Journal of Mathematical Logic
2250010 (22 pages)

DOI: 10.1142/S0219061322500106

Strongly Compact Cardinals and Ordinal Definability

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Received 4 August 2021
Accepted 4 January 2022
Published 15 May 2023

This paper explores several topics related to Woodin's HOD conjecture. We improve the large cardinal hypothesis of Woodin's HOD dichotomy theorem from an extendible cardinal to a strongly compact cardinal. We show that assuming there is a strongly compact cardinal and the HOD hypothesis holds, there is no elementary embedding from HOD to HOD, settling a question of Woodin. We show that the HOD hypothesis is equivalent to a uniqueness property of elementary embeddings of levels of the cumulative hierarchy. We prove that the HOD hypothesis holds if and only if every regular cardinal above the first strongly compact cardinal carries an ordinal definable ω-Jónsson algebra. We show that if the HOD hypothesis holds and HOD satisfies the Ultrapower Axiom, then every supercompact cardinal is supercompact in HOD.

Keywords: Strongly compact cardinal; supercompact cardinal; ordinal definability; HOD conjecture.

Mathematics Subject Classification 2020: 03E45, 03E55

1. Introduction

The Jensen covering theorem [2] states that if $0^\#$ does not exist, then every uncountable set of ordinals is covered by a constructible set of the same cardinality. This leads to a strong dichotomy for the cardinal correctness of the constructible universe L.

Theorem (Jensen). *Exactly one of the following holds*:

(1) *For all singular cardinals* λ, λ *is singular in* L *and* $(\lambda^+)^L = \lambda^+$.
(2) *Every infinite cardinal is strongly inaccessible in* L.

The proof of this theorem and its many generalizations tend to make heavy use of the special structure of canonical inner models. By completely different techniques, however, Woodin [10] showed that such a dichotomy holds for the inner model HOD under large cardinal hypotheses despite the fact that HOD is not a canonical inner model.

Theorem (Woodin's HOD dichotomy). *If κ is extendible, exactly one of the following holds*:

(1) *For all singular cardinals $\lambda > \kappa$, λ is singular in* HOD *and* $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$.
(2) *Every regular cardinal greater than or equal to κ is measurable in* HOD.

Woodin's HOD conjecture states that conclusion (1), which in this context is known as the HOD *hypothesis*,[a] is provable from large cardinal axioms.

The first few theorems of this paper (Sec. 2) show that the HOD dichotomy in fact takes hold far below the least extendible cardinal.

Theorem. *If κ is strongly compact, exactly one of the following holds*:

(1) *For all singular cardinals $\lambda > \kappa$, λ is singular in* HOD *and* $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$.
(2) *All sufficiently large regular cardinals are measurable in* HOD.

We prove this as a corollary of a theorem that establishes a similar dichotomy for a fairly broad class of inner models. An inner model N of ZFC is *ω-club amenable* if for all ordinals δ of uncountable cofinality, the ω-club filter F on δ satisfies $F \cap N \in N$.

Theorem 2.9. *Assume N is ω-club amenable and κ is ω_1-strongly compact. Either all singular cardinals $\lambda > \kappa$ are singular in N and $(\lambda^+)^N = \lambda^+$, or all sufficiently large regular cardinals are measurable in N.*

Note that we use a somewhat weaker large cardinal hypothesis than strong compactness: a cardinal κ is ω_1-strongly compact if every κ-complete filter extends to an ω_1-complete ultrafilter.

The main theorem of [4] states that if $j_0, j_1 : V \to M$ are elementary embeddings, then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$. Section 3 turns to the relationship between local forms of this theorem and Woodin's HOD conjecture. If P and Q are transitive sets and $\delta \in \mathrm{Ord} \cap P$, then $j_0, j_1 : P \to Q$ are *δ-similar* if $j_0(\delta) = j_1(\delta)$ and $\sup j_0[\delta] = \sup j_1[\delta]$.

Theorem 3.1. *Assume κ is a supercompact cardinal. Then the following are equivalent*:

(1) *The* HOD *hypothesis.*
(2) *For all regular $\delta \geq \kappa$ and all sufficiently large ordinals $\lambda > \delta$, if $j_0, j_1 : V_\lambda \to M$ are δ-similar, then $j_0 \restriction \delta = j_1 \restriction \delta$.*

[a]In the context of bare ZFC (without large cardinal axioms), the HOD hypothesis states that there is a proper class of regular cardinals that are not ω-strongly measurable in HOD. Conceivably the HOD hypothesis is provable in ZFC.

We also prove a related theorem that connects the failure of the HOD hypothesis to definable infinitary partition properties, choiceless large cardinals and sharps.

A cardinal λ is *Jónsson* if for all $f : [\lambda]^{<\omega} \to \lambda$, there is a set $A \subseteq \lambda$ of cardinality λ such that $\operatorname{ran}(f \restriction [A]^{<\omega})$ is a proper subset of λ. A cardinal λ is *constructibly Jónsson* if this holds for all constructible functions $f : [\lambda]^{<\omega} \to \lambda$; that is, there is a set $A \subseteq \lambda$ of cardinality λ such that $\operatorname{ran}(f \restriction [A]^{<\omega})$ is a proper subset of λ. (The set A need not be constructible itself.)

Proposition. *The following are equivalent*:

- $0^{\#}$ *exists.*
- *Every uncountable cardinal is constructibly Jónsson.*
- *Some uncountable cardinal is constructibly Jónsson.*

A cardinal λ is *ω-Jónsson* if for all $f : [\lambda]^{\omega} \to \lambda$, there is a set $A \subseteq \lambda$ of cardinality λ such that $f[A]^{\omega}$ is a proper subset of λ. Recall Kunen's theorem that there is no elementary embedding $j : V \to V$. Kunen proved his theorem by showing that if $j(\lambda) = \lambda$, then λ is ω-Jónsson. He then cites the following combinatorial theorem.

Theorem (Erdös-Hajnal). *There are no ω-Jónsson cardinals.*

A cardinal λ is *definably ω-Jónsson* if for all ordinal definable functions $f : [\lambda]^{\omega} \to \lambda$, there is a set $A \subseteq \lambda$ of cardinality λ such that $f[A]^{\omega}$ is a proper subset of λ.

Theorem 3.5. *Assume κ is strongly compact. Then the following are equivalent*:

- *The* HOD *hypothesis fails.*
- *All sufficiently large regular cardinals are definably ω-Jónsson.*
- *Some regular cardinal above κ is definably ω-Jónsson.*

In Sec. 3.3, we use techniques drawn from the proof of the main theorem of [4] to answer a decade-old question of Woodin (implicit in [10]). Recall Kunen's famous theorem relating the rigidity of L to $0^{\#}$:

Theorem (Kunen). *If* $0^{\#}$ *does not exist*, *then there is no nontrivial elementary embedding from L to L.*

Here we prove an analog for HOD, replacing the assumption that $0^{\#}$ does not exist with the HOD hypothesis.

Theorem 3.7. *Assume there is an ω_1-strongly compact cardinal. If the* HOD *hypothesis holds*, *then there is no nontrivial elementary embedding from* HOD *to* HOD.

Section 4.1 turns to the question of the structure of strongly compact cardinals in the HOD of a model of determinacy. In this context, something close to the failure of the HOD hypothesis holds, and yet we will show that HOD still has certain local covering properties in regions of strong compactness. One consequence of our analysis is the following theorem on the equivalence of strong compactness and ω_1-strong compactness in HOD assuming Woodin's HOD *ultrafilter conjecture*: under $\text{AD} + V = L(P(\mathbb{R}))$, every ω_1-complete ultrafilter of HOD generates an ω_1-complete filter in V. (The HOD ultrafilter conjecture is true in $L(\mathbb{R})$ and more generally in all inner models of determinacy amenable to the techniques of contemporary inner model theory.)

Theorem 4.2. *Assume* AD^+, $V = L(P(\mathbb{R}))$, *and the* HOD-*ultrafilter conjecture. Suppose* $\kappa < \Theta$ *is a regular cardinal and* $\delta \geq \kappa$ *is a* HOD-*regular ordinal. Then* κ *is* δ-*strongly compact in* HOD *if and only if* κ *is* (ω_1, δ)-*strongly compact in* HOD.

This should be compared with a theorem of [3] stating that under the Ultrapower Axiom, for any regular cardinal δ, the least (ω_1, δ)-strongly compact cardinal is δ-strongly compact. It is natural to conjecture that assuming AD, if $V = L(P(\mathbb{R}))$, then HOD is a model of the Ultrapower Axiom.

The final section (Sec. 4.2) explores the absoluteness of large cardinals to HOD in the context of a strongly compact cardinal. Woodin [10] showed.

Theorem (Woodin). Suppose κ is extendible and the HOD hypothesis holds. Then κ is extendible in HOD.

By results of Cheng *et al.* [1], the least supercompact cardinal need not be weakly compact in HOD. Here we show that nevertheless, HOD is very close to an inner model with a supercompact cardinal.

Theorem 4.12. *Suppose* κ *is supercompact and the* HOD *hypothesis holds. Then* κ *is supercompact in an inner model* N *of* ZFC *such that* $\text{HOD} \subseteq N$ *and* $\text{HOD}^{<\kappa} \cap N \subseteq \text{HOD}$.

As a corollary, we show the following theorem.

Theorem 4.13. *Suppose* κ *is supercompact, the* HOD *hypothesis holds, and* HOD *satisfies the Ultrapower Axiom. Then* κ *is supercompact in* HOD.

2. On the HOD Dichotomy

2.1. *Strongly measurable cardinals*

For any regular cardinal γ and any ordinal δ of cofinality greater than γ, $\mathcal{C}_\delta$ denotes the closed unbounded filter on δ, S^δ_γ denotes the set of ordinals less than δ of cofinality γ and $\mathcal{C}_{\delta,\gamma}$ denotes the filter generated by $\mathcal{C}_\delta \cup \{S^\delta_\gamma\}$, or equivalently the filter generated by the γ-closed unbounded subsets of δ. Similarly, $S^\delta_{<\gamma} = \bigcup_{\lambda<\gamma} S^\delta_\lambda$ and $\mathcal{C}_{\delta,<\gamma}$ denotes the filter generated by $\mathcal{C}_\delta \cup \{S^\delta_{<\gamma}\}$.

A filter F is γ-*saturated* if any collection of F-almost disjoint F-positive sets has cardinality less than γ. The filters considered in this paper are typically sufficiently complete that the distinction between F-almost disjointness and disjointness is irrelevant: if F is γ-complete, then F is γ-saturated if and only if any collection of disjoint F-positive sets has cardinality less than γ. The saturation of a filter is a measure of how close the filter is to being an ultrafilter.

Lemma 2.1 (Ulam). *Suppose F is a $(2^\gamma)^+$-complete γ-saturated filter. Then F is the intersection of fewer than γ-many ultrafilters.*

Note that if a γ-complete filter F is the intersection of fewer than γ-many ultrafilters, then the underlying set of F can be partitioned into atoms of F.

If $\nu < \delta$ are regular cardinals, δ is *ν-strongly measurable* in an inner model N if $\mathcal{C}_{\delta,\nu} \cap N$ belongs to N and is γ-saturated in N for some N-cardinal γ such that $(2^\gamma)^N < \delta$. Thus, a cardinal δ is *not* ν-strongly measurable in N if and only if for each γ such that $(2^\gamma)^N < \delta$, there is in N a γ-sequence of pairwise disjoint stationary sets whose union is $\mathcal{C}_{\delta,\nu}$-positive.

Lemma 2.2. *Suppose N is an inner model of* ZFC, *$\nu < \delta$ are regular cardinals, and η is the least N-cardinal such that $(2^\eta)^N \geq \delta$. Then the following are equivalent*:

- *δ is ν-strongly measurable in N.*
- *For some $\gamma < \eta$, $\mathcal{C}_{\delta,\nu} \cap N$ can be written as an intersection of fewer than γ-many ultrafilters of N.*
- *For some $\gamma < \eta$, δ can be partitioned into γ-many $\mathcal{C}_{\delta,\nu}$-positive sets $\langle S_\alpha : \alpha < \nu\rangle \in N$ such that $(\mathcal{C}_{\delta,\nu} \restriction S_\alpha) \cap N$ is an ultrafilter in N.*

2.2. *The HOD dichotomy*

Suppose M is an inner model, λ is a cardinal, and λ' is an M-cardinal. An inner model M has the *(λ, λ')-cover property* if for every set $\sigma \subseteq M$ of cardinality less than λ, there is a set $\tau \in M$ of M-cardinality less than λ' such that $\sigma \subseteq \tau$. (Note that if λ' is a cardinal in V, then $\tau \in M$ has M-cardinality less than λ' if and only if τ has cardinality less than λ'.) The (λ, λ)-cover property will be referred to as the *λ-cover property* and the (λ^+, λ')-cover property as the *$(\leq\lambda, \lambda')$-cover property.*

Proposition 2.3. *Suppose κ is strongly compact. Then exactly one of the following holds*:

(1) *All sufficiently large regular cardinals are measurable in* HOD.
(2) HOD *has the κ-cover property.*

Proof. There are two cases.

Case 1. There is a HOD-cardinal η such that for all regular cardinals δ, $\mathcal{C}_{\delta,<\kappa} \cap \text{HOD}$ is η-saturated in HOD.

In this case, Lemma 2.1 implies that all regular cardinals $\delta > (2^\eta)^{\text{HOD}}$ are measurable in HOD.

Case 2. For every HOD-cardinal η, there is a regular cardinal δ such that $\mathcal{C}_{\delta,<\kappa} \cap \text{HOD}$ is not η-saturated in HOD.

We claim that HOD has the κ-cover property. For this, it suffices to show that for all η, there is a κ-complete fine ultrafilter $\mathcal{U}$ on $P_\kappa(\eta)$ such that $\text{HOD} \cap P_\kappa(\eta) \in \mathcal{U}$. Then for each $\sigma \subseteq \eta$ with $|\sigma| < \kappa$, the set $\{\tau \in P_\kappa(\eta) : \sigma \subseteq \tau\}$ belongs to $\mathcal{U}$ by fineness and κ-completeness. Thus, this set meets $\text{HOD} \cap P_\kappa(\eta)$, which yields a set $\tau \in \text{HOD}$ such that $|\tau|^{\text{HOD}} < \kappa$ and $\sigma \subseteq \tau$.

Fix an ordinal η. Let δ be a regular cardinal such that $\mathcal{C}_{\delta,<\kappa} \cap \text{HOD}$ is not η-saturated in HOD. Let $\mathcal{S} = \{S_\alpha : \alpha < \eta\}$ be an ordinal definable partition of $S^\delta_{<\kappa}$ into stationary sets.

Since κ is strongly compact, there is an elementary embedding $j : V \to M$ with critical point κ such that M has the $(\delta^+, j(\kappa))$-cover property. Let $\delta_* = \sup j[\delta]$. This implies that $\text{cf}^M(\delta_*) < j(\kappa)$. Fix a closed unbounded set $C \subseteq \delta_*$ such that $C \in M$ and $|C|^M < j(\kappa)$.

Let $\mathcal{T} = \langle T_\beta : \beta < j(\eta)\rangle = j(\mathcal{S})$. Let

$$\sigma = \{\beta < j(\eta) : T_\beta \cap \delta_* \text{ is stationary in } M\}.$$

Then σ is ordinal definable in M since $\mathcal{T}$ is. Notice that for all $\alpha < \eta$, $j[S_\alpha]$ is a stationary subset of δ_*: j is continuous on $S^\delta_{<\kappa}$, and so there is a continuous increasing cofinal function $f : \delta \to \delta_*$ such that $j[S_\alpha] = f[S_\alpha]$. Since $j[S_\alpha] \subseteq T_{j(\alpha)}$, $j(\alpha) \in \sigma$. On the other hand, for each $\beta \in \sigma$, let $f(\beta) = \min(T_\beta \cap C)$. Then $f : \sigma \to C$ is an injection, so $|\sigma|^M < j(\kappa)$. In other words, $\sigma \in j(P_\kappa(\eta))$.

Let $\mathcal{U}$ be the ultrafilter on $P_\kappa(\eta)$ derived from j using σ. Then $\text{HOD} \cap P_\kappa(\eta) \in \mathcal{U}$ since σ is ordinal definable in M. Moreover $\mathcal{U}$ is fine since $j[\eta] \subseteq \sigma$. Finally $\mathcal{U}$ is κ-complete since $\text{crit}(j) = \kappa$.

This finishes the proof that HOD has the κ-cover property. □

The proof of Proposition 2.3 shows that if HOD does not cover V, then sufficiently large regular cardinals are measurable in HOD in a strong sense.

Proposition 2.4. *Suppose κ is strongly compact. Then exactly one of the following holds*:

(1) *For some γ, for all ordinals δ with* $\text{cf}(\delta) \geq \gamma$, $\mathcal{C}_{\delta,<\kappa} \cap \text{HOD}$ *is the intersection of fewer than γ-many ultrafilters in* HOD.
(2) HOD *has the κ-cover property.*

For any set A, let $\theta^{\text{OD}}(A)$ denote the least ordinal θ such that there is no partial surjective function from A to θ. (The fact that we consider partial functions will not be relevant since in the case of interest, $A \in \text{OD}$ and A is nonempty.) For any cardinal ν and any set X, let $\beta_\nu(X)$ denote the set of ν-complete ultrafilters on X.

Proposition 2.5. *Suppose κ is strongly compact and* HOD *has the κ-cover property. Then for any cardinal λ,* HOD *has the $(\leq\lambda, \theta)$-cover property where $\theta = \theta^{\mathrm{OD}}(\beta_\kappa(\lambda))$.*

Proof. We first consider the case that λ is regular. Assume σ is a set of ordinals of cardinality less than or equal to λ. We will construct an ordinal definable partial function on $\beta_\kappa(\lambda)$ whose range contains σ.

Suppose U is a κ-complete (κ, λ)-regular ultrafilter on λ. In other words, M_U has the $(\leq\lambda, j_{\mathcal{U}}(\kappa))$-cover property, and so there is a set $\sigma' \in M_U$ such that $|\sigma'|^{M_U} < j_U(\kappa)$ and $\sigma \subseteq \sigma'$. By elementarity, HOD^{M_U} has the $j_U(\kappa)$-cover property in M_U, and so σ' is covered by a set $\sigma_U \in \mathrm{HOD}^{M_U}$ with $|\sigma_U|^{\mathrm{HOD}^{M_U}} < j_U(\kappa)$. Since $\mathrm{HOD}^{M_U} \subseteq \mathrm{HOD}_U$, we have $\sigma_U \in \mathrm{HOD}_U$ and $|\sigma_U|^{\mathrm{HOD}_U} < j_U(\kappa)$. Let $\gamma_U = |\sigma_U|^{\mathrm{HOD}_U}$, and let $f_U : \gamma_U \to \mathrm{Ord}$ be the OD_U-least enumeration of σ_U. Let $\gamma = \sup_{U \in \beta_\kappa(\lambda)} \gamma_U$, and define a partial function $g : \beta_\kappa(\lambda) \times \gamma \to \mathrm{Ord}$ by setting $g(U, \alpha) = f_U(\alpha)$. Then $\sigma \subseteq \mathrm{ran}(g)$ and g is ordinal definable.

To finish, note that there is an ordinal definable partial surjection from $\beta_\kappa(\lambda)$ to an ordinal larger than γ. For example, consider the function sending $U \in \beta_\kappa(\lambda)$ to the least element of the canonical decomposition of $[\mathrm{id}]_U$ into additively indecomposable ordinals.

We now consider the case that λ is singular. Let $\delta = \mathrm{cf}(\lambda)$. If σ is a set of size λ, then $\sigma = \bigcup_{\alpha<\delta} \sigma_\alpha$ where $|\sigma_\alpha| < \lambda$ is regular. Let $\theta = \theta^{\mathrm{OD}}(\beta_\kappa(\lambda))$. By our previous work, for each $\alpha < \delta$, σ_α can be covered by a set $\tau_\alpha \in \mathrm{HOD}$ of cardinality at most θ. On the other hand, $\{\tau_\alpha : \alpha < \delta\} \subseteq \mathrm{HOD}$ can be covered by a set $A \in \mathrm{HOD}$ with $|A| < \theta$, and without loss of generality we may assume that for all $\tau \in A$, $|\tau| < \theta$. It is easy to check that for any ordinal definable set X, the ordinal $\theta^{\mathrm{OD}}(X)$ is OD-regular in the sense that if $f : \gamma \to \theta^{\mathrm{OD}}(X)$ is ordinal definable where $\gamma < \theta^{\mathrm{OD}}(X)$, then f is bounded. Therefore, $\bigcup A$ covers σ and $|\bigcup A| < \theta$. $\square$

The ordinal θ lies somewhere in the half-open interval $((2^\lambda)^+, (2^{2^\lambda})^+]$, which yields the following more quotable corollary.

Corollary 2.6. *If κ is strongly compact, either all sufficiently large regular cardinals are measurable in* HOD *or* HOD *has the λ-cover property for all strong limit cardinals $\lambda \geq \kappa$.*

In particular, for all strong limit singular cardinals $\lambda \geq \kappa$, λ is singular in HOD and $\lambda^{+\mathrm{HOD}} = \lambda^+$. But in fact one can prove this for arbitrary singular cardinals above κ.

The proof is more general in two ways. First, it uses a weaker large cardinal hypothesis: a cardinal κ is *ω_1-strongly compact* if every κ-complete filter extends to a countably complete ultrafilter. Second, it applies to a broader class of models than just HOD: an inner model N of ZFC is *ω-club amenable* if for every ordinal δ of uncountable cofinality, $\mathcal{C}_{\delta,\omega} \cap N \in N$.

The proof of the following lemma is almost identical to that of Proposition 2.3.

Lemma 2.7. *Suppose N is ω-club amenable and κ is ω_1-strongly compact. Then one of the following holds*:

(1) *All sufficiently large regular cardinals are measurable in N.*
(2) *N has the (ω_1, κ)-cover property.*

Proof. The proof splits into cases as in Proposition 2.3.

Case 1. There is an N-cardinal η such that for all regular cardinals δ, $\mathcal{C}_{\delta,\omega} \cap N$ is η-saturated in N.

Then for all sufficiently large regular cardinals δ, δ is measurable in N by Lemma 2.1, applied in N to $\mathcal{C}_{\delta,\omega} \cap N$ in N.

Case 2. For every N-cardinal η, there is a regular cardinal δ such that $\mathcal{C}_{\delta,\omega} \cap N$ is not η-saturated in N.

In this case, one shows that for all $\lambda \geq \kappa$, there is a countably complete fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ such that $N \cap P_\kappa(\lambda) \in \mathcal{U}$. It follows that every countable set $\sigma \subseteq \lambda$ belongs to some $\tau \in N \cap P_\kappa(\lambda)$. □

Theorem 2.8. *Suppose ν is a regular cardinal and N is an ω-club amenable inner model with the (ω_1, ν)-cover property. Then for all N-regular $\delta \geq \nu$,* $\text{cf}(\delta) = |\delta|$.

Proof. Let $A = (S^\delta_{<\nu})^N$. The (ω_1, ν)-cover property of N implies that $S^\delta_\omega \subseteq A$, which is all we will use. Fix $\langle c_\xi : \xi \in A\rangle \in N$ such that c_ξ is a closed cofinal subset of ξ of ordertype less than ν. For each $\alpha < \delta$, let β_α denote the least ordinal such that for a stationary set of $\xi \in S^\delta_\omega$, $c_\xi \cap [\alpha, \beta_\alpha) \neq \emptyset$. One can prove that β_α exists for all $\alpha < \delta$ by applying Fodor's lemma to the function $f(\xi) = \min(c_\xi \setminus \alpha)$. Define a continuous increasing sequence $\langle \epsilon_\alpha : \alpha < \delta\rangle$ by setting $\epsilon_{\alpha+1} = \beta_{\epsilon_\alpha}$, taking suprema at limit steps; these suprema are always below δ because the sequence belongs to N and δ is regular in N.

For each $\xi \in A$, let

$$\sigma_\xi = \{\alpha < \delta : c_\xi \cap [\epsilon_\alpha, \epsilon_{\alpha+1}) \neq \emptyset\}.$$

Note that $|\sigma_\xi| < \nu$ since $|c_\xi| < \nu$. Moreover for all α, the set $S_\alpha = \{\xi \in A : \alpha \in \sigma_\xi\}$ is stationary. Let $C \subseteq \delta$ be a closed cofinal set of ordertype $\text{cf}(\delta)$. For any $\alpha < \delta$, there is some $\xi \in S_\alpha \cap C$, which means that $\alpha \in \sigma_\xi$. This implies that $\delta = \bigcup_{\xi \in C} \sigma_\xi$. Thus, $|\delta| = |C| \cdot \sup_{\xi \in C} |\sigma_\xi|$. If $|C| < \nu$, then since ν is regular, $\sup_{\xi \in C} |\sigma_\xi| < \nu$ and hence $|\delta| < \nu$, contradicting that $\delta \geq \nu$. Therefore, $|C| \geq \nu$. Therefore, $|\delta| = |C| \cdot \sup_{\xi \in C} |\sigma_\xi| = |C| \cdot \nu = |C| = \text{cf}(\delta)$. □

Corollary 2.9. *Suppose κ is ω_1-strongly compact and N is an ω-club amenable inner model. Either all sufficiently large regular cardinals are measurable in N or every singular cardinal λ greater than κ is singular in N and $\lambda^{+N} = \lambda^+$.*

Proof. If N is an inner model of ZFC such that $\text{cf}(\delta) = |\delta|$ for all cardinals that are regular in N, then by a standard argument one can conclude that every singular cardinal λ greater than κ is singular in N and $\lambda^{+N} = \lambda^+$. First of all, $\text{cf}(\lambda) < |\lambda|$, and so λ is singular in N. Second, since λ^{+N} is regular in N, $\text{cf}(\lambda^{+N}) = |\lambda^{+N}| \geq \lambda$, and so since λ^+ is the least regular cardinal greater than or equal to λ, $\text{cf}(\lambda^{+N}) \geq \lambda^+$. It follows that $\lambda^{+N} = \lambda^+$. $\square$

We state some straightforward reformulations of the failure of the HOD hypothesis for future reference.

Theorem 2.10. *If κ is strongly compact, each of the following statements equivalent to the failure of the* HOD *hypothesis*:

(1) *There is a regular cardinal $\delta \geq \kappa$ with a stationary subset $S \subseteq S^\delta_{<\kappa}$ that admits no ordinal definable partition into δ-many stationary sets.*
(2) *There is an ω-strongly measurable cardinal above κ.*
(3) *For some regular $\gamma < \kappa$, there is a γ-strongly measurable cardinal above κ.*
(4) *For all regular $\gamma < \kappa$, all sufficiently large regular cardinals are γ-strongly measurable.*
(5) *For some cardinal λ, for all regular $\gamma < \kappa$ and all ν with $\text{cf}(\nu) > \gamma$, $\mathcal{C}_{\nu,\gamma} \cap \text{HOD}$ is λ-saturated in* HOD.

It is easy to see that if the HOD hypothesis fails, then every regular cardinal δ contains an ω-club of ordinals that are strongly inaccessible in HOD. In fact, this alone implies a stronger conclusion.

Theorem 2.11. *Suppose N is an inner model and δ is a regular cardinal such that $\text{Reg}^N \cap \delta$ contains an ω-club. Then $\text{Reg}^N \cap \delta$ contains a club.*

Proof. Assume towards a contradiction that $\text{Sing}^N \cap \delta$ is stationary. Since the cofinality function as computed in N is regressive, there is an ordinal γ that is regular in N such that $(S^\delta_\gamma)^N$ is stationary. Let γ be the least such ordinal and let $S = (S^\delta_\gamma)^N$. We claim $\text{cf}(\gamma) = \omega$.

Choose $\langle c_\xi : \xi \in S\rangle \in N$ such that c_ξ is a closed unbounded subset of ξ with ordertype γ. Consider the set T of $\nu \in S^\delta_\omega$ such that there is some $\xi \geq \nu$ in S such that $c_\xi \cap \nu$ is cofinal in ν. Then T is stationary. To see this, fix a closed unbounded set $C \subseteq \delta$, and we will show that $C \cap T \neq \emptyset$. Fix $\xi \in S \cap \text{acc}(C)$. Since $\text{cf}(\xi) > \omega$, $C \cap \text{acc}(c_\gamma)$ is closed unbounded in γ, and so there is some $\nu \in C \cap \text{acc}(c_\gamma) \cap S^\delta_\omega$. By definition, $\nu \in T$. So $C \cap T \neq \emptyset$, as claimed.

Note that $T \subseteq (S^\delta_{<\gamma})^N$, so $(S^\delta_{<\gamma})^N$ is stationary. Again applying Fodor's lemma, there is an N-regular ordinal $\gamma' < \gamma$ such that $(S^\delta_{\gamma'})^N$ is stationary, and this contradicts the minimality of γ.

Since $(S^\delta_\gamma)^N$ is a stationary subset of S^δ_ω, its intersection with any ω-club is stationary. This implies that $(S^\delta_\gamma)^N \cap \text{Reg}^N$ is stationary, which is a contradiction. $\square$

Corollary 2.12. *Assume there is an ω_1-strongly compact cardinal κ and N is an ω-club amenable model such that $\lambda^{+N} < \lambda^+$ for some singular $\lambda > \kappa$. Then all sufficiently large regular cardinals contain a closed unbounded set of N-inaccessible cardinals.*

Proof. Suppose δ is ω-strongly measurable in N, and we will show that there is an ω-club of N-regular ordinals below δ. Suppose not. Then Sing^N is $\mathcal{C}_{\delta,\omega}$-positive. It follows that there is a set $S \in N$ such that $S \subseteq \text{Sing}^N$ and $\mathcal{C}_{\delta,\omega} \restriction S$ is a normal ultrafilter in N. This is a contradiction since $\text{Reg}^N \cap \delta$ belongs to any normal ultrafilter of N on δ.

Since all sufficiently large regular cardinals are ω-strongly measurable in N by Corollary 2.9, the desired conclusion follows from Theorem 2.11. □

Corollary 2.13. *Assume there is an ω_1-strongly compact cardinal and the* HOD *hypothesis fails. Then all sufficiently large regular cardinals contain a closed unbounded set of* HOD*-inaccessible cardinals.*

3. Embeddings of HOD

3.1. *Uniqueness of elementary embeddings*

The main theorem of [4] states that if $j_0, j_1 : V \to M$ are elementary embeddings, then $j_0 \restriction \text{Ord} = j_1 \restriction \text{Ord}$. We prove here that the HOD Hypothesis is equivalent to a similar uniqueness property of elementary embeddings. Notably this is a structural property of large cardinals that makes no reference to ordinal definability. If P and Q are transitive sets and $\delta \in \text{Ord} \cap P$, then a pair of elementary embeddings $j_0, j_1 : P \to Q$ are *δ-similar* if $j_0(\delta) = j_1(\delta)$ and $\sup j_0[\delta] = \sup j_1[\delta]$.

Theorem 3.1. *Suppose κ is a supercompact cardinal. Then the following are equivalent*:

(1) *The* HOD *hypothesis holds.*
(2) *For all regular cardinals δ, for all sufficiently large ordinals $\lambda > \delta$, if $j_0, j_1 : V_\lambda \to M$ are δ-similar elementary embeddings,*[b] *then $j_0 \restriction \delta = j_1 \restriction \delta$.*

Proof. Assume (1) holds. Let $\langle S_\alpha : \alpha < \delta \rangle$ be the least partition of S^δ_ω into stationary sets in the canonical well-order of HOD. (Note that this exists by Theorem 2.10.) For sufficiently large ordinals λ, $\langle S_\alpha : \alpha < \delta \rangle$ is definable from δ in V_λ. Suppose $j_0, j_1 : V_\lambda \to M$ are δ-similar. Then since $j_0(\delta) = j_1(\delta)$ and $\langle S_\alpha : \alpha < \delta \rangle$ is definable from δ in V_λ,

$$j_0(\langle S_\alpha : \alpha < \delta \rangle) = j_1(\langle S_\alpha : \alpha < \delta \rangle).$$

[b]Here M denotes a transitive set.

We let $\langle T_\alpha : \alpha < j_0(\delta)\rangle = j_0(\langle S_\alpha : \alpha < \delta\rangle)$. By Solovay's lemma [9], for $n = 0, 1$,

$$j_0[\delta] = \{\alpha < j_0(\delta) : T_\alpha \cap \sup j_0[\delta] \text{ is stationary}\},$$

$$j_1[\delta] = \{\alpha < j_1(\delta) : T_\alpha \cap \sup j_1[\delta] \text{ is stationary}\}$$

and so since j_0 and j_1 are δ-similar, $j_0[\delta] = j_1[\delta]$, proving (2).

Now assume (2). Let δ be a regular cardinal, and we will show that there is a normal fine ultrafilter on $P_\kappa(\delta)$ that concentrates on $\text{HOD} \cap P_\kappa(\delta)$. This implies the HOD hypothesis as in Case 2 of the proof of Proposition 2.3. Applying (2), let λ be large enough that if $j_0, j_1 : V_\lambda \to M$ are δ-similar elementary embeddings, then $j_0 \restriction \delta = j_1 \restriction \delta$. Let $j : V \to M$ be an elementary embedding such that $M^{V_\lambda} \subseteq M$. Then in M, $j[\delta]$ is ordinal definable, since it is equal to $i[\delta]$ for any elementary embedding $i : V_\lambda \to V_{j(\lambda)} \cap M$ such that $i(\delta) = j(\delta)$ and $\sup i[\delta] = \sup j[\delta]$. (The point is that $V_{j(\lambda)} \cap M$ is ordinal definable in M, although perhaps not in V.) Let $\mathcal{U}$ be the normal fine ultrafilter on $P_\kappa(\delta)$ derived from j using $j[\delta]$. Then $\text{HOD} \cap P_\kappa(\delta) \in \mathcal{U}$ since $j[\delta] \in \text{HOD}^M = j(\text{HOD})$. □

We prove two propositions that suggest that the failure of Theorem 3.1(2) is quite strong.

Proposition 3.2. *Suppose that $V_\lambda \preceq_{\Sigma_2} V$, $\delta < \lambda$ is regular, and there exist δ-similar elementary embeddings $j_0, j_1 : V_\lambda \to M$ such that $j_0[\delta] \neq j_1[\delta]$. Then there is an ω_1-strongly compact cardinal.*

Proof. Fix $\alpha < \delta$ such that $j_0(\alpha) \neq j_1(\alpha)$. Without loss of generality, assume $j_0(\alpha) < j_1(\alpha)$. Let γ be the αth regular cardinal that does not carry a countably complete uniform ultrafilter. Note that $\gamma < \lambda$ since $V_\lambda \preceq_{\Sigma_2} V$. Fix $S \subseteq \delta$ such that $|S| = \gamma$ and $j_0[S] = j_1[S]$. Then $j_1[S] \subseteq j_0(S)$, and so $j_1[S]$ is covered by a set in M of size less than $j_1(\gamma)$. It follows easily that $j_1[\gamma]$ is covered by a set in M of size less than $j_1(\gamma)$, and this implies that $\sup j_1[\gamma] < j_1(\gamma)$. Letting D be the ultrafilter on γ derived from j_1 using $\sup j_1[\gamma]$, we contradict the assumption that there is no uniform countably complete ultrafilter on γ. □

As a consequence, the failure of Theorem 3.1(2) has consistency strength beyond all hypotheses for which core model theory has been developed.

Proposition 3.3. *Suppose that $V_\lambda \preceq_{\Sigma_2} V$, $\delta < \lambda$ is regular, and there exist δ-similar elementary embeddings $j_0, j_1 : V_\lambda \to M$ such that $j_0[\delta] \subsetneq j_1[\delta]$. Then there is a nontrivial elementary embedding from* $\text{HOD} \cap V_\delta$ *to* $\text{HOD} \cap V_\delta$.

Proof. Define $\pi : \text{HOD} \cap V_\delta \to \text{HOD} \cap V_\delta$ by $\pi(x) = j_1^{-1}(j_0(x))$. □

3.2. *ω-Jónsson cardinals*

Recall that a cardinal λ is *ω-Jónsson* if for all functions $f : [\lambda]^\omega \to \lambda$, there is some $H \subseteq \lambda$ such that $\text{ot}(H) = \lambda$ and $\text{ran}(f \restriction [H]^\omega)$ is a proper subset of λ. The

Erdös-Hajnal theorem states that there is no such cardinal (at least if one assumes the Axiom of Choice). One can ask what happens when one places definability constraints on f.

Definition 3.4. A cardinal λ is *definably ω-Jónsson* if for all ordinal definable functions $f : [\lambda]^\omega \to \lambda$, there is some $H \subseteq \lambda$ such that $\mathrm{ot}(H) = \lambda$ and $\mathrm{ran}(f \restriction [H]^\omega)$ is a proper subset of λ.

Theorem 3.5. *Suppose κ is strongly compact. Then the following are equivalent*:

(1) *The* HOD *hypothesis fails.*
(2) *All sufficiently large regular cardinals are definably ω-Jónsson.*
(3) *Some regular cardinal above κ is definably ω-Jónsson.*

Proof. (1) implies (2): We prove the contrapositive using an argument due to Solovay [9]. Let R be the class of regular cardinals δ that are not definably ω-Jónsson. Fix $\delta \in R$, and we will show that there is a κ-complete fine ultrafilter on $P_\kappa(\delta)$ that concentrates on HOD. This completes the proof, because assuming (2) fails, R is a proper class, and hence HOD has the κ-cover property, which, by Proposition 2.3, is equivalent to the HOD hypothesis given a strongly compact cardinal.

To show that there is a κ-complete fine ultrafilter on $P_\kappa(\delta)$ that concentrates on HOD, it suffices to show that there is an elementary embedding $j : V \to M$ such that $j[\delta]$ is covered by a set of size less than $j(\kappa)$ that is ordinal definable in M. Since κ is strongly compact, there is an elementary embedding $j : V \to M$ such that in M, $j[\delta]$ is covered by a set of size less than $j(\kappa)$; we will show that the cover can be taken to be ordinal definable in M. Since δ is not definably ω-Jónsson, there is an ordinal definable function $f : [\delta]^\omega \to \delta$ such that for all $A \subseteq \delta$ such that $|A| = \delta$, $\mathrm{ran}(f \restriction [A]^\omega) = \delta$. We claim that any ω-closed unbounded subset C of $\sup j[\delta]$ such that $j(f)[C] \subseteq C$ contains $j[\delta]$. Let $B = j^{-1}[C]$. Then B is unbounded below δ and B is closed under f, and therefore $B \supseteq f[B] = \delta$. In other words, $j[\delta] \subseteq C$.

Working in M, let A be the intersection of all ω-closed unbounded subsets of $\sup j[\delta]$ that are closed under $j(f)$. Then $A \in M$ and $j[\delta] \subseteq A$ and A is ordinal definable from $j(f)$ in M. Therefore, $A \in \mathrm{HOD}^M$.

(2) implies (3): Trivial.

(3) implies (1): Suppose $\delta \geq \kappa$ is regular and definably ω-Jónsson, and assume towards a contradiction that the HOD hypothesis holds. Applying Theorem 2.10, let $\langle S_\alpha : \alpha < \delta \rangle \in \mathrm{HOD}$ partition S^δ_ω into stationary sets. Let $f : [\delta]^\omega \to \delta$ be defined by $f(s) = \alpha$ where $\alpha < \delta$ is the unique ordinal such that $\sup(s) \in S_\alpha$. Then for any unbounded set $T \subseteq \delta$, for each $\alpha < \delta$, there is some $s \in [T]^\omega$ such that $\sup s \in S_\alpha$, and therefore $f(s) = \alpha$. It follows that $\mathrm{ran}(f \restriction [T]^\omega) = \delta$. ◻

The usual characterizations of Jónsson cardinals in terms of elementary embeddings yields the following equivalence.

Corollary 3.6. *Assume there is a strongly compact cardinal. Then exactly one of the following holds*:

(1) *The* HOD *hypothesis.*
(2) *For all sufficiently large regular cardinals* δ, *for all ordinals* $\alpha > \delta$, *there is an elementary embedding* $j : M \to \mathrm{HOD} \cap V_\alpha$ *such that* $\mathrm{crit}(j) < \delta$, $j(\delta) = \delta$, *and* j *is continuous at ordinals of cofinality* ω.

3.3. *The rigidity of HOD*

This section is devoted to a proof of the following theorem.

Theorem 3.7. *Assume there is an* ω_1*-strongly compact cardinal. If the* HOD *hypothesis holds*, *then there is no nontrivial elementary embedding from* HOD *to* HOD.

We need to prove some preliminary lemmas. If P and Q are transitive models of ZFC^- and $j : P \to Q$ is an elementary embedding, an ordinal $\nu \in Q$ is a *generator of* j if ν is not definable in Q from elements of $j[P]$ and ordinals less than ν. If $\langle \nu_\beta : \beta < \alpha \rangle$ enumerates the generators of j, then j can be reconstructed from the directed system $\langle D_s : s \in [\alpha]^{<\omega} \rangle$ where D_s is the P-ultrafilter derived from j using $\langle \nu_\beta : \beta \in s \rangle$.

Lemma 3.8. *If* $j : \mathrm{HOD} \to \mathrm{HOD}$ *is a nontrivial elementary embedding, then* j *has a proper class of generators.*

Proof. If j had only a set of generators, then letting λ be their supremum, j is definable from the HOD-extender E of length λ derived from j. But Hamkins *et al.* [5] show that no nontrivial elementary embedding from HOD to HOD is definable from parameters over V. □

Lemma 3.9. *Suppose* κ *is* ω_1*-strongly compact and* H *is an* ω*-club amenable model such that arbitrarily large regular cardinals are not measurable in* H. *Then the Singular Cardinals Hypothesis holds in* H *above* $\kappa^+ \cdot (\kappa^\omega)^H$.

Proof. By Silver's theorem [8], it suffices to show that in H, for all singular cardinals $\lambda > 2^{<\kappa}$ with $\mathrm{cf}(\lambda) = \omega$, $\lambda^\omega = \lambda^+$. In fact, we will show that for all H-regular $\delta \geq \kappa^+$, there is a set $C \subseteq P_\kappa(\delta) \cap H$ in H such that $|C|^H = \delta$ and every countable $\sigma \subseteq \delta$ is contained in some $\tau \in C$. Then in particular, $([\delta]^\omega)^H \subseteq \bigcup_{\tau \in C} ([\tau]^\omega)^H$ and hence $(\delta^\omega)^H = (\kappa^\omega)^H \cdot \delta$.

To define C, we first split $S = (S^\delta_{<\kappa})^H$ into δ-many $\mathcal{C}_{\delta,\omega}$-positive sets in H. Note that S contains S^δ_ω by Lemma 2.7, and hence S has full measure with respect to $\mathcal{C}_{\delta,\omega}$. Now one can use a standard argument to partition S into δ-many $\mathcal{C}_{\delta,\omega}$-positive sets in H. Working in H, for each $\alpha \in S$, fix a club $c_\alpha \subseteq \alpha$ of ordertype less than κ. For $\xi < \kappa$, let $f_\xi(\alpha) = c_\alpha(\xi)$ for any α such that $\xi < \mathrm{ot}(c_\alpha)$.

We claim that there is some $\xi < \kappa$ such that for unboundedly many $\beta < \delta$, $f_\xi^{-1}\{\beta\}$ is $\mathcal{C}_{\delta,\omega}$-positive. Otherwise, using that $C_{\delta,\omega}$ is weakly normal and f_ξ is regressive, for each $\xi < \kappa$, there is an ordinal γ_ξ such that $f_\xi(\alpha) < \gamma_\xi$ for an ω-club of $\alpha \in S$. Since $\text{cf}(\delta) > \kappa$, there is then a single ω-club of α such that for all $\xi < \kappa$, $f_\xi(\alpha) < \gamma_\xi$ for all appropriate α. Letting $\gamma = \sup_{\xi<\kappa} \gamma_\xi < \delta$, we see that for an ω-club of α, for all appropriate $\xi < \kappa$, $f_\xi(\alpha) < \gamma$. But if $\alpha > \gamma$ belongs to this club, then the fact that $c_\alpha(\xi) = f_\xi(\alpha) < \gamma$ for all $\xi < \text{ot}(c_\alpha)$ contradicts that c_α was chosen to be unbounded in α.

Now fix $\xi < \kappa$ such that for unboundedly many $\beta < \delta$, $f_\xi^{-1}\{\beta\}$ is $\mathcal{C}_{\delta,\omega}$-positive. Since H is ω-club amenable and f_ξ belongs to H, the set B of such β belongs to H, and so since δ is regular in H, the ordertype of B is δ. Finally, let $S_\beta = f_\xi^{-1}\{\beta\}$. Then $\langle S_\beta : \beta \in B\rangle$ is the desired stationary partition.

Finally, for each $\xi < \delta$ of uncountable cofinality, let σ_ξ be the set of $\beta \in B$ such that S_β is $\mathcal{C}_{\delta,\xi}$-positive. Then $\langle \sigma_\xi : \xi < \delta\rangle$ is in H by ω-club amenability, and the argument of Lemma 2.7 shows that every countable subset of δ is contained in σ_ξ for some $\xi < \delta$. Therefore, $C = \{\sigma_\xi : \xi < \delta\}$ is as desired. □

Lemma 3.10. *Suppose δ is a cardinal, N and H are models with the (ω_1, δ)-cover property, and $(2^\lambda)^H = \lambda^{+H}$ for all sufficiently large strong limit cardinals λ of countable cofinality. Then any elementary embedding $k : N \to H$ with $\text{crit}(k) \geq \delta$ is an extender embedding.*

Proof. Note that k is continuous at ordinals of cofinality ω, since these have N-cofinality less than δ. We will show that there cannot exist a singular strong cardinal λ with the following properties:

- $\text{cf}(\lambda) = \omega$.
- λ is a limit of generators of k.
- $k(\lambda) = \lambda$.
- $(2^\lambda)^H = \lambda^{+H}$.

We first point out that $\lambda^{+H} = \lambda^{+N} = \lambda^+$. Using the (ω_1, δ)-cover property, $\lambda^+ \leq |P_\delta(\lambda) \cap H| \leq (2^\lambda)^H = \lambda^{+H}$. The argument for N is similar, using that $(2^\lambda)^N = \lambda^{+N}$ by elementarity.

Let $\sigma \subseteq \lambda$ be a countable cofinal set, and let $\tau \in H$ be a cover of σ of size less than δ. Let U be the N-ultrafilter on $P_\delta(\lambda) \cap N$ derived from k using τ. Let $j_U : N \to P$ be the ultrapower embedding and $i : P \to H$ be the factor embedding. Then $\tau = i([\text{id}]_U) \in \text{ran}(i)$, and so since $|\tau| < \delta$ and $\text{crit}(i) < \delta$, $\tau \subseteq \text{ran}(i)$. As a consequence, λ is a limit of generators of j_U, which implies $\lambda_U \geq \lambda$. But $(2^\lambda)^H = \lambda^{+H}$, and so by elementarity $(2^\lambda)^N = \lambda^{+N}$. Thus, $\lambda_U \leq |P_\delta(\lambda) \cap N| \leq (2^\lambda)^N \leq \lambda^{+N}$. Since $\lambda \leq \lambda_U \leq \lambda^{+N}$, we must have $j_U(\lambda) > \lambda$ or $j_U(\lambda^{+N}) > \lambda^{+N}$. This implies $k(\lambda) > \lambda$ or $k(\lambda^{+N}) > \lambda^{+N}$, contradicting either that $k(\lambda) = \lambda$ (by assumption) or that $k(\lambda^{+N}) = \lambda^{+H} = \lambda^{+N}$ (by the previous paragraph).

Since there is no such λ, the class of limits of generators of k does not intersect the ω-closed unbounded class of singular strong limit fixed points η of k of countable cofinality satisfying $(2^\eta)^H = \eta^{+H}$. It follows that the class of limits of generators of k is not closed unbounded, which means that the class of generators of k is bounded, or in other words, k is an extender embedding. □

Lemma 3.11. *Suppose $i : H \to N$ is an extender embedding of length at most δ such that $i[\delta] \subseteq \delta$ where δ is an H-cardinal of uncountable cofinality. If H has the (ω_1, δ)-cover property, then so does N.*

Proof. Suppose $\sigma \subseteq N$ is countable. Choose $f_n \in H$ and $a_n \in [\delta]^{<\omega}$ such that $\sigma = \{i(f_n)(a_n) : n < \omega\}$. Using the (ω_1, δ)-cover property of H, fix a family of functions $\mathcal{F} \in H$ such that $|\mathcal{F}| < \delta$ and $f_n \in \mathcal{F}$ for all $n < \omega$. Let $\gamma = \sup(\bigcup_{n<\omega} a_n)$, so $\gamma < \delta$ since δ has uncountable cofinality. Let $\mathcal{G} = i(\mathcal{F})$, so $\mathcal{G} \in N$ and $|\mathcal{G}| < \delta$ since $i[\delta] \subseteq \delta$. Then

$$\tau = \{g(a) : g \in \mathcal{G},\, a \in [\gamma]^{<\omega}\}$$

belongs to N, covers σ, and has cardinality less than δ. □

Proof of Theorem 3.7. Suppose $j : \mathrm{HOD} \to \mathrm{HOD}$ is an elementary embedding and assume towards a contradiction that it is nontrivial. Let κ be the least ω_1-strongly compact cardinal. By Lemma 2.7, HOD has the (ω_1, κ)-cover property. Let $\delta \geq \kappa$ be a cardinal of uncountable cofinality such that $j[\delta] \subseteq \delta$, let $i : \mathrm{HOD} \to N$ be given by the extender of length δ derived from j, and let $k : N \to \mathrm{HOD}$ be the factor embedding, so $\mathrm{crit}(k) \geq \delta$.

By Lemma 3.11, N has the (ω_1, δ)-cover property. Lemma 3.9 implies that HOD satisfies the cardinal arithmetic condition of Lemma 3.10, and so k is an extender embedding. Thus, $j = k \circ i$ is an extender embedding, contradicting Lemma 3.8. □

4. Large Cardinals in HOD

4.1. *HOD under AD*

Recall that a cardinal κ is (ν, δ)-*strongly compact* if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) \geq \nu$ and M has the $(\delta^+, j(\kappa))$-cover property. An ultrafilter U *witnesses that κ is (ν, δ)-strongly compact* if $\mathrm{crit}(j_U) \geq \nu$ and M_U has the $(\delta^+, j(\kappa))$-cover property. If N is an inner model and U is an ultrafilter, then U *witnesses that κ is δ-strongly compact in* N if $U \cap N \in N$ and $U \cap N$ witnesses that κ is δ-strongly compact in N.

We say an inner model N has the κ-*cover property below* δ if $P_\kappa(\delta) \cap N$ is cofinal in $P_\kappa(\delta)$; N has the κ-*approximation property below* δ if N contains every $A \subseteq \delta$ such that $A \cap \sigma \in N$ for all $\sigma \in P_\kappa(\delta)$.

Theorem 4.1. *Assume* AD^+ *and* $V = L(P(\mathbb{R}))$*. Suppose* $\kappa < \Theta$ *is a regular cardinal and* $\delta \geq \kappa$ *is a* HOD*-regular ordinal. Then the following are equivalent*:

(1) $(S^\delta_{<\kappa})^{\mathrm{HOD}}$ *is stationary.*
(2) HOD *has the* κ*-cover property below* δ*.*
(3) HOD *has the* κ*-cover and* κ*-approximation properties below* δ*.*
(4) $\mathcal{C}_{\delta,\omega}$ *witnesses that* κ *is* δ*-strongly compact in* HOD*.*
(5) *Some countably complete ultrafilter witnesses* κ *is* (ω_1, δ)*-strongly compact in* HOD*.*
(6) HOD *has the* (ω_1, κ)*-cover property below* δ*.*
(7) $S^\delta_\omega \subseteq (S^\delta_{<\kappa})^{\mathrm{HOD}}$*.*

Proof. (1) implies (2) Let $T = (S^\delta_{<\kappa})^{\mathrm{HOD}}$. We claim that $S^\delta_\omega \cap T$ is stationary. Fix a sequence $\langle c_\alpha : \alpha \in T\rangle \in \mathrm{HOD}$ such that c_α is a closed cofinal subset of α of ordertype $\mathrm{cf}^{\mathrm{HOD}}(\alpha)$. Let

$$S = \{\beta \in S^\delta_\omega : \exists\, \alpha \in T \sup(c_\alpha \cap \beta) = \beta\}.$$

Suppose C is closed unbounded, and we will show $C \cap S \neq \emptyset$. Since T is stationary, there is some β in $\mathrm{acc}(C)$ such that $\sup(c_\alpha \cap \beta) = \beta$. We claim that the least such β has countable cofinality. If not, then β has uncountable cofinality, so $C \cap c_\alpha \cap \beta$ is closed unbounded. But fixing any $\beta' \in \mathrm{acc}(C \cap c_\alpha \cap \beta)$, we have $\beta' \in \mathrm{acc}(C)$ and $\sup(c_\alpha \cap \beta') = \beta'$, contrary to the minimality of β.

Next, we construct a sequence $\langle \sigma_\xi : \xi \in T\rangle$ such that $\sigma_\xi \in P_\kappa(\xi)$ and for all α, for $\mathcal{C}_{\delta,\omega}$-almost all ξ, $\alpha \in \sigma_\xi$. This proceeds as in Theorem 2.8, using however that $\mathcal{C}_{\delta,\omega}$ is an ultrafilter. As in Theorem 2.8, one can use this sequence to show $\mathrm{cf}(\delta) = |\delta|$. Let $C \subseteq \delta$ be a closed cofinal set of ordertype $\mathrm{cf}(\delta)$. Then $\delta = \bigcup_{\xi \in C} \sigma_\xi$. Therefore, $|\delta| = |C| \cdot \sup_{\xi \in C} \sigma_\xi$, so as in Theorem 2.8, $|\delta| = |C| = \mathrm{cf}(\delta)$.

Finally, suppose $\sigma \in P_\kappa(\delta)$. The set $\{\xi \in T : \sigma \subseteq \sigma_\xi\}$ is the intersection of fewer than κ-many sets in $\mathcal{C}_{\delta,\omega}$, and so it belongs to $\mathcal{C}_{\delta,\omega}$. It follows that there is some $\xi \in T$ such that $\sigma \subseteq \sigma_\xi$. This shows that HOD has the κ-cover property below δ.

(2) implies (3) In HOD, let $\langle X_\alpha : \alpha < \delta\rangle$ be a κ-independent family of subsets of some set S. Suppose $A \subseteq \delta$ and $A \cap \sigma \in \mathrm{HOD}$ for all $\sigma \in P_\kappa(\delta) \cap \mathrm{HOD}$. For $\alpha < \delta$, let

$$Y_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \in A,\\ S \backslash X_\alpha & \text{otherwise.}\end{cases}$$

Suppose $\sigma \in P_\kappa(\delta)$. We will show that $\bigcap_{\alpha\in\sigma} Y_\alpha \neq \emptyset$. First, let $\tau \in \mathrm{HOD}$ cover σ. Then by our assumption on A, $\tau \cap A \in \mathrm{HOD}$. It follows that $\langle Y_\alpha : \alpha \in \tau\rangle \in \mathrm{HOD}$, and therefore since $\langle X_\alpha : \alpha < \delta\rangle$ is κ-independent in HOD, $\bigcap_{\alpha\in\tau} Y_\alpha \neq \emptyset$. Since $\sigma \subseteq \tau$, $\bigcap_{\alpha\in\sigma} Y_\alpha \neq \emptyset$. Let F be the filter generated by $\{Y_\alpha : \alpha < \delta\}$. Then F is a κ-complete filter on a well-orderable set, and so F extends to a countably complete ultrafilter U. Applying AD, $U \cap \mathrm{HOD} \in \mathrm{HOD}$, and therefore $A = \{\alpha < \delta : X_\alpha \in U\}$ belongs to HOD as well.

(3) implies (4) $(S^{\delta}_{<\kappa})^{\text{HOD}} \in \mathcal{C}_{\delta,\omega}$ and $C_{\delta,\omega}$ is κ-complete since $\text{cf}(\delta) \geq \kappa$. Therefore in HOD, $\mathcal{C}_{\delta,\omega} \cap \text{HOD}$ is a κ-complete, weakly normal ultrafilter on δ concentrating on ordinals of cofinality less than κ, and so it witnesses that κ is δ-strongly compact by a theorem of Ketonen [6].

(4) implies (5). Trivial.

(5) implies (6). Let U be a countably complete ultrafilter on δ such that $U \cap \text{HOD}$ witnesses that κ is (ω_1, δ)-strongly compact in HOD. Let $f : \delta \to P_\kappa(\delta) \cap \text{HOD}$ push $U \cap \text{HOD}$ forward to a fine ultrafilter in HOD. Let $\mathcal{U} = f_*(U)$. Then $\mathcal{U}$ is a fine countably complete ultrafilter, and therefore for all $\sigma \in P_{\omega_1}(\delta)$, the set $\{\tau \in P_\kappa(\delta) : \sigma \subseteq \tau\}$ belongs to $\mathcal{U}$. Since $P_\kappa(\delta) \cap \text{HOD} \in \mathcal{U}$, it follows that there is some $\tau \in P_\kappa(\delta) \cap \text{HOD}$ such that $\sigma \subseteq \tau$. This shows that HOD has the (ω_1, δ)-cover property.

(6) implies (7) Trivial.

(7) implies (1). Trivial. □

In the context of $\text{AD}^+ + V = L(P(\mathbb{R}))$, Woodin's HOD-*ultrafilter conjecture* asserts that every countably complete ultrafilter of $\text{HOD} \cap V_\Theta$ extends to a countably complete ultrafilter.

Theorem 4.2. *Assume* AD^+, $V = L(P(\mathbb{R}))$, *and the* HOD-*ultrafilter conjecture. Suppose $\kappa < \Theta$ is a regular cardinal and $\delta \geq \kappa$ is a* HOD-*regular ordinal. Then κ is δ-strongly compact in* HOD *if and only if κ is (ω_1, δ)-strongly compact in* HOD.

4.2. *Weak extender models*

An ultrafilter U on an ordinal λ is κ-complete if and only if it is locally principal in the sense that for each set $\sigma \subseteq P(\lambda)$ with $|\sigma| < \kappa$, U coincides on σ with a principal filter. This allows us to assign to each κ-complete ultrafilter U on an ordinal λ a *trace function* $\text{tr}_U : P_\kappa(P(\lambda)) \to \lambda$: for each set $\sigma \subseteq P(\lambda)$ with $|\sigma| < \kappa$, one can let $\text{tr}_U(\sigma)$ denote the least $\alpha < \lambda$ such that U agrees on σ with the principal ultrafilter concentrated at α. One can compare the complexity of two ultrafilters by comparing the growth rates of their trace functions: for example, one may set $U < W$ if there is a closed unbounded set of $\sigma \subseteq P(\lambda)$ such that $\text{tr}_U(\sigma) < \text{tr}_W(\sigma)$. More generally, any fine filter on $P_\kappa(P(\lambda))$ can be used to define a similar order.

Definition 4.3. Suppose $\mathcal{F}$ is a fine filter on $P_\kappa(P(\lambda))$ and U and W are κ-complete ultrafilters on λ. Then $U <_{\mathcal{F}} W$ if $\text{tr}_U(\sigma) < \text{tr}_W(\sigma)$ for $\mathcal{F}$-almost all $\sigma \in P_\kappa(P(\lambda))$.

Lemma 4.4. *Suppose $\mathcal{F}$ is a fine filter on $P_\kappa(P(\lambda))$. Then $<_{\mathcal{F}}$ is a strict partial order. If $\mathcal{F}$ is countably complete, then $<_{\mathcal{F}}$ is well-founded. If $\mathcal{F}$ is an ultrafilter, then $<_{\mathcal{F}}$ is linear.*

Thus, $<_{\mathcal{F}}$ is a well-order if $\mathcal{F}$ is a countably complete fine ultrafilter on $P_\kappa(P(\lambda))$. Therefore, under $\text{AD}_{\mathbb{R}}$, $<_{\mathcal{F}}$ is a well-order where $\mathcal{F}$ is the closed unbounded filter on $P_\kappa(P(\lambda))$. The Ultrapower Axiom [3] implies that if $\mathcal{F}$ is the closed unbounded filter on $P_{\omega_1}(P(\lambda))$, then $<_{\mathcal{F}}$ is a well-order, even though in this context (that is, assuming the Axiom of Choice) the closed unbounded filter is not an ultrafilter.

We will use the order $<_{\mathcal{F}}$ in the proof of the following theorem.

Theorem 4.5. *Suppose κ is strongly compact and N is an inner model of* ZFC *with the κ-cover property. Then there is a minimum extension of N to a model of* ZFC *with the κ-approximation property.*

The proof uses the following facts.

Lemma 4.6. *Suppose δ is a regular cardinal, κ is δ-strongly compact, N is an inner model of* ZFC *with the κ-cover property, and M is an inner model containing N such that every κ-complete N-ultrafilter on δ belongs to M. Then every subset of δ that is κ-approximated by N belongs to M.*

Proof. In N, fix a κ-independent family $\mathscr{F}$ of subsets of a set X with $|X|^N = \delta$. It suffices to show that every set $A \subseteq \mathscr{F}$ that is κ-approximated by N belongs to M. Let $B = \{X \setminus Y : Y \in \mathscr{F} \setminus A\}$. Then any $<\kappa$-sized intersection of sets in $A \cup B$ is nonempty: if $\sigma \in P_\kappa(A \cup B)$, let $\tau \in N$ cover σ, and note that since A is κ-approximated by N, $\tau \cap A \in N$ and $\tau \cap B \in N$. By the κ-independence of $\mathscr{F}$ in N, $\bigcap \tau \cap (A \cup B) \neq \emptyset$, and so $\bigcap \sigma \neq \emptyset$. It follows that $A \cup B$ extends to a κ-complete ultrafilter U on X, and so since every κ-complete N-ultrafilter on δ belongs to M and $|X|^N = \delta$, $U \cap N \in M$. But $A = \{Y \in \mathscr{F} : Y \in U \cap N\}$, and so $A \in M$. $\square$

The following theorem is an important part of Hamkins's proof of the Laver–Woodin theorem on the definability of the ground model over a forcing extension [7]; the proof is included since it is short.

Theorem 4.7 (Hamkins). *Suppose $\kappa < \lambda$ are cardinals,* $\mathrm{cf}(\lambda) \geq \kappa$ *and M and M' are inner models with the κ-approximation and cover properties below λ such that $P_\kappa(\kappa^+) \cap M = P_\kappa(\kappa^+) \cap M'$. Then $P(\lambda) \cap M = P(\lambda) \cap M'$.*

Proof. It suffices to show that $P_\kappa(\lambda) \cap M = P_\kappa(\lambda) \cap M'$. Suppose $\sigma \in P_\kappa(\lambda)$. We will show that $\sigma \in M$ if and only if $\sigma \in M'$.

There is a set $\tau \in P_{\kappa^+}(\lambda)$ such that $\tau \in M \cap M'$ and $\sigma \subseteq \tau$. To see this, use the κ-cover property to construct an increasing continuous sequence $\langle \tau_\alpha \rangle_{\alpha < \kappa}$ sequence of sets in $P_\kappa(\lambda)$ such that $\sigma \subseteq \tau_0$, $\tau_{\alpha+2n} \in M$ and $\tau_{\alpha+2n+1} \in M'$ for limit ordinals $\alpha < \kappa$. Then $\tau = \bigcup_{\alpha<\kappa} \tau_\alpha$ is as desired since τ is κ-approximated by both M and M'.

Let $\nu < \kappa^+$ be the ordertype of τ, and let $f : \nu \to \tau$ be the increasing enumeration of τ. Then f is in both M and M', and so $\sigma \in M$ if and only if $f^{-1}[\sigma] \in M$ if and only if $f^{-1}[\sigma] \in M'$ if and only if $\sigma \in M'$. $\square$

Proof of Theorem 4.5. Suppose $\mathcal{U}$ is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$ where λ is a strong limit cardinal λ of cofinality at least κ and $P_\kappa(\lambda) \cap N \in \mathcal{U}$. Suppose

$f : \lambda \to P_{\text{bd}}(\lambda) \cap N$ is a surjection in N. Let $\tilde{\mathcal{U}}$ denote the pushforward of $\mathcal{U}$ by the function $\tilde{f} : P_\kappa(\lambda) \to P_\kappa(P_{\text{bd}}(\lambda))$ given by $\tilde{f}(\sigma) = f[\sigma]$. Let $\vec{U} = \vec{U}_{\mathcal{U},f} = \langle U_\alpha : \alpha < \lambda \rangle$ enumerate the κ-complete N-ultrafilters on ordinals less than λ in the well-order $<_{\tilde{\mathcal{U}}}$. Let N' be the inner model $L[\vec{U}, f]$ and let f' be the increasing enumeration of $P_{\text{bd}}(\lambda) \cap N'$ in the following order: for $a, b \in P_{\text{bd}}(\lambda) \cap N'$, set $a < b$ if either $\sup(a) < \sup(b)$ or a precedes b in the canonical well-order of $L[\vec{U}, f]$ and $\sup(a) = \sup(b)$.

Note that any model M with the κ-approximation property that contains N must contain the set of κ-complete N-ultrafilters on ordinals less than λ and the order on it induced by $\tilde{\mathcal{U}}$. Hence $N' \subseteq M$.

Iterating this procedure yields, for each ordinal γ, an inner model $N_\gamma = N_{\mathcal{U},f,\gamma}$ of ZFC, a function f_γ, and a λ-sequence $\vec{U}_\gamma$ enumerating the κ-complete N_γ-ultrafilters on ordinals less than λ. Specifically (although still somewhat informally), let $N_\gamma = L[\langle \vec{U}_\xi, f_\xi : \xi < \gamma \rangle]$, let $f_\gamma : \lambda \to P_{\text{bd}}(\lambda) \cap N_\gamma$ be the increasing enumeration of $P_{\text{bd}}(\lambda) \cap N_\gamma$ in a well-order similar to the one described in the first paragraph, and finally define $\vec{U}_\gamma = \vec{U}_{\mathcal{U},f_\gamma}$ as in the first paragraph.

The uniformity of this procedure guarantees that any model M such that $N \subseteq M$ and $\mathcal{U} \cap M \in M$ contains N_γ for all ordinals γ.

The sequence $\langle N_\gamma : \gamma < \kappa^+ \rangle$ is increasing, so let $\gamma = \gamma_{\mathcal{U},f}$ be the least ordinal $\alpha < \kappa^+$ such that $N_\alpha \cap P_{\text{bd}}(\kappa) = N_{\alpha+1} \cap P_{\text{bd}}(\kappa)$. We claim that $N_{\gamma+1}$ has the κ-approximation property below λ.

We first show that $P_\kappa(\lambda) \cap N_\gamma = P_\kappa(\lambda) \cap N_{\gamma+1}$. If $\sigma \in P_\kappa(\lambda) \cap N_{\gamma+1}$, then there is some $\tau \in P_\kappa(\lambda) \cap N$ such that $\sigma \subseteq \tau$. Now $\sigma \in N_\gamma$, and since $P_{\text{bd}}(\kappa) \cap N_\gamma = P_{\text{bd}}(\kappa) \cap N_{\gamma+1}$, $P(\tau) \cap N_\gamma = P(\tau) \cap N_{\gamma+1}$; the latter set contains σ, and hence $\sigma \in N_\gamma$ as desired. By Lemma 4.6 and the definition of $N_{\gamma+1}$, every bounded subset of λ that is κ-approximated by N_γ belongs to $N_{\gamma+1}$. Since $P_\kappa(\lambda) \cap N_\gamma = P_\kappa(\lambda) \cap N_{\gamma+1}$, every subset of λ that is κ-approximated by $N_{\gamma+1}$ is κ-approximated by N_γ. Thus, $N_{\gamma+1}$ has the κ-approximation property below λ.

Now for each pair $(\mathcal{U}, f)$ such that there is a strong limit cardinal λ of cofinality at least κ such that $\mathcal{U}$ is a κ-complete fine ultrafilters on $P_\kappa(\lambda)$ and $f : \lambda \to P_{\text{bd}}(\lambda)$ is a surjection in N, let $M_{\mathcal{U},f} = N_{\mathcal{U},f,\gamma} \cap H(\lambda)$, where $\gamma = \gamma_{\mathcal{U},f}$. Reiterating what we have proved above, any inner model M of ZFC such that $N \subseteq M$ and $\mathcal{U} \cap M \in M$ must contain $M_{\mathcal{U},f}$, and therefore any inner model M with the κ-approximation property contains $\bigcup_{\mathcal{U},f} M_{\mathcal{U},f}$.

Fix $S \subseteq P_\kappa(\kappa^+)$ such that for a proper class of appropriate $\mathcal{U}$ and f, $M_{\mathcal{U},f} \cap P_\kappa(\kappa^+) = S$. Let C be the class of pairs $(\mathcal{U}, f)$ such that $M_{\mathcal{U},f} \cap P_\kappa(\kappa^+) = S$. Then by Hamkins's theorem (Theorem 4.7), for all $u, v \in C$, either $M_u \subseteq M_v$ or $M_v \subseteq M_u$. Thus, $M = \bigcup_{u \in C} M_u$ is an inner model of ZFC, and since each M_u has the κ-approximation property below λ, M has the κ-approximation property. Finally, $N \subseteq M$ since for all but a set of $u \in C$, $N \cap H(\lambda) \subseteq M_u$. □

The construction of the previous theorem is much simpler when N is an inner model that not only has the κ-cover property (i.e. is positive for the fine filter) but also is positive for the supercompactness filters $\mathcal{N}_{\kappa,\lambda}$, defined for all $\kappa \leq \lambda$ as the intersection of the κ-complete normal fine ultrafilters on $P_\kappa(\lambda)$.

Definition 4.8. An inner model N is *(κ, λ)-supercompact* if $N \cap P_\kappa(\lambda)$ is $\mathcal{N}_{\kappa,\lambda}$-positive; N is (κ, ∞)-supercompact if it is (κ, λ)-supercompact for all cardinals λ.

Note that N is (κ, λ)-supercompact if and only if $N \cap P_\kappa(\lambda)$ belongs to some κ-complete normal fine ultrafilter on $P_\kappa(\lambda)$, and if N is (κ, λ)-supercompact for some ordinal λ, then N is (κ, α)-supercompact for all $\alpha < \lambda$. The notion bears an obvious resemblance to Woodin's weak extender models.

Definition 4.9. An inner model N of ZFC is a *weak extender model of κ is λ-supercompact* if there is a κ-complete normal fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ such that $\mathcal{U} \cap N \in N$ and $P_\kappa(\lambda) \cap N \in \mathcal{U}$; N is a *weak extender model of κ is supercompact* if it is a weak extender model of κ is λ-supercompact for all cardinals λ.

The substantive part of the following characterization of weak extender models is due to Woodin and Usuba independently.

Lemma 4.10. *An inner model of* ZFC *is a weak extender model of κ is supercompact if and only if it is (κ, ∞)-supercompact and has the κ-approximation property.*

An inner model M is *$<\kappa$-closed* if every subset of M of cardinality less than κ belongs to M. The following theorem shows that the (κ, ∞)-supercompact inner models are the inner models that appear to be $<\kappa$-closed in some weak extender model.

Theorem 4.11. *If N is a (κ, ∞)-supercompact inner model of* ZFC, *then there is a weak extender model of κ is supercompact that contains N as a $<\kappa$-closed inner model.*

Proof. The reverse direction is obvious, so we focus on the forwards direction. Let λ be a cardinal and let $\mathcal{U}$ be a κ-complete normal fine ultrafilter on $P_\kappa(\lambda)$ such that $P_\kappa(\lambda) \cap N \in N$, and let $j : V \to M$ be the associated ultrapower embedding. Let $W = j(N)$. We claim W contains every subset A of λ that is κ-approximated by N.

To see this, note that since W has the $j(\kappa)$-cover property in M, we can find $\sigma \in P_{j(\kappa)}(j(\lambda))$ such that $j[A] \subseteq \sigma$. But $j(A)$ is $j(\kappa)$-approximated by W, so $j(A) \cap \sigma \in W$. Since $P_\kappa(\lambda) \cap N \in \mathcal{U}$, $j[\lambda] \in W$, and this implies $A = j^{-1}[j(A) \cap \sigma] \in W$, as desired.

Next, a familiar argument shows that since $N \cap P(\lambda) \subseteq W$, $N \cap V_\kappa = W \cap V_\kappa$, and N has the κ-cover property, in fact, $P_\kappa(\lambda) \cap W \subseteq N$. This implies that W has the κ-approximation property below λ.

It follows that for any κ-complete normal fine ultrafilters $\mathcal{U}$ and $\mathcal{U}'$ with $P_\kappa(\lambda) \cap N \in \mathcal{U} \cap \mathcal{U}'$, $j_\mathcal{U}(N) \cap P(\lambda) = j_{\mathcal{U}'}(N) \cap P(\lambda)$ by the preceding remarks and the Hamkins uniqueness theorem. Letting X_λ be the union of $j_\mathcal{U}(N) \cap P(\lambda)$

for all normal fine κ-complete $\mathcal{U}$ such that $P_\kappa(\lambda) \cap N \in \mathcal{U}$, it follows that $M = L(\bigcup_{\lambda \in \mathrm{Ord}} X_\lambda)$ is a model of ZFC with the κ-approximation and cover properties and N is a $<\kappa$-closed inner model of M. Since N is (κ, ∞)-supercompact and M has the κ-approximation property, Lemma 4.10 implies that M is a weak extender model of κ is supercompact. □

A cardinal κ is *distributively supercompact* if for all cardinals λ, there is a κ-distributive partial order $\mathbb{P}$ such that κ is λ-supercompact in $V^{\mathbb{P}}$.

Corollary 4.12 (HOD Hypothesis). *Suppose κ is supercompact. Then there is a unique weak extender model of κ is supercompact that contains* HOD *as a $<\kappa$-closed inner model. In particular,* HOD *satisfies that κ is distributively supercompact.*

The following corollary shows that the first-order theory of HOD has an influence on the question of whether the least supercompact cardinal is supercompact in HOD. One could actually replace UA in the argument with any Π_2 sentence that implies that every countably complete ultrafilter on an ordinal is ordinal definable.

Theorem 4.13. *Suppose κ is a supercompact cardinal and the* HOD *hypothesis holds. If $V_\kappa \cap \mathrm{HOD}$ satisfies the Ultrapower Axiom, then κ is supercompact in* HOD.

Proof. Let N be the inner model of Corollary 4.12. Then N is definable without parameters and $V_\kappa \cap \mathrm{HOD} = V_\kappa \cap N \preceq_{\Sigma_2} N$ since κ is supercompact in N. Since UA is Π_2, N satisfies UA. Now working in N, we apply the following consequence of UA: if κ is supercompact and A is a set such that $V_\kappa \subseteq \mathrm{HOD}_A$, then $V = \mathrm{HOD}_A$. Therefore, $N = (\mathrm{HOD}_A)^N$ for any set of ordinals $A \in N$ such that $V_\kappa \cap N \subseteq (\mathrm{HOD}_A)^N$. Let $A \in \mathrm{HOD}$ be a set of ordinals such that $V_\kappa \cap \mathrm{HOD} \subseteq L[A]$. Then $A \in N$ and $N = (\mathrm{HOD}_A)^N$, so there is a well-order of N definable over N from A. Since N is definable without parameters and A is ordinal definable, this well-order of N is definable from an ordinal parameter. Any ordinal definably well-ordered transitive class is contained in HOD, so $N \subseteq \mathrm{HOD}$. Therefore, $\mathrm{HOD} = N$, and so κ is supercompact in HOD. □

References

[1] Y. Cheng, S.-D. Friedman and J. D. Hamkins, Large cardinals need not be large in HOD, *Ann. Pure Appl. Logic* **166**(11) (2015) 1186–1198.

[2] K. Devlin and R. Jensen, Marginalia to a theorem of silver, in *ISILC Logic Conference* (Springer, 1975), pp. 115–142.

[3] G. Goldberg, The Ultrapower Axiom, Ph.D. thesis, Harvard University (2019).

[4] G. Goldberg, The uniqueness of elementary embeddings, to appear.

[5] J. D. Hamkins, G. Kirmayer and N. L. Perlmutter, Generalizations of the Kunen inconsistency, *Ann. Pure Appl. Logic* **163**(12) (2012) 1872–1890.

[6] J. Ketonen, Strong compactness and other cardinal sins, *Ann. Math. Logic* **5** (1972/1973) 47–76.

[7] R. Laver, Certain very large cardinals are not created in small forcing extensions, *Ann. Pure Appl. Logic* **149**(1–3) (2007) 1–6.

[8] J. Silver, On the singular cardinals problem, in *Proc. Int. Congr. Mathematicians* (*Vancouver, B. C.,* 1974), Vol. 1 (Canadian Mathematical Congress, Montreal, Quebec, 1975), pp. 265–268.

[9] R. M. Solovay, Strongly compact cardinals and the GCH, in *Proc. Tarski Symposium* (*Proc. Sympos. Pure Math.*, *Vol. XXV, University of California*, *Berkeley*, *California,* 1971) (American Mathematical Society, Providence, RI, 1974), pp. 365–372.

[10] W. H. Woodin, In search of Ultimate-L: The 19th Midrasha Mathematicae Lectures, *Bull. Symb. Log.* **23**(1) (2017) 1–109.