Graphs of scramble number two

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Abstract

The scramble number of a graph provides a lower bound for gonality and an upper bound for treewidth, making it a graph invariant of interest. In this paper we study graphs of scramble number at most two, and give a classification of all such graphs with a finite list of forbidden topological minors. We then prove that there exists no finite list of forbidden topological minors to characterize graphs with scramble number at most k for any fixed $k \geq 3$.

1 Introduction

Chip firing games on graphs provide a combinatorial analog for divisor theory on algebraic curves [1]. In chip firing games, a *divisor* is a placement of integer numbers of chips on the vertices of a graph. A vertex can then be fired to donate one chip along each incident edge, rearranging the chips into a new divisor. This brings us to the following question: how many chips do we need in a divisor so that, given any vertex, we can perform chip-firing moves to place a positive number of chips on that vertex, with a nonnegative number of chips on all other vertices? We call this minimum number the *gonality* of G. Gonality has been extensively studied as a graph invariant and was proved to be NP-hard to compute in [5]. Because computing gonality is computationally difficult, bounds on it are of particular interest. Scramble number, developed in [7], is one such graph invariant that serves as a lower bound for gonality: for any connected multigraph G, we have

$$\mathrm{sn}(G) \leq \mathrm{gon}(G).$$

One of the first theorems on scramble number was a complete characterization of the (connected) graphs of scramble number 1, which are precisely the trees [7]. In this paper we push further to characterize all graphs of scramble number 2.

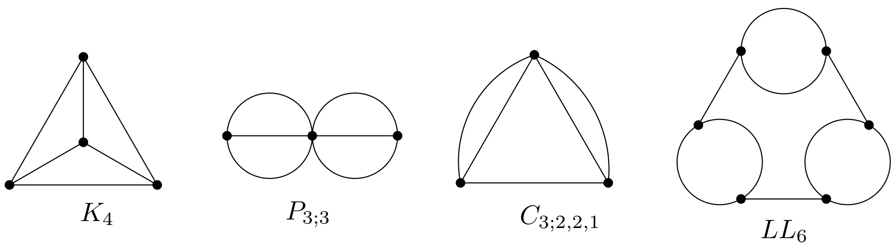

Figure 1: The four graphs from Theorem 1.1

Our main theorem refers to the four graphs in Figure 1, which we denote by K_4 (the complete graph on 4 vertices), $P_{3;3}$ (a multipath), $C_{3;2,2,1}$ (a multicycle), and LL_6 (the loop-of-loops on 6 vertices).

Theorem 1.1. A graph has scramble number at most 2 if and only if it has none of K_4, $P_{3;3}$, $C_{3;2,2,1}$, and LL_6 as a topological minor.

Since all graphs have a positive scramble number, and since the (connected) graphs of scramble number 1 are precisely the trees, a (connected) graph has scramble number exactly 2 if and only if it is not a tree and has none of the four graphs in Figure 1 as a topological minor.

For our other primary result, let $\mathscr{S}_m$ denote the set of all connected graphs of scramble number at most m.

Theorem 1.2. The set $\mathscr{S}_m$ admits a characterization by a finite list of forbidden topological minors if and only if $m \leq 2$.

Our paper is organized as follows. In Section 2 we present background material and useful lemmas. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 by constructing, for each $k \geq 3$, an infinite family of graphs that are topological minor minimal among those with scramble number k. In Section 5 we present some applications of our results.

2 Background and preliminaries

A *graph* G is a finite set of *vertices*, $V(G)$, and a finite multiset of *edges*, $E(G)$, such that every edge connects exactly two vertices. We allow for multiple edges to connect the same pair of vertices, but we do not allow an edge to connect a vertex to itself. A graph is *connected* if there exist a path of edges and vertices between every pair of vertices in G, and is *disconnected* if it is not connected. The *edge connectivity* of a graph G, denoted $\lambda(G)$, is the minimum number of edges we must delete from G to obtain a disconnected subgraph. The edge connectivity of a graph offers more insight into the structure of the graph through the following integral graph theory theorem.

Theorem 2.1 (Menger's Theorem [9]). A graph G has $\lambda(G) \geq k$ if and only if there exist k edge-disjoint paths between every pair of vertices in G.

We define the *degree* of a vertex v to be the number of edges incident to v, i.e. the number of edges that have v as an endpoint. A vertex u of degree two, incident to distinct vertices v and w via edges e_1 and e_2, can be "smoothed", meaning we delete u, e_1, and e_2 and add an edge between v and w. If a graph H can be obtained from a graph G by deleting vertices, deleting edges, and smoothing vertices, H is called a *topological minor* of G, denoted $H \preceq G$. See Figure 2 for an example.

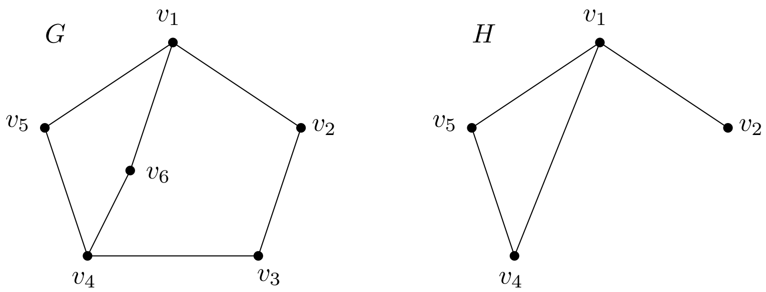

Figure 2: A topological minor H of a graph G, obtained by deleting v_5 and its incident edges, and smoothing v_6.

If H can be obtained from G by deleting edges and vertices, we call H a *subgraph* of G. Given a subset $S \subset V(G)$, we let $G[S]$ denote the *subgraph induced by* S; that is, the subgraph with vertex set S and edge multiset consisting of all edges from $E(G)$ with both endpoints in S. If $G[S]$ is a connected graph, we refer to S as a *connected subset* of $V(G)$.

Later on we will use work from [8] applied to graphs of edge-connectivity 3. These results are most easily phrased in the language of *pseudographs*, in which we do allow loops connecting vertices to themselves. A graph G is called 3-*edge-minimal* if $\lambda(G) = 3$, but for every edge in G we have $\lambda(G - e) < 3$, where $G - e$ denotes G with the edge e deleted. We describe two operations on a graph G: in the operation O_1^+, an edge of G is subdivided and an edge is added to connect the new vertex to another vertex; and in the operation $O_1^{(2)}$, an edge of G is subdivided to yield a new vertex z, and then another edge (not adjacent to z) is subdivided to yield another new vertex w, and then an edge is added to connect z and w. We remark that these operations depend on the choice of vertices and edges. For examples of these operations, we refer the reader to Figure 3. The leftmost graph can be turned into the upper adjacent graph through the operation O_1^+ by subdividing the loop and attaching the new vertex to the middle vertex; alternatively, that graph can be turned into the lower adjacent graph through the operation $O_1^{(2)}$ by adding a vertex to the loop, subdividing one of the previously existing edges, and connecting the two new vertices them with an edge. Two more operations are illustrated on the right, both instances of $O_1^{(2)}$.

Theorem 2.2 (Corollary 17 in [8], $k = 1$ case). Let G be a 3-edge minimal multigraph with at least two vertices. Then either there is a pseudograph G_1 with $\lambda(G_1) \geq 3$ and $|V(G_1)| = |V(G)| - 1$, such that G is obtained from G_1 by performing the operation O_1^+; or else there is a pseudograph G_2 with $\lambda(G_2) \geq 3$ and $|V(G_2)| = |G| - 2$, such that G is obtained from G_2 by performing the operation $O_1^{(2)}$.

We use this to prove the following corollary, which will be useful in the proof of Theorem 1.1.

Corollary 2.3. If G is a 3-edge-connected graph on three or more vertices, then G contains one of K_4, $P_{3;3}$, and $C_{3;2,2,1}$ as a topological minor.

Proof. We prove this by induction on the number of vertices $|V(G)|$. If $|V(G)| = 3$, then the underlying simple graph of G is either the path on three vertices P_3 or the cycle on three vertices C_3. In the first case, each edge must appear as 3 or more parallel copies to ensure 3-edge-connectivity, so $P_{3;3}$ is a subgraph. In the second case, at least two edges must appear as 2 parallel copies to ensure 3-edge-connectivity, so $C_{3;2,2,1}$ is a subgraph.

Now let $|V(G)| \geq 4$, and assume the claim holds for all 3-edge-connected graphs with between 3 and $|V(G)| - 1$ vertices. By Theorem 2.2, we can obtain G from a pseudograph G_1 on one fewer vertex using O_1^+, or from a pseudograph G_2 on two fewer vertex using $O_1^{(2)}$.

We now deal with several cases.

(i) The relevant G_i is a graph on 3 or more vertices. By the inductive hypothesis, G_i has one of the three graphs as a topological minor; and G_i is itself a topological minor of G, giving us the desired claim.

(ii) The relevant G_i is a graph on 2 vertices; this occurs only when $|V(G)| = 4$, and when G is obtained via $O_1^{(2)}$ from $P_{2;n}$ (a graph with two vertices connected by n edges) for some $n \geq 3$, as these are all 3-edge-connected graphs on 2-vertices. Up to symmetry there is only one way to perform $O_1^{(2)}$ on $P_{2;n}$, and it yields K_4 as a subgraph. Thus G has K_4 as a topological minor.

(iii) The relevant G_i is a pseudograph that is not a graph, i.e. G_i has at least one loop. Since G is a graph, the operation must eliminate any loops. If G is obtained from G_1 by O_1^+, then the edge that is subdivided must be the loop, say l rooted at a vertex $v \in V(G_1)$, where $G_1 - l$ is a simple graph. We know that $|V(G_1 - l)| = |V(G_1)| = |V(G)| - 1$, which is between 3 and $|V(G)| - 1$, and since removing l does not change edge-connectivity we know that $\lambda(G_1 - l) \geq 3$. This allows us to apply our inductive hypothesis to show that $G_1 - l$, and thus G, has one of the three graphs as a topological minor.

If G is obtained from G_2 by $O_1^{(2)}$, there are several subcases to consider. The operation $O_1^{(2)}$ must eliminate any loops in G_2, of which there can be at most 2 by the structure of $O_1^{(2)}$. Let L be the set of all loops in G_2. By the same logic as the previous argument, if $|V(G)| \geq 5$ we have $G_2 - L$ is a 3-edge-connected simple graph on 3 or more vertices, giving it (and G) one of the three graphs as a topological minors by the inductive hypothesis. The last case to deal with is if $|V(G)| = 4$, and therefore $|V(G_2)| = 2$. Since $\lambda(G_2) \geq 3$, we know that G_2 is of the form $P_{2,n}$ with either 1 or 2 loops attached, possibly on the same vertex or on different vertices. If one loop, there are two ways to perform $O_1^{(2)}$ to eliminate the loop, one of which yields $P_{3;3}$ and the other of which yields $C_{3;2,2,1}$ as a topological minor. If two loops, there is a unique way to perform $O_1^{(2)}$ to eliminate both of them; regardless of the placement of the loops, this yields $C_{3;2,2,1}$ as a topological minor. These operations are illustrated in Figure 3.

We conclude by induction that every 3-edge connected graph on three or more vertices has one of our three graphs as a topological minor. □

We now move on to scramble number. An *egg* on a graph G is a connected subset of vertices. A *scramble* on a graph G is a collection of eggs on G.

Every scramble has an order, which requires several steps to calculate. An *egg-cut* for a scramble is a collection of edges in $E(G)$ that, when deleted, disconnect the graph into two components, each of which contains an egg. The *egg-cut number* of a scramble $\mathcal{S}$, denoted $e(\mathcal{S})$, is the minimum size of an egg-cut for $\mathcal{S}$. A *hitting set* for a scramble is a set of vertices in $V(G)$ such that every egg contains at least one vertex in that set. The *hitting number* of a scramble $\mathcal{S}$, denoted $h(\mathcal{S})$, is the minimum size of a hitting set for $\mathcal{S}$.

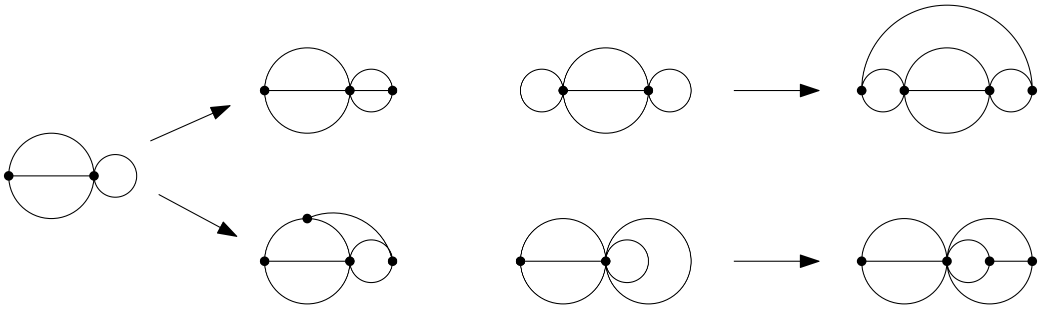

Figure 3: Possible operations to turn G_2 into G, one yielding $P_{3;3}$ and the others yielding $C_{3;2,2,1}$ as a topological minor

These two definitions bring us to the *order* of a scramble $\mathcal{S}$, which is defined as

$$||\mathcal{S}|| = \min\{h(S), e(S)\}.$$

The *scramble number of a graph* G is then the maximum possible order of a scramble $\mathcal{S}$ on G. That is,

$$\text{sn}(G) = \max_{\mathcal{S} \text{ on } G}\{||\mathcal{S}||\}.$$

Example 2.4. Figure 4 presents a scramble on each of four graphs. In each scramble, the eggs are denoted by circled collections of vertices. Since for these examples all the eggs are disjoint, the hitting number for each scramble is the number of eggs, i.e. 4 for the scramble on K_4 and 3 for the other three. Each of the scrambles has an egg-cut number of 3; taking the minimum of the two relevant numbers, each scramble has order 3. We remark that the egg-cut number may be smaller than the minimum number of edges incident to an egg; for instance, in the scramble on LL_6, each egg is incident to 4 edges, but an egg-cut of size 3 can be obtained by deleting two parallel edges and the edge opposite them in the underlying cycle graph C_6. Thus for any graph G in this figure we have $\text{sn}(G) \geq 3$; later, in Example 2.12, we give an argument that $\text{sn}(G) = 3$ for each.

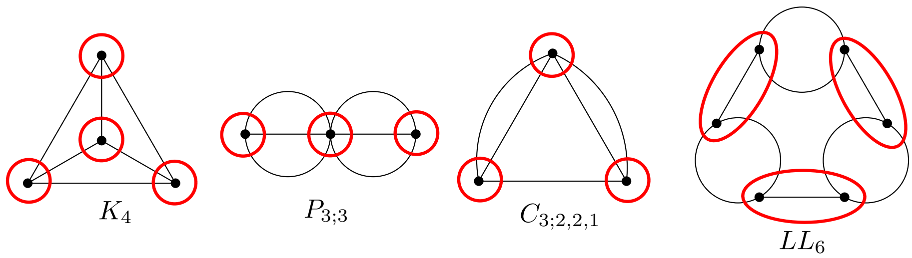

Figure 4: Example of scrambles of order three.

We now present some previous results. Recall a graph G is a *tree* if there is exactly one path between each pair of vertices of G, or equivalently if G is connected and acyclic.

Lemma 2.5 ([7, Corollary 4.2])**.** A graph G has $\text{sn}(G) = 1$ if and only if G is a tree.

Lemma 2.6 ([4, Lemma 2.5])**.** Let G be a graph. Then, $\text{sn}(G) \geq \min\{\lambda(G), |V(G)|\}$.

This is a powerful result for us as it is often easier to compute $\lambda(G)$ than trying to compute $\text{sn}(G)$ directly. For the next lemma, we define a *bridge* to be an edge in G such that deleting the edge disconnects G.

Lemma 2.7 ([4, Lemma 2.4])**.** If G is a graph with bridge e and the two connected components of $G - e$ are G_1 and G_2, then $\text{sn}(G) = \max\{\text{sn}(G_1), \text{sn}(G_2)\}$.

These next two lemmas will allow us to conclude that scramble number is *topological minor monotone*; that is, that scramble number can only decrease or remain unchanged when taking a topological minor. This is an important step in using topological minors to study scramble numbers.

Lemma 2.8 ([7, Proposition 4.5])**.** If H is a subgraph of G, $\mathrm{sn}(H) \leq \mathrm{sn}(G)$.

We remark that the same result does not hold if G' is a minor of G; that is, it is possible for scramble number to increase when an edge is contracted [7, Example 4.4].

Lemma 2.9 ([7, Proposition 4.6])**.** If H is obtained from G by smoothing vertices, $\mathrm{sn}(H) = \mathrm{sn}(G)$.

We can now provide a result on topological minors.

Corollary 2.10. If H is a topological minor of G, then $\mathrm{sn}(H) \leq \mathrm{sn}(G)$.

Proof. If H is a topological minor of G, then there exists a subgraph G' of G such that H is obtained from G' by smoothing vertices. By Lemma 2.9, we have $\mathrm{sn}(H) = \mathrm{sn}(G')$, and by Lemma 2.8, we have $\mathrm{sn}(G') \leq \mathrm{sn}(G)$. Thus $\mathrm{sn}(H) \leq \mathrm{sn}(G)$. □

It follows that if we can find the scramble number of a topological minor of G, we have also found a lower bound on the scramble number of G itself. This process provides the underlying structure for proving Theorem 1.1 in Section 3.

The final topic we recall from a previous paper is the *screewidth* of a graph, introduced in [3]. Given a graph G, a *tree-cut decomposition* is a pair $(T, \mathcal{X})$ such that T is a tree and $\mathcal{X}$ is a set of subsets X_b of $V(G)$, one for each $b \in V(T)$, such that

- $\bigcup_{b \in V(T)} X_b = V(G)$, and
- $X_{b_1} \cap X_{b_2} = \emptyset$ for $b_1 \neq b_2$.

That is, $\mathcal{X}$ forms a *near partition* of $V(G)$, which is a partition with empty sets allowed. For clarity, we will refer to the vertices and edges of T as *nodes* and *links*, respectively, reserving the terms *vertices* and *edges* for G.

For $l \in E(T)$, deleting l from T partitions the vertices of $V(T)$, and thus the vertices of $V(G)$, into two sets. The *(link) adhesion of* l, denoted $\mathrm{adh}(l)$, is the set of edges in $E(G)$ with one endpoint in each of the two sets. Similarly, for $b \in V(T)$ not a leaf, deleting b from $V(T)$ partitions the set $V(T) - b$, and thus the set $V(G) - \mathcal{X}_b$, into at least two subsets. The *(node) adhesion of* b, denoted $\mathrm{adh}(b)$, is the set of edges in $E(G)$ connecting two vertices in these different sets.

These adhesions admit an intuitive description. Given a tree-cut decomposition of a graph G, draw G in a thickened copy of T so that a vertex $v \in V(G)$ is within the node $b \in T$ such that $v \in X_b$; and so that an edge connecting $u, v \in V(G)$ is drawn along the unique path from b_1 to b_2 in T, where $u \in X_{b_1}$ and $v \in X_{b_2}$. The adhesion of a link l is then the number of edges from $E(G)$ drawn passing through l; and the adhesion of a node b is the number of edges from $E(G)$ passing through b with neither endpoint in X_b.

The *width* of the tree-cut decomposition $\mathcal{T} = (T, \mathcal{X})$ is defined to be the maximum of the following numbers:

- $\max_{l \in E(T)} |\mathrm{adh}(l)|$;
- $\max_{b \in V(T)} |\mathrm{adh}(b)| + |X_b|$.

Finally, the *screewidth* of a graph G, denoted $\mathrm{scw}(G)$, is the minimum possible width of a tree-cut decomposition of G.

Theorem 2.11 (Theorem 1.1 in [3])**.** For any graph G, we have $\mathrm{sn}(G) \leq \mathrm{scw}(G)$.

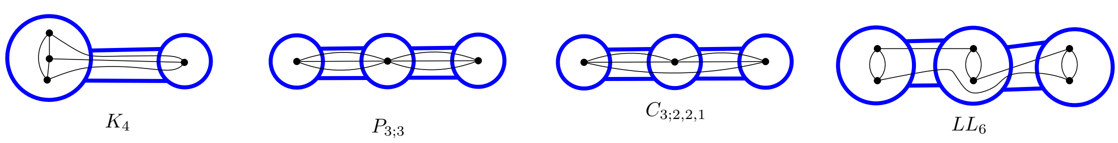

Figure 5: Tree-cut decompositions of width 3

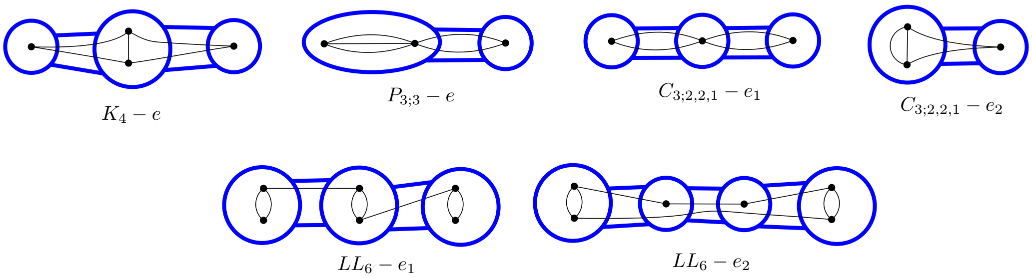

Figure 6: Tree-cut decompositions of width 2

Example 2.12. Figure 5 illustrates a tree-cut decomposition for each of four graphs; the width of each decomposition is 3, so every graph G in the figure satisfies $\text{scw}(G) \le 3$. From Example 2.4, we already had that each graph satisfied $\text{sn}(G) \ge 3$, which combined with Theorem 2.11 gives us that $\text{sn}(G) = \text{scw}(G) = 3$ for all four graphs G.

In fact, we can say even more about these graphs: they are minimal, with respect to the topological minor relation, among all graphs of scramble number 3 or more. Since none has a degree 2 vertex, it suffices to show that deleting any edge will lower the scramble number. Illustrated in Figure 6 are tree-cut decompositions of all subgraphs, up to symmetry, of these four graphs obtained by deleting a single edge. Since each decomposition has width 2, each of these graphs H satisfies $\text{sn}(H) \le \text{scw}(H) \le 2 < 3$, as desired.

3 Characterizing Graphs of Scramble number Two

We say a graph G is *scramble minimal* if for any proper topological minor $H \preceq G$, $\text{sn}(H) < \text{sn}(G)$. We remark that since smoothing a vertex does not change scramble number, a graph is scramble minimal if and only if any degree 2 node is incident to two parallel edges (preventing a smoothing), and for every edge $e \in E(G)$ we have $\text{sn}(G - e) < \text{sn}(G)$. A graph G is *k-scramble minimal* if $\text{sn}(G) \ge k$ and any proper topological minor $H \preceq G$ has $\text{sn}(H) < k$.

Lemma 3.1. If G is k-scramble minimal with k or more vertices, then $\lambda(G) \le k$.

Proof. Let G be a k-scramble minimal graph, and suppose for the sake of contradiction that $\lambda(G) > k$. Now delete any edge $e \in E(G)$ to create G'. Note that the graph G' has $\lambda(G') \ge k$, and $|V(G')| \ge k$, so by Lemma 2.6 we have that $\text{sn}(G) \ge \min\{\lambda(G), |V(G)|\} \ge \min\{k, k\} = k$. This contradicts the k-scramble minimality of G. Therefore, if G is k-scramble minimal with k or more vertices, $\lambda(G) \le k$. □

Frequently in upcoming proofs it will be helpful to consider what happens to a scramble when a vertex or edge is deleted from a graph. Because scrambles are defined to be specific to a graph, there is a priori no way of "transferring" a scramble to a subgraph. To this end, for a scramble $\mathcal{S}$ on a graph G with subgraph H, we construct $\mathcal{S}|_H$ on H, which we refer to as *the restriction of the scramble $\mathcal{S}$ to H*. For each egg $E \in \mathcal{S}$, if $E \cap V(H)$ forms a nonempty connected subset in H, we let $E \cap V(H) \in \mathcal{S}|_H$. Using this definition, we can extract the following result.

Proposition 3.2. Let $\mathcal{S}$ be a scramble of order at least 2 on a graph G, let $e \in E(G)$ be an edge such that $H = G - e$ is connected, and let $\mathcal{S}' = \mathcal{S}|_H$. Then, $||\mathcal{S}'|| \ge ||\mathcal{S}|| - 1$.

Proof. Note that $\mathcal{S}'$ must contain at least one egg; if deleting e had disconnected every egg in $\mathcal{S}$, then $||\mathcal{S}|| \le h(\mathcal{S}) = 1$. Now, we claim that $e(\mathcal{S}') \ge e(\mathcal{S}) - 1$. If all eggs in $\mathcal{S}'$ overlap, then $e(\mathcal{S}') = \infty$ and the claim holds. Otherwise, choose an egg-cut T of size $e(\mathcal{S}')$ for $\mathcal{S}'$ on H. Then $T \cup \{e\}$ is an egg-cut for $\mathcal{S}$ on G, separating the same pair of eggs as T did. Thus we have $e(\mathcal{S}') \ge e(\mathcal{S}) - 1$.

Now let S' be a minimum hitting set of $\mathcal{S}'$. Letting u denote an endpoint of e, note that $S' \cup \{u\}$ forms a hitting set of $\mathcal{S}$, since any egg of $\mathcal{S}$ that is not hit by S' must have contained the edge e, and thus can be hit by the vertex u. Therefore, we have found a hitting set of $\mathcal{S}$ of size $h(\mathcal{S}') + 1$, so $h(\mathcal{S}') \ge h(\mathcal{S}) - 1$. It follows that $||\mathcal{S}'|| \ge ||\mathcal{S}|| - 1$. □

Corollary 3.3. If $H = G - e$ for some non-bridge edge $e \in E(G)$, then

$$\mathrm{sn}(G) - 1 \leq \mathrm{sn}(H) \leq \mathrm{sn}(G).$$

Proof. Since H is a subgraph of G, we have by Lemma 2.8 that $\mathrm{sn}(H) \leq \mathrm{sn}(G)$. Let $\mathcal{S}$ be a scramble on G with $||\mathcal{S}|| = \mathrm{sn}(G)$, and let $\mathcal{S}' = \mathcal{S}|_H$. By Proposition 3.2, we have $||\mathcal{S}'|| \geq ||\mathcal{S}|| - 1$, and so $\mathrm{sn}(H) \geq \mathrm{sn}(G) - 1$, as desired. □

Corollary 3.4. Let $k \geq 2$. If G is k-scramble minimal, then $\mathrm{sn}(G) = k$.

Proof. By definition, if G is k-scramble minimal, $\mathrm{sn}(G) \geq k$ and for any proper topological minor H we have $\mathrm{sn}(H) < k$. Since $\mathrm{sn}(G) \geq 2$, G contains some edge such that $H = G - e$ is connected. By the previous corollary, $\mathrm{sn}(G) - 1 \leq \mathrm{sn}(H) \leq k - 1$. Combined with $\mathrm{sn}(G) \geq k$, it follows that $\mathrm{sn}(G) = k$. □

We now have all of the necessary tools to characterize all graphs of scramble number two with our list of four forbidden topological minors.

Proof of Theorem 1.1. We know by Example 2.12 that all four graphs from Figure 1 have scramble number 3. By Corollary 2.10, we therefore have that any graph with one of them as a topological minor has scramble number at least 3. Contrapositively, a graph of scramble at most 2 has none of the four graphs as a topological minor.

Let G be a graph of scramble number at least 3; we must show it has one of the four graphs as a topological minor. Without loss of generality, we will assume G is 3-scramble minimal; if it is not, then delete edges and vertices and smooth degree 2 vertices until it is. By Corollary 3.4, $\mathrm{sn}(G) = 3$. Note that G must have at least three vertices as $\mathrm{sn}(G) = 3$. We have that $1 \leq \lambda(G) \leq 3$ from Lemma 3.1. However, if $\lambda(G) = 1$, then G has a bridge. From Lemma 2.7, G has a subgraph with the same scramble number, a contradiction to the minimality of G. Thus, $\lambda(G) > 1$. If $\lambda(G) = 3$, then since $|V(G)| \geq 3$ we may apply Corollary 2.3 to conclude that G has one of K_4, $P_{3;3}$, or $C_{3;2,2,1}$ as a topological minor (indeed, G must equal one of these graphs, since G is 3-scramble minimal).

It remains to handle the case of $\lambda(G) = 2$. If $\lambda(G) = 2$, we know that there exist two edges $e_1, e_2 \in E(G)$ that form an edge-cut of G. We will refer to the connected graphs obtained by deleting these edges as G_1 and G_2, with labels as illustrated in Figure 7. Note that a priori, it is possible that $v_1 = v_2$, and that $v_3 = v_4$.

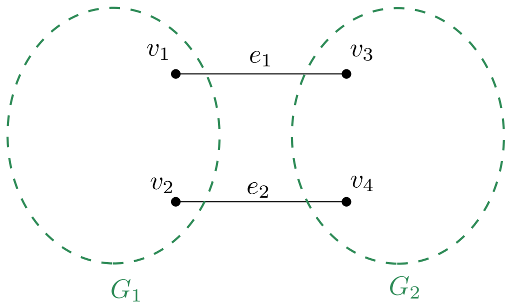

Figure 7: The structure of G with $\lambda(G) = 2$

Let $\mathcal{S}$ be a scramble on G with $||\mathcal{S}|| = 3$. As $\{e_1, e_2\}$ forms a 2-edge cut of G, $\mathcal{S}$ cannot have one egg completely contained in G_1 and one egg completely contained in G_2. Therefore, one subgraph (without loss of generality, G_1) does not completely contain any egg. We now argue that $v_1 \neq v_2$. Suppose for the sake of contradiction that $v_1 = v_2$. Let H be obtained from G by deleting all vertices in G_1 besides v_1. Then consider the scramble $\mathcal{S}|_H$ on H. Since no egg in $\mathcal{S}$ is completely contained within G_1, and as all paths from G_1 to the rest of G travel through v_1, the hitting number and egg-cut number remain unchanged. Since G is minimal, it follows that $G = H$. Then, v_1 is a degree 2 vertex, which either contradicts the scramble minimality of G (if v_1 can be smoothed), or implies that $v_3 = v_4$, so that e_1 and e_2 are parallel edges connecting G_2 to an isolated vertex v_1. In this case $\mathcal{S}|_{G_2}$ has the same order as $\mathcal{S}$, and we still have a contradiction to G being minimal. Thus we know that $v_1 \neq v_2$.

We now argue that there at least two distinct paths between v_1 and v_2 within G_1. There must exist at least one path between v_1 and v_2 within G_1; otherwise $\lambda(G) = 1$, with $\{e_1\}$ forming an edge-cut. If there were only one path, we could choose any edge e on that path; since $\{e, e_1\}$ forms an edge-cut, we could redefine our subgraphs G_1 and G_2 so that G_1 is smaller, and repeat. This would eventually lead us to $v_1 = v_2$, which as we argued before is impossible. Thus there are two distinct paths between v_1 and v_2, implying that the graph illustrated in Figure 8 is a topological minor of G.

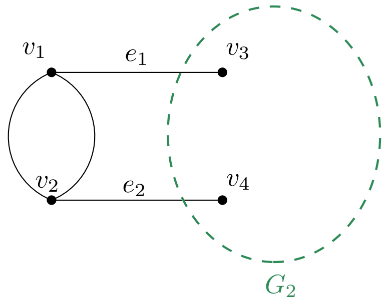

Figure 8: A topological minor of G, which ends up equalling G

We claim that the graph H in Figure 8 has scramble number equal to 3 and therefore must actually be G by the scramble minimality of G. Consider $\mathcal{S}|_H$, the scramble obtained by intersecting the eggs of $\mathcal{S}$ with $V(H)$. We remark that every egg $E \in \mathcal{S}$ remains connected when we restrict to H: any path between two vertices in $E \cap V(H)$ remains a path H, unless that path used edges from G_1 to get from v_1 to v_2; but the edges between v_1 and v_2 in H can replace that portion of the path. Any hitting set for $\mathcal{S}|_H$ is also a hitting set for $\mathcal{S}$, since every egg of $\mathcal{S}$ intersecting G_1 must contain v_1 or v_2; thus $h(\mathcal{S}|_H) \geq h(\mathcal{S})$. Now let $E_1, E_2 \in \mathcal{S}$, with $E_i' = E_i \cap V(H) \in \mathcal{S}|_H$. Suppose for the sake of contradiction that we may disconnect H by deleting a set T with fewer than 3 edges so that E_1' is in one component and E_2' is in the other. Since $\lambda(G) = 2$, we know $\lambda(H) = 2$ as well, so $|T| = 2$. Note that deleting the two edges connecting v_1 and v_2 does not disconnect the graph; nor does deleting one of those edges and any other edge e (otherwise $\{e\}$ would be an edge-cut of size 1). Thus $T \subset E(G_2) \cup \{e_1, e_2\}$. By the structure of our graphs, it follows that T is also an egg-cut for E_1 and E_2, contradicting $e(\mathcal{S}) \geq 3$. Thus $e(\mathcal{S}|_H) \geq 3$, and we have $||\mathcal{S}|_H|| \geq 3$. It follows that $3 \leq \text{sn}(H) \leq \text{sn}(G) = 3$, so $\text{sn}(H) = 3$. As G is 3-scramble minimal, we have $G = H$.

Now, since G is 3-scramble minimal, deleting any edge decreases the scramble number of G to two. Consider G', the subgraph of G obtained by deleting one of the edges connecting v_1 and v_2. No eggs of $\mathcal{S}$ are disconnected by this, so $\mathcal{S}|_{G'}$ consists of the same eggs as $\mathcal{S}$. Since this scramble has the same hitting number, we know that $\mathcal{S}|_{G'}$ must have an egg-cut of size at most two, since $\text{sn}(G') \leq 2$. The graph G does not have an egg-cut of size two, so the deleted edge must have been part of a minimal egg-cut of size 3 in G; it follows that the other edge from v_1 to v_2 is part of the egg-cut in G'. So, this remaining parallel edge is part of this cut in G, as well as some edge in G_2, which must form a bridge of G_2. Let this third egg-cut edge in G_2 be named e_3. Now we can refine G_2 into two subgraphs, G_A as the subgraph between e_1 and e_3 and G_B as the subgraph between e_2 and e_3. This structure is illustrated in Figure 9.

Since $\{v_1v_2, v_1v_2, e_3\}$ is an egg-cut, say separating the eggs E_1 and E_2, we claim without loss of generality that E_1 contains the edge e_1 and E_2 contains the edge e_2. If this were not the case, the parallel edges in G_1 could be replaced with one of e_1 or e_2, creating an egg-cut of size two for $\mathcal{S}$, which is impossible as $|\mathcal{S}| \geq 3$. The eggs E_1 and E_2 do not intersect, so we know $V(E_1) \subset V(G_A) \cup \{v_1\}$ and $E_2 \subset V(G_B) \cup \{v_2\}$.

We now remark that no egg in $\mathcal{S}$ can be contained in $V(G_A)$, as $\{e_1, e_3\}$ would form an egg cut separating it from E_2; and similarly no egg can be contained in G_B. We have already assumed that no egg can be contained in $V(G_1) = \{v_1, v_2\}$. Thus every egg contains at least one of e_1, e_2, and e_3. We note that some egg contains e_3 without intersecting e_1 or e_2: otherwise $\{v_3, v_4\}$ would be a hitting set of size two. It follows that $v_3 \neq v_5$, and $v_4 \neq v_6$. We now claim that there must exist two distinct (but not necessarily edge-disjoint) paths between v_3 and v_5 in G_A. Suppose not, so that there exists a unique path connecting them. If some egg F_1 containing e_1 and some egg F_3 containing e_3 don't intersect, then e_2 together with some edge on that path forms an egg-cut of size two. If every pair of such eggs intersects, then they must all intersect at a common vertex v of

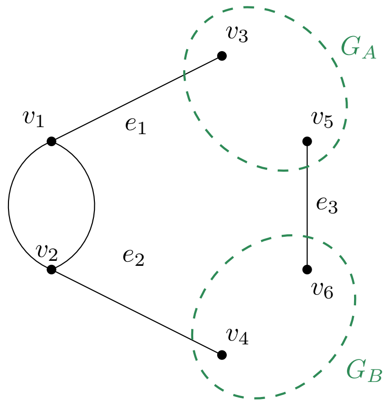

Figure 9: Further structure of G

that path. Since every other egg must contain the edge e_2, we have that $\{v, v_2\}$ is a hitting set of size 2, a contradiction. Thus there are two distinct paths from v_3 to v_5 in G_A, and by symmetry two distinct paths from v_4 to v_6 in G_B. This leads to G containing LL_6 as a topological minor, illustrated below. (In fact, since $\text{sn}(LL_6) = 3$ and G is 3-scramble minimal, we can conclude that $G = LL_6$.)

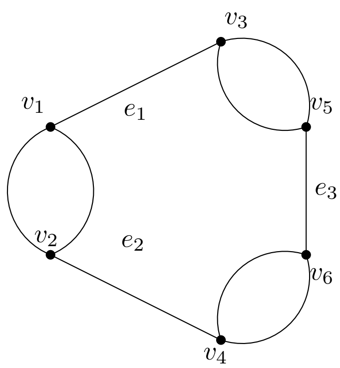

Figure 10: A topological minor of G, which ends up equalling G

Thus in all cases, our 3-scramble minimal graph G contains (in fact, is equal to) one of the four claimed topological minors. This completes the proof. □

4 Scrambles of Higher Orders

In this section we build families of graphs to prove that no finite forbidden topological minor characterization exists for $\mathscr{S}_m$, the set of graphs of scramble number at most m, when $m \geq 3$. Our first lemma gives us a family of scramble minimal graphs with even scramble number.

Lemma 4.1. Let $k \geq 2$ and $n \geq 2k$, and let $C_{n;k}$ denote the cycle graph on n vertices, where each edge has k parallel copies. Then $C_{n;k}$ is $2k$-scramble minimal.

Proof. Let $\mathcal{S}$ denote the scramble whose eggs are the vertices of $C_{n;k}$. This scramble has order $\min\{|V(C_{n;k})|, \lambda(C_{n;k})\} = \min\{n, 2k\} = 2k$, so $\text{sn}(C_{n;k}) \geq 2k$. See the left of Figure 11.

To see that $C_{n;k}$ is $2k$-scramble minimal, we must show that any proper topological minor of $C_{n;k}$ has scramble number at most $2k - 1$. Since $k \geq 2$, there are no degree 2 nodes. Thus it suffices to show that deleting any edge gives a graph with scramble number at most $2k - 1$. Let $v_1, \ldots, v_n$ denote the vertices of $C_{n;k}$, ordered cyclically. By the symmetry of $C_{n;k}$, we may consider $H = C_{n;k} - e$, where e is an edge connecting v_1 and v_n. Construct a tree-cut decomposition $\mathcal{T} = (T, \mathcal{X})$ of H with $T = P_n$, the path graph on n vertices, with nodes $b_1, \ldots, b_n$ and $X_{b_i} = \{v_i\}$ for all i. See the right of Figure 11 for an example. The adhesion of the i^{th} link consists of $2k - 1$ edges, namely the $k - 1$ connecting v_1 with v_n together with the k edges connecting v_i and v_{i+1}.

Each node similarly has adhesion size $k-1$, and the size of each X_b is 1. Thus $w(\mathcal{T}) = 2k-1$, so $\text{sn}(H) \leq \text{scw}(H) \leq 2k-1$. We conclude that $C_{n;k}$ is $2k$-scramble minimal. □

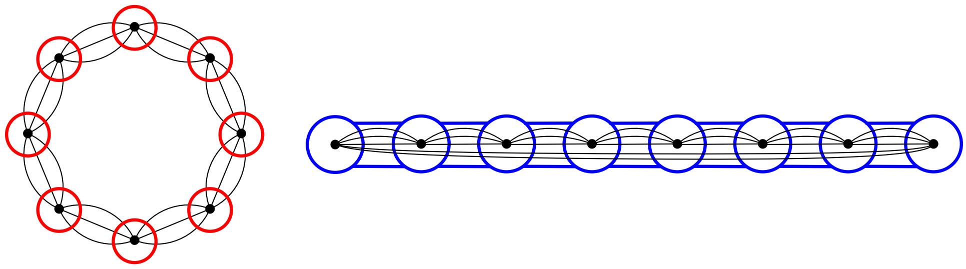

Figure 11: The graph $C_{8;3}$ with a scramble of order 6, and a tree-cut decomposition of $C_{8;3} - e$ of width 5

Our next lemma gives us a family of scramble-minimal graphs with odd scramble number.

Lemma 4.2. Let $k \geq 2$ and $n \geq 3k$, and let $\tilde{C}_{n;k}$ denote a cycle graph on n vertices where $2k$ consecutive edges have $k+1$ parallel copies (say from v_1 through v_{2k+1}), and all others have k. Then $\tilde{C}_{n;k}$ is $(2k+1)$-scramble minimal.

Proof. Consider the scramble

$$\mathcal{S} = \{\{v_2\}, \ldots, \{v_{2k+1}\}, \{v_{2k+2}, \ldots, v_n, v_1\}\}.$$

An example of this scramble is illustrated in Figure 12. Since the eggs are disjoint, we have $h(\mathcal{S}) = 2k+1$. For an egg-cut, at least two collections of parallel edges must be deleted, as this is necessary to disconnect the graph. However, no two bundles of k edges can be deleted to disconnect two eggs; while one bundle of k edges and another bundle of $k+1$ can. Thus $e(\mathcal{S}) = 2k+1$, and so $\text{sn}(\tilde{C}_{n;k}) \geq ||\mathcal{S}|| = \min\{n-k+1, 2k+1\} = 2k+1$.

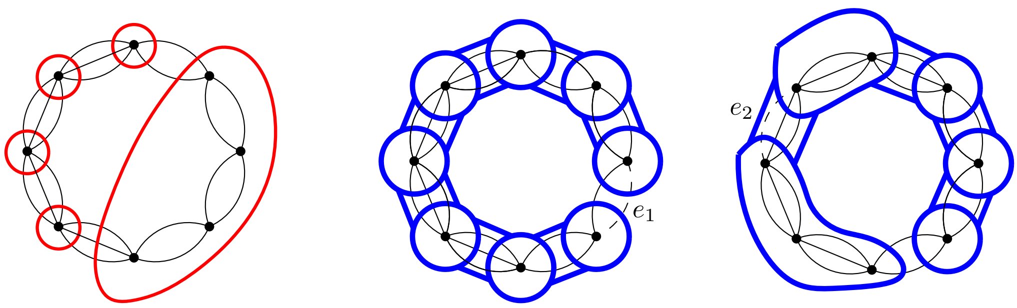

Figure 12: The graph $\tilde{C}_{8;2}$ with a scramble of order 5, and tree-cut decompositions of $\tilde{C}_{8;2} - e_1$ and $\tilde{C}_{8;2} - e_2$, both of width 4

To see that $\tilde{C}_{n;k}$ is $(2k+1)$-scramble minimal, we first note that since $k \geq 2$, there are no degree 2 nodes. Thus it suffices to show that deleting any edge yields a graph with scramble number at most $2k$. First we deal with the case where the deleted edge e_1 is from a bundle of k edges, say between v_i and v_{i+1} where $2k+2 \leq i \leq n$. We construct a tree-cut decomposition similar to that from the proof of Lemma 4.1, except with the nodes in the path corresponding to $v_{i+1}, v_{i+2}, \ldots, v_i$, ordered cyclically. Then every (non-leaf) node and every link has as part of its adhesion the $k-1$ edges connecting v_i and v_{i+1}. See the middle of Figure 12 for an example. The nodes have no other adhesion (and correspond to sets of size 1), and every link has either k or $k+1$ more edges in its adhesion. Thus $\text{sn}(\tilde{C}_{n;k} - e_1) \leq \text{scw}(\tilde{C}_{n;k} - e_1) \leq 2k$.

Now we handle the case where the deleted edge e_2 came from a bundle of $k+1$ nodes, say between v_i and v_{i+1} where $1 \leq i \leq 2k+1$; by the symmetry of the graph, we may assume $k+1 \leq i \leq 2k+1$. Construct a tree-cut decomposition with $T = P_{n-i+1}$ on nodes $b_1, \ldots, b_{n-2}$ where $X_{b_1} = \{v_1, \ldots, v_i\}$, $X_{b_2} = \{v_{i+1}, \ldots, v_{2k+2}\}$, and $X_{b_j} = \{v_{2k+j}\}$ for $3 \leq j \leq n-2k$. See the right of Figure 12 for an example. By our choice of i, $|X_{b_1}| \leq 2k$, and as b_i is a leaf node it has no adhesion. Similarly, $|X_{b_2}| \leq k$, and its adhesion consists of the k edges connecting v_1 and v_n.

Every other b_i has $|X_{b_i}| = 1$, with the non-leaves having the same adhesion of k edges. Finally, every link has size $k + k = 2k$. Thus $\text{sn}(\tilde{C}_{n;k} - e_2) \leq \text{scw}(\tilde{C}_{n;k} - e_2) \leq 2k$. We conclude that $\tilde{C}_{n;k}$ is $(2k + 1)$-scramble minimal. □

We are ready to prove that $\mathscr{S}_m$, the set of all connected graphs of scramble number at most m, admits a characterization by a finite list of forbidden topological minors if and only if $m \leq 2$.

Proof of Theorem 1.2. Note that $\mathscr{S}_1$ is precisely the set of trees. A connected graph is a tree if and only if it does not contain a cycle as a subgraph if and only if it does not have as a topological minor the graph $P_{2;2}$, consisting of a pair of vertices connected by a pair of edges. Thus $\mathscr{S}_1$ admits a characterization by a finite list of forbidden topological minors. We also have such a characterization for $\mathscr{S}_2$ given by Theorem 1.1.

Now let $m \geq 3$. Any complete list of forbidden topological minors for $\mathscr{S}_m$ must include all $(m + 1)$-scramble minimal graphs. By Lemmas 4.1 and 4.2, there are infinitely many $(m + 1)$-scramble minimal graphs, namely $C_{n;(m+1)/2}$ for $m + 1$ even and $\tilde{C}_{n,\lfloor (m+1)/2 \rfloor}$ for $m + 1$ odd, as n varies above the prescribed minimum. Thus there can exist no characterization of $\mathscr{S}_m$ by a finite list of forbidden topological minors. □

Although we have shown that there exists no finite list of forbidden topological minors for $\mathscr{S}_m$ with $m \geq 3$, we might still hope for some nice characterization. For instance, perhaps we could classify all $(m + 1)$-scramble minimal graphs as nicely described infinite families, with a finite collection of special cases. Another possibility comes from generalizing the notion of a topological minor, an approach we detail here.

Let u be a vertex in a graph G adjacent to exactly two other vertices, say v by m edges and w by n edges. A *multi-smoothing at u* removes u from the graph and adds $\min\{m, n\}$ edges between v and w. We say that a graph H is a *multi-topological minor* of a graph G if it can be obtained from G by deleting vertices and edges and performing multi-smoothings. For example, the graph $C_{n;k}$ is a multi-topological minor of $C_{n+1;k}$, obtained by performing a multi-smoothing at any vertex.

Proposition 4.3. If H is a multi-topological minor of G, then $\text{sn}(H) \leq \text{sn}(G)$.

Proof. It suffices to assume that H can be obtained from G by multi-smoothing a single vertex, say u with adjacent vertices v (connected by m edges) and w (connected by n edges). After smoothing u, the number of edges between v and w has increased by $\min\{m, n\}$. Let e' be one such edge.

Now, let $\mathcal{S}'$ be some scramble on H. We will define the scramble $\mathcal{S}$ on G as follows. For each egg $E'_i \in \mathcal{S}'$, include $E_i \in \mathcal{S}$, where

$$E_i = \begin{cases} E'_i & \text{if } v \notin E'_i \\ E'_i \cup \{u\} & \text{if } v \in E'_i. \end{cases} \tag{1}$$

Consider a minimal hitting set K for $\mathcal{S}$. If $u \notin K$, then K must also be a hitting set for $\mathcal{S}'$. If $u \in K$, then $K' = (K \cup \{v\}) - \{u\}$ is a hitting set for $\mathcal{S}'$. It follows that $h(\mathcal{S}') \leq h(\mathcal{S})$.

Now consider a set $A \subseteq V(G)$ such that A and A^c both contain eggs of $\mathcal{S}$, and let $A' = A \cap V(H)$. Without loss of generality, $u \notin A$. It follows that A' and $(A')^c$ both contain eggs of $\mathcal{S}'$ in H. Then, we have the three possible situations, illustrated in Figure 13. We could have $v, w \in A$, one of v or w in A, or both $v, w \notin A$. We claim that in passing from G to H, the number of edges in our egg-cut could only decrease. If $v, w \in A$, then we lose $m + n$; if $v \in A$ and $w \notin A$ we lose $m - \min\{m, n\}$ (and similar if $w \in A$ and $v \notin A$); and if $v, w \notin A$ we lose no edges. Thus for any egg-cut for $\mathcal{S}$, there is an egg-cut with at most that many edges for $\mathcal{S}'$, meaning $e(\mathcal{S}') \leq e(\mathcal{S})$.

It follows that $||\mathcal{S}'|| \leq ||\mathcal{S}||$, implying that $\text{sn}(H) \leq \text{sn}(G)$, as desired. □

Since scramble number is multi-topological minor monotone, we could hope for a finite forbidden multi-topological minor characterization of $\mathscr{S}_m$ for $m \geq 3$. For instance, for $m + 1$ even and $n \geq 2(m + 1)$, the graphs $C_{n,(m+1)/2}$ all have $C_{2(m+1),(m+1)/2}$ as a multi-topological minor. Moreover, $C_{2(m+1),(m+1)/2}$ is minimal among graphs of scramble number $2(m + 1)$ with respect to the multi-topological minor relation: performing any multi-smoothing would decrease the number of vertices to $2m + 1$, bringing the scramble number down. For $m + 1$ odd, the graph $\tilde{C}_{2(m+1)+1;\lfloor (m+1)/2 \rfloor}$ plays a similar role. An interesting direction for future work would be to determine if this multi-topological minor relation could lead to finite characterizations of $\mathscr{S}_m$.

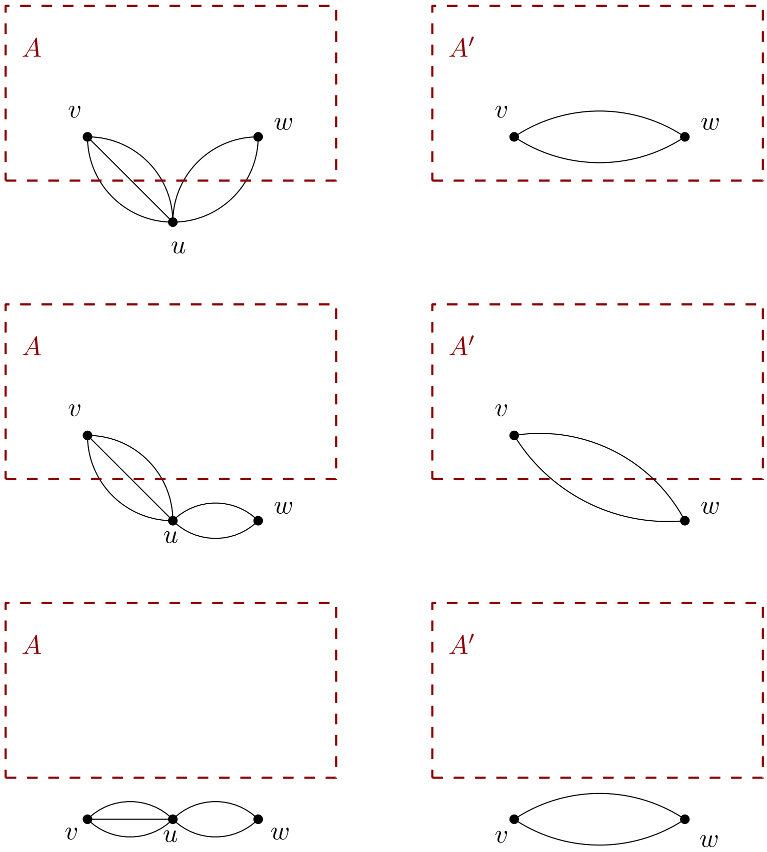

Figure 13: Possible relations of u, v, w to an egg-cut during a multi-smoothing

5 Applications

In this section we present several applications of Theorem 1.1.

The first result we present is on a variant of scramble number. We say that a scramble $\mathcal{S}$ on a graph G is *disjoint* if $E_i \cap E_j = \emptyset$ for all distinct eggs $E_i, E_j \in \mathcal{S}$. The *disjoint scramble number of a graph* G, denoted $\operatorname{dsn}(G)$, is the maximum possible order a disjoint scramble on G. Note that every disjoint scramble on a graph G is also a scramble on a graph G, so $\operatorname{dsn}(G) \leq \operatorname{sn}(G)$. In practice, disjoint scrambles are easier to work with, since the computation of $h(\mathcal{S})$ is immediate (in particular, it equals the number of eggs). As we will see in the following example, we may have $\operatorname{dsn}(G) \neq \operatorname{sn}(G)$.

Example 5.1. Consider the wheel graph W_5, consisting of a cycle on 5 vertices together with a vertex connected to all other vertices (pictured on the left in Figure 14). We claim that $\operatorname{sn}(W_5) = 4$. Pictured in the middle of Figure 14 is a tree-cut decomposition of W_5 of width 4, so $\operatorname{sn}(G) \leq \operatorname{scw}(G) \leq 4$. For the lower bound, we consider the 2-uniform scramble $\mathcal{E}_2$ on W_5, whose eggs are all sets of the form $\{u, v\}$ with $u \neq v$ and $W_5[u, v]$ connected (that is, with $uv \in E(W_5)$). By [2, Theorem 3.1], we have

$$||\mathcal{E}_2|| = \min\{\lambda_2(W_5), |V(W_5)| - \alpha(W_5)\},$$

where $\lambda_2(G)$ denotes the smallest number of edges necessary to disconnect a graph G into two connected components, each with at least 2 vertices; and $\alpha(G)$ denotes the independence number of G, the maximum size of a subset of $V(G)$ with no two elements adjacent in G. Inspecting all connected subgraphs of W_5 with 2 or 3 vertices, we find $\lambda_2(W_5) = 4$; and we find $\alpha(W_5) = 2$. Thus we have $\operatorname{sn}(W_5) \geq ||\mathcal{E}_2|| = 4$, so $\operatorname{sn}(W_5) = 4$.

We now claim that $\operatorname{dsn}(W_5) = 3$. Let $\mathcal{S}$ be a disjoint scramble on W_5. If $\mathcal{S}$ has three eggs or fewer, $||\mathcal{S}|| \leq h(\mathcal{S}) \leq 3$. Otherwise, $\mathcal{S}$ has at least four eggs. Since there are six vertices in W_5, at least two of the eggs must consist of a single vertex, meaning at least one of the eggs consists of a single vertex of degree 3. The set of edges incident to that vertex then forms an egg-cut, so $||\mathcal{S}|| \leq e(\mathcal{S}) \leq 3$. Therefore if $\mathcal{S}$ is a disjoint scramble, it has order at most 3, meaning that

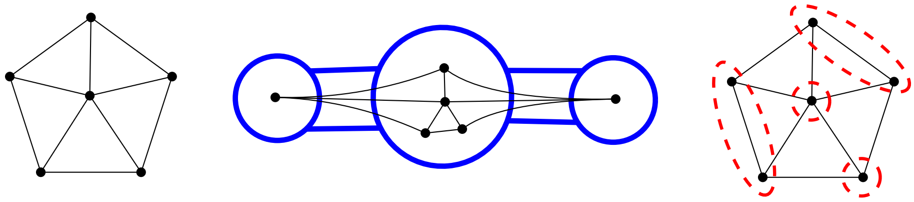

Figure 14: The wheel graph W_5, with a tree-cut decomposition of width 4 and a disjoint scramble of order 3

$\mathrm{dsn}(W_5) \leq 3$. There do indeed disjoint scrambles of order 3, such as the one on the right in Figure 14, so $\mathrm{dsn}(W_5) = 3$.

In the following proposition we demonstrate precisely when it is possible for scramble number and disjoint scramble number to be distinct.

Proposition 5.2. For all graphs G with $\mathrm{sn}(G) \leq 3$, we have $\mathrm{dsn}(G) = \mathrm{sn}(G)$. However, for $n \geq 4$, there exists a graph G with $n = \mathrm{sn}(G) > \mathrm{dsn}(G)$.

Proof. If G is a graph with $\mathrm{sn}(G) = 1$, then letting $v \in V(G)$ the scramble $\mathcal{S} = \{\{v\}\}$ has $||\mathcal{S}|| = 1$ and so $\mathrm{dsn}(G) = \mathrm{sn}(G)$. If G is a graph with $\mathrm{sn}(G) = 2$, G is not a tree. Thus, G contains some cycle C. Let u and v be two distinct vertices on C, and let $\mathcal{S} = \{\{u\}, \{v\}\}$. An egg-cut must cut C in two places, and so $e(\mathcal{S}) \geq 2$. Additionally, as the two eggs of $\mathcal{S}$ are disjoint, $h(\mathcal{S}) = 2$ and thus $||\mathcal{S}|| = 2$. It follows that $\mathrm{dsn}(G) \geq 2$ and so $\mathrm{sn}(G) = \mathrm{dsn}(G)$.

If $\mathrm{sn}(G) = 3$, then by Theorem 1.1 we know that G contains one of the graphs from Figure 1 as a topological minor; call that graph M. Let H be a subgraph of G that is a subdivision of M. Build a scramble on H as follows: start with the scramble $\mathcal{S}_M$ on M illustrated in Figure 4, and modify the eggs to include any vertices in edges that are subdivided in passing from M to H, thereby obtaining a scramble $\mathcal{S}_H$ on H. The eggs have remained disjoint, and the number of edge-disjoint paths connecting each pairs of eggs has remained unchanged, so $||\mathcal{S}_H|| = ||\mathcal{S}_M|| = 3$. Finally, we may consider our scramble as a scramble $\mathcal{S}_G$ on G, and we have $\mathrm{dsn}(G) \geq ||\mathcal{S}_G|| \geq ||\mathcal{S}_H|| = 3$ (in passing from H to G, the hitting number remains unchanged, and the egg-cut number could only increase). Since $\mathrm{sn}(G) = 3$, we have $\mathrm{dsn}(G) = 3$, as desired.

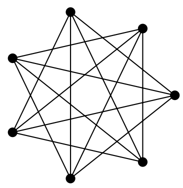

Figure 15: A graph with scramble number 5 and disjoint scramble number 4

We already know that there exists a graph G with $\mathrm{dsn}(G) < \mathrm{sn}(G) = 4$, namely the wheel graph W_5 from Example 5.1. Now let $n \geq 5$, and let G be the graph obtained from a complete graph K_{n+2} by deleting a cycle of length $n+2$; this graph is illustrated for $n = 5$ in Figure 15. Note that $\alpha(G) = 2$: every vertex is connected to all but two other vertices, which are connected to each other. The minimum degree $\delta(G)$ equals $n-1$, which since $n \geq 5$ is at least $\lfloor (n+2)/2 \rfloor - 1 = \lfloor |V(G)|/2 \rfloor - 1$. This allows us to apply [4, Corollary 3.2] to deduce that $\mathrm{sn}(G) = |V(G)| - \alpha(G) = (n+2) - 2 = n$. To see that $\mathrm{dsn}(G) < n$, suppose that $\mathcal{S}$ is a disjoint scramble on G. If $h(\mathcal{S}) \geq n$, then at least one of the n or more eggs has at most one vertex in it. But deleting the $n-1$ edges incident to that vertex would then give an egg-cut, meaning $||\mathcal{S}|| \leq n-1$. Thus $\mathrm{dsn}(G) \leq n - 1 < n = \mathrm{sn}(G)$. □

We close with a result concerning the computational complexity of scramble number. It is known that computing the scramble number of a graph is NP-hard [4]. We can ask if, for a fixed k, there exists a polynomial-time algorithm to check whether $\mathrm{sn}(G) \leq k$. For $k = 1$, the answer is yes, since there are efficient algorithms to check if a graph is a tree or not. For $k = 2$, Theorem 1.1 allows us to determine the answer is still yes. We first recall the following useful result.

Theorem 5.3 ([6, Theorem 1.1])**.** For every graph H, there is a $O(|V(G)|^3)$ time algorithm that decides if H is a topological minor of G.

Corollary 5.4. There is a $O(|V(G)|^3)$ time algorithm that decides whether $\text{sn}(G) \leq 2$.

Proof. Let G be a graph. From Theorem 5.3 we know there is a $O(|V(G)|^3)$ time algorithm to check if any fixed graph H is a topological minor of G. To check if G has $\text{sn}(G) \leq 2$ we must simply run this algorithm four times to check for graphs K_4, $P_{3;3}$, $C_{3;2,2,1}$, and LL_6. If G contains one of these graphs as a topological minor, then $\text{sn}(G) > 2$. If G does not, then $\text{sn}(G) \leq 2$. □

For fixed $k \geq 3$, the question is to our knowledge open.

Question 5.5. For what fixed values of $k \geq 3$ does there exist a polynomial time algorithm to determine whether $\text{sn}(G) \leq k$?

We remark that for $k = 3$, the answer will be the same if we replace $\text{sn}(G)$ with $\text{dsn}(G)$ (which may be simpler to work with).

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