

Global existence for the stochastic Navier-Stokes equations with small L^p data

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ABSTRACT. We consider the stochastic Navier-Stokes equations in $\mathbb{T}^3$ with multiplicative white noise. We construct a unique local strong solution with initial data in L^p , where $p > 5$. We also address the global existence of solutions when the initial data is small in L^p , with the same range of p .

1. Introduction

In this paper we address the global well-posedness of the stochastic Navier-Stokes equation (SNSE)

$$du(t, x) = \nu \Delta u(t, x) dt - \mathcal{P}((u(t, x) \cdot \nabla)u(t, x)) dt + \sigma(u(t, x)) d\mathbb{W}(t), \quad (1.1)$$

$$\nabla \cdot u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{T}^3, \quad (1.2)$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{T}^3, \quad (1.3)$$

on the 3D torus $\mathbb{T}^3 = [-\pi, \pi]^3$. Here, u is the velocity field of a stochastic flow, ν is the viscosity, and $\mathcal{P}$ is the Leray (also called Helmholtz-Hodge) orthogonal projection onto the mean zero divergence-free fields. The stochastic term $\sigma(u)d\mathbb{W}(t)$ denotes an infinite-dimensional and possibly degenerate multiplicative white noise which is understood in the Itô sense. Note that in the formulation (1.1) the pressure term has been eliminated by utilization of the projector $\mathcal{P}$.

The stochastic forcing driving the Navier-Stokes equation represents a perturbation during the flow evolution and thus the SNSE may be argued to be a realistic model for fluids. Consequently, much effort has been devoted to studying its well-posedness theory; cf. [BT, ZBL1, ZBL2, FRS] for results on mild solutions of the SNSE with Lévy-type jump noise, and [BCF, CC, DZ, MS] for results on mild formulation subject to white noise. The existence of a global L^2 martingale solution to the SNSE with Stratonovich noise in $\mathbb{R}^3$ was proven in [MR], and the existence of a martingale solution in $L^{8/d}(0, T; L^4)$, where $d = 2, 3$, was shown in [BF] for the SNSE with noise that is colored in space.

There are fewer available results on the existence of strong (pathwise) solutions, and most were established in a Hilbert space setting. For example, within Hilbert setting, [MeS] proved the global existence of a strong solution for the 2D SNSE with additive white noise. Also, Flandoli [F] proved the result in the 3D case. For the SNSE with multiplicative noise, Fernando and Sritharan showed in [FS] the existence of a global strong solution in 2D unbounded domain in Hilbert spaces, while in [GZ] Glatt-Holtz and the third author established the existence of a maximal strong solution in a 3D bounded domain by assuming the H^1 regularity for the initial data. In [Ki], Kim proved the existence with large probability of a global strong solution to the SNSE with non-degenerate noise and assuming smallness in $H^s(\mathbb{R}^3)$.

Inspired by results on the deterministic Navier-Stokes equation in L^p Banach spaces [FJR, K], we aim to find a global L^p strong solution to (1.1)–(1.3) in three space dimensions. We show that a unique

solution exists for small initial velocity. To be precise, we prove

$$\mathbb{P}(\tau_M < \infty) \geq 1 - C_p^{-1} M^{-p} \mathbb{E}[\|u_0\|_p^p],$$

where $\tau_M = \inf\{t > 0 : \|u(s)\|_p > M\}$ and u evolves in $L^p(\Omega, C((0, \infty), L^p))$. Note that we do not impose any structural assumptions on the multiplicative noise, allowing it to be degenerate. In [BR], Barbu and Röckner obtained the existence and uniqueness of a global mild solution in L^p ($3/2 < p < 2$) to the vorticity equation associated with the SNSE. They worked with a convolution-type finite-dimensional noise and a small initial vorticity. The convolution structure is needed for obtaining a commutative C_0 noise operator, which is essential for transforming the vorticity equation into a random equation. Also, in [GV], Glatt-Holtz and Vicol used linear noise to treat the 3D stochastic Euler equation.

As has been shown in many existing results, a major obstacle when seeking global solutions is a combination of a multiplicative noise and the convective nonlinearity $(u \cdot \nabla)u$. To overcome this difficulty, authors usually introduce stopping times of ascending u -norms to in a sense linearize this term in a specified function space. The stopping time argument proves to be a powerful tool for obtaining local existence, but showing the non-degeneracy of these stopping times is a major problem. In this paper, we truncate $(u \cdot \nabla)u$ at $\|u\|_p = \delta_0$ for some $\delta_0 > 0$. To obtain a global solution we use a stochastic heat equation (SHE) with additive noise (see [R]) to prove the existence of a global solution. Then we fix $\delta_0 > 0$ sufficiently small and estimate the probability distribution of $e^{at}\|u\|_p^p$ for a small $a > 0$. We show that (1.1) agrees with a truncated SNSE for all time on a large part of the probability space if the initial velocity is small, obtaining thus a global solution to (1.1)–(1.3). To prove the existence of a local strong solution up to a stopping time, we send $\delta \rightarrow \infty$ along integer values and use uniqueness of solutions.

Working in a function space of low regularity imposes several challenges. First, we need to obtain a global L^p solution to the SHE exists and then adapt it to the stochastic truncated SNSE. Considering that [R] only provides a W_p^m solution to the SHE (see Chapter 4 of [R]) and the W_p^m estimate obtained in [R] does not support the L^p convergence of approximating solutions, we extend [R] to obtain a L^p a priori estimate for the SHE. Next, the regularity of the drift function in this L^p estimate must be strictly less than L^p , because the drift corresponds to $(u \cdot \nabla)u$ when one relates the truncated SNSE to the SHE in the fixed point argument, and $(u \cdot \nabla)u$ is less regular than u itself. We utilize the dissipative term to make such an estimate possible. But at a cost, the use of the dissipative term generates a non-linear term $|\nabla(|u|^{p/2})|^2$, which prevents convergence in the strong topology. Hence, we resort to the weak lower-semicontinuity of Hilbert norms and fulfill the requirement of passing the limit for this term (cf. Lemma 4.4 below). The third difficulty is due to the structure of $(u \cdot \nabla)u$ and the introduced truncation. To overcome this, we apply the fixed point iteration twice. The main trick is to introduce a square of the cut-off, which allows us to treat the difference via a special splitting (cf. (5.19)–(5.20) below). Note that a high-regularity truncation on the SNSE is required by the iteration, while a low-regularity norm is preferable for showing convergence of the iterated solutions. Overall, we are able to obtain convergence when $p > 5$. It would be desirable to obtain our theorems in the range $p > 3$ and for $p \geq 3$ for small data (as in [K] in the deterministic case), but this remains open (for the case of additive noise, see however [MS]).

The paper is organized as follows. In Section 2, we introduce the notation and preliminaries on stochastic calculus. In Section 3, we state our assumptions and main results. Theorems on the SHE are collected in Section 4. The global existence and uniqueness of a strong solution to the truncated SNSE is established in Section 5, where we also obtain the local existence of solutions up to a stopping time. The global existence of solutions for small data is obtained in Section 6.

2. Notation and Preliminaries

2.1. Basic Notation. Let $T \in (0, \infty]$. For a scalar function $u = u(t, x)$ on $[0, T) \times \mathbb{T}^3$, we denote its partial derivatives by $\partial_t u = \partial u / \partial t$, $\partial_i u = \partial u / \partial x_i$, and $\partial_{ij}^2 u = \partial^2 u / \partial x_i \partial x_j$. Also, we denote its gradient with respect to x by $\nabla u = (\partial_1 u, \dots, \partial_d u)$, while we write $D^\gamma u = \partial^{|\gamma|} u / \partial x_1^{\gamma_1} \dots \partial x_d^{\gamma_d}$ for a multi-index $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}_0^d$.

We use $C^\infty(\mathbb{T}^3)$ for the set of infinitely differentiable functions on $\mathbb{T}^3$ and $\mathcal{D}'(\mathbb{T}^3)$ for the space of distributions $(C^\infty(\mathbb{T}^3))'$. Note that we have $C^\infty(\mathbb{T}^3) \subseteq L^p(\mathbb{T}^3) \subseteq \mathcal{D}'(\mathbb{T}^3)$ for $1 \leq p \leq \infty$. The usual L^p norms are denoted by $\|\cdot\|_p$.

The m -th Fourier coefficient of an L^1 function f on $\mathbb{T}^3$ is defined as

$$\mathcal{F}f(m) = \hat{f}(m) := \int_{\mathbb{T}^3} f(x) e^{-2\pi i m \cdot x} dx, \quad m \in \mathbb{Z}^3,$$

and the corresponding Fourier series (Fourier inversion) of g at $x \in \mathbb{T}^3$ is

$$(\mathcal{F}^{-1}g)(x) = \sum_{m \in \mathbb{Z}^3} g(m) e^{2\pi i m \cdot x}.$$

Recall that $\mathcal{F}$ can be extended to $\mathcal{D}'(\mathbb{T}^3)$ and $\mathcal{F}^{-1}\mathcal{F} = \text{Id}$ on $\mathcal{D}'(\mathbb{T}^3)$. For $s \in \mathbb{R}$ and $f \in \mathcal{D}'(\mathbb{T}^3)$, we denote

$$J^s f(x) = \sum_{m \in \mathbb{Z}^3} (1 + 4\pi^2 |m|^2)^{s/2} \hat{f}(m) e^{2\pi i m \cdot x}, \quad x \in \mathbb{T}^3,$$

and

$$\partial^s f(x) = \sum_{m \in \mathbb{Z}^3} |m|^s \hat{f}(m) e^{2\pi i m \cdot x}, \quad x \in \mathbb{T}^3,$$

and we define $W^{s,p}(\mathbb{T}^3)$ to be the class of functions $f \in \mathcal{D}'(\mathbb{T}^3)$ such that

$$\|f\|_{s,p} := \|J^s f\|_p < \infty, \quad s \in \mathbb{R}, \quad p > 1.$$

For the L^2 based spaces, we abbreviate $H^s(\mathbb{T}^3) = W^{s,2}(\mathbb{T}^3)$. Recall that there exists a positive constant C independent of f such that

$$\frac{1}{C} \|f\|_{s,p} \leq \|f\|_p + \|\partial^s f\|_p \leq C \|f\|_{s,p}, \quad s \geq 0, \quad 1 < p < \infty.$$

The Leray orthogonal projection $\mathcal{P}$ is defined by

$$\widehat{(\mathcal{P}u)}_j(m) = \sum_{k=1}^d \left(\delta_{jk} - \frac{m_j m_k}{|m|^2} \right) \widehat{u}_k(m), \quad j = 1, 2, \dots, d. \quad (2.1)$$

Using the Riesz transforms

$$R_j = -\frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}}, \quad j = 1, 2, \dots, d,$$

the equation (2.1) for $\mathcal{P}$ may be rewritten as

$$(\mathcal{P}u)_j(x) = \sum_{k=1}^d (\delta_{jk} + R_j R_k) u_k(x), \quad j = 1, 2, \dots, d,$$

from where

$$((I - \mathcal{P})u)_j(x) = -\sum_{k=1}^d R_j R_k u_k(x), \quad j = 1, 2, \dots, d.$$

For convenience we write

$$W_{\text{sol}}^{s,p} = \{\mathcal{P}f : f \in W^{s,p}\}. \quad (2.2)$$

As usual, C represents a generic positive constant, whose value may increase from line to line, with explicit dependence indicated when necessary. We consider p fixed, so C is allowed to depend on p without an explicit mention.

2.2. Preliminaries on stochastic analysis. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a complete probability space with an augmented filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by a cylindrical Brownian motion $\mathbb{W}$ on $(\mathcal{F}_t)_{t \geq 0}$. We assume that $\mathbb{W}$ is an $\mathcal{H}$ -valued process for some real separable Hilbert space $\mathcal{H}$, which may be infinite dimensional. Choosing a complete orthonormal basis $\{\mathbf{e}_k\}_{k \geq 1}$ for $\mathcal{H}$, we may formally write $\mathbb{W}(t, \omega) = \sum_{k \geq 1} W_k(t, \omega) \mathbf{e}_k$, where $\{W_k : k \in \mathbb{N}\}$ is a collection of mutually independent 1D Brownian motions.

Let $\mathcal{Y}$ be another real separable Hilbert space. Denote by $l^2(\mathcal{H}, \mathcal{Y})$ the set of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{Y}$, i.e., $G \in l^2(\mathcal{H}, \mathcal{Y})$ if and only if G is a linear bounded operator mapping from $\mathcal{H}$ to $\mathcal{Y}$ such that

$$\|G\|_{l^2(\mathcal{H}, \mathcal{Y})}^2 := \sum_{k=1}^{\dim \mathcal{H}} |G\mathbf{e}_k|_{\mathcal{Y}}^2 < \infty.$$

In our context, $\mathcal{Y}$ denotes either $\mathbb{R}$ or $\mathbb{R}^d$, and $\|\cdot\|_{l^2}$ is used interchangeably for $\|\cdot\|_{l^2(\mathcal{H}, \mathbb{R})}$ and $\|\cdot\|_{l^2(\mathcal{H}, \mathbb{R}^d)}$ when there is no risk of confusion. Note that any operator in $l^2(\mathcal{H}, \mathcal{Y})$ is compact and $l^2(\mathcal{H}, \mathcal{Y})$ is a separable Hilbert space endowed with a scalar product defined by

$$(A, B)_{l^2(\mathcal{H}, \mathcal{Y})} := \sum_{k=1}^{\dim \mathcal{H}} (A\mathbf{e}_k, B\mathbf{e}_k)_{\mathcal{Y}}, \quad A, B \in l^2(\mathcal{H}, \mathcal{Y}).$$

Next, by the Burkholder-Davis-Gundy (BDG) inequality, for $G \in l^2(\mathcal{H}, \mathcal{Y})$ and $1 \leq p < \infty$,

$$\mathbb{E} \left(\sup_{s \in [0, t]} \left| \int_0^s G d\mathbb{W}_r \right|_{\mathcal{Y}}^p \right) \leq C \mathbb{E} \left(\int_0^t \|G\|_{l^2(\mathcal{H}, \mathcal{Y})}^2 dr \right)^{p/2}.$$

Using this fact and letting $(J^s f)\mathbf{e}_k = J^s(f\mathbf{e}_k)$, we introduce Banach spaces

$$\mathbb{W}^{s, p} := \left\{ f : \mathbb{T}^3 \rightarrow l^2(\mathcal{H}, \mathcal{Y}) : f\mathbf{e}_k \in W^{s, p}(\mathbb{T}^3) \text{ for each } k, \text{ and } \int_{\mathbb{T}^3} \|J^s f\|_{l^2(\mathcal{H}, \mathcal{Y})}^p dx < \infty \right\},$$

with respect to the norm

$$\|f\|_{\mathbb{W}^{s, p}} := \left(\int_{\mathbb{T}^3} \|J^s f\|_{l^2(\mathcal{H}, \mathcal{Y})}^p dx \right)^{1/p},$$

for $s \geq 0$ and $1 < p < \infty$. Also, $\mathbb{W}^{0, p}$ is abbreviated as $\mathbb{L}^p$. Letting $(\mathcal{P}f)\mathbf{e}_k = \mathcal{P}(f\mathbf{e}_k)$, where $\mathcal{P}$ is the Leray projector, we have $\mathcal{P}f \in \mathbb{W}^{s, p}$ if $f \in \mathbb{W}^{s, p}$. Define

$$\mathbb{W}_{\text{sol}}^{s, p} = \{\mathcal{P}f : f \in \mathbb{W}^{s, p}\}.$$

We assume for (1.1) that σ maps $W_{\text{sol}}^{s, p}$ into $\mathbb{W}_{\text{sol}}^{s, p}$, where $W_{\text{sol}}^{s, p}$ was introduced in (2.2).

3. Assumptions and Main Results

We seek a strong (pathwise) solution to (1.1)–(1.3) in $L^p(\mathbb{T}^3)$ for p sufficiently large. Here, we say a solution to a stochastic partial differential equation (SPDE) is strong if, with probability one, it satisfies the SPDE in the distribution sense and it evolves in the designated function space (cf. [GZ, GV, Kr] and references therein). This notion demonstrates a pathwise behavior rather than a law property, which distinguishes it from the martingale solution whose probability law fits the equation.

Suppose σ and g are $(l^2(\mathcal{H}, \mathbb{R}))^d$ -valued operators, namely, σ and g have d components and each component is an $l^2(\mathcal{H}, \mathbb{R})$ -valued. Let A be an operator that is usually unbounded and

$$u(t, x) = u_0(x) + \int_0^t (Au(r, x) + f(r, x)) dr + \int_0^t (\sigma(u(r, x)) + g(r, x)) d\mathbb{W}(r), \quad (3.1)$$

a d -dimensional stochastic evolution partial differential equation on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Different notions of solutions are defined as follows.

DEFINITION 3.1 (Local Strong Solution). A pair (u, τ) is a local strong L^p solution to (3.1) if τ is a positive stopping time $\mathbb{P}$ -almost surely, the stochastic process u is adapted with respect to $\mathcal{F}_t$ and is in $L^p(\Omega; C([0, \tau \wedge T], L^p))$, and u satisfies

$$(u(t), \phi) = (u_0, \phi) + \int_0^t ((Au(r) + f(r)), \phi) dr + \int_0^t ((\sigma u(r) + g(r)), \phi) d\mathbb{W}(r) \text{ a.e. } (t, \omega), \quad (3.2)$$

for all $\phi \in C^\infty(\mathbb{T}^3)$.

The term $(Au(r), \phi)$ in our application below is interpreted using integration by parts.

DEFINITION 3.2 (Maximal Strong Solution). A pair (u, τ) is said to be a maximal strong L^p solution to (3.1) if there exists an increasing sequence of stopping times τ_n with $\tau_n \uparrow \tau$ a.s. such that each pair (u, τ_n) is a local strong solution,

$$\sup_{0 \leq t \leq \tau_n} \|u(t)\|_p + \int_0^{\tau_n} \int_{\mathbb{T}^3} |\nabla(|u(t)|^{p/2})|^2 dx dt < \infty,$$

and

$$\sup_{0 \leq t \leq \tau} \|u(t)\|_p + \int_0^\tau \int_{\mathbb{T}^3} |\nabla(|u(t)|^{p/2})|^2 dx dt = \infty,$$

on the set $\{\tau \leq T\}$.

For the local existence, we assume

$$\sum_{i=1}^3 \|\sigma_i(u)\|_{\mathbb{L}^p} = \sum_{i=1}^3 \left(\int_{\mathbb{T}^3} \|\sigma_i(u)\|_{l_2^p}^p dx \right)^{1/p} \leq C(\|u\|_p + 1) \quad (3.3)$$

and

$$\sum_{i=1}^3 \|\sigma_i(u) - \sigma_i(v)\|_{\mathbb{L}^p} \leq C\|u - v\|_p. \quad (3.4)$$

The following statement is the main result on the local existence of strong solutions.

THEOREM 3.1. (*Local strong solution up to a stopping time*) Let $p > 5$ and $u_0 \in L^p(\Omega; L^p)$. Then there exists a unique maximal strong solution (u, τ) to (1.1)–(1.3) such that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} \|u(s, \cdot)\|_p^p + \int_0^T \sum_j \int_{\mathbb{T}^3} |\partial_j(|u_j(s, x)|^{p/2})|^2 dx ds \right] \leq C\mathbb{E}[\|u_0\|_p^p + 1],$$

where $C > 0$ is a constant depending on p .

The theorem is proven at the end of Section 5 below.

In the next statement, we address the global existence of solutions, for which we impose, in addition to (3.4), a superlinearity assumption

$$\sum_{i=1}^3 \|\sigma_i(u)\|_{\mathbb{L}^p} \leq \epsilon_0 \|u\|_p, \quad (3.5)$$

where $\epsilon_0 > 0$.

THEOREM 3.2. (*Global strong solution for small data*) Suppose that (3.4) and (3.5) hold with $\epsilon_0 \in (0, 1]$ sufficiently small. Let (u, τ) be the solution provided in Theorem 3.1. For every $\epsilon \in (0, 1]$ there exists $\delta > 0$ such that if

$$\mathbb{E}[\|u_0\|_p^p] \leq \delta, \quad (3.6)$$

then

$$\mathbb{P}(\tau = \infty) \geq 1 - \epsilon.$$

The proof of Theorem 3.2 is given in Section 6.

4. Stochastic heat equation on the torus

In this section, we prove the global existence of an L^p solution to the stochastic heat equation

$$du(t, x) = \Delta u(t, x) dt + f(t, x) dt + g(t, x) d\mathbb{W}(t), \quad (4.1)$$

$$u(0, x) = u_0(x) \text{ a.s.}, \quad t \in (0, T), \quad x \in \mathbb{T}^d \quad (4.2)$$

on $\mathbb{T}^d$, where $d \in \mathbb{N}$. The white noise $\mathbb{W}$ was introduced above, the drift f is a predictable process evolving in $W^{-1,p}$, the noise coefficient g takes values in $l^2(\mathcal{H}, \mathbb{R})$, and u_0 is $\mathcal{F}_0$ -measurable.

Using the terminology in [R], the equation (4.1) is super-parabolic. Also, the a priori estimates for Theorems 4.1.2 and 4.1.4 in [R] remain true on the torus without change. Thus if $u_0 \in L^{p'}(\Omega; W^{m,p'})$, $f \in L^{p'}(\Omega \times [0, T], W^{m,p'})$, and $g \in L^{p'}(\Omega \times [0, T], \mathbb{W}^{m,p'})$ for some $m \in \mathbb{N}$ and $p' \geq 2$, then there exists $u \in L^{p'}(\Omega \times [0, T]; C_{\text{weak}} W^{m,p'})$ satisfying (4.1)–(4.2) in the sense of (3.2). If in addition $(m - k)p > d$, then u has a version that belongs to $C_b^{0,k}([0, T] \times \mathbb{T}^d)$ $\mathbb{P}$ -almost surely. This conclusion of global existence relies on a high regularity of the forcing term, which needs to be relaxed to apply to the Navier-Stokes equations.

THEOREM 4.1. *Let $2 \leq p < \infty$ and $0 < T \leq \infty$. Suppose that $u_0 \in L^p(\Omega, L^p(\mathbb{T}^d))$, $f \in L^p(\Omega \times [0, T], W^{-1,p}(\mathbb{T}^d))$, and $g \in L^p(\Omega \times [0, T], \mathbb{L}^p(\mathbb{T}^d))$, where*

$$\frac{dp}{p + d - 2} < q \leq p, \quad (4.3)$$

provided $d \geq 2$ or $1 < q \leq p$ if $d = 1$. Then there exists a unique maximal solution $u \in L^p(\Omega; C([0, T], L^p))$ to (4.1)–(4.2) such that

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_p^p + \int_0^T \int_{\mathbb{T}^d} |\nabla(|u(t, x)|^{p/2})|^2 dx dt \right] \\ & \leq C \mathbb{E} \left[\|u_0\|_p^p + \int_0^T \|f(t, \cdot)\|_{-1,q}^p dt + \int_0^T \|g^\varepsilon(r)\|_{\mathbb{L}^p}^p dt \right], \end{aligned} \quad (4.4)$$

where $C > 0$ depends on p .

Recall that we use the notation

$$\|g^\varepsilon(r)\|_{\mathbb{L}^p}^p = \int_{\mathbb{T}^d} \|g(t, x)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx.$$

Introduce the standard convolution function $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\text{supp } \rho \subseteq \{x \in \mathbb{R}^d : |x| \leq \pi/2\}$ and $\int_{\mathbb{R}^d} \rho(x) dx = 1$. Assume also that ρ is nonnegative and radial. Set $\rho_\varepsilon = \varepsilon^{-d} \rho(\cdot/\varepsilon)$.

The next lemma is needed when approximating the forcing term in (4.1).

LEMMA 4.2. *Let $q \in (1, \infty)$. If $f \in W^{-1,q}(\mathbb{T}^d)$, then $f * \rho_\varepsilon \rightarrow f$ in $W^{-1,q}(\mathbb{T}^d)$ as $\varepsilon \rightarrow 0$.*

PROOF OF LEMMA 4.2. The mapping $S = -\Delta + I$ is a Banach space isomorphism $S: W^{1,q}(\mathbb{T}^d) \rightarrow W^{-1,q}(\mathbb{T}^d)$, which commutes with the convolution operator. Thus the statement follows by applying S to $(S^{-1}f) * \rho_\varepsilon \rightarrow S^{-1}f$ in $W^{1,q}$. $\square$

REMARK 4.3. Note that the above proof implies that if $f \in L^p(\Omega \times [0, T], W^{-1,p})$, then $f * \rho_\varepsilon \rightarrow f$ in $L^p(\Omega \times [0, T], W^{-1,p})$.

The following lemma is essential when passing to the limit in the inequality (4.4).

LEMMA 4.4. *Let $p \geq 2$. If*

$$u_n \rightarrow u \text{ in } L^p(\Omega; L^\infty([0, T], L^p)) \text{ as } n \rightarrow \infty$$

and

$$\nabla(|u_n(\omega, t, x)|^{p/2}) \text{ are uniformly bounded in } L^2(\Omega \times [0, T], L^2), \quad (4.5)$$

then

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_n(\omega, t, x)|^{p/2})|^2 dx dt \right] \geq \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u(\omega, t, x)|^{p/2})|^2 dx dt \right]. \quad (4.6)$$

PROOF OF LEMMA 4.4. First, there exists a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ of $\{u_n\}_{n \in \mathbb{N}}$ such that

$$\lim_k \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_{n_k}(\omega, t, x)|^{p/2})|^2 dx dt \right] = \liminf_n \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_n(\omega, t, x)|^{p/2})|^2 dx dt \right]. \quad (4.7)$$

Observe that, by (4.7), it suffices to prove (4.6) for a subsequence of $\{u_{n_k}\}_k$. For simplicity of notation, relabel $\{u_{n_k}\}_k$ as $\{u_n\}_n$. Note that $|u_n|^{p/2} \rightarrow |u|^{p/2}$ in $L^2(\Omega; L^\infty([0, T], L^2))$ and thus also

$$|u_n|^{p/2} \rightarrow |u|^{p/2} \text{ in } L^2(\Omega \times [0, T], L^2).$$

By (4.5), we may pass to a subsequence and assume that

$$\nabla(|u_n(\omega, t, x)|^{p/2}) \rightarrow g \text{ weakly in } L^2(\Omega \times [0, T], L^2) \text{ as } n \rightarrow \infty,$$

for some $g \in L^2(\Omega \times [0, T], L^2)$, which also implies

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_n(\omega, t, x)|^{p/2})|^2 dx dt \right] \geq \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |g|^2 dx dt \right]$$

by the weak lower-semicontinuity of the Hilbert norm. In order to obtain (4.6), we only need to prove that g and $\nabla(|u|^{p/2})$ agree as elements in $L^2(\Omega \times [0, T], L^2)$. To prove this, let $\varphi \in C^\infty(\mathbb{T}^d)$ be arbitrary. Then, for all $j = 1, \dots, d$, we have

$$(g_j, \varphi) = \lim_n (\partial_j(|u_n|^{p/2}), \varphi) = -\lim_n (|u_n|^{p/2}, \partial_j \varphi) = -(|u|^{p/2}, \partial_j \varphi) = (\partial_j(|u|^{p/2}), \varphi),$$

where $(\cdot, \cdot)$ represents the inner product on $L^2(\Omega \times [0, T], L^2)$. Thus we obtain that g and $\nabla(|u|^{p/2})$ agree in $L^2(\Omega \times [0, T], L^2)$, as claimed. $\square$

PROOF OF THEOREM 4.1. Denote $u_0^\varepsilon = u_0 * \rho_\varepsilon$, $f^\varepsilon = f * \rho_\varepsilon$, and $g^\varepsilon = g * \rho_\varepsilon$. By Young's convolution inequality and the Sobolev embedding, we have $u_0^\varepsilon \in L^{p'}(\Omega; W^{m, p'})$ and $f^\varepsilon \in L^{p'}(\Omega \times [0, T], W^{m, p'})$ for $m \in \mathbb{N}$ and $2 \leq p' \leq q$ as well as $g^\varepsilon \in L^{p'}(\Omega \times [0, T], \mathbb{W}^{m, p'})$ for $m \in \mathbb{N}$ and $2 \leq p' < \infty$. Note that $u_0^\varepsilon \rightarrow u_0$ in $L^p(\Omega, L^p)$, $f^\varepsilon(t, \cdot) \rightarrow f(t, \cdot)$ in $L^p(\Omega \times [0, T], W^{-1, q})$, and $g^\varepsilon(t, \cdot) \rightarrow g(t, \cdot)$ in $L^p(\Omega \times [0, T], \mathbb{L}^p)$ as $\varepsilon \rightarrow 0$. Now, consider

$$du^\varepsilon(t, x) = \Delta u^\varepsilon(t, x) dt + f^\varepsilon(t, x) dt + g^\varepsilon(t, x) d\mathbb{W}(t), \quad (4.8)$$

$$u^\varepsilon(0, x) = u_0^\varepsilon(x) \text{ a.s.} \quad (4.9)$$

Clearly, assumptions of Theorem 4.1.4 in [R] are fulfilled. Therefore, there exists $u^\varepsilon \in L^p(\Omega \times [0, T], W^{m, p})$ satisfying (4.8)–(4.9) in the sense of Definition 3.1. By Corollary 4.1.4 in [R], u^ε has a modification that belongs to $C_b^{0, n}([0, T] \times \mathbb{T}^d)$ $\mathbb{P}$ -a.s. if $m > n + d/p$. We shall choose m sufficiently large and use the continuously differentiable modification for u^ε .

Applying the Itô formula to $h(y) = |y|^p$ with $y = u^\varepsilon(t, x)$ and $p \geq 2$, we get

$$\begin{aligned} |u^\varepsilon(t)|^p &= |u_0^\varepsilon|^p + p \int_0^t |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) (\Delta u^\varepsilon(r) + f^\varepsilon(r)) dr + p \int_0^t |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) g^\varepsilon(r) d\mathbb{W}_r \\ &\quad + \frac{p(p-1)}{2} \int_0^t \int |u^\varepsilon(r)|^{p-2} \|g^\varepsilon(r)\|_{l^2(\mathcal{H}, \mathbb{R})}^2 dx dr. \end{aligned}$$

We integrate both sides of the equation with respect to x and apply the stochastic Fubini theorem obtaining

$$\begin{aligned} \|u^\varepsilon(t)\|_p^p &= \|u_0^\varepsilon\|_p^p + p \int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) (\Delta u^\varepsilon(r) + f^\varepsilon(r)) dx dr \\ &\quad + p \int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) g^\varepsilon(r) dx d\mathbb{W}_r \\ &\quad + \frac{p(p-1)}{2} \int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} \|g_i^\varepsilon(r)\|_{l^2}^2 dx dr. \end{aligned} \quad (4.10)$$

For the dissipative term, we have

$$\begin{aligned} p \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) \Delta u^\varepsilon(r) dx \\ = -p(p-1) \int_{\mathbb{T}^d} |u^\varepsilon|^{p-2} |\nabla u^\varepsilon(r)|^2 dx = -\frac{4(p-1)}{p} \int_{\mathbb{T}^d} |\nabla |u^\varepsilon|^{p/2}|^2 dx. \end{aligned} \quad (4.11)$$

It then follows from (4.10) and (4.11) that

$$\begin{aligned} \|u^\varepsilon(t)\|_p^p &+ \frac{4(p-1)}{p} \int_0^t \int_{\mathbb{T}^d} |\nabla (|u^\varepsilon(r)|^{p/2})|^2 dx dr \\ &\leq \|u_0^\varepsilon\|_p^p + p \int_0^t \left| \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) f^\varepsilon(r) dx \right| dr \\ &\quad + \frac{p(p-1)}{2} \int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} \|g^\varepsilon(r)\|_{l^2}^2 dx dr + p \left| \int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) g^\varepsilon(r) dx d\mathbb{W}_r \right| \\ &= \|u_0^\varepsilon\|_p^p + I_1 + I_2 + I_3. \end{aligned} \quad (4.12)$$

With $q' = q/(q-1)$, we have

$$I_2 \leq C \|f^\varepsilon(r)\|_{-1,q} \| |u^\varepsilon|^{p-2} u^\varepsilon \|_{1,q'} \leq C \|f^\varepsilon(r)\|_{-1,q} (\| |u^\varepsilon|^{p-2} u^\varepsilon \|_{q'} + \|\nabla (|u^\varepsilon|^{p-2} u^\varepsilon)\|_{q'}), \quad (4.13)$$

where, recall, we allow that C depends on p throughout. Note that

$$\| |u^\varepsilon|^{p-2} u^\varepsilon \|_{1,q'} \leq C \| |u^\varepsilon|^{p-2} u^\varepsilon \|_{q'} + C \|\nabla (|u^\varepsilon|^{p-2} u^\varepsilon)\|_{q'}. \quad (4.14)$$

Since $\int_{\mathbb{T}^d} u^\varepsilon = 0$, we have, as in [KZ], a Poincaré type inequality

$$\| |u^\varepsilon|^{p-2} u^\varepsilon \|_{q'} \leq C \|\nabla (|u^\varepsilon|^{p-2} u^\varepsilon)\|_{q'} \quad (4.15)$$

when $p, q \in (1, \infty)$. In [KZ, Lemma 3] the inequality

$$\| |u^\varepsilon|^{p-1} \|_{q'} \leq C \|\nabla (|u^\varepsilon|^{p-1})\|_{q'} \quad (4.16)$$

was proven, but the same proof works for (4.15) as well. By (4.13), (4.14) and (4.15), we have

$$I_2 \leq C \|f^\varepsilon(r)\|_{-1,q} \| |u^\varepsilon|^{p-2} u^\varepsilon \|_{1,q'} \leq C \|f^\varepsilon(r)\|_{-1,q} \|\nabla (|u^\varepsilon|^{p-2} u^\varepsilon)\|_{q'}. \quad (4.17)$$

Now, note that

$$\begin{aligned} C \|\nabla (|u^\varepsilon|^{p-2} u^\varepsilon)\|_{q'} &\leq C \| |u^\varepsilon|^{p/2-1} \nabla (|u^\varepsilon|^{p/2}) \|_{q'} \leq C \| |u^\varepsilon|^{p/2-1} \|_{\bar{r}} \|\nabla (|u^\varepsilon|^{p/2})\|_2 \\ &= C \|u^\varepsilon\|_{\bar{r}(p-2)/2}^{(p-2)/2} \|\nabla (|u^\varepsilon|^{p/2})\|_2 = C \| |u^\varepsilon|^{p/2} \|_{\bar{r}(p-2)/p}^{(p-2)/p} \|\nabla (|u^\varepsilon|^{p/2})\|_2, \end{aligned} \quad (4.18)$$

where $1/\bar{r} + 1/2 = 1/q'$, i.e.,

$$\frac{1}{\bar{r}} + \frac{1}{q} = \frac{1}{2}. \quad (4.19)$$

(The assumptions on the exponents p and q imply $q > 2$.) It is easy to check that the condition (4.3) implies

$$2 \leq \frac{\bar{r}(p-2)}{p} < \frac{2d}{d-2},$$

when $d \geq 2$, and thus we get from (4.18) and the Gagliardo-Nirenberg inequality

$$\|w\|_{\bar{r}(p-2)/p} \leq C \|w\|_2^{1-\alpha} \|\nabla w\|_2^\alpha,$$

where $\alpha = d(1/2 - p/\bar{r}(p-2))$, with $w = |u^\varepsilon|^{p/2}$, we get

$$C \|\nabla(|u^\varepsilon|^{p-2} u^\varepsilon)\|_{q'} \leq C \| |u^\varepsilon|^{p/2} \|_{\bar{r}(p-2)/p}^{(1-\alpha)(p-2)/p} \|\nabla(|u^\varepsilon|^{p/2})\|_2^{1+\alpha(p-2)/p}. \quad (4.20)$$

From (4.17)–(4.20), we thus obtain

$$\begin{aligned} I_2 &\leq C \left| \int \|f^\varepsilon(r)\|_{-1,q} \| |u^\varepsilon|^{p/2} \|_{\bar{r}(p-2)/p}^{(1-\alpha)(p-2)/p} \|\nabla(|u^\varepsilon|^{p/2})\|_2^{1+\alpha(p-2)/p} dr \right| \\ &\leq \delta \int_0^t \|\nabla(|u^\varepsilon|^{p/2})\|_2^2 dr + \delta \sup_{0 \leq r \leq t} \|u(t, \cdot)\|_p^p + C_\delta \int_0^t \|f^\varepsilon(r)\|_{-1,q}^p dr \end{aligned} \quad (4.21)$$

with $\delta > 0$ arbitrarily small, where we applied Young's inequality in the last step. Next, for the term I_2 in (4.12), we write

$$I_2 = \frac{p(p-1)}{2} \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} \|g^\varepsilon(r)\|_{l^2}^2 dx \leq \delta \|u^\varepsilon(r)\|_p^p + C_\delta \|g^\varepsilon(r)\|_{\mathbb{L}^p}^p.$$

Finally, we consider the last term in (4.12). Using Minkowski's inequality, we have

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^T \left\| \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) g^\varepsilon(r) dx \right\|_{l^2}^2 dr \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\left(\int_0^T \left(\int_{\mathbb{T}^d} \| |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) g^\varepsilon(r) \|_{l^2} dx \right)^2 dr \right)^{1/2} \right] \\ &= \mathbb{E} \left[\left(\int_0^T \left(\int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-1} \|g^\varepsilon(r)\|_{l^2} dx \right)^2 \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\left(\sup_{r \in [0,T]} \|u^\varepsilon(r)\|_p^{p/2} \int_0^T \left(\int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p/2-1} \|g^\varepsilon(r)\|_{l^2} dx \right)^2 \right)^{1/2} \right], \end{aligned} \quad (4.22)$$

where we abbreviated $l^2 = l^2(\mathcal{H}, \mathbb{R}^d)$. Therefore,

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^T \left\| \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) g^\varepsilon(r) dx \right\|_{l^2}^2 dr \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\left(\sup_{r \in [0,T]} \|u^\varepsilon(r)\|_p^{p/2} \int_0^T \left(\int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} \|g^\varepsilon(r)\|_{l^2}^2 dx \right) \right)^{1/2} \right] \\ &\leq \frac{1}{8p} \mathbb{E} \left[\sup_{r \in [0,T]} \|u^\varepsilon(r)\|_p^p \right] + C \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} \|g(t, x)\|_{l^2}^2 dx dt \right] \\ &\leq \frac{1}{4p} \mathbb{E} \left[\sup_{r \in [0,T]} \|u^\varepsilon(r)\|_p^p \right] + C \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} \|g(t, x)\|_{l^2}^p dx dt \right] + C, \end{aligned}$$

where we used Young's inequality and the sublinearity assumption (3.3) in the last step. Note that the far right side is finite by (4.1.21) in [R]. Thus, from the BDG inequality

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(r)|^{p-2} u^\varepsilon(r) g^\varepsilon(r) dx d\mathbb{W}_r \right| \right] \\ & \leq \frac{1}{4p} \mathbb{E} \left[\sup_{r \in [0, T]} \|u^\varepsilon(r)\|_p^p \right] + C \mathbb{E} \left[\int_0^T \|g^\varepsilon(r)\|_{\mathbb{L}^p}^p dr \right]. \end{aligned} \quad (4.23)$$

Now, setting δ in (4.21) to be sufficiently small, taking the supremum over $t \in [0, T]$ on both sides of (4.12), and then compute the expectation, we obtain

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left(\|u^\varepsilon(t)\|_p^p + \frac{1}{p} \int_0^t \int_{\mathbb{T}^d} |\nabla(|u^\varepsilon(r)|^{p/2})|^2 dx dr \right) \right] \\ & \leq \frac{1}{2} \mathbb{E} \left[\sup_{r \in [0, T]} \|u^\varepsilon(r)\|_p^p \right] + \mathbb{E}[\|u_0^\varepsilon\|_p^p] + C \mathbb{E} \left[\int_0^T (\|f^\varepsilon(r)\|_{-1, q}^p + \|g^\varepsilon(r)\|_{\mathbb{L}^p}^p) dr \right], \end{aligned}$$

which implies

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u^\varepsilon(t)\|_p^p \right] \leq 2 \mathbb{E}[\|u_0^\varepsilon\|_p^p] + C \mathbb{E} \left[\int_0^T (\|f^\varepsilon(r)\|_{-1, q}^p + \|g^\varepsilon(r)\|_{\mathbb{L}^p}^p) dr \right] \quad (4.24)$$

and

$$\begin{aligned} & \frac{1}{2p} \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u^\varepsilon(r)|^{p/2})|^2 dx dr \right] \\ & \leq \frac{1}{4} \mathbb{E} \left[\sup_{r \in [0, T]} \|u^\varepsilon(r)\|_p^p \right] + \frac{1}{2} \mathbb{E}[\|u_0^\varepsilon\|_p^p] + C \mathbb{E} \left[\int_0^T (\|f^\varepsilon(r)\|_{-1, q}^p + \|g^\varepsilon(r)\|_{\mathbb{L}^p}^p) dr \right]. \end{aligned}$$

In summary,

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \|u^\varepsilon(t)\|_p^p + \int_0^T \int_{\mathbb{T}^d} |\nabla(|u^\varepsilon(r)|^{p/2})|^2 dx dr \right] \\ & \leq C \mathbb{E} \left[\|u_0^\varepsilon\|_p^p + \int_0^T (\|f^\varepsilon(r)\|_{-1, q}^p + \|g^\varepsilon(r)\|_{\mathbb{L}^p}^p) dr \right]. \end{aligned} \quad (4.25)$$

Note that (4.24) does not depend on ε . Thus, we may apply the same procedure to $u^\varepsilon - u^{\varepsilon'}$ and obtain

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \|u^\varepsilon(t) - u^{\varepsilon'}(t)\|_p^p \right] \\ & \leq C \mathbb{E} \left[\|u_0^\varepsilon - u_0^{\varepsilon'}\|_p^p + \int_0^T (\|f^\varepsilon(r) - f^{\varepsilon'}(r)\|_{-1, q}^p + \|g^\varepsilon(r) - g^{\varepsilon'}(r)\|_{\mathbb{L}^p}^p) dr \right]. \end{aligned}$$

Since each u^ε is in $L^p(\Omega; C([0, T], L^p))$ and they converge in $L^p(\Omega; L^\infty([0, T], L^p))$, they have a limit in $L^p(\Omega; C([0, T], L^p))$, and there exists a subsequence u^{ε_n} converges to that limit in $L^\infty([0, T], L^p)$ almost surely. We shall denote this limit by u and prove that it is a strong L^p solution to (4.1)–(4.2). Indeed, since

$$(u^\varepsilon(t), \phi) = (u_0^\varepsilon, \phi) + \int_0^t ((\Delta u^\varepsilon(r) + f^\varepsilon(r)), \phi) dr + \int_0^t (g^\varepsilon(r), \phi) d\mathbb{W}(r), \quad (t, \omega)\text{-a.e.},$$

for all $\phi \in C^\infty(\mathbb{T}^d)$ and all $\varepsilon > 0$, by the Hölder inequality and the dominated convergence theorem, we have

$$(u^{\varepsilon_n}(t), \phi) - (u_0^{\varepsilon_n}, \phi) \rightarrow (u(t), \phi) - (u_0, \phi)$$

and

$$\int_0^t ((u^{\varepsilon_n}(r), \Delta\phi) + (f^{\varepsilon_n}(r), \phi)) dr \rightarrow \int_0^t ((u(r), \Delta\phi) + (f(r), \phi)) dr$$

for a.e. (t, ω) as $n \rightarrow \infty$. By the BDG inequality,

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t (g^{\varepsilon_n}(r) - g(r), \phi) d\mathbb{W}(r) \right| \right] \\ & \leq \mathbb{E} \left[\left(\int_0^T \|g^{\varepsilon_n}(r) - g(r), \phi\|_{l^2}^2 dr \right)^{1/2} \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that for a further subsequence, which we still denote by u^{ε_n} , the following holds.

$$\int_0^t (g^{\varepsilon_n}(r), \phi) d\mathbb{W}(r) \xrightarrow{n \rightarrow \infty} \int_0^t (g(r), \phi) d\mathbb{W}(r), \quad (t, \omega)\text{-a.e.}$$

Using Lemma 4.4 and letting $n \rightarrow \infty$ in (4.25), we obtain (4.4).

Suppose u_1, u_2 are two strong L^p solutions to (4.1)–(4.2). Then $v := u_1 - u_2$ satisfies

$$\begin{aligned} dv(t, x) &= \Delta v(t, x) dt \\ v(0, x) &= 0 \text{ a.s.} \end{aligned}$$

on $[0, T] \times \mathbb{T}^d$. Then $v \equiv 0$ $\mathbb{P}$ -a.s. □

For convenience we also state the vector-valued version of the previous theorem. Thus, consider (4.1)–(4.2) on $\mathbb{T}^d$ but with u, f, g , and u_0 $\mathbb{R}^D$ -valued, where $D \in \mathbb{N}$. Then, under the assumptions of Theorem 4.1, we have

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_p^p + \sum_{j=1}^D \int_0^T \int_{\mathbb{T}^d} |\nabla(|u_j(t, x)|^{p/2})|^2 dx dt \right] \\ & \leq C \mathbb{E} \left[\|u_0\|_p^p + \int_0^T \|f(t, \cdot)\|_{-1, q}^p dt + \sum_{j=1}^D \int_0^T \int_{\mathbb{T}^d} \|g_j(t, x)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx dt \right]. \end{aligned} \quad (4.26)$$

5. Stochastic Truncated Navier-Stokes Equation

From here on, we restrict ourselves to the space dimension 3, although all the theorems can be adjusted to any dimension $d \geq 2$. Also, with a constant $\delta_0 > 0$, which is not necessarily small, denote by $\varphi: [0, \infty) \rightarrow [0, 1]$ a decreasing smooth function such that $\varphi \equiv 1$ on $[0, \delta_0/2]$ and $\varphi \equiv 0$ on $[\delta_0, \infty)$. In addition, we assume

$$|\varphi(t_1) - \varphi(t_2)| \leq \frac{C}{\delta_0} |t_1 - t_2|, \quad t_1, t_2 \geq 0.$$

We consider a stochastic Navier-Stokes equations truncated by this function, which reads

$$\begin{aligned} du(t, x) &= \Delta u(t, x) dt - \varphi(\|u(t)\|_p)^2 \mathcal{P}((u(t, x) \cdot \nabla)u(t, x)) dt \\ &\quad + \varphi(\|u(t)\|_p)^2 \sigma(u(t, x)) d\mathbb{W}(t), \\ \nabla \cdot u(t, x) &= 0, \quad (t, x) \in (0, T) \times \mathbb{T}^3 \\ u(0, x) &= u_0(x) \text{ a.s.,} \quad x \in \mathbb{T}^3, \end{aligned} \quad (5.1)$$

where σ is $(l^2(\mathcal{H}, \mathbb{R}))^3$ -valued, $u_0 \in L^p(\Omega; L^p)$ is $\mathcal{F}_0$ -measurable with $p > 5$, and $\nabla \cdot u_0 = 0$. Our goal in this section is to find the unique global solution for (5.1) by applying a fixed point argument.

THEOREM 5.1. *Let $p > 5$ and $u_0 \in L^p(\Omega; L^p)$. For every $T > 0$, there exists a unique solution $u \in L^p(\Omega; C([0, T], L^p))$ to (5.1) such that*

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} \|u(s, \cdot)\|_p^p + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla(|u_j(s, x)|^{p/2})|^2 dx ds \right] \leq C\mathbb{E}[\|u_0\|_p^p] + C_{p,T}. \quad (5.2)$$

In order to solve (5.1), we use the iteration algorithm

$$\begin{aligned} du^{(n)} - \Delta u^{(n)} dt &= -\varphi(\|u^{(n)}\|_p) \varphi(\|u^{(n-1)}\|_p) \mathcal{P}((u^{(n-1)} \cdot \nabla) u^{(n-1)}) dt \\ &\quad + \varphi(\|u^{(n)}\|_p) \varphi(\|u^{(n-1)}\|_p) \sigma(u^{(n-1)}) d\mathbb{W}(t), \\ \nabla \cdot u^{(n)} &= 0, \quad (t, x) \in (0, T) \times \mathbb{T}^3, \\ u^{(n)}(0) &= u_0 \text{ a.s.,} \quad x \in \mathbb{T}^3, \end{aligned} \quad (5.3)$$

where $u^{(0)}$ is the strong solution to

$$\begin{aligned} du^{(0)}(t, x) - \Delta u^{(0)}(t, x) dt &= 0, \\ \nabla \cdot u^{(0)}(t, x) &= 0, \quad (t, x) \in (0, T) \times \mathbb{T}^3, \\ u^{(0)}(0, x) &= u_0(x) \text{ a.s.,} \quad x \in \mathbb{T}^3. \end{aligned}$$

Utilizing the results from the previous section, we conclude that $u^{(0)} \in L^p(\Omega; C([0, T], L^p))$, and

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|u^{(0)}(t, \cdot)\|_p^p + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla(|u_j^{(0)}(t, x)|^{p/2})|^2 dx dt \right] \leq C\mathbb{E}[\|u_0\|_p^p]. \quad (5.4)$$

We need to prove that at each step n , there exists a unique solution $u^{(n)} \in L^p(\Omega; C([0, T], L^p))$ to (5.3), which is uniformly bounded in an consistent manner with (5.4). Thus we first consider the equation

$$\begin{aligned} du - \Delta u dt &= -\varphi(\|u\|_p) \varphi(\|v\|_p) \mathcal{P}((v \cdot \nabla) v) dt + \varphi(\|u\|_p) \varphi(\|v\|_p) \sigma(v) d\mathbb{W}(t), \\ \nabla \cdot u &= 0, \quad (t, x) \in (0, T) \times \mathbb{T}^3, \\ u(0) &= u_0, \quad \text{a.s.,} \quad x \in \mathbb{T}^3, \end{aligned} \quad (5.5)$$

where v is divergence-free and satisfies

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|v(t, \cdot)\|_p^p + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla(|v(t, x)|^{p/2})|^2 dx dt \right] \leq C\mathbb{E}[\|u_0\|_p^p] + C_T. \quad (5.6)$$

In order to solve (5.5), we employ the iteration procedure

$$\begin{aligned} du^{(n)} - \Delta u^{(n)} dt &= -\varphi(\|u^{(n-1)}\|_p) \varphi(\|v\|_p) \mathcal{P}((v \cdot \nabla) v) dt \\ &\quad + \varphi(\|u^{(n-1)}\|_p) \varphi(\|v\|_p) \sigma(v) d\mathbb{W}(t), \\ \nabla \cdot u^{(n)} &= 0, \quad (t, x) \in (0, T) \times \mathbb{T}^3, \\ u^{(n)}(0) &= u_0, \quad \text{a.s.,} \quad x \in \mathbb{T}^3, \end{aligned} \quad (5.7)$$

for v which is divergence-free and satisfies (5.6). Note that $u^{(n)}$ in (5.7) is not the same as in (5.3).

We shall prove the existence by obtaining an exponential rate of convergence for the fixed point iteration and by claiming that a sequence of random variables converges to zero a.s. if their expectation approaches zero rapidly. Thus, we will need the following auxiliary result.

LEMMA 5.2. *Let ξ_n be a sequence of nonnegative random variables such that $\mathbb{E}[\xi_n] \leq \eta^n$, for $n \in \mathbb{N}$, where $\eta \in (0, 1)$. Then, $\xi_n \rightarrow 0$ almost surely.*

PROOF OF LEMMA 5.2. Denote the probability event $\{\omega \in \Omega : \xi_n(\omega) \geq 1/m\}$ by A_n^m . If $\xi_n(\omega)$ does not go to zero as $n \rightarrow \infty$, then $\omega \in \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^m$. For each fixed $m \in \mathbb{N}$,

$$\sum_n \mathbb{P}(A_n^m) \leq m \sum_n \mathbb{E}[\xi_n] < \infty,$$

then $\mathbb{P}(\limsup_{n \rightarrow \infty} A_n^m) = 0$ by the Borel-Cantelli Lemma. Hence,

$$\mathbb{P}(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^m) = \lim_{m \rightarrow \infty} \mathbb{P}(\limsup_{n \rightarrow \infty} A_n^m) = 0.$$

This ends the proof. $\square$

REMARK 5.3. This conclusion can be extended from expectation and the probability measure to integration with respect to any finite measure. In particular, the integration on $\Omega \times [0, T]$ with respect to the product measure.

For convenience, we abbreviate

$$\begin{aligned} \varphi^{(n)} &= \varphi(\|u^{(n)}\|_p), & n \in \mathbb{N} \\ \varphi_v &= \varphi(\|v\|_p), \end{aligned}$$

in the rest of the section. The next lemma asserts uniform boundedness of $u^{(n)}$, which is needed in the fixed point argument.

LEMMA 5.4. *Let $p > 5$ and $n \in \mathbb{N}$. Suppose $u_0 \in L^p(\Omega; L^p)$ and assume that for each $j \in \{1, 2, \dots, n\}$, there exists a sufficiently small time $T > 0$ and a unique solution $u^{(n)} \in L^p(\Omega; C([0, T], L^p))$ to the initial value problem (5.5), where v satisfies (5.6). Then,*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|u^{(n)}(t, \cdot)\|_p^p + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla(|u_j^{(n)}(t, x)|^{p/2})|^2 dx dt \right] \leq C_{p,T} + C\mathbb{E}[\|u_0\|_p^p]. \quad (5.8)$$

PROOF. We apply Theorem 4.1 (cf. the inequality (4.26)) to the equation

$$\begin{aligned} du_j^{(n)} - \Delta u_j^{(n)} dt &= -\varphi^{(n-1)} \varphi_v (\mathcal{P}((v \cdot \nabla)v))_j dt \\ &\quad + \varphi^{(n-1)} \varphi_v \sigma_j(v) d\mathbb{W}(t), \quad x \in \mathbb{T}^3, \quad j = 1, 2, 3. \end{aligned} \quad (5.9)$$

We write the first term on the right side of (5.9) as

$$-\sum_i \varphi^{(n-1)} \varphi_v \partial_i (\mathcal{P}(v_i v))_j dt.$$

In order to apply (4.26), we need to estimate (assuming $l \leq p$)

$$\begin{aligned} C\mathbb{E} \left[\int_0^t \|\varphi^{(n-1)} \varphi_v v_i v\|_q^p ds \right] &\leq C\mathbb{E} \left[\int_0^t \varphi^{(n-1)} \varphi_v \|v_i\|_r^p \|v\|_l^p ds \right] \\ &\leq C\mathbb{E} \left[\int_0^t \varphi^{(n-1)} \varphi_v \|v_i\|_r^p \|v\|_p^p ds \right] \leq C\delta_0^p \mathbb{E} \left[\int_0^t \varphi^{(n-1)} \varphi_v \|v_i\|_r^p ds \right], \end{aligned} \quad (5.10)$$

where

$$\frac{3p}{p+1} < q \leq p \quad (5.11)$$

and

$$\frac{1}{r} + \frac{1}{l} = \frac{1}{q}.$$

For the last step in (5.10) we require

$$l \leq p \quad (5.12)$$

and then use $\varphi_v \|v\|_p^p \leq C \delta_0^p \varphi_v$. When we consider below the differences of iterates (cf. (5.21)–(5.22) below), we need a stronger inequality

$$r \leq p. \quad (5.13)$$

For the sake of exposition, we fix the exponents at this point as

$$q = (3p + \eta_0)/(p + 1) \text{ and } r = l = 2q. \quad (5.14)$$

The parameter $\eta_0 > 0$ is chosen so that

$$\frac{3p + \eta_0}{p + 1} < p, \quad (5.15)$$

which is possible when $p > 5$. It remains to estimate the last term in (5.9) (cf. (4.26)), i.e.,

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|\varphi^{(n-1)} \varphi_v \sigma(v)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx ds \right] \leq C \int_0^t \varphi_v (\|v\|_p^p + 1) ds \leq C(t + 1),$$

using sub-linear growth of the noise (3.4), and we obtain (5.8). $\square$

LEMMA 5.5. *Let $p > 5$ and suppose that $u_0 \in L^p(\Omega; L^p)$. Then there exists $t \in [0, T]$ such that the initial value problem (5.5), where v satisfies (5.6), has a unique solution $u \in L^p(\Omega; C([0, t], L^p))$, which satisfies*

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \|u(s, \cdot)\|_p^p + \sum_j \int_0^t \int_{\mathbb{T}^3} |\nabla(|u_j(s, x)|^{p/2})|^2 dx ds \right] \leq C \mathbb{E}[\|u_0\|_p^p] + C_t. \quad (5.16)$$

PROOF OF LEMMA 5.5. We employ the fixed point argument on the iteration (5.7). The difference $z^{(n)} = u^{(n+1)} - u^{(n)}$ satisfies

$$dz_j^{(n)} - \Delta z_j^{(n)} dt = \sum_i \partial_i f_{ij} dt + g_j d\mathbb{W}(t), \quad j = 1, 2, 3, \quad (5.17)$$

where

$$f_{ij} = (\varphi^{(n)} - \varphi^{(n-1)}) \varphi_v \mathcal{P}(v_i v_j)$$

and

$$g_j = (\varphi^{(n)} - \varphi^{(n-1)}) \varphi_v \sigma_j(v) d\mathbb{W}(t).$$

In addition to (5.17), we have

$$\begin{aligned} \nabla \cdot z^{(n)} &= 0, & (t, x) &\in (0, T) \times \mathbb{T}^3, \\ z^{(n)}(0) &= 0 \text{ a.s., } & x &\in \mathbb{T}^3. \end{aligned}$$

Note that

$$|\varphi^{(n)} - \varphi^{(n-1)}| \leq \frac{C}{\delta_0} \left| \|u^{(n)}\|_p - \|u^{(n-1)}\|_p \right| \leq \frac{C}{\delta_0} \|u^{(n)} - u^{(n-1)}\|_p = \frac{C}{\delta_0} \|z^{(n-1)}\|_p. \quad (5.18)$$

Now, we apply (4.26). The second term on the right side of (4.26) is estimated as

$$\begin{aligned}
C\mathbb{E} \left[\int_0^t \|(\varphi^{(n)} - \varphi^{(n-1)})\varphi_v v_i v_j\|_q^p ds \right] &\leq \frac{C}{\delta_0^p} \mathbb{E} \left[\int_0^t \varphi_v^p \|z^{(n-1)}\|_p^p \|v_i\|_r^p \|v_j\|_l^p ds \right] \\
&\leq C\mathbb{E} \left[\int_0^t \|z^{(n-1)}\|_p^p ds \right] \\
&\leq C_{\delta, \delta_0} \mathbb{E} \left[\int_0^t \|z^{(n-1)}\|_p^p ds \right] + \delta \mathbb{E} \left[\sum_j \int_0^t \int_{\mathbb{T}^3} |\nabla(|z_j^{(n-1)}(s, x)|^{p/2})|^2 dx ds \right] \\
&\leq C_{\delta, \delta_0} t \mathbb{E} \left[\sup_{s \in [0, t]} \|z^{(n-1)}\|_p^p ds \right] + \delta \mathbb{E} \left[\sum_j \int_0^t \int_{\mathbb{T}^3} |\nabla(|z_j^{(n-1)}(s, x)|^{p/2})|^2 dx ds \right],
\end{aligned}$$

where we used (5.12) and (5.13) in the second inequality. For the third term in (4.26), we estimate

$$\begin{aligned}
C\mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|g(t, x)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx ds \right] &\leq C_{\delta_0} \mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|z^{(n-1)}\|_p^p dx ds \right] \\
&\leq C_{\delta, \delta_0} \mathbb{E} \left[\int_0^t \|z^{(n-1)}\|_p^p ds \right] + \delta \mathbb{E} \left[\sum_j \int_0^t \int_{\mathbb{T}^3} |\nabla(|z_j^{(n-1)}(s, x)|^{p/2})|^2 dx ds \right] \\
&\leq C_{\delta, \delta_0} t \mathbb{E} \left[\sup_{s \in [0, t]} \|z^{(n-1)}\|_p^p ds \right] + \delta \mathbb{E} \left[\sum_j \int_0^t \int_{\mathbb{T}^3} |\nabla(|z_j^{(n-1)}(s, x)|^{p/2})|^2 dx ds \right].
\end{aligned}$$

This concludes the proof of existence of a fixed point for (5.5) on $[0, t]$ in $L_\omega^p L_t^\infty L_x^p \cap L_\omega^p L_t^p L_x^{3p}$ if $t > 0$ is sufficiently small. We denote this fixed point by u . Observing the exponential rate of convergence, we apply Lemma 5.2 and also obtain $\varphi(\|u^{(n)}(t)\|_p) \rightarrow \varphi(\|u(t)\|_p)$ for a.e. (ω, t) . Then, we obtain that u is indeed a solution to (5.7) by applying the dominated convergence theorem. By Lemma 4.4, (5.16) holds. $\square$

PROOF OF THEOREM 5.1. Consider the iteration (5.3), i.e.,

$$\begin{aligned}
du^{(n)} - \Delta u^{(n)} dt &= -\varphi^{(n)} \varphi^{(n-1)} \mathcal{P}((u^{(n-1)} \cdot \nabla) u^{(n-1)}) dt \\
&\quad + \varphi^{(n)} \varphi^{(n-1)} \sigma(u^{(n-1)}) d\mathbb{W}(t), \\
\nabla \cdot u^{(n)} &= 0, \quad (t, x) \in (0, T) \times \mathbb{T}^3, \\
u^{(n)}(0) &= u_0, \quad \text{a.s.,} \quad x \in \mathbb{T}^3.
\end{aligned}$$

Lemma 5.5 implies the existence of a unique maximal solution $u^{(n)}$, which satisfies

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|u^{(n)}(t, \cdot)\|_p^p + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla(|u_j^{(n)}(t, x)|^{p/2})|^2 dx dt \right] \leq C_{p, T, \delta_0} + C\mathbb{E}[\|u_0\|_p^p].$$

In order to apply the fixed point technique, we consider the difference

$$v^{(n)} = u^{(n+1)} - u^{(n)}$$

for which

$$\begin{aligned} dv^{(n)} - \Delta v^{(n)} ds + \left(\varphi^{(n+1)} \varphi^{(n)} \mathcal{P}((u^{(n)} \cdot \nabla) u^{(n)}) - \varphi^{(n)} \varphi^{(n-1)} \mathcal{P}((u^{(n-1)} \cdot \nabla) u^{(n-1)}) \right) ds \\ = \left(\varphi^{(n+1)} \varphi^{(n)} \sigma(u^{(n)}) - \varphi^{(n)} \varphi^{(n-1)} \sigma(u^{(n-1)}) \right) d\mathbb{W}(s), \\ \nabla \cdot v^{(n)} = 0, \quad (s, x) \in (0, t) \times \mathbb{T}^3, \\ v^{(n)}(0) = 0 \text{ a.s.} \end{aligned}$$

We rewrite the first equation as

$$dv_j^{(n)} - \Delta v_j^{(n)} dt = \partial_i f_{ij} dt + g_j d\mathbb{W}(t), \quad j = 1, 2, 3,$$

where

$$\begin{aligned} f_{ij} &= - \left(\varphi^{(n+1)} \varphi^{(n)} (u_i^{(n)} u_j^{(n)}) - \varphi^{(n)} \varphi^{(n-1)} (u_i^{(n-1)} u_j^{(n-1)}) \right) \\ &= - \varphi^{(n)} (\varphi^{(n+1)} - \varphi^{(n)}) u_i^{(n)} u_j^{(n)} - \varphi^{(n)} (\varphi^{(n)} - \varphi^{(n-1)}) u_i^{(n)} u_j^{(n)} \\ &\quad - \varphi^{(n)} \varphi^{(n-1)} v_i^{(n-1)} u_j^{(n)} - \varphi^{(n)} \varphi^{(n-1)} u_i^{(n-1)} v_j^{(n-1)} \\ &= f_{ij}^{(1)} + f_{ij}^{(2)} + f_{ij}^{(3)} + f_{ij}^{(4)} \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} g_j &= (\varphi^{(n+1)} \varphi^{(n)} \sigma(u^{(n)}) - \varphi^{(n)} \varphi^{(n-1)} \sigma(u^{(n-1)})) \\ &= - \varphi^{(n)} (\varphi^{(n+1)} - \varphi^{(n)}) \sigma(u^{(n)}) - \varphi^{(n)} (\varphi^{(n)} - \varphi^{(n-1)}) \sigma(u^{(n)}) \\ &\quad - \varphi^{(n)} \varphi^{(n-1)} (\sigma(u^{(n)}) - \sigma(u^{(n-1)})) \\ &= g_j^{(1)} + g_j^{(2)} + g_j^{(3)}. \end{aligned} \tag{5.20}$$

We first apply (4.26) on all the terms on the far right side of (5.19). Now, choose the exponents q , r , and s as in (5.11)–(5.15). Regarding the first term in (5.19), we have

$$\begin{aligned} \mathbb{E} \left[\int_0^t \|f^{(1)}\|_q^p ds \right] &\leq C \mathbb{E} \left[\int_0^t (\varphi^{(n)} - \varphi^{(n-1)})^p (\varphi^{(n)})^p \|u_i^{(n)} u_j^{(n)}\|_q^p ds \right] \\ &\leq \frac{C}{\delta_0^p} \mathbb{E} \left[\int_0^t \|v^{(n)}\|_p^p (\varphi^{(n)})^p \|u^{(n)}\|_r^p \|u^{(n)}\|_t^p ds \right] \\ &\leq C \delta_0^p \mathbb{E} \left[\int_0^t \|v^{(n)}\|_p^p ds \right] \end{aligned} \tag{5.21}$$

by

$$|\varphi^{(n+1)} - \varphi^{(n)}| \leq \frac{C}{\delta_0} \left| \|u^{(n+1)}\|_p - \|u^{(n)}\|_p \right| \leq \frac{C}{\delta_0} \|u^{(n+1)} - u^{(n)}\|_p = \frac{C}{\delta_0} \|v^{(n)}\|_p,$$

as in (5.18), and where we also used (5.12) and (5.13) in the last inequality in (5.21). As in (5.21), we have

$$\mathbb{E} \left[\int_0^t \|f^{(2)}\|_q^p ds \right] \leq C \delta_0^p \mathbb{E} \left[\int_0^t \|v^{(n-1)}\|_p^p ds \right].$$

Similarly,

$$\mathbb{E} \left[\int_0^t \|f^{(3)}\|_q^p ds \right] + \mathbb{E} \left[\int_0^t \|f^{(4)}\|_q^p ds \right] \leq C \delta_0^p \mathbb{E} \left[\int_0^t \|v^{(n-1)}\|_p^p ds \right]. \tag{5.22}$$

Summarizing (5.21), (5.22), and (5.22), we get

$$\mathbb{E} \left[\int_0^t \|f\|_q^p ds \right] \leq C_{\delta, \delta_0} t \mathbb{E} \left[\sup_{s \in [0, t]} \|v^{(n-1)}\|_p^p \right] + C_{\delta, \delta_0} t \mathbb{E} \left[\sup_{s \in [0, t]} \|v^{(n)}\|_p^p \right].$$

Now, we turn to the three terms in (5.20). For the first one, we have

$$\begin{aligned}
& C\mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|g^{(1)}(t, x)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx ds \right] \\
& \leq C\mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} (\varphi^{(n)})^p (\varphi^{(n+1)} - \varphi^{(n)})^p \|\sigma(u^{(n)})\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx ds \right] \\
& \leq \frac{C}{\delta_0^p} \mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} (\varphi^{(n)})^p \|v^{(n)}\|_p^p (\|u^{(n)}\|_p^p + 1) dx ds \right] \\
& \leq C_{\delta_0} \mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|v^{(n)}\|_p^p dx ds \right]
\end{aligned} \tag{5.23}$$

and similarly,

$$\begin{aligned}
& C\mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|g^{(2)}(t, x)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx ds \right] + C\mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|g^{(3)}(t, x)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx ds \right] \\
& \leq C_{\delta_0} \mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|v^{(n-1)}\|_p^p dx ds \right].
\end{aligned} \tag{5.24}$$

We may summarize (5.23) and (5.24) as

$$C\mathbb{E} \left[\int_0^t \int_{\mathbb{T}^d} \|g(t, x)\|_{l^2(\mathcal{H}, \mathbb{R}^d)}^p dx ds \right] \leq C_{\delta, \delta_0} t \mathbb{E} \left[\sup_{s \in [0, t]} \|v^{(n-1)}\|_p^p \right].$$

Therefore, we obtain the existence of a unique fixed point u of (5.1) in $L_\omega^p L_t^\infty L_x^p \cap L_\omega^p L_t^p L_x^{3p}$ on $[0, S]$, where $S > 0$ is a sufficiently small constant. Since each $u^{(n)} \in L^p(\Omega; C([0, S], L^p))$, so is u . By Lemma 5.5,

$$\begin{aligned}
(u^{(n)}(s), \phi) &= (u_0, \phi) + \int_0^s (u^{(n)}(r), \Delta \phi) dr \\
&+ \sum_j \int_0^s (\varphi^{(n)} \varphi^{(n-1)} \mathcal{P}(u_j^{(n-1)} u^{(n-1)}), \partial_j \phi) dr \\
&+ \int_0^s (\varphi^{(n)} \varphi^{(n-1)} \sigma(u^{(n-1)}), \phi) d\mathbb{W}(r), \quad (s, \omega)\text{-a.e.},
\end{aligned}$$

for all $\phi \in C^\infty(\mathbb{T}^3)$. The exponential convergence rate and Remark 5.3 implies that $\varphi(\|u^{(n)}(s)\|_p)$, $\varphi(\|u^{(n-1)}(s)\|_p) \rightarrow \varphi(\|u(s)\|_p)$ for a.e. (s, ω) . Together with the divergence free condition, the Hölder inequality, and the dominated convergence theorem, we get

$$\begin{aligned}
& \int_0^s (u^{(n)}(r), \Delta \phi) dr + \sum_j \int_0^s (\varphi^{(n)} \varphi^{(n-1)} \mathcal{P}(u_j^{(n-1)} u^{(n-1)}), \partial_j \phi) dr \\
& \rightarrow \int_0^s ((u(r), \Delta \phi) + (\varphi^2 \mathcal{P}(uu_j), \phi)) dr
\end{aligned}$$

for a.e. (s, ω) as $n \rightarrow \infty$. Also, by the BDG inequality and assumptions on σ ,

$$\begin{aligned}
& \mathbb{E} \left[\sup_{s \in [0, S]} \left| \int_0^s (\varphi^{(n)} \varphi^{(n-1)} \sigma(u^{(n-1)}) - \varphi^2 \sigma(u), \phi) d\mathbb{W}(r) \right| \right] \\
& \leq \mathbb{E} \left[\left(\int_0^S \|(\varphi^{(n)} \varphi^{(n-1)} \sigma(u^{(n-1)}) - \varphi^2 \sigma(u), \phi)\|_{l^2}^2 dr \right)^{1/2} \right].
\end{aligned}$$

Moreover, the right side goes to zero exponentially fast as $n \rightarrow \infty$. This implies that

$$\int_0^s (\varphi^{(n)} \varphi^{(n-1)} \sigma(u^{(n-1)}), \phi) d\mathbb{W}(r) \xrightarrow{n \rightarrow \infty} \int_0^s (\varphi^2 \sigma(u), \phi) d\mathbb{W}(r), \quad (s, \omega)\text{-a.e.}$$

Using Lemma 4.4 and letting $n \rightarrow \infty$ in (5.16), we obtain that u solves (5.1). It is standard to obtain the uniqueness from the contraction argument and for the sake of brevity, we omit the details.

Note that the time t chosen above is deterministic and positive; denote this time by t^* . Let n^* be a positive integer such that $T/n^* < t^*$. Denote $t_i = iT/n^*$ for $i \in \{0, 1, \dots, n^*\}$. Applying the existence and uniqueness inductively on $[t_i, t_{i+1}]$, $i \in \{0, 1, \dots, n^*\}$, we obtain a strong solution to (5.1) on $[0, T]$ and (5.2) holds. $\square$

PROOF OF THEOREM 3.1. For $n = 1, 2, \dots$, denote by $u^{(n)}$ the solution of the truncated SNSE (5.1) with $\delta_0 = n$. Also, introduce the stopping times

$$\tau_n = \inf \left\{ t \geq 0 : \|u^{(n)}(t)\|_p > \frac{n}{2} \right\}.$$

By uniqueness, the sequence is non-decreasing a.s. and $u^{(m)} = u^{(n)}$ on the set $\{\tau^{(m)} \leq \delta_0/2\} \cap \{\tau^{(n)} \leq \delta_0/2\}$ a.s. Let $\tau = \lim_n \tau_n$. Also, for any integer $n \in \mathbb{N}$, define $u = u^{(n)}$ on the set $\{\tau \geq 2n\}$. It is easy to check that (u, τ) satisfies all the required properties. $\square$

6. Global solution and energy decay

The truncated stochastic Navier-Stokes equations read

$$\begin{aligned} du(t, x) &= \Delta u(t, x) dt - \varphi(\|u(t)\|_p)^2 \mathcal{P}((u(t, x) \cdot \nabla)u(t, x)) dt \\ &\quad + \varphi(\|u(t)\|_p)^2 \sigma(u(t, x)) d\mathbb{W}(t), \\ \nabla \cdot u(t, x) &= 0, \quad (t, x) \in (0, T) \times \mathbb{T}^3, \\ u(0, x) &= u_0(x) \text{ a.s.,} \quad x \in \mathbb{T}^3. \end{aligned} \tag{6.1}$$

Note that in the previous section, we have proved the global well-posedness of this initial value problem. Recall that $\delta_0 > 0$ and that $\varphi: [0, \infty) \rightarrow [0, 1]$ is a decreasing smooth function with $\varphi \equiv 1$ on $[0, \delta_0/2]$ and $\varphi \equiv 0$ on $[\delta_0, \infty)$. In addition, we assumed

$$|\varphi(t_1) - \varphi(t_2)| \leq \frac{C}{\delta_0} |t_1 - t_2|, \quad t_1, t_2 \geq 0.$$

We shall set $\delta_0 > 0$ sufficiently small. Note that when $\|u\|_p$ is below $\delta_0/2$, the initial value problem (1.1)–(1.3) coincides with this cut-off model. Hence, an estimate of the likelihood that $\|u\|_p$ exceeds $\delta_0/2$ determines the time of existence for the solution to (1.1)–(1.3). The next result is essential for estimating that likelihood.

THEOREM 6.1. *Let $p > 5$. Then the global solution $u \in L^p(\Omega; C([0, T], L^p))$ to (6.1) satisfies*

$$\mathbb{E} \left[\sup_{t \in [0, \infty)} e^{at} \|u(t)\|_p^p \right] \leq C \mathbb{E}[\|u_0\|_p^p],$$

provided $a, \delta_0, \epsilon_0 > 0$ are sufficiently small constants.

Note that the constant $\epsilon_0 > 0$ is in the condition (3.5).

PROOF. Applying the Itô-Wentzel formula to $F_i(t) = e^{at} \|u_i(t)\|_p^p$, for a fixed $i \in \{1, 2, 3\}$, we obtain

$$d(e^{at} \|u_i(t)\|_p^p) = a e^{at} \|u_i(t)\|_p^p dt + e^{at} d(\|u_i(t)\|_p^p). \tag{6.2}$$

Utilizing the Itô expansion in the proof of Lemma 4.4 (cf. (4.12)) and (6.2), we have

$$\begin{aligned}
& e^{at} \|u_i(t)\|_p^p + \frac{4(p-1)}{p} \int_0^t e^{as} \int_{\mathbb{T}^3} |\nabla(|u_i(s)|^{p/2})|^2 dx ds \\
&= \|u_{0i}\|_p^p - p \int_0^t e^{as} \varphi^2 \int_{\mathbb{T}^3} |u_i|^{p-2} u_i (\mathcal{P}(u \cdot \nabla) u)_i dx ds \\
&\quad + p \int_0^t e^{as} \varphi^2 \int_{\mathbb{T}^3} |u_i|^{p-2} u_i \sigma_i(u) dx d\mathbb{W}_s \\
&\quad + \frac{p(p-1)}{2} \int_0^t e^{as} \varphi^4 \int_{\mathbb{T}^3} |u_i|^{p-2} \|\sigma_i(u)\|_{l^2}^2 dx ds + a \int_0^t e^{as} \|u_i(s)\|_p^p ds.
\end{aligned} \tag{6.3}$$

Now, choose q, r, l as in (5.11)–(5.15) and $\bar{r}$ as in (4.19). Using integration by parts, we have

$$\begin{aligned}
& pe^{at} \varphi^2 \left| \int_{\mathbb{T}^3} |u_i(t)|^{p-2} u_i(t) (\mathcal{P}(u \cdot \nabla) u)_i dx \right| \\
&= pe^{at} \varphi^2 \left| \sum_j \int_{\mathbb{T}^3} \partial_j (|u_i(t)|^{p-2} u_i(t)) \mathcal{P}(u_i u_j) dx \right| \\
&\leq Ce^{at} \varphi^2 \|\nabla(|u_i(t)|^{p/2})\|_2 \| |u_i|^{(p-2)/2} \|_{\bar{r}} \|u_i\|_r \|u\|_l \\
&\leq C\delta_0 e^{at} \varphi \|\nabla(|u_i(t)|^{p/2})\|_2 \| |u_i|^{(p-2)/2} \|_{\bar{r}} \|u_i\|_p,
\end{aligned}$$

using (5.12) and (5.13) and $\phi\|u\|_p \leq \delta_0$ in the last step. As in the proof of Theorem 4.1 above, we get

$$\begin{aligned}
& pe^{at} \varphi^2 \left| \int_0^t \int_{\mathbb{T}^3} |u_i(s)|^{p-2} u_i(s) \mathcal{P}((u \cdot \nabla) u_i) dx ds \right| \\
&\leq \delta \int_0^t e^{as} \sum_i \|\nabla(|u_i(s)|^{p/2})\|_2^2 ds + C\delta\delta_0^\kappa \int_0^t e^{as} \|u(s)\|_p^p ds,
\end{aligned} \tag{6.4}$$

where $\delta > 0$ is arbitrary and where $\kappa > 0$ is a constant depending on p . Note that the first term on the right side may be absorbed in the dissipative term if $\delta > 0$ is sufficiently small. Also, using the Poincaré type inequality

$$\| |v|^{p/2} \|_2 \leq C \|\nabla(|v|^{p/2})\|_2, \tag{6.5}$$

for v such that $\int_{\mathbb{T}^d} v dx = 0$, as in (4.16), the second term in (6.4) may also be absorbed if $\delta_0 > 0$ is sufficiently small. Regarding the fourth term in (6.3), we use the superlinearity assumption on the noise (3.5) and obtain

$$\frac{p(p-1)}{2} e^{at} \varphi^4 \int_{\mathbb{T}^3} |u_i(t)|^{p-2} \|\sigma_i(u)\|_{l^2}^2 dx \leq C\epsilon_0^2 e^{at} \varphi^4 \|u(t)\|_p^p.$$

Note that the last term in (6.3) may also be absorbed in the dissipative part if $a > 0$ is sufficiently small constant (independent of p). Combining the estimates above and absorbing the second, fourth, and fifth terms on the right-hand side of (6.3), we arrive at

$$\begin{aligned}
& \frac{1}{2} e^{at} \|u_i(t)\|_p^p + \frac{1}{2} \int_0^t e^{as} \|\nabla(|u_i(s)|^{p/2})\|_2^2 ds \\
&\leq \|u_i(0)\|_p^p + \int_0^t \int_{\mathbb{T}^3} e^{as} \|u_i\|_p^p dx ds + pe^{at} \varphi^2 \int_{\mathbb{T}^3} |u_i(s)|^{p-2} u_i(s) \sigma_i(u) dx d\mathbb{W}_s
\end{aligned}$$

since $4(p-1)/p \geq 1/2$. Hence,

$$\begin{aligned} & \mathbb{E} \left[\sup_{r \in [0, t]} e^{at} \|u_i(t)\|_p^p \right] + \mathbb{E} \left[\int_0^t e^{as} \|\nabla(|u_i(s)|^{p/2})\|_2^2 ds \right] \\ & \leq 2\mathbb{E} [\|u_i(0)\|_p^p] + p\mathbb{E} \left[\sup_{r \in [0, t]} \left| \int_0^r e^{as} \varphi^2 \int_{\mathbb{T}^3} |u_i(s)|^{p-2} u_i(s) \sigma_i(u) dx d\mathbb{W}_s \right| \right]. \end{aligned} \quad (6.6)$$

For the last term in (6.6), we apply the same approach as in (4.22)–(4.23), except that we use the assumption (3.5). We thus obtain

$$p\mathbb{E} \left[\sup_{r \in [0, t]} \left| \int_0^r e^{as} \varphi^2 \int_{\mathbb{T}^3} |u_i(s)|^{p-2} u_i(s) \sigma_i(u) dx d\mathbb{W}_s \right| \right] \leq C_\delta \epsilon_0 \mathbb{E} \left[\int_0^t \varphi^2 \|u(s)\|_p^p ds \right].$$

Using also (6.5), by taking $\epsilon_0 > 0$ sufficiently small, the right-hand side may be absorbed in the left side of (6.6). Therefore,

$$\mathbb{E} \left[\sup_{r \in [0, T]} e^{ar} \|u(r)\|_p^p + \int_0^T e^{as} \sum_i \|\nabla(|u_i(s)|^{p/2})\|_2^2 ds \right] \leq C\mathbb{E}[\|u_0\|_p^p],$$

whence

$$\mathbb{E} \left[\sup_{r \in [0, \infty)} e^{ar} \|u(r)\|_p^p + \int_0^\infty e^{as} \sum_i \|\nabla(|u_i(s)|^{p/2})\|_2^2 ds \right] \leq C\mathbb{E}[\|u_0\|_p^p],$$

and the proof is concluded. $\square$

Now, we are ready to prove the main theorem on the global existence of solutions for small data.

PROOF OF THEOREM 3.2. Let $\epsilon_0, \delta_0, a > 0$ be as in Theorem 6.1. Assume that (3.6) holds for some $\delta > 0$. By Markov's inequality, we have

$$\mathbb{P} \left(\sup_{t \in [0, \infty)} e^{at} \|u(t)\|_p \geq \frac{\delta_0}{2} \right) \leq \frac{C}{\delta_0} \mathbb{E}[\|u_0\|_p^p] \leq \frac{C\delta}{\delta_0}.$$

The assertion is then obtained by choosing $\delta > 0$ sufficiently small. $\square$

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References

- [BR] V. Barbu and M. Röckner, *Global solutions to random 3D vorticity equations for small initial data*, J. Differential Equations **263** (2017), no. 9, 5395–5411.
- [BT] P. Benner and C. Trautwein, *Optimal control problems constrained by the stochastic Navier-Stokes equations with multiplicative Lévy noise*, Math. Nachr. **292** (2019), no. 7, 1444–1461.
- [BCF] Z. Brzeźniak, M. Capiński, and F. Flandoli, *Stochastic Navier-Stokes equations with multiplicative noise*, Stochastic Anal. Appl. **10** (1992), no. 5, 523–532.
- [BF] Z. Brzeźniak and B. Ferrario, *A note on stochastic Navier-Stokes equations with not regular multiplicative noise*, Stoch. Partial Differ. Equ. Anal. Comput. **5** (2017), no. 1, 53–80.
- [CC] M. Capiński and N.J. Cutland, *Navier-Stokes equations with multiplicative noise*, Nonlinearity **6** (1993), no. 1, 71–78.
- [DZ] G. Da Prato and J. Zabczyk, *Ergodicity for infinite-dimensional systems*, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996.
- [F] F. Flandoli, *An introduction to 3D stochastic fluid dynamics*, SPDE in hydrodynamic: recent progress and prospects, Lecture Notes in Math., vol. 1942, Springer, Berlin, 2008, pp. 51–150.
- [FJR] E.B. Fabes, B.F. Jones, and N.M. Rivière, *The initial value problem for the Navier-Stokes equations with data in L^p* , Arch. Rational Mech. Anal. **45** (1972), 222–240.

- [FRS] B.P.W. Fernando, B. Rüdiger, and S.S. Sritharan, *Mild solutions of stochastic Navier-Stokes equation with jump noise in $\mathcal{V}L^p$ -spaces*, Math. Nachr. **288** (2015), no. 14-15, 1615–1621.
- [FS] B.P.W. Fernando and S.S. Sritharan, *Nonlinear filtering of stochastic Navier-Stokes equation with Itô-Lévy noise*, Stoch. Anal. Appl. **31** (2013), no. 3, 381–426.
- [MeS] J.-L. Menaldi and S.S. Sritharan, *Stochastic 2-D Navier-Stokes equation*, Appl. Math. Optim. **46** (2002), no. 1, 31–53.
- [MS] M.T. Mohan and S.S. Sritharan, *$\mathbb{L}^p$ -solutions of the stochastic Navier-Stokes equations subject to Lévy noise with $\mathbb{L}^m(\mathbb{R}^m)$ initial data*, Evol. Equ. Control Theory **6** (2017), no. 3, 409–425.
- [R] B.L. Rozovskiĭ, *Stochastic evolution systems*, Mathematics and its Applications (Soviet Series), vol. 35, Kluwer Academic Publishers Group, Dordrecht, 1990, Linear theory and applications to nonlinear filtering, Translated from the Russian by A. Yarkho.
- [GV] N.E. Glatt-Holtz and V.C. Vicol, *Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise*, Ann. Probab. **42** (2014), no. 1, 80–145.
- [GZ] N. Glatt-Holtz and M. Ziane, *Strong pathwise solutions of the stochastic Navier-Stokes system*, Adv. Differential Equations **14** (2009), no. 5-6, 567–600.
- [K] T. Kato, *Strong L^p -solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions*, Math. Z. **187** (1984), no. 4, 471–480.
- [Ki] J.U. Kim, *Strong solutions of the stochastic Navier-Stokes equations in $\mathbb{R}^3$* , Indiana Univ. Math. J. **59** (2010), no. 4, 1417–1450.
- [Kr] N.V. Krylov, *On L_p -theory of stochastic partial differential equations in the whole space*, SIAM J. Math. Anal. **27** (1996), no. 2, 313–340.
- [MR] R. Mikulevicius and B.L. Rozovskii, *Global L_2 -solutions of stochastic Navier-Stokes equations*, Ann. Probab. **33** (2005), no. 1, 137–176.
- [KZ] I. Kukavica and M. Ziane, *Regularity of the Navier-Stokes equation in a thin periodic domain with large data*, Discrete Contin. Dyn. Syst. **16** (2006), no. 1, 67–86.
- [ZBL1] J. Zhu, Z. Brzeźniak, and W. Liu, *$\mathbb{L}^p$ -solutions for stochastic Navier-Stokes equations with jump noise*, Statist. Probab. Lett. **155** (2019), 108563, 9pp.
- [ZBL2] J. Zhu, Z. Brzeźniak, and W. Liu, *Maximal inequalities and exponential estimates for stochastic convolutions driven by Lévy-type processes in Banach spaces with application to stochastic quasi-geostrophic equations*, SIAM J. Math. Anal. **51** (2019), no. 3, 2121–2167.

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