{1.1}$$

with the initial data $u(0) = u_0$ and ν, σ, and $\lambda > 0$ on the torus $[0, 2\pi]^2$. This model is constructed from the velocity formulation of the Kuramoto-Sivashinky equation (KSE)

$$\begin{aligned} &\partial_t u_1 + \Delta^2 u_1 + \Delta u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 = 0 \\ &\partial_t u_2 + \Delta^2 u_2 + \Delta u_2 + u_1\partial_x u_2 + u_2\partial_y u_2 = 0 \\ &\partial_y u_1 = \partial_x u_2. \end{aligned}$$

The KSE is a well-known model introduced in the 1970s in [Kur, S, T] to address several important physical problems, particularly flame fronts, ion plasmas, and low dimensional chaos. Key results have been discovered for the one-dimensional equation, including the global existence, dissipation, and existence of an attractor. The size of the absorbing ball relative to the size of the domain was found for one-dimension [CEES, I, G, GO, NST, O]. For further results on one-dimensional regularity of KSE, see [BS, GK, RK, SS, T, TP].

However, the two-dimensional results that are of particular interest for the physical models have proven more challenging due to the lack of conservation of energy. Additionally, the methods treating KSE as a perturbation of the Burgers equation as in [I, G, GJO] do not extend to higher dimensions. Nevertheless, some results have still been achieved. In [AM], the global existence was shown for small initial data on the torus $[0, 2\pi]^2$ along with decay and analyticity of solutions. For domains where one dimension is small relative to the other and for initial data further controlled by the longer scale, global existence was shown in [ST] using methods inspired by the work on 3D Navier-Stokes by Raugel and Sell [RS1, RS2, RS3] (cf. later works in [A, HS, KRZ, KZ, H1, H2]). This work was adapted by Molinet in [M1, M2] to show local dissipation and global existence for a suitably chosen, thin domain. This domain was expanded in [BKRZ, KMa]

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to show that the global solution exists on thin, rectangular domains $[0, L_1] \times [0, L_2]$ under the condition $L_2 \leq 1/CL_1^{3/5}$ with initial data bounded in L^2.

The anisotropically-reduced equation was devised to address core difficulties with the two dimensional problem. In particular, by modifying the model in the first dimension to resemble Navier-Stokes, the reduced equation acquires something akin to a maximum principle. In [LY], for sufficiently regular initial data, H^1, and L^∞ in the first dimension, Larios and Yamazaki proved well-posedness on the torus. They utilized Galerkin approximation to show local existence and uniqueness followed by energy methods and a maximum-like principle to prove the global existence.

In this paper, we reduce the required regularity of the initial data for global well-posedness by utilizing a method involving semi-groups. Namely, we show that for initial data contained in $L^2 \times H^{-1+\eta}$, for $\eta \in (0, 3/2)$ a unique, strong, global solution exists. This allows for the initial conditions to lie in negative Sobolev spaces. We first introduce an operator based on semi-groups and the r-KSE. This semi-group based approach provides a convenient approach to the negative Sobolev spaces, and additionally yields instantaneous smoothing properties. We then prepare a norm to prove our chosen operator is a closed, bounded, contraction mapping with respect to it, which ensures local existence of a unique solution. The norm needs to be selected such that it balances the relative regularity for u_1 and u_2. Once the correct choice is made, we ensure the operator is bounded with respect to the norm. The uniform bound and subsequently continuity of the operator requires control of the norm, which we achieve through local time, smallness of the norm of the semi-group operator, and by utilizing a large constant to balance the borderline case. We apply bootstrapping through a similar argument to prove the solution is smooth. With smooth solutions, we can then utilize the proof for the global existence of a solution as provided in [LY] to show existence of unique, global solutions past an initial time. Overlapping the two unique solutions ensures global existence and uniqueness of the solution for all time. At the end, we also provide a summary of a simpler, alternative proof to the main result of [LY] by utilizing semi-groups to prove local existence instead of Galerkin approximation for the same initial regularity assumptions.

2. Main Theorem and Supporting Results

The r-KSE equation

$$\begin{aligned}
&\partial_t u_1 - \nu\Delta u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 = \sigma u_1\\
&\partial_t u_2 + \Delta^2 u_2 + \lambda\Delta u_2 + u_1\partial_x u_2 + u_2\partial_y u_2 = 0\\
&\partial_y u_1 = \partial_x u_2
\end{aligned}$$

on the torus $\mathbb{T} = [0, 2\pi]^2$ has been shown in [LY] to have a strong, unique, global solution $u = (u_1, u_2)$ for initial data $u_0 = (u_{01}, u_{02}) \in (H^1)^2$ and $u_{01} \in L^\infty$. We define a strong solution u as satisfying

$$\begin{aligned}
&\int_{\mathbb{T}} \partial_t u_1\phi_1 dx + \nu\int_{\mathbb{T}} \nabla u_1 \cdot \nabla\phi_1 dx + \int_{\mathbb{T}} (u\cdot\nabla)u_1\phi_1 dx = \sigma\int_{\mathbb{T}} u_1\phi_1 dx\\
&\int_{\mathbb{T}} \partial_t u_2\phi_2 dx + \int_{\mathbb{T}} \Delta u_2\Delta\phi_2 dx + \lambda\int_{\mathbb{T}} \nabla u_2 \cdot \nabla\phi_2 dx + \int_{\mathbb{T}} (u\cdot\nabla)u_2\phi_2 dx = 0
\end{aligned}$$

for $\phi = (\phi_1, \phi_2) \in C^\infty(\mathbb{T})^2$ on $[0, T]$. In addition, we require

$$u \in C_b((0, T]; H^k(\mathbb{T}))^2 \cap C_b([0, T]; L^2(\mathbb{T})) \times C_b([0, T]; H^{-1+\eta}(\mathbb{T})),$$

for all $k > 0$ and $\eta \in (0, 3/2)$ where we define the negative Sobolev spaces $H^{-1+\eta}(\mathbb{T})$ as the dual space of $H^{1-\eta}(\mathbb{T})$ when $\eta < 1$. Then, we show that for weaker initial data than in [LY], we still have a strong, global solution.

Theorem 2.1. *For $u_{01} \in L^2$ and $u_{02} \in H^{-1+\eta}$ where $\eta \in (0, 3/2)$, there exists a unique, strong, global solution.*

For $\eta \geq 3/2$, an argument using boostrapping shows that the same result holds as well. We next recall known properties of Sobolev spaces that are used in the proof.

Lemma 2.1. *Let $0 < s_1, s_2 \leq s_3$ be such that $1 + s_3 \leq s_1 + s_2$ with a strict inequality if $s_3 = s_1$ or $s_3 = s_2$. Then, we have*

$$\|uv\|_{H^{s_3}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}},$$

for $u \in H^{s_1}$ and $v \in H^{s_2}$.

The Sobolev multiplicative inequality for two dimensions follows directly from Hölder and embedding inequalities. Furthermore, we have the following well known lemma.

Lemma 2.2. *If $s \geq \tilde{s}$, $t > 0$, and $\alpha \in \mathbb{N}$, then we have*

$$\|e^{-t(-\Delta)^\alpha} f\|_{H^s} \lesssim t^{-\frac{s-\tilde{s}}{\alpha}} \|f\|_{H^{\tilde{s}}},$$

for $f \in H^{\tilde{s}}$.

This allows us to interchange between derivatives in space and powers in time. From [W], we recall the following property.

Lemma 2.3. *For $f(t) = e^{t\Delta} f_0$ and $f_0 \in H^{\tilde{s}}$ where $\tilde{s} \in \mathbb{R}$, we have*

$$\lim_{T \to 0} \sup_{0 < t \leq T} (t^{\frac{\alpha}{2}} \|f(t)\|_{H^\alpha}) = 0,$$

for $\alpha > \tilde{s}$.

The proof follows from a density argument as in [W]. This helps us to show continuity in borderline cases.

We suspect that the main theorem continues to hold if we change the domain from the torus $\mathbb{T}^2$ to the whole space $\mathbb{R}^2$ or to a bounded domain Ω with u subject to appropriate boundary conditions. In both of these cases, what qualities the r-KSE displays comparatively with the KSE remains an open problem.

3. Proof of Main Theorem

The proof of the main theorem utilizes a contraction mapping argument to show existence, uniqueness and regularity of the solution for local time. Then, due to the smoothness, we can extend by arguments made in [LY] our local solution to a global solution.

Proof of Theorem 2.1. We begin with proving the local existence of a solution and its uniqueness. Thus, we define a sequence $u^{(m)} = (u_1^{(m)}, u_2^{(m)})$ such that for $m \in \mathbb{N}_0$ we have

$$u^{(m+1)} = e^{tA} u_0 - \int_0^t e^{(t-s)A} (u^{(m)} \cdot \nabla u^{(m)})(s) ds + \int_0^t e^{(t-s)A} L(u^{(m)})(s) ds \tag{3.1}$$

where $A = (\nu\Delta, -\Delta^2)^T$, and $L((u_1, u_2)) = (\sigma u_1, -\lambda\Delta u_2)^T$, with the starting term $u^{(0)} = e^{tA}u_0$. Furthermore, define the norm

$$\|(u_1,u_2)\|_X = \sup_{t\le T}\|u_1\|_{L^2} + K\sup_{t\le T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big) + \sup_{t\le T}\|u_2\|_{H^{-\alpha+\epsilon}} + K\sup_{t\le T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)$$

where $\alpha \in [1/2, 1)$, $\epsilon \in (0, 1]$, and $K \ge 1$ is a large constant to be determined. Denote $\eta = 1 - \alpha + \epsilon$ so that $-\alpha+\epsilon = -1+\eta$. Here, $X = C_b((0,T]; H^k(\mathbb{T}))^2 \cap \big(C_b([0,T]; L^2(\mathbb{T})) \times C_b([0,T]; H^{-1+\eta}(\mathbb{T}))\big)$. We claim that $u^{(m)}$ is uniformly bounded in X by M where $M = 8\max\{1, \|u_{01}\|_{L^2}, \|u_{02}\|_{H^{-1+\eta}}\}$. To show this, we begin by considering the base case in an induction argument to find that

$$\begin{aligned}\|u^{(0)}(t)\|_X \le \sup_{t\le T}\|e^{\nu t\Delta}u_{01}\|_{L^2} + K\sup_{t\le T}t^{\frac{\alpha}{2}}\|e^{\nu t\Delta}u_{01}\|_{H^\alpha} + \sup_{t\le T}\|e^{-t\Delta^2}u_{02}\|_{H^{-\alpha+\epsilon}}\\ + K\sup_{t\le T}t^{\frac{1}{4}(2-\epsilon)}\|e^{-t\Delta^2}u_{02}\|_{H^{2-\alpha}}.\end{aligned}$$

Using Lemma 2.3 and choosing $T > 0$ to be sufficiently small, we determine

$$\sup_{t\le T}t^{\frac{\alpha}{2}}\|e^{\nu t\Delta}u_{01}\|_{H^\alpha} \le \frac{M}{8K}$$

and

$$\sup_{t\le T}t^{\frac{1}{4}(2-\epsilon)}\|e^{-t\Delta^2}u_{02}\|_{H^{2-\alpha}} \le \frac{M}{8K}.$$

Then, by the prior observations, $\|u^{(0)}\|_X \le M/2$. By the induction hypothesis, assume that $\|u^{(m)}\|_X \le M$. Then, observe that

$$\|u^{(m+1)}\|_X \le \|e^{tA}u_0\|_X + \left\|\int_0^t e^{(t-s)A}(u^{(m)}\cdot\nabla u^{(m)})ds\right\|_X + \left\|\int_0^t e^{(t-s)A}L(u^{(m)})ds\right\|_X. \tag{3.2}$$

Note that the bound on $u^{(m+1)}$ is determined by the bound on the preceding term in the sequence. Thus, for notational simplicity, we denote $u = u^{(m)}$. Examining the middle term, we observe that it expands as

$$\begin{aligned}&\sup_{t\le T}\int_0^t\|e^{\nu(t-s)\Delta}(u_1\partial_x u_1)\|_{L^2}ds + \sup_{t\le T}\int_0^t\|e^{\nu(t-s)\Delta}(\partial_y(u_1u_2) - u_1\partial_y u_2)\|_{L^2}ds\\ &\quad+\sup_{t\le T}\int_0^t\|e^{-(t-s)\Delta^2}(u_1\partial_x u_2)\|_{H^{-\alpha+\epsilon}}ds + \sup_{t\le T}\int_0^t\|e^{-(t-s)\Delta^2}(u_2\partial_y u_2)\|_{H^{-\alpha+\epsilon}}ds\\ &\quad+K\sup_{t\le T}t^{\frac{\alpha}{2}}\left(\int_0^t\|e^{\nu(t-s)\Delta}(u_1\partial_x u_1)\|_{H^\alpha}ds + \int_0^t\|e^{\nu(t-s)\Delta}(\partial_y(u_1u_2) - u_1\partial_y u_2)\|_{H^\alpha}ds\right)\\ &\quad+K\sup_{t\le T}t^{\frac{1}{4}(2-\epsilon)}\left(\int_0^t\|e^{-(t-s)\Delta^2}(u_1\partial_x u_2)\|_{H^{2-\alpha}}ds + \int_0^t\|e^{-(t-s)\Delta^2}(u_2\partial_y u_2)\|_{H^{2-\alpha}}ds\right).\end{aligned} \tag{3.3}$$

Neglecting the supremum in time for now, for the first term in (3.3), using Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\int_0^t\|e^{\nu(t-s)\Delta}(u_1\partial_x u_1)\|_{L^2}ds &\lesssim \int_0^t\|e^{\nu(t-s)\Delta}\big(\partial_x(u_1^2)\big)\|_{L^2}ds \lesssim \int_0^t(t-s)^{-\frac{1}{2}(1-(2\alpha-1))}\|u_1^2\|_{H^{2\alpha-1}}ds\\ &\lesssim \int_0^t(t-s)^{-(1-\alpha)}\|u_1\|_{H^\alpha}^2ds \lesssim \int_0^t(t-s)^{-(1-\alpha)}s^{-\alpha}(s^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha})^2ds\\ &\lesssim \big(\sup_{t\le T}t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)^2\int_0^t(t-s)^{-(1-\alpha)}s^{-\alpha}ds,\end{aligned} \tag{3.4}$$

noting that $\alpha \in [1/2, 1)$, which ensures $0 \le 2\alpha - 1 < \alpha$. Thus, we determine that

$$\int_0^t \|e^{\nu(t-s)\Delta}(u_1\partial_x u_1)\|_{L^2} ds \le C_T \sup_{t\le T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)^2 \le \frac{C_T M^2}{K^2} \tag{3.5}$$

where C_T is a constant depending only on T. The second term in (3.3), containing both u_1 and u_2, is bounded by

$$\begin{aligned}
\int_0^t \|e^{\nu(t-s)\Delta}(\partial_y(u_1u_2) - u_1\partial_y u_2)\|_{L^2} ds &\lesssim \int_0^t \big(\|e^{\nu(t-s)\Delta}\partial_y(u_1u_2)\|_{L^2} + \|e^{\nu(t-s)\Delta}(u_1\partial_y u_2)\|_{L^2}\big) ds \\
&\lesssim \int_0^t \big((t-s)^{-\frac{1}{2}(1-\alpha)}\|u_1u_2\|_{H^\alpha} + \|u_1\partial_y u_2\|_{L^2}\big) ds \\
&\lesssim \int_0^t \big((t-s)^{-\frac{1}{2}(1-\alpha)}\|u_1\|_{H^\alpha}\|u_2\|_{H^{2-\alpha}} + \|u_1\|_{H^\alpha}\|\partial_y u_2\|_{H^{1-\alpha}}\big) ds \\
&\lesssim \sup_{t\le T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)\sup_{t\le T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)\int_0^t \big((t-s)^{-\frac{1}{2}(1-\alpha)} + 1\big)s^{-\frac{\alpha}{2}}s^{-\frac{1}{4}(2-\epsilon)} ds,
\end{aligned} \tag{3.6}$$

which implies that we have

$$\int_0^t \|e^{\nu(t-s)\Delta}(\partial_y(u_1u_2) - u_1\partial_y u_2)\|_{L^2} ds \le C_T \sup_{t\le T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)\sup_{t\le T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big) \le \frac{C_T M^2}{K^2}. \tag{3.7}$$

The third term in (3.3) reads

$$\begin{aligned}
\int_0^t \|e^{-(t-s)\Delta^2}(u_1\partial_x u_2)\|_{H^{-\alpha+\epsilon}} ds &\lesssim \int_0^t \|u_1\partial_x u_2\|_{L^2} ds \lesssim \int_0^t \|u_1\|_{H^\alpha}\|\partial_x u_2\|_{H^{1-\alpha}} ds \\
&\lesssim \sup_{t\le T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)\sup_{t\le T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)\int s^{-\frac{\alpha}{2}}s^{-\frac{1}{4}(2-\epsilon)} ds.
\end{aligned} \tag{3.8}$$

Thus, we find that

$$\int_0^t \|e^{-(t-s)\Delta^2}u_1\partial_x u_2\|_{H^{-\alpha+\epsilon}} ds \le \frac{C_T M^2}{K^2}. \tag{3.9}$$

The fourth term in (3.3) is estimated as

$$\begin{aligned}
\int_0^t \|e^{-(t-s)\Delta^2}\partial_y(u_2^2)\|_{H^{-\alpha+\epsilon}} ds &\lesssim \int_0^t \|u_2^2\|_{H^{1-\alpha+\epsilon}} ds \lesssim \int_0^t \|u_2\|_{H^{2-\alpha}}^2 ds \\
&\lesssim \sup_{t\le T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)^2 \int_0^t s^{-\frac{1}{2}(2-\epsilon)} ds,
\end{aligned} \tag{3.10}$$

where we again used Lemmas 2.1, 2.2 and $\epsilon \le 1$. Therefore, we can bound the term as

$$\int_0^t \|e^{-(t-s)\Delta^2}\partial_y(u_2^2)\|_{H^{-\alpha+\epsilon}} ds \le C_T \sup_{t\le T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)^2 \le \frac{C_T M^2}{K^2}. \tag{3.11}$$

To bound the remaining four terms in (3.3) we note that

$$t^{\frac{\alpha}{2}} \lesssim (t-s)^{\frac{\alpha}{2}} + s^{\frac{\alpha}{2}}. \tag{3.12}$$

We use arguments in (3.4) to find that

$$\begin{aligned} Kt^{\frac{\alpha}{2}} &\int_0^t \|e^{\nu(t-s)\Delta}(u_1\partial_x u_1)\|_{H^\alpha} ds \\ &\leq CK\int_0^t \big((t-s)^{\frac{\alpha}{2}} + s^{\frac{\alpha}{2}}\big)(t-s)^{-\frac{\alpha}{2}}(t-s)^{-(1-\alpha)}\|u_1^2\|_{H^{2\alpha-1}} ds \\ &\leq CK\int_0^t \big((t-s)^{\frac{\alpha}{2}} + s^{\frac{\alpha}{2}}\big)(t-s)^{-(1-\frac{\alpha}{2})}s^{-\alpha}\|u_1\|_{H^\alpha}^2 ds \\ &\leq CK\sup_{t\leq T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)^2 \\ &\qquad\times\int_0^t \big((t-s)^{-(1-\frac{\alpha}{2})}s^{-\frac{\alpha}{2}} + (t-s)^{-(1-\alpha)}s^{-\alpha}\big) ds. \end{aligned} \tag{3.13}$$

As in (3.6), we observe the sixth term is bounded as

$$\begin{aligned} Kt^{\frac{\alpha}{2}} &\int_0^t \|e^{\nu(t-s)\Delta}(\partial_y(u_1u_2) - u_1\partial_y u_2)\|_{H^\alpha} ds \\ &\leq CK\sup_{t\leq T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)\sup_{t\leq T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big) \\ &\qquad\times\int_0^t \big(s^{-\frac{\alpha}{2}} + (t-s)^{-\frac{\alpha}{2}}\big)\big((t-s)^{-\frac{1}{2}(1-\alpha)} + 1\big)s^{-\frac{1}{4}(2-\epsilon)} ds. \end{aligned} \tag{3.14}$$

Recalling (3.8) for the seventh term, we find that

$$\begin{aligned} Kt^{\frac{1}{4}(2-\epsilon)} &\int_0^t \|e^{-(t-s)\Delta^2}u_1\partial_x u_2\|_{H^{2-\alpha}} ds \\ &\leq CK\sup_{t\leq T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)\sup_{t\leq T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big) \\ &\qquad\times\int_0^t \big((t-s)^{\frac{1}{4}(2-\epsilon)} + s^{\frac{1}{4}(2-\epsilon)}\big)s^{-\frac{\alpha}{2}}s^{-\frac{1}{4}(2-\epsilon)}(t-s)^{-\frac{1}{4}(2-\alpha)} ds \\ &\leq CK\sup_{t\leq T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)\sup_{t\leq T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big) \\ &\qquad\times\int_0^t \big((t-s)^{-\frac{1}{4}(\epsilon-\alpha)}s^{-\frac{1}{4}(2-\epsilon)} + (t-s)^{-\frac{1}{4}(2-\alpha)}\big)s^{-\frac{\alpha}{2}} ds. \end{aligned} \tag{3.15}$$

For the eighth term, combining Lemmas 2.1 and 2.2 with (3.10), we observe that

$$\begin{aligned} \frac{1}{2}Kt^{\frac{1}{4}(2-\epsilon)} &\int_0^t \|e^{-(t-s)\Delta^2}\partial_y(u_2^2)\|_{H^{2-\alpha}} ds \leq CK\int_0^t \big((t-s)^{\frac{1}{4}(2-\epsilon)} + s^{\frac{1}{4}(2-\epsilon)}\big)(t-s)^{-\frac{1}{4}}\|u_2^2\|_{H^{2-\alpha}} ds \\ &\leq CK\int_0^t \big((t-s)^{\frac{1}{4}(2-\epsilon)} + s^{\frac{1}{4}(2-\epsilon)}\big)(t-s)^{-\frac{1}{4}}\|u_2\|_{H^{2-\alpha}}^2 ds \\ &\leq CK\sup_{t\leq T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)^2\int_0^t \big(\big((t-s)^{\frac{1}{4}(2-\epsilon)} + s^{\frac{1}{4}(2-\epsilon)}\big)(t-s)^{-\frac{1}{4}}s^{-\frac{1}{2}(2-\epsilon)}\big) ds \\ &\leq CK\sup_{t\leq T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)^2\int_0^t \big((t-s)^{\frac{1}{4}(1-\epsilon)}s^{-\frac{1}{2}(2-\epsilon)} + (t-s)^{-\frac{1}{4}}s^{-\frac{1}{4}(2-\epsilon)}\big) ds \end{aligned} \tag{3.16}$$

noting that $2-\alpha > 1$ ensures that Lemma 2.1 applies. In each case, the sum of the exponents under the integral is greater than or equal to -1. Therefore, these four terms can be bounded by $C_T M^2/K$

using (3.13)–(3.16). Lastly we consider the linear term in (3.2) where we have the inequality

$$\begin{aligned}\int_0^t \|e^{(t-s)A}L(u)\|_X ds &\lesssim \int_0^t \|e^{\nu(t-s)\Delta}\sigma u_1\|_{L^2} ds + \int_0^t \|e^{-(t-s)\Delta^2}\lambda\Delta u_2\|_{H^{-\alpha+\epsilon}} ds \\ &\quad + t^{\frac{\alpha}{2}}\int_0^t \|e^{\nu(t-s)\Delta}\sigma u_1\|_{H^\alpha} ds + t^{\frac{1}{4}(2-\epsilon)}\int_0^t \|e^{-(t-s)\Delta^2}\lambda\Delta u_2\|_{H^{2-\alpha}} ds \\ &\lesssim \sup_{t\le T}\big(t^{\frac{\alpha}{2}}\|u_1\|_{H^\alpha}\big)\int_0^t \big(s^{-\frac{\alpha}{2}} + (t-s)^{\frac{\alpha}{2}}s^{-\frac{\alpha}{2}} + 1\big) ds \\ &\quad + \sup_{t\le T}\big(t^{\frac{1}{4}(2-\epsilon)}\|u_2\|_{H^{2-\alpha}}\big)\int_0^t \big((t-s)^{-\frac{\epsilon}{2}}s^{-\frac{1}{4}(2-\epsilon)} + (t-s)^{-\frac{1}{2}}\big) ds,\end{aligned} \tag{3.17}$$

which we can bound by $M/12$ by the induction hypothesis and choosing T appropriately small. So, combining (3.4)–(3.11) with our bounds on (3.13)–(3.17) and choosing $K \ge 12C_T\max\{1, M\}$ we determine, with the bounds on the initial data and re-introducing supremums for $t \le T$, that the equation

$$\begin{aligned}\|u^{(m+1)}\|_X &\le \|e^{tA}u_0\|_X + \int_0^t \|e^{(t-s)A}(u\cdot\nabla u)\|_X ds + \int_0^t \|e^{(t-s)A}L(u)\|_X ds \\ &\le \frac{M}{2} + \frac{5M}{12} + \frac{M}{12} = M\end{aligned}$$

holds. Therefore, we conclude by induction that $\|u^{(m)}\|_X \le M$ for all $m \in \mathbb{N}_0$.

Regarding continuity, we claim that if

$$u^{(m)} \in C_b([0,T]; L^2) \times C_b([0,T]; H^{-\alpha+\epsilon})$$

and

$$\big(t^{\frac{\alpha}{2}}u_1^{(m)}, t^{\frac{2-\alpha}{4}}u_2^{(m)}\big) \in C_b([0,T]; H^\alpha) \times C([0,T]; H^{2-\alpha}),$$

then $u^{(m+1)}$ also resides in these spaces. We assume the second term to equal zero at $t = 0$. We omit the proof for continuity for $T > 0$ since the argument is analogous to the one at $T = 0$. To prove continuity at $T = 0$, we examine the inequalities

$$\|u_1^{(m+1)}(t) - u_1^{(m+1)}(0)\|_{L^2} \le \|e^{\nu t\Delta}u_{01} - u_{01}\|_{L^2} + \int_0^t \|e^{\nu(t-s)\Delta}(u_1\partial_x u_1 + u_2\partial_y u_1 + \sigma u_1)\|_{L^2} ds \tag{3.18}$$

and

$$\begin{aligned}&\|u_1^{(m+1)}(t) - u_1^{(m+1)}(0)\|_{H^{-\alpha+\epsilon}} \\ &\quad \le \|e^{-t\Delta^2}u_{02} - u_{02}\|_{H^{-\alpha+\epsilon}} + \int_0^t \|e^{-(t-s)\Delta^2}(u_1\partial_x u_2 + u_2\partial_y u_2 + \lambda\Delta u_2)\|_{H^{-\alpha+\epsilon}} ds\end{aligned} \tag{3.19}$$

with

$$t^{\frac{\alpha}{2}}\|u_1^{(m+1)}(t) - u_1^{(m+1)}(0)\|_{H^\alpha} \le t^{\frac{\alpha}{2}}\|e^{\nu t\Delta}u_{01}\|_{H^\alpha} + t^{\frac{\alpha}{2}}\int_0^t \|e^{\nu(t-s)\Delta}(u_1\partial_x u_1 + u_2\partial_y u_1 + \sigma u_1)\|_{H^\alpha} ds \tag{3.20}$$

and

$$\begin{aligned}&t^{\frac{1}{4}(2-\epsilon)}\|u_2^{(m+1)}(t) - u_2^{(m+1)}(0)\|_{H^{2-\alpha}} \\ &\quad \le t^{\frac{1}{4}(2-\epsilon)}\|e^{-t\Delta^2}u_{02}\|_{H^{2-\alpha}} + t^{\frac{1}{4}(2-\epsilon)}\int_0^t \|e^{-(t-s)\Delta^2}(u_1\partial_x u_2 + u_2\partial_y u_2 + \lambda\Delta u_2)\|_{H^{2-\alpha}} ds.\end{aligned} \tag{3.21}$$

Considering the bounds in (3.4) and (3.13), we observe that for terms involving only u_1, the sum of the exponents under the integrals in the bounds is equal to -1, and thus the integrals are constants in time. By Lemma 2.3, these upper bounds decrease to zero when T converges. In (3.6)–(3.10) and (3.14)–(3.17) the sum of the time exponents is strictly greater than -1, so these integrals integrate to t^δ for some $\delta > 0$, which also converges to zero as T converges to zero. Combined with initial conditions being bounded in $L^2 \times H^{-\alpha+\epsilon}$, we conclude that the right hand sides of (3.18)–(3.21) each decrease to 0 as T decreases to 0.

Finally, we claim that the sequence $u^{(m)}$ is contracting. Considering for $n \in \mathbb{N}_0$, we observe that

$$\begin{aligned}\|u^{(m+1)} - u^{(m)}\|_X \leq &\left\| \int_0^t e^{(t-s)A}(u^{(m)} \cdot \nabla u^{(m)} - u^{(m-1)} \cdot \nabla u^{(m-1)})ds \right\|_X \\ &+ \left\| \int_0^t e^{(t-s)A}(L(u^{(m)}) - L(u^{(m-1)}))ds \right\|_X .\end{aligned} \tag{3.22}$$

Expanding out the norms, we find another eight terms analogous to (3.3). Thus, by using similar arguments to (3.4) and neglecting the supremum in time, we have for the first term

$$\begin{aligned}&\int_0^t \|e^{\nu(t-s)\Delta}(u_1^{(m)}\partial_x u_1^{(m)} - u_1^{(m-1)}\partial_x u_1^{(m-1)})\|_{L^2} ds \\ &\quad\lesssim \int_0^t (t-s)^{\alpha-1}\|(u_1^{(m)} - u_1^{(m-1)})(u_1^{(m)} + u_1^{(m-1)})\|_{H^{2\alpha-1}} ds \\ &\quad\lesssim \int_0^t (t-s)^{\alpha-1}\|u_1^{(m)} - u^{(m-1)}\|_{H^\alpha}\|u_1^{(m)} + u_1^{(m-1)}\|_{H^\alpha} ds \\ &\quad\lesssim \sup_{t\leq T}\big(t^{\frac{\alpha}{2}}\|u_1^{(m)} - u_1^{(m-1)}\|_{H^\alpha}\big)\sup_{t\leq T}\big(t^{\frac{\alpha}{2}}\|u_1^{(m)} + u_1^{(m-1)}\|_{H^\alpha}\big)\int_0^t (t-s)^{\alpha-1}s^{-\alpha} ds,\end{aligned} \tag{3.23}$$

which we may bound by $C_T\|u^{(m)} - u^{(m-1)}\|_X/K^2$ allowing the implicit constants to depend on M. Referring to (3.6), we find for the second term that

$$\begin{aligned}&\int_0^t \|e^{\nu(t-s)\Delta}(u_2^{(m)}\partial_y u_1^{(m)} - u_2^{(m-1)}\partial_y u_1^{(m-1)})\|_{L^2} ds \\ &\quad= \int_0^t \|e^{\nu(t-s)\Delta}\big(\partial_y(u_1^{(m)}u_2^{(m)} - u_1^{(m-1)}u_2^{(m-1)}) - (u_1^{(m)}\partial_y u_2^{(m)} - u_1^{(m-1)}\partial_y u_2^{(m-1)})\big)\|_{L^2} ds \\ &\quad\lesssim \int_0^t (t-s)^{-\frac{1}{2}(1-\alpha)}\|(u_1^{(m)}u_2^{(m)} - u_1^{(m-1)}u_2^{(m-1)})\|_{H^\alpha} ds \\ &\qquad+ \int_0^t \|u_1^{(m)}\partial_y u_2^{(m)} - u_1^{(m-1)}\partial_y u_2^{(m-1)}\|_{L^2} ds,\end{aligned}$$

and by adding and subtracting terms and factoring, we conclude that the last expression equals

$$\begin{aligned}&\int_0^t (t-s)^{-\frac{1}{2}(1-\alpha)}\|u_2^{(m)}(u_1^{(m)} - u_1^{(m-1)}) - u_1^{(m-1)}(u_2^{(m)} - u_2^{(m-1)})\|_{H^\alpha} ds \\ &\qquad+ \int_0^t \|(\partial_y u_2^{(m)}(u_1^{(m)} - u_1^{(m-1)}) + u_1^{(m-1)}\partial_y(u_2^{(m)} - u_2^{(m-1)}))\|_{L^2} ds \\ &\quad\lesssim \int_0^t (t-s)^{-\frac{1}{2}(1-\alpha)}\big(\|u_2^{(m)}\|_{H^{2-\alpha}}\|u_1^{(m)} - u_1^{(m-1)}\|_{H^\alpha} + \|u_1^{(m-1)}\|_{H^\alpha}\|(u_2^{(m)} - u_2^{(m-1)})\|_{H^{2-\alpha}}\big) ds \\ &\qquad+ \int_0^t \|(\partial_y u_2^{(m)}\|_{H^{1-\alpha}}\|u_1^{(m)} - u_1^{(m-1)}\|_{H^\alpha} + \|u_1^{(m-1)}\|_{H^\alpha}\|\partial_y(u_2^{(m)} - u_2^{(m-1)}))\|_{H^{1-\alpha}} ds\end{aligned}$$

where we applied Lemma 2.1. We can then bound this by the equation

$$\begin{aligned}\sup_{t\le T}\big(t^{\frac{\alpha}{2}}\|u_1^{(m)}-u_1^{(m-1)}\|_{H^\alpha}\big)\int_0^t (t-s)^{-\frac12(1-\alpha)}s^{-\frac{\alpha}{2}}s^{-\frac14(2-\epsilon)}ds\\ +\sup_{t\le T}\big(t^{\frac14(2-\epsilon)}\|u_2^{(m)}-u_2^{(m-1)}\|_{H^{2-\alpha}}\big)\int_0^t s^{-\frac{\alpha}{2}}s^{-\frac14(2-\epsilon)}ds.\end{aligned} \tag{3.24}$$

The second term may then be bounded by $C_T\|u^{(m)}-u^{(m-1)}\|_X/K^2$. We refer to (3.10) for the fourth term to deduce that

$$\begin{aligned}\int_0^t \|e^{-(t-s)\Delta^2}\partial_y((u_2^{(m)})^2-(u_2^{(m-1)})^2)\|_{H^{-\alpha+\epsilon}}ds &\lesssim \int_0^t \|(u_2^{(m)}-u_2^{(m-1)})(u_2^{(m)}+u_2^{(m-1)})\|_{H^{1-\alpha+\epsilon}}ds\\ &\lesssim \int_0^t \|u_2^{(m)}-u_2^{(m-1)}\|_{H^{2-\alpha}}\|(u_2^{(m)}+u_2^{(m-1)})\|_{H^{2-\alpha}}ds\\ &\lesssim \sup_{t\le T}\big(t^{\frac14(2-\epsilon)}\|u_2^{(m)}-u_2^{(m-1)}\|_{H^{2-\alpha}}\big)\sup_{t\le T}\big(t^{\frac14(2-\epsilon)}\|u_2^{(m)}+u_2^{(m-1)}\|_{H^{2-\alpha}}\big)\int_0^t s^{-\frac12(2-\epsilon)}ds,\end{aligned} \tag{3.25}$$

which is also bounded by $C_T\|u^{(m)}-u^{(m-1)}\|_X/K^2$. A similar result holds for the third term using (3.8). By using arguments found in (3.13)–(3.17), we can also bound the last four terms by $C_T\|u^{(m)}-u^{(m-1)}\|_X/K$. Finally, the linear term in (3.22) is also bounded by $1/12$ for small T by arguments analogous to (3.17). By choosing $K\ge 12C_T$, the constants can be made small. These new choices of K and T are made such that they still satisfy the choices made for uniform boundedness. Therefore, using (3.22)–(3.25) with the analogous results for the remaining terms, and combining with our choices of K and T, we have

$$\|u^{(m+1)}-u^{(m)}\|_X\le\frac12\|u_1^{(m)}-u^{(m-1)}\|_X.$$

Therefore, the sequence is contracting, and $u^{(m)}$ converges by the Contraction Mapping Theorem to a unique, local solution to (1.1), which we denote as u. Since the solution is a fixed point, we furthermore have the solution satisfying

$$u\in C_b([0,T];L^2)\times C_b([0,T];H^{-\alpha+\epsilon})$$

and

$$\big(t^{\frac{\alpha}{2}}u_1,t^{\frac{2-\alpha}{4}}u_2\big)\in C_b([0,T];H^\alpha)\times C([0,T];H^{2-\alpha}).$$

By using bootstrapping, we can repeat the above arguments to show that $u\in C_b((0,T],H^k(\mathbb{T}))^2$ for $k>0$. Thus, the local solution is strong.

Up to this point, we have shown the existence of a strong, unique solution on $[0,T]$. To prove the solution is global, we choose $t_0=T-\delta$ for some $\delta\in(0,T)$. We have shown that $u(t_0)\in H^k(\mathbb{T})^2$ for $k>0$, which satisfies the initial condition regularity requirements of [LY]. Thus, we have a unique, global solution $\tilde u$ with initial condition $\tilde u(t_0)=u(t_0)$. Since $u=\tilde u$ on $[t_0,t_0+\delta]$ by uniqueness the solution u is global. □

For completeness, we also provide a summary of an alternative proof to the main theorem of [LY] using semi-groups instead of Galerkin approximation. In [LY], it was found using approximations that when $u_{01}\in L^\infty$ and $u_0\in(H^1)^2$, a strong, unique, local solution exists. This was then extended to a global

solution using a maximum principle. Under the same assumptions, we define the sequence as in (3.1), but utilize a new space-time norm

$$\|u\|_X = \sup_{t\leq T}\|u\|_{H^1} + \sup_{t\leq T}\left(t^{\frac{1}{2}}\|u_1\|_{H^2}\right) + \sup_{t\leq T}\left(t^{\frac{1}{4}}\|u_2\|_{H^2}\right)$$

to prove the sequence is uniformly bounded with respect to the X-norm. The techniques to prove this have to be adapted however, as while we have stronger regularity, we also have lower exponents under the integral. Attempting to proceed as before results in the sum of the exponents under the integrals being less than -1. The primary alteration to our argument is to use the stronger assumptions on our initial conditions to put less weight onto the higher regularity norms. We expand $u^{(m+1)}$ in a similar manner to (3.2) to find that

$$\|u^{(m+1)}\|_X \leq \|e^{tA}u_0\|_X + \left\|\int_0^t e^{(t-s)A}(u^{(m)}\cdot\nabla u^{(m)})ds\right\|_X + \left\|\int_0^t e^{(t-s)A}L(u^{(m)})ds\right\|_X.$$

Therefore, denoting $u = u^{(m)}$, we find that the center term expands as

$$\begin{aligned}
&\sup_{t\leq T}\int_0^t \|e^{\nu(t-s)\Delta}(u_1\partial_x u_1)\|_{H^1}ds + \sup_{t\leq T}\int_0^t \|e^{\nu(t-s)\Delta}(\partial_y(u_1u_2) - u_1\partial_y u_2)\|_{H^1}ds\\
&\quad + \sup_{t\leq T}\int_0^t \|e^{-(t-s)\Delta^2}(u_1\partial_x u_2)\|_{H^1}ds + \sup_{t\leq T}\int_0^t \|e^{-(t-s)\Delta^2}(u_2\partial_y u_2)\|_{H^1}ds\\
&\quad + \sup_{t\leq T} t^{\frac{1}{2}}\left(\int_0^t \|e^{\nu(t-s)\Delta}(u_1\partial_x u_1)\|_{H^2}ds + \int_0^t \|e^{\nu(t-s)\Delta}(\partial_y(u_1u_2) - u_1\partial_y u_2)\|_{H^2}ds\right)\\
&\quad + \sup_{t\leq T} t^{\frac{1}{4}}\left(\int_0^t \|e^{-(t-s)\Delta^2}(u_1\partial_x u_2)\|_{H^2}ds + \int_0^t \|e^{-(t-s)\Delta^2}(u_2\partial_y u_2)\|_{H^2}ds\right).
\end{aligned} \tag{3.26}$$

We then, for brevity, consider the second term, which is of highest order. Using Hölder and Agmon's inequalities, the second term satisfies

$$\begin{aligned}
\int_0^t \|e^{\nu(t-s)\Delta}u_2\partial_y u_1\|_{H^1}ds &\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\|u_2\partial_y u_1\|_{L^2}ds\\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\|u_2\|_{L^2}^{\frac{1}{3}}\|\nabla u_2\|_{L^2}^{\frac{2}{3}}\|\nabla u_1\|_{L^2}^{\frac{2}{3}}\|\Delta u_1\|_{L^2}^{\frac{1}{3}}ds\\
&\lesssim \sup_{t\leq T}\|u_2\|_{H^1}\sup_{t\leq T}\|u_1\|_{H^1}^{\frac{2}{3}}\sup_{t\leq T}\left(t^{\frac{1}{2}}\|u_1\|_{H^2}\right)^{\frac{1}{3}}\int_0^t (t-s)^{\frac{1}{2}}s^{-\frac{1}{6}}ds.
\end{aligned}$$

Thus, we have

$$\int_0^t \|e^{\nu(t-s)\Delta}u_2\partial_y u_1\|_{H^1}ds \leq C_T\|u\|_X^2, \tag{3.27}$$

where C_T decreases to 0 with T. We observe that the base case is bounded by $M/2$ using Lemma 2.2, for some $M > 0$. Since the remaining terms from (3.26) all proceed similarly, using (3.27), and assuming by induction, the prior step is bounded by M, we find that

$$\|u^{(m+1)}\|_X \leq C_T M^2.$$

Thus, $\|u^{(m)}\|_X \leq M$ for T sufficiently small such that $C_T \leq 1/M$. Continuity follows using arguments as in (3.18)–(3.21). To prove that the sequence is contracting, we observe that

$$\begin{aligned}
&\int_0^t \|e^{\nu(t-s)\Delta}(u_2^{(m)}\partial_y u_1^{(m)} - u_2^{(m-1)}\partial_y u_1^{(m-1)})\|_{H^1} ds \\
&\quad\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\big(\|\partial_y u_1^{(m)}(u_2^{(m)} - u_2^{(m-1)})\|_{L_2} + \|u_2^{(m-1)}\partial_y(u_1^{(m)} - u_1^{(m-1)}))\|_{L_2}\big) ds \\
&\quad\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\big(\|\partial_y u_1^{(m)}\|_{L^3}\|u_2^{(m)} - u_2^{(m-1)}\|_{L^6} + \|u_2^{(m-1)}\|_{L^6}\|\partial_y(u_1^{(m)} - u_1^{(m-1)})\|_{L^3}\big) ds \\
&\quad\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\big(\|\nabla u_1^{(m)}\|_{L_2}^{\frac{2}{3}}\|\Delta u_1^{(m)}\|_{L_2}^{\frac{1}{3}}\|u_2^{(m)} - u_2^{(m-1)}\|_{L_2}^{\frac{1}{3}}\|\nabla(u_2^{(m)} - u_2^{(m-1)})\|_{L_2}^{\frac{2}{3}} \\
&\qquad + \|u_2^{(m-1)}\|_{L_2}^{\frac{1}{3}}\|\nabla u_2^{(m-1)}\|_{L_2}^{\frac{2}{3}}\|\nabla(u_1^{(m)} - u_1^{(m-1)})\|_{L_2}^{\frac{2}{3}}\|\Delta(u_1^{(m)} - u_1^{(m-1)})\|_{L_2}^{\frac{1}{3}}\big) ds,
\end{aligned}$$

which follows from Lemma 2.2, Hölder and Agmon's inequalities. This can be bounded by

$$\begin{aligned}
&\big(\sup_{t\leq T}\big(\|u_1^{(m)}\|_{H^1}\big)^{\frac{2}{3}}\sup_{t\leq T}\big(t^{\frac{1}{2}}\|u_1^{(m)}\|_{H_2}\big)^{\frac{1}{3}}\sup_{t\leq T}\|u_2^{(m)} - u_2^{(m-1)}\|_{H_1} \\
&\quad + \sup_{t\leq T}\|u_2^{(m-1)}\|_{H^1}\sup_{t\leq T}\|u_1^{(m)} - u_1^{(m-1)}\|_{H_1}^{\frac{2}{3}}\sup_{t\leq T}\big(t^{\frac{1}{2}}\|u_1^{(m)} - u_1^{(m-1)}\|_{H_2}\big)^{\frac{1}{3}}\big)\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{6}} ds
\end{aligned}$$

from which we conclude that the term is bounded by

$$\delta(T)M\|u^{(m)} - u^{(m-1)}\|_X,$$

where $\delta(T)$ may be made small for T close to 0. Thus, using analogous results for the remaining terms from the expansion (3.26), we conclude the sequence is contracting for T sufficiently small. Therefore using the Contraction Mapping Principle, we conclude the existence of a unique, strong solution on the torus for local time. Using bootstrapping, the definition of strong solution is satisfied locally. This provides an alternative to the Galerkin approximations for existence of local, strong solutions. In order to obtain the global solution we then proceed as in [LY] by showing that the boundedness of $\|u_1\|_{L^\infty}$ achieved through a maximum principle implies that the solution does not blow up in the H^1 norm.

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