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SMALL A_∞ RESULTS FOR DAHLBERG-KENIG-PIPHER OPERATORS IN SETS WITH UNIFORMLY RECTIFIABLE BOUNDARIES

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ABSTRACT. In the present paper we consider elliptic operators $L = -\operatorname{div}(A\nabla)$ in a domain bounded by a chord-arc surface Γ with small enough constant, and whose coefficients A satisfy a weak form of the Dahlberg-Kenig-Pipher condition of approximation by constant coefficient matrices, with a small enough Carleson norm, and show that the elliptic measure with pole at infinity associated to L is A_∞-absolutely continuous with respect to the surface measure on Γ, with a small A_∞ constant. In other words, we show that for relatively flat uniformly rectifiable sets and for operators with slowly oscillating coefficients the elliptic measure satisfies the A_∞ condition with a small constant and the logarithm of the Poisson kernel has small oscillations.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Recent developments have showed that for the class of Dahlberg-Kenig-Pipher operators the elliptic measure is absolutely continuous with respect to the Hausdorff measure on all uniformly rectifiable sets with some mild topological conditions (we will provide the relevant list of references below). The conditions on the operator and on the geometry are both essentially necessary and in fact, one can establish the equivalence of the aforementioned geometric features of the domain to the property of absolute continuity of harmonic measure [AHM+20]. The present paper focuses on the "small constant" version of these results, showing, roughly speaking,

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that for relatively flat domains and for slowly oscillating coefficients, the elliptic measure is A_∞ with respect to the Hausdorff measure with a small norm and that the oscillations of the logarithm of the Poisson kernel are small. This turned out to be surprisingly non-trivial primarily because the standard methods of work with Carleson measures, uniform rectifiability and related notions "bootstrap" and hence, enlarge the constants.

To be precise, in this paper we prove small A_∞ estimates for the elliptic measure, with pole at infinity, of a divergence form elliptic operators $L = -\operatorname{div}(A\nabla)$, with a matrix of coefficients A that satisfies a weak Dahlberg-Kenig-Pipher condition (see the definitions below) with a small corresponding Carleson norm. We will do this both in the upper half-space $\mathbb{R}^{d+1}_+$ and in any of the two connected components Ω bounded by a Chord-arc surface with small constant $\Gamma \subset \mathbb{R}^{d+1}$.

Concerning the special case of $\mathbb{R}^{d+1}_+$, this paper can be viewed as a continuation of a paper of Bortz, Toro, and Zhao [BTZ23], where a similar result was proved in the vanishing case. That is, they assume in addition that the Carleson measure associated to A has a vanishing trace, as in Definition 1.3, and get that the elliptic measure with pole at infinity is A_∞ with respect to the Lebesgue measure on $\mathbb{R}^d$, with a vanishing constant. We can use their result, plus a compactness argument, to say that if our result were not true on $\mathbb{R}^{d+1}_+$, we would be able to construct a counterexample to their result too. But one can also deduce this from [BTZ23] and a more recent result of Bortz, Egert and Saari [BES21]. Let us say a bit more about this. The result of [BTZ23] is based on estimates for the Green function for L (with a pole at infinity), which come from [DLM22], and, to go from the Green function to the elliptic measure, a lemma of Korey [Kor98] on weights, which itself is a vanishing constant variant of a result of Fefferman, Kenig and Pipher [FKP91]. Surprisingly, due to the nonlinearity of the problem, the small constant variant of these results on weights cannot apparently be deduced from [FKP91] and [Kor98], but it was finally proved by Bortz, Egert and Saari [BES21]. In view of this, our proof by compactness is no longer needed, but we still present it as an appendix, because we believe that it is interesting in its own right, in particular because it contains the necessary tools for taking care of the dependence on A in the weak DKP context. By contrast, [BES21] is more direct and concerns a much more general situation with weights. After an earlier version of this article was completed, Joseph Feneuil pointed out to us that the small A_∞ result for the half-space case could be also obtained by combining the argument in [Fen22] and the perturbation result of [MPT14], which would give a shorter and more direct proof of the result for operators satisfying a slightly stronger condition, namely, the DKP condition (see the remark after Definition 1.10) with sufficiently small constant.

Now let us discuss the case of Chord-arc surfaces with small constant ($CASSC$). These were introduced by Semmes in [Sem89, Sem90b, Sem90a], and they can also be seen as the correct notion of uniformly rectifiable sets with a small constant, a small generalization of Lipschitz graphs with small constant. The most convenient definition for us will be by "very big pieces of small Lipschitz graphs" (see Definition 1.13), but many other definitions are equivalent. We will say more about $CASSC$ in Section 4, and in particular we will check that if Γ is a $CASSC$, then $\mathbb{R}^{d+1} \setminus \Gamma$ has exactly two connected components, which are both NTA domains. Our main result is that if Ω is one of these components, and the elliptic operator $L = -\operatorname{div}(A\nabla)$ satisfies the weak DKP condition on Ω (see Definition 1.10), with a small enough norm, then the corresponding elliptic measure on Γ, with pole at infinity, is absolutely continuous with respect to the Hausdorff measure on Γ, with a small A_∞ density (see Definition 1.15).

The main case will be when Γ is a Lipschitz graph with small constant, and will be deduced from the case of $\mathbb{R}^{d+1}$ with a suitable change of variable from Ω to $\mathbb{R}^{d+1}_+$. Here the point is that the distortion coming from the change of variable is compatible with the weak DKP condition. After this, we will go from small Lipschitz graphs to $CASSC$ with a comparison argument, where we put small Lipschitz domains *outside of* Ω with a large boundary intersection. See Corollary 4.20 for the precise statement. Typically, one might want to approach Ω from *inside* by a Lipschitz graph (see Lemma 4.10) as the comparison of elliptic measures in two domains would be easier. However, this approach would not work in our situation as the weak DKP condition does not cooperate well when restricting to subdomains. Therefore, we approximate Ω by small Lipschitz graphs from outside and need, somewhat unconventionally in this context, to extend the coefficients to the Lipschitz domains.

Let us now give more precise definitions and state the main results. We will be working on the domain $\Omega \subset \mathbb{R}^{d+1}$, bounded by Γ, which will either be a $CASSC$ (defined below) or $\mathbb{R}^d$. Points of Γ will be denoted by lower-case letters (like x), and points of Ω by upper-case letters (like X). For $x \in \Gamma$ and $r > 0$, we shall use the Carleson boxes

$$T(x,r) = \Omega \cap B_r(x), \tag{1.1}$$

where $B_r(x) = B(x,r)$ denotes the open ball in $\mathbb{R}^{d+1}$, the surface balls $\Delta(x,r) = B(x,r) \cap \Gamma$, and the Whitney boxes

$$W_\Omega(x,r) = \big\{X \in \Omega \cap B_r(x)\,;\, \operatorname{dist}(X,\Gamma) \geq r/2\big\} \subset T(x,r). \tag{1.2}$$

We now introduce Carleson measures on $\Gamma \times (0,+\infty)$. There is a similar notion of Carleson measures on Ω, which will not be needed here, but notice that in the special case of $\mathbb{R}^{d+1}_+$, the two notions are the same.

Definition 1.3 (Carleson measures on $\Gamma \times (0,+\infty)$)**.** A Carleson measure on $\Gamma \times (0,+\infty)$ is a nonnegative Borel measure μ on $\Gamma \times (0,+\infty)$ whose Carleson norm

$$\|\mu\|_{\mathcal{C}} := \sup_{(x,r)\in\Gamma\times(0,+\infty)} r^{-d}\mu(\Delta(x,r)\times(0,r)) \tag{1.4}$$

is finite. We use $\mathcal{C}$, or $\mathcal{C}_\Omega$, or $\mathcal{C}(\Gamma\times(0,+\infty))$ to denote the set of Carleson measures on $\Gamma \times (0,+\infty)$.

We say that μ is a Carleson measure on $\Gamma\times(0,+\infty)$ with vanishing trace when μ is a Carleson measure on $\Gamma\times(0,+\infty)$ and

$$\lim_{r_0\to 0} \sup_{0<r\leq r_0} \sup_{x\in\Gamma} r^{-d}\mu(\Delta(x,r)\times(0,r)) = 0. \tag{1.5}$$

For any pair $(x_0,r_0) \in \Gamma\times(0,+\infty)$, we use $\mathcal{C}(x_0,r_0)$ or sometimes $\mathcal{C}(\Delta_0)$ when we work with $\Delta_0 = \Delta(x_0,r_0)$, to denote the set of Borel measures satisfying the Carleson condition restricted to $\Delta_0 \times (0,r_0)$, i.e., such that

$$\|\mu\|_{\mathcal{C}(x_0,r_0)} := \sup_{(x,r)\in\Gamma\times(0,r_0),\, B(x,r)\subset B(x_0,r_0)} r^{-d}\mu(\Delta(x,r)\times(0,r)) < +\infty. \tag{1.6}$$

We shall consider operators in divergence form $L = -\operatorname{div}(A\nabla)$, where $A = \big[a_{ij}(X)\big] : \Omega \to M_{d+1}(\mathbb{R})$ is a matrix-valued function, which is elliptic, as follows.

Definition 1.7 (Elliptic operators)**.** Let $\mu_0 \geq 1$ be given. We say that $L = -\operatorname{div}(A\nabla)$ (and by extension the matrix-valued function A) is $\underline{\mu_0\text{-elliptic}}$ on Ω when

$$\langle A(X)\xi, \zeta\rangle \leq \mu_0 |\xi| |\zeta| \text{ and } \langle A(X)\xi, \xi\rangle \geq \mu_0^{-1} |\xi|^2 \text{ for } X \in \Omega \text{ and } \xi, \eta \in \mathbb{R}^{d+1}. \tag{1.8}$$

We say that L (or A) is elliptic when it is μ_0-elliptic for some $\mu_0 \geq 1$. Finally we denote by $\mathfrak{A}_0(\mu_0)$ the collection of all *constant* μ_0-elliptic matrices.

We are ready to define the weak DKP condition. Let $\mu_0 \geq 1$ be given. We use the following quantity to measure the closeness of a μ_0-elliptic matrix $A = A(X)$ to constant coefficient matrices. For $x \in \Gamma$ and $r > 0$, we define

$$\alpha_A(x, r) = \inf_{A_0 \in \mathfrak{A}_0(\mu_0)} \left\{ \fint\!\!\fint_{X \in W(x,r)} |A(X) - A_0|^2 dX \right\}^{1/2}. \tag{1.9}$$

Notice that we only integrate on the Whitney box, but it turns out that since Γ is nice, a Carleson measure condition on the $\alpha_A(x, r)$ also yields a similar control on the larger numbers $\gamma_A(x, r)$ where you integrate on the full box $T(x, r)$; see Lemma A.29. Also, we required the constant matrix A_0 to be μ_0-elliptic, but if we did not, we would still get an equivalent number $\widetilde{\alpha}_A(x, r)$; use Chebyshev or see [DLM22].

Definition 1.10 (Weak DKP condition, vanishing weak DKP condition)**.** Let A be a μ_0-elliptic matrix-valued function on Ω.

- We say that A satisfies the weak DKP condition with constant $M > 0$, when the measure μ defined by $d\mu(x, r) = \alpha_A(x, r)^2 \frac{d\sigma(x)dr}{r}$ is a Carleson measure on $\Gamma \times (0, +\infty)$, with norm

$$\mathfrak{N}(A) := \left\| \alpha_A(x, r)^2 \frac{d\sigma(x)dr}{r} \right\|_{\mathcal{C}} \leq M, \tag{1.11}$$

  and where we called $\sigma = \mathcal{H}^d_{|\Gamma}$ the surface measure on Γ.
- We say that A satisfies the vanishing weak DKP condition if μ defined by $d\mu(x, r) = \alpha_A(x, r)^2 \frac{d\sigma(x)dr}{r}$ is a Carleson measure on $\Gamma \times (0, +\infty)$ with norm vanishing trace.

The weak DKP condition is a weaker version of the celebrated Carleson condition popularized by Dahlberg, Kenig, and Pipher (DKP), which instead demands that $\widetilde{\alpha}_A(x, r)^2 \frac{dxdr}{r}$ satisfies a Carleson measure estimate, where

$$\widetilde{\alpha}_A(x, r) = r \sup_{(y,s) \in W(x,r)} |\nabla A(y, s)|.$$

In 1984, Dahlberg first introduced this condition, and conjectured that such a Carleson condition guarantees the absolute continuity of the elliptic measure with respect to the Lebesgue measure in the upper half-space. In 2001, Kenig and Pipher [KP01] proved Dahlberg's conjecture. More precisely, they show that the DKP condition guarantees that the corresponding elliptic measure is A_∞ with respect to the surface measure in Lipschitz domains. Let μ and ν be two non-negative Borel measures on Γ. We say that $\mu \in A_\infty(\nu)$ if there exists $C, \eta, \theta > 0$ such that for any surface ball $\Delta \subset \Gamma$ and any $E \subset \Delta$, we have

$$\frac{\mu(E)}{\mu(\Delta)} \leq C\left(\frac{\nu(E)}{\nu(\Delta)}\right)^{\theta} \quad \text{and} \quad \frac{\nu(E)}{\nu(\Delta)} \leq C\left(\frac{\mu(E)}{\mu(\Delta)}\right)^{\eta}. \tag{1.12}$$

We refer the readers to [CF74] for equivalent definitions of A_∞.

The weak DKP condition was first introduced in [DLM22], in connection with the regularity of the Green function. In the aforementioned recent work of Bortz, Toro and Zhao [BTZ23], the authors show that the weak DKP condition is sufficient for A_∞ of the elliptic measure in the upper half-space (see Lemma A.2).

Next we recall our main condition on Γ. We use the simplest definition of $CASSC$ for our purposes, and will provide further comments later in Section 4.

Definition 1.13 (Chord-arc surfaces with small constant)**.** Let $\varepsilon \in (0, 10^{-1})$ be given. Let Γ be a closed, unbounded set in $\mathbb{R}^{d+1}$. We say that Γ is a chord-arc surface with constant ε, and we write $\Gamma \in CASSC(\varepsilon)$, when for $x \in \Gamma$ and $r > 0$, we can find an ε-Lipschitz graph $G_{x,r}$ that meets $B(x, r/2)$ and such that

$$\mathcal{H}^d(\Gamma \cap B(x,r) \setminus G_{x,r}) + \mathcal{H}^d(G_{x,r} \cap B(x,r) \setminus \Gamma) \leq \varepsilon r^d. \tag{1.14}$$

By ε-Lipschitz graph, we mean a set $G = \big\{x + A(x)\,;\, x \in P\big\}$, where $P \subset \mathbb{R}^{d+1}$ is a vector hyperplane, $A : P \to P^\perp$ is a Lipschitz function with norm at most ε, and $P^\perp$ is the vector line orthogonal to P.

In this paper, Ω will always be one of the two connected components of Γ, where either Γ is the hyperplane $\mathbb{R}^d \subset \mathbb{R}^{d+1}$, or $\Gamma \in CASSC(\varepsilon)$ for a small enough $\varepsilon > 0$. In both cases, Γ is uniformly rectifiable, Ω satisfies the (two-sided) NTA condition, and for $X \in \Omega$ there is a standard definition of the L-elliptic measure ω^X with pole at X and the Green function $G_L(X, Y)$, $Y \in \Omega$, as long as L is elliptic. As we mentioned before, since in addition A satisfies the weak DKP condition, ω^X is even absolutely continuous with respect to the surface measure $\sigma = \mathcal{H}^d_{|\Gamma}$, and the density $\frac{\partial\omega}{d\sigma}$ is an A_∞ weight when $\Gamma = \mathbb{R}^d$ by [BTZ23].[1] We will see that this is also true when $\Gamma \in CASSC(\varepsilon)$. With the usual DKP condition, this is known and is stated in [HMM+21], but factually is a straightforward combination of [KP01] with [DJ90] as the domains in question are NTA. However, the main point of the paper is not the weaker DKP condition, but the fact that the A_∞ constant is as small as we want.

Our main statement will be easier to state with the elliptic measure ω^∞ with a pole at infinity which we will define in Section 2 as a normalized limit of measures ω^X, with $|X|$ tending to infinity. One advantage, compared to the usual collection of measures $\{\omega^X\}$, $X \in \Omega$, is that A_∞ is easier to define. Appropriately modified statements are valid for the case of a finite pole as well, and will be stated below. Recall that $\Gamma \in CASSC(\varepsilon)$ for a small enough $\varepsilon > 0$, and that $\sigma = \mathcal{H}^d_{|\Gamma}$.

Definition 1.15 (A_∞ with small constant)**.** Let $\delta \in (0, 10^{-1})$ be given; we say that the positive Borel measure ω on Γ lies in $A_\infty(\sigma, \delta)$ when

$$\left|\frac{\omega(E)}{\omega(\Delta)} - \frac{\sigma(E)}{\sigma(\Delta)}\right| < \delta \tag{1.16}$$

for every surface ball $\Delta = B(x,r) \cap \Gamma$ and every Borel subset $E \subset \Delta$.

[1]In fact, they prove the result for C^1-square domains.

We will see later that this is also equivalent (modulo the precise value of δ) to saying that ω is absolutely continuous with respect to σ, and the corresponding density $k = \frac{d\omega}{d\sigma}$ is such that $\| \log(k)\|_{BMO} \leq \tau$, where τ tends to 0 with δ. Notice that none of the two conditions change when we multiply σ or ω by a constant.

The main result of the present paper can be formulated as follows.

Theorem 1.17. *For every $\delta > 0$, we can find $\varepsilon > 0$ that depends only on δ, d, and μ_0, such that if $\Gamma \in CASSC(\varepsilon)$, $L = -\operatorname{div} A\nabla$ is an elliptic operator with ellipticity μ_0, which satisfies the weak DKP condition with norm $\mathfrak{N}(A) < \varepsilon$, then the associated harmonic measure ω^∞ lies in $A_\infty(\sigma, \delta)$.*

As we shall see in Corollary 5.24, we can replace the conclusion with the essentially equivalent fact that ω is absolutely continuous with respect to $\sigma = \mathcal{H}^d_{|\Gamma}$, and its density $\frac{d\omega}{d\sigma}$ satisfies

$$\left\| \log \frac{d\omega}{d\sigma} \right\|_{BMO(\Gamma)} \leq \delta. \tag{1.18}$$

We also have analogous local results, where we consider the elliptic measure ω^X with a finite pole, but then need to restrict to small enough balls.

Theorem 1.19. *For every choice of $\delta > 0$ and $\kappa > 1$, we can find $\varepsilon = \varepsilon(\delta, d, \mu_0) > 0$ and $\tau = \tau(\delta, d, \mu_0, \kappa) \in (0, 1)$ with the following properties. Suppose $\Gamma \in CASSC(\varepsilon)$, and $L = -\operatorname{div} A\nabla$ is an elliptic operator with ellipticity μ_0 which satisfies the weak DKP condition with norm $\mathfrak{N}(A) < \varepsilon$. Let $X \in \Omega$ be given, and denote by ω^X the elliptic measure on Γ associated to L and with pole at X. Then*

$$\left| \frac{\omega^X(E)}{\omega^X(\Delta)} - \frac{\sigma(E)}{\sigma(\Delta)} \right| < \delta \tag{1.20}$$

for every surface ball $\Delta = B(x, r) \cap \Gamma$ such that $\Delta \subset B(X, \kappa \operatorname{dist}(X, \Gamma))$ and $0 < r \leq \tau \operatorname{dist}(X, \Gamma)$, and every Borel subset $E \subset \Delta$.

Theorem 1.19 will be deduced from Theorem 1.17 at the end of Section 2, and we will see in Corollary 5.24 that (1.20) can be replaced by the fact that $k = \frac{d\omega^X}{d\sigma}$ is defined on Δ, and $\| \log(k)\|_{BMO(\Delta)} \leq \delta$.

The rest of this paper is organized as follows. In Section 2, we recall definitions and some well-known properties of the elliptic measures and the Green function. We also deduce the local version (Theorem 1.19) of Theorem 1.17 using Lemma 2.17 that allows us to compare elliptic measures with different poles. In Section 3, we define a change of variables that allows us to prove Theorem 1.17 for small Lipschitz graphs. In Section 4, we treat the $CASSC$ case, using the result for small Lipschitz graphs. In Section 5, we define BMO and VMO, and show the relations between BMO (with a small semi-norm) and A_∞ (with a small constant). In Appendices A and B, we give a proof of Theorem 1.17 in the special case of the upper half-space; we assume the theorem is wrong and construct coefficients with the vanishing weak DKP condition that would make the theorem of [BTZ23] fail. We check in Appendix B the lemma that allows to compute the harmonic measure for a limit of weak DKP elliptic operators.

2. Green functions and harmonic measure with a pole at infinity

Let $\Gamma \subset \mathbb{R}^{d+1}$ be a $CASSC$, and $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator on Ω, one of the components of $\mathbb{R}^{d+1} \setminus \Gamma$. In this section, we only use the ellipticity of L, the Ahlfors regularity of Γ, and the NTA character of Ω (see Section 4), and recall the definition and basic properties of the Green function and elliptic measure with a pole at infinity.

We start with the L-elliptic measure with a pole in Ω. Recall that for $X \in \Omega$, there exists a unique Borel probability measure $\omega^X = \omega_L^X$ on $\partial\Omega$, such that for $f \in C_c^\infty(\partial\Omega)$ the function

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y)$$

is the unique weak solution to the Dirichlet problem

$$(D)_L \begin{cases} Lu = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f \end{cases}$$

satisfying $u \in C(\overline{\Omega})$ and $u(X) \to 0$ as $|X| \to \infty$ in Ω. We call ω_L^X the L-elliptic measure (or the elliptic measure corresponding to L) with pole at X.

We can also associate to L a unique Green function in Ω, $G_L(X,Y) : \Omega \times \Omega \setminus \operatorname{diag}(\Omega) \to \mathbb{R}$ with the following properties (see e.g. [HMT17]): $G_L(\cdot, Y) \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{Y\}) \cap C(\overline{\Omega} \setminus \{Y\})$, $G_L(\cdot,Y)\big|_{\partial\Omega} \equiv 0$ for any $Y \in \Omega$, and $LG_L(\cdot,Y) = \delta_Y$ in the weak sense in Ω, that is,

$$\iint_\Omega A(X)\, \nabla_X G_L(X,Y) \cdot \nabla\varphi(X)\, dX = \varphi(Y), \qquad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{2.1}$$

In particular, $G_L(\cdot,Y)$ is a weak solution to $LG_L(\cdot,Y) = 0$ in $\Omega \setminus \{Y\}$. Moreover,

$$G_L(X,Y) \le C\, |X-Y|^{1-d} \quad \text{for } X, Y \in \Omega\,, \tag{2.2}$$

$$c_\theta\, |X-Y|^{1-d} \le G_L(X,Y)\,, \quad \text{if } |X-Y| \le \theta \operatorname{dist}(X,\partial\Omega)\,,\ \theta \in (0,1)\,, \tag{2.3}$$

$$G_L(X,Y) \ge 0\,, \quad G_L(X,Y) = G_{L^\top}(Y,X), \qquad \text{for all } X, Y \in \Omega\,,\ X \neq Y. \tag{2.4}$$

Here, and in the sequel, $L^\top$ is the adjoint of L defined by $L^\top = -\operatorname{div} A^\top \nabla$, where $A^\top$ denotes the transpose matrix of A.

We have the following Caffarelli-Fabes-Mortola-Salsa estimates (cf. [CFMS81], and [JK82] for NTA domains).

Lemma 2.5. *Let Ω be an NTA domain in $\mathbb{R}^{d+1}$ and $L = -\operatorname{div} A\nabla$ be a μ_0-elliptic operator. Let $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$. Then for $X \in \Omega \setminus B(x, 2r)$ we have*

$$\frac{1}{C}\omega_L^X(\Delta(x,r)) \le r^{d-1} G_L(X, X_{x,r}) \le C\omega_L^X(\Delta(x,r)), \tag{2.6}$$

where $X_{x,r}$ is a corkscrew point for x at scale r (see Definition 4.1). The constant in (2.6) depends only on μ_0, dimension and on the constants in the NTA character.

Lemma 2.7 (Bourgain's estimate [Bou87, Ken94])**.** *Let Ω and L be as in Lemma 2.5. Let $x \in \partial\Omega$ and $0 < 2r < \operatorname{diam}(\partial\Omega)$. Then*

$$\omega_L^{X_{x,r}}(\Delta(x,r)) \ge \frac{1}{C},$$

where $C > 1$ depends on d, μ_0 and the NTA constant of Ω.

Lemma 2.8. *Let Ω be an NTA domain. There is a constant $M > 1$ that depends on the dimension and the NTA constants for Ω, such that when $0 < Mr < \operatorname{diam}(\partial\Omega)$, and u, v are non-trivial functions which vanish continuously on $\Delta(x, Mr)$ for some $x \in \partial\Omega$, $u, v \geq 0$ and $Lu = Lv = 0$ in $B(x, Mr) \cap \Omega$, then*

$$\frac{u(X)}{v(X)} \approx \frac{u(X_{x,r})}{v(X_{x,r})}, \quad \textit{for all } X \in B(x,r) \cap \Omega,$$

where the implicit constants depend on n, μ_0 and the NTA constants for Ω.

We now consider poles at infinity. With minor modifications only, one can prove Lemma 2.9 as in [KT99, Lemma 3.7 and Corollary 3.2].

Lemma 2.9. *Let $L = -\operatorname{div} A\nabla$ be an elliptic operator on Ω with Green function $G_L(X,Y)$. Then for any fixed $Z_0 \in \Omega$, there exists a unique function $U \in C(\overline{\Omega})$ such that*

$$\begin{cases} L^\top U = 0 & \textit{in } \Omega, \\ U > 0 & \textit{in } \Omega, \\ U = 0 & \textit{on } \partial\Omega, \end{cases}$$

and $U(Z_0) = 1$. In addition, for any sequence $\{X_k\}_k$ of points in Ω such that $|X_k| \to \infty$, there exists a subsequence (which we still denote by $\{X_k\}_k$) such that if we set

$$u_k(Y) := \frac{G_L(X_k, Y)}{G_L(X_k, Z_0)},$$

then

$$\lim_{k\to\infty} u_k(Y) = U(Y) \qquad \textit{for any } Y \in \Omega. \tag{2.10}$$

Moreover, there exists a unique locally finite positive Borel measure ω_L^∞ on $\partial\Omega$ such that the Riesz formula

$$\int_{\partial\Omega} f(y)\, d\omega_L^\infty(y) = -\iint A^\top(Y)\nabla_Y U(Y) \cdot \nabla_Y F(Y)\, dY \tag{2.11}$$

holds whenever $f \in C_c^\infty(\partial\Omega)$ and $F \in C_c^\infty(\Omega)$ are such that $F|_{\partial\Omega} = f$.

Definition 2.12 (Elliptic measure and Green function with pole at infinity)**.** Let L, Z_0, U and ω_L^∞ be as in Lemma 2.9. We call ω_L^∞ the elliptic measure with pole at infinity (normalized at Z_0), and U the Green function with pole at infinity (normalized at Z_0).

We will systematically fix Z_0 at the beginning (and take $Z_0 = (0,1)$ when $\Omega = \mathbb{R}^{d+1}_+$), so we drop the term "normalized at Z_0" for the elliptic measure and the Green function with pole at infinity.

Now we state estimates that allow to go from poles $X \in \Omega$ to poles at infinity. Here there is a single operator L, so we drop L from the notation, and for instance write ω^X for ω_L^X. Recall that the different measures ω^X are absolutely continuous with respect to each other, as a consequence of Harnack's inequality. Similarly, ω^X and ω^∞ are mutually absolutely continuous too, as a result of the local Hôlder continuity of solutions near Γ, and we even have the estimates stated below on the density, which we take from [BTZ23]. For $X \in \Omega$, call

$$H_\infty(X, z) := \frac{d\omega^X}{d\omega^\infty}(z) \tag{2.13}$$

the Radon-Nykodym density of ω^X with respect to ω^∞, evaluated at $z \in \Gamma$; the existence comes with Lemma 2.14.

Lemma 2.14 ([BTZ23, Lemma 5.8]). *Let Γ, Ω, and L be as above; in particular $\Gamma \in CASSC(\varepsilon)$ for a small enough $\varepsilon > 0$, and $L - \operatorname{div} A\nabla$ is an elliptic operator with ellipticity μ_0. Given $X \in \Omega$, set $\delta(X) = \operatorname{dist}(X, \Gamma)$ and pick a point $X_1 \in \Omega \cap B(X, \operatorname{dist}(X, \Gamma))$ such that $\operatorname{dist}(X_1, \Gamma) = \delta(X)/4$. The density $H_\infty(X, z)$ of (2.13) exists and is locally Hölder continuous of order $\gamma = \gamma(d, \mu_0)$. Moreover, for every $\kappa > 1$ there exists $C_\kappa = C_\kappa(\kappa, d, \mu_0)$ such that*

$$|H_\infty(X,z) - H_\infty(X,z')| \leq C_\kappa \frac{G_L(X,X_1)}{U(X)} \left(\frac{|z-z'|}{\delta(X)}\right)^\gamma \tag{2.15}$$

for all $z, z' \in \Gamma \cap B(X, 6\kappa\delta(X))$, $|z - z'| < \delta(X)/4$ and

$$(C_\kappa)^{-1} \frac{G_L(X,X_1)}{U(X)} \leq H_\infty(X,z) \leq C_\kappa \frac{G_L(X,X_1)}{U(X)} \tag{2.16}$$

for all $z \in \Gamma \cap B(X, 6\kappa\delta(X))$. Here U is the Green function for L with pole at infinity.

In fact Lemma 2.14 is proved in [BTZ23] in the special case when $\Gamma = \mathbb{R}^d$, but the proof goes through. The existence of $X_1 \in \Omega \cap B(X, \operatorname{dist}(X, \Gamma))$ such that $\operatorname{dist}(X_1, \Gamma) = \delta(X)/4$ is easy because Γ is a $CASSC$, but a nearby corkscrew point (not so close to X) would do the job in a more general setting. Notice also that we will only use the case when $\Gamma \neq \mathbb{R}^d$ to deduce Theorem 1.19 from Theorem 1.17 at the end of the section.

We now use the estimates (2.15) and (2.16) to compare $\omega^{X_0}(E)$ and $\omega^\infty(E)$.

Lemma 2.17. *Let Γ, Ω, L, and $X \in \Omega$ be as in Lemma 2.14. Let $L = -\operatorname{div} A\nabla$ be an elliptic operator with ellipticity μ_0. Then for any $\kappa > 1$, for $x \in \Gamma \cap B(X, 5\kappa\delta(X))$ and $0 < r \leq \delta(X)/4$, and any Borel set $E \subset \Delta = \Gamma \cap B(x, r)$, there holds*

$$\left|\frac{\omega^X(E)}{\omega^X(\Delta)} - \frac{\omega^\infty(E)}{\omega^\infty(\Delta)}\right| \leq C_\kappa \left(\frac{r}{\delta(X)}\right)^\gamma, \tag{2.18}$$

where $C_\kappa = C_\kappa(\kappa, d, \mu_0)$ and $\gamma = \gamma(d, \mu_0)$.

Proof. By the definition (2.13) of H_∞,

$$\frac{\omega^X(E)}{\omega^{X_0}(\Delta)} = \frac{\int_E H_\infty(X,z)d\omega^\infty(z)}{\int_\Delta H_\infty(X,z)d\omega^\infty(z)}.$$

Next

$$\frac{\inf_{z\in E} H_\infty(X,z)}{\sup_{z\in\Delta} H_\infty(X,z)} \frac{\omega^\infty(E)}{\omega^\infty(\Delta)} \leq \frac{\omega^X(E)}{\omega^X(\Delta)} \leq \frac{\sup_{z\in E} H_\infty(X,z)}{\inf_{z\in\Delta} H_\infty(X,z)} \frac{\omega^\infty(E)}{\omega^\infty(\Delta)},$$

which implies that

$$\begin{aligned}\left(\frac{\inf_{z\in E} H_\infty(X,z)}{\sup_{z\in\Delta} H_\infty(X,z)} - 1\right) \frac{\omega^\infty(E)}{\omega^\infty(\Delta)} &\leq \frac{\omega^X(E)}{\omega^X(\Delta)} - \frac{\omega^\infty(E)}{\omega^\infty(\Delta)} \\ &\leq \left(\frac{\sup_{z\in E} H_\infty(X,z)}{\inf_{z\in\Delta} H_\infty(X,z)} - 1\right) \frac{\omega^\infty(E)}{\omega^\infty(\Delta)}.\end{aligned} \tag{2.19}$$

Notice that

$$\left(\frac{\sup_{z\in E} H_\infty(X,z)}{\inf_{z\in\Delta} H_\infty(X,z)}-1\right)\frac{\omega_L^\infty(E)}{\omega_L^\infty(\Delta)}\le\frac{\sup_{z\in\Delta} H_\infty(X,z)}{\inf_{z\in\Delta} H_\infty(X,z)}-1,$$

and that

$$\frac{\inf_{z\in E} H_\infty(X,z)}{\sup_{z\in\Delta} H_\infty(X,z)}-1\le\left(\frac{\inf_{z\in E} H_\infty(X,z)}{\sup_{z\in\Delta} H_\infty(X,z)}-1\right)\frac{\omega_L^\infty(E)}{\omega_L^\infty(\Delta)}\le 0,$$

so (2.19) gives that

$$\left|\frac{\omega_L^X(E)}{\omega^X(\Delta)}-\frac{\omega^\infty(E)}{\omega^\infty(\Delta)}\right|\le\sup_{z,z'\in\Delta}\left|\frac{H_\infty(X,z)}{H_\infty(X,z')}-1\right|. \tag{2.20}$$

But Lemma 2.14 says that for all $z,z'\in\Gamma\cap B(X,6\kappa\delta(X))$ such that $|z-z'|<\frac{\delta(X)}{4}$,

$$\left|\frac{H_\infty(X,z)}{H_\infty(X,z')}-1\right|=\frac{|H_\infty(X,z)-H_\infty(X,z')|}{H_\infty(X,z')}\le C_\kappa\left(\frac{|z-z'|}{\delta(X)}\right)^\gamma.$$

Then the desired estimate (2.18) follows from (2.20) and our assumptions on Δ. $\square$

We now use Lemma 2.17 to deduce Theorem 1.19 from Theorem 1.17.

Proof of Theorem 1.19 given Theorem 1.17. Let Γ, Ω, and L be as in both theorems. By Theorem 1.17 we can find $\varepsilon>0$, depending on δ, d, and μ_0, such that if $\Gamma\in CASSC(\varepsilon)$ and $\mathfrak{N}(A)<\varepsilon$, we have

$$\left|\frac{\omega^\infty(E)}{\omega^\infty(\Delta)}-\frac{\sigma(E)}{\sigma(\Delta)}\right|<\frac{\delta}{2}$$

for any surface ball $\Delta\subset\Gamma$ and any Borel set $E\subset\Delta$. We now have to replace ω^∞ with ω^X. Let $\Delta=\Gamma\cap B(x,r)$ be such that $\Delta\le B(X,\kappa\delta(X))$ and $0<r<\tau\delta(X)$ be as in the statement. Then we can apply Lemma 2.17, and we get (2.18), i.e.,

$$\left|\frac{\omega^X(E)}{\omega^X(\Delta)}-\frac{\omega^\infty(E)}{\omega^\infty(\Delta)}\right|\le C_\kappa\left(\frac{r}{\delta(X)}\right)^\gamma\le C_\kappa\tau^\gamma.$$

We now choose τ so small, depending on δ, d, μ_0, and κ, that $C_\kappa\tau^\gamma\le\delta/2$, and (1.20) follows from triangle inequality. $\square$

3. Small Lipschitz graphs

In this section we use a change of variable to prove Theorem 1.17 in when Γ is the graph of a Lipschitz function $\varphi:\mathbb{R}^d\to\mathbb{R}$, with small Lipschitz constant. More precisely, we assume that

$$\|\nabla\varphi\|_\infty\le\varepsilon, \tag{3.1}$$

set

$$\Gamma=\left\{(x,\varphi(x))\in\mathbb{R}^{d+1};\,x\in\mathbb{R}^d\right\}, \tag{3.2}$$

and denote by Ω the connected component of $\mathbb{R}^{d+1}\setminus\Gamma$ that lies above Γ. We then let A and L be as in Theorem 1.17, and we need to show that if ε is small enough, depending on δ, d, and the ellipticity constant μ_0, the associated elliptic measure ω_L^∞ lies in $A_\infty(\sigma,\delta)$.

We now define a mapping from $\mathbb{R}^{d+1}_+$ to Ω which is the same as the one that is used in [KP01], originally due to Dahlberg, Kenig, and Stein. For $(x,t)\in\mathbb{R}^{d+1}_+$, let $\eta_t(x) = t^{-d}\eta(x/t)$, where η is a nonnegative radial C^∞ function supported in $\{|x| \le 1/2\}$ and such that $\int_{\mathbb{R}^d} \eta = 1$. Define a mapping

$$\rho : \mathbb{R}^{d+1}_+ \to \Omega, \quad \rho(x,t) = (x, c_0 t + F(x,t)) \tag{3.3}$$

with $F(x,t) = \eta_t * \varphi(x) = \int \eta(z)\varphi(x - tz)dz$, and $c_0 = 1 + c_d \|\nabla\varphi\|_\infty$, where

$$c_d = \int_{\mathbb{R}^d} |\eta(y)|\,|y|\,dy. \tag{3.4}$$

This choice of c_0 guarantees that ρ is a one-to-one bi-Lipschitz mapping of $\mathbb{R}^{d+1}_+$ onto Ω. In fact, we have

$$\nabla\rho(x,t) = \begin{pmatrix} I & \nabla_x F(x,t) \\ 0 & c_0 + \partial_t F(x,t) \end{pmatrix}, \tag{3.5}$$

and $\det \nabla\rho(x,t) = c_0 + \partial_t F(x,t)$. Since

$$|\partial_t F(x,t)| = \left| - \int_{\mathbb{R}^d} \eta(z) z \cdot (\nabla\varphi)(x - tz)dz \right| \le c_d \|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}, \tag{3.6}$$

we get that

$$1 \le \det \nabla\rho(x,t) \le 1 + 2c_d \|\nabla\varphi\|_\infty. \tag{3.7}$$

Incidentally, we have that

$$|\nabla_x F(x,t)| = \left| \int_{\mathbb{R}^d} \eta(z)(\nabla\varphi)(x - zt)dz \right| \le c'_d \|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}, \tag{3.8}$$

where $c'_d = \int |\eta(y)|\,dy$.

Another important property of the mapping ρ is that $\left|\nabla^2\rho(x,t)\right|^2 t\,dxdt$ is a Carleson measure on $\mathbb{R}^{d+1}_+$. This property has been mentioned in [KP01], but no proof nor an estimate for the Carleson norm has been given. We give the control of the Carleson norm and sketch the easy proof.

Lemma 3.9. *For $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, the measure μ defined by the density*

$$d\mu(x,t) = \left|\nabla^2_{x,t} F(x,t)\right|^2 t\,dxdt$$

is a Carleson measure on $\mathbb{R}^{d+1}_+$, and

$$\|\mu\|_{\mathcal{C}} \le C \|\nabla\varphi\|^2_{BMO(\mathbb{R}^d)} \le C \|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^d)},$$

where the constant C depends only η and d.

Sketch of proof. The lemma follows from a Carleson measure characterization of BMO functions (see e.g. [Gra14, Theorem 3.3.8]). In fact, all the second derivatives of $F(x,t)$ can be expressed in the form of $t^{-1}\theta_t * \nabla\varphi(x)$, where $\theta_t(x) = t^{-d}\theta(x/t)$ satisfies $\int_{\mathbb{R}}^d \theta(x)dx = 0$, $|\theta(x)| \le C(1 + |x|)^{-n-\delta}$ for some C, $\delta \in (0,\infty)$, and

$$\sup_{\xi\in\mathbb{R}^d} \int_0^\infty \left| \hat{\theta}(s\xi)\frac{ds}{s} \right| < \infty.$$

Therefore,

$$\left\| \left|\theta_t * \nabla\varphi(x)\right|^2 \frac{dxdt}{t} \right\|_{\mathcal{C}} \le C \|\nabla\varphi\|^2_{BMO(\mathbb{R}^d)},$$

which gives the desired estimate. □

One can check (see for instance, in a slightly different context, (1.42) and its proof in Lemma 6.17 in [DFM19]) that if u is a solution of $Lu = -\operatorname{div} A\nabla u = 0$ in Ω, then $v = u \circ \rho$ is a solution of $\widetilde{L}v = -\operatorname{div} \widetilde{A}\nabla v = 0$ in $\mathbb{R}^{d+1}_+$, where

$$\widetilde{A}(x,t) = \det(\nabla\rho(x,t))(\nabla\rho(x,t)^{-1})^T A(\rho(x,t))\nabla\rho(x,t)^{-1}. \tag{3.10}$$

It will be useful to write out the matrix $\widetilde{A}(x,t)$. A simple computation shows that

$$\begin{aligned}\widetilde{A}(x,t) = \begin{pmatrix} I & \mathbf{0} \\ -\nabla_x F(x,t) & 1 \end{pmatrix} A(\rho(x,t)) \begin{pmatrix} I & -\frac{\nabla_x F(x,t)}{c_0+\partial_t F(x,t)} \\ 0 & (c_0+\partial_t F(x,t))^{-1} \end{pmatrix} \\ =: P(x,t)A(\rho(x,t))Q(x,t).\end{aligned} \tag{3.11}$$

Lemma 3.12. *If $\|\nabla\varphi\|_\infty < \varepsilon < \frac{1}{100(1+c_d)}$, where c_d is as in (3.4), then for $\widetilde{A}$ defined in (3.10),*

$$\mathfrak{N}(\widetilde{A}) \le C\,\|\nabla\varphi\|_\infty^2 + C\mathfrak{N}(A),$$

where C depends on d and the ellipticity constant of A.

Proof. For $x \in \mathbb{R}^d$ and $r > 0$, let us use the notation $W(x,r) = \Delta_{\mathbb{R}^d}(x,r) \times (5r/6, r)$ for Whitney regions in $\mathbb{R}^{d+1}_+$, where $\Delta_{\mathbb{R}^d}(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$. Notice that they are slightly smaller than the Whitney regions defined in (A.3), but as we said earlier, this change is insignificant in terms of Theorems 1.17 and 1.19 in $\mathbb{R}^{d+1}_+$. We denote by $W_\Omega(x_\rho, r)$ the Whitney region $\{Y \in \Omega \cap B(x_\rho, r) : \operatorname{dist}(Y,\Gamma) > 3r/10\}$ in Ω, where $x_\rho = (x, \varphi(x)) \in \Gamma$ for $x \in \mathbb{R}^d$. Notice that we replace the constant $1/2$ in (1.2) with $3/10$, but we know that this does not matter. It will be convenient to work with the following different Whitney regions in Ω: for $x \in \mathbb{R}^d$ and $r > 0$, set

$$W_\rho(x,r) := \rho(W(x,r)).$$

Since $1 \le \partial_t(c_0 t + F(y,t)) \le 1 + 2c_d\,\|\nabla\varphi\|_\infty$ for all $(y,t) \in \mathbb{R}^{d+1}_+$,

$$t \le c_0 t + F(y,t) - \varphi(x) \le (1 + 2c_d\varepsilon)t \quad \text{for } (y,t) \in \mathbb{R}^{d+1}_+. \tag{3.13}$$

Let $(y,s) = W_\rho(x,r)$ be given, and write $(y,s) = \rho(y,t) = (y, c_0 t + F(y,t))$ for some $(y,t) \in W(x,r)$. Thus $|y-x| < r$ and $5r/6 < t < r$. By (3.13), $5r/6 < s - \varphi(y) \le (1+2c_d\varepsilon)r$. This means that $|(y,s) - y_\rho| > 5r/6$, and hence $\operatorname{dist}((y,s),\Gamma) \ge 3r/4$ if we choose ε sufficiently small. In addition,

$$|(y,s) - x_\rho| \le |y-x| + |s - \varphi(y)| + |\varphi(y) - \varphi(x)| \le (2 + (2c_d+1)\varepsilon)\,r \le 5r/2,$$

if we choose $(2c_d+1)\varepsilon < 1/2$. So

$$W_\rho(x,r) \subset \{Y \in \Omega \cap B(x_\rho, 5r/2) : \operatorname{dist}(Y,\Gamma) > 3r/4)\} = W_\Omega(x_\rho, 5r/2). \tag{3.14}$$

Now we fix $x \in \mathbb{R}^d$ and $r > 0$, and estimate $\alpha_{\widetilde{A}}(x,r)$. Let A_0 be a constant coefficient matrix which achieves the infimum for $\alpha_A(x_\rho, \frac{3r}{2})$, set

$$a = a(x,r) = \fint\!\!\fint_{W(x,r)} \nabla_y F(y,s)dyds$$

and

$$b = b(x,r) = \fint\!\!\fint_{W(x,r)} \partial_s F(y,s)dyds,$$

and define $\widetilde{A}_0$ to be the constant coefficient matrix

$$\widetilde{A}_0 := \begin{pmatrix} I & \mathbf{0} \\ -a & 1 \end{pmatrix} A_0 \begin{pmatrix} I & -a\,(c_0+b)^{-1} \\ 0 & (c_0+b)^{-1} \end{pmatrix} =: P_0 A_0 Q_0,$$

which should be compared to the expression of $\widetilde{A}(x,t)$ in (3.11). Then

$$\alpha_{\widetilde{A}}(x,r)^2 \leq \fint\!\!\fint_{W(x,r)} \left|\widetilde{A}(y,s) - \widetilde{A}_0\right|^2 dyds$$
$$= \fint\!\!\fint_{W(x,r)} \left|P(y,s)A(\rho(y,s))Q(y,s) - P_0A_0Q_0\right|^2 dyds.$$

By the triangle inequality,

$$\begin{aligned}|P(y,s)A(\rho(y,s))Q(y,s) - P_0A_0Q_0| &\leq |(P(y,s)-P_0)A(\rho(y,s))Q(y,s)| \\ &\quad + |P_0A(\rho(y,s))(Q(y,s)-Q_0)| + |P_0\,(A(\rho(y,s))-A_0)\,Q_0|\,.\end{aligned}$$

Using the definitions of P, Q, P_0, Q_0, the ellipticity of A and A_0, and (3.7), (3.8), we obtain that

$$\begin{aligned}|P(y,s)A(\rho(y,s))Q(y,s) - P_0A_0Q_0| &\lesssim \left|\nabla_y F(y,s) - a\right| \\ &\quad + \left|\partial_s F(y,s) - b\right| + |A(\rho(y,s)) - A_0|\,.\end{aligned}$$

Then

$$\begin{aligned}(3.15)\quad & \alpha_{\widetilde{A}}(x,r)^2 \\ &\lesssim \fint\!\!\fint_{W(x,r)} \left|\nabla F(y,s) - \fint\!\!\fint_{W(x,r)} \nabla F(z,\tau)dzd\tau\right|^2 dyds + \fint\!\!\fint_{W(x,r)} |A(\rho(y,s)) - A_0|^2\, dyds \\ &\lesssim r^2 \fint\!\!\fint_{W(x,r)} \left|\nabla^2 F(y,s)\right|^2 dyds + \fint\!\!\fint_{\rho(W(x,r))} |A(Y) - A_0|^2\, dY.\end{aligned}$$

We fix any $x_0 \in \mathbb{R}^d$ and $r_0 > 0$, and estimate $\iint_{T(x_0,r_0)} \alpha_{\widetilde{A}}(x,r)^2 dxdr/r$. By Fubini's theorem (recall also that $r \sim s$ when $(y,s) \in W(x,r)$),

$$\iint_{T(x_0,r_0)} r^2 \fint_{W(x,r)} \left|\nabla^2 F(y,s)\right|^2 dyds \frac{dxdr}{r} \leq C \int_{\Delta_{\mathbb{R}^d}(x_0,2r_0)} \int_0^{r_0} \left|\nabla^2 F(y,s)\right|^2 s\, dyds,$$

which is bounded by $C\,\|\nabla\varphi\|_\infty^2\, r_0^d$ thanks to Lemma 3.9. By (3.14) and our choice of A_0,

$$\begin{aligned}&\iint_{(x,r)\in T(x_0,r_0)} \fint\!\!\fint_{Y\in\rho(W(x,r))} |A(Y) - A_0|^2\, dY \frac{dxdr}{r} \\ &\qquad\leq C \int_{x\in\Delta_{\mathbb{R}^d}(x_0,r_0)} \int_{r=0}^{r_0} \fint_{Y\in W_\Omega(x_\rho,5r/2)} |A(Y) - A_0|^2\, dY \frac{dxdr}{r} \\ &\qquad\leq C \int_{x\in\Delta_{\mathbb{R}^d}(x_0,r_0)} \int_{r=0}^{r_0} \alpha_A(x_\rho, 5r/2)^2 \frac{dxdr}{r},\end{aligned}$$

where in fact $\alpha_A(x,r)$ is defined as in (1.9), but in terms of the larger Whitney boxes $W_\Omega(x,r)$; it is easy to see this does not alter the weak DKP condition (1.11). Now observe that if $x \in \Delta_{\mathbb{R}^d}(x_0,r_0)$, then

$$x_\rho \in \Delta((x_0)_\rho,(1+2\varepsilon)r_0) \subset \Delta((x_0)_\rho, 3r_0/2)$$

if ε is small enough, and now, setting $\xi = x_\rho$,

$$\int_{x\in\Delta_{\mathbb{R}^d}(x_0,r_0)} \int_{r=0}^{r_0} \alpha_A(x_\rho,5r/2)^2 \frac{dxdr}{r} \\ \leq \int_{\xi\in\Delta((x_0)_\rho,3r_0/2)} \int_{r=0}^{r_0} \alpha_A(\xi,5r/2)^2 \frac{d\sigma(\xi)dr}{r} \leq C\mathfrak{N}(A)r_0^d$$

by the weak DKP condition on the $\alpha_A(\xi,r)$. Recalling (3.15), we have proved that for any $x_0 \in \mathbb{R}^d$ and $r_0 > 0$,

$$\iint_{T(x_0,r_0)} \alpha_{\widetilde{A}}(x,r)^2 \frac{dxdr}{r} \leq C\left(\|\nabla\varphi\|_\infty^2 + \mathfrak{N}(A)\right) r_0^d,$$

which gives the desired the estimate for $\mathfrak{N}(\widetilde{A})$. □

Let ω^{X_0} (or ω^∞) denote the elliptic measure with pole at $X_0 \in \Omega$ (or at infinity) corresponding to the operator $L = -\operatorname{div} A\nabla$ in Ω. Let $\widetilde{\omega}^{X_0}$ (or $\widetilde{\omega}^\infty$) denote the elliptic measure with pole at $X_0 \in \mathbb{R}^{d+1}_+$ (or at infinity) corresponding to the operator $\widetilde{L} = -\operatorname{div}\widetilde{A}\nabla$ in $\mathbb{R}^{d+1}_+$. A change of variables shows that for any set $E \subset \Gamma$, and any $X_0 \in \Omega$,

$$\omega^{X_0}(E) = \widetilde{\omega}^{\rho^{-1}(X_0)}(\rho^{-1}(E)). \tag{3.16}$$

Similarly, one can show that for any X, $Y \in \Omega$, $\widetilde{G}(\rho^{-1}(X),\rho^{-1}(Y)) = G(X,Y)$, where $\widetilde{G}$ is the Green function for $\widetilde{L}$ in $\mathbb{R}^{d+1}_+$, and G is the Green function for L in Ω. This implies, by the definition of the elliptic measure with a pole at infinity in Lemma 2.9, that for any $E \subset \Gamma$,

$$\omega^\infty(E) = \widetilde{\omega}^\infty(\rho^{-1}(E)). \tag{3.17}$$

Now we are ready to prove Theorems 1.17 and 1.19 for Γ defined in (3.2) with small Lipschitz constant.

Proof of Theorem 1.17 for small Lipschitz graphs. Let $\Delta = B\cap\Gamma$ be any surface ball on $\Gamma = \{(x,\varphi(x)) : x\in\mathbb{R}^d\}$ and let $E \subset \Delta$ be any Borel set. We write $B = B(x_\rho, r)$, with $x \in \mathbb{R}^d$ and $x_\rho = (x,\varphi(x))$, and set $\widetilde{\Delta} = B(x,r)\cap\mathbb{R}^d$. Also call $\pi : \mathbb{R}^{d+1} \to \partial\mathbb{R}^{d+1}_+ \cong \mathbb{R}^d$ the projection onto $\mathbb{R}^d$. Observe also that $\pi(\Delta) \subset \widetilde{\Delta}$ and, due to the definition of ρ and the fact that Γ is a graph,

$$\rho^{-1}(E) = \pi(E) \quad\text{and}\quad \rho^{-1}(\Delta) = \pi(\Delta). \tag{3.18}$$

Next we compare the Hausdorff measures. Since Γ is an ε-Lipschitz graph,

$$\sigma(\pi(E)) \leq \sigma(E) \leq (1+C\varepsilon)\sigma(\pi(E)), \tag{3.19}$$

and similarly

$$\sigma(\pi(\Delta)) \leq \sigma(\Delta) \leq (1+C\varepsilon)\sigma(\pi(\Delta)). \tag{3.20}$$

In addition, $\pi(\Delta)$ contains $\mathbb{R}^d \cap B(x,(1-\varepsilon)r)$, so

$$\sigma(\pi(\Delta)) \geq (1-C\varepsilon)\sigma(\widetilde{\Delta}), \tag{3.21}$$

and hence

$$1 - C\varepsilon \leq \frac{\sigma(\pi(\Delta))}{\sigma(\widetilde{\Delta})} \leq 1 \tag{3.22}$$

because $\pi(\Delta) \subset \widetilde{\Delta}$.

Lemma 3.12 and Theorem 1.17 for $\mathbb{R}^{d+1}_+$ say that if ε and $\mathfrak{N}(A)$ are small enough,

$$\left|\frac{\widetilde{\omega}^\infty(\pi(E))}{\widetilde{\omega}^\infty(\widetilde{\Delta})} - \frac{\sigma(\pi(E))}{\sigma(\widetilde{\Delta})}\right| < \frac{\delta}{10} \quad \text{and} \quad \left|\frac{\widetilde{\omega}^\infty(\pi(\Delta))}{\widetilde{\omega}^\infty(\widetilde{\Delta})} - \frac{\sigma(\pi(\Delta))}{\sigma(\widetilde{\Delta})}\right| < \frac{\delta}{10}. \tag{3.23}$$

Because of (3.17) and (3.18), this becomes

$$\left|\frac{\omega^\infty(E)}{\widetilde{\omega}^\infty(\widetilde{\Delta})} - \frac{\sigma(\pi(E))}{\sigma(\widetilde{\Delta})}\right| < \frac{\delta}{10} \quad \text{and} \quad \left|\frac{\omega^\infty(\Delta)}{\widetilde{\omega}^\infty(\widetilde{\Delta})} - \frac{\sigma(\pi(\Delta))}{\sigma(\widetilde{\Delta})}\right| < \frac{\delta}{10}, \tag{3.24}$$

and by (3.22) the second part yields

$$1 - C\varepsilon - \frac{\delta}{10} \leq \frac{\omega^\infty(\Delta)}{\widetilde{\omega}^\infty(\widetilde{\Delta})} \leq 1 + \frac{\delta}{10}. \tag{3.25}$$

Since by (3.19) $|\sigma(\pi(E)) - \sigma(E)| \leq C\varepsilon\sigma(\pi(E)) \leq C\varepsilon\sigma(\widetilde{\Delta})$, we deduce from the first part of (3.24) that

$$\left|\frac{\omega^\infty(E)}{\widetilde{\omega}^\infty(\widetilde{\Delta})} - \frac{\sigma(E))}{\sigma(\widetilde{\Delta})}\right| < \frac{\delta}{10} + C\varepsilon. \tag{3.26}$$

Finally, we may as well assume that we took $\delta < 10^{-1}$, and then we deduce from (3.25) and (3.26) that

$$\left|\frac{\omega^\infty(E)}{\omega^\infty(\Delta)} - \frac{\sigma(E))}{\sigma(\widetilde{\Delta})}\right| < \frac{\delta}{2}, \tag{3.27}$$

and since

$$\left|\frac{\sigma(E))}{\sigma(\widetilde{\Delta})} - \frac{\sigma(E))}{\sigma(\Delta)}\right| = \frac{\sigma(E))}{\sigma(\Delta)}\,\frac{\sigma(\widetilde{\Delta}) - \sigma(\Delta)}{\sigma(\widetilde{\Delta})} \leq C\varepsilon \tag{3.28}$$

by (3.22) and (3.20), we get that

$$\left|\frac{\omega^\infty(E)}{\omega^\infty(\Delta)} - \frac{\sigma(E))}{\sigma(\widetilde{\Delta})}\right| < \delta, \tag{3.29}$$

as needed. □

Proof of Theorem 1.19 for small Lipschitz graphs. It follows from Theorem 1.17 for small Lipschitz graph and Lemma 2.17. Or we can work directly and apply (3.16) and Theorem 1.19 for $\mathbb{R}^{d+1}_+$. □

4. Chord-arc surfaces with small constants

4.1. **Geometric properties of the *CASSC*.** In Definition 1.13 we defined the *CASSC* by the fact that they have very big pieces of ε-Lipschitz graphs. This is convenient for us (and also extends well to higher co-dimensions), but we could have used different definitions, a popular one being that the unit normal to Γ (say, with values in $\mathbb{S}/\{\pm 1\}$) has a small BMO norm. We refer to the early papers of S. Semmes [Sem89, Sem90b, Sem90a] for details.

Let us first give some geometric definitions.

Definition 4.1 (Two-sided Corkscrew condition [JK82])**.** We say a domain $\Omega \subset \mathbb{R}^{d+1}$ satisfies the two-sided corkscrew condition if there exists a uniform constant $M \geq 2$ such that for all $x \in \partial\Omega$ and $r \in (0, \operatorname{diam} \partial\Omega)$ there exists $X_1, X_2 \in \mathbb{R}^{d+1}$ such that

$$B(X_1, r/M) \subset B(x,r) \cap \Omega, \quad B(X_2, r/M) \subset B(x,r) \setminus \overline{\Omega}.$$

We write $X_{x,r} := X_1$, for the interior corkscrew point for x at scale r. We also use the notation $X_\Delta := X_{x,r}$ when $\Delta = \Delta(x,r) = B(x,r) \cap \partial\Omega$.

Definition 4.2 (Harnack chain condition [JK82])**.** We say a domain $\Omega \subset \mathbb{R}^{d+1}$ satisfies the Harnack chain condition if there exists a uniform constant $M \geq 2$ such that if $X_1, X_2 \in \Omega$ with $\operatorname{dist}(X_i, \partial\Omega) > \varepsilon > 0$ and $|X_1 - X_2| < 2^k \varepsilon$ then there exists a 'chain' of open balls $B_1, \ldots, B_N$ with $N < Mk$ such that $X_1 \in B_1$, $X_2 \in B_N$, $B_j \cap B_{j+1} \neq \varnothing$ for $j = 1, \ldots, N-1$ and $M^{-1} \operatorname{diam} B_j \leq \operatorname{dist}(B_j, \partial\Omega) \leq M \operatorname{diam} B_j$ for $j = 1, \ldots, N$.

Definition 4.3 (NTA domains [JK82])**.** We say a domain $\Omega \subset \mathbb{R}^{d+1}$ is an NTA domain if it satisfies the two-sided corkscrew condition and the Harnack chain condition.

Let $\Gamma \in CASSC(\varepsilon)$ be given. We systematically assume that ε is small enough, depending on d when needed. We start with simple geometric consequences of the definition of the *CASSC*.

Lemma 4.4. *The set Γ is Ahlfors regular and Reifenberg flat. More precisely, there is a constant $C \geq 1$ that only depends on d, such that for $x \in \Gamma$ and $r > 0$,*

$$(1 - C\varepsilon^{1/d}) c_d r^d \leq \mathcal{H}^d(\Gamma \cap B(x,r)) \leq (1 + C\varepsilon) c_d r^d \tag{4.5}$$

and there is a hyperplane P through x such that

$$\operatorname{dist}(y, P) \leq C\varepsilon^{1/d} r \text{ for } y \in \Gamma \cap B(x,r) \text{ and } \operatorname{dist}(y, \Gamma) \leq C\varepsilon^{1/d} r \text{ for } y \in P \cap B(x,r). \tag{4.6}$$

Proof. We first prove the upper bound in (4.5). Let $x \in \Gamma$ and $r > 0$ be given, and apply the definition to find an ε-Lipschitz graph $G = G_{x,r}$ that meets $B(x, r/2)$ and such that (1.14) holds. Notice that $\mathcal{H}^d(G \cap B(x,r)) \leq (1 + C\varepsilon) c_d r^d$, and then the right-hand side of (4.5), with a slightly larger C, follows. Since G meets $B(x, r/2)$ at some point y, we also have that $\mathcal{H}^d(G \cap B(x,r)) \geq \mathcal{H}^d(G \cap B(y, r/2)) \geq (1 - C\varepsilon) c_d (r/2)^d$ and then, if ε is small enough,

$$\mathcal{H}^d(\Gamma \cap B(x,r)) \geq \mathcal{H}^d(G \cap B(x,r)) - \varepsilon r^d \geq c_d (r/3)^d. \tag{4.7}$$

This already gives the Ahlfors regularity, but we want to improve the lower bound and prove the Reifenberg-flatness. Set $\delta = \operatorname{dist}(x, G)$; then $\Gamma \cap B(x, \delta) \subset \Gamma \setminus G$, so by (1.14) $\mathcal{H}^d(\Gamma \cap B(x,\delta)) \leq \varepsilon r^d$, and (4.7) (for (x, δ)) implies that $\delta \leq C\varepsilon^{1/d} r$.

We may now revise our proof of (4.7), because if we pick $y \in G \cap \overline{B}(x,\delta)$, then $\mathcal{H}^d(G \cap B(x,r)) \geq \mathcal{H}^d(G \cap B(y,(1-C\varepsilon^{1/d})r)) \geq (1-C\varepsilon)(1-C\varepsilon^{1/d})c_d r^d$, and the proof of (4.7) yields the lower bound in (4.5).

Incidentally, we added the constraint that $G = G_{x,r}$ that meets $B(x,r/2)$ in Definition 1.13 to avoid the case of a very small $\Gamma \cap B(x,r)$ and a $G_{x,r}$ that does not meet $B(x,r)$. The $\varepsilon^{1/d}$ is a little ugly, but requiring G to go through x sounded a little too much. And also we could use the fact that we deal with ε-Lipschitz graphs to prove (4.5) and (4.6) with the better constants $C\varepsilon$, but we won't care.

Return to G. Since both G and Γ are Ahlfors-regular, (1.14) now implies that

$$\text{(4.8)} \quad \operatorname{dist}(y,G) \leq C\varepsilon^{1/d} r \text{ for } y \in \Gamma \cap B(x,r/2) \\ \text{and } \operatorname{dist}(y,\Gamma) \leq C\varepsilon^{1/d} r \text{ for } y \in G \cap B(x,r/2)$$

(otherwise we can find a ball of size $C\varepsilon^{1/d}r$ centered on Γ (respectively G) that is contained in $B(x,r) \cap \Gamma \setminus G$ (respectively $B(x,r) \cap G \setminus \Gamma$). But we can also find a hyperplane P that contains x and which is $C\varepsilon^{1/d}r$-close to G in $B(x,r)$, and then

$$\text{(4.9)} \quad \operatorname{dist}(y,P) \leq C\varepsilon^{1/d} r \text{ for } y \in \Gamma \cap B(x,r/3) \\ \text{and } \operatorname{dist}(y,\Gamma) \leq C\varepsilon^{1/d} r \text{ for } y \in P \cap B(x,r/3).$$

This is the same as (4.6), but for $B(x,r/3)$; the lemma follows. □

Recall that since Γ is Reifenberg flat, it separates exactly two domains (there is even a bi-Hölder mapping of $\mathbb{R}^{d+1}$ that maps $\mathbb{R}^d$ to Γ), and it is easy to check that both domains are NTA. Thus Lemma 4.4 allows us to apply the usual estimates for elliptic operators. We let Ω be one of these domains.

Similarly, the Ahlfors regularity of Γ allows us to define A_∞ (with respect to $\sigma = \mathcal{H}^d_{|\Gamma}$, which is doubling) and $A_\infty(\sigma,\delta)$ (as in Definition 1.15).

Finally, we can even construct an analogue of dyadic cubes on $(\Gamma, d\sigma)$, which is pleasant because it simplifies the theory of A_∞ weights, and it implies that the John-Nirenberg inequality on the exponential integrability of BMO functions holds on $(\Gamma, d\sigma)$ essentially as on $\mathbb{R}^d$.

We now approximate Ω by small Lipschitz domains that are contained in Ω.

Lemma 4.10. *Suppose as above that $\Gamma \in CASSC(\varepsilon)$, with ε small enough (depending on d). Set $\eta = \varepsilon^{\frac{1}{2d}}$. Then for each $x \in \Gamma$ and $r > 0$, we can find an η-Lipschitz graph G such that for one of the connected components U of $\mathbb{R}^{d+1} \setminus G$, we have*

$$\text{(4.11)} \qquad U \cap B(x,r) \subset \Omega$$

and

$$\text{(4.12)} \qquad \mathcal{H}^d(\Gamma \cap B(x,r) \setminus G) + \mathcal{H}^d(G \cap B(x,r) \setminus \Gamma) \leq C\varepsilon^{1/2} r^d.$$

As usual, C depends only on d.

Proof. Again we make no attempt to get the best constants here. Let $x \in \Gamma$ and $r > 0$ be given, and let $G_0 = G_{x,2r}$ be the ε-Lipschitz graph given by Definition 1.13. After a rotation if needed, we can assume that G_0 is the graph of $A_0 : \mathbb{R}^d \to \mathbb{R}$ (even if A was not initially defined on the whole $\mathbb{R}^d$, we could extend it).

Write $x = (x_0, t_0)$. Notice that $\mathrm{dist}(x, G_0) \le C\varepsilon^{1/d} r$ by (4.8), and since G_0 is almost horizontal (and if ε is small enough), this means that

$$|A_0(x_0) - t_0| \le C\varepsilon^{1/d} r. \tag{4.13}$$

Then consider the points $X_\pm = (x_0, t_0 \pm r/2)$; since $X_\pm$ is far from G_0, (4.8) says that $X_\pm$ is also far from Γ; without loss of generality, we can assume that $X_+ \in \Omega$, because otherwise $X_- \in \Omega$ (recall that Γ is Reifenberg flat, so Ω lies at least on one side) and we could replace U below with the lower component of $\mathbb{R}^{d+1} \setminus G_0$.

Set $\mathcal{Z} = \Gamma \cap \overline{B}(x,r) \setminus G_0$; we want to hide $\mathcal{Z}$ below the new graph G. The simplest is to take

$$A(y) = \max\big(A_0(y), \sup_{(z,t)\in\mathcal{Z}} t - \eta|y-z|\big). \tag{4.14}$$

Obviously A is η-Lipschitz. Call G the graph of A, and U the component of $\mathbb{R}^{d+1} \setminus G$ above G.

Our next step is to prove that $U \cap B(x,r) \subset \Omega$, as in (4.11). First we claim that $U \cap B(x,r)$ does not meet Γ. Clearly it does not meet G_0, because $A \ge A_0$, so it is enough to exclude points of $B(x,r) \cap \Gamma \setminus G_0 \subset \mathcal{Z}$. But if (z,t) is such a point, then by (4.14), $A(z) \ge t - \eta|z-z| = t$, so $(z,t) \notin U$; this proves our claim.

Let us now check that $X_+ = (x_0, t_0 + r/2) \in U$, or in other words $A(x_0) < t_0 + r/2$. But $A_0(x_0) \le t_0 + C\varepsilon^{1/d} r < t_0 + r/2$ by (4.13), and for $(z,t) \in \mathcal{Z}$, we have that $t \le A(x_0) + C\varepsilon^{1/d} r$ because $\mathrm{dist}((z,t), G_0) \le \varepsilon^{1/d} r$ by (4.8), and G_0 is almost horizontal. Then $t - \eta|y-z| \le t \le A(x_0) + C\varepsilon^{1/d} r < t_0 + r/2$, so $X_+ \in U$. At this point $U \cap B(x,r)$ is a connected set that contains $X_+ \in \Omega$ and that does not meet Γ, and this implies that $U \cap B(x,r) \subset \Omega$, as needed for (4.11).

Finally we estimate the size of the two bad sets in (4.12). For each $Z = (z,t) \in \mathcal{Z}$, consider the ball $B_Z = B(Z, r(Z))$, with $r(Z) = \mathrm{dist}(Z, G_0)/100$. By the 5-covering lemma of Vitali, we can find a countable collection of balls B_Z, $Z \in I$, such that the B_Z are disjoint but the $5B_Z$ cover $\mathcal{Z}$. First consider the set

$$H = \big\{y \in \mathbb{R}^d \cap B(x_0, r)\,;\, A(y) \ne A_0(y)\big\}. \tag{4.15}$$

We want to cover H by multiples of the B_Z, so let $y \in H$ be given. By definition, $A(y) \ne A_0(y)$ and so we can find $Z' = (z', t') \in \mathcal{Z}$ such that $A_0(y) < t' - \eta|y - z'|$ and hence, since A_0 is ε-Lipschitz, $A_0(z') < t' - \eta|y-z'| + \varepsilon|y-z'| < t' - \eta|y-z'|/2$. Then we can find $Z = (z,t) \in I$ such that $(z', t') \in 5B_Z$.

$$|y - z| \le |y - z'| + |z' - z| \le 2\eta^{-1}(t' - A_0(z')) + 5r(Z). \tag{4.16}$$

Now let $W = (w, A_0(w)) \in G_0$ minimize the distance to Z. Then $|w - z| \le |W - Z| = 100r(Z)$, so $|A_0(z) - A_0(w)| \le \varepsilon|z - w| \le 100\varepsilon r(Z)$. Now

$$\begin{aligned} t' - A_0(z') &\le |t' - t| + |A_0(z') - A_0(w)| + |A_0(w) - t| \\ &\le 5r(Z) + \varepsilon|z' - w| + |W - Z| \\ &\le 5r(Z) + \varepsilon(|Z' - Z| + |Z - W|) + 100r(Z) \le 106r(Z), \end{aligned} \tag{4.17}$$

so (4.16) implies that $|y - z| \le 150\eta^{-1} r(Z)$, and altogether

$$H \subset \bigcup_{Z=(z,t)\in I} B(z, 200r(Z)). \tag{4.18}$$

This yields

$$\begin{aligned}(4.19)\qquad \mathcal{H}^d(H) &\leq C\eta^{-d}\sum_{Z\in I} r(Z)^d \leq C\eta^{-d}\sum_{Z\in I}\mathcal{H}^d(\Gamma\cap B_Z)\\ &\leq C\eta^{-d}\mathcal{H}^d(\Gamma\cap B(x,2r)\setminus G_0) \leq C\eta^{-d}\varepsilon r^d = C\varepsilon^{1/2}r^d\end{aligned}$$

because the B_Z are disjoint by construction, contained in $B(x, 2r)\setminus G_0$ because $r(Z) = \mathrm{dist}(Z, G_0)/100 << r$, by the Ahlfors regularity of Γ, by (1.14), and because $\eta = \varepsilon^{1/2d}$. So we control the measure of the difference between the two graphs G and G_0, and at this point (4.12) follows from (1.14). $\square$

As a corollary of Lemma 4.10, the inclusion (4.11) can be reversed. That is, given any ball $B(x, r)$ centered on Γ, Ω can be approximated by a Lipschitz domain that contains $\Omega\cap B(x, r)$. We state this result as follows.

Corollary 4.20. *Suppose as above that $\Gamma\in CASSC(\varepsilon)$, with ε small enough (depending on d). Set $\eta = \varepsilon^{\frac{1}{2d}}$. Then for each $x\in\Gamma$ and $r > 0$, we can find an η-Lipschitz graph G such that for one of the connected components U of $\mathbb{R}^{d+1}\setminus G$, we have*

$$(4.21)\qquad \Omega\cap B(x, r)\subset U$$

and

$$(4.22)\qquad \mathcal{H}^d(\Gamma\cap B(x, r)\setminus G) + \mathcal{H}^d(G\cap B(x, r)\setminus\Gamma)\leq C\varepsilon^{1/2}r^d.$$

As usual, C depends only on d.

Proof. Let G be the η-Lipschitz graph as in Lemma 4.10. Then we only need to prove (4.21) since (4.22) is the same as (4.12). Let the two connected components of $\mathbb{R}^{d+1}\setminus G$ be $U_\pm$, and the two connected components of $\mathbb{R}^{d+1}\setminus\Gamma$ be $\Omega_\pm$. Then by Lemma 4.10, $U_-\cap B(x, r)\subset\Omega_-$. Taking complement, we get that $\Omega_+\subset U_+\cup B(x, r)^{\mathsf{c}}$. Therefore, $\Omega_+\cap B(x, r)\subset U_+\cap B(x, r)\subset U_+$, as desired. $\square$

4.2. **Small A_∞ result in the case of $CASSC$.** In this section we deduce Theorem 4.23 from the case of small Lipschitz graphs, through a standard comparison argument.

Throughout the section, we let $\Gamma\in CASSC(\varepsilon)$ with ε small enough, and Ω be one of the components of $\mathbb{R}^{d+1}\setminus\Gamma$. We shall approximate Ω from outside by a Lipschitz graph as in Corollary 4.20.

Theorem 4.23. *For every choice of $\delta > 0$, there exist $\varepsilon_0 = \varepsilon_0(\delta, d, \mu_0) > 0$ with the following properties. Let $0 < \varepsilon\leq\varepsilon_0$. Suppose $\Gamma\in CASSC(\varepsilon)$, and $L = -\operatorname{div} A\nabla$ is an elliptic operator with ellipticity μ_0 which satisfies the weak DKP condition with norm $\mathfrak{N}(A) < \varepsilon$. Then for every surface ball $\Delta = B(x, r)\cap\Gamma$ and every Borel subset $E\subset\Delta$,*

$$\left|\frac{\omega_\Omega^{X_0}(E)}{\omega_\Omega^{X_0}(\Delta)} - \frac{\sigma(E)}{\sigma(\Delta)}\right| < \delta,$$

whenever $X_0 = X_{x,\varepsilon^{-\beta}r}\in\Omega$ is a corkscrew point for x at scale $\varepsilon^{-\beta}r$, with $0 < \beta\leq\beta_0$ for some β_0 that depends only on d and μ_0.

Remark 4.24. Although Theorem 4.23 is stated differently from Theorem 1.19, one can check that the latter can be deduced from the former. In fact, let $\delta > 0$, $\kappa > 1$ and $X\in\Omega$ be fixed. Take $\Delta = \Delta(x, r)\subset B(X, \kappa\,\delta(X))$ with $r\leq\tau\,\delta(X)$ and $\tau > 0$

to be determined, where $\delta(X) = \operatorname{dist}(X, \Gamma)$, and let $E \subset \Delta$. By Theorem 4.23, there exists an $\varepsilon_0 > 0$ such that

$$\left| \frac{\omega^{X_0}(E)}{\omega^{X_0}(\Delta)} - \frac{\sigma(E)}{\sigma(\Delta)} \right| < \delta/2 \tag{4.25}$$

whenever $X_0 \in \Omega$ is a corkscrew point for x at scale $\varepsilon_0^{-\beta_0} r$, for some $\beta_0 = \beta_0(d, \mu_0) > 0$. By a change-of-pole argument similar to Lemma 2.17, one can show that

$$\left| \frac{\omega^{X_0}(E)}{\omega^{X_0}(\Delta)} - \frac{\omega^{X}(E)}{\omega^{X}(\Delta)} \right| \le C \left(\frac{r}{d}\right)^\gamma \quad \text{for } r < \frac{d}{10} \tag{4.26}$$

for some $\gamma = \gamma(d, \mu_0) > 0$, where $d = \min\{\delta(X), \delta(X_0)\}$. Note that we can assume that $d = \delta(X_0)$ by taking τ sufficiently small (for instance, let $\tau = \varepsilon_0^{\beta_0}/2$). Now we take ε_0 sufficiently small so that $\varepsilon_0^{\beta_0 \gamma} < \frac{\delta}{10C}$, then Theorem 1.19 follows from (4.25), (4.26), and the triangle inequality. Theorem 1.17 can be deduced from Theorem 1.19 using Lemma 2.17.

We shall need Lemmata 4.27 and 4.30 that enable us to localize the elliptic measures.

Lemma 4.27. *There exists a constant $M \ge 1$ depending on the dimension so that the following holds. Let $x \in \Gamma$ and $r > 0$ be given, and $L = -\operatorname{div} A\nabla$ be a μ_0-elliptic operator. Let $M' \ge M^2$. Let ω be the L-elliptic measure of Ω with pole at $X_0 = X_{x,2M'r}$ and let $\widetilde{\omega}$ be the L-elliptic measure of $\Omega \cap B(x, 4M'r)$ with pole at X_0. Then $\omega|_{\Delta(x,r)}$ and $\widetilde{\omega}|_{\Delta(x,r)}$ are mutually absolutely continuous, and*

$$\frac{d\widetilde{\omega}}{d\omega}(y) \approx 1, \quad \omega \text{ a.e. } y \in \Delta(x, r), \tag{4.28}$$

where the implicit constants depend on d, μ_0, and the NTA constant. Therefore,

$$\widetilde{\omega}(E) \approx \omega(E) \quad \text{for any Borel set } E \subset \Delta(x, r), \tag{4.29}$$

where the implicit constants depend on d, μ_0, and the NTA constant.

Proof. Let $y \in \Gamma$ and $s > 0$ be such that $B(y, s) \subset B(x, r)$. By Lemma 2.5, and (2.4),

$$\frac{\widetilde{\omega}(\Delta(y,s))}{\omega(\Delta(y,s))} \approx \frac{\widetilde{G}(X_0, X_{y,s})}{G(X_0, X_{y,s})} \approx \frac{\widetilde{G}^\top(X_{y,s}, X_0)}{G^\top(X_{y,s}, X_0)},$$

where $\widetilde{G}(\cdot, Y)$ is the L-Green function for $\Omega \cap B(x, 4M'r)$ with pole at Y, and $G(\cdot, Y)$ is the L-Green function for Ω with pole at Y. Since $|X_0 - x| \ge \operatorname{dist}(X_0, \partial\Omega) \ge 2M'r/M$ and we have chosen $M' \ge M^2$, $\widetilde{G}^\top(\cdot, X_0)$ and $G^\top(\cdot, X_0)$ are both solutions to $L^\top u = 0$ in $B(x, 2Mr) \cap \Omega$ that vanish on $\Delta(x, 2Mr)$. By the comparison principle (Lemma 2.8) and the estimates (2.2), (2.3) for the Green function,

$$\frac{\widetilde{G}^\top(X_{y,s}, X_0)}{G^\top(X_{y,s}, X_0)} \approx \frac{\widetilde{G}^\top(X_{x,r}, X_0)}{G^\top(X_{x,r}, X_0)} \approx 1,$$

and thus

$$\frac{\widetilde{\omega}(\Delta(y,s))}{\omega(\Delta(y,s))} \approx 1.$$

Since the elliptic measures are regular and doubling, this implies that $\widetilde{\omega}|_{\Delta(x,r)}$ and $\omega|_{\Delta(x,r)}$ are mutually absolutely continuous. Then the Lebesgue differentiation theorem asserts that

$$\frac{d\widetilde{\omega}}{d\omega}(y)=\lim_{s\to 0}\frac{\widetilde{\omega}(\Delta(y,s))}{\omega(\Delta(y,s))}\approx 1 \quad \omega \text{ a.e. } y\in\Delta(x,r),$$

which proves (4.28). Then (4.29) is an immediate consequence of (4.28) since we can write $\widetilde{\omega}(E)=\int_E \frac{d\widetilde{\omega}}{d\omega}(y)d\omega(y)$. □

Lemma 4.30. *There exists a constant $M\geq 1$ depending on the dimension so that the following holds. Let $M'\geq M^2$, $r>0$, $x\in\Gamma$, and L be as in Lemma 4.27. Let ω be the L-elliptic measure of Ω with pole at $X_0=X_{x,M'r}$, and let $\widetilde{\omega}$ be the L-elliptic measure of $\Omega\cap B(x,M'r)$ with pole at X_0. Then*

$$\frac{d\widetilde{\omega}}{d\omega}(y)=\lim_{Y\to y}\frac{\widetilde{G}(X_0,Y)}{G(X_0,Y)} \quad \textit{for } \omega \textit{ a.e. } y\in\Delta(x,r), \tag{4.31}$$

where $\widetilde{G}(\cdot,Y)$ is the L-Green function for $\Omega\cap B(x,M'r)$, $G(\cdot,Y)$ is the L-Green function for Ω, and the limit is taken in $B(x,M'r)\cap\Omega$. Moreover, for E, $E'\subset\Gamma\cap B(x,r)$,

$$\frac{1-\frac{C}{M'^\alpha}}{1+\frac{C}{M'^\alpha}}\frac{\widetilde{\omega}(E)}{\widetilde{\omega}(E')}\leq\frac{\omega(E)}{\omega(E')}\leq\frac{1+\frac{C}{M'^\alpha}}{1-\frac{C}{M'^\alpha}}\frac{\widetilde{\omega}(E)}{\widetilde{\omega}(E')}, \tag{4.32}$$

where C and α are positive constants that depend on d, μ_0, and the NTA constant of Ω.

Sketch of proof. By Lemma 4.27, $\widetilde{\omega}|_{\Delta(x,r)}$ and $\omega|_\Delta(x,r)$ are mutually absolutely continuous, and

$$\frac{d\omega}{d\widetilde{\omega}}(y)=\lim_{s\to 0}\frac{\omega(\Delta(y,s))}{\widetilde{\omega}(\Delta(y,s))} \quad \widetilde{\omega} \text{ a.e. } y\in\Delta(x,r).$$

Then (4.31) can be proved using the Riesz formula, CFMS estimates, the ellipticity of the operator and the boundary Caccioppoli inequality. We refer the readers to [BTZ20, Lemma 5.1] for details. Now (4.32) is a consequence of (4.31), as one can write

$$\omega(E)=\int_E\frac{d\omega}{d\widetilde{\omega}}(y)d\widetilde{\omega}(y)=\int_E\mathcal{L}(y)d\widetilde{\omega}(y),$$

where $\mathcal{L}(y)=\lim_{Y\to y}\frac{G(X_0,Y)}{\widetilde{G}(X_0,Y)}$. By the comparison principle, one can show that

$$\left(1-\frac{C}{M'^\alpha}\right)\mathcal{L}(x)\leq\mathcal{L}(y)\leq\left(1+\frac{C}{M'^\alpha}\right)\mathcal{L}(x) \quad \text{for } y\in\Delta(x,r).$$

From here, (4.32) follows. Details can be found in e.g. [KT97, Corollary 4.2]. □

Now we start our proof of Theorem 4.23. Let $\delta\in(0,\frac{1}{10})$, $x_0\in\Gamma$ and $r_0>0$ be given. Let $\Delta_0=B(x_0,r_0)\cap\Gamma$, and let $E\subset\Delta_0$. Let M be the largest of the two constants M in Lemmata 4.27 and 4.30, and take $K\geq M^2$ to be determined. Set $X_0=X_{x_0,Kr_0}\in\Omega\cap B(x_0,Kr_0)$ to be a corkscrew point for x_0 at scale Kr_0.

By Corollary 4.20, there exists an $\varepsilon^{\frac{1}{2d}}$-Lipschitz graph $G=\{(y,A(y)):y\in\mathbb{R}^d\}$ such that for one of the connected components U of $\mathbb{R}^{d+1}\setminus G$, we have

$$\Omega\cap B(x_0,10Kr_0)\subset U \tag{4.33}$$

and

$$\mathcal{H}^d(\Gamma \cap B(x_0, 10Kr_0) \setminus G) + \mathcal{H}^d(G \cap B(x_0, 10Kr_0) \setminus \Gamma) \leq CK^d \varepsilon^{1/2} r_0^d. \tag{4.34}$$

Similar to (4.8), we have that

$$\begin{aligned} &\operatorname{dist}(y, G) \leq CK\varepsilon^{\frac{1}{2d}} r_0 \text{ for } y \in \Gamma \cap B(x_0, 10Kr_0/2), \\ &\operatorname{dist}(y, \Gamma) \leq CK\varepsilon^{\frac{1}{2d}} r_0 \text{ for } y \in G \cap B(x_0, 10Kr_0/2). \end{aligned} \tag{4.35}$$

We shall choose

$$K = K(\varepsilon) = \varepsilon^{-\beta}, \tag{4.36}$$

with a small constant $\beta > 0$ that will be chosen near the end of the argument, and then $\varepsilon = \varepsilon(\delta, d, \mu_0) > 0$ sufficiently small. Thus K is as large as we want, and in particular we shall choose ε so small that $K \geq M$. We intend to show that for $\varepsilon = \varepsilon(\delta, d, \mu_0) > 0$ sufficiently small,

$$\frac{\omega_\Omega^{X_0}(E)}{\omega_\Omega^{X_0}(\Delta_0)} \geq \frac{\sigma(E)}{\sigma(\Delta_0)} - \delta. \tag{4.37}$$

Once we have (4.37) for any set $E \subset \Delta_0$, we obtain that $\frac{\omega_\Omega^{X_0}(E)}{\omega_\Omega^{X_0}(\Delta_0)} \leq \frac{\sigma(E)}{\sigma(\Delta_0)} + \delta$ by taking E to be $\Delta_0 \setminus E$ in (4.37). Therefore, it suffices to show (4.37).

Set $B = B(x_0, Kr_0)$ to lighten the notation. We first transfer elliptic measures from Ω to $\Omega \cap B$; this will be convenient because $\Omega \cap B$ is contained in U, while Ω may not be.

Recall that $L = -\operatorname{div} A\nabla$. In Section 4.3, we will show that there is a μ_0-elliptic operator $\widetilde{L} = -\operatorname{div} \widetilde{A}\nabla$ that satisfies the weak DKP condition in U with constant $\mathfrak{N}(\widetilde{A}) \leq C\varepsilon$, and such that

$$\widetilde{A}(X) = A(X) \text{ on } \Omega \cap B(x_0, 2Kr_0) \subset U. \tag{4.38}$$

Denote by ω the L-elliptic measure of $\Omega \cap B$ with pole at X_0 and by ω' the $\widetilde{L}$-elliptic measure of $U \cap B$ with pole at X_0. Define

$$\mathcal{Z} := \Gamma \cap B \setminus G. \tag{4.39}$$

Observe that by (4.34),

$$\mathcal{H}^d(\mathcal{Z}) \leq CK^d \varepsilon^{1/2} r_0^d. \tag{4.40}$$

Our first claim is that for any set $H \subset \Gamma \cap G \cap B(x_0, r_0)$,

$$\omega'(H) \leq \omega(\mathcal{Z}) + \omega(H). \tag{4.41}$$

Proof of (4.41). Let $H_n \subset H$ be closed sets in $G \cap \Gamma$ such that $H_n \uparrow H$ as $n \to \infty$. Let $O_n \subset G \cap B(x_0, 2r_0)$ be open sets such that $O_n \downarrow H$. Let $g_n \in C_c^\infty(O_n)$ be such that $\mathbb{1}_{H_n} \leq g_n \leq \mathbb{1}_{O_n}$. For $X \in U \cap B$, define $u_n(X) = \int g_n(y) d\omega'^X(y)$, where ω'^X is the $\widetilde{L}$-elliptic measure of $U \cap B$ with pole at X. Then by the definition of elliptic measures, $\widetilde{L}u_n = 0$ in $U \cap B$ and u is continuous in $\overline{U \cap B}$. Since $\partial(\Omega \cap B) \subset \overline{U \cap B}$, u_n is continuous on $\partial(\Omega \cap B)$. Let $v_n(X) = \int u_n(y) d\omega^X(y)$, where ω^X is the L-elliptic measure of $\Omega \cap B$ with pole at X. Then $Lv_n = 0$ in $\Omega \cap B$, with $v_n = u_n$ on $\partial(\Omega \cap B)$. Since $\widetilde{L} = L$ in $\Omega \cap B$, the maximum principle implies that $v_n = u_n$

in $\Omega\cap B$. Observe that $\partial(\Omega\cap B)=(\Gamma\cap G\cap B)\cup\mathcal{Z}\cup(\partial B\cap\Omega)$, and that $u_n=g_n$ on $(\Gamma\cap G\cap B)\cup(\partial B\cap\Omega)$. So for $X\in\Omega\cap B$,

$$\begin{aligned}u_n(X)=v_n(X)&=\int_{\Gamma\cap G\cap B}g_n(y)d\omega^X(y)+\int_{\mathcal{Z}}u_n(y)d\omega^X(y)+\int_{\partial B\cap\Omega}g_n(y)d\omega^X(y)\\&=\int_{\Gamma\cap O_n}g_n(y)d\omega^X(y)+\int_{\mathcal{Z}}u_n(y)d\omega^X(y)\le\omega^X(O_n\cap\Gamma)+\omega^X(\mathcal{Z}),\end{aligned}$$

where in the last inequality we have used $g_n\le\mathbb{1}_{O_n}$ and $u_n\le1$. On the other hand, since $g_n\ge\mathbb{1}_{H_n}$,

$$u_n(X)\ge\int_{H_n}d\omega'^X(y)=\omega'^X(H_n).$$

So we get that

$$\omega'^X(H_n)\le\omega^X(O_n\cap\Gamma)+\omega^X(\mathcal{Z}).$$

Then (4.41) follows from taking $n\to\infty$ and by the regularity of elliptic measures. □

Now we define another set F, which we think of as the large shadow of $\mathcal{Z}$ on G, which is defined as

$$F:=\bigcup_{Z\in\mathcal{Z}}G\cap B(Z,20\,\delta(Z)). \tag{4.42}$$

Here, and in the sequel, $\delta(Z)=\operatorname{dist}(Z,G)$. Notice that by (4.35), $\delta(Z)\le C\varepsilon^{\frac{1}{2d}}Kr_0$ for all $Z\in\mathcal{Z}$. We show that

$$\mathcal{H}^d(F)\le CK^d\varepsilon^{1/2}r_0^d. \tag{4.43}$$

Proof of (4.43). For each $Z\in\mathcal{Z}$, let $B_Z=B(Z,r(Z))$ with $r(Z)=\operatorname{dist}(Z,G)/100$. Then by the Vitali covering lemma, we can find a countable collection of balls B_Z, $Z\in I$, such that the B_Z are disjoint but $\mathcal{Z}\subset\bigcup_{Z\in I}5B_Z$. Recall that G is the graph of $A:\mathbb{R}^d\to\mathbb{R}$. Write $x_0=(x,t_0)$, where $x\in\mathbb{R}^d$ and $t_0\in\mathbb{R}$. By (4.35), for ε sufficiently small, we have that $F\subset\{(y,A(y)):y\in\mathbb{R}^d\cap B(x,2Kr_0)\}$. Set

$$H:=\{y\in\mathbb{R}^d\cap B(x,2Kr_0):(y,A(y))\in F\}.$$

Observe that for any fixed $y\in H$, there exists a $Z=(z,t)\in\mathcal{Z}$ such that $|(y,A(y))-(z,t)|\le 20\delta(Z)$. But there must be a $Z'=(z',t')\in I$ such that $Z\in 5B_{Z'}$. So

$$\delta(Z)\le\delta(Z')+|Z-Z'|\le\delta(Z')+5r(Z')\le2\delta(Z').$$

Then

$$|y-z'|\le|y-z|+|z-z'|\le20\delta(Z)+5r(Z')\le21\delta(Z'),$$

which implies that $y\in B(z',21\delta(Z'))$. Since $y\in H$ is arbitrary, we obtain that

$$H\subset\bigcup_{Z=(z,t)\in I}B(z,21\delta(Z)).$$

Since Γ is d-Ahlfors regular,

$$\mathcal{H}^d(H)\le C\sum_{Z\in I}\delta(Z)^d\le C\sum_{Z\in I}\mathcal{H}^d(\Gamma\cap B_Z).$$

Recall that the B_Z are disjoint and that $B_Z \subset B(x_0, 2Kr_0) \setminus G$, so

$$\sum_{Z \in I} \mathcal{H}^d(\Gamma \cap B_Z) \leq C\mathcal{H}^d(\Gamma \cap B(x_0, 2Kr_0) \setminus G) \leq CK^d \varepsilon^{1/2} r_0^d,$$

where the last inequality follows from (4.34). Therefore,

$$\mathcal{H}^d(F) \leq \int_H \sqrt{1 + |\nabla A|^2} dy \leq C(1 + 2\varepsilon^{1/d}) K^d \varepsilon^{1/2} r_0^d \leq CK^d \varepsilon^{1/2} r_0^d,$$

as desired. □

We now consider the $\widetilde{L}$-elliptic measure of U. We claim that for some $C > 1$ depending on d and μ_0,

$$\omega_U^Z(F) \geq \frac{1}{C} \quad \text{for all } Z \in \mathcal{Z}. \tag{4.44}$$

Proof of (4.44). Let $Z = (z, t) \in \mathcal{Z}$ be given. We first show that

$$(y, A(y)) \in F \quad \text{for any } y \in \mathbb{R}^d \text{ that satisfies } |y - z| \leq 10\delta(Z). \tag{4.45}$$

Let $Z' = (z', A(z')) \in G$ be such that $|Z - Z'| = \delta(Z)$. Then

$$|z - z'| + |t - A(z')| \leq \sqrt{2\left|Z - Z'\right|^2} < 2\delta(Z).$$

For any $y \in \mathbb{R}^d$ with $|y - z| \leq 10\delta(Z)$, we have that

$$\begin{aligned}
|(y, A(y)) - Z| &\leq |y - z| + |A(y) - t| \\
&\leq |y - z| + |A(y) - A(z)| + |A(z) - A(z')| + |A(z') - t| \\
&\leq \left(1 + \varepsilon^{\frac{1}{2d}}\right)|y - z| + \varepsilon^{\frac{1}{2d}}|z - z'| + |A(z') - t| \\
&\leq \left(12 + 3\varepsilon^{\frac{1}{2d}}\right)\delta(Z) \leq 20\delta(Z),
\end{aligned}$$

which proves (4.45).

By (4.45), $Z' \in F$ and $B(Z', 5\delta(Z)) \cap G \subset F$. Therefore,

$$\omega_U^Z(F) \geq \omega_U^Z(B(Z', 5\delta(Z)) \cap G) \geq \frac{1}{C},$$

where in the last inequality we have used Bourgain's estimate (Lemma 2.7). □

With (4.44), we are ready to show that

$$\omega_U^{X_0}(F) \geq \frac{1}{C}\omega(\mathcal{Z}). \tag{4.46}$$

Proof of (4.46). Observe that $\mathcal{Z} \subset \Gamma$ is open in Γ, and that $F \subset G$ is open in G. Let $F_n \subset F$ be a sequence of closed sets such that $F_n \uparrow F$, and let $\mathcal{Z}_m \subset \mathcal{Z}$ be a sequence of closed sets such that $\mathcal{Z}_m \uparrow \mathcal{Z}$. Let $f_n \in C_c^\infty(F)$ and $\mathbb{1}_{F_n} \leq f_n \leq \mathbb{1}_F$. Let $g_m \subset C_c^\infty(\mathcal{Z})$ and $\mathbb{1}_{\mathcal{Z}_m} \leq g_m \leq \mathbb{1}_{\mathcal{Z}}$. For $X \in U$, set $v_n(X) = \int_G f_n(y) d\omega_U^X$. Then by the definition of elliptic measures, $\widetilde{L}v_n = 0$ in U and $v_n = f_n$ on G. Also,

$$\omega_U^X(F) \geq v_n(X) \geq \omega_U^X(F_n). \tag{4.47}$$

For $X \in \Omega \cap B$, set

$$u_m^{(n)}(X) = \int_{\partial(\Omega \cap B)} g_m(y) v_n(y) d\omega_{\Omega \cap B}^X(y).$$

Then $Lu_m^{(n)} = 0$ in $\Omega \cap B$ and $u_m^{(n)} = g_m v_n$ on $\partial(\Omega \cap B)$. Since $\widetilde{L} = L$ in $\Omega \cap B$, the maximum principle implies that

$$v_n(X) \geq u_m^{(n)}(X) \quad \text{for any } X \in \Omega \cap B, \text{ for any } m, n \in \mathbb{N}.$$

Then by $g_m \geq \mathbb{1}_{\mathcal{Z}_m}$ and (4.47),

$$\omega_U^X(F) \geq \int_{\mathcal{Z}_m} v_n(y)\, d\omega_{\Omega\cap B}^X(y) \geq \int_{\mathcal{Z}_m} \omega_U^y(F_n)\, d\omega_{\Omega\cap B}^X(y).$$

Letting $n \to \infty$, by the regularity of ω_U and (4.46), we get that

$$\omega_U^X(F) \geq \int_{\mathcal{Z}_m} \omega_U^y(F)\, d\omega_{\Omega\cap B}^X(y) \geq \frac{1}{C}\omega_{\Omega\cap B}^X(\mathcal{Z}_m).$$

Now letting $m \to \infty$, the regularity of $\omega_{\Omega\cap B}$ gives that

$$\omega_U^X(F) \geq \frac{1}{C}\omega_{\Omega\cap B}^X(\mathcal{Z}) \quad \text{for any } X \in \Omega \cap B.$$

This proves (4.44) since $X_0 \in \Omega \cap B$. □

We are now ready to prove (4.37). By (4.32),

$$\frac{\omega_\Omega^{X_0}(E)}{\omega_\Omega^{X_0}(\Delta_0)} \geq \frac{1-\frac{C}{K^\alpha}}{1+\frac{C}{K^\alpha}}\,\frac{\omega(E)}{\omega(\Delta_0)} \geq \frac{\omega(E)}{\omega(\Delta_0)} - C\varepsilon^{\alpha\beta}, \tag{4.48}$$

where, according to (4.36), we have chosen $K = \varepsilon^{-\beta}$ for some $\beta > 0$ to be determined later. We now compute a lower bound for $\frac{\omega(E)}{\omega(\Delta_0)}$. Since E and Δ_0 might not be contained in G, we write

$$\frac{\omega(E)}{\omega(\Delta_0)} \geq \frac{\omega(E\cap G)}{\omega(\Delta_0 \cap G) + \omega(\Delta_0 \setminus G)}.$$

By (4.41),

$$\omega(E\cap G) \geq \omega'(E\cap G) - \omega(\mathcal{Z}).$$

Since $\Omega \cap B \subset U \cap B$ and $L = \widetilde{L}$ on $\Omega \cap B$, the maximum principle implies that $\omega(\Delta_0 \cap G) \leq \omega'(\Delta_0 \cap G)$. Also, from the definition of $\mathcal{Z}$, it follows that $\omega(\Delta_0 \setminus G) \leq \omega(\mathcal{Z})$. So

$$\frac{\omega(E)}{\omega(\Delta_0)} \geq \frac{\omega'(E\cap G) - \omega(\mathcal{Z})}{\omega'(\Delta_0\cap G) + \omega(\mathcal{Z})} \geq \frac{\omega'(E\cap G)}{\omega'(\Delta_0\cap G)} - \frac{2\omega(\mathcal{Z})}{\omega'(\Delta_0\cap G)} =: I_1 - 2I_2. \tag{4.49}$$

Let us first show that I_2 can be arbitrary small. By (4.46) and (4.29),

$$I_2 \leq C\,\frac{\omega_U^{X_0}(F)}{\omega_U^{X_0}(\Delta_0\cap G)}. \tag{4.50}$$

By Section 3, the $\widetilde{L}$-elliptic measure of U is in $A_\infty(d\mathcal{H}^d|_G)$ (we even have A_∞ with small constant). Therefore,

$$\omega_U^{X_0}(F) \leq C\omega_U^{X_0}(B\cap G)\left(\frac{\mathcal{H}^d(F)}{\mathcal{H}^d(B\cap G)}\right)^\theta,$$

and

$$\omega_U^{X_0}(\Delta_0\cap G) \geq \frac{1}{C}\omega_U^{X_0}(B(x_0,r_0)\cap G)\left(\frac{\mathcal{H}^d(\Delta_0\cap G)}{\mathcal{H}^d(B(x_0,r_0)\cap G)}\right)^\eta,$$

where $\theta, \eta > 0$ are A_∞ constants, which are independent of the sets. Again by A_∞, we have that

$$\frac{\omega_U^{X_0}(B\cap G)}{\omega_U^{X_0}(B(x_0,r_0)\cap G)} \le C\left(\frac{\mathcal{H}^d(B(x_0,r_0)\cap G)}{\mathcal{H}^d(B\cap G)}\right)^{-\eta}.$$

Substituting these estimates in (4.50), we get that

$$I_2 \le C\left(\frac{\mathcal{H}^d(F)}{\mathcal{H}^d(B\cap G)}\right)^{\theta} \Big/ \left(\frac{\mathcal{H}^d(\Delta_0\cap G)}{\mathcal{H}^d(B\cap G)}\right)^{\eta}. \tag{4.51}$$

Observe that

$$\mathcal{H}^d(\Delta_0\cap G) = \mathcal{H}^d(\Gamma\cap B(x_0,r_0)\cap G) \ge \mathcal{H}^d(\Gamma\cap B(x_0,r_0)) - \mathcal{H}^d(\mathcal{Z}).$$

Thus, if we choose $K = K(\varepsilon)$ according to (4.36) and ε so small that

$$CK^d\varepsilon^{1/2} < \frac{(1-C\varepsilon^{1/d})c_d}{2}, \tag{4.52}$$

then by (4.5) and (4.40),

$$\mathcal{H}^d(\Delta_0\cap G) \ge \frac{(1-C\varepsilon^{1/d})c_d}{2}\, r_0^d \ge C^{-1}r_0^d. \tag{4.53}$$

We now check that

$$\frac{c_d}{2}K^d r_0^d \le \mathcal{H}^d(B\cap G) \le c_d(1+C\varepsilon^{\frac{1}{2d}})K^d r_0^d. \tag{4.54}$$

The second inequality follows directly from the fact that G is an $\varepsilon^{\frac{1}{2d}}$-Lipschitz graph. To see the first inequality, we want to use a ball centered on Γ. By (4.35), there exists a point $x^* \in G$ such that $\delta(x_0) = |x_0 - x^*| \le C\varepsilon^{\frac{1}{2d}}Kr_0$. Since $B(x^*, Kr_0 - \delta(x_0)) \subset B$, we get that

$$\mathcal{H}^d(B\cap G) \ge \mathcal{H}^d(B(x^*, Kr_0-\delta(x_0))) \ge (1-C\varepsilon^{\frac{1}{2d}})(Kr_0-\delta(x_0))^d \ge \frac{c_d}{2}K^d r_0^d$$

because G is an $\varepsilon^{\frac{1}{2d}}$-Lipschitz graph and if ε is small enough; (4.54) follows. Using (4.43), (4.53) and (4.54) in (4.51), we obtain that

$$I_2 \le CK^{d\eta}\varepsilon^{\frac{\theta}{2}}. \tag{4.55}$$

We return to I_1 in (4.49). By (4.32),

$$I_1 \ge \frac{1-\frac{C}{K^\alpha}}{1+\frac{C}{K^\alpha}}\,\frac{\omega_U^{X_0}(E\cap G)}{\omega_U^{X_0}(\Delta_0\cap G)}. \tag{4.56}$$

We are in a position to apply the $A_\infty(\sigma,\delta)$ result for small Lipschitz graph, which says that for $\varepsilon = \varepsilon(d,\mu_0,\delta)$ sufficiently small and

$$K \ge \frac{M}{\tau(d,\delta,\mu_0)}, \tag{4.57}$$

where $\tau = \tau(d,\delta,\mu_0)$ is as in Theorem 1.19 for small Lipschitz graphs,

$$\frac{\omega_U^{X_0}(E\cap G)}{\omega_U^{X_0}(\Delta_0\cap G)} \ge \frac{\mathcal{H}^d(E\cap G)}{\mathcal{H}^d(\Delta_0\cap G)} - \frac{\delta}{8}. \tag{4.58}$$

Notice that we do not have a zero denominator on the right-hand side of (4.58), and even $\mathcal{H}^d(\Delta_0 \cap G) \geq c_d r_0^d$, because $B(x^*, r_0/2) \subset B(x_0, r_0)$ for the same x^* as above, as soon as we choose $\beta < \frac{1}{4d}$ in (4.36) and ε so small that

$$CK\varepsilon^{\frac{1}{2d}} \leq 1/2. \tag{4.59}$$

Since we want $\frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)}$ in (4.37), we need to compare $\frac{\mathcal{H}^d(E\cap G)}{\mathcal{H}^d(\Delta_0\cap G)}$ and $\frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)}$. We use (4.40) and the Ahlfors regularity of Γ to get that

$$\frac{\mathcal{H}^d(E\cap G)}{\mathcal{H}^d(\Delta_0\cap G)} \geq \frac{\mathcal{H}^d(E) - \mathcal{H}^d(\mathcal{Z})}{\mathcal{H}^d(\Delta_0)} \geq \frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)} - \frac{CK^d\varepsilon^{1/2}}{(1+C\varepsilon)c_d} \geq \frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)} - \frac{\delta}{8},$$

where we have chosen $\beta < \frac{1}{4d}$ in (4.36) and $\varepsilon = \varepsilon(d, \delta, \mu_0)$ so small that

$$\frac{CK^d\varepsilon^{1/2}}{(1+C\varepsilon)c_d} \leq \frac{\delta}{8} \tag{4.60}$$

in the last inequality. Returning to (4.56), we obtain that for ε small enough,

$$I_1 \geq \frac{1-\frac{C}{K^\alpha}}{1+\frac{C}{K^\alpha}}\left(\frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)} - \frac{\delta}{4}\right) \geq \frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)} - \frac{\delta}{4} - C\varepsilon^{\alpha\beta} \geq \frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)} - \frac{\delta}{2}. \tag{4.61}$$

Altogether, by (4.61), (4.55), (4.49) and (4.48), we get that

$$\frac{\omega_\Omega^{X_0}(E)}{\omega_\Omega^{X_0}(\Delta_0)} \geq \frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)} - \frac{\delta}{2} - CK^{d\eta}\varepsilon^{\frac{\theta}{2}} - C\varepsilon^{\alpha\beta} \geq \frac{\mathcal{H}^d(E)}{\mathcal{H}^d(\Delta_0)} - \delta$$

if we choose β so that $d\eta\beta < \frac{\theta}{2}$ and ε so small that

$$CK^{d\eta}\varepsilon^{\frac{\theta}{2}} + C\varepsilon^{\alpha\beta} \leq \delta/2 \tag{4.62}$$

in the last inequality.

Finally, we check that all the conditions ((4.52), (4.57), (4.59), (4.60), (4.62)) on K and ε can be satisfied if we choose $0 < \beta < \min\left\{\frac{1}{2d}, \frac{\theta}{2d\eta}\right\}$ and $\varepsilon = \varepsilon(d, \mu_0, \delta)$ sufficiently small. This completes the proof of (4.37).

4.3. **We extend to U the matrix of coefficients.** In this subsection, we extend the matrix A of coefficients of $L = -\operatorname{div}(A\nabla)$ to U so that the extension satisfies the weak DKP condition on $G\times(0,\infty)$, with a constant smaller than $C\mathfrak{N}(A) \leq C\varepsilon$.

We shall use the setup in Section 4.2, that is, given $x_0 \in \Gamma$ and $r_0 > 0$, we have an $\varepsilon^{\frac{1}{2d}}$-Lipschitz graph G such that (4.33) and (4.34) hold.

We have seen in Section 4.2 that if we can find a μ_0-elliptic operator $\widetilde{L} = -\operatorname{div}(\widetilde{A}\nabla)$ on U that satisfies the weak DKP condition in U with constant $\mathfrak{N}(\widetilde{A}) \leq C\varepsilon$, and such that

$$\widetilde{A} = A \text{ on } \Omega \cap B(x_0, 2Kr_0) \subset U, \tag{4.63}$$

as in (4.38), then we can prove Theorem 4.23. We also required that (4.33) and (4.34) hold in the larger ball $B(x_0, 10Kr_0)$, and we shall use the extra space to build the desired extension $\widetilde{A}$ with a small enough norm $\mathfrak{N}(\widetilde{A})$. The construction will be similar to the usual proof of the Whitney extension theorem with partitions of unity.

Set $B = B(x_0, Kr_0)$ as before. Because of (4.63), we want to keep $\widetilde{A} = A$ on the closure of $\Omega \cap 2B$, and we are free to define $\widetilde{A}$ as we please on $V = \mathbb{R}^{d+1} \setminus \overline{\Omega \cap 2B}$. So we cover V by Whitney cubes, which we construct as in [Ste70, Chapter VI].

That is, let $\{Q_i\}$, $i \in I$, denote the collection of maximal dyadic cubes $Q_i \subset V$ such that, say,

$$\operatorname{diam}(Q_i) \leq 10^{-1} \operatorname{dist}(Q_i, \Omega \cap 2B). \tag{4.64}$$

The Q_i, $i \in I$, cover V, and they have overlap properties that we will recall when we need them. We can also construct a partition of unity $\{\chi_i\}_{i\in I}$ on V, adapted to $\{Q_i\}_{i\in I}$, such that $\sum_i \chi_i = \mathbb{1}_V$ and for each i, $\chi_i \in C_c^\infty(\frac{6}{5}Q_i)$, $0 \leq \chi_i \leq 1$, and $|\nabla\chi_i| \leq C \operatorname{diam}(Q_i)^{-1}$. See for instance [Ste70] for details. As usual in these instances, we shall keep $\widetilde{A} = A$ on $\overline{\Omega \cap 2B}$ (as suggested above) and set

$$\widetilde{A}(X) = \sum_{i\in I} A_i\, \chi_i(X) \quad \text{for } X \in V, \tag{4.65}$$

where we take

$$A_i = \fint_{W_i} A(X) dX, \tag{4.66}$$

where the average is componentwise, for some set $W_i \subset \Omega$ (so that A is defined on W_i) that we shall now choose carefully. Notice that A_i and $\widetilde{A}(x)$ are μ_0-elliptic matrices, because they are averages of μ_0-elliptic matrices. Set $d_i = \operatorname{dist}(Q_i, \Omega \cap 2B)$. We start with the case when

$$\operatorname{dist}(Q_i, \Gamma) \leq d_i \leq K r_0 \tag{4.67}$$

(and then we say that $i \in I_1$). Then we pick $\xi_i \in \Gamma$ such that $\operatorname{dist}(\xi_i, Q_i) = \operatorname{dist}(Q_i, \Gamma) \leq d_i$, denote by W_i the Whitney cube $W(\xi_i, d_i) \subset \Omega$ associated to Ω as in (1.2). When

$$d_i < \operatorname{dist}(Q_i, \Gamma) \text{ and } d_i \leq K r_0, \tag{4.68}$$

we say that $i \in I_2$ and we choose $W_i = Q_i$. Notice that Q_i does not meet Γ, so Q_i is either contained in Ω or in $\mathbb{R}^{d+1} \setminus \Omega$. The second case is impossible because $\operatorname{dist}(Q_i, \Omega \cap B) \leq d_i < \operatorname{dist}(Q_i, \Gamma)$, so $W_i \subset \Omega$ as needed. We are left with the case when

$$d_i > K r_0. \tag{4.69}$$

Then we say that $i \in I_3$, and we decide to take $W_i = W(x_0, K r_0)$, which is contained in $\Omega \cap B$ by definition. This will conveniently kill the variations of $\widetilde{A}$ far from B.

Notice that $W_i \subset \Omega \cap 5B$ for all $i \in I$. This is obvious by definition when $i \in I_3$ and because $d_i \leq K r_0$ otherwise, so that $Q_i \subset 3B$, which proves the case when $i \in I_2$. When $i \in I_1$, $\xi_i \in \frac{7}{2}B$, so $W_i = W(\xi_i, d_i) \subset \frac{9}{2}B$.

It will also be good to know that

$$\operatorname{dist}(X, Q_i) \leq 3d_i = 3 \operatorname{dist}(Q_i, \Omega \cap 2B) \quad \text{for } X \in W_i, \quad i \in I. \tag{4.70}$$

When $i \in I_1$, this is clear because $\operatorname{dist}(\xi_i, Q_i) \leq d_i$. When $i \in I_2$ this is trivial. Finally, when $i \in I_3$, set $D = \operatorname{dist}(Q_i, x_0)$, observe that $W_i = W(x_0, K r_0) \subset B$ lies in a $D + K r_0$-neighborhood of Q_i, while $D \leq \operatorname{dist}(Q_i, \Omega \cap 2B) + 2K r_0 = d_i + 2K r_0 \leq 3d_i$ by definition of I_3.

So we have a function $\widetilde{A}$ defined on the whole $\mathbb{R}^{d+1}_+$, and our next task is to evaluate the numbers $\overline{\alpha}(y, r)$ associated to (the restriction to U of) $\widetilde{A}$, defined for $y \in G$ and $r > 0$. Recall that they are defined by

$$\overline{\alpha}(y, r) = \inf_{A_0} \Big\{ \fint_{W(y,r)} |\widetilde{A} - A_0|^2 \Big\}^{1/2}, \tag{4.71}$$

where the infimum is taken over constant μ_0-elliptic matrices A_0, and we use the Whitney boxes

$$\overline{W}(y,r) = \big\{X \in U \cap B(y,r)\,;\, \operatorname{dist}(X,G) \geq r/2\big\}. \tag{4.72}$$

We claim that

$$\overline{\alpha}(y,r) = 0 \ \text{ when } r > 6Kr_0, \tag{4.73}$$

and

$$\overline{\alpha}(y,r) = 0 \ \text{ when } \operatorname{dist}(y,\Gamma \cap 2B) > 2r \text{ and } \operatorname{dist}(y,\Gamma \cap 2B) > 6Kr_0. \tag{4.74}$$

To see this, let us first prove that

$$\widetilde{A}(X) = A_{00} := \fint_{W(x_0,Kr_0)} A(Y)dY \ \text{ for } X \in \mathbb{R}^{d+1} \setminus 3B. \tag{4.75}$$

Indeed for any Whitney cube Q_i, $i \in I_1 \cup I_2$, we have that $\operatorname{dist}(Q_i, \Omega \cap 2B) = d_i \leq Kr_0$, and then $\frac{6}{5}Q_i \subset 3B$ because $\operatorname{diam}(Q_i) \leq d_i/10$. Now if $X \in \mathbb{R}^{d+1} \setminus 3B$, all the indices i such that $\chi_i(X) \neq 0$ lie in I_3, and since $A_i = A_{00}$ for $i \in I_3$, (4.65) yields $\widetilde{A}(X) = A_{00}$.

If $r > 6K_0$, we have that $\operatorname{dist}(X,G) > r/2 \geq 3Kr_0$ for $X \in \overline{W}(y,r)$, but then $\operatorname{dist}(X,x_0) \geq \operatorname{dist}(X,G) > 3Kr_0$ and $\widetilde{A}(X) = A_{00}$. That is, $\widetilde{A}$ is constant on $\overline{W}(y,r)$, and (4.73) follows. If $\operatorname{dist}(y,\Gamma \cap 2B) > 6Kr_0$ and $r < \operatorname{dist}(y,\Gamma \cap 2B)/2$, then for any $Z \in \overline{W}(y,r)$, $|Z - x_0| \geq |y - x_0| - |Z - y| \geq \operatorname{dist}(y,\Gamma \cap 2B)/2 > 3Kr_0$, which shows that $\overline{W}(y,r) \subset \mathbb{R}^{d+1} \setminus 3B$. Then (4.74) follows from (4.75).

Let $y \in G$ and $r > 0$ be given. We shall distinguish between cases, and the most interesting is probably when

$$\operatorname{dist}(y,\Gamma \cap 2B) \leq 2r \text{ and } r \leq 6Kr_0. \tag{4.76}$$

We claim that in this case

$$\overline{\alpha}(y,r) \leq C\gamma_A(x,80r) \text{ for every } x \in \Gamma \cap B(y,4r), \tag{4.77}$$

where $\gamma_A(x,r)$ is the variant of $\alpha_A(x,r)$, but defined with balls. That is,

$$\gamma_A(x,r) = \inf_{A_0} \Big\{ \fint_{\Omega \cap B(x,r)} |A - A_0|^2 \Big\}^{1/2}. \tag{4.78}$$

Let $x \in \Gamma \cap B(y,4r)$ be as in the claim. We will use the same constant matrix A_0 as in the definition of $\gamma_A(x,80r)$ to evaluate $\overline{\alpha}(y,r)$. First write

$$\overline{\alpha}(y,r)^2 \leq \big|\overline{W}(y,r)\big|^{-1} \left(\int_{\overline{W}(y,r) \cap \overline{\Omega \cap 2B}} |A - A_0|^2\, dX + \int_{\overline{W}(y,r) \cap V} \Big|\widetilde{A} - A_0\Big|^2 dX \right),$$

and notice that the first part is in order, because $\Omega \cap \overline{W}(y,r) \subset \Omega \cap B(y,r) \subset \Omega \cap B(x,80r)$ and the denominators $\big|\overline{W}(y,r)\big|$ and $|B(x,80r)|$ are comparable. We are left with the integral $J = \int_{\overline{W}(y,r) \cap V} |\widetilde{A} - A_0|^2$. Let $I(y,r)$ be the collection of indices such that Q_i meets $\overline{W}(y,r) \cap V$; then $J \leq \sum_{i \in I(y,r)} \big|\widetilde{A} - A_0\big|^2 dX$, and the definition (4.65) of $\widetilde{A}$ yields

$$J \leq \sum_{i \in I(y,r)} \int_{Q_i} \Big| \sum_j A_j\, \chi_j(X) - A_0 \Big|^2 dX = \sum_{i \in I(y,r)} \int_{Q_i} \Big| \sum_j (A_j - A_0)\, \chi_j(X) \Big|^2 dX$$

because $\sum_j \chi_j(X) = 1$. Observe that for $X \in Q_i$, we only sum over the set J_i of those $j \in I$ such that $\frac{6}{5}Q_j$ contains X and hence meets Q_i. There are only finitely many of such indices j, and they are all such that $\frac{1}{4}\operatorname{diam}(Q_j) \le \operatorname{diam}(Q_i) \le 4\operatorname{diam}(Q_j)$. We use Minkowski's inequality to get that

$$\Big(\int_{Q_i}\Big|\sum_j (A_j - A_0)\chi_j(X)\Big|^2 dX\Big)^{1/2} \le \sum_{j\in J_i}\Big(\int_{Q_i}|(A_j - A_0)\chi_j(X)|^2\,dX\Big)^{1/2}.$$

Since $\chi_j \le 1$ and by definition of A_j, the right-hand side is at most

$$\sum_{j\in J_i}|Q_i|^{1/2}|A_j - A_0| \le C\sum_{j\in J_i}\Big(\frac{|Q_j|}{|W_j|}\int_{W_j}|A(Z)-A_0|^2\,dZ\Big)^{1/2},$$

where in the last inequality we have used the fact that $|Q_j| \approx |Q_i|$ when $\frac{6}{5}Q_j$ meets Q_i. By construction $|W_j| \approx |Q_j|$ for $j \in I_1 \cup I_2$. Recall from (4.76) that $\operatorname{dist}(y, \Gamma\cap 2B) \le 2r \le 12Kr_0$, hence for $j \in I_i$ where $i \in I_3 \cap I(y,r)$, $\operatorname{diam}(Q_j) \approx CKr_0$. Therefore $|W_j| \approx |Q_j|$ holds for all $j \in I_i$, $i \in I(y,r)$, and so we may drop $\frac{|Q_j|}{|W_j|}$. Altogether

$$J \le C\sum_{i\in I(y,r)}\Big\{\sum_{j\in J_i}\Big(\int_{W_j}|A(Z)-A_0|^2\,dZ\Big)^{1/2}\Big\}^2 \le C\sum_{i\in I(y,r)}\sum_{j\in J_i}\int_{W_j}|A(Z)-A_0|^2\,dZ, \tag{4.79}$$

where the last inequality follows from Jensen's inequality. We claim that

$$W_j \subset B(x, 80r) \text{ for } j \in J_i. \tag{4.80}$$

To see this, let $Z \in W_j$ be given; by (4.70), $\operatorname{dist}(Z, Q_j) \le 3d_j$. But $d_j = \operatorname{dist}(Q_j, \Omega\cap 2B) \le 20\operatorname{diam}(Q_j)$ by definition of our Whitney cubes, so $\operatorname{dist}(Z, Q_j) \le 60\operatorname{diam}(Q_j)$. Next $\frac{6}{5}Q_j$ meets Q_i, so $\operatorname{dist}(Z, Q_i) \le 62\operatorname{diam}(Q_j) \le 244\operatorname{diam}(Q_i)$. Pick $X \in Q_i \cap \overline{W}(y,r)$; it exists because $i \in I(y,r)$; then $|X-x| \le |X-y|+|y-x| \le 5r$, and so $|Z-x| \le 244\operatorname{diam}(Q_i) + 5r$. But the definition of our cubes yields $\operatorname{diam}(Q_i) \le 10^{-1}\operatorname{dist}(Q_i, \Omega\cap 2B)$, and $\operatorname{dist}(Q_i, \Omega\cap 2B) \le \operatorname{dist}(Q_i, \Gamma\cap 2B) \le 3r$ by (4.76). So $244\operatorname{diam}(Q_i) \le 244\frac{3r}{10} < 75r$, and (4.80) follows.

Return to (4.79); by (4.80) and the fact that the W_j have finite overlap,

$$J \le C\int_{B(x,80r)\cap\Omega}|A(Z)-A_0|^2\,dZ \le Cr^{d+1}\gamma_A(x,80r)^2,$$

which completes the proof of our claim (4.77).

Let us now use the claim to do the part of our Carleson measure estimate that comes from

$$\Xi = \big\{(y,r) \in G\times(0,+\infty)\,;\, \operatorname{dist}(y,\Gamma\cap 2B) \le 2r\big\}.$$

We need to know that the set $\Gamma\cap B(y,r)$ of the claim is not too small. We use (4.76) to pick $y^* \in \Gamma\cap 2B$ such that $|y-y^*| \le \operatorname{dist}(y,\Gamma\cap 2B) \le 2r$; then by (4.5) for balls centered at y^* and since $B(y^*,2r) \subset B(y,4r) \subset B(y^*,6r)$,

$$c_d r^d \le (1-C\varepsilon^{1/d})c_d(2r)^d \le \mathcal{H}^d(B(y,4r)\cap\Gamma) \le (1+C\varepsilon^{1/d})c_d(6r)^d \le 7^d c_d r^d. \tag{4.81}$$

Let $y_0 \in G$ and $s_0 > 0$ be given, and estimate

$$J(\Xi) = \int_{y\in G\cap B(y_0,s_0)}\int_{0<r\le s_0}\mathbb{1}_\Xi(y,r)\,\overline{\alpha}(y,r)^2\frac{dr}{r}d\mathcal{H}^d(y).$$

Denote $\min\{s_0, 6Kr_0\}$ by $\widehat{s_0}$. We use (4.73), (4.77), (4.81), the Ahlfors regularity of Γ and G, and a Fubini argument, to get that

(4.82)
$$\begin{aligned} J(\Xi) &\leq C \int_{y\in G\cap B(y_0,s_0)} \int_{0<r\leq \widehat{s_0}} \Big\{ r^{-d} \int_{x\in\Gamma\cap B(y,4r)} \gamma_A(x,80r)^2 d\mathcal{H}^d(x) \Big\} \frac{dr}{r} d\mathcal{H}^d(y) \\ &\leq C \int_{x\in\Gamma\cap B(y_0,5s_0)} \int_{0<r\leq\widehat{s_0}} \gamma_A(x,80r)^2 \frac{dr}{r} d\mathcal{H}^d(x) \\ &= C \int_{x\in\Gamma\cap B(y_0,5s_0)} \int_{0<r\leq 80 s_0} \gamma_A(x,r)^2 \frac{dr}{r} d\mathcal{H}^d(x). \end{aligned}$$

In [DLM22], we proved that the weak DKP condition implies the stronger estimate that $\gamma(x,r)^2 \frac{dxdr}{r}$ is a Carleson measure in $\mathbb{R}^d \times (0,\infty)$. The same thing holds here, with almost the same proof, even though the geometry is a bit different because we work with Γ and Ω. That is, for any $x_0 \in \Gamma$ and $r_0 > 0$, we have that
$$\left\| \gamma_A(x,r)^2 \frac{d\mathcal{H}^d(x)dr}{r} \right\|_{\mathcal{C}(x_0,r_0)} \leq C \left\| \alpha_A(x,r)^2 \frac{d\mathcal{H}^d(x)dr}{r} \right\|_{\mathcal{C}(x_0,r_0)}.$$
Therefore, (4.82) now implies that
$$J(\Xi) \leq C s_0^d \mathfrak{N}(A) = C s_0^d \varepsilon \tag{4.83}$$
and concludes this part of the argument.

Return to our main Carleson estimate. We are left with the pairs (y,r) that belong to the set
$$\Xi' = \big\{(y,r) \in G\times(0,+\infty)\,;\, \operatorname{dist}(y,\Gamma\cap 2B) > 2r\big\}. \tag{4.84}$$
Let $(y,r)\in\Xi'$ be given, and assume additionally that $\operatorname{dist}(y,\Gamma\cap 2B) \leq 6Kr_0$. Set $d(y) = \operatorname{dist}(y,\Gamma\cap 2B)$, then $2r < d(y) \leq 6Kr_0$. Obviously
$$\operatorname{dist}(\overline{W}(y,r),\Gamma\cap 2B) \geq \operatorname{dist}(B(y,r),\Gamma\cap 2B) > d(y) - r \geq d(y)/2. \tag{4.85}$$

Let Q_i be any of the Whitney cubes such that $\frac{6}{5}Q_i$ meets $\overline{W}(y,r)$, and pick $X\in\overline{W}(y,r)\cap\frac{6}{5}Q_i$. Then by (4.85) $d(y)/2 \leq \operatorname{dist}(X,\Gamma\cap 2B) \leq \operatorname{dist}(Q_i,\Gamma\cap 2B) + \frac{6}{5}\operatorname{diam}(Q_i) \leq 12\operatorname{diam}(Q_i)$, so $\operatorname{diam}(Q_i) \geq d(y)/24$. Next pick any $X\in\overline{W}(y,r)$ and any constant matrix A_0, and observe that by (4.65)
$$\nabla\widetilde{A}(X) = \sum_{i\in I} A_i\,\nabla\chi_i(X) = \sum_{i\in I}[A_i - A_0]\,\nabla\chi_i(X) \tag{4.86}$$
because $\sum\nabla\chi_i = 0$. We sum only over the set $I(X)$ of indices i such that $X\in\frac{6}{5}Q_i$, and since $|\nabla\chi_i(X)| \leq C\operatorname{diam}(Q_i)^{-1} \leq Cd(y)^{-1}$ for $i\in I(X)$, we get that
$$|\nabla\widetilde{A}(X)| \leq Cd(y)^{-1}\sum_{i\in I(X)}|A_i - A_0|. \tag{4.87}$$

Recall that A_i is the average of A on W_i; we want to compare A_i to some average of A on a large ball centered on Γ, so we choose $y^*\in\Gamma\cap\overline{2B}$ such that $|y^*-y|\leq d(y)$ and we check that
$$W_i \subset B(y^*, 12d(y)). \tag{4.88}$$
Indeed, for $Z\in W_i$, (4.70) says that $\operatorname{dist}(Z,Q_i) \leq 3\operatorname{dist}(Q_i,\Omega\cap 2B) \leq 60\operatorname{diam}(Q_i)$, and all points of Q_i lie within $2\operatorname{diam}(Q_i)$ from $\overline{W}(y,r)$ (because $\frac{6}{5}Q_i$ meets $\overline{W}(y,r)$)

hence within $2\operatorname{diam}(Q_i)+r$ from y. So $|Z-y^*|\le|Z-y|+d(y)\le 60\operatorname{diam}(Q_i)+3d(y)/2$.

Now $\frac{6}{5}Q_i$ meets $\overline{W}(y,r)$, and if $X\in\overline{W}(y,r)\cap\frac{6}{5}Q_i$ we both have that $\operatorname{dist}(X,\Gamma\cap 2B)\le d(y)+r\le 3d(y)/2$ by (4.84), and $\operatorname{dist}(X,\Gamma\cap 2B)\ge\operatorname{dist}(\frac{6}{5}Q_i,\Gamma\cap 2B)\ge\operatorname{dist}(Q_i,\Gamma\cap 2B)-\frac{1}{5}\operatorname{diam}(Q_i)\ge 9\operatorname{diam}(Q_i)$, so $60\operatorname{diam}(Q_i)\le 10d(y)$ and (4.88) follows. Therefore, choosing $A_0=\fint_{\Omega\cap B(y^*,12d(y))}A(Y)dY$ (which does not depend on $i\in I(X)$, and not even on X),

$$\begin{aligned}|A_i-A_0|&\le|W_i|^{-1}\int_{W_i}|A(Y)-A_0|dY\le|W_i|^{-1}\int_{B(y^*,12d(y))}|A(Y)-A_0|dY\\&\le C|W_i|^{-1}|B(y^*,12d(y))|\fint_{B(y^*,12d(y))}|A(Y)-A_0|dY.\end{aligned}$$

Observe that $|W_i|^{-1}|B(y^*,12d(y))|\le C$. For $i\in I_1\cup I_2$, this follows from $\operatorname{diam}(Q_i)\ge d(y)/24$. For $i\in I_3$, this is because $d(y)\le 6Kr_0$. Hence,

$$|A_i-A_0|\le Cd(y)^{-1}\gamma_A(y^*,12d(y)) \tag{4.89}$$

by Cauchy-Schwarz. We notice that all the W_i, $i\in I(Z)$, stay at distance at least $C^{-1}\operatorname{diam}(Q_i)\ge C^{-1}d(y)$ from Γ, so we can replace $\gamma(y^*,12d(y))$ with $\widehat{\alpha}(y^*,d(y))$, the larger variant of $\alpha(x,r)$, defined by

$$\widehat{\alpha}(x,r)=\Big\{\inf_{A_0}|\widehat{W}(x,r)|^{-1}\int_{\widehat{W}(x,r)}|A-A_0|^2\Big\}^{1/2},$$

with $\widehat{W}(x,r)=\big\{X\in\Omega\cap B(x,20r)\,;\,\operatorname{dist}(X,\Gamma)\ge C^{-1}r\big\}$. Altogether,

$$|\nabla\widetilde{A}(X)|\le Cd(y)^{-1}\sum_{i\in I(X)}|A_i-A_0|\le Cd(y)^{-1}\widehat{\alpha}(y^*,d(y)). \tag{4.90}$$

Because of this, the oscillation of $\widetilde{A}(X)$ on $\overline{W}(y,r)$ is at most $Cd(y)^{-1}r\widehat{\alpha}(y^*,d(y))$, and hence $\overline{\alpha}(y,r)\le Cd(y)^{-1}r\widehat{\alpha}(y^*,d(y))$. By this and (4.74),

$$\begin{aligned}J(\Xi'):=&\int_{y\in G\cap B(y_0,s_0)}\int_{0<r\le s_0}\mathbb{1}_{\Xi'}(y,r)\,\mathbb{1}_{d(y)\le 6Kr_0}(y)\,\overline{\alpha}(y,r)^2\frac{dr}{r}d\mathcal{H}^d(y)\\&\le C\int_{y\in G\cap B(y_0,s_0)}\int_{0<r\le d(y)/2}\widehat{\alpha}(y^*,d(y))\,d(y)^{-2}r\,dr\,d\mathcal{H}^d(y)\\&\le C\int_{y\in G\cap B(y_0,s_0)}\widehat{\alpha}(y^*,d(y))^2\,d\mathcal{H}^d(y)\\&\le Cs_0^d\sup_{y\in G\cap B(y_0,s_0)}\widehat{\alpha}(y^*,d(y))^2.\end{aligned} \tag{4.91}$$

Now the weak DKP condition implies that $\widehat{\alpha}(z,r)^2\le C\mathfrak{N}(A)\le C\varepsilon$ for all $z\in\Gamma$ and $r>0$, where the Carleson norm $\mathfrak{N}(A)$ is as in (1.11). The verification is also easy (and is done in [DLM22]). Thus we also get that $J(\Xi')\le Cs_0^d\varepsilon$, and since (4.83) provides the estimate for the other piece $J(\Xi')$, we finally get that the extension $\widetilde{A}$ satisfies the weak DKP condition on $G\times(0,\infty)$, with a norm at most $C\varepsilon$.

5. Small and vanishing A_∞ and BMO

At this point, our proof of Theorems 4.23 and 1.19 is complete. However, since we like to work with chord-arc surface with small constant, we want to explain a few rather simple relations between A_∞ weights and BMO estimates for their logarithm. We want to do this in $(\Gamma, d\sigma)$, where either $\Gamma \subset \mathbb{R}^{d+1}$ is a chord-arc surface with small constant and $\sigma = \mathcal{H}^d_{|\Gamma}$, or $\Gamma = \mathbb{R}^d$ and σ is the Lebesgue measure on Γ. Because of the first case, we define BMO with balls.

We first define the mean oscillation on a set E such that $0 < \sigma(E) < +\infty$, of the function $f \in L^1(E, d\sigma)$ by

$$mo(f,E) = \sigma(E)^{-1} \int_E \left| f(y) - \Big(\fint_E f d\sigma \Big) \right| d\sigma(y). \tag{5.1}$$

We say that $f \in BMO(\Gamma)$ when $f \in L^1_{loc}(\Gamma, d\sigma)$ and

$$\|f\|_{BMO} := \sup_{r>0} \sup_{x\in\Gamma} mo(f, \Delta(x,r)) < +\infty, \tag{5.2}$$

and that $f \in VMO(\Gamma)$ when $f \in BMO(\Gamma)$ and in addition

$$\lim_{r\to 0_+} \sup_{x\in\Gamma} mo(f, \Delta(x,r)) = 0. \tag{5.3}$$

It is easy to see that in the case of $\mathbb{R}^d$, we can replace the collection of balls $\Delta(x,r)$ with the collection of cubes Q with faces parallel to the axes, and get the same space BMO, with a slightly different but equivalent semi-norm

$$\|f\|_* := \sup_Q mo(f,Q); \tag{5.4}$$

also $VMO(\Gamma)$ could be defined with cubes. Finally, we can localize: if $\Delta = \Delta(x_0, r_0)$ is a surface ball in Γ, we say that $f \in BMO(\Delta)$ when $f \in L^1(\Delta, d\sigma)$ and

$$\|f\|_{BMO(\Delta)} := \sup_{(x,r)\in\Delta\times(0,r_0), \Delta(x,r)\subset\Delta} mo(f, \Delta(x,r)) < +\infty. \tag{5.5}$$

This may be a little ugly near the boundary, but often we only care about the restriction of f to $\Delta(x_0, r_0/2)$ anyway. In $\mathbb{R}^d$, the version with cubes is a little better. If $Q_0 \subset \mathbb{R}^d$ is a cube (with faces parallel to the axes, we won't repeat), we say that $f \in BMO_*(Q_0)$ when $f \in L^1(Q_0, d\sigma)$ and

$$\|f\|_{BMO_*(Q_0)} := \sup_{Q\subset Q_0} mo(f,Q) < +\infty, \tag{5.6}$$

where the supremum is over the cubes $Q \subset Q_0$.

With all these definitions at hand, we are ready to state a first result about A_∞ absolute continuity with vanishing constant and VMO.

Lemma 5.7. *Let Γ be a chord-arc surface (or just an Ahlfors regular set of dimension d) and let σ be an Ahlfors regular measure on Γ. Let ω be a positive locally finite Borel measure on Γ, suppose that it is absolutely continuous with respect to σ, and denote by $k = \frac{d\omega}{d\sigma}$ its Radon-Nikodym density. Assume $\log k \in VMO(\Gamma)$. Then for any $\delta > 0$, there exists $\gamma > 0$ such that for any surface ball $\Delta = \Delta(x,r) \subset \Gamma$ with $0 < r \leq \gamma$, and any Borel set $E \subset \Delta$,*

$$\left| \frac{\omega(E)}{\omega(\Delta)} - \frac{\sigma(E)}{\sigma(\Delta)} \right| < \delta. \tag{5.8}$$

There is a similar "small constant" statement that says that for each $\delta > 0$, we can find $\varepsilon = \varepsilon(\delta, \Gamma) > 0$ such that if $\|\log k\|_{BMO} \leq \varepsilon$, then $\omega \in A_\infty(\sigma, \delta)$. But we leave the details to the reader.

For the comfort of the reader, we are only going to prove this in the special case when $\Gamma = \mathbb{R}^d$ and σ is the Lebesgue measure, but in a way that is easy to follow in the general case, with simple modifications, by observing that we can cut Γ into collections of "M. Christ dyadic cubes" that have roughly the same properties as the dyadic cubes in $\mathbb{R}^d$. We don't even need the slightly unpleasant "small boundary property", just size and nesting properties of cubes are enough.

We start the proof with the simplest version of the John and Nirenberg theorem, with cubes in $\mathbb{R}^d$.

Lemma 5.9. *There exist constants $c = c_d > 0$ and $C = C_d \geq 1$ that depend only on d, such that if Q_0 is a cube and $f \in BMO_*(Q_0)$, then*

$$\int_{Q_0} \exp\left(c\|f\|_{BMO_*(Q_0)}^{-1} \left| f(x) - \left(\fint_{Q_0} f d\sigma \right) \right| \right) d\sigma(x) \leq C\sigma(Q_0). \tag{5.10}$$

In fact, the (very standard) proof with a stopping time on dyadic cubes only requires a control on $mo(f, Q)$ when Q is a dyadic subcube of Q_0 (i.e., obtained from Q_0 by the usual dyadic partitions).

Proof of Lemma 5.7. Let ω and $k = \frac{d\omega}{d\sigma}$ be as in the statement, and let $\delta > 0$ be given. Let $\varepsilon > 0$ be given, and choose $r_0 > 0$ so small that $mo(\log k, \Delta(x, r)) < \varepsilon$ for every surface ball $\Delta(x, r)$ such that $0 < r \leq r_0$. By standard manipulations of covering cubes with balls of roughly the same sizes, we also see that $\| \log k\|_{BMO_*(Q_0)} \leq C\varepsilon$ for every cube Q_0 such that $\mathrm{diam}(Q_0) \leq r_0$.

We choose $\gamma = d^{-1/2} r_0$, consider a surface ball $\Delta = \Delta(x, r)$ such that $r \leq \gamma$, and try to prove (5.8) for any Borel set $E \subset \Delta$. Call Q_0 the smallest cube that contains Δ; we chose γ so that $\mathrm{diam}(Q_0) \leq r_0$, and so $\| \log(k)\|_{BMO_*(Q_0)} \leq C\varepsilon$. This allows us to apply Lemma 5.9 and get

$$\int_{Q_0} \exp\left(a\varepsilon^{-1} \left| \log(k(x)) - \left(\fint_{Q_0} \log(k) d\sigma \right) \right| \right) d\sigma(x) \leq C, \tag{5.11}$$

with a new constant $a = C^{-1}c > 0$ that depends only on d. Since (5.8) does not change when we multiply ω by a positive constant, we may assume that $\fint_{Q_0} \log(k) d\sigma = 0$, which will simplify our computations slightly. We write

$$\omega(Q) = \int_Q k(x) d\sigma(x) \ \text{ and } \omega(E) = \int_E k(x) d\sigma(x) \tag{5.12}$$

and decide to cut each of these integrals in three, corresponding to the three regions

$$\begin{aligned} Q_1 &= \{x \in Q_0\,;\, \log(k(x)) \leq -A\varepsilon\}, \\ Q_2 &= \{x \in Q_0\,;\, |\log(k(x))| \leq A\varepsilon\}, \\ Q_3 &= \{x \in Q_0\,;\, \log(k(x)) \geq A\varepsilon\}, \end{aligned} \tag{5.13}$$

where the large constant A will be chosen soon. Notice that $a\varepsilon^{-1}|\log(k)| \geq aA$ on $Q_1 \cup Q_3$, so by Chebyshev

$$\omega(Q_1) \leq \sigma(Q_1) \leq \sigma(Q_1) + \sigma(Q_3) \leq Ce^{-aA}\sigma(Q_0). \tag{5.14}$$

Similarly, we cut Q_3 into the regions $D_j = \big\{x \in Q_3\,;\, j + A\varepsilon \le \log(k(x)) < j+1+A\varepsilon\big\}$ and get that

$$\begin{aligned}(5.15)\quad \omega(Q_3) &\le \int_{Q_3} k(x)d\sigma(x) \le \sum_{j\ge 0}\int_{D_j} k(x)d\sigma(x)\\ &\le \sum_{j\ge 0} e^{j+1+A\varepsilon}\exp\big(-a\varepsilon^{-1}(j+A\varepsilon)\big)\int_{D_j}\exp\big(a\varepsilon^{-1}\log(k(x))\big)d\sigma(x)\\ &\le C\sigma(Q_0)e^{1+A\varepsilon}e^{-aA}\sum_{j\ge 0} e^j e^{-a\varepsilon^{-1}j}\\ &\le C\sigma(Q_0)e^{1+A\varepsilon}e^{-aA} \le Ce^{-aA/2}\end{aligned}$$

if we choose $A \ge 2a^{-1}$ and $\varepsilon \le A^{-1}$. Now let F be a Borel subset of Q_0; we think about E or Δ. Then

(5.16)

$$\begin{aligned}|\omega(F)-\sigma(F)| &\le \omega(F\cap(Q_1\cup Q_3)) + \sigma(F\cap(Q_1\cup Q_3)) + \int_{F\cap Q_2}|k(x)-1|d\sigma\\ &\le \omega(Q_1\cup Q_3)+\sigma(Q_1\cup Q_3)+\int_{Q_2}\big|e^{A\varepsilon}-1\big|d\sigma \le (Ce^{-aA/2}+2A\varepsilon)\sigma(Q_0)\end{aligned}$$

by (5.13)-(5.15). By definition of Q_0, we also have that $\sigma(\Delta) \ge \alpha\sigma(Q_0)$ for some constant $\alpha > 0$ that depends only on d. Then we choose A, and then ε, so that $Ce^{-aA/2} + 2A\varepsilon \le 10^{-2}\alpha\delta$ in (5.16). Then

$$(5.17)\qquad |\omega(\Delta)-\sigma(\Delta)| \le 10^{-2}\alpha\delta\sigma(Q_0) \le 10^{-3}\delta\sigma(\Delta)$$

and similarly

$$(5.18)\qquad |\omega(E)-\sigma(E| \le 10^{-2}\delta\sigma(\Delta),$$

from which it is easy to see that $\left|\frac{\omega(E)}{\omega(\Delta)} - \frac{\sigma(E)}{\sigma(\Delta)}\right| < \delta$ as in (5.8). □

There is a converse to Lemma 5.7 and its small constant variant, which we discuss now.

Lemma 5.19. *There is a constant $\varepsilon > 0$ that depends only on d, and for each $\delta > 0$, a constant $\varepsilon_1 = \varepsilon_1(\delta, d)$ with the following property. Let $\Gamma \in CASSC(\varepsilon)$ be a chord-arc surface with small enough constant in $\mathbb{R}^{d+1}$ (as in Definition 1.13), and let ω be another measure on Γ such that $\omega \in A_\infty(\sigma, \varepsilon_1)$ (see Definition 1.15). Then ω is absolutely continuous with respect to σ, and its density $k = \frac{d\omega}{d\sigma}$ satisfies $\|\log(k)\|_{BMO} \le \delta$.*

Proof. Let $\Gamma \in CASSC(\varepsilon)$ and $\omega \in A_\infty(\sigma, \varepsilon_1)$ satisfy the assumption. It is a general fact about A_∞ that then ω is absolutely continuous with respect to σ, so k is well defined and locally integrable. Let $a > 0$ be small, to be chosen later (depending on d). We want to show that (if ε and ε_1 are small enough), for every surface ball Δ, there is a constant C_Δ such that

$$(5.20)\qquad \sigma\big(\big\{x \in \Delta\,;\, |\log(k(x)) - c_\Delta| \ge \delta\big\}\big) \le a\sigma(\Delta).$$

In fact, the simplest is to take $c_\Delta = \log\left(\frac{\omega(\Delta)}{\sigma(\Delta)}\right)$. Write $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$, with

$$\begin{aligned}(5.21)\quad \Delta_1 &= \{x \in \Delta\,;\, k(x) < e^{c_\Delta - \delta}\},\\ \Delta_2 &= \{x \in \Delta\,;\, e^{c_\Delta - \delta} \le k(x) \le e^{c_\Delta + \delta}\},\\ \Delta_3 &= \{x \in \Delta\,;\, k(x) > e^{c_\Delta + \delta}\};\end{aligned}$$

we want to show that $\sigma(\Delta_1)+\sigma(\Delta_3) \le \sigma(\Delta)/100$. We take $E = \Delta_1$ in the definition (1.16); we get that $\left|\frac{\omega(\Delta_1)}{\omega(\Delta)} - \frac{\sigma(\Delta_1)}{\sigma(\Delta)}\right| < \varepsilon_1$ or, since $\omega(\Delta) = e^{c_\Delta}\sigma(\Delta)$,

$$|e^{-c_\Delta}\omega(\Delta_1) - \sigma(\Delta_1)| \le \varepsilon_1 \sigma(\Delta). \tag{5.22}$$

But $\omega(\Delta_1) = \int_{\Delta_1} k d\sigma \le e^{c_\Delta - \delta}\sigma(\Delta)$ so $\sigma(\Delta_1) - e^{-c_\Delta}\omega(\Delta_1) \ge (1-e^{-\delta})\sigma(\Delta_1)$, so the comparison with (5.22) yields $\sigma(\Delta_1) \le \varepsilon_1(1-e^{-\delta})^{-1}\sigma(\Delta) < a\sigma(\Delta)/2$ if ε_1 is chosen small enough. Similarly,

$$|e^{-c_\Delta}\omega(\Delta_3) - \sigma(\Delta_3)| \le \varepsilon_1 \sigma(\Delta) \tag{5.23}$$

by the proof of (5.22), and the definition yields $\omega(\Delta_3) = \int_{\Delta_3} k d\sigma \ge e^{c_\Delta + \delta}\sigma(\Delta)$, so that $e^{-c_\Delta}\omega(\Delta_3) - \sigma(\Delta_3) \ge (e^{-\delta} - 1)\sigma(\Delta_3)$ and the comparison yields $\sigma(\Delta_3) \le \varepsilon_1(e^\delta - 1)^{-1}\sigma(\Delta) < a\sigma(\Delta)/2$. This proves (5.20).

Now we claim that a well known result proved by F. John [Joh65], improved by J. O. Strömberg [Str79], allows one to deduce from (5.20) that $\log(k) \in BMO$, with a norm less than $C\delta$. Here C depends on d, and of course the difference between $C\delta$ and δ could be easily compensated by taking ε_1 smaller. The advantage of the result of John and Strömberg is that it is directly valid for a doubling measure. For the sake of the reader that would know the usual proof of the John-Nirenberg theorem (with simple stopping times), let us say two words about how the underlying ideas could be applied. In $\mathbb{R}^d$, if you have (5.20) for balls, you also get it for cubes Q, by the usual trick of using (5.20) for the smallest ball containing Q, and at the price of starting with the smaller $\widetilde{a} = a\sigma(Q)/\sigma(\Delta)$. Then, if a is small enough, we deduce from (5.20) that if Q and R are cubes such that $Q \subset R$ and $\operatorname{diam}(Q) \ge \operatorname{diam}(R)/2$, then $|c_Q - c_R| \le 2\delta$. Then the proof of the John-Nirenberg theorem can then be applied, and this yields $\|\log(k)\|_{BMO} \le C\delta$. So the result is valid when $\Gamma \in CASSC(\varepsilon)$, ε small enough (so that σ is doubling), with the usual fake dyadic cubes on Γ. We need to take a even smaller, so that in particular we can control $|c_Q - c_R|$ for fake cubes, when Q is a child of R, but the proof goes through. □

Corollary 5.24. *In Theorem 1.17, we can replace the conclusion that $\omega^\infty \in A_\infty(\sigma, \delta)$ with the conclusion that ω^∞ is absolutely continuous with respect to σ, with a density $k = \frac{d\omega}{d\sigma}$ such that $\|\log(k)\|_{BMO} \le \delta$.*

In Theorem 1.19, we can replace the conclusion by the fact that for each surface ball $\Delta = B(x,r) \cap \Gamma$ such that $\Delta \subset B(X, \kappa \operatorname{dist}(X,\Gamma))$ and $0 < r \le \tau \operatorname{dist}(X,\Gamma)$, ω^X is absolutely continuous with respect to σ on Δ, and the density $k = \frac{d\omega^X}{d\sigma}$ is such that $\|\log(k)\|_{BMO(\Delta)} \le \delta$.

Proof. The first part follows at once from Theorem 1.17 and Lemma 5.19. The second part follows Theorem 1.19 (applied with a larger κ for security) and the proof of Lemma 5.19. □

Appendix A. Construction of an example in $\mathbb{R}^{d+1}_+$

In this section, we give a different proof of the special case of Theorem 1.17 where $\Omega = \mathbb{R}^{d+1}_+$. We came up with this proof in the period when the result of [BES21] for small weights was not yet available. As was said in the introduction, we shall assume that the theorem fails, and get the desired contradiction by constructing an operator L that satisfies the assumption of the main theorem of [BTZ23], but not the conclusion. We recall the main theorem of [BTZ23].

Lemma A.1 ([BTZ23, Theorem 1.2])**.** *Let $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator on $\mathbb{R}^{d+1}_+$, whose coefficient matrix A satisfies the vanishing weak DKP condition. If k^∞_L is the elliptic kernel associated to L (in $\mathbb{R}^{d+1}_+$) with pole at infinity then $\log k^\infty_L \in VMO(\mathbb{R}^d)$ (see (5.3) for the definition of VMO).*

The authors of [BTZ23] also show the big constant version of the result above.

Lemma A.2 ([BTZ23, Theorem 5.2])**.** *Let $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator on $\mathbb{R}^{d+1}_+$, whose coefficient matrix A satisfies the weak DKP condition. Let ω^∞_L be the elliptic measure with pole at infinity. Then $\omega^\infty_L \ll \mathcal{L}^d$, where $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$, $\omega^\infty_L \in A_\infty(dx)$, and $k^\infty_L(y) := \frac{d\omega^\infty_L}{dx}(y)$ has the property that $\log k^\infty_L \in BMO(\mathbb{R}^d)$. The implicit constants in the statements $\omega^\infty_L \in A_\infty(dx)$ and $\log k \in BMO(\mathbb{R}^d)$ are each bounded by a constant depending on d, the ellipticity constant μ_0, and $\left\|\alpha_A(x,r)^2\,\frac{dx\,dr}{r}\right\|_{\mathcal{C}}$.*

In this section, $\Gamma = \mathbb{R}^d \subset \mathbb{R}^{d+1}$, $\Omega = \mathbb{R}^{d+1}_+$, and we study operators $L = -\operatorname{div}(A\nabla)$ that are μ_0-elliptic. Points of $\mathbb{R}^{d+1}_+$ will be written $X = (x,t)$, with $x \in \mathbb{R}^d$ and $t > 0$, and surface balls will be denoted by $\Delta(x,r) = B_r(x) \cap \{t = 0\} \subset \mathbb{R}^d$, with $x \in \mathbb{R}^d$ and $r > 0$.

If $\Delta = \Delta(x,r)$ is a surface ball, we may also denote by T_Δ the Carleson box $T(x,r) = B_r(x) \cap \mathbb{R}^{d+1}_+$ over $\Delta(x,r)$, but our notation will be slightly simpler if we switch to

$$W(x,r) := \Delta(x,r) \times \left(\frac{r}{2}, r\right] \subset \mathbb{R}^{d+1}_+ \tag{A.3}$$

for the Whitney cube. This makes no real difference (compared to (1.2)), and in particular the definition of Carleson measures and the weak DKP condition do not change in any significant way.

So we start the proof of Theorem 1.17 in $\mathbb{R}^{d+1}_+$ by contradiction: we assume the theorem to be false, so we can find an ellipticity constant μ_0, a small constant $\delta_0 > 0$, and for each integer $j \geq 0$, a μ_0-elliptic matrix A_j such that

$$\mathfrak{N}(A_j) \leq \varepsilon_j^2, \quad \text{where } \varepsilon_j = 2^{-j}, \tag{A.4}$$

but for which the conclusion of the theorem fails for $L_j = -\operatorname{div}(A_j\nabla)$. That is, $\omega^\infty_{L_j} \notin A_\infty(\sigma, \delta_0)$, where by habit we still denote by σ the Lebesgue measure on $\mathbb{R}^d$. Notice that changing the normalization point for $\omega^\infty_{L_j}$ would only multiply $\omega^\infty_{L_j}$ by a constant, and not affect the fact that $\omega^\infty_{L_j} \notin A_\infty(\sigma, \delta_0)$.

This means that for each j we can find a surface ball $\Delta_j \subset \mathbb{R}^d$ and a Borel set $E_j \subset \Delta_j$ such that

$$\left|\frac{\omega^\infty_{L_j}(E_j)}{\omega^\infty_{L_j}(\Delta_j)} - \frac{\sigma(E_j)}{\sigma(\Delta_j)}\right| \geq \delta_0. \tag{A.5}$$

We want to use this to derive a contradiction, but before we start for good, it will be good to record how our various objects behave under dilations that preserve the boundary.

A.1. **Some scaling properties.** In this subsection we are only interested in the changes of variable induced by the linear transformations

$$T_\lambda^z : \mathbb{R}_+^{d+1} \to \mathbb{R}_+^{d+1}, \quad X \mapsto \lambda X + (z,0) = \lambda X + Z, \tag{A.6}$$

where $\lambda > 0$, $z \in \mathbb{R}^d$, and we set $Z = (z,0) \in \mathbb{R}^{d+1}$ to prevent confusion between $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$. In the following computations, λ and z are fixed, we are given an elliptic matrix valued function A, and we consider

$$\widetilde{A}(X) = A(T_\lambda^z(X)). \tag{A.7}$$

Lemma A.8. *With the notation above,*

$$\alpha_{\widetilde{A}}(x,r) = \alpha_A(T_\lambda^z(x,r)) \qquad \text{for any } x \in \mathbb{R}^d, r > 0, \tag{A.9}$$

and

$$\mathfrak{N}(\widetilde{A}) = \mathfrak{N}(A). \tag{A.10}$$

Proof. Both (A.9) and (A.10) follow from direct computations; we omit the proof. □

Lemma A.11. *Now let $\widetilde{L} = -\operatorname{div} \widetilde{A}\nabla$. Then for any Borel set $E \in \mathbb{R}^d$,*

$$\omega_L^\infty(E) = \lambda^{d-1} U(z,\lambda)\, \omega_{\widetilde{L}}^\infty\left(\frac{E-(z,0)}{\lambda}\right) = \lambda^{d-1} U(z,\lambda)\, \omega_{\widetilde{L}}^\infty((T_\lambda^z)^{-1}(E)), \tag{A.12}$$

where U is the Green function with pole at infinity associated to L (and normalized at the point $Z_0 = (0,1)$).

Proof. By (2.1), one can show that

$$G_{\widetilde{L}}(X,Y) = \lambda^{1-d} G_L(T_\lambda^z(X), T_\lambda^z(Y)) \qquad \text{for } X, Y \in \mathbb{R}_+^{d+1}. \tag{A.13}$$

Denote by $\widetilde{U}$ the Green function with pole at infinity for $\widetilde{L}$. Then by (2.10) and (A.13),

$$\widetilde{U}(Y) = \lim_{k\to\infty} \frac{G_{\widetilde{L}}(X_k, Y)}{G_{\widetilde{L}}(X_k, Z_0)} = \lim_{k\to\infty} \frac{G_L(T_\lambda^z(X_k), T_\lambda^z(Y))}{G_L(T_\lambda^z(X_k), T_\lambda^z(Z_0))},$$

where $\{X_k\}$ is some sequence of points in $\mathbb{R}_+^{d+1}$ such that $|X_k| \to \infty$ as $k \to \infty$. On the other hand, we can write

$$\begin{aligned} U(Y) = \lim_{k\to\infty} \frac{G_L(T_\lambda^z(X_k), Y)}{G_L(T_\lambda^z(X_k), Z_0)} &= \lim_{k\to\infty} \frac{G_L(T_\lambda^z(X_k), Y)}{G_L(T_\lambda^z(X_k), T_\lambda^z(Z_0))} \frac{G_L(T_\lambda^z(X_k), T_\lambda^z(Z_0))}{G_L(T_\lambda^z(X_k), Z_0)} \\ &= \lim_{k\to\infty} \frac{G_L(T_\lambda^z(X_k), Y)}{G_L(T_\lambda^z(X_k), T_\lambda^z(Z_0))} U(T_\lambda^z(Z_0)). \end{aligned}$$

Comparing it to $\widetilde{U}(Y)$, we obtain that

$$U(T_\lambda^z(Y)) = \widetilde{U}(Y) U(T_\lambda^z(Z_0)) \qquad \text{for any } Y \in \mathbb{R}_+^{d+1}. \tag{A.14}$$

Recall that the Riesz formula (2.11) for $\widetilde{L}$ asserts that

$$\int_{\partial\Omega} \widetilde{f}(y)\, d\omega_{\widetilde{L}}^\infty(y) = -\iint \widetilde{A}^\top(Y) \nabla_Y \widetilde{U}(Y) \cdot \nabla_Y \widetilde{F}(Y)\, dY \tag{A.15}$$

for $\widetilde{f} \in C_c^\infty(\mathbb{R}^d)$ and $\widetilde{F} \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\widetilde{F}(y,0) = \widetilde{f}(y)$. Set $F(T_\lambda^z(Y)) = \widetilde{F}(Y)$, $f(y) = F(y,0)$. Then $f \in C_c^\infty(\mathbb{R}^d)$ and $F \in C_c^\infty(\mathbb{R}^{d+1})$. Using (A.14), the change of variable $Y' = T_\lambda^z(Y)$, and then the Riesz formula for L, the right-hand side of (A.15) is equal to

$$- \lambda^{1-d} U(T_\lambda^z(Z_0))^{-1} \iint A^\top(Y')\nabla U(Y') \cdot \nabla F(Y')\, dY' \\ = \lambda^{1-d} U(z,\lambda)^{-1} \int_{\mathbb{R}^d} f(y) d\omega_L^\infty(y),$$

where we have also used the fact that $U(T_\lambda^z(Z_0)) = U(z,\lambda)$. Since $\widetilde{f}(y) = \widetilde{F}(y,0) = F(T_\lambda^z(y,0)) = f(\lambda y + z)$, (A.15) asserts that

$$\int_{\mathbb{R}^d} f(\lambda y + z) d\omega_{\widetilde{L}}^\infty(y) = \lambda^{1-d} U(z,\lambda)^{-1} \int_{\mathbb{R}^d} f(y) d\omega_L^\infty(y)$$

for $f \in C_c^\infty(\mathbb{R}^d)$. By a limiting argument, this implies that for any Borel set $E \subset \mathbb{R}^d$, and if we set $E' = \lambda^{-1}(E - (z,0)) = (T_\lambda^z)^{-1}(E)$,

$$\omega_{\widetilde{L}}^\infty(E) = \int_{\mathbb{R}^d} \mathbb{1}_E(\lambda y + z) d\omega_{\widetilde{L}}^\infty(y) = \lambda^{1-d} U(z,\lambda)^{-1} \int_{\mathbb{R}^d} \mathbb{1}_{E'}(y) d\omega_L^\infty(y) \\ = \lambda^{1-d} U(z,\lambda)^{-1} \omega_L^\infty(E'),$$

as desired. □

A.2. **We can modify A_j away from a band.** Return to the bad operator $L_j = -\operatorname{div}(A_j \nabla)$ and its bad set $E_j \subset \Delta_j$. We claim that we can assume that for all j, $\Delta_j = \Delta_0$, the unit ball in $\mathbb{R}^d$. Write $\Delta_j = \Delta(z,\lambda)$, and notice that $T_\lambda^z(\Delta_0) = \Delta_j$. Then replace A_j with the function $\widetilde{A}_j = A \circ T_\lambda^z$ given by (A.7). By Lemma A.8, $\mathfrak{N}(\widetilde{A}_j) = \mathfrak{N}(A_j)$. In addition, set $\widetilde{E}_j = (T_\lambda^z)_{-1}(E_j) \subset (T_\lambda^z)_{-1}(\Delta_j) = \Delta_0$; then by Lemma A.11,

$$\frac{\omega_{L_j}^\infty(E_j)}{\omega_{L_j}^\infty(\Delta_j)} = \frac{\omega_{\widetilde{L}_j}^\infty(\widetilde{E}_j)}{\omega_{\widetilde{L}_j}^\infty(\Delta_0)},$$

where the extra factors $\lambda^{d-1}U(z,\lambda)$ are the same on the numerator and denominator and cancel. So let us assume that $\Delta_j = \Delta_0$ (and that (A.4) and (A.5) hold).

Our goal is to use the A_j to construct a matrix-valued function A^* that satisfies the vanishing weak DKP condition and keeps the bad properties of each A_j. Lemma A.16 will allow us to modify A_j outside of some strip, but essentially maintain (A.5).

Lemma A.16. *There exist $0 < \rho_j < 1$ and $M_j > 1$ such that if A_j^* is any μ_0-elliptic matrix that satisfies $A_j^*(x,t) = A_j(x,t)$ for $t \in (\rho_j, M_j)$, and $\mathfrak{N}(A_j^*) \le C_0$ for some $C_0 < \infty$, then*

$$\left| \frac{\omega_{L_j*}^\infty(E_j)}{\omega_{L_j*}^\infty(\Delta_0)} - \frac{\sigma(E_j)}{\sigma(\Delta_0)} \right| \ge \frac{\delta_0}{2}, \tag{A.17}$$

where $\omega_{L_j*}^\infty$ corresponds to the operator $L_j* = -\operatorname{div} A_j^*\nabla$.

We could also have allowed modifications of A_j for $|x|$ very large, but since this is not necessary for our construction, we will not do that.

Proof. Notice that we allow ρ_j and M_j to depend wildly on j. We prove the lemma by contradiction. Fix j and suppose that the statement is false; then there exists a sequence of μ_0-elliptic matrices $\{A_{j,k}\}_{k\in\mathbb{N}}$, such that $A_{j,k}(x,t) = A_j(x,t)$ for $t \in (2^{-k}, 2^k)$, $\mathfrak{N}(A_{j,k}) \le C_0$, and

$$\left| \frac{\omega^\infty_{L_{j,k}}(E_j)}{\omega^\infty_{L_{j,k}}(\Delta_0)} - \frac{\sigma(E_j)}{\sigma(\Delta_0)} \right| < \frac{\delta_0}{2}, \qquad k \in \mathbb{N}. \tag{A.18}$$

Let $X = (0, t_0)$, with $t_0 > 4$ to be chosen soon. Then by Lemma 2.17 (applied with $\kappa = 2$, $x = 0$, and $r = 1$),

$$\left| \frac{\omega^X_{L_j}(E_j)}{\omega^X_{L_j}(\Delta_0)} - \frac{\omega^\infty_{L_j}(E_j)}{\omega^\infty_{L_j}(\Delta_0)} \right| \le C\left(\frac{1}{t_0}\right)^\gamma = C{t_0}^{-\gamma},$$

where C and γ are positive constants that depend only on d and μ_0. Choose t_0 so large (depending on d and μ_0) that $C{t_0}^{-\gamma} \le \frac{\delta_0}{8}$; then

$$\left| \frac{\omega^X_{L_j}(E_j)}{\omega^X_{L_j}(\Delta_0)} - \frac{\omega^\infty_{L_j}(E_j)}{\omega^\infty_{L_j}(\Delta_0)} \right| \le \frac{\delta_0}{8}. \tag{A.19}$$

By the triangle inequality, (A.5) (recall that $\Delta_j = \Delta_0$) and (A.19) yield

$$\left| \frac{\omega^X_{L_j}(E_j)}{\omega^X_{L_j}(\Delta_0)} - \frac{\sigma(E_j)}{\sigma(\Delta_0)} \right| \ge \frac{7}{8}\delta_0, \tag{A.20}$$

while (A.18) and the analogue of (A.19) for $L_{j,k}$ (which is valid with uniform bounds) give that

$$\left| \frac{\omega^X_{L_{j,k}}(E_j)}{\omega^X_{L_{j,k}}(\Delta_0)} - \frac{\sigma(E_j)}{\sigma(\Delta_0)} \right| < \frac{3}{8}\delta_0 \qquad \text{for } k \in \mathbb{N}. \tag{A.21}$$

Now we want to let k tend to $+\infty$. We claim that for any bounded Borel set $E \subset \mathbb{R}^d$,

$$\lim_{k\to\infty} \omega^X_{L_{j,k}}(E) = \omega^X_{L_j}(E). \tag{A.22}$$

Once we prove this, we let k tend to $+\infty$ in (A.21) and get a contradiction with (A.20). So Lemma A.16 follows from Lemma A.23 (proved later). □

Lemma A.23. *Let $L = -\operatorname{div} A\nabla$ be a μ_0-elliptic operator, let $L_k = -\operatorname{div} A_k\nabla$ be a μ_0-elliptic operator that satisfies $A_k(x,t) = A(x,t)$ for $t \in (2^{-k}, 2^k)$, and $\mathfrak{N}(A_k) \le C_0$ for some $C_0 < \infty$ for all $k \in \mathbb{Z}_+$. Then for any bounded Borel set $E \subset \mathbb{R}^d$, and any $X_0 \in \mathbb{R}^{d+1}_+$, there holds*

$$\lim_{k\to\infty} \omega^{X_0}_{L_k}(E) = \omega^{X_0}_L(E). \tag{A.24}$$

Lemma A.23 should not surprise the reader, but something like the uniform bound $\mathfrak{N}(A_k) \le C_0$ that guarantees A_∞ is needed. The weak convergence of the elliptic measures, on the other hand, does not require anything for the operator apart from ellipticity. In fact, it is well known that assuming $A_k \to A$ a.e. as $k \to \infty$, A_k and A elliptic, then for any fixed $X_0 \in \mathbb{R}^{d+1}_+$, $\int f\, d\omega^{X_0}_{L_k}$ tends to $\int f\, d\omega^{X_0}_L$ as $k \to \infty$ for any continuous function f with compact support on $\mathbb{R}^d$; see e.g. [KP93]. However, we do not seem to find a proof written for the convergence (A.24), and so we give the proof in Section B.

A.3. **We neutralize the coefficients of A_j away from a band.** Now we continue with a given $j \geq 0$ and construct a matrix A_j^* that coincides with A_j on the large band $\left\{(x,t)\,;\, \frac{\rho_j}{2} < t \leq 2M_j\right\}$ and with the identity matrix outside of a much larger band (so that further gluing will be easier). The main point of the construction is to do it without increasing the norm $\mathfrak{N}(A_j)$ too much.

Definition A.25 (The matrix A_j^*)**.** Let A_j be the matrix that we fixed earlier. Let ρ_j and M_j be as in Lemma A.16. Set $N_j = \varepsilon_j^{-1} = 2^j$ and denote by I is the identity matrix of size $d+1$. We define A_j^* by

$$A_j^*(x,t) := \begin{cases} I, & t > 2^{N_j} M_j \text{ or } t \leq 2^{-N_j}\rho_j, \\ \frac{l}{N_j} I + \left(1 - \frac{l}{N_j}\right) A_j(x,t), & 2^l M_j < t \leq 2^{l+1} M_j, \text{ for } 1 \leq l \leq N_j - 1, \\ A_j(x,t), & \frac{\rho_j}{2} < t \leq 2M_j, \\ \frac{l}{N_j} I + \left(1 - \frac{l}{N_j}\right) A_j(x,t), & 2^{-l-1}\rho_j < t \leq 2^{-l}\rho_j, \text{ for } 1 \leq l \leq N_j - 1. \end{cases}$$

Notice that we only care about the values of t here, and we do not modify $A_j(x,t)$ for $|x|$ large, because this is not needed for the construction. We need a very large interval around $[\frac{\rho_j}{2}, 2M_j]$ because we want the coefficients to vary slowly, so that some ℓ^2-norm is small. We now evaluate the α-numbers for A_j^*.

Lemma A.26. *For any $x \in \mathbb{R}^d$,*

$$\alpha_{A_j^*}(x,r) \begin{cases} = 0, & r \leq 2^{-N_j}\rho_j \text{ or } r \geq 2^{N_j+1} M_j, \\ \leq C\varepsilon_j, & 2^{-N_j}\rho_j \leq r \leq \rho_j \text{ or } 2M_j \leq r \leq 2^{N_j+1} M_j, \\ = \alpha_{A_j}(x,r), & \rho_j < r \leq 2M_j. \end{cases}$$

Here, the constant C depends only on d and μ_0.

Proof. Let us discuss four cases.

Case 1. $r \leq 2^{-N_j}\rho_j$ or $r \geq 2^{N_j+1} M_j$.

In this case, observe that for any $x \in \mathbb{R}^d$,

$$W(x,r) = \Delta(x,r) \times (r/2, r] \subset \mathbb{R}^d \times \left\{(0, 2^{-N_j \rho_j}] \cup (2^{N_j} M_j, \infty)\right\}.$$

By the definition of A_j^*, $A_j^*(x,t) = I$ for $(x,t) \in \mathbb{R}^d \times \left\{(0, 2^{-N_j\rho_j}] \cup (2^{N_j} M_j, \infty)\right\}$. So we can just take $A_0 = I$ in the definition of $\alpha_{A_j^*}$ and get that

$$\alpha_{A_j^*}(x,r) = 0 \qquad \text{for } x \in \mathbb{R}^d,\ r \leq 2^{-N_j}\rho_j \text{ or } r \geq 2^{N_j+1} M_j.$$

Case 2. $2^{-N_j}\rho_j \leq r \leq \rho_j$.

Let k be such that $2^{-k-1}\rho_j < r \leq 2^{-k}\rho_j$; then $0 \leq k \leq N_j - 1$ and for $x \in \mathbb{R}^d$,

$$\begin{aligned} &\alpha_{A_j^*}(x,r)^2 \\ &\quad \leq \frac{1}{|W(x,r)|} \int_{\Delta(x,r)} \int_{r/2}^{2^{-k-1}\rho_j} \left| \frac{k+1}{N_j} I + \left(1 - \frac{k+1}{N_j}\right) A_j(y,t) - A_0 \right|^2 dtdy \\ &\qquad + \frac{1}{|W(x,r)|} \int_{\Delta(x,r)} \int_{2^{-k-1}\rho_j}^{r} \left| \frac{k}{N_j} I + \left(1 - \frac{k}{N_j}\right) A_j(y,t) - A_0 \right|^2 dtdy \\ &\qquad =: I_1 + I_2, \end{aligned}$$