Optimal Merging Control of an Autonomous Vehicle in Mixed Traffic: an Optimal Index Policy ⋆

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Abstract: We consider the problem of a single Autonomous Vehicle (AV) merging into traffic consisting only of Human Driven Vehicles (HDVs) with the goal of minimizing both the travel time and energy consumption of the entire group of vehicles involved in the merging process. This is done by controlling only the AV and determining both the optimal merging sequence and the optimal AV trajectory associated with it. We derive an optimal index policy which prescribes the merging position of the AV within the group of HDVs. We also specify conditions under which the optimal index corresponds to the AV merging before all HDVs or after all HDVs, in which case no interaction of the AV with the HDVs is required. Simulation results are included to validate the optimal index policy and demonstrate cases where optimal merging can be achieved without requiring any explicit assumptions regarding human driving behavior.

Keywords: Intelligent transportation, Autonomous vehicle, Optimal control

1. INTRODUCTION

The emergence of Connected and Automated Vehicles (CAVs) has the potential to drastically impact transportation systems in terms of increased safety, as well as reducing congestion, energy consumption, and air and noise pollution, [Li et al. (2013)]. The main approach has been to formulate and solve optimal control problems for CAVs seeking to minimize travel times and fuel consumption while always satisfying safety constraints, e.g., [Zhang and Cassandras (2019)]. The most important benefit of CAVs lies in the fact that they can *cooperate* by sharing information and coordinating their respective motion. In contrast, a transportation system consisting entirely of Human Driven Vehicles (HDVs) operates based on the vehicle-centric (selfish) behavior of each driver who competes, rather than cooperating, with other drivers. This prevents a traffic network from achieving a much more efficient system-centric (socially optimal) equilibrium based on CAVs. To date, most of the research involving the optimal control of CAVs has been based on the assumption that all vehicles on a traffic network are CAVs and can cooperate with each other. Since such 100% CAV penetration rate is likely to take a few decades before it can be realized [Alessandrini et al. (2015)], it is crucial to investigate the ways in which the benefits of CAVs can be realized while they co-exist with HDVs in a "mixed traffic" environment. It is, for instance, possible that a partial presence of CAVs may even worsen the problem of congestion, since HDV behavior is stochastic and selfish by nature [Mehr and Horowitz (2020)]. It is therefore important to adapt the controllers or algorithms developed under the assumption of 100% CAV penetration so that they can capitalize on the ability of CAVs to control their own motion in ways that minimize the unpredictability or selfish behavior of HDVs with which they interact.

A common way to deal with CAVs in mixed traffic settings is to either make assumptions on the behavior of HDVs, such as maintaining constant speed in free flow traffic [Omidvar et al. (2020)] or adopting a specific car following model, [Sun et al. (2020)]. Alternatively, the problem may be approached through data-driven control [Wang et al. (2022)] or learning-based algorithms to handle HDV stochasticity, [Kreidieh and M (2018)], [Liu et al. (2021)] . Despite some encouraging results, such trial-and-error methods are not yet applicable to real-time settings.

A key question in a mixed traffic setting is "what should the CAV penetration rate be to start seeing their beneficial effect on the existing transportation network?" [Zhang and Cassandras (2018)]. Motivated by this question the goal of this paper is to investigate the extent to which a single CAV (which we will henceforth refer to as just an AV since it is not connected to any other vehicles) can have a positive impact when interacting with a group of HDVs in terms of travel time and energy consumption for the HDVs and the AV itself [Bethge et al. (2020)],[Giammarino et al. (2021)]. We consider the merging problem in such mixed traffic situations, specifically a group of N HDVS approaching a merging point from one road while an AV approaches it from another road. The goal is to establish a "building block" for how a single AV can interact with HDV groups in more general traffic control settings where we can have two or more CAVs which can further enhance performance through connectivity.

Our contribution is to solve the problem of determining the optimal merging sequence in the above setting that benefits all vehicles in terms of both energy and time. In particular,

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the AV, indexed by 0, can select one of $N+1$ possible ways to merge relative to the HDVs, i.e., $\{0,1,\ldots,N\},\ldots,\{1,\ldots,N,0\}$. We provide an optimal trajectory for the AV to merge in the optimal order while guaranteeing safe merging. Moreover, we derive conditions under which the optimal merging sequence is either $\{0,1,\ldots,N\}$ or $\{1,\ldots,N,0\}$, i.e, the AV either proceeds ahead of or waits to merge behind the whole group. The importance of such a sequence is that it involves no disruption of the HDV group and requires no assumptions by the AV as to the behavior of the HDVs.

The paper is organized as follows. In Section 2, we define the merging problem framework and formulate the optimal control problem for minimizing both travel time and energy consumption of all $N+1$ vehicles. In Section 3, the problem is reformulated so as to find the optimal merging sequence of the vehicles, as well as the associated optimal trajectories. Conditions for the two special sequences mentioned above to be optimal are also derived. In Section 4, simulation results validate the general solution along with the special sequences that require no knowledge of HDV behavior.

2. PROBLEM FORMULATION

Similar to the merging problem studied in [Xiao and Cassandras (2021)], we consider traffic arriving from two roads joined at a Merging Point (MP) M where a collision might occur. In this paper, we consider situations where a single AV on one road merges with a group of HDVs coming from the other road as shown in Fig. 1 by green and red vehicles, respectively. A *Control Zone* (CZ) is defined to be an area within which vehicles can see other vehicles (if any) and obtain or possibly estimate their states using their on-board sensing equipment. The range L_{CZ} of such zones can vary for different vehicles as it depends on the geometry of the road and the vehicle sensing capabilities. Unlike the case of a CZ with 100% CAVs and full communication capabilities, e.g., [Xiao and Cassandras (2021)], here HDVs do not have any connectivity. After crossing the MP, vehicles join a road segment called *Acceleration Lane* (AL) of length L_{AL}, which is assumed to be long enough so that vehicles can reach their desired speed.

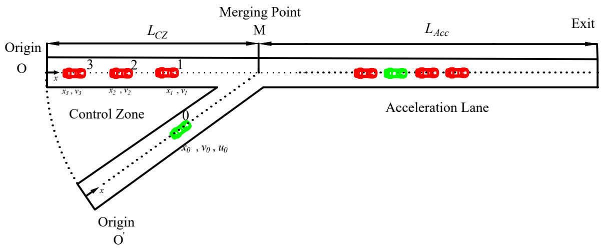

Fig. 1. The merging problem

Throughout the paper, the index 0 is reserved for the AV and let t_0^a be the time when the AV arrives at the CZ from one road and detects any HDVs traveling on the other road. Then, $\mathcal{N}(t_0^a)=\{0,1,...,N(t_0^a)\}$ is the set of all (unique) indices in the CZ, where $N(t_0^a)\in\mathbb{N}$ is the total number of HDVs. If the AV detects an additional HDV entering the CZ, it is assigned the index $N(t_0^a)+1$ and added to $\mathcal{N}(t_0^a)$. To ease notation, we will set $N=N(t_0^a)$ and $\mathcal{N}=\mathcal{N}(t_0^a)$ for the rest of the paper.

The AV dynamics are assumed to be of the form

$$\dot{x}_0(t)=v_0(t),\ \ \dot{v}_o(t)=u_0(t) \tag{1}$$

where $x_0(t)$ denotes the distance from the origin at which AV 0 arrives, $v_0(t)$ denotes the velocity, and $u_0(t)$ denotes the control input (acceleration). The vehicle dynamics for the HDVs within the CZ are unknown to the AV, but we assume that their position $x_i(t)$ and velocity $v_i(t)$, $i\in\mathcal{N}-\{0\}$, can be directly obtained or estimated by the AV's on-board sensors.

The objective of the AV is to derive an optimal acceleration/deceleration profile so as to *jointly* minimize the energy consumption and travel time of the total $N+1$ vehicles present in the CZ (including the AV itself). Therefore two objectives are defined as follows:

Objective 1 (Minimize travel time): Let t_i^f denotes the time vehicle i, $i\in\mathcal{N}$ leaves AL at the exit point. We wish to minimize the total travel time of all vehicles $\sum_{i=0}^{N}(t_i^f-t_0^a)$. It is important to note that the travel times of the HDVs are measured relative to the time that the AV observes them. Since HDVs cannot be controlled and there is no communication with them, their performance cannot be optimized without the presence of at least one AV which can decide when to reach the MP.

Objective 2 (Minimize energy consumption): we wish to minimize the total energy consumption of all vehicles in the CZ, $\sum_{i=0}^{N}E_i$, where

$$E_i(t_i^f,u_i(t))=\int_{t_0^a}^{t_i^f}\mathscr{C}(u_i(t))dt, \tag{2}$$

where u_i is the acceleration of each vehicle i and $\mathscr{C}(\cdot)$ is a strictly increasing function of its argument. For simplicity, we limit ourselves to the case where $\mathscr{C}(u_i(t))=\frac{1}{2}u_i^2(t)$.

Constraint 1 (Safety constraints): Let $p\in\mathcal{N}-\{0\}$ denotes the index of the HDV which physically immediately precedes the AV (if one is present). We require that the distance $z_{0,p}(t):=x_p(t)-x_0(t)$ be constrained by:

$$z_{0,p}(t)\geq\varphi_c v_0(t)+\delta,\ \ \forall t\in[t_0^a,t_0^f], \tag{3}$$

The reaction time for the AV here is denoted by φ_c (as a rule, $\varphi_c=1.8s$ is used, e.g., [Vogel (2003)]) and δ is a minimum safe distance. If we define $z_{0,p}$ to be the distance from the center of the AV to the center of HDV p, then δ depends on the length of both vehicles (generally dependent on the AV and HDV p but here is constant for all vehicles for simplicity).

Constraint 2 (Safe merging): Whenever the AV crosses a MP, there must be adequate safe space for the AV at this MP to avoid a possible lateral collision, i.e.,

$$z_{0,j}(t_0^m)\geq\varphi_c v_0(t_0^m)+\delta, \tag{4}$$

where j is the index of the HDV that may collide with AV 0 at the merging point. The choice of j depends on the merging sequence that the AV determines which will be discussed in the next section. It is worth pointing out that HDV j is also the one that will precede the AV in the AL, therefore, $j=p$ if such p exists. Note that this constraint only applies at time t_0^m and it provides a safe distance between AV and HDV j; however, in addition, the AV must ensure that the HDV $j+1$ (i.e. the HDV trailing the AV after merging) is willing and able to provide adequate space to avoid any collision.

Constraint 3 (Vehicle physical limitations): There are constraints on the speed and acceleration of vehicles:

$$v_{min}\leq v_i(t)\leq v_{max},\ \forall t\in[t_0^a,t_i^f] \tag{5}$$

$$u_{min}\leq u_i(t)\leq u_{max},\ \forall t\in[t_0^a,t_i^f], \tag{6}$$

where $v_{max} > 0$, $v_{min} \geq 0$ denote the maximum and minimum speed allowed for vehicles, and $u_{min} < 0$, $u_{max} > 0$ denote the minimum and maximum control for vehicles.

Optimal control problem formulation. We use the weight $\alpha \in [0,1]$ to construct a convex combination of time and energy metrics as follows:

$$\min_{u_0(t), t_0^m} J(u_0(t), t_0^m) = \sum_{i=0}^{N} (\alpha(t_i^m - t_0^a) + (1-\alpha)E_i), \;\; t \in [t_0^a, t_0^m] \tag{7}$$

subject to (1), (3), (4), (5) and, (6). The weight α also nondimensionalize each term in the objective function. Note that the only decision variables here are the AV control input and AV merging time, since there is no control over the HDVs. This problem is complicated by the fact that obtaining an expression for each HDV's t_i^m and E_i, is a difficult task. To address this issue, we reformulate the optimal control problem so as to first derive an optimal merging sequence for all vehicles to minimize the objectives and determine the optimal control for the AV.

3. OPTIMAL INDEX POLICY

In this section, we derive an optimal index policy for the AV with the goal of minimizing the overall travel time and energy consumption of all vehicles as defined in (7). In other words, we seek the optimal among all possible merging sequences of the AV and N HDVs expressed as $\mathcal{K} = \Big\{\{0,1,...,N\},\{1,0,...,N\},...,\{1,...,N,0\}\Big\}$, where, for example, the first sequence in $\mathcal{K}$ stands for the AV crossing the MP ahead of all N HDVs. In the sequel, we use the index $k \in \{1,\ldots,N+1\}$ to denote the kth element of the set $\mathcal{K}$. Thus, $k=1$ denotes the sequence in which the AV crosses the MP ahead of all N HDVs and $k = N+1$ is the sequence where the AV is the last to cross the MP behind all HDVs. Therefore, we can rewrite the objective function in (7) as follows:

$$J_k = \alpha\left(t_{0,k}^m - t_0^a + \sum_{i=1}^{N} t_{i,k}^m\right) + (1-\alpha)\left(E_{0,k} + \sum_{i=1}^{N} E_{i,k}\right), \tag{8}$$

where $t_{i,k}^m$ and $E_{i,k}$ stand for the travel time and energy consumption (starting from t_0^a) of vehicle $i \in \mathcal{N}$, respectively, when the AV crosses the MP as the k^{th} vehicle.

Assumption 1 : All HDVs have reached their desired speed $v_{i,k}(t) = v_i^d$, $t \in [t_0^a, t_0^f]$, $i \in \mathcal{N} - \{0\}$ and intend to pass the MP cruising at v_i^d if not constrained by other road traffic.

Based on Assumption 1, the road leading to the CZ is long enough to ensure that each HDV has reached its steady-state velocity in the vicinity of MP. Moreover, any deceleration by a HDV is perceived as a "disruption". This allows us to express the travel time $t_{i,k}^m$ in (8) in terms of its minimal value and a disruption term $D_{i,k}^t$. Observe that the minimal value of $t_{i,k}^m$ corresponds to the index $k = N+1$, i.e., the AV crosses the MP after all N HDVs, so we write $t_{i,k}^m = t_{i,N+1}^m + D_{i,k}^t$, where $D_{i,k}^t \geq 0$, $\forall k \in \mathcal{K} - \{1, N+1\}$.

Similarly, for $E_{i,k}$ in (8), we write $E_{i,k} = E_{i,N+1} + D_{i,k}^E$ where $E_{i,N+1}$ is the amount of energy spent by HDV i in the merging sequence $N+1$ where its disruption is zero, and $D_{i,k}^E$ is the excess energy that HDV i spends because of a potential disruption in the merging sequence k. Thus, $D_{i,k}^E \geq 0$, $\forall k \in \mathcal{K} - \{1, N+1\}$. Further, since $\mathcal{C}(u_i(t)) = \frac{1}{2}u_i^2(t)$, we have $E_{i,N+1} = 0$. Therefore, $E_{i,k} = D_{i,k}^E$.

As a result, by replacing $t_{i,k}^m$ and $E_{i,k}$ in (8) by the disruption-based expressions above and by separating the AV's objective function J_{A_k} from the objective function J_{H_k} of the N HDVs, (8) for $k \in \mathcal{K}$ can be rewritten as:

$$J_k = \underbrace{\alpha(t_{0,k}^m - t_0^a) + (1-\alpha)E_{0,k}}_{J_{A_k}} + \underbrace{\sum_{i=1}^{N} \alpha\left(t_{i,N+1}^m + D_{i,k}^t\right) + (1-\alpha)D_{i,k}^E}_{J_{H_k}} \tag{9}$$

HDV objective function (J_{H_k}) evaluation : To obtain an expression for J_{H_k} we need to evaluate $t_{i,N+1}^m$, $D_{i,k}^t$ and, $D_{i,k}^E$ for $k \in \mathcal{K}$. The undisrupted travel time $t_{i,N+1}^m$ under Assumption 1 can be written as, $t_{i,N+1}^m = \frac{L_{CZ} - x_i(t_0^a)}{v_i^d}$. The values of $D_{i,k}^t$ and $D_{i,k}^E$ depend on the ith HDV's behavior for all $i \geq k$ which are disrupted by the AV in merging sequence k. To approximate these values, we approximate the disruption of HDV k, the first HDV that provides space for the AV to merge ahead of it and cross the MP, and then we use a discount factor $\gamma_{i,k} \in (0,1)$ to propagate the effect of the AV through the rest of the HDVs $i > k$. Let us assume that any HDV acceleration or deceleration rate is constant and given by $|\bar{u}|$. Given the speed of the AV at the MP, $v_0(t_{0,k}^m)$, and that of HDV k, v_k^d, and based on Assumption 1, we can estimate $D_{i,k}^t$ and $D_{i,k}^E$ based on the difference between the HDV's desired velocity and the AV merging velocity and the HDV's deceleration $-\bar{u}$ to provide enough space for merging. In particular, $D_{k,k}^t(v_0(t_{0,k}^m)) = \frac{\max\left(v_k^d - v_0(t_{0,k}^m), 0\right)^2}{2\bar{u}v_k^d}$, $D_{k,k}^E(v_0(t_{0,k}^m)) = \frac{1}{2}\bar{u}\max\left(v_k^d - v_0(t_{0,k}^m), 0\right)$, (details are omitted but can be found in [Sabouni and Cassandras (2022)]). Now $D_{k,k}^t$ and $D_{k,k}^E$ can be used as the basis to calculate the disruption for HDVs $i > k$ as follows:

$$D_{i,k}^t(v_0(t_{0,k}^m)) = \gamma_{i,k} D_{k,k}^t(v_0(t_{0,k}^m)) \tag{10}$$

$$D_{i,k}^E(v_0(t_{0,k}^m)) = \gamma_{i,k} D_{k,k}^E(v_0(t_{0,k}^m)) \tag{11}$$

where $\gamma_{i,k} \in (0,1)$ is the discount factor. We model this as a function of the distance $z_{i,k}(t_{0,k}^m)$. Here, we set $\gamma_{i,k} = \exp(-\beta z_{i,k})$ where $\beta \in (0,1)$ and note that for $i < k$ there is no disruption, therefore we only define (10) and (11) for $i \geq k$.

AV objective function (J_{A_k}) evaluation: To evaluate J_{A_k} in (9) we need to obtain an expression for the AV merging time $t_{0,k}^m$ and energy consumption $E_{0,k}$ in the merging sequence k. Since J_{A_k} pertains to the AV objective function when it selects the sequence $k \in \mathcal{K}$, the associated optimal control problem, denoted by P_k, is as follows:

Problem P_k:

$$\min_{u_0(t), t_{0,k}^m} J_{A_k}(u_0(t), t_{0,k}^m) = \alpha(t_{0,k}^m - t_0^a) + (1-\alpha)\int_{t_0^a}^{t_{0,k}^m} \frac{1}{2}u_0^2(t)dt, \tag{12}$$

subject to (1), $x_0(t_{0,k}^m) = L_{CZ}$, (4), (5) and (6). Note that this formulation does not include any rear-end safety constraint since there is no preceding vehicle for the AV.

The solution of P_k is premised on HDV k allowing the AV to merge ahead of it. We assume that this is the case as long as the AV selects a safe merging time $t_{0,k}^m$ and merging velocity,

$v_0(t_{0,k}^m)$ to ensure that it positions itself safely between HDVs $k-1$ and k (if they exist).

Safe merging time and merging velocity sets: When both HDVs k and $k-1$ exist, HDV k has to provide space for the AV while the AV also has to satisfy the safe merging constraint (4) with HDV $k-1$. As a result, the AV must not violate either the safety constraint of HDV k or the safe merging constraint (4) at the same time. By applying (3) and (4), this implies $z_{0,k-1}(t_{0,k}^m) \geq \varphi_c v_0(t_{0,k}^m) + \delta$ and $z_{k,0}(t_{0,k}^m) \geq \varphi_h v_k(t_{0,k}^m) + \delta$. By adding these two constraints, we obtain the following:

$$z_{k,k-1}(t_{0,k}^m) \geq \varphi_c v_0(t_{0,k}^m) + \varphi_h v_k(t_{0,k}^m) + 2\delta, \tag{13}$$

where $z_{k,k-1}(t_{0,k}^m)$ is the distance between HDVs $k-1$ and k at the merging time, $v_k(t_{0,k}^m)$ is the velocity of HDV k at the merging time, and φ_h is the reaction time for the HDVs (which can be estimated). Note that (13) depends on the velocity of HDV k at the merging time, $v_k(t_{0,k}^m)$ which the AV needs to estimate. Since we assume that the HDVs maintain a constant speed prior to the MP, we set $v_k(t_{0,k}^m) = v_k^d$. Therefore, the constraint (13) can be rewritten as follows:

$$z_{k,k-1}(t_{0,k}^m) \geq \varphi_c v_0(t_{0,k}^m) + \varphi_h v_k^d + 2\delta. \tag{14}$$

Since $v_k(t_{0,k}^m) = v_k^d$, it is easy to derive the positions of HDVs k and $k-1$ at the merging time $t_{0,k}^m$ as follows:

$$\begin{aligned} x_k(t_{0,k}^m) &= x_k(t_0^a) + (t_{0,k}^m - t_0^a) v_k^d, \\ x_{k-1}(t_{0,k}^m) &= x_{k-1}(t_0^a) + (t_{0,k}^m - t_0^a) v_{k-1}^d. \end{aligned} \tag{15}$$

By combining (14) and (15), we can define the following safe sets for the AV merging velocity $v_0(t_{0,k}^m)$ and merging time $t_{0,k}^m$:

$$S_{0,k}^v = \left\{ v_0(t_{0,k}^m) \in \mathscr{R}^+ : v_0(t_{0,k}^m) \leq v_{0,k}^u \right\} \tag{16}$$

$$S_{0,k}^t(v_0(t_{0,k}^m)) = \left\{ t_{0,k}^m \in \mathscr{R}^+ : t_{0,k}^l(v_0(t_{0,k}^m)) \leq t_{0,k}^m \leq t_{0,k}^u \right\}, \tag{17}$$

where $v_{0,k}^u = \frac{1}{\varphi_c}\left(z_{k,k-1}(t_{0,k}^m) - \varphi_h v_k^d - 2\delta\right)$, $t_{0,k}^l(v_0(t_{0,k}^m)) = t_0^a + \frac{1}{v_{k-1}^d}\left(x_0(t_{0,k}^m) + \varphi_c v_0(t_{0,k}^m) + \delta - x_{k-1}(t_0^a)\right)$, and $t_{0,k}^u = t_0^a + \frac{1}{v_k^d}\left(x_0(t_{0,k}^m) - \varphi_h v_k^d - \delta - x_k(t_0^a))\right)$.
The satisfaction of (14) implies that there exists a safe gap between HDVs $k-1$ and k and there will be a safe merging time and merging velocity for the AV in merging sequence k.

Solution of problem P_k: We now return to problem P_k with (4) replaced by (16), (17). The final states in this problem are constrained by $x_0(t_{0,k}^m) = L_{CZ}$ and the merging velocity is chosen from the set $S_{0,k}^v$. Ignoring the vehicle limitations in (5) and (6), this problem can be readily solved [Bryson and Ho (2018)] to give:

$$u_0^*(t) = a_0 t + b_0 \tag{18}$$

$$v_0^*(t) = \frac{1}{2} a_0 t^2 + b_0 t + c_0 \tag{19}$$

$$x_0^*(t) = \frac{1}{6} a_0 t^3 + \frac{1}{2} b_0 t^2 + c_0 t + d_0, \tag{20}$$

The four coefficients above can be computed by using the initial and final conditions of P_k, i.e., $x_0(t_0^a)$, $v_0(t_0^a)$, L_{CZ}, and $v_0(t_{0,k}^m)$. The first three are given, but $v_0(t_{0,k}^m)$ and $t_{0,k}^m$ are still undetermined and will be derived in the next subsection. Nonetheless, we can solve for these coefficients as functions of $v_0(t_{0,k}^m)$ and $t_{0,k}^m$ through the following system of four linear equations of the form $\mathbf{T}_0 \mathbf{b}_0 = \mathbf{q}_0$ (dropping time arguments):

$$\begin{bmatrix} \frac{1}{6}(t_0^a)^3 & \frac{1}{2}(t_0^a)^2 & t_0^a & 1 \\ \frac{1}{2}(t_0^a)^2 & t_0^a & 1 & 0 \\ \frac{1}{6}(t_{0,k}^m)^3 & \frac{1}{2}(t_{0,k}^m)^2 & t_{0,k}^m & 1 \\ \frac{1}{2}(t_{0,k}^m)^2 & t_{0,k}^m & 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \\ L_{CZ} \\ v_{0,k}^m \end{bmatrix}, \tag{21}$$

whose solution is of the form,

$$\mathbf{b}_0(v_0(t_{0,k}^m)), t_{0,k}^m) = \mathbf{T_0}^{-1} \mathbf{q_0}, \tag{22}$$

emphasizing the fact that the solution still depends on $v_0(t_{0,k}^m)$ and $t_{0,k}^m$. We can now use $u_0^*(t)$ in (18) with a_0, b_0 from above to obtain $E_{0,k}$ in (9):

$$E_{0,k}(v_0(t_{0,k}^m), t_{0,k}^m) = \int_{t_0^a}^{t_{0,k}^m} \frac{1}{2} {u_0^*}^2(t)\,dt = \int_{t_0^a}^{t_{0,k}^m} \frac{1}{2}(a_0 t + b_0)^2 dt \tag{23}$$

Solution of problem (9) **and optimal index determination:** We can now obtain a complete solution to the optimal merging problem (7) in three steps:

Step 1: Determine the optimal values of $v_0(t_{0,k}^m)$ and $t_{0,k}^m$ which minimize the cost $J_k(t_{0,k}^m, v_{0,k}^m)$ for each $k \in \mathscr{K}$:

$$\min_{t_{0,k}^m, v_{0,k}^m} J_k(t_{0,k}^m, v_{0,k}^m), \quad k \in \mathscr{K} \tag{24}$$

subject to (16), (17), and (5). The solution provides the optimal cost J_k^* for each $k \in \mathscr{K}$.

Step 2: Determine the optimal index k^* (the optimal merging sequence). This is a simple comparison problem:

$$k^* = \mathrm{argmin}_{k \in \mathscr{K}} J_k^* \tag{25}$$

Step 3: Given k^* and $t_{0,k^*}^{m^*}$, $v_{0,k^*}^{m^*}$, the values of a_0, b_0, c_0, d_0 are determined through (22) and provide the complete AV optimal trajectory through (18),(19), and (20).

An analytical solution to problem (24) for each $k \in \mathscr{K}$ is difficult to obtain in general due to the complex form of $J_k(t_{0,k}^m, v_{0,k}^m)$. However, solutions are easily obtained numerically, as long as the number N of HDVs in the CZ is relatively small (in fact, it is always bounded by the length L_{CZ} of the CZ). This limits the number of comparisons required in (25) to the cardinality of the set $\mathscr{K}$.

In what follows, we consider cases in which the optimal index is easily determined by bypassing Step 1 and show that either $k^* = 1$ or $k^* = N+1$.

Assumption 2: The HDVs observed by the AV at t_0^a move with the same speed $v_i^d(t) = v^d$ and maintain an equal distance from each other $z_{i,i-1}(t) = z$, $\forall t \in [t_0^a, t_0^f]$, $i \in \mathscr{N} - \{0\}$.

Under Assumption 2, $S_{0,k}^v = S_0^v$ since $v_{0,k}^u = v_0^u$ therefore $v_{0,k}^m = v_m$. Moreover, since the distance between the HDVs and the merging velocity are the same among all $k \in \mathscr{K}$, we have $t_{0,k+1}^m - t_{0,k}^m = \frac{z}{v^d}$ for any two consecutive merging options. Further, Assumption 2 implies that $\gamma_{i,k} = \gamma^{i-k}$, $D_{i,k}^t = \gamma^{i-k} D^t$ and $D_{i,k}^E = \gamma^{i-k} D^E$.

3.1.Time-optimal merging sequence : We consider the case where $\alpha = 1$ in (9), corresponding to the time-optimal merging sequence. We show that this sequence remains optimal for α satisfying $\alpha_l \leq \alpha \leq 1$ where $\alpha_l = \frac{E^{max}}{E^{max} + \frac{z}{v^d} + \gamma^{N-1} D^t}$, with $E^{max} = \frac{1}{2}\left((v_{max} - v_0)u_{max} + N(v^d - v_{min})\bar{u}\right)$ (all proofs are omitted and can be found in [Sabouni and Cassandras (2022)]).

Theorem 1. Under Assumptions 1-2 and $\alpha_l \leq \alpha \leq 1$,

(i): If $v_m^* \geq v^d$, then $k^* = 1$.

(ii): If $v_m^* < v^d$, then $k^* = q$, where q is determined from

$$\frac{1-\gamma^{N-q+1}}{(1-\gamma)(N-q+1)} v^d D^t \leq z \leq \frac{\gamma^{N-q+1}(1-\gamma^{q-1})}{(1-\gamma)(q-1)} v^d D^t, \quad (26)$$

where z is the distance between HDVs.

3.2.Energy-optimal merging sequence : When $\alpha = 0$ in (9), it can be shown that the energy-optimal sequence is unchanged for all α such that $0 \leq \alpha \leq \alpha_u$ where, $\alpha_u = \frac{E^{min}}{E^{min} + \frac{Nz}{v^d} + \frac{1-\gamma^N}{1-\gamma} D^t}$, and $E^{min} = \gamma^{N-1} D^E$. Next it is shown that under easy-to-check conditions the optimal index is either $k^* = 1$ or $k^* = N+1$, i.e., the AV either merges ahead of or behind the entire HDV group. Before proceeding, let us take a closer look at the AV energy function derived in (23), which depends on the AV's travel time and merging velocity. Under Assumption 2, however, the expression in (23) depends, for all $k \in \mathcal{K}$, only on the AV travel time. Thus, the energy expression in (23) is a rational function of the AV's travel time, $t_k = t_{0,k}^m - t_0^a$ with the general form:

$$E_0(t_k) = \frac{A_1 t_k^2 + A_2 t_k + A_3}{t_k^3}, \quad \forall k \in \mathcal{K}, \quad (27)$$

where $A_1 = 2(v_0^2 + v_0 v_m + v_m^2) > 0$, $A_2 = -6(L_{CZ} - x_0)(v_0 + v_m) < 0$ and $A_3 = 6(L_{CZ} - x_0) > 0$ are all constants. It can be shown that this function has both vertical and horizontal asymptotes at $E_0(t_k) = 0$ and $E_0(t_k) = \infty$.

Lemma 1: The function $E_0(t_k)$ in (27) is monotonically decreasing in $k \in \mathcal{K}$ only if $t_{N+1} \leq \tau_1$, $\tau_1 = \frac{-A_2 - \sqrt{A_2^2 - 3A_1A_3}}{A_1}$.

Theorem 2. Under Assumptions 1-2 and $0 \leq \alpha \leq \alpha_u$, if $t_{0,N+1}^{m^*} - t_0^a \leq \frac{3(L_{CZ} - x_0)}{v_0 + v_m^* + \sqrt{v_0 v_m^*}}$,

$$\min J_k^* = \min(J_1^*, J_{N+1}^*) \quad (28)$$

Note that in Theorem 1, we managed to obtain a condition on z for the optimality over all merging sequences, whereas in Theorem 2 we are only able to derive a sufficient condition dependent on the optimal AV travel time when $k = N+1$ not exceeding a fixed threshold τ_1 given by Lemma 1. In this case, the AV can ignore all but the first and last merging sequence, since either of these two is optimal.

4. SIMULATION RESULTS

All simulations of the merging problem shown in Fig. 1 are based on PTV VISSIM and all algorithms are implemented using MATLAB and ODE45 to integrate the AV dynamics and CasADi for solving the minimization problem in (24). The parameters used are: $L_{CZ} = 400\text{m}, \varphi_c = 1.8\text{s}, \varphi_h = 1.5\text{s}, \delta = 3.78\text{m}, \bar{u} = 2, \gamma = 0.4, \beta = 0.1, u_{max} = 4.905\text{m/s}^2, u_{min} = -5.886\text{m/s}^2, v_{max} = 30\text{m/s}, t_0^a = 0, v_{min} = 0\text{m/s}, v^d = 16.66$, $v_0 = 16.66$. The simulations have been performed for a variety of initial positions of the AV and HDVs.

Time-optimal and energy-optimal cases: Table 1 summarizes the results of 14 simulations for 4 HDVs and 1 AV performing the merging process. To validate Theorems 1 and 2,

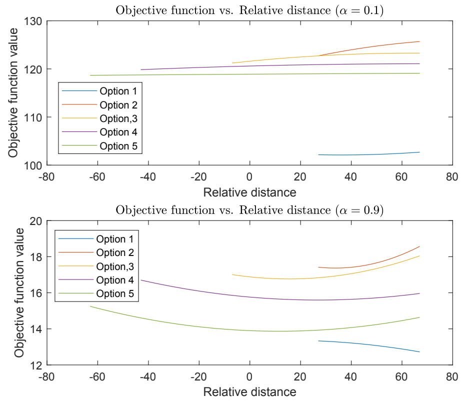

Fig. 2. Objective function value with respect to relative distance ($z = 22$)

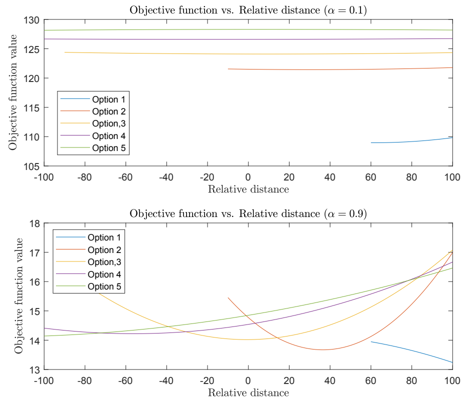

Fig. 3. Objective function value with respect to relative distance ($z = 40$)

the objective value corresponding to each option has been provided in the table for sufficiently small $\alpha = 0.1$ and large $\alpha = 0.9$ (costs corresponding to the optimal options are shown in red and blue, respectively). In the table, "inf" indicates the infeasibility of a particular merging sequence due to the AV's limitations in terms of maximum/minimum acceleration and speed. The table data are visually depicted in Figs. 2 and 3 (for inter-HDV distances $z = 22$ and $z = 40$ respectively) showing the objective value as a function of the relative distance between the HDV group and the AV for all sequences. The relative distance is measured from the projection of the AV onto the other road to the center of the whole HDV group: $x_0'(t_0^a) = x_0(t_0^a) - \frac{x_1(t_0^a) + x_r(t_0^a)}{2}$.

Theorem 1 implies that the time-optimal merging sequence may be independent of the AV's initial position relative to the HDVs, leading to $k^* = 1$. This can be seen in the figures, as long as this sequence is feasible. When this is not feasible, the AV selects the optimal sequence depending on the relative distance. Theorem 2 is also illustrated for energy-optimal sequences when z is sufficiently small so that a disruption would be too costly, therefore the energy-optimal index is either $k = 1$ or $k = 5$.

General case: When neither Theorem 1 or 2 applies, we need to use a numerical method to calculate the optimal merging sequence through (24) and (25). Thus we choose $\alpha = 0.5$ so that neither time or energy optimality dominates. As seen in

HDV initial position	x_0	J_1		J_2		J_3		J_4		J_5	
		$\alpha=0.1$	$\alpha=0.9$	$\alpha=0.1$	$\alpha=0.9$	$\alpha=0.1$	$\alpha=0.9$	$\alpha=0.1$	$\alpha=0.9$	$\alpha=0.1$	$\alpha=0.9$
$x_1(t_0^a)=330$ $x_2(t_0^a)=308$ $x_3(t_0^a)=286$ $x_4(t_0^a)=264$	254	inf	inf	inf	inf	inf	inf	119.49	16.76	118.72	14.60
	274	inf	inf	inf	inf	inf	inf	120.18	16.08	118.81	14.16
	294	inf	inf	inf	inf	121.35	17.0	120.51	15.80	118.88	13.92
	314	inf	inf	inf	inf	122.28	15.56	120.74	15.53	119.0	13.88
	334	102.09	13.26	123.47	17.25	123.0	17.04	121.02	15.66	119.02	14.02
	364	102.41	12.92	125.53	18.21	123.16	17.69	121.11	16.07	119.06	14.38
$x_1(t_0^a)=330$ $x_2(t_0^a)=290$ $x_3(t_0^a)=250$ $x_4(t_0^a)=210$	220	inf	inf	inf	inf	124.21	14.72	126.64	14.16	128.25	14.33
	240	inf	inf	121.62	17.45	124.15	14.15	126.55	14.16	128.27	14.47
	260	inf	inf	121.58	15.43	124.12	13.90	126.57	14.39	128.29	14.67
	280	inf	inf	121.53	14.22	124.09	14.01	126.59	14.59	128.32	14.94
	300	inf	inf	121.46	13.58	124.17	14.41	126.68	14.97	128.36	15.28
	320	inf	inf	121.48	13.80	124.18	14.87	126.69	15.49	128.40	15.68
	340	109.0	13.82	121.59	14.74	124.28	15.74	126.77	16.13	128.26	15.98
	360	109.44	13.44	121.67	16.25	124.15	16.55	126.51	16.26	128.17	16.01

Table 1. Objective function values of merging sequences corresponding to sufficiently large and sufficiently small α.

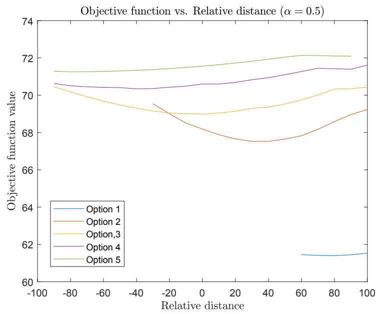

Fig. 4. Objective function value with respect to relative distance ($z = 40$)

Fig. (4) for $z = 40$, different values of the index k are optimal depending on the relative distance shown.

Finally, in terms of computational complexity, the 3-step solution approach in (24) and (25) takes approximately 50 msec to evaluate each option, therefore a total of approximately $50 \times N$ msec is needed for all possible merging sequences.

5. CONCLUSIONS

We have studied the merging problem that the integration of a single AV with a group of HDVs as a potential "building block" for more complex mixed traffic situations involving a relatively small number of CAVs (since AVs may now communicate among them) interacting with HDVs. An index optimal policy is derived to determine a joint time and energy-optimal merging sequence for the AV so as to benefit the whole system. Future work is dedicated to extend this framework to multiple AVs and compare it with other existing benchmarks.

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