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Strong compactness and the ultrapower axiom I: The least strongly compact cardinal

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The Ultrapower Axiom is a combinatorial principle concerning the structure of large cardinals that is true in all known canonical inner models of set theory. A longstanding test question for inner model theory is the equiconsistency of strongly compact and supercompact cardinals. In this paper, it is shown that under the Ultrapower Axiom, the least strongly compact cardinal is supercompact. A number of stronger results are established, setting the stage for a complete analysis of strong compactness and supercompactness under UA that will be carried out in the sequel to this paper.

Keywords: Supercompact cardinal; strongly compact cardinal; ultrafilters; Ultrapower Axiom.

1. Introduction

How large is the least strongly compact cardinal?[a] Keisler-Tarski [19] asked whether it must be larger than the least measurable cardinal, and Solovay later conjectured that it is much larger: in fact, he conjectured that every strongly compact cardinal is supercompact.[b] His conjecture was refuted by Menas [15], who showed that the least strongly compact cardinal that is a limit of strongly compact cardinals is not supercompact. Still, Tarski's original question was left unresolved until in a remarkable pair of independence results, Magidor showed that the size of the least strongly compact cannot be determined using only the standard axioms of set theory (ZFC). More precisely, it is consistent with ZFC that the least strongly compact cardinal is the least measurable cardinal, but it is also consistent that the least strongly compact cardinal is the least supercompact cardinal. Since Magidor's

[a]See Definition 2.55.
[b]See Definition 2.36.

result, an extensive literature has sprung up concerning what Magidor termed the "identity crisis" for the first strongly compact cardinal, showing that there is essentially nothing one can prove about the size of the first strongly compact cardinal in ZFC.

The relationship between strongly compact cardinals and supercompact cardinals in terms of consistency strength remains a complete mystery, and in fact, one of the most prominent open questions in set theory is whether the existence of a strongly compact cardinal is equiconsistent with the existence of a supercompact cardinal. There are good reasons to believe that this question cannot be answered without generalizing inner model theory to the level of strongly compact and supercompact cardinals. One reason is that there are many analogous equiconsistency results lower in the large cardinal hierarchy, and all of them require inner model theory: for example, the equiconsistency of weakly inaccessible cardinals and strongly inaccessible cardinals, of weakly compact cardinals and the tree property, of Jonsson cardinals and Ramsey cardinals [16], of tall cardinals and strong cardinals [4]. In fact, almost every nontrivial equiconsistency theorem in set theory involves inner model theory.

While there is a great deal of evidence that inner model theory is required for a solution to the equiconsistency problem, this evidence does not bear on whether the equiconsistency actually holds. There are a number of striking similarities between the theories of strong compactness and supercompactness that seem to provide some evidence that there is some deeper connection between the two concepts. But there are plenty of similar analogies between large cardinal notions that are *not* equiconsistent.

The main theorem of this paper roughly shows that in any canonical inner model built by anything like today's inner model theoretic methodology, the least strongly compact cardinal is supercompact. This suggests that strong compactness and supercompactness really are equiconsistent: with perhaps a few exceptions, the consistency order on large cardinal axioms coincides with the size of their least instance in a canonical inner model.[c]

The precise statement of our theorem involves a combinatorial principle called the Ultrapower Axiom (UA). This principle holds in all known canonical inner models and is expected to hold in any canonical inner model built in the future. We prove:

Theorem 5.17 (UA). *The least strongly compact cardinal is supercompact.*

[c]For example, Woodin cardinals exceed strong cardinals in consistency strength, and accordingly the least ordinal that is strong in a Mitchell-Steel extender model is strictly smaller than the least ordinal that is Woodin in a Mitchell-Steel extender model (even though in V, the least Woodin cardinal is smaller than the least strong cardinal: the existence of a Woodin cardinal is equivalent to a Σ_2-sentence and if κ is strong, then $V_\kappa \preceq_{\Sigma_2} V$). A possible counterexample: by a theorem of Shelah-Mekler [14], the consistency strength of a cardinal every stationary subset of which reflects is strictly less than a weakly compact, even though a theorem of Jensen [9] states that in the constructible universe, all such cardinals are weakly compact.

To say more about the Ultrapower Axiom, we must discuss the fundamental theorem of inner model theory: the Comparison Lemma. The key to inner model theory at the level of Woodin cardinals is a fine structural analysis of models of a weak set theory approximating the inner model under construction, which for obscure reasons, are known as *mice.* There is a natural way to attempt to form iterated ultrapowers of mice, and a mouse is said to be iterable if this process never breaks down. The class of mice is constructed by recursion and analyzed by induction. To verify that the mice constructed so far are canonical, and to analyze them well enough to keep the induction going, one must prove the Comparison Lemma. Very roughly, this is the statement that any two iterable mice M_0 and M_1 have iterated ultrapowers N_0 and N_1 such that either $N_0 \subseteq N_1$ or $N_1 \subsetneq N_0$. From this lemma and its variants flow all the other structural properties of mice. The inner model one produces is the union of a proper class of mice. The known inner model constructions are best viewed as an attempt to produce models that contain large cardinals and satisfy the Comparison Lemma. The tension between these two constraints is the driving force behind the whole subject.

The Ultrapower Axiom is an abstract form of the Comparison Lemma that can be stated without reference to the fine structural details of a particular inner model construction. UA roughly asserts that any two wellfounded ultrapowers of the universe of sets have a common wellfounded ultrapower. More precisely, UA states that for any pair of countably complete ultrafilters U_0 and U_1, there are linear iterations (U_0, W_0) and (U_1, W_1) such that the associated iterated ultrapowers $\text{Ult}(\text{Ult}(V, U_0), W_0)$ and $\text{Ult}(\text{Ult}(V, U_1), W_1)$ are equal and the associated iterated ultrapower embeddings $j_{W_0} \circ j_{U_0}$ and $j_{W_1} \circ j_{U_1}$ coincide. Obviously this is formally similar to the Comparison Lemma, and in fact there is a very general argument [6, Theorem 2.3.10] showing that UA *follows* from the Comparison Lemma. Any canonical inner model that can be constructed and analyzed using anything like the current methodology of inner model theory will therefore satisfy UA.

The inner models that have been constructed to date cannot contain supercompact cardinals. Whether canonical inner models with supercompact cardinals exist has been one of the main open problems in inner model theory for almost half a century. Given the intractability of this problem, it is natural to wonder whether the existence of a supercompact cardinal is compatible with the Comparison Lemma. Having formulated the Ultrapower Axiom, one can state a perfectly precise version of this question.

Question 1.1. Is the Ultrapower Axiom consistent with the existence of a supercompact cardinal?

This paper and its sequel show that the Ultrapower Axiom has fairly deep interactions with supercompactness. The coherence of the theory developed so far suggests that the answer to Question 1.1 is yes. It seems unlikely, however, that a positive answer can be established without extending inner model theory to the level of supercompact cardinals. We are optimistic that studying the consequences of UA with a supercompact cardinal will shed some light on how this can be done (or else lead to a proof that it is impossible).

Finally, let us mention two stronger theorems we will prove here, and some further results that will appear in the sequel to this paper. For any cardinal κ, τ_κ denotes the least ordinal α such that for any β, there is a κ-complete ultrafilter U such that $j_U(\alpha) > \beta$. (The cardinal τ_κ is a hybrid of Hamkins's notion of a strongly tall cardinal [8] and Bagaria-Magidor's compactness principles [2].)

Theorem 5.18 (UA). *The cardinal τ_{ω_1} is supercompact.*

Finally, we prove a very local form of Theorem 5.17.

Theorem 5.14 (UA). *If a successor cardinal λ carries a countably complete uniform ultrafilter, then there is some $\kappa \leq \lambda$ that is λ-supercompact.*

In fact, the easiest proof we know of Question 1.1 necessarily establishes this stronger theorem. If λ is a regular limit cardinal, it is open whether the theorem remains true.

Of course this raises an obvious question: what about the other strongly compact and supercompact cardinals? This is the subject of the sequel to this paper, in which it will be shown that under UA, every strongly compact cardinal is either supercompact or a limit of supercompact cardinals, which by Menas's theorem is best possible. The techniques leading to this result enable the analysis of many large cardinal notions beyond supercompactness under UA. One also obtains some powerful consequences of UA for the general theory of countably complete ultrafilters.

This paper is fairly self-contained. Section 2 contains many standard definitions and self-explanatory notions from the theory of ultrafilters and some less well-known material. It is intended to be used as a reference and need not be read from beginning to end. Section 3 contains a number of the author's results on the general theory of countably complete ultrafilters under UA that are used in the analysis of strong compactness. The reader can skim this, and start in earnest with Sec. 4, which contains a proof that the least strongly compact cardinal is supercompact under a technical assumption about the size of the least strongly compact cardinal. This assumption is proved in Sec. 5, completing the proof.

2. Preliminaries

2.1. *Ultrapowers*

Suppose N is a model of set theory (which the reader may assume to be transitive) and $X \in N$. An *N-ultrafilter over X* is a maximal filter on the partial order $P^N(X)$ under inclusion. The *ultrapower of N by U*, denoted M_U^N, consists of mod U equivalence classes of those functions of N that are defined U-almost everywhere. If $f \in N$ is defined U-almost everywhere, $[f]_U^N$ denotes the equivalence class of f modulo U.

The class M_U^N is endowed with the structure of a model of set theory under the relation $\in_U$ defined by setting $[f]_U^N \in_U [g]_U^N$ if $\{x \in X : f(x) \in^N g(x)\} \in U$.

Los's Theorem states that $M_U^N \vDash \varphi([f_1]_U, \ldots, [f_n]_U)$ if and only if $\{x \in X : N \vDash \varphi(f_1(x), \ldots, f_n(x))\} \in U$. The *ultrapower embedding of* N *associated to* U, denoted $j_U^N : N \to M_U^N$, is the embedding defined by $j_U^N(a) = [c_a]_U$, where $c_a : X \to \{a\}$ is the constant function. This embedding is elementary as an immediate consequence of Los's Theorem.

If M_U^N is wellfounded (as it almost always will be in this paper), we adhere to the standard set-theoretic convention of identifying M_U^N with its transitive collapse. We also adopt two nonstandard conventions in order to declutter our notation. First, we omit the superscript "N"s whenever they can be inferred from context, writing, for example, M_U instead of M_U^N. Second, we make the following definition.

Definition 2.1. If U is an N-ultrafilter, $a_U = [\mathrm{id}]_U^N$.

Suppose N is a model of set theory. We say U is an *ultrafilter of* N if U is an N-ultrafilter that belongs to N. We say $j : N \to P$ is an *ultrapower embedding* and P is an *ultrapower of* N if for some N-ultrafilter U, there is an isomorphism $k : M_U \to P$ such that $j = k \circ j_U$; we say $j : N \to P$ is an *internal ultrapower embedding* if one can find such an N-ultrafilter U that in addition belongs to N.

In the case of interest, when N and P are transitive, $j : N \to P$ is an ultrapower embedding of N if and only if $j = j_U$ for some N-ultrafilter U.

Not every elementary embedding is an ultrapower embedding, but the derived ultrafilter construction reduces many "local" properties of elementary embeddings to properties of ultrapowers.

Definition 2.2. Suppose $j : N \to P$ is an elementary embedding, $X \in N$, and $a \in j(X)$. Then the *N-ultrafilter over X derived from j using a* is the N-ultrafilter $\{A \in P^N(X) : a \in j(A)\}$.

Lemma 2.3. *Suppose $j : N \to P$ is an elementary embedding, $X \in N$, and $a \in j(X)$. The N-ultrafilter over X derived from j using a is the unique N-ultrafilter U such that there is an elementary embedding $k : M_U^N \to P$ such that $k \circ j_U^N = j$ and $k(a_U) = a$.*

Sketch. The embedding k is defined by $k([f]_U) = j(f)(a)$. It is routine to check that k is well-defined and elementary. □

We call the (unique) embedding k of Lemma 2.3 the *canonical factor embedding* associated to the derived ultrafilter U.

There is an ultrafilter-free characterization of (internal) ultrapower embeddings that helps establish some of their basic properties. An elementary embedding $j : M \to N$ is said to be *cofinal* if for all $a \in N$, there is some $X \in M$ such that $a \in j(X)$.

Lemma 2.4. *A cofinal elementary embedding $j : N \to P$ is an ultrapower embedding if and only if there is some $a \in P$ such that every element of P is definable in P from a and parameters in $j[N]$.*

Proof. For the forwards direction, fix an N-ultrafilter U such that $j = j_U$. Then for all appropriate $f \in N$, $[f]_U = j(f)(a_U)$, and hence $[f]_U$ is definable in P from $j(f)$ and a_U.

Conversely, suppose every element of P is definable in P from a and parameters in $j[N]$. Let U be the N-ultrafilter derived from j using a. Let $k : M_U \to P$ be the canonical factor embedding. Then $j[N] \cup \{a\} \subseteq k[M_U]$, so in fact, $P \subseteq k[M_U]$. In other words, k is an isomorphism. Hence $j = k \circ j_U$ is an ultrapower embedding. □

Thus, an ultrapower of N is a finitely generated elementary extension of N.

Definition 2.5. An elementary embedding $j : N \to P$ is *close to* N if for all $a \in P$, $j^{-1}[a] \in N$.

Obviously any elementary embedding that is definable over N is close to N, so in particular, internal ultrapower embeddings are close embeddings.

Lemma 2.6. *An ultrapower embedding $j : N \to P$ is internal if and only if it is close.*

Lemma 2.7. *An elementary embedding $j : N \to P$ is close to N if and only if j is cofinal and every N-ultrafilter derived from j belongs to N.*

Proof. For the forwards direction, assume j is close. Then for any $X \in N$ and $a \in j(X)$, the N-ultrafilter derived from j using a is equal to $j^{-1}[p_a]$ where $p_a = \{A \in P(j(X)) \cap P : a \in A\}$. Obviously $p_a \in P$, so $j^{-1}[p_a] \in N$ by closeness. To see that j is cofinal, fix $a \in P$. Let Y be the set of elements of P that have rank less than or equal to that of a. Then $j^{-1}[Y] \in N$. Let $X = j^{-1}[Y]$. Then $a \in j(X)$.

For the converse, assume j is cofinal and every derived N-ultrafilter of j belongs to N. Fix $a \in P$, and we will show that $j^{-1}[a] \in N$. Since j is cofinal, there is some $X \in N$ such that $a \in j(X)$. Let U be the N-ultrafilter derived from j using a, and let $k : M_U^N \to P$ be the canonical factor embedding. Then $j_U : N \to M_U^P$ is an internal ultrapower embedding, $k \circ j_U = j$ and $k(a_U) = a$. Therefore, $j^{-1}[a] = j_U^{-1}[a_U] \in N$, as desired. □

Lemma 2.6 follows immediately from Lemma 2.7.

Proof of Lemma 2.6. Note that U is the N-ultrafilter derived from j_U using a_U, so if j_U is close, then $U \in N$, and so j_U is an internal ultrapower embedding. Conversely, if U is an internal ultrapower embedding, then $M_U \subseteq N$ and j_U is definable over N, so obviously $j_U^{-1}[a] \in N$ for every $a \in M_U$. □

These ultrafilter-free characterizations make various properties of ultrapowers completely transparent.

Lemma 2.8. *Suppose $M \xrightarrow{i} N \xrightarrow{j} P$ are elementary embeddings and $j \circ i$ is an ultrapower embedding. Then j is an ultrapower embedding.*

Proof. This is an immediate consequence of Lemma 2.4 since $j \circ i[M] \subseteq j[N]$. □

Our next lemma follows directly from the definition of a close embedding.

Lemma 2.9. *Suppose $N \xrightarrow{j} P \xrightarrow{k} Q$ are elementary embeddings and $k \circ j$ is a close embedding. Then j is close to N.*

Lemmas 2.9 and 2.6 lead to a useful criterion for an elementary embedding to be an internal ultrapower embedding.

Corollary 2.10. *Suppose $M \xrightarrow{i} N \xrightarrow{j} P \xrightarrow{k} Q$ are elementary embeddings, $j \circ i$ is an ultrapower embedding, and $k \circ j$ is a close embedding. Then j is an internal ultrapower embedding.*

One also easily obtains that compositions of (internal) ultrapower embeddings are again (internal) ultrapower embeddings.

Lemma 2.11. *The composition of two ultrapower embeddings is an ultrapower embedding.*

Proof. Fix ultrapower embeddings $M \xrightarrow{i} N \xrightarrow{j} P$.

Fix $b \in N$ such that every element of N is definable in N from b and parameters in $i[M]$. By elementarity, every element of $j[N]$ is definable in P from $j(b)$ and parameters in $j \circ i[M]$.

Fix $c \in P$ such that every element of P is definable in P from c and parameters in $j[N]$. Thus, every element of P is definable from c, $j(b)$, and parameters in $j \circ i[M]$. It follows from Lemma 2.4 (with $a = (j(b), c)$) that $j \circ i$ is an ultrapower embedding. □

Again the following lemma is immediate from the definition of a close embedding:[d]

Lemma 2.12. *The composition of two close embeddings is a close embedding.*

Applying Lemmas 2.6, 2.11 and 2.12, we obtain:

Corollary 2.13. *The composition of two internal ultrapower embeddings is an internal ultrapower embedding.*

[d]The lemma is originally due to Woodin, who took Lemma 2.7 as the definition of a close embedding, so that Lemma 2.12 is not as obvious.

2.2. *Uniform and fine ultrafilters*

This section defines two different ways in which an ultrafilter can be said to concentrate on large sets. The first is a constraint on cardinality.

Definition 2.14. An ultrafilter U over a set X of cardinality λ is *uniform* if for all $A \in U$, $|A| = \lambda$.

Given an ultrafilter U over a set X, there is always some $Y \in U$ such that $U \cap P(Y)$ is uniform: let $\lambda = \min\{|A| : A \in U\}$ and let Y be any set in U of cardinality λ. The ultrafilters U and $U \cap P(Y)$ are essentially the same object, so in a sense, uniform ultrafilters are just as "general" as ultrafilters are. Since there is no *canonical* choice of Y, however, there are often good reasons for considering ultrafilters that are not uniform.

The second notion is a constraint on cofinality.

Definition 2.15. An ultrafilter $\mathcal{U}$ over a family of sets $\mathcal{X}$ is *fine* if for all $x \in \bigcup \mathcal{X}$, $\{\sigma \in \mathcal{X} : x \in \sigma\} \in \mathcal{U}$.

The only fine ultrafilters that are important in this paper are the fine ultrafilters over ordinals. As in the case of uniform ultrafilters, for any ultrafilter U over an ordinal γ, there is an ordinal δ such that $U \cap P(\delta)$ is a fine ultrafilter: let $\delta = \min_{\alpha \leq \gamma} \alpha \in U$.

Lemma 2.16. *Suppose U is an ultrafilter over an ordinal δ. Then the following are equivalent*:

- *U is fine.*
- *Every set in U is cofinal in δ.*
- *δ is the least ordinal such that $j_U(\delta) > a_U$.*

Both uniformity and fineness amount to the requirement that an ultrafilter extend a certain filter naturally associated to its underlying set: an ultrafilter U over a set X of cardinality λ is uniform if and only if it extends the generalized Fréchet filter over X defined by $\{A \subseteq X : |X \setminus A| < \lambda\}$, while an ultrafilter $\mathcal{U}$ over a family of sets $\mathcal{X}$ is fine if and only if it extends the tail filter over $\mathcal{X}$, generated by sets of the form $\{\sigma \in X : x \in \sigma\}$ for $x \in \bigcup \mathcal{X}$.

Take, for example, the class of principal ultrafilters.

Definition 2.17. Suppose X is a set and a is an element of X. The *principal ultrafilter over X concentrated at a* is the set $p_a[X] = \{A \subseteq X : a \in A\}$.

We write p_a instead of $p_a[X]$ when the choice of X is obvious or irrelevant, as is almost always the case. If $\alpha \leq \beta$ are ordinals, then the principal ultrafilter $p_\alpha[\beta]$ is fine if and only if $\beta = \alpha + 1$.

By Lemma 2.16, fineness and uniformity coincide at regular cardinals.

Lemma 2.18. *An ultrafilter over a regular cardinal is fine if and only if it is uniform.*

We briefly discuss weakly normal ultrafilters.

Definition 2.19. A uniform ultrafilter U over a regular cardinal δ is *weakly normal* if for any function f such that $f(\alpha) < \alpha$ for U-almost all ordinals α, there is some $\nu < \delta$ such that $f(\alpha) < \nu$ for U-almost all α.

In other words, U is weakly normal if every U-regressive function is U-bounded. Weak normality can be expressed in terms of the ultrapower.

Lemma 2.20. *If U is a uniform ultrafilter over a regular cardinal δ, then U is weakly normal if and only if $a_U = \sup j_U[\delta]$.*

We will use the following lemma, which follows easily from Lemma 2.20.

Lemma 2.21. *If $j : V \to M$ is an elementary embedding and $\sup j[\delta] < j(\delta)$, then the ultrafilter U derived from j using $\sup j[\delta]$ is weakly normal.*

2.3. *Pushforwards and limits*

We now turn to the concept of a pushforward ultrafilter and the more general concept of a limit of ultrafilters.

Definition 2.22. Suppose U is an ultrafilter over a set X and $f : X \to Y$ is a function. Then the *pushforward of U by f* is the ultrafilter

$$f_*(U) = \{A \subseteq Y : f^{-1}[A] \in U\}.$$

Equivalently, $f_*(U)$ is the ultrafilter generated by sets of the form $f[A]$ where $A \in U$.

Pushforwards are closely related to derived ultrafilters.

Lemma 2.23. *Suppose U is an ultrafilter over a set X and $f : X \to Y$ is a function. Then $f_*(U)$ is the ultrafilter D derived from j_U using $[f]_U$. The canonical factor embedding $k : M_D \to M_U$ is given by $k([g]_D) = [g \circ f]_U$.*

We now turn to ultrafilter limits.

Definition 2.24. Suppose U is an ultrafilter over a set X and $\langle W_x : x \in X\rangle$ is a sequence of ultrafilters over a set Y. Then the *U-limit of* $\langle W_x : x \in X\rangle$ is the ultrafilter

$$U\text{-}\lim_{x\in X} W_x = \{A \subseteq Y : \{x \in X : A \in W_x\} \in U\}.$$

If U is an ultrafilter over X and $f : X \to Y$ is a function, then the pushforward $f_*(U)$ is the U-limit of the principal ultrafilters $\langle p_{f(x)}[Y] : x \in X\rangle$.

We now generalize Lemma 2.23 to ultrafilter limits.

Lemma 2.25. *Suppose U is an ultrafilter over a set X and $\langle W_x : x \in X\rangle$ is a sequence of ultrafilters over a set Y. Then $U\text{-}\lim_{x\in X} W_x = j_U^{-1}[W_*]$ where $W_* = [x \mapsto W_x]_U$.*

The proof is straightforward. There is also an analog of the canonical factor embedding (Lemma 2.3).

Definition 2.26. Suppose $j : N \to P$ is an elementary embedding, W is an N-ultrafilter over a set $Y \in N$, and W_* is a P-ultrafilter such that $W = j^{-1}[W_*]$. Then the associated *shift embedding* is the embedding $k : M_W \to M_{W_*}$ defined by $k([f]_W) = [j(f)]_{W_*}$.

The following lemma is a special case of the *shift lemma* from the theory of iterated ultrapowers.

Lemma 2.27. *Suppose $j : N \to P$ is an elementary embedding, W is an N-ultrafilter over a set $Y \in N$, and W_* is a P-ultrafilter such that $W = j^{-1}[W_*]$. Then the associated shift embedding $k : M_W \to M_{W_*}$ is a well-defined elementary embedding satisfying $k \circ j_W = j_{W_*} \circ j$ and $k(a_W) = a_{W_*}$.*

2.4. *Completeness, supercompactness and Kunen's theorem*

In this section, we exposit the basic results involving κ-complete ultrafilters, although the reader is likely familiar with them. We also discuss the concept of supercompactness and its relationship with completeness.

Definition 2.28. Suppose κ is a cardinal. An ultrafilter U is *κ-complete* if for all $\gamma < \kappa$, for any sequence $\langle A_\alpha \rangle_{\alpha<\gamma}$ with $A_\alpha \in U$ for all $\alpha < \gamma$, $\bigcap_{\alpha<\gamma} A_\alpha \in U$.

We will say an ultrafilter U is *countably complete* if it is ω_1-complete.

Definition 2.29. If $j : N \to P$ is a nontrivial elementary embedding, then the *critical point of j*, denoted $\text{crit}(j)$, is the least N-ordinal α such that $j(\alpha) \neq j[\alpha]$.

If N is illfounded, then j may have no critical point. (This issue does not come up here.) Every nontrivial elementary embedding of a wellfounded model of ZFC has a critical point.

Theorem 2.30 (Scott). *Suppose U is an ultrafilter. Then $\text{crit}(j_U)$ is the largest cardinal κ such that U is κ-complete.*

This inspires some obvious notation.

Definition 2.31. If U is an ultrafilter, the *completeness of U*, denoted $\text{crit}(U)$, is the largest cardinal κ such that U is κ-complete.

The completeness of an ultrafilter turns out to have an intimate relationship with the closure properties of its ultrapower.

Lemma 2.32. *Suppose $j : V \to M$ is an elementary embedding and λ is a cardinal. Then the following are equivalent*:

(1) *For some set X of cardinality λ, $j[X] \in M$.*
(2) *For any set X of cardinality at most λ, $j[X] \in M$.*

If j is an ultrapower embedding, one can add to the list:

(3) *M is closed under λ-sequences.*

As a consequence of Theorem 2.30, one obtains the following fact.

Proposition 2.33 (Scott). *Suppose U is a κ-complete ultrafilter. Then M_U is closed under γ-sequences for all $\gamma < \kappa$.*

Proof. If $\gamma < \kappa$, then $j_U(\gamma) = j_U[\gamma]$ by Theorem 2.30. In particular, $j_U[\gamma] \in M_U$, so M_U is closed under γ-sequences by Lemma 2.32. □

A model of set theory that is closed under countable sequences is necessarily wellfounded, which yields the most important corollary of Proposition 2.33.

Corollary 2.34. *An ultrafilter is countably complete if and only if its ultrapower is wellfounded.*

Proof. The forwards direction follows from Proposition 2.33. For the converse, suppose U is an ultrafilter and assume M_U is wellfounded (or just $\omega^M \cong \omega$). Obviously for all $n < \omega$, $j(n) = j[n]$. Therefore, $j_U(\omega) = j_U[\omega]$ since any $n \in j_U(\omega) \setminus j_U[\omega]$ would be nonstandard. It follows that $\text{crit}(j_U) > \omega$, and so U is ω_1-complete by Theorem 2.30. □

From Corollary 2.34, we can deduce a strengthening of Proposition 2.33.

Theorem 2.35 (Scott). *Suppose κ is an uncountable cardinal and U is a κ-complete ultrafilter. Then M_U is closed under κ-sequences.*

Proof. Corollary 2.34 implies M_U is wellfounded, and so we identify it with its transitive collapse, which is an inner model. Then for all $\gamma < \kappa$, $j_U[\gamma] = j_U(\gamma)$ is an ordinal, and hence $j_U[\kappa] = \bigcup_{\gamma<\kappa} j_U[\gamma]$ is an ordinal. Since M_U contains every ordinal, $j_U[\kappa] \in M_U$, and hence M_U is closed under κ-sequences by Lemma 2.32. □

It is natural to wonder whether the converse to Theorem 2.35 is true: if U is an ultrafilter such that M_U is closed under λ-sequences, must U be λ-complete? Clearly this is true when $\lambda = \aleph_1$ or when λ is the first measurable cardinal.[e] At the level of a supercompact cardinal, this implication breaks down.

Definition 2.36. A cardinal κ is λ-*supercompact* if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. If κ is λ-supercompact for all cardinals λ, then κ is said to be *supercompact*.

[e] This is best possible. Let κ be the least measurable cardinal. By an unpublished theorem of Woodin (later generalized by Apter [1, Theorem 1]), it is consistent that there is an ultrafilter U such that M_U is closed under κ^+-sequences but U is not κ^+-complete.

Even granting the consistency of supercompact cardinals (which we do), one can still prove a partial converse to Theorem 2.35. This is a corollary of the Kunen inconsistency theorem.

Theorem 2.37 (Kunen). *Suppose M is an inner model and $j : V \to M$ is a nontrivial elementary embedding. Then $j[\lambda] \notin M$ where λ is the least ordinal strictly larger than* $\mathrm{crit}(j)$ *such that $j[\lambda] \subseteq \lambda$.*

Note that since an elementary embedding from V to an inner model is continuous at ordinals of cofinality ω, in the context of Theorem 2.37, λ is also the least fixed point of j above $\mathrm{crit}(j)$. Moreover, $\lambda = \sup_{n<\omega} j^{(n)}(\mathrm{crit}(j))$.

Here is the partial converse to Theorem 2.35.

Corollary 2.38. *Suppose U is an ultrafilter and λ is an infinite cardinal such that M_U is closed under δ-sequences for all $\delta < \lambda$. If $j_U[\lambda] \subseteq \lambda$, then U is λ-complete.*

Proof. Assume not. Without loss of generality, we may assume λ is the least ordinal strictly larger than $\mathrm{crit}(j_U)$ such that $j_U[\lambda] \subseteq \lambda$. By the remarks above, λ has countable cofinality. Since M_U is closed under ω-sequences and δ-sequences for all $\delta < \lambda$, in fact, M_U is closed under λ-sequences, contrary to Theorem 2.37. □

2.5. *Indecomposable ultrafilters*

Keisler and Príkry [17] introduced a spectrum of cardinals associated to an ultrafilter U generalizing the notion of completeness.

Definition 2.39. Suppose γ is a cardinal and U is an ultrafilter over X. Then a *γ-decomposition of U* is a function $f : X \to \gamma$ such that $|f[A]| = \gamma$ for all $A \in U$. The ultrafilter U is said to be *γ-decomposable* if there is a γ-decomposition of U.

The completeness of U is the least element of its decomposability spectrum.

Lemma 2.40. *For any ultrafilter U, the completeness of U is the least cardinal κ such that U is κ-decomposable.*

In general, the decomposability spectrum of an ultrafilter U consists of those cardinals over which U projects to a uniform ultrafilter.

Lemma 2.41. *Suppose U is an ultrafilter over a set X and γ is a cardinal. Then U is γ-decomposable if and only if there is a function $f : X \to \gamma$ such that $f_*(U)$ is a uniform ultrafilter over γ; in fact, for any cardinal γ, a function $f : X \to \gamma$ is a γ-decomposition of U if and only if $f_*(U)$ is a uniform ultrafilter over γ.*

For regular cardinals γ, γ-indecomposability has a simple ultrapower theoretic characterization.

Proposition 2.42. *Suppose γ is a regular cardinal and U is an ultrafilter over a set X. Then U is γ-indecomposable if and only if $j_U(\gamma) = \sup j_U[\gamma]$.*

Sketch. Using the regularity of γ, one shows that a function $f : X \to \gamma$ is a γ-decomposition if and only if $j_U(\alpha) < [f]_U$ for all $\alpha < \gamma$. $\square$

There is also a combinatorial characterization of indecomposability in terms of partitions.

Lemma 2.43. *An ultrafilter U over a set X is γ-indecomposable if and only if for every partition $\langle A_\xi : \xi < \gamma\rangle$ of X, there is some $S \subseteq \gamma$ with $|S| < \gamma$ such that $\bigcup_{\xi\in S} A_\xi \in U$.*

One can also characterize γ-indecomposable ultrafilters as those that are closed under intersections of descending γ-sequences. The following definition generalizes this observation.

Definition 2.44. Suppose U is an ultrafilter and $\kappa \leq \lambda$ are cardinals. Then U is (κ, λ)-*indecomposable* if for all $\gamma < \lambda$, if $\langle A_\alpha : \alpha < \gamma\rangle$ is a collection of sets with $\bigcap_{\alpha\in\sigma} A_\alpha \in U$ for all $\sigma \in P_\kappa(\gamma)$, then $\bigcap_{\alpha<\gamma} A_\alpha \in U$.

Proposition 2.45. *An ultrafilter U is (κ, λ)-indecomposable if and only if U is γ-indecomposable for every cardinal γ such that $\kappa \leq \gamma < \lambda$.*

Proof. Note that U is γ-indecomposable if and only if $\bigcap_{\alpha<\gamma} A_\alpha \in U$ for any $\langle A_\alpha : \alpha < \gamma\rangle$ such that $\bigcap_{\alpha\in\sigma} A_\alpha \in U$ for all $\sigma \in P_\gamma(\gamma)$. The proposition follows immediately. $\square$

In Sec. 5, we analyze the class of cardinals carrying uniform countably complete ultrafilters. For this, the following theorem is quite useful.

Theorem 2.46 (Príkry-Kunen, [11]). *Suppose λ is a regular cardinal and U is a uniform ultrafilter over λ^+. Then U is λ-decomposable.*

The corresponding fact is not true for singular cardinals λ. For example, if κ is $\kappa^{+\omega+1}$-strongly compact then $\kappa^{+\omega+1}$ carries a uniform κ-complete ultrafilter U, which, being countably complete, cannot be $\kappa^{+\omega}$-decomposable. (A λ-decomposable ultrafilter is always $\text{cf}(\lambda)$-decomposable, while countable completeness is equivalent to $\aleph_0$-indecomposability.) There is, however, a reasonable generalization of Theorem 2.46 that one might hope to prove (Conjecture 2.53). This generalization involves the concept of a regular ultrafilter (Definition 2.47), so it will be discussed at the end of the following section.

2.6. *Regular ultrafilters*

Definition 2.47. Suppose $\kappa \leq \lambda$ are cardinals. An ultrafilter U is (κ, λ)-*regular* if every subset of M_U of cardinality at most λ is contained in an element of M_U of M_U-cardinality less than $j_U(\kappa)$.

Given Lemma 2.41, the following proposition shows that (κ, λ)-regularity is the analog of γ-decomposability in which uniform ultrafilters are replaced by fine ones.

Proposition 2.48. *An ultrafilter U over X is (κ, λ)-regular if and only if there is a function $f : X \to P_\kappa(\lambda)$ such that $f_*(U)$ is a fine ultrafilter.*

Combining this with Lemmas 2.18 and 2.41, if γ is a regular cardinal, U is (γ, γ)-regular if and only if U is γ-decomposable. (On the other hand, if γ is singular, (γ, γ)-regularity neither implies γ-decomposability nor follows from it.)

The following proposition is an immediate consequence of the definition of (κ, λ)-regularity.

Proposition 2.49. *Suppose $\kappa \leq \kappa' \leq \lambda' \leq \lambda$ are cardinals. If U is a (κ, λ)-regular ultrafilter, then U is (κ', λ')-regular.*

Corollary 2.50. *Suppose U is (κ, λ)-regular. Then U is γ-decomposable for every regular cardinal γ in the interval $[\kappa, \lambda]$.*

Proof. By Proposition 2.49, U is (γ, γ)-regular for all γ in the interval $[\kappa, \lambda]$. If γ is regular, our comments above imply that U is γ-decomposable. □

The following lemma is in spirit due to Ketonen, but the proof we give is based on an argument due to Woodin that we learned from [3].

Lemma 2.51. *Suppose δ is a cardinal, $j : V \to M$ is an elementary embedding, and λ is an M-cardinal. Then the following are equivalent:*

(1) *For some set X of cardinality δ, there is some $Y \in M$ such that $j[X] \subseteq Y$ and $|Y|^M < \lambda$.*
(2) *For any set A of cardinality at most δ, there is some $B \in M$ such that $j[A] \subseteq B$ and $|B|^M < \lambda$.*

If δ is regular and M is wellfounded, then one can add to the list:

(3) $\mathrm{cf}^M(\sup j[\delta]) < \lambda$.

If j is a wellfounded ultrapower embedding, one can add to the list:

(4) *Every subset of M of cardinality at most δ is contained in a set in M of M-cardinality less than λ.*

Proof. (1) implies (2): Fix a set X of cardinality δ and a set $Y \in M$ such that $j[X] \subseteq Y$ and $|Y|^M < \lambda$. Fix a set A of cardinality at most δ. We must find $B \in M$ such that $j[A] \subseteq B$ and $|B|^M < \lambda$. Let $f : X \to A$ be a surjection. Then $B = j(f)[Y]$ is as desired.

(2) implies (1): Trivial.

(2) implies (3): Take $B \in M$ such that $j[\delta] \subseteq B$ and $|B|^M < \lambda$. Then $B \cap \sup j[\delta]$ is a cofinal subset of $\sup j[\delta]$ that belongs to M and has M-cardinality less than λ, so $\sup j[\delta]$ has M-cofinality less than λ.

(3) implies (1) assuming δ is regular: Let $Y \in M$ be a closed unbounded subset of $\sup j[\delta]$ of M-cardinality less than λ. Note that $Y \cap j[\delta]$ is an ω-closed unbounded subset of $\sup j[\delta]$. Letting $X = j^{-1}[Y]$, it follows that X is an ω-closed unbounded subset of δ. Since δ is regular, this implies that X has cardinality δ. Since $j[X] \subseteq Y$, this establishes (1).

(4) implies (2): Trivial.

(2) implies (4) assuming j is an ultrapower embedding: Fix a set $S \subseteq M$ of cardinality at most δ. We must show that there is some $T \in M$ of M-cardinality less than λ such that $S \subseteq T$.

Since M is an ultrapower embedding, there is some a in M such that every element of M is of the form $j(f)(a)$ for some function f. One can therefore find a set of functions F of cardinality at most λ such that $S = \{j(f)(a) : f \in F\}$. Fix $G \in M$ such that $j[F] \subseteq G$ and $|G|^M < \lambda$. Without loss of generality we may assume that G consists only of functions that are defined at a. Setting $T = \{g(a) : g \in G\}$, we have $S \subseteq T$ and $|T|^M < \lambda$, as desired. □

The wellfoundedness assumptions in Lemma 2.51 are unnecessary, although this is not clear from the proof above. Ketonen's proof [10] works assuming only that $\sup j[\delta]$ exists.

Corollary 2.52 (Ketonen). *Suppose κ is a cardinal and λ is a regular cardinal. A countably complete ultrafilter U is (κ, λ)-regular if and only if*

$$\mathrm{cf}^{M_U}(\sup j_U[\lambda]) < j_U(\kappa).$$

Therefore, a countably complete weakly normal ultrafilter over λ is (κ, λ)-regular if and only if it concentrates on the set $S^\lambda_\kappa = \{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\}$.

Returning to the question of generalizing Theorem 2.46 to singular cardinals, we consider the following conjecture of Lipparini.

Conjecture 2.53 (Lipparini, [12, Conjecture 2.12]). *Suppose λ is a cardinal and U is a uniform ultrafilter over λ^+. Must one of the following hold?*

- *U is (κ, λ^+)-regular for some $\kappa < \lambda$.*
- *U is λ-decomposable.*

There is an approximate answer to this question that will be quite important in our analysis of strongly compact cardinals under UA. Recall that if W is an ultrafilter over X and Z is an ultrafilter over Y then

$$W \otimes Z = \{A \subseteq X \times Y : \forall^W x\ \forall^Z y\ (x, y) \in A\}.$$

Theorem 3.24. *Suppose λ is a cardinal and U is a uniform ultrafilter over λ^+. Then either U is (κ, λ^+)-regular for some $\kappa < \lambda$ or else $U \otimes U$ is λ-decomposable.*

If λ is regular, then U is λ-decomposable if and only if $U \otimes U$ is λ-decomposable. (This is obvious from Proposition 2.42.) Therefore, Theorem 3.24 generalizes Príkry's Theorem (Theorem 2.46). Lipparini pointed out in personal correspondence that Theorem 3.24 follows from results of [13]. We will give a more direct proof at the end of Sec. 3.4.

2.7. *Compactness principles*

In this subsection, we define various strong compactness principles: classical strong compactness, due to Tarski [19], and a number of variants introduced by Bagaria-Magidor [2].

Definition 2.54. Suppose $\delta \leq \kappa \leq \lambda$ are cardinals. Then κ is (δ, λ)-*strongly compact* if for some inner model M, there is an elementary embedding $j : V \to M$ with critical point at least δ such that every subset of M of cardinality at most λ is contained in an element of M of M-cardinality less than $j(\kappa)$.

This principle is degenerate in the sense that if κ is (δ, λ)-strongly compact, then all cardinals in the interval $[\kappa, \lambda]$ are (δ, λ)-strongly compact.

Definition 2.55. Suppose $\delta \leq \kappa \leq \lambda$ are cardinals.

- κ is (δ, ∞)-*strongly compact* if it is (δ, γ)-strongly compact for all cardinals $\gamma \geq \kappa$.
- κ is λ-*strongly compact* if it is (κ, λ)-strongly compact.
- κ is *strongly compact* if it is (κ, ∞)-strongly compact.

There are many alternate characterizations of strong compactness.

Theorem 2.56 (Solovay, Ketonen). *Suppose $\delta \leq \kappa \leq \lambda$ are cardinals. Then the following are equivalent*:

(1) *κ is (δ, λ)-strongly compact.*
(2) *There is a δ-complete fine ultrafilter over the set $P_\kappa(\lambda) = \{\sigma \subseteq \lambda : |\sigma| < \kappa\}$.*
(3) *There is a δ-complete (κ, λ)-regular ultrafilter.*

If $\mathrm{cf}(\lambda) \geq \kappa$, *one can add to the list:*

(4) *Every κ-complete filter generated by λ sets extends to a δ-complete ultrafilter.*
(5) *Every regular cardinal γ with $\kappa \leq \gamma \leq \lambda$ carries a δ-complete uniform ultrafilter.*

If λ is regular, one can add to the list:

(6) *There is a δ-complete weakly normal fine ultrafilter over λ concentrating on the set $S^\lambda_\kappa = \{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\}$.*

Finally, we will use Solovay's theorem that the Singular Cardinals Hypothesis holds above a strongly compact cardinal in the following form.

Theorem 2.57 (Solovay, [18]). *Assume $\kappa \leq \lambda$ are cardinals*, $\mathrm{cf}(\lambda) \geq \kappa$, *and κ is λ-strongly compact. Then $\lambda^{<\kappa} = \lambda$.*

A proof of this local form of Solovay's theorem in the case that λ is regular appears in the author's thesis [6, Theorem 7.2.16], although of course it is not due to the author. The case that $\text{cf}(\lambda) \geq \kappa$ follows since $\lambda^{<\kappa} = \sup_{\delta<\lambda} \delta^{<\kappa}$ if λ is a limit cardinal and $\text{cf}(\lambda) \geq \kappa$.

3. The Ultrapower Axiom

In this section, we state the Ultrapower Axiom and prove some basic results from the general theory of countably complete ultrafilters under UA that are required to prove the supercompactness of the least strongly compact cardinal.

Since we never consider countably incomplete ultrafilters, we will only be interested in *wellfounded* ultrapowers, so we adopt the following convention.

Convention 3.1. All the models of set theory under consideration are from now on assumed to be transitive.

3.1. *Comparisons and the Ultrapower Axiom*

By Corollary 2.13, ultrapowers of V form a category under internal ultrapower embeddings.

Definition 3.2. The *category of internal ultrapowers* is the category whose objects are ultrapowers of the universe of sets[f] and whose morphisms are internal ultrapower embeddings.

Our next definition is standard in model theory.

Definition 3.3. A category $\mathbf{C}$ has the *Amalgamation Property* if for all morphisms $j_0 : P \to M_0$ and $j_1 : P \to M_1$ of $\mathbf{C}$ there exist morphisms $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ of $\mathbf{C}$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

Definition 3.4. The *Ultrapower Axiom* states that the category of internal ultrapowers has the Amalgamation Property.

Let us put down an equivalent formulation of the Ultrapower Axiom which the categoraphobic among us may take as the definition.

Definition 3.5. Suppose M_0 and M_1 are models of set theory. We say

$$(i_0, i_1) : (M_0, M_1) \to N$$

is a *comparison of* (M_0, M_1) if $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ are elementary embeddings. If P is a model of set theory $j_0 : P \to M_0$ and $j_1 : P \to M_1$ are elementary embeddings, we say $(i_0, i_1) : (M_0, M_1) \to N$ is a *comparison of* (j_0, j_1).

[f] But only transitive ultrapowers; see Convention 3.1.

We will say a comparison is an *internal ultrapower comparison* if its constituent embeddings are internal ultrapower embeddings.

Lemma 3.6. *The Ultrapower Axiom holds if and only if every pair of ultrapower embeddings of the universe of sets admits an internal ultrapower comparison.*

Proof. The forwards direction is clear. For the converse, suppose P, M_0, M_1 are ultrapowers of V and $j_0 : P \to M_0$ and $j_1 : P \to M_1$ are internal ultrapower embeddings. To verify the Amalgamation Property, we must show that there exists an internal ultrapower comparison $(i_0, i_1) : (M_0, M_1) \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$. Thus, the only difficulty is that P may not be equal to V.

Note however that the statement that every pair of ultrapower embeddings of the universe of sets admits an internal ultrapower comparison is equivalent to a first-order statement in the language of set theory. Therefore, since $P \equiv V$, P satisfies that every pair of ultrapower embeddings of the universe of sets admits an internal ultrapower comparison. But in P, $j_0 : P \to M_0$ and $j_1 : P \to M_1$ are ultrapower embeddings of the universe of sets since j_0 and j_1 are internal. It follows that in P there is an internal ultrapower comparison $(i_0, i_1) : (M_0, M_1) \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$. The fact that (i_0, i_1) is an internal ultrapower comparison of (j_0, j_1) is easily upwards absolute, so in fact (i_0, i_1) really is an internal ultrapower comparison of (j_0, j_1), and this completes the proof. □

3.2. *The Ketonen order*

One of the most important structural consequences of the Ultrapower Axiom is the existence of a natural wellorder of the class of countably complete fine ultrafilters over ordinals. The order coincides with a partial order introduced in a somewhat restricted setting by Ketonen [10]. Ketonen's definition of this order was expressed in terms of limits of ultrafilters (Definition 2.24).

Definition 3.7. Suppose U and W are countably complete ultrafilters over ordinals ϵ and δ. Then the *Ketonen order* is defined by setting $U <_{\Bbbk} W$ (respectively, $U \leq_{\Bbbk} W$) if $U = W\text{-}\lim_{\alpha<\delta} U_\alpha$ for a sequence $\langle U_\alpha : \alpha < \delta \rangle$ of countably complete ultrafilters over ϵ such that $\alpha \cap \epsilon \in U_\alpha$ (respectively, $(\alpha + 1) \cap \epsilon \in U_\alpha$) for W-almost all α.

A complete exposition of the basic theory of the Ketonen order appears in [7]. Here we will just state some facts and sketch some easy proofs.

Lemma 3.8. *Suppose U and W are countably complete ultrafilters over ordinals ϵ and δ. Then the following are equivalent*:

(1) $U <_{\Bbbk} W$ (respectively, $U \leq_{\Bbbk} W$).
(2) *For some ultrafilter U_* of M_W over $j_W(\epsilon)$, $j_W^{-1}[U_*] = U$ and $a_W \cap j_W(\epsilon) \in U_*$* (respectively, $(a_W + 1) \cap j_W(\epsilon) \in U_*$).

(3) *There is a comparison* $(k, \ell) : (M_U, M_W) \to N$ *of* (j_U, j_W) *such that* ℓ *is close to* M_W *and* $k(a_U) < \ell(a_W)$ (respectively, $k(a_U) \leq \ell(a_W)$).

Proof. We prove the lemma for the strict Ketonen order $<_{\Bbbk}$; the proof for $\leq_{\Bbbk}$ is a trivial modification.

The equivalence of (1) and (2) is an immediate application of Łoś's Theorem and Lemma 2.25.

(2) implies (3): Assume (2). Let $k : M_U \to M_{U_*}$ be the associated shift embedding (Definition 2.26), and let $\ell : M_W \to M_{U_*}$ be the ultrapower embedding associated to U_*. Then by Lemma 2.27, (k, ℓ) is a comparison of (j_U, j_W). Since ℓ is an internal ultrapower embedding, ℓ is close to M_W.

Finally, we show $k(a_U) < \ell(a_W)$. First note that $k(a_U) = a_{U_*}$ by Lemma 2.27. Since $a_W \cap j_W(\epsilon) \in U_*$, $a_{U_*} \in \ell(a_W) \cap \ell(j_W(\epsilon))$, and in particular $a_{U_*} < \ell(a_W)$. Therefore, $k(a_U) = a_{U_*} < \ell(a_W)$, as desired.

(3) implies (2): Assume (3), and let U_* be the M_W-ultrafilter over $j_W(\epsilon)$ derived from ℓ using $k(a_U)$. Since ℓ is close to M_W, Lemma 2.7 implies that $U_* \in M_W$. Since $k(a_U) < \ell(a_W)$, we have $\ell(a_W) \cap j_W(\epsilon) \in U_*$. Finally,

$$j_W^{-1}[U_*] = j_W^{-1}[\ell^{-1}[p_{k(a_U)}]] = j_U^{-1}[k^{-1}[p_{k(a_U)}]] = j_U^{-1}[p_{a_U}] = U. \qquad \square$$

The following is the first place where countable completeness is used in the theory of the Ketonen order.

Proposition 3.9. $\leq_{\Bbbk}$ *is a wellfounded preorder.*

The preorder $\leq_{\Bbbk}$ is almost antisymmetric.

Lemma 3.10. *Suppose* U *and* W *are countably complete ultrafilters over ordinals* ϵ *and* δ*. Then* $U \leq_{\Bbbk} W$ *and* $W \leq_{\Bbbk} U$ *if and only if there is an ordinal* $\alpha \in U \cap W$ *such that* $U \cap P(\alpha) = W \cap P(\alpha)$.

Corollary 3.11. *The restriction of the Ketonen order to fine ultrafilters, or to ultrafilters over a fixed ordinal, is antisymmetric.*

The relationship between $\leq_{\Bbbk}$ and $<_{\Bbbk}$ is simple.

Lemma 3.12. *If* U *and* W *are countably complete ultrafilters, then* $U <_{\Bbbk} W$ *if and only if* $U \leq_{\Bbbk} W$ *and* $W \not\leq_{\Bbbk} U$.

The following lemma allows us to view the Ketonen order as an extension of the natural wellorder of the class of ordinals to the class of countably complete ultrafilters over ordinals.

Lemma 3.13. *If* U *and* W *are countably complete ultrafilters over ordinals and* $\min(\text{Ord} \cap U) < \min(\text{Ord} \cap W)$, *then* $U <_{\Bbbk} W$.

Lemma 3.13 implies that for any ordinals α and β, $p_\alpha <_{\Bbbk} p_\beta$ if and only if $\alpha < \beta$. Moreover, for any countably complete ultrafilter U, $U <_{\Bbbk} p_\alpha$ if and only if

U concentrates on α. In particular, the set of fine ultrafilters below p_α is precisely the set of fine ultrafilters over ordinals less than α.

The following theorem is the fundamental consequence of the Ultrapower Axiom.

Theorem 3.14 (UA). *The Ketonen order wellorders the class of countably complete fine ultrafilters over ordinals.*

Proof. Given Proposition 3.9 and Lemma 3.10, it suffices to show that the Ketonen order is total, meaning that for any countably complete fine ultrafilters U and W, either $U \leq_{\Bbbk} W$ or $W <_{\Bbbk} U$. To see this, let $(k, \ell) : (M_U, M_W) \to N$ be an internal ultrapower comparison of (j_U, j_W). If $k(a_U) \leq \ell(a_W)$, then (k, ℓ) witnesses $U \leq_{\Bbbk} W$ by Lemma 3.8. Otherwise, $\ell(a_W) < k(a_U)$ so (ℓ, k) witnesses $W <_{\Bbbk} U$ by Lemma 3.8. □

3.3. *Translation functions*

In this section, we sketch the proofs of some slightly deeper structural consequences of UA for countably complete ultrafilters.

The irreflexivity of the Ketonen order yields the following useful lemma.[g]

Lemma 3.15. *Suppose U is a countably complete ultrafilter over an ordinal and i_0 and i_1 are elementary embeddings from M_U to a common model N such that i_1 is close to M_U and $i_0 \circ j_U = i_1 \circ j_U$. Then $i_0(a_U) \geq i_1(a_U)$.*

Proof. Otherwise by Lemma 3.8, $(i_0, i_1) : (M_U, M_U) \to N$ witnesses that $U <_{\Bbbk} U$, contradicting the wellfoundedness of the Ketonen order. □

Using Lemma 3.15, we obtain a key consequence of UA.

Theorem 3.16 (UA). *Suppose $j : V \to M$ is an ultrapower embedding and U is an ultrafilter over an ordinal δ. Let $U_* \in M$ be a countably complete M-ultrafilter over $j(\delta)$ such that $j^{-1}[U_*] = U$. Let $k : M_U \to M_{U_*}$ be defined by $k([f]_U) = [j(f)]_{U_*}$. Then the following are equivalent:*

(1) *U_* is $<_{\Bbbk}^M$-minimal among all countably complete ultrafilters U' of M over $j(\delta)$ such that $j^{-1}[U'] = U$.*
(2) *k is an internal ultrapower embedding of M_U.*

Recall that the shift lemma (Lemma 2.27) implies that the embedding k defined above is elementary. Moreover, Lemmas 2.8 and 2.11 imply that k is an ultrapower embedding. (We do include the details in the proof of Theorem 3.16.) Therefore, all that is in question in (2) is whether k is internal.

[g]We make two unrelated comments. First, the lemma does not actually require the hypothesis that U is countably complete. Second, a significant strengthening of this lemma appears as [6, Theorem 3.5.11]: given two inner models M and N and elementary embeddings $i_0, i_1 : M \to N$, if i_1 is definable over M from parameters, then $i_0(\alpha) \geq i_1(\alpha)$ for every ordinal α.

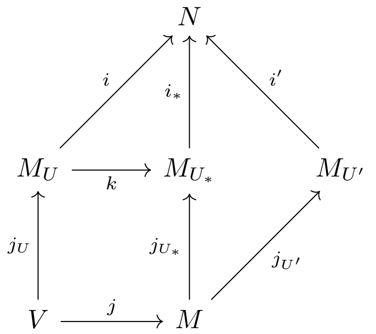

Fig. 1. The proof of Theorem 3.16.

Proof of Theorem 3.16. (1) implies (2): Let $(i, i_*) : (M_U, M_{U_*}) \to N$ be an internal ultrapower comparison of $(j_U, j_{U_*} \circ j)$.

The main claim is that $i_* \circ k = i$. Since every element of M_U is definable in M_U from parameters in $j_U[V] \cup \{a_U\}$, it suffices to show that $i_* \circ k$ and i coincide on $j_U[V] \cup \{a_U\}$. By the definition of a comparison,

$$i \circ j_U = i_* \circ j_{U_*} \circ j = i_* \circ k \circ j_U$$

so $i_* \circ k$ and i coincide on $j_U[V]$.

We finish the proof of the main claim by showing $i_* \circ k(a_U) = i(a_U)$. To see $i_* \circ k(a_U) \geq i(a_U)$, notice that $i_* \circ k$ and i are elementary embeddings from M_U to a common model N, i is close to M_U and $i_* \circ k \circ j_U = i \circ j_U$. Therefore, applying Lemma 3.15 with $i_0 = i_* \circ k$ and $i_1 = i$, we can conclude that $i_* \circ k(a_U) \geq i(a_U)$. On the other hand, the minimality of U_* implies that $i_* \circ k(a_U) \leq i(a_U)$. To see this, let U' be the M-ultrafilter derived from $i_* \circ j_{U^*}$ using $i(a_U)$. We claim that $j^{-1}[U'] = U$: indeed, for any $A \subseteq \delta$,

$$\begin{aligned} j(A) \in U' &\Leftrightarrow i(a_U) \in i_* \circ j_{U^*}(j(A)) \\ &\Leftrightarrow i(a_U) \in i(j_U(A)) \\ &\Leftrightarrow a_U \in j_U(A) \\ &\Leftrightarrow A \in U. \end{aligned}$$

The second equivalence follows from the fact that $i \circ j_U = i_* \circ j_{U^*} \circ j$ by the definition of a comparison.

By the minimality of U_*, $U_* \leq_{\Bbbk} U'$ in M. Let $i' : M_{U'} \to N$ be the canonical factor embedding associated to the derived ultrafilter U', so $i' \circ j_{U'} = i_* \circ j_{U^*}$ and $i'(a_{U'}) = i(a_U)$ by Lemma 2.3. Then $(i', i_*) : (M_{U'}, M_{U_*}) \to N$ is a comparison of $(j_{U'}, j_{U_*})$ and i_* is internal to M_{U_*}. Therefore, since $U_* \leq_{\Bbbk} U'$, we must have $i_*(a_{U_*}) \leq i'(a_{U'})$ But $i_*(a_{U_*}) = i_* \circ k(a_U)$ and $i'(a_{U'}) = i(a_U)$, so this implies $i_* \circ k(a_U) \leq i(a_U)$, as claimed.

This proves $i_* \circ k(a_U) = i(a_U)$, and concludes our proof of the claim that $i_* \circ k = i$. In particular, $i_* \circ k$ is an internal ultrapower embedding of M_U. We can therefore apply Corollary 2.10 to the sequence

$$V \xrightarrow{j_U} M_U \xrightarrow{k} M_{U_*} \xrightarrow{i_*} N$$

and deduce that k is an internal ultrapower embedding.

(2) implies (1): The proof does not use the Ultrapower Axiom. Suppose U' is another countably complete ultrafilter of M such that $j^{-1}[U'] = U$. We will show that $U' \not<_{\Bbbk} U_*$.

Let $(i', i_*) : (M_{U'}, M_{U_*}) \to N$ be a comparison of $(j_{U'}, j_{U_*})$ where i_* is close to M_{U_*}. We must show that $i'(a_{U'}) \geq i_*(a_{U_*})$. Let $k' : M_U \to M_{U'}$ be the shift embedding defined by $k'([f]_U) = [j(f)]_{U'}$. Since $i' \circ k'$ and $i_* \circ k$ are elementary embeddings from M_U to N such that $i' \circ k' \circ j_U = i_* \circ k \circ j_U$ and $i_* \circ k$ is close to M_U, $i' \circ k'(a_U) \geq i_* \circ k(a_U)$ by Lemma 3.15. But $i' \circ k'(a_U) = i'(a_{U'})$ and $i_* \circ k(a_U) = i_*(a_{U_*})$, so we have shown $i'(a_{U'}) \geq i_*(a_{U_*})$, as desired. □

Since Theorem 3.16 comes up so often, it is useful to introduce the following definition.

Definition 3.17 (UA). Suppose U is a countably complete ultrafilter over an ordinal δ, M is an inner model, and $j : V \to M$ is an elementary embedding. Then the *translation of U by j*, denoted $t_j(U)$, is the $<_{\Bbbk}^M$-least ultrafilter U' of M over $j(\delta)$ such that $j^{-1}[U'] = U$. If D is a countably complete ultrafilter, $t_D(U) = t_{j_D}(U)$.

Since $j^{-1}[j(U)] = U$, $t_j(U)$ is always defined, and moreover, $t_j(U) \leq_{\Bbbk} j(U)$ in M.

We make some comments on translation functions that are not needed in our applications. Assuming UA, it is not hard to show that for any countably complete ultrafilters U and W over ordinals,

$$(j_{t_U(W)}, j_{t_W(U)}) : (M_U, M_W) \to N$$

is a comparison of (j_U, j_W). This comparison is the *minimum* comparison of (j_U, j_W) in the sense that for any other internal ultrapower comparison

$$(k, \ell) : (M_U, M_W) \to P$$

of (j_U, j_W), there is a unique internal ultrapower embedding $h : N \to P$ such that $k = h \circ j_{t_U(W)}$ and $\ell = h \circ j_{t_W(U)}$. In other words, $(j_{t_U(W)}, j_{t_W(U)})$ is the *pushout* of (j_U, j_W) in the category of internal ultrapowers, the category theoretic analog of a least upper bound.

3.4. *The internal relation*

Probably the most important application of Theorem 3.16 concerns the case in which $t_j(W) = j(W)$.

Corollary 3.18 (UA). *Assume $j : V \to M$ is an ultrapower embedding and W is a countably complete ultrafilter over an ordinal. Then $t_j(W) = j(W)$ if and only if $j \restriction M_W$ is an internal ultrapower embedding.*

Proof. Let $k : M_W \to M_{j(W)}$ be the shift embedding defined by $k([f]_W) = [j(f)]_{j(W)}$. By Theorem 3.16, k is an internal ultrapower embedding of M_W if and only if $j(W) = t_j(W)$. But $k([f]_W) = [j(f)]_{j(W)} = j([f]_W)$, so $k = j$. The corollary follows. □

This corollary motivates the introduction of a variant of the Mitchell order.

Definition 3.19. The *internal relation* is defined on ultrafilters U and W by setting $U \sqsubset W$ if $j_U \restriction M_W$ is an internal ultrapower embedding of M_W.

Corollary 3.18 amounts to a computation of $t_U(W)$ in the case that $U \sqsubset W$. What about $t_W(U)$?

Definition 3.20. Suppose U is a countably complete ultrafilter over a set X, M is an inner model, and $j : V \to M$ is an elementary embedding. Then the *pushforward of U to M via j*, denoted $s_j(U)$, is the M-ultrafilter over $j(X)$ defined by $s_j(U) = \{A \in j(P(X)) : j^{-1}[A] \in U\}$. If W is a countably complete ultrafilter, then $s_W(U) = s_{j_W}(U)$.

Note that $j^{-1}[s_j(U)] = U$.

Notice that $s_j(U) = j_*(U) \cap M$ where $j_*(U)$ denotes the pushforward of U by j (Definition 2.22), so $s_j(U)$ really is a kind of pushforward. The following proposition reinforces our contention that the internal relation is a variant of the Mitchell order.

Proposition 3.21. *Suppose U is an ultrafilter over a set X and $j : V \to M$ is an ultrapower embedding. Then the following hold*:

(1) *$s_j(U)$ is the M-ultrafilter over $j(X)$ derived from $j_U \restriction M$ using $j_U(j)(a_U)$.*
(2) *$j^M_{s_j(U)} = j_U \restriction M$.*
(3) *$j_U \restriction M$ is an internal ultrapower embedding of M if and only if $s_j(U) \in M$.*

Proof. Towards (1), let U' be the M-ultrafilter over $j(X)$ derived from $j_U \restriction M$ using $j_U(j)(a_U)$. Then for any $A \in j(P(X))$,

$$\begin{aligned} A \in U' &\Leftrightarrow j_U(j)(a_U) \in j_U(A) \\ &\Leftrightarrow \{x \in X : j(x) \in A\} \in U \\ &\Leftrightarrow j^{-1}[A] \in U. \end{aligned}$$

Thus, $U' = s_j(U)$, proving (1).

Towards (2), let $k : M_{s_j(U)} \to j_U(M)$ be the canonical factor embedding with $k \circ j_{s_j(U)} = j_U \restriction M$ and $k(a_{s_j(U)}) = j_U(j)(a_U)$. We must show that k is an isomorphism. Since k is an elementary embedding, it suffices to show that k is surjective.

Since j is an ultrapower embedding, Lemma 2.4 yields some $a \in M$ such that every element of M is definable in M from a and parameters in $j[V]$. Hence by Los's

Theorem, every element of $j_U(M)$ is definable in $j_U(M)$ from $j_U(a)$ and parameters in $j_U(j)[M_U]$.

On the other hand, every element of M_U is definable in M_U from a_U and parameters in $j_U[V]$. Therefore by elementarity, every element of $j_U(j)[M_U]$ is definable in $j_U(M)$ from $j_U(j)(a_U)$ and parameters in $j_U(j) \circ j_U[V] = j_U \circ j[V]$.

It follows that every element of $j_U(M)$ is definable in $j_U(M)$ from $j_U(a)$, $j_U(j)(a_U)$, and parameters in $j_U \circ j[V]$. Therefore, every element of $j_U(M)$ is definable in $j_U(M)$ from $j_U(j)(a_U)$ and parameters in $j_U[M]$. Since $j_U(j)(a_U) \in k[M_{s_j(U)}]$ and $j_U[M] \subseteq k[M_{s_j(U)}]$, every element of $j_U(M)$ is definable in $j_U(M)$ from parameters in $k[M_{s_j(U)}]$. Since k is elementary, it follows that $k[M_{s_j(U)}] = j_U(M)$. Thus, k is surjective, proving (2).

The forwards direction of (3) is an immediate consequence of (2), and the reverse direction is an immediate consequence of (1). □

Proposition 3.22 (UA). *Suppose U is a countably complete ultrafilter over an ordinal, M is an inner model, and $j : V \to M$ is an elementary embedding. Then $j_U \restriction M$ is an internal ultrapower embedding if and only if $t_j(U) = s_j(U)$.*

Proof. If $t_j(U) = s_j(U)$, then $s_j(U) \in M$, so $j_U \restriction M$ is an internal ultrapower embedding by Proposition 3.21.

Conversely assume $s_j(U) \in M$. Consider the shift map $k : M_U \to M_{s_j(U)}$ defined by $k([f]_U) = [j(f)]_{s_j(U)}$. (This is well-defined because $j^{-1}[s_j(U)] = U$.) By Proposition 3.21,

$$k([f]_U) = j_U(j(f))(j_U(j)(a_U)) = j_U(j)([f]_U).$$

Hence $k = j_U(j)$, and in particular, k is an internal ultrapower embedding of M_U. Therefore by Theorem 3.16, $s_j(U) = t_j(U)$. □

We now use $s_U(U)$ to prove Theorem 3.24. First, we need a lemma which is essentially a combinatorial restatement of Proposition 3.21. For $A \subseteq V \times V$, let $A_x = \{y : (x, y) \in A\}$ and $A^y = \{x : (x, y) \in A\}$.

Lemma 3.23. *Suppose W is an ultrafilter over X and Z is an ultrafilter over Y. Then for any $A \subseteq X \times Y$,*

$$A \in W \otimes Z \Leftrightarrow [y \mapsto A^y]_Z \in s_Z(W). \tag{1}$$

Proof. The lemma is a consequence of the following computation:

$$\begin{aligned} A \in W \otimes Z &\Leftrightarrow \{x \in X : A_x \in Z\} \in W \\ &\Leftrightarrow \{x \in X : a_Z \in j_Z(A_x)\} \in W \\ &\Leftrightarrow \{x \in X : (j_Z(x), a_Z) \in j_Z(A)\} \in W \\ &\Leftrightarrow \{x \in X : j_Z(x) \in j_Z(A)^{a_Z}\} \in W \end{aligned}$$

$$\Leftrightarrow \{x \in X : j_Z(x) \in [y \mapsto A^y]_Z\} \in W$$

$$\Leftrightarrow j_Z^{-1}([y \mapsto A^y]_Z) \in W$$

$$\Leftrightarrow [y \mapsto A^y]_Z \in s_Z(W). \qquad \square$$

Theorem 3.24. *Suppose λ is a cardinal and U is a uniform ultrafilter over λ^+. Then either U is (κ, λ^+)-regular for some $\kappa < \lambda$ or else $U \otimes U$ is λ-decomposable.*

Proof. Assume first that for some $\kappa < \lambda$, $s_U(U)$ contains a set Y of M_U-cardinality less than $j_U(\kappa)$. Let $X = j_U^{-1}[Y]$. Then by the definition of $s_U(U)$, $X \in U$, so since U is uniform, $|X| = \lambda^+$. It follows from Lemma 2.51 that U is (κ, λ^+)-regular.

Assume instead that for all $\kappa < \lambda$, no set in $s_U(U)$ has M_U-cardinality less than $j_U(\kappa)$. Note that there is a set $Y \in s_U(U)$ of M_U-cardinality at most $j_U(\lambda)$: indeed, any ordinal ξ with $j_U(\alpha) < \xi$ for all $\alpha < \lambda^+$ belongs to $s_U(U)$, and there is such an ordinal ξ strictly below $j_U(\lambda^+)$ since U is λ^+-decomposable (Proposition 2.42). Clearly, $|\xi|^{M_U} \leq j_U(\lambda)$.

Let $f : j_U(\lambda^+) \to j_U(\lambda)$ be one-to-one on Y. Let $g : \lambda^+ \to {}^{\lambda^+}\lambda$ be such that $[g]_U = f$. We claim that the function $F(\alpha, \beta) = g(\beta)(\alpha)$ is a λ-decomposition of $U \otimes U$. Suppose not, and fix $A \in U \otimes U$ such that $|F[A]| < \kappa$ for some $\kappa < \lambda$. Let $A^\beta = \{\alpha : (\alpha, \beta) \in A\}$. Then $|g(\beta)[A^\beta]| < \kappa$ for all $\beta < \lambda$. Hence letting $B = [\beta \mapsto A^\beta]_U$, and recalling that $f = [g]_U$, we have that $|f[B]|^{M_U} < j_U(\kappa)$ by Los's Theorem. Moreover since $A \in U \otimes U$, $B \in s_U(U)$ by Eq. (1). Therefore, $B \cap Y \in s_U(U)$. But f is one-to-one on $B \cap Y$, and therefore $|B \cap Y|^{M_U} = |f[B \cap Y]|^{M_U} < j_U(\kappa)$, contrary to our case hypothesis. $\square$

3.5. *Commuting ultrapowers*

Given the wellfoundedness of the Mitchell order, it is natural to ask whether the internal relation is wellfounded. Kunen's Commuting Ultrapowers Lemma shows that it is not, and in fact, it contains 2-cycles:[h]

Theorem 3.25 (Kunen). *Suppose U is a κ-complete ultrafilter and W is an ultrafilter over a set of size less than κ. Then $j_U(j_W) = j_W \restriction M_U$ and $j_W(j_U) = j_U \restriction M_W$. In particular $U \sqsubset W$ and $W \sqsubset U$.*

A proof appears in the author's thesis [6, Theorem 5.5.21].

Definition 3.26. Two countably complete ultrafilters U and W are said to *commute* if $j_U(j_W) = j_W \restriction M_U$ and $j_W(j_U) = j_U \restriction M_W$.

Proposition 3.22 and Corollary 3.18 combine to yield the following converse to Theorem 3.25.

Theorem 3.27 (UA). *Suppose U and W are countably complete ultrafilters. Then*

[h] If U and W are fine ultrafilters over an ordinal δ and $U \sqsubset W$, then $U <_{\Bbbk} W$. Hence the internal relation *is* wellfounded when restricted to fine ultrafilters over a fixed ordinal δ.

the following are equivalent:

(1) $U \sqsubset W$ *and* $W \sqsubset U$.
(2) U *and* W *commute.*

Proof. (1) implies (2): Applying Proposition 3.22 and Corollary 3.18, $j_U(W) = t_U(W) = s_U(W)$. Therefore, $j_U(j_W) = j_{j_U(W)} = j_{s_U(W)} = j_W \restriction M_U$, where the final equality follows from Proposition 3.21. By symmetry, this establishes (2).

(2) implies (1): Trivial. □

One can ask whether Theorem 3.25 has a converse in another sense: if U and W commute, must U contain a set of size less than $\text{crit}(W)$ or vice versa? The answer is no, and the reader can no doubt provide a counterexample with a little bit of thought. It will be important, however, to establish a positive answer for ultrafilters with the following property.

Definition 3.28. Suppose λ is a cardinal and U is a countably complete ultrafilter. Then U is λ-*internal* if $D \sqsubset U$ for any ultrafilter over a set of size less than λ and U is *uniformly internal* if U is λ-internal where $\lambda = \min_{A\in U} |A|$.

For example, a κ-complete ultrafilter over κ is uniformly internal.

Theorem 3.29. *Suppose U and W are countably complete uniformly internal ultrafilters. Then the following are equivalent*:

(1) U *and* W *commute.*
(2) U *contains a set of cardinality less than* $\text{crit}(W)$ *or* W *contains a set of cardinality less than* $\text{crit}(U)$.

The proof requires the following fact.

Proposition 3.30. *Suppose κ is a strong limit cardinal and U is a countably complete κ-internal ultrafilter such that $j_U[\kappa] \subseteq \kappa$. Then U is κ-complete.*

The idea of the proof of Proposition 3.30 is to show that for every $\delta < \kappa$, there is an ultrafilter D over a set of size less than κ such that $j_D[\delta] = j_U[\delta]$. Since $D \sqsubset U$, $j_U[\delta] = j_D[\delta] \in M_U$. Now using the Kunen inconsistency theorem (Corollary 2.38), $\text{crit}(j_U) \geq \kappa$.

Thus, we must show that an ultrafilter U can be "approximated" by ultrafilters over smaller sets.

Lemma 3.31. *Suppose I, X and Y are sets, U is an ultrafilter over X, and $p : I \to j_U(Y)$ is a function. Then for some ultrafilter D over Y^I, there is an elementary embedding $k : M_D \to M_U$ such that $k \circ j_D = j_U$ and $p[I] \subseteq k[M_D]$.*

Proof. Choose $q : I \to {}^XY$ so that $[q(i)]_U = p(i)$. Define $f : X \to Y^I$ by $f(x)(i) = q(i)(x)$. Let $D = f_*(U)$ be the pushforward of U by f (Definition 2.22), and let

$k : M_D \to M_U$ be the canonical factor embedding with $k \circ j_D = j_U$ and $k(a_D) = [f]_U$ (Lemmas 2.3 and 2.23).

An easy computation shows that for any function g on Y^I, $k([g]_D) = [g \circ f]_U$. Note that for any $i \in I$, $q(i) = \text{eval}_i \circ f$ where $\text{eval}_i : Y^I \to Y$ is defined by $\text{eval}_i(g) = g(i)$. It follows that $k([\text{eval}_i]_D) = [\text{eval}_i \circ f]_U = [q(i)]_U = p(i)$. Hence $p[I] \subseteq k[M_D]$. □

Corollary 3.32. *Suppose M is an inner model, $j : V \to M$ is an ultrapower embedding, and κ is a cardinal. Then for some ultrafilter D over a set of size 2^κ, there is an elementary embedding $k : M_D \to M$ such that $k \circ j_D = j$ and $\text{crit}(k) > \kappa$.*

Proof. Apply Lemma 3.31 with $Y = \kappa + 1$, $I = \kappa$, and $p : \kappa \to j(Y)$ such that $\kappa + 1 \subseteq p[\kappa]$. Then one obtains an ultrafilter D over $(\kappa + 1)^\kappa$ such that there is an elementary embedding $k : M_D \to M$ with $\kappa + 1 \subseteq k[M_D]$, or in other words, $\text{crit}(k) > \kappa$. □

Proof of Proposition 3.30. By Corollary 3.32, for any $\delta < \kappa$, there is an ultrafilter D over a set of cardinality less than κ such that $j_D \restriction \delta = j_U \restriction \delta$. Since $D \sqsubset U$, $j_U[\delta] \in M_U$.

This shows that $j_U[\delta] \in M_U$ for all $\delta < \kappa$. This contradicts the Kunen inconsistency theorem for ultrapowers (Corollary 2.38), unless $\text{crit}(j_U) \geq \kappa$, or in other words, U is κ-complete. □

Proof of Theorem 3.29. (1) implies (2): Since U and W commute, $j_U(\text{crit}(W)) = \text{crit}(W)$. In particular $\text{crit}(U) \neq \text{crit}(W)$, so assume without loss of generality that $\text{crit}(U) < \text{crit}(W)$.

If U does not contain a set of size less than $\text{crit}(W)$, then since U is uniformly internal, U is $\text{crit}(W)$-internal. But then by Proposition 3.30, U is $\text{crit}(W)$-complete. This contradicts that $\text{crit}(U) < \text{crit}(W)$.

(2) implies (1): This is a special case of Theorem 3.25. □

4. The Least Supercompact Cardinal

In the course of the next two sections, we will prove that the least strongly compact cardinal is supercompact (Theorem 5.17). In this section, we prove a conditional result in this direction, a coarse version of which can be described in terms of the following cardinal.

Definition 4.1. For every cardinal γ, let τ_γ denote the least ordinal τ such that for all ordinals α, there is an ultrapower embedding $j : V \to M$ such that $\text{crit}(j) \geq \gamma$ and $j(\tau) > \alpha$.

Theorem 4.13 (UA). *If τ_{ω_1} is strongly compact, then it is supercompact.*

In Sec. 5 we will prove Theorem 5.18, which shows that τ_{ω_1} is supercompact. Notice that τ_{ω_1} is less than or equal to the least strongly compact which is less than or equal to the least supercompact, so if τ_{ω_1} is supercompact, then the least strongly compact cardinal is supercompact.

4.1. *Ketonen ultrafilters*

It is quite easy to construct an example of an ultrafilter U whose associated ultrapower embedding $j_U : V \to M_U$ witnesses that κ is strongly compact but does not witness that κ is supercompact. Therefore, to prove the supercompactness of the least strongly compact cardinal, we will have to define special ultrafilters that do witness supercompactness. The ultrafilters we use are called *Ketonen ultrafilters*.

Definition 4.2. A countably complete ultrafilter U over a cardinal λ is said to be a *Ketonen ultrafilter* if U is minimal in the Ketonen order among all countably complete uniform ultrafilters over λ.

In this section, we will only discuss Ketonen ultrafilters over regular cardinals, which have a simple combinatorial characterization.

Theorem 4.3 (Ketonen). *A countably complete ultrafilter U over a regular cardinal δ is a Ketonen ultrafilter if and only if U is weakly normal and concentrates on ordinals that carry no countably complete fine ultrafilter.*

Proof. Let U be a Ketonen ultrafilter over δ.

To see that U is weakly normal, consider the ultrafilter W over δ derived from j_U using $\sup j_U[\delta]$. By Lemma 2.21, W is weakly normal. Let $k : M_W \to M_U$ be the canonical factor embedding with $k(a_W) = \sup j_U[\delta]$. Then $(k, \mathrm{id}) : (M_W, M_U) \to M_U$ is a comparison witnessing $W \leq_{\Bbbk} U$. Since W is uniform, the minimality of U implies $U = W$. Hence U is weakly normal.

We now show that the set A of ordinals $\alpha < \delta$ carrying a countably complete fine ultrafilter is U-null. Assume not, and for each $\alpha \in A$, let U_α be a fine ultrafilter over α. Let W_α be the ultrafilter over δ induced by U_α, meaning $W_\alpha = \{B \subseteq \delta : B \cap \alpha \in U_\alpha\}$. Then $U\text{-}\lim_{\alpha \in A} W_\alpha <_{\Bbbk} U$ by definition, but clearly $U\text{-}\lim_{\alpha\in A} W_\alpha$ is a fine ultrafilter over δ.

Conversely, suppose U is weakly normal and concentrates on ordinals that carry no countably complete fine ultrafilter. Suppose W is an ultrafilter over δ such that $W <_{\Bbbk} U$. We must show that W is not fine. By the definition of the Ketonen order, $W = U\text{-}\lim_{\alpha<\delta} W_\alpha$ where and $\alpha \in W_\alpha$ for U-almost all $\alpha \in A$. For each $\alpha < \delta$, let $f(\alpha)$ be the least ordinal in W_α. Since U-almost every α carries no countably complete fine ultrafilter, $f(\alpha) < \alpha$ for U-almost all α. Since U is weakly normal and f is regressive modulo U, there is some $\nu < \delta$ such that $f(\alpha) < \nu$ for U-almost all $\alpha < \delta$. It follows that $\nu \in W_\alpha$ for U-almost all $\alpha < \delta$, and hence $\nu \in W$. Since $\nu < \delta$, this means that W is not fine. □

The key ZFC theorem on Ketonen ultrafilters is a variant of a theorem due to Ketonen himself, which he used to prove Theorem 2.56.

Theorem 4.4 (Ketonen). *Suppose κ is a cardinal, δ is a regular cardinal, and U is a Ketonen ultrafilter over δ. Assume that every regular cardinal in the interval $[\kappa, \delta]$ carries a countably complete fine ultrafilter. Then U is (κ, δ)-regular.*

Proof. By Lemma 2.51, it suffices to show

$$\mathrm{cf}^{M_U}(\sup j_U[\delta]) < j_U(\kappa).$$

By Theorem 4.3, $\sup j_U[\delta]$ carries no countably complete fine ultrafilter in M_U, and hence $\mathrm{cf}^{M_U}(\sup j_U[\delta])$ carries no countably complete fine ultrafilter in M_U. By elementarity, every M_U-regular cardinal in the interval $[j_U(\kappa), j_U(\delta)]$ carries a countably complete fine ultrafilter in M_U. Thus, $\mathrm{cf}^{M_U}(\sup j_U[\delta])$ does not lie in this interval, and so since $\mathrm{cf}^{M_U}(\sup j_U[\delta]) \leq j_U(\delta)$, it must be the case that $\mathrm{cf}^{M_U}(\sup j_U[\delta]) < j_U(\kappa)$. □

4.2. *The ultrafilter $\mathscr{K}_\delta$*

Under the Ultrapower Axiom, the Ketonen order is linear, and therefore there is at most one Ketonen ultrafilter over any ordinal.

Definition 4.5 (UA). For any cardinal λ, let $\mathscr{K}_\lambda$ denote the unique Ketonen ultrafilter over λ, if it exists.

The ultrafilter $\mathscr{K}_\lambda$ is the key to understanding strong compactness and supercompactness under UA, especially when λ is a regular cardinal. For example, Theorems 5.13 and 5.20 show that if δ is either a successor cardinal or an inaccessible cardinal, then $\mathscr{K}_\delta$ witnesses the δ-strong compactness of $\mathrm{crit}(\mathscr{K}_\delta)$ (assuming $\mathscr{K}_\delta$ exists). That is, $\mathscr{K}_\delta$ is $(\mathrm{crit}(\mathscr{K}_\delta), \delta)$-regular. Moreover, Corollary 5.14 shows that for any successor cardinal δ, $\mathscr{K}_\delta$ witnesses the δ-supercompactness of $\mathrm{crit}(\mathscr{K}_\delta)$; that is, the ultrapower $M_{\mathscr{K}_\delta}$ is closed under δ-sequences.

These properties of $\mathscr{K}_\delta$ trace back to the *universal property of $\mathscr{K}_\delta$*.

Theorem 4.6 (UA). *Suppose δ is a regular and carries a countably complete fine ultrafilter. Then for any wellfounded ultrapower embedding $j : V \to M$, the following are equivalent:*

(1) *There is an internal ultrapower embedding $k : M_{\mathscr{K}_\delta} \to M$ such that $k \circ j_{\mathscr{K}_\delta} = j$ and $k(\sup j_{\mathscr{K}_\delta}[\delta]) = \sup j[\delta]$.*
(2) $\sup j[\delta]$ *carries no countably complete fine ultrafilter in M.*

Proof. (1) implies (2): Since $\mathscr{K}_\delta$ is a Ketonen ultrafilter, $\sup j_{\mathscr{K}_\delta}[\delta]$ carries no countably complete fine ultrafilter in $M_{\mathscr{K}_\delta}$. Therefore by elementarity, $k(\sup j_{\mathscr{K}_\delta}[\delta]) = \sup j[\delta]$ carries no countably complete fine ultrafilter in M.

(2) implies (1). Let $(i_0, i_1) : (M_{\mathscr{K}_\delta}, M) \to N$ be an internal ultrapower comparison of $(j_{\mathscr{K}_\delta}, j)$. Since i_0 is an internal ultrapower embedding of $M_{\mathscr{K}_\delta}$, and there is no

countably complete fine ultrafilter over $\sup j_{\mathscr{K}_\delta}[\delta]$ in $M_{\mathscr{K}_\delta}$, $i_0(\sup j_{\mathscr{K}_\delta}[\delta]) = \sup i_0 \circ j_{\mathscr{K}_\delta}[\delta]$. Similarly, $i_1(\sup j[\delta]) = \sup i_1 \circ j[\delta]$. Therefore, $i_0(\sup j_{\mathscr{K}_\delta}[\delta]) = i_1(\sup j[\delta])$. Since $\mathscr{K}_\delta$ is weakly normal, every element of $M_{\mathscr{K}_\delta}$ is of the form $j_{\mathscr{K}_\delta}(f)(\sup j_{\mathscr{K}_\delta}[\delta])$ for some function $f : \delta \to V$. But $i_0(j_{\mathscr{K}_\delta}(f)(\sup j_{\mathscr{K}_\delta}[\delta])) = i_1(j(f))(i_1(\sup j[\delta])) \in i_1[M]$. It follows that $i_0[M_{\mathscr{K}_\delta}] \subseteq i_1[M]$.

Let $k = i_1^{-1} \circ i_0$. Since $i_0[M_{\mathscr{K}_\delta}] \subseteq i_1[M]$, $k : M_{\mathscr{K}_\delta} \to M$ is a (total) elementary embedding. Clearly, $k \circ j_{\mathscr{K}_\delta} = j$ and $k(\sup j_{\mathscr{K}_\delta}[\delta]) = \sup j[\delta]$.

It remains to show that k is an internal ultrapower embedding, but this follows immediately from Corollary 2.10 applied to the sequence

$$V \xrightarrow{j_{\mathscr{K}_\delta}} M_{\mathscr{K}_\delta} \xrightarrow{k} M \xrightarrow{i_1} N. \qquad \square$$

Theorem 4.6 yields a simple characterization of the internal ultrapower embeddings of $M_{\mathscr{K}_\delta}$.

Theorem 4.7 (UA). *Suppose δ is regular and carries a countably complete fine ultrafilter. An ultrapower embedding $h : M_{\mathscr{K}_\delta} \to N$ is internal if and only if $h(\sup j_{\mathscr{K}_\delta}[\delta]) = \sup h \circ j_{\mathscr{K}_\delta}[\delta]$.*

Proof. The forwards implication is trivial. For the converse, apply Theorem 4.6 with $j_1 = h \circ j_{\mathscr{K}_\delta}$ to obtain an internal ultrapower embedding $k : M_{\mathscr{K}_\delta} \to N$ such that $k \circ j_{\mathscr{K}_\delta} = h \circ j_{\mathscr{K}_\delta}$ and $k(\sup j_{\mathscr{K}_\delta}[\delta]) = \sup k \circ j_{\mathscr{K}_\delta}[\delta]$. Since $\sup k \circ j_{\mathscr{K}_\delta}[\delta] = \sup h \circ j_{\mathscr{K}_\delta}[\delta] = h(\sup j_{\mathscr{K}_\delta}[\delta])$, $k(\sup j_{\mathscr{K}_\delta}[\delta]) = h(\sup j_{\mathscr{K}_\delta}[\delta])$. Every element of $M_{\mathscr{K}_\delta}$ is of the form $j_{\mathscr{K}_\delta}(f)(\sup j_{\mathscr{K}_\delta}[\delta])$ so since $k \circ j_{\mathscr{K}_\delta} = h \circ j_{\mathscr{K}_\delta}$ and $k(\sup j_{\mathscr{K}_\delta}[\delta]) = h(\sup j_{\mathscr{K}_\delta}[\delta])$, $k = h$. It follows that h is an internal ultrapower embedding. $\square$

4.3. *Supercompactness conditioned on strong compactness*

The main result of this section is the following implication.

Theorem 4.8 (UA). *Suppose δ is a regular cardinal. Let $\kappa = \text{crit}(\mathscr{K}_\delta)$. Assume κ is δ-strongly compact. Then*

- *$M_{\mathscr{K}_\delta}$ is closed under γ-sequences for all $\gamma < \delta$.*
- *Every subset of $M_{\mathscr{K}_\delta}$ of cardinality δ is contained in a set in $M_{\mathscr{K}_\delta}$ that has cardinality δ in $M_{\mathscr{K}_\delta}$.*
- *$M_{\mathscr{K}_\delta}$ is closed under δ-sequences unless δ is strongly inaccessible.*

Theorem 4.7 shows that many $M_{\mathscr{K}_\delta}$-ultrafilters necessarily belong to $M_{\mathscr{K}_\delta}$. The idea behind Theorem 4.8 is that given a set $S \subseteq M_{\mathscr{K}_\delta}$, one can try to code S by an ultrafilter U, and if this can be done, one can attempt to use Theorem 4.7 to show $U \in M_{\mathscr{K}_\delta}$. Since S is coded into U, one can then conclude that $S \in M_{\mathscr{K}_\delta}$.

It is open whether (4) can be generalized to the case that δ is strongly inaccessible. The proof of (4) involves coding subsets of δ into ultrafilters on a cardinal $\gamma < \delta$, which is impossible if δ is strongly inaccessible.

This coding of sets by ultrafilters requires some infinite combinatorics; namely, the concept of an independent family of sets.

Definition 4.9. Suppose κ is a cardinal. A family of sets $\mathcal{F}$ is *κ-independent* if for any disjoint subfamilies σ and τ of $\mathcal{F}$, each of cardinality less than κ, there is a point that belongs to all the sets in σ and none of the sets in τ.

Suppose κ is a regular cardinal. Then a family of sets $\mathcal{F}$ is κ-independent if and only if given any disjoint subfamilies B_0 and B_1 of $\mathcal{F}$, there is a κ-complete proper filter G such that $B_0 \subseteq G$ and $B_1 \subseteq G^*$ where G^* denotes the ideal dual to G. For the forwards direction, consider the family B consisting of $<\kappa$-sized intersections of sets in B_0 and complements of sets in B_1. Every set in B is nonempty since $\mathcal{F}$ is independent, and since κ is regular, B is closed under $<\kappa$-sized intersections. Therefore, the filter G generated by B is proper and κ-complete. The reverse direction is an easy exercise.

Let us give a simple example of a κ-independent family. Suppose X is a set. For each $x \in X$, let $A_x = \{\sigma \in P_\kappa(X) : x \in \sigma\}$. Then $\mathcal{F} = \{A_x : x \in X\}$ is a κ-independent family. Notice that $|\mathcal{F}| = |X|$. Hausdorff proved that there are much larger κ-independent families.

Theorem 4.10 (Hausdorff). *For any set X, there is a κ-independent family of subsets of $P_\kappa(X)$ of cardinality $2^{|X|}$.*

Proof. Let λ be the cardinality of X. Since κ-independence is preserved by relabelings of the underlying set, it suffices to exhibit a κ-independent family $\mathcal{F}$ of subsets of some set S such that $|\mathcal{F}| = 2^\lambda$ and $|S| = \lambda^{<\kappa}$.

Let

$$S = \coprod_{\sigma \in P_\kappa(X)} P_\kappa(P(\sigma)) = \{(\sigma, t) : t \in P_\kappa(P(\sigma))\}.$$

Note that for all $\sigma \in P_\kappa(X)$, $|P_\kappa(P(\sigma))| \leq (2^{|\sigma|})^{<\kappa} = 2^{<\kappa}$. Therefore,

$$|S| = |P_\kappa(\lambda)| \cdot \sup_{\sigma \in P_\kappa(\lambda)} |P_\kappa(P(\sigma))| = \lambda^{<\kappa}.$$

For each $A \subseteq X$, let

$$S_A = \{(\sigma, t) \in S : A \cap \sigma \in t\}$$

and let

$$\mathcal{F} = \{S_A : A \subseteq X\}.$$

Then $|\mathcal{F}| = 2^\lambda$: the function $A \mapsto S_A$ is injective since $x \in A$ if and only if $(\{x\}, \{\{x\}\}) \in S_A$.

To finish, we show that $\mathcal{F}$ is κ-independent. Suppose $\mathcal{T}_0$ and $\mathcal{T}_1$ are disjoint collections of subsets of X, each of cardinality less than κ. We will find a point in S that belongs to S_A for every $A \in \mathcal{T}_0$ and does not belong to S_B for any $B \in \mathcal{T}_1$, which clearly implies that $\mathcal{F}$ is κ-independent.

Choose $\sigma \in P_\kappa(X)$ large enough that for any $A \in \mathcal{T}_0$ and $B \in \mathcal{T}_1$, $A\cap\sigma \neq B\cap\sigma$. Now let

$$t = \{A \cap \sigma : A \in \mathcal{T}_0\}.$$

Obviously, $A\cap\sigma \in t$ if $A \in \mathcal{T}_0$. Moreover, $B\cap\sigma \notin t$ if $B \in \mathcal{T}_1$ since $B\cap\sigma \neq A\cap\sigma$ for all $A \in \mathcal{T}_0$ by our choice of σ. It follows from the definition of S_A that $(\sigma, t) \in S_A$ for all $A \in \mathcal{T}_0$ and $(\sigma, t) \notin S_B$ for all $B \in \mathcal{T}_1$. Thus, (σ, t) is as desired. □

Lemma 4.11. *Suppose κ is a λ-strongly compact cardinal. Suppose M is an inner model of* ZFC *with $P_\kappa(\lambda) \subseteq M$. Suppose $\gamma \leq \lambda$ is an M-cardinal of cofinality at least κ such that $\lambda \leq (2^\gamma)^M$ and $U \cap M \in M$ for every κ-complete ultrafilter U over γ. Then $P(\lambda) \subseteq M$.*

Proof. We first note that κ is δ-strongly compact in M for all M-regular cardinals $\delta \leq \gamma$. Indeed, since $P_\kappa(\gamma) \subseteq M$, every regular cardinal δ of M with $\delta \leq \gamma$ has cofinality at least κ in V and hence carries a κ-complete fine ultrafilter W. By assumption $W \cap M \in M$, so W is a κ-complete uniform ultrafilter over δ in M. By Ketonen's Theorem (Theorem 2.56), this implies κ is δ-strongly compact in M for all regular $\delta \leq \gamma$.

In particular, it follows from Solovay's Theorem (Theorem 2.57) applied in M that $|P_\kappa(\gamma)|^M = \gamma$. Therefore, $U \cap M \in M$ for every κ-complete ultrafilter U over $P_\kappa(\gamma)$.

To prove the lemma, it suffices to show that there is a set $\mathcal{F} \in M$ such that $|\mathcal{F}|^M = \lambda$ and $P(\mathcal{F}) \subseteq M$.

Working in M, apply Theorem 4.10 to obtain a κ-independent family $\mathcal{F}$ of subsets of $P_\kappa(\gamma)$ of cardinality λ. Since $P_\kappa(\lambda) \in M$, M contains every subfamily of $\mathcal{F}$ of cardinality less than κ. It follows that $\mathcal{F}$ is truly κ-independent.

Suppose $S \subseteq \mathcal{F}$. We will show that $S \in M$. Let $T = \mathcal{F}\setminus S$. By the remarks following Definition 4.9, there is a κ-complete filter G over $P_\kappa(\gamma)$ such that $S \subseteq G$ and $T \subseteq G^*$ where G^* denotes the ideal dual to G. Moreover, the least such filter G is generated by λ-many sets; namely, the $<\kappa$-sized intersections of sets in S and complements of sets in T. Since κ is λ-strongly compact, G extends to a κ-complete ultrafilter U over $P_\kappa(\gamma)$. (See Theorem 2.56.) By assumption $U \cap M \in M$. But $S = U \cap \mathcal{F}$, so $S \in M$. □

[6, Theorem 7.3.22] generalizes Lemma 4.11 in a way that clarifies the underlying combinatorics.

We can now prove the main result of this section.

Proof of Theorem 4.8. Let $j : V \to M$ denote the ultrapower embedding associated to $\mathscr{K}_\delta$.

We first show that for all ordinals $\alpha < \delta$, $P(\alpha) \subseteq M$. Assume towards a contradiction that this fails, and let γ be the least ordinal such that $P(\gamma) \nsubseteq M$. Clearly,

γ is an M-cardinal, and moreover since M is closed under κ-sequences, $\text{cf}(\gamma) \geq \kappa$. Since $\gamma < \delta$, Theorem 4.7 implies that for all countably complete ultrafilters U over γ, $U \sqsubset \mathscr{K}_\delta$ and hence $U \cap M \in M$. Therefore by Lemma 4.11, $P(\gamma) \subseteq M$. This is a contradiction, establishing that $P(\alpha) \subseteq M$ for all $\alpha < \delta$.

Next, we show that $\text{cf}^M(\sup j[\delta]) = \delta$. Suppose not, towards a contradiction. Then for every countably complete ultrafilter U over δ, $U \cap M \in M$. To see this, note that $j^M_{U\cap M} : M \to M^M_U$ is continuous at $\text{cf}^M(\sup j[\delta])$. Therefore by Theorem 4.6, $j^M_{U\cap M}$ is an internal ultrapower embedding of M, and hence $U \cap M \in M$.

Note that in particular, $\mathscr{K}_\delta \cap M \in M$. This is quite strange given the fact that no countably complete ultrafilter belongs to its own ultrapower, but it is not in and of itself a contradiction.

Since $U \cap M \in M$ for every countably complete ultrafilter U over δ, however, we can apply Lemma 4.11 with $\gamma = \lambda = \delta$ to obtain that $P(\delta) \subseteq M$. Now $\mathscr{K}_\delta = \mathscr{K}_\delta \cap M \in M$. This contradicts that an ultrafilter never belongs to its own ultrapower, and this contradiction proves that $\text{cf}^M(\sup j[\delta]) = \delta$. By Lemma 2.51, it follows that every subset of M of cardinality at most δ is contained in a set in M of M-cardinality at most δ.

We now show M is closed under γ-sequences for all $\gamma \leq \delta$ such that $P(\gamma) \subseteq M$. By Lemma 2.32, it suffices to show that $j[\gamma] \in M$ for all $\gamma < \delta$.

Assume towards a contradiction that $\gamma < \delta$ is the least ordinal such that M is not closed under γ-sequences. Clearly, γ is a regular cardinal. By the covering property of M established above, $\sup j[\gamma]$ has cofinality at most δ in M. In fact, since $\sup j[\gamma]$ has cofinality γ in V, its cofinality in M is *strictly less* than δ. Applying Lemma 2.51 again, we obtain a set $A \in M$ containing $j[\gamma]$ such that $|A|^M < \delta$. Since $P(|A|^M) \subseteq M$, $P(A) \subseteq M$, and so $j[\gamma] \in M$. Therefore, M is closed under γ-sequences. This is a contradiction, establishing that M is closed under γ-sequences for all $\gamma < \delta$.

Finally, assume δ is not strongly inaccessible, and we will show that M is closed under δ-sequences. There are two cases.

Suppose first that $\delta \leq \gamma^{<\kappa}$ for some cardinal $\gamma < \delta$. Then by Theorem 2.57, $\delta = \gamma^+$ and γ has cofinality less than κ. Since $j[\gamma] \in M$, $j[P_\kappa(\gamma)] = P_\kappa(j[\gamma]) \in M$. Since $|P_\kappa(\gamma)| = \delta$, $j[\delta] = M$. This shows that M is closed under δ-sequences in this case.

Suppose instead that $\gamma^{<\kappa} < \delta$ for all $\gamma < \delta$. Let γ be least such that $2^\gamma \geq \delta$. Then since $\gamma < \delta$, $P(\gamma) \in M$, and so $(2^\gamma)^M \geq \delta$. Therefore, applying Lemma 4.11, $P(\delta) \subseteq M$. Since we showed that M is closed under α-sequences for all $\alpha \leq \delta$ such that $P(\alpha) \subseteq M$, it follows that M is closed under δ-sequences in this case as well. □

We now prove the conditional result on the cardinal τ_{ω_1} described at the beginning of this section.

Theorem 4.12. *Suppose τ_{ω_1} is δ-strongly compact where $\delta \geq \tau_{\omega_1}$ is a regular cardinal. Then* $\text{crit}(\mathscr{K}_\delta) = \tau_{\omega_1}$.

Proof. The fact that $\text{crit}(\mathscr{K}_\delta) \leq \tau_{\omega_1}$ is a consequence of Theorem 4.4. More explicitly, note that since τ_{ω_1} is δ-strongly compact, every regular cardinal in the interval $[\tau_{\omega_1}, \delta]$ carries a uniform countably complete ultrafilter. Applying Theorem 4.4, this means that $\mathscr{K}_\delta$ is $(\tau_{\omega_1}, \delta)$-regular. In particular, $\mathscr{K}_\delta$ is τ_{ω_1}-decomposable (by Corollary 2.50), and hence $\text{crit}(\mathscr{K}_\delta) \leq \tau_{\omega_1}$.

To show that $\tau_{\omega_1} \leq \text{crit}(\mathscr{K}_\delta)$, we first show that for any ultrapower embedding $j : V \to M$, $j[\tau_{\omega_1}] \subseteq \tau_{\omega_1}$. Suppose $j : V \to M$ is an ultrapower embedding and κ is an ordinal such that $j(\kappa) \geq \tau_{\omega_1}$. We will show $\kappa \geq \tau_{\omega_1}$. We claim that for any α, there is an ultrapower embedding $i : V \to N$ such that $i(\kappa) > \alpha$. To see this, one just composes j with an ultrapower embedding sending τ_{ω_1} above α, (The fact that the composed embedding is also an ultrapower embedding follows from Lemma 2.11 and Proposition 3.21.) By the minimality of τ_{ω_1}, it follows that $\kappa \geq \tau_{\omega_1}$, as claimed.

In particular, $j_{\mathscr{K}_\delta}[\tau_{\omega_1}] \subseteq \tau_{\omega_1}$. But $\mathscr{K}_\delta$ is τ_{ω_1}-internal and τ_{ω_1} is a strong limit cardinal, so by Proposition 3.30, $\text{crit}(\mathscr{K}_\delta) \geq \tau_{\omega_1}$. □

Theorem 4.13. *If the cardinal τ_{ω_1} is strongly compact, it is supercompact.*

Proof. Let $\delta \geq \tau_{\omega_1}$ be a successor cardinal. Since τ_{ω_1} is δ-strongly compact, $\mathscr{K}_\delta$ exists. By Theorem 4.12, $\text{crit}(\mathscr{K}_\delta) = \tau_{\omega_1}$. Therefore by assumption, $\text{crit}(\mathscr{K}_\delta)$ is δ-strongly compact, and so we can apply Theorem 4.8 to obtain that $M_{\mathscr{K}_\delta}$ is closed under δ-sequences. This yields that τ_{ω_1} is δ-supercompact, and so since δ was an arbitrary successor cardinal, τ_{ω_1} is supercompact. □

5. Strong Compactness and Uniform Ultrafilters

In this section, we study the pattern of cardinals that carry countably complete uniform ultrafilters under UA, culminating in our main theorem.

Theorem 5.17 (UA). *The least strongly compact cardinal is supercompact.*

In fact, this will follow as a consequence of various stronger and more local theorems (Corollary 5.14 and Theorem 5.21).

5.1. *Fréchet cardinals*

Definition 5.1. A cardinal λ is *Fréchet* if it carries a countably complete uniform ultrafilter.

A cardinal λ is Fréchet if and only if the Fréchet filter $\{A \subseteq \lambda : |\lambda \backslash A| < \lambda\}$ extends to a countably complete ultrafilter.

We will use a number of characterizations of Fréchet cardinals that are very easy to prove.

Lemma 5.2. *For any cardinal λ, the following are equivalent:*

(1) *λ is Fréchet.*

(2) *There is a countably complete ultrafilter* U *such that* $\min_{A\in U} |A| = \lambda$.
(3) *There is a countably complete* λ*-decomposable ultrafilter.*

If λ *is regular, one can add to the list*:

(4) *There is an elementary embedding* $j : V \to M$ *such that* $\sup j[\lambda] < j(\lambda)$.

Proof. The equivalence of (1) and (2) is trivial, and the equivalence of (1) and (3) follows immediately from Lemma 2.41. (4) follows from (1) by taking an ultrapower, and (1) follows from (4) by considering the ultrafilter over λ derived from j using $\sup j[\lambda]$, which is fine, and hence uniform if λ is regular (Lemma 2.18). □

Given an ordinal γ, how large is the least Fréchet cardinal above γ?

Definition 5.3. For any ordinal γ, γ^σ denotes the least Fréchet cardinal above γ.

Of course, γ^σ may not be defined. Note that 1^σ is equal to the least measurable cardinal (if there is one). On the other hand, if κ is κ^+-strongly compact, then $\kappa^\sigma = \kappa^+$. It is natural to conjecture that these are the only possibilities. It is consistent, however, that this is not the case: for example, Gitik [5] shows that if there are two measurable cardinals and $\kappa < \lambda$ are the least ones, then it is possible to build a forcing extension in which λ is strongly inaccessible but not weakly compact and $\kappa^\sigma = \lambda$. It is therefore natural to make the following revised conjecture.

Conjecture 5.4. *Assuming the Ultrapower Axiom, for any ordinal* γ, *either* $\gamma^\sigma = \gamma^+$ *or* γ^σ *is measurable.*

While Conjecture 5.4 remains open, we will show in Proposition 5.19 that it holds assuming GCH. (This is by far the most important case given that our interest is not really in *arbitrary* models of UA.)

The notion of an *isolated cardinal* naturally arises from the attempt to prove Conjecture 5.4.

Definition 5.5. A cardinal λ is isolated if λ is a limit cardinal and there is some $\gamma < \lambda$ such that $\lambda = \gamma^\sigma$.

The following lemma will be quite important.

Lemma 5.6. *Suppose* λ *is a cardinal and* λ^+ *is Fréchet. Either* λ *is Fréchet or* λ *is a singular limit of Fréchet cardinals.*

Proof. Theorem 3.24 implies that there is a countably complete ultrafilter W that is either λ-decomposable or (κ, λ^+)-regular for some $\kappa < \lambda$.

If W is λ-decomposable, then by Lemma 5.2, λ is a Fréchet cardinal. Assume instead that W is (κ, λ^+)-regular. Then by Corollary 2.50, W is δ-decomposable for every regular cardinal δ in the interval $[\kappa, \lambda^+]$. If λ is regular, then λ itself is a regular cardinal in the interval $[\kappa, \lambda^+]$, and so λ is Fréchet. Otherwise, λ is a singular cardinal, so λ is a singular limit of Fréchet cardinals. □

By the following proposition, to prove Conjecture 5.4, one only has to show that every isolated cardinal is measurable.

Proposition 5.7. *Suppose γ^σ is a successor cardinal. Then $\gamma^\sigma = \gamma^+$.*

Proof. Let λ be the cardinal predecessor of γ^σ. Then $\gamma \geq \lambda$: otherwise, Lemma 5.6 implies that there is a Fréchet cardinal strictly between γ and γ^σ, which contradicts the definition of γ^σ. Now $\gamma^\sigma = \lambda^+ \leq \gamma^+$, proving the proposition. □

The following corollary is the true explanation for our interest in isolated cardinals.

Proposition 5.8. *Suppose δ is a regular cardinal, U is a Ketonen ultrafilter over δ and $\kappa = \text{crit}(U)$. If there are no isolated cardinals λ such that $\kappa < \lambda \leq \delta$, then κ is δ-strongly compact and U is (κ, δ)-regular.*

Proof. Fix an ordinal γ such that $\kappa \leq \gamma < \delta$.

Let $\lambda = \gamma^\sigma$. We claim $\lambda = \gamma^+$. Assume towards a contradiction that $\lambda > \gamma^+$. Then λ is an isolated cardinal by Proposition 5.7. Since δ is a Fréchet cardinal above γ, $\lambda \leq \delta$. This contradicts that there are no isolated cardinals such that $\kappa < \lambda \leq \delta$.

This proves that every successor cardinal in the interval (κ, δ) is Fréchet. By Lemma 5.6, it follows that every regular cardinal in the interval $[\kappa, \delta]$ is Fréchet. Therefore by Ketonen's theorem (Theorem 4.4), U is (κ, δ)-strongly compact. In particular, it follows (from Theorem 2.56, say, but essentially by definition) that κ is δ-strongly compact. □

5.2. *Ketonen ultrafilters over isolated cardinals*

In this section, we study $\mathscr{K}_\lambda$ when λ is isolated. The key is a weak form of Theorem 4.7.

Lemma 5.9. *Suppose λ is an isolated cardinal. Then for any countably complete ultrafilter D over a cardinal $\delta < \lambda$, $j_D \restriction M_{\mathscr{K}_\lambda}$ is an internal ultrapower embedding.*

The lemma is only of interest when λ is singular, since we have already proved a much stronger result in the regular case (Theorem 4.7). In the singular case, the assumption that λ is isolated is essential, since one can show that if λ is a nonisolated singular Fréchet cardinal of cofinality δ, then $j_{\mathscr{K}_\delta} \restriction M_{\mathscr{K}_\lambda}$ is *not* an internal ultrapower embedding.

Lemma 5.9 is a consequence of Corollary 3.18, our general UA technique for proving that the restriction of an ultrapower embedding to another ultrapower is internal. The proof also uses the following easy lemma.

Lemma 5.10. *Suppose λ is a cardinal, X is a set of cardinality less than λ, and D is an ultrafilter over X. Suppose Y is a set and U_* is an ultrafilter of M_D over*

$j_D(Y)$ that is $(j_D(\gamma), j_D(\lambda^+))$-indecomposable for some $\gamma < \lambda$. Then $j_D^{-1}[U_*]$ is λ-indecomposable.

Proof. Let $U = j_D^{-1}[U_*]$. Let $\langle U_x : x \in X\rangle$ be a sequence of (γ, λ^+)-indecomposable ultrafilters such that $U_* = [\langle U_x : x \in X\rangle]_D$. Suppose $\langle A_\alpha : \alpha < \lambda\rangle$ is a partition of Y. We must find a set $S \subseteq \lambda$ of cardinality less than λ such that $\bigcup_{\alpha\in S} A_\alpha \in U$.

For each $x \in X$, let $S_x \subseteq \lambda$ be a set of cardinality less than γ such that $\bigcup_{\alpha\in S_x} A_\alpha \in U_x$. Let $S = \bigcup_{x\in X} S_x$. Since $|X| < \lambda$ and $|S_x| < \gamma$ for all x, $|S| < \lambda$. Moreover, $\bigcup_{x\in S} A_x \in U_x$ for all $x \in X$ since $S_x \subseteq S$. By Los's Theorem, this implies that $j_D(\bigcup_{x\in S} A_x) \in U_*$, or in other words, $\bigcup_{x\in S} A_x \in U$. Therefore, S is as desired. □

Proof of Lemma 5.9. Let U_* be the $<_{\Bbbk}^{M_D}$-least ultrafilter of M_D over $j_D(\lambda)$ such that $j_D^{-1}[U_*] = \mathscr{K}_\lambda$.

We claim that U_* is $j_D(\lambda)$-decomposable in M_D. To see this, fix $\gamma < \lambda$ such that $\lambda = \gamma^\sigma$. Then U_* (and indeed any countably complete ultrafilter of M_D) is $(j_D(\gamma), j_D(\lambda))$-indecomposable in M_D, since $j_D(\lambda)$ is isolated in M_D. Assume towards a contradiction that U_* is $j_D(\lambda)$-indecomposable. Then by Proposition 2.45, U_* is $(j_D(\gamma), j_D(\lambda^+))$-indecomposable. Therefore by Lemma 5.10, $j_D^{-1}[U_*] = \mathscr{K}_\lambda$ is λ-indecomposable. This contradicts that $\mathscr{K}_\lambda$ is a uniform ultrafilter over λ.

Note that for any cardinal η, $\mathscr{K}_\eta$ is the Ketonen least fine ultrafilter that is η-decomposable. Therefore, since U_* is $j_D(\lambda)$-decomposable in M_D, $j_D(\mathscr{K}_\lambda) \leq_{\Bbbk} U_*$ in M_D. Obviously $U_* \leq_{\Bbbk} j_D(\mathscr{K}_\lambda)$, so equality holds. Hence by Corollary 3.18, $j_D \restriction M_{\mathscr{K}_\lambda}$ is an internal ultrapower embedding. □

The following criterion for nonisolation will prove quite useful.

Lemma 5.11 (UA). *Suppose λ is a Fréchet cardinal and there is a countably complete ultrafilter D such that $\mathscr{K}_\lambda \sqsubset D$ but $D \not\leq \mathscr{K}_\lambda$. Then λ is not isolated.*

Proof. Assume towards a contradiction that λ is isolated. Assume without loss of generality that the underlying set of D is a cardinal.

Since $\mathscr{K}_\lambda \sqsubset D$, $t_D(\mathscr{K}_\lambda) = s_D(\mathscr{K}_\lambda)$ by Proposition 3.22. Since $D \not\leq \mathscr{K}_\lambda$, Corollary 3.18 implies that $t_D(\mathscr{K}_\lambda) <_{\Bbbk} j_D(\mathscr{K}_\lambda)$. Since $j_D(\mathscr{K}_\lambda)$ is the least countably complete uniform ultrafilter of M_D over $j_D(\lambda)$, it follows that $s_D(\mathscr{K}_\lambda)$ is not a uniform ultrafilter in M_D.

Fix a set $Y \in s_D(\mathscr{K}_\lambda)$ of minimal cardinality. Let δ be the least cardinal less than λ such that there are no Fréchet cardinals γ with $\delta \leq \gamma < \lambda$. Then since $s_D(\mathscr{K}_\lambda) \cap P^{M_D}(Y)$ is uniform, $|Y|^{M_D} < j_D(\delta)$. Let $X = j_D^{-1}[Y]$. Then $X \in \mathscr{K}_\lambda$ since $Y \in s_D(\mathscr{K}_\lambda)$. Now applying Lemma 2.51, D is (δ, λ)-regular: we have found a set X of cardinality λ such that $j_D[X]$ is covered by a set Y in M_D of cardinality less than $j_D(\delta)$, and so since Lemma 2.51(1) implies Lemma 2.51(4), the (δ, λ)-regularity D follows by definition. Now by Corollary 2.50, λ is a limit of Fréchet cardinals, and hence λ is not isolated. This is a contradiction. □

As a consequence, we obtain the following theorem.

Theorem 5.12 (UA). *Suppose $\lambda_0 < \lambda_1$ are Fréchet cardinals, λ_0 is isolated, and λ_1 is either isolated or regular. Then $\lambda_0 < \text{crit}(\mathscr{K}_{\lambda_1})$.*

Proof. Note that $\mathscr{K}_{\lambda_0} \sqsubset \mathscr{K}_{\lambda_1}$ by Theorem 4.7 or Lemma 5.9. Since λ_0 is isolated, Lemma 5.11 implies that $\mathscr{K}_{\lambda_1} \sqsubset \mathscr{K}_{\lambda_0}$. By Theorem 3.27, it follows that $\mathscr{K}_{\lambda_0}$ and $\mathscr{K}_{\lambda_1}$ commute.

By Theorem 4.7 and Lemma 5.9, $\mathscr{K}_{\lambda_0}$ and $\mathscr{K}_{\lambda_1}$ are uniformly internal, and so Theorem 3.29 implies that $\mathscr{K}_{\lambda_0}$ contains a set of size less than $\text{crit}(\mathscr{K}_{\lambda_1})$. Since $\mathscr{K}_{\lambda_0}$ is a uniform ultrafilter over λ_0, this means that $\lambda_0 < \text{crit}(\mathscr{K}_{\lambda_1})$. $\square$

5.3. *Strong compactness*

The most important application of Theorem 5.12 is the following theorem.

Theorem 5.13 (UA). *If δ is a regular Fréchet cardinal that is not isolated. Let $\kappa = \text{crit}(\mathscr{K}_\delta)$. Then κ is δ-strongly compact and $\mathscr{K}_\delta$ is (κ, δ)-regular.*

Proof. Let $\kappa = \text{crit}(\mathscr{K}_\delta)$. By Theorem 5.12, there are no isolated cardinals in the interval (κ, δ). Therefore by Proposition 5.8, κ is δ-strongly compact. $\square$

Corollary 5.14 (UA). *Suppose δ is a successor cardinal that carries a countably complete uniform ultrafilter. Then some cardinal $\kappa < \delta$ is δ-supercompact.*

Proof. Since successor cardinals are not isolated, this corollary follows immediately from Theorems 4.8 and 5.13. $\square$

The proof of Theorem 5.13 yields two characterizations of the least δ-strongly compact cardinal.

Corollary 5.15 (UA). *Suppose δ is a regular Fréchet cardinal that is not isolated. Let κ be the supremum of all isolated cardinals less than δ. Let κ' be the least (ω_1, δ)-strongly compact cardinal. Let κ'' be the least δ-strongly compact cardinal. Let $\kappa''' = \text{crit}(\mathscr{K}_\delta)$. Then $\kappa = \kappa' = \kappa'' = \kappa'''$.*

Proof. Clearly, $\kappa \leq \kappa'$ since every cardinal in the interval $[\kappa', \delta]$ is Fréchet, and so no cardinal in the interval (κ', δ) is isolated. Trivially, $\kappa' \leq \kappa''$. Theorem 5.13 shows that $\text{crit}(\mathscr{K}_\delta)$ is δ-strongly compact. Hence

$$\kappa \leq \kappa' \leq \kappa'' \leq \kappa'''.$$

Since there are no isolated cardinals in the interval (κ, δ), the argument of Proposition 5.8 shows that for any ordinal $\gamma \in (\kappa, \delta)$, $\gamma^\sigma = \gamma^+$. As in Proposition 5.8, it follows that every regular cardinal in the interval $[\kappa, \delta]$ is Fréchet. Therefore by Theorem 4.4, $\mathscr{K}_\delta$ is (κ, δ)-regular, and hence $\kappa''' = \text{crit}(\mathscr{K}_\delta) \leq \kappa$. This proves the corollary. $\square$

Corollary 5.16 (UA). *For any successor cardinal δ, the least (ω_1, δ)-strongly compact cardinal is δ-supercompact.*

Proof. This is an immediate consequence of Theorem 4.8 and Corollary 5.15. □

This immediately implies our main theorem.

Theorem 5.17. *The least strongly compact cardinal is supercompact.*

Proof. Indeed, the least (ω_1, ∞)-strongly compact cardinal κ is supercompact. This follows from Corollary 5.16, noting that for all $\delta \geq \kappa$, κ is the least (ω_1, δ)-strongly compact cardinal. (If $\bar{\kappa} \leq \kappa$ is (ω_1, κ)-strongly compact, then $\bar{\kappa}$ is (ω_1, ∞)-strongly compact by Theorem 2.56.) □

Let us also make good on our promise to prove that the cardinal τ_{ω_1} defined at the beginning of Sec. 4 is strongly compact.

Theorem 5.18 (UA). *The cardinal τ_{ω_1} is supercompact.*

Proof. The existence of τ_{ω_1} easily implies that there is a proper class of Fréchet cardinals. We will show that there are no isolated cardinals above τ_{ω_1}.

Fix a Fréchet cardinal $\lambda \geq \tau_{\omega_1}$. Let $\kappa = \text{crit}(\mathscr{K}_\lambda)$. Let ξ be the least ordinal greater than κ such that $j_{\mathscr{K}_\lambda}(\xi) = \xi$. Let W be the $<_\Bbbk$-least countably complete fine ultrafilter over an ordinal such that $j_W(\tau_{\omega_1}) > \xi$.

By the proof of Theorem 4.12, $\tau_{\omega_1} \leq \kappa \leq \lambda$. Therefore, $j_W(\kappa) > \kappa$. In particular, it cannot be that $\mathscr{K}_\lambda \sqsubset W$ and $W \sqsubset \mathscr{K}_\lambda$: otherwise $j_W(j_{\mathscr{K}_\lambda}) = j_{\mathscr{K}_\lambda} \restriction M_W$ by Theorem 3.27, and hence $j_W(\kappa) = \kappa$.

We now show that $\mathscr{K}_\lambda \sqsubset W$. Let $W_* = t_{\mathscr{K}_\lambda}(W)$. Let $k : M_W \to M_{W_*}$ be the shift embedding (Definition 2.26). Then

$$j_{W_*}(j_{\mathscr{K}_\lambda}(\tau_{\omega_1})) = k(j_W(\tau_{\omega_1})) \geq j_W(\tau_{\omega_1}) > \xi = j_{\mathscr{K}_\lambda}(\xi).$$

In $M_{\mathscr{K}_\lambda}$, $j_{\mathscr{K}_\lambda}(W)$ is the $<_\Bbbk$-least countably complete ultrafilter W' over $j_{\mathscr{K}_\lambda}(\delta)$ such that $j_{W'}(j_{\mathscr{K}_\lambda}(\tau_{\omega_1})) > j_{\mathscr{K}_\lambda}(\xi)$, so $j_{\mathscr{K}_\lambda}(W) \leq_\Bbbk W_*$. It follows that $j_{\mathscr{K}_\lambda}(W) = W_*$, and therefore $\mathscr{K}_\lambda \sqsubset W$ by Corollary 3.18, as claimed.

Since $\mathscr{K}_\lambda \sqsubset W$, we can conclude that $W \not\sqsubset \mathscr{K}_\lambda$. Therefore by Lemma 5.11, λ is not isolated.

Since there are no isolated cardinals above τ_{ω_1}, it must be that $\gamma^\sigma = \gamma^+$ for all $\gamma \geq \tau_{\omega_1}$. It follows that every successor cardinal above τ_{ω_1} is Fréchet, and hence by Theorem 2.46 or Theorem 3.24, every regular cardinal above τ_{ω_1} is Fréchet. Thus, τ_{ω_1} is the least (ω_1, ∞)-strongly compact cardinal. By Corollary 5.16, it follows that τ_{ω_1} is supercompact. □

5.4. *Inaccessible Fréchet cardinals*

The following proposition shows that Conjecture 5.4 follows from the Generalized Continuum Hypothesis.

Proposition 5.19 (UA). *Suppose λ is an isolated strong limit cardinal. Then λ is measurable.*

Proof. Let δ be the strict supremum of all Fréchet cardinals less than λ. Then $\mathscr{K}_\lambda$ is (δ, λ)-indecomposable. Since λ is a strong limit cardinal, Corollary 3.32 implies that there is an ultrafilter D on a set of size less than δ such that $j_D \restriction \lambda = j_{\mathscr{K}_\lambda} \restriction \lambda$. By Lemma 5.9, it follows that $j_{\mathscr{K}_\lambda} \restriction \lambda \in M_{\mathscr{K}_\lambda}$. Therefore by the Kunen inconsistency theorem (Corollary 2.38), $\mathscr{K}_\lambda$ is λ-complete. Therefore, $\mathscr{K}_\lambda$ witnesses that λ is measurable. □

Using Proposition 5.19, one can generalize Theorem 5.13 to inaccessible cardinals.

Theorem 5.20 (UA). *Suppose λ is strongly inaccessible and carries a countably complete uniform ultrafilter. Then some cardinal $\kappa \leq \lambda$ is λ-strongly compact.*

Proof. If λ is isolated, then Proposition 5.19 implies that λ is λ-supercompact, which implies the theorem. Otherwise, λ is not isolated, so Theorem 5.13 yields that some $\kappa < \lambda$ is λ-strongly compact. □

As a corollary of Theorem 4.8 and the proof of Theorem 5.20, one can *almost* generalize Corollary 5.14 to inaccessible cardinals.

Theorem 5.21 (UA). *Suppose λ is strongly inaccessible and carries a countably complete uniform ultrafilter. Then there is an elementary embedding $j : V \to M$ with the following properties*:

- *The critical point of j is κ and $j(\kappa) > \lambda$.*
- *For all $\delta < \lambda$, M is closed under δ-sequences.*
- *Every $A \subseteq M$ with $|A| = \lambda$ is contained in some $B \in M$ with $|B|^M = \lambda$.*

Question 5.22 (UA). Suppose λ is strongly inaccessible and carries a countably complete uniform ultrafilter. Is there a cardinal $\kappa \leq \lambda$ that is λ-supercompact?

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