Journal of Symbolic Computation 115 (2023) 223–247

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

A certified iterative method for isolated singular roots

Angelos Mantzaflaris [a], Bernard Mourrain [a], Agnes Szanto [b]

[a] Inria Sophia Antipolis, Université Côte d'Azur, 2004 route des Lucioles, B.P. 93, Sophia Antipolis, 06902, France
[b] Dept. of Mathematics, North Carolina State University, Campus Box 8205, Raleigh, NC 27965, USA

ARTICLE INFO

Article history:
Available online 4 August 2022

Keywords:
Polynomial equations
Isolated solution
Multiple root
Certification
Newton iteration
Deflation

ABSTRACT

In this paper we provide a new method to certify that a nearby polynomial system has a singular isolated root and we compute its multiplicity structure. More precisely, given a polynomial system $\mathbf{f} = (f_1, \ldots, f_N) \in \mathbb{C}[x_1, \ldots, x_n]^N$, we present a Newton iteration on an extended deflated system that locally converges, under regularity conditions, to a small deformation of f such that this deformed system has an exact singular root. The iteration simultaneously converges to the coordinates of the singular root and the coefficients of the so-called inverse system that describes the multiplicity structure at the root. We use α-theory test to certify the quadratic convergence, and to give bounds on the size of the deformation and on the approximation error. The approach relies on an analysis of the punctual Hilbert scheme, for which we provide a new description. We show in particular that some of its strata can be rationally parametrized and exploit these parametrizations in the certification. We show in numerical experimentation how the approximate inverse system can be computed as a starting point of the Newton iterations and the fast numerical convergence to the singular root with its multiplicity structure, certified by our criteria.

E-mail addresses: angelos.mantzaflaris@inria.fr (A. Mantzaflaris), bernard.mourrain@inria.fr (B. Mourrain), aszanto@ncsu.edu (A. Szanto).

https://doi.org/10.1016/j.jsc.2022.08.006
0747-7171/

1. Introduction

Local numerical methods such as Newton iterations have proved their efficiency to approximate and certify the existence of simple roots. However, for multiple roots they dramatically fail to provide fast numerical convergence and certification. The motivation for this work is to find a method with fast convergence to an exact singular point and its multiplicity structure for a small perturbation of the input polynomials, and to give numerical tests that can certify it. The knowledge of the multiplicity structure together with a high precision numerical approximation of a singular solution can be valuable information in many problems.

In Mourrain (1997) a method called later *integration method* is devised to compute the so-called *inverse system* or multiplicity structure at a multiple root. It is used in Mantzaflaris and Mourrain (2011) to compute an approximation of the inverse system, given an approximation of that root and to obtain a perturbed system that satisfies the duality property. However, this method did not give a way to improve the accuracy of the initial approximation of the root and the corresponding inverse system. In Hauenstein et al. (2016) a new one-step deflation method is presented that gives an overdetermined polynomial system in the coordinates of the roots and the corresponding inverse system, serving as a starting point for the present paper. However, for certification, (Hauenstein et al., 2016) refers to the symbolic-numeric method in Ayyildiz Akoglu et al. (2018) that only works if the input system is given exactly with rational coefficients and have a multiple root with the prescribed multiplicity structure.

In the present paper we give a solution for the following problem:

Problem 1.1. Given a polynomial system $\mathbf{f} = (f_1, \ldots, f_N) \in \mathbb{C}[\mathbf{x}]^N$ and a point $\xi \in \mathbb{C}^n$, deduce an iterative method that converges quadratically to the triple $(\xi^*, \mu^*, \epsilon^*)$ such that $\xi^* \in \mathbb{C}^n$, μ^* defines the coefficients of a basis $\mathbf{\Lambda}^* = \{\Lambda_1^*, \ldots, \Lambda_r^*\} \subset \mathbb{C}[\mathbf{d}_{\xi^*}]$ dual to the set $B_{\xi^*} = \{(\mathbf{x} - \xi^*)^{\beta_1}, \ldots, (\mathbf{x} - \xi^*)^{\beta_r}\} \subset \mathbb{C}[\mathbf{x}]$ and ϵ^* defines a perturbed polynomial system $\mathbf{f}_{\epsilon^*} := \mathbf{f} + \epsilon^* B_{\xi^*}$ with the property that ξ^* is an exact multiple root of $\mathbf{f}_{\epsilon^*}$ with inverse system $\mathbf{\Lambda}^*$. Furthermore, certify this property and give an upper bound on the size of the perturbation $\|\epsilon^*\|$.

The difficulty in solving Problem 1.1 is that known polynomial systems defining the coordinates of the roots and the inverse system are overdetermined, and we need a square subsystem of it in the Newton iterations to guarantee the existence of a root (and not merely a local minimum of the norm of the function value), which at the same time ensures a quadratic convergence to the root. Thus, roots of this square subsystem may not be exact roots of the complete polynomial system, and we cannot certify numerically that they are approximations of a root of the complete system. This is the reason why we introduce the variables ϵ that allow perturbation of the input system. One of the goals of the present paper is to understand what kind of perturbations are needed and to bound their magnitude.

Certifying the correctness of the multiplicity structure that the numerical iterations converge to poses a more significant challenge: the set of parameter values describing an affine point with multiplicity r forms a projective variety called the *punctual Hilbert scheme*. The goal is to certify that we converge to a point on this variety. We study an affine subset of the punctual Hilbert scheme and give a new description using multilinear quadratic equations that have a triangular structure. These equations appear in our deflated polynomial system, have integer coefficients, and have to be satisfied exactly without perturbation, otherwise the solution does not define a proper inverse system, closed under derivation. Fortunately, the structure allowed us to define a rational parametrization of a strata of the punctual Hilbert scheme, called the *regular* strata. In turn, this rational parametrization allows certification when converging to a point on this regular strata.

Our method comprises three parts: first, we apply the Integration Method (Algorithm 1) with input $\mathbf{f}$ and ξ to compute an approximation of the multiplicity structure, second, an analysis and certification part (see Section 6 and Algorithm 2), and third, a numerical iteration part converging to the exact multiple root with its multiplicity structure for an explicit perturbation of the input system (see Section 5).

This paper is an extended version of the paper (Mantzaflaris et al., 2020). The present version contains a new result, presented in Subsection 5.2, on how the updates of our Newton iteration can be evaluated efficiently from the previous iterates, without resorting to the symbolic expression of the dual basis in terms of parameters. Furthermore, we give a more detailed explanation of our examples and numerical experimentation in Section 7. Moreover, in the present version we included all proofs that were left aside in the proceedings version (Proofs of Propositions 2.3 and 4.4; Lemma 2.5; Theorems 2.8, 3.3, 5.1 and 6.1).

Related work There are many works in the literature studying the certification of isolated singular roots of polynomial systems. One approach is to give *separation bounds* for isolated roots, i.e. a bound that guarantees that there is exactly one root within a neighborhood of a given point. Worst case separation bounds for square polynomial systems with support in given polytopes and rational coefficients are presented in Emiris et al. (2010). In the presence of singular roots, turned into root clusters after perturbations, these separation bounds separate the clusters from each other and bound the cluster size. Yakoubsohn (2000, 2002); Giusti et al. (2005) give separation bounds and numerical algorithms to compute clusters of zeroes of univariate polynomials. (Dedieu and Shub, 2001) extends α-theory and gives separation bounds for simple double zeroes of polynomial systems, Giusti et al. (2007) extend these results to zeroes of embedding dimension one.

Another approach, called deflation, comprises of transforming the singular root into a regular root of a new system and to apply certification techniques on the new system. Kanzawa et al. (1997) uses a square deflated system to prove the existence of singular solutions. Leykin et al. (2006) devises a deflation technique that adds new variables to the systems for isolated singular roots that accelerates Newton's method and Leykin et al. (2008) modifies this to compute the multiplicity structure. Rump and Graillat (2010) computes error bounds that guarantee the existence of a simple double root within that error bound from the input, Li and Zhi (2013, 2014) generalizes Rump and Graillat (2010) to the breadth one case and give an algorithm to compute such error bound. Li and Sang (2015) gives verified error bounds for isolated and some non-isolated singular roots using higher order deflations. Dayton and Zeng (2005); Wu and Zhi (2008a); Zeng (2009); Wu and Zhi (2008b); Dayton et al. (2011); Hao et al. (2013) give deflation techniques based on numerical linear algebra on the Macaulay matrices that compute the coefficients of the inverse system, with improvements using the closedness property of the dual space. Giusti and Yakoubsohn (2013, 2020) give a new deflation method that does not introduce new variables and extends α-theory to general isolated multiple roots for the certification to a simple root of a subsystem of the overdetermined deflated system. In Hauenstein et al. (2016) a new deflated system is presented, its simple roots correspond to the isolated singular points with their multiplicity structure. A somewhat different approach is given in Ayyildiz Akoglu et al. (2018), where they use a symbolic-numeric certification techniques that certify that polynomial systems with rational coefficients have exact isolated singular roots. More recently, Lee et al. (2019) design a square Newton iteration and provide separation bounds for roots when the deflation method of Leykin et al. (2006) terminates in one iteration, and give bounds for the size of the clusters.

The certification approach that we propose is based on an algebraic analysis of some strata of the punctual Hilbert scheme. Some of its geometric properties have been investigated long time ago, for instance in Briançon (1977); Iarrobino (1977); Briançon and Iarrobino (1978) or more recently in the plane (Bejleri and Stapleton, 2017). However, as far as we know, the effective description that we use and the rational parametrization of the regular strata that we compute have not been developed previously.

The paper is structured as follows. In the next Section we recall the main definitions and algorithms regarding isolated multiple points. In Section 3 we define the punctual Hilbert Scheme and in Section 4 we show that it admits a rational parametrization for its regular part, which can be obtained algorithmically. Then in Sections 5 and 6 we describe the construction and the certification of a Newton procedure for computing a multiple point to high accuracy. Finally in Section 7 we develop some examples and benchmarks of the proposed approach.

2. Preliminaries

Let $\mathbf{f} := (f_1, \ldots, f_N) \in \mathbb{C}[\mathbf{x}]^N$ with $\mathbf{x} = (x_1, \ldots, x_n)$. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ be an isolated multiple root of $\mathbf{f}$. Let $I = \langle f_1, \ldots, f_N \rangle$, $\mathfrak{m}_\xi$ be the maximal ideal at ξ and Q be the primary component of I at $\boldsymbol{\xi}$ so that $\sqrt{Q} = \mathfrak{m}_\xi$. The shifted monomials at ξ will be denoted for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ by

$$\mathbf{x}_\xi^\alpha := (x_1 - \xi_1)^{\alpha_1} \cdots (x_1 - \xi_n)^{\alpha_n}.$$

2.1. Duality and differential polynomials

Consider the ring of power series $\mathbb{C}[[\mathbf{d}_\xi]] := \mathbb{C}[[d_{1,\xi}, \ldots, d_{n,\xi}]]$ and we denote $\mathbf{d}_\xi^\beta := d_{1,\xi}^{\beta_1} \cdots d_{n,\xi}^{\beta_n}$, with $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. We identify $\mathbb{C}[[\mathbf{d}_\xi]]$ with the dual space $\mathbb{C}[\mathbf{x}]^*$ by considering the action of $\mathbf{d}_\xi^\beta$ on polynomials as derivations and evaluations at $\boldsymbol{\xi}$, defined as

$$\langle \mathbf{d}_\xi^\beta \mid p \rangle_\xi := \mathbf{d}_\xi^\beta(p) = \boldsymbol{\partial}^\beta(p)\big|_{\boldsymbol{\xi}} = \frac{\partial^{|\beta|} p}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(\boldsymbol{\xi}) \quad \text{for } p \in \mathbb{C}[\mathbf{x}]. \tag{1}$$

More generally, for $\Lambda = \sum_\alpha \Lambda_\alpha \mathbf{d}_\xi^\alpha \in \mathbb{C}[[\mathbf{d}_\xi]]$ and $p \in \mathbb{C}[\mathbf{x}]$, we denote $\langle \Lambda | p \rangle_\xi := \Lambda(\mathbf{d}_\xi)(p) = \sum_\alpha \Lambda_\alpha \boldsymbol{\partial}^\alpha(p)\big|_{\boldsymbol{\xi}}$. Hereafter, we reserve the notation $\mathbf{d}$ and d_i for the dual variables while $\boldsymbol{\partial}$ and ∂_{x_i} for derivation. We indicate the evaluation at $\xi \in \mathbb{C}^n$ by writing $d_{i,\xi}$ and $\mathbf{d}_\xi$, and for $\xi = 0$ it will be denoted by $\mathbf{d}$. The derivation with respect to the variable $d_{i,\xi}$ in $\mathbb{C}[[\mathbf{d}_\xi]]$ is denoted $\partial_{d_{i,\xi}}$ $(i = 1, \ldots, n)$. Observe that

$$\frac{1}{\beta!} \mathbf{d}_\xi^\beta((\mathbf{x} - \boldsymbol{\xi})^\alpha) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta! = \beta_1! \cdots \beta_n!$.

For $p \in \mathbb{C}[\mathbf{x}]$ and $\Lambda \in \mathbb{C}[[\mathbf{d}_\xi]] = \mathbb{C}[\mathbf{x}]^*$, let $p \star \Lambda : q \mapsto \Lambda(p\,q)$. We check that $p = (x_i - \xi_i)$ acts as a derivation on $\mathbb{C}[[\mathbf{d}_\xi]]$: $(x_i - \xi_i) \star \mathbf{d}_\xi^\beta = \partial_{d_{i,\xi}}(\mathbf{d}_\xi^\beta) = \beta_i \mathbf{d}_\xi^{\beta - \mathbf{e}_i}$. Throughout the paper we use the notation $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for the standard basis of $\mathbb{C}^n$ or for a canonical basis of any vector space V of dimension n. We will also use integrals of polynomials in $\mathbb{C}[[\mathbf{d}_\xi]]$ as follows: for $\Lambda \in \mathbb{C}[[\mathbf{d}_\xi]]$ and $k = 1, \ldots, n$, $\int_k \Lambda$ denotes the polynomial $\Lambda^* \in \mathbb{C}[[\mathbf{d}_\xi]]$ such that $\partial_{d_{k,\xi}}(\Lambda^*) = \Lambda$ and Λ^* has no constant term. We introduce the following shorthand notation

$$\overline{\int_k} \Lambda := \int_k \Lambda(d_{1,\xi}, \ldots, d_{k,\xi}, 0, \ldots, 0). \tag{2}$$

For an ideal $I \subset \mathbb{C}[\mathbf{x}]$, let $I^\perp = \{\Lambda \in \mathbb{C}[[\mathbf{d}_\xi]] \mid \forall p \in I, \Lambda(p) = 0\}$. The vector space $I^\perp$ is naturally identified with the dual space of $\mathbb{C}[\mathbf{x}]/I$. We check that $I^\perp$ is a vector subspace of $\mathbb{C}[[\mathbf{d}_\xi]]$ which is closed under the derivations $\partial_{d_{i,\xi}}$ for $i = 1, \ldots, n$.

Lemma 2.1. *If Q is a $\mathfrak{m}_\xi$-primary isolated component of I, then $Q^\perp = I^\perp \cap \mathbb{C}[\mathbf{d}_\xi]$.*

This lemma shows that to compute $Q^\perp$, it suffices to compute all polynomials of $\mathbb{C}[\mathbf{d}_\xi]$ which are in $I^\perp$. Let us denote this set $\mathcal{D} = I^\perp \cap \mathbb{C}[\mathbf{d}_\xi]$. It is a vector space stable under the derivations $\partial_{d_{i,\xi}}$. Its dimension is the dimension of $Q^\perp$ or $\mathbb{C}[\mathbf{x}]/Q$, that is the *multiplicity* of ξ, denoted $r_\xi(I)$, or simply r if ξ and I is clear from the context.

For an element $\Lambda(\mathbf{d}_\xi) \in \mathbb{C}[\mathbf{d}_\xi]$ we define the degree or *order* $\operatorname{ord}(\Lambda)$ to be the maximal $|\beta|$ s.t. $\mathbf{d}_\xi^\beta$ appears in $\Lambda(\mathbf{d}_\xi)$ with non-zero coefficient.

For $t \in \mathbb{N}$, let $\mathcal{D}_t$ be the elements of $\mathcal{D}$ of order $\leq t$. As $\mathcal{D}$ is of dimension r, there exists a smallest $t \geq 0$ s.t. $\mathcal{D}_{t+1} = \mathcal{D}_t$. Let us call this smallest t, the *nil-index* of $\mathcal{D}$ and denote it by $\delta_\xi(I)$, or simply by

δ. As $\mathcal{D}$ is stable by the derivations $\partial_{d_{i,\xi}}$, we easily check that for $t \geq \delta_\xi(I)$, $\mathcal{D}_t = \mathcal{D}$ and that $\delta_\xi(I)$ is the maximal degree of elements of $\mathcal{D}$.

Let $B = \{\mathbf{x}_\xi^{\beta_1}, \ldots, \mathbf{x}_\xi^{\beta_r}\}$ be a basis of $\mathbb{C}[\mathbf{x}]/Q$. We can identify the elements of $\mathbb{C}[\mathbf{x}]/Q$ with the elements of the vector space $\mathrm{span}_{\mathbb{C}}(B)$. We define the normal form $N(p)$ of a polynomial p in $\mathbb{C}[\mathbf{x}]$ as the unique element b of $\mathrm{span}_{\mathbb{C}}(B)$ such that $p - b \in Q$. Hereafter, we are going to identify the elements of $\mathbb{C}[\mathbf{x}]/Q$ with their normal form in $\mathrm{span}_{\mathbb{C}}(B)$. For $\alpha \in \mathbb{N}^n$, we will write the normal form of $\mathbf{x}_\xi^\alpha$ as

$$N(\mathbf{x}_\xi^\alpha) = \sum_{i=1}^{r} \mu_{\beta_i,\alpha}\, \mathbf{x}_\xi^{\beta_i}. \tag{3}$$

2.2. The multiplicity structure

We start this subsection by recalling the definition of graded primal-dual pairs of bases for the space $\mathbb{C}[\mathbf{x}]/Q$ and its dual. The following lemma defines the same dual space as in e.g. (Dayton and Zeng, 2005; Dayton et al., 2011; Li and Zhi, 2014), but we emphasize on a primal-dual basis pair to obtain a concrete isomorphism between the factor ring and the dual space.

Lemma 2.2 *(Graded primal-dual basis pair). Let* $\mathbf{f}$, I, $\boldsymbol{\xi}$, Q, $\mathcal{D}$, $r = r_\xi(I)$ *and* $\delta = \delta_\xi(I)$ *be as above. Then there exists a primal-dual basis pair* $(B, \boldsymbol{\Lambda})$ *of the local ring* $\mathbb{C}[\mathbf{x}]/Q$ *with the following properties:*

1. *The* primal basis *of the local ring* $\mathbb{C}[\mathbf{x}]/Q$ *has the form*
$$B := \left\{\mathbf{x}_\xi^{\beta_1}, \mathbf{x}_\xi^{\beta_2}, \ldots, \mathbf{x}_\xi^{\beta_r}\right\}. \tag{4}$$
We can assume that $\beta_1 = 0$ *and that the ordering of the elements in* B *by increasing degree. Define the set of exponents in* B *as* $E := \{\beta_1, \ldots, \beta_r\} \subset \mathbb{N}^n$.
2. *The unique* dual basis $\boldsymbol{\Lambda} = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_r\}$ *of* $\mathcal{D} \subset \mathbb{C}[\mathbf{d}_\xi]$ *dual to* B *has the form*
$$\Lambda_i = \frac{1}{\beta_i!}\mathbf{d}_{\boldsymbol{\xi}}^{\beta_i} + \sum_{\substack{|\alpha| \leq |\beta_i| \\ \alpha \notin E}} \mu_{\beta_i,\alpha}\, \frac{1}{\beta!}\mathbf{d}_{\boldsymbol{\xi}}^{\alpha}.$$
3. *We have* $0 = \mathrm{ord}(\Lambda_1) \leq \cdots \leq \mathrm{ord}(\Lambda_r)$, *and for all* $0 \leq t \leq \delta$ *we have* $\mathcal{D}_t = \mathrm{span}\left\{\Lambda_j \,:\, \mathrm{ord}(\Lambda_j) \leq t\right\}$, *where* $\mathcal{D}_t$ *denotes the elements of* $\mathcal{D}$ *of order* $\leq t$, *as above.*

A graded primal-dual basis pair $(B, \boldsymbol{\Lambda})$ of $\mathcal{D}$ as described in Lemma 2.2 can be obtained from any basis $\tilde{\boldsymbol{\Lambda}}$ of $\mathcal{D}$ by first choosing pivot elements that are the leading monomials with respect to a graded monomial ordering on $\mathbb{C}[\mathbf{d}]$, these leading monomials define B, then transforming the coefficient matrix of $\tilde{\boldsymbol{\Lambda}}$ into row echelon form using the pivot leading coefficients, defining $\boldsymbol{\Lambda}$.

A monomial set B is called a *graded primal basis* of $\mathbf{f}$ at ξ if there exists $\boldsymbol{\Lambda} \subset \mathbb{C}[\mathbf{d}_\xi]$ such that $(B, \boldsymbol{\Lambda})$ is a graded primal-dual basis pair and $\boldsymbol{\Lambda}$ is complete for $\mathbf{f}$ at ξ.

Next we describe the so-called *integration method* introduced in Mourrain (1997); Mantzaflaris and Mourrain (2011) that computes a graded pair of primal-dual bases as in Lemma 2.2 if the root ξ is given. The integration method performs the computation of a basis order by order. We need the following proposition, a new version of (Mourrain, 1997, Theorem 4.2):

Proposition 2.3. *Let* $\Lambda_1, \ldots, \Lambda_s \in \mathbb{C}[\mathbf{d}_\xi]$ *and assume that* $\mathrm{ord}(\Lambda_i) \leq t$ *for some* $t \in \mathbb{N}$. *Suppose that the subspace* $\mathcal{D} := \mathrm{span}(\Lambda_1, \ldots, \Lambda_s) \subset \mathbb{C}[\mathbf{d}_\xi]$ *is closed under derivation. Then* $\Delta \in \mathbb{C}[\mathbf{d}_\xi]$ *with no constant term satisfies* $\partial_{d_k}(\Delta) \in \mathcal{D}$ *for all* $k = 1, \ldots, n$ *if and only if* Δ *is of the form*

$$\Delta = \sum_{i=1}^{s}\sum_{k=1}^{n} \nu_i^k \overline{\int_k} \Lambda_i \tag{5}$$

for some $\nu_i^k \in \mathbb{C}$ satisfying

$$\sum_{i=1}^{s} \nu_i^k \partial_{d_l}(\Lambda_i) - \nu_i^l \partial_{d_k}(\Lambda_i) = 0 \text{ for } 1 \le k < l \le n. \tag{6}$$

Furthermore, (5) and (6) implies that

$$\partial_{d_k}(\Delta) = \sum_{i=1}^{s} \nu_i^k \Lambda_i \quad \text{for } k = 1, \dots, n. \tag{7}$$

Proof. Suppose $\Lambda \in \mathbb{C}[\mathbf{d}]$ with no constant term satisfies $\partial_{d_k}(\Lambda) \in \mathcal{D}$ for all $k = 1, \dots, n$. To prove (5), we can proceed exactly as in the proof of (Mourrain, 1997, Theorem 4.2): we write Δ uniquely as

$$\Delta = \Delta_1(d_1, \dots, d_n) + \Delta_2(d_2, \dots, d_n) + \cdots, + \Delta_n(d_n)$$

with $\Delta_i \in \mathbb{C}[d_i, \dots, d_n] \setminus \mathbb{C}[d_{i+1}, \dots, d_n]$. Then $\int_i \partial_{d_i} \Delta_i = \Delta_i$. Then we prove that by induction on k that if $\sigma_k := \Delta_1 + \cdots + \Delta_k$ then

$$\Delta_k = \sum_{j=1}^{s} \nu_j^k \int_k \Lambda_j - \left(\sigma_{k-1} - \sigma_{k-1}|_{d_k=0}\right)$$

and

$$\begin{aligned}\sigma_k = \Delta_k + \sigma_{k-1} &= \sum_{j=1}^{s} \nu_j^k \int_k \Lambda_j + \sigma_{k-1}|_{d_k=0} \\ &= \sum_{j=1}^{s} \nu_j^k \int_k \Lambda_j + \sum_{j=1}^{s} \nu_j^{k-1} \int_k \Lambda_j|_{d_k=0} + \cdots + \sum_{j=1}^{s} \nu_j^1 \int_k \Lambda_j|_{d_k=0,\cdots d_2=0}.\end{aligned}$$

Conversely, suppose that $\Lambda \in \mathbb{C}[\mathbf{d}]$ with no constant term is of the form (5) satisfying (6). Define $\bar{\Delta}_1 = \bar{\sigma}_1 := \sum_{j=1}^{s} \nu_j^1 \int_1 \Lambda_j$ and for $k = 2, \dots n$ define

$$\bar{\Delta}_k := \sum_{j=1}^{s} \nu_j^k \int_k \Lambda_j - \left(\sigma_{k-1} - \sigma_{k-1}|_{d_k=0}\right)$$

and $\bar{\sigma}_k := \bar{\Delta}_1 + \cdots + \bar{\Delta}_k$. Then in the proof of (Mourrain, 1997, Theorem 4.2) it is shown that $\bar{\Delta}_k \in \mathbb{C}[d_k, \dots, d_n] \setminus \mathbb{C}[d_{k+1}, \dots, d_n]$ and

$$\bar{\sigma}_k = \sum_{j=1}^{s} \nu_j^k \int_k \Lambda_j + \sum_{j=1}^{s} \nu_j^{k-1} \int_k \Lambda_j|_{d_k=0} + \cdots + \sum_{j=1}^{s} \nu_j^1 \int_k \Lambda_j|_{d_k=0,\cdots d_2=0}$$

so we get that $\partial_{d_k}(\Lambda) = \partial_{d_k}(\bar{\sigma}_k) = \sum_{j=1}^{s} \nu_j^k \Lambda_j \in \mathcal{D}_t$ as claimed. □

Let Q be a $\mathfrak{m}_\xi$-primary ideal. Proposition 2.3 implies that if $\mathbf{\Lambda} = \{\Lambda_1, \dots, \Lambda_r\} \subset \mathbb{C}[\mathbf{d}_\xi]$ with $\Lambda_1 = 1_\xi$ is a basis of $Q^\perp$, dual to the basis $B = \{\mathbf{x}_\xi^{\beta_1}, \dots, \mathbf{x}_\xi^{\beta_r}\} \subset \mathbb{C}[\mathbf{x}]$ of $\mathbb{C}[\mathbf{x}]/Q$ with $\mathrm{ord}(\Lambda_i) = |\beta_i|$, then there exist $\nu_{i,j}^k \in \mathbb{C}$ such that

$$\partial_{d_k}(\Lambda_i) = \sum_{|\beta_j| < |\beta_i|} \nu_{i,j}^k \Lambda_j.$$

Therefore, the matrix $\mathtt{M}_k$ of the multiplication map M_k by $x_k - \xi_k$ in the basis B of $\mathbb{C}[\mathbf{x}]/Q$ is

$$\mathrm{M}_k = [\nu^k_{j,i}]^T_{1\le i,j\le r} = [\mu_{\beta_i,\beta_j+\mathbf{e}_k}]_{1\le i,j\le r}$$

using the notation (3) and the convention that $\nu^k_{i,j} = \mu_{\beta_i,\beta_j+\mathbf{e}_k} = 0$ if $|\beta_i| \ge |\beta_j|$. Consequently,

$$\nu^k_{i,j} = \mu_{\beta_i,\beta_j+e_k} \quad i, j, = 1, \ldots, r, k = 1, \ldots, n,$$

and we have

$$\Lambda_i = \sum_{|\beta_j|<|\beta_i|} \sum_{k=1}^{n} \mu_{\beta_i,\beta_j+\mathbf{e}_k} \overline{\int_k} \Lambda_j$$

where $\mu_{\beta_i,\beta_j+\mathbf{e}_k}$ is the coefficient of $\mathbf{x}_\xi^{\beta_i}$ in the normal form of $\mathbf{x}_\xi^{\beta_j+\mathbf{e}_k}$ in the basis B of $\mathbb{C}[\mathbf{x}]/Q$.

Next we give a result that allows to simplify the linear systems involved in the integration method. We first need a definition:

Definition 2.4. Let $E \subset \mathbb{N}^n$ be a set of exponents. We say that E is *closed under division* if $\beta = (\beta_1, \ldots, \beta_n) \in E$ implies that $\beta - \mathbf{e}_k \in E$ as long as $\beta_k > 0$ for all $k = 1, \ldots, n$. We also call the corresponding primal basis $B = \{\mathbf{x}_\xi^{\beta_1}, \ldots, \mathbf{x}_\xi^{\beta_r}\}$ closed under division.

The following lemma provides a simple characterization of dual bases of inverse systems closed under derivation, that we will use in the integration algorithm.

Lemma 2.5. *Let $B = \{\mathbf{x}_\xi^{\beta_1}, \ldots, \mathbf{x}_\xi^{\beta_r}\} \subset \mathbb{C}[\mathbf{x}]$ be closed under division and ordered by degree. Let $\boldsymbol{\Lambda} = \{\Lambda_1, \ldots, \Lambda_r\} \subset \mathbb{C}[\mathbf{d}_\xi]$ be a linearly independent set such that*

$$\Lambda_i = \sum_{|\beta_j|<|\beta_i|} \sum_{k=1}^{n} \mu_{\beta_i,\beta_j+\mathbf{e}_k} \overline{\int_k} \Lambda_j. \tag{8}$$

Then $\mathcal{D} = \mathrm{span}\{\Lambda_1, \ldots, \Lambda_r\}$ is closed under derivation iff for all $i, s = 1, \ldots, r$, $|\beta_s| < |\beta_i|$ and $k \ne l \in \{1, \ldots, n\}$ we have

$$\sum_{j:|\beta_s|<|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k}\mu_{\beta_j,\beta_s+\mathbf{e}_l} - \mu_{\beta_i,\beta_j+\mathbf{e}_l}\mu_{\beta_j,\beta_s+\mathbf{e}_k} = 0. \tag{9}$$

Furthermore, $(B, \boldsymbol{\Lambda})$ is a graded primal-dual basis pair iff they satisfy (9) and

$$\mu_{\beta_i,\beta_j+\mathbf{e}_k} = \begin{cases} 1 & \text{for } \beta_i = \beta_j + \mathbf{e}_k \\ 0 & \text{for } \beta_j + \mathbf{e}_k \in E,\ \beta_i \ne \beta_j + \mathbf{e}_k. \end{cases} \tag{10}$$

Proof. Assume $\boldsymbol{\Lambda} = \{\Lambda_1, \ldots, \Lambda_r\}$ is linearly independent and $\mathcal{D} = \mathrm{span}(\Lambda)$ is closed under derivation. For $t \in \{0, \ldots, \delta\}$ denote by $\{\Lambda_1, \ldots, \Lambda_{r_t}\} = \boldsymbol{\Lambda} \cap \mathbb{C}[\mathbf{d}_\xi]_t$ and $\mathcal{D}_t = \mathrm{span}(\Lambda_1, \ldots, \Lambda_{r_t})$. Then by Proposition 2.3, $\boldsymbol{\Lambda}$ satisfy equations (7) for $t = 0, \ldots, \delta$ and for $j = 1, \ldots, r$, $k = 1, \ldots, n$, we have $\partial_{d_k}(\Lambda_j) = \sum_{|\beta_s|<|\beta_j|} \mu_{\beta_j,\beta_s+\mathbf{e}_k}\Lambda_s$. Substituting this to (6) we get for $i = 1, \ldots, r$

$$\begin{aligned} &\sum_{|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k} \sum_{|\beta_s|<|\beta_j|} \mu_{\beta_j,\beta_s+\mathbf{e}_l}\Lambda_s \\ &-\mu_{\beta_i,\beta_j+\mathbf{e}_l} \sum_{|\beta_s|<|\beta_j|} \mu_{\beta_j,\beta_s+\mathbf{e}_k}\Lambda_s = 0. \end{aligned} \tag{11}$$

Then using linear independence and collecting the coefficients of Λ_s we get (9).
Conversely, assume that (9) is satisfied. Then (11) is also satisfied. We use induction on t to prove that $\mathcal{D}_t$ is closed under derivation. For $t = 0$ there is nothing to prove. Assume $\mathcal{D}_{t-1}$ is closed under

derivation. Then by Proposition 2.3 if $|\beta_j| < t$ then $\partial_{d_k}(\Lambda_j) = \sum_{|\beta_s|<|\beta_j|} \mu_{\beta_j,\beta_s+\mathbf{e}_k}\Lambda_s$ for $k = 1, \ldots, n$. Thus for $|\beta_i| = t$, (11) implies that

$$\sum_{|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k}\partial_{d_l}(\Lambda_j) - \mu_{\beta_i,\beta_j+\mathbf{e}_l}\partial_{d_k}(\Lambda_j) = 0.$$

Again, by Proposition 2.3 we get that $\mathcal{D}_t$ is closed under derivation.

Next, assume first that (B, Λ) is a graded primal-dual basis pair. This means that for $i = 1, \ldots, r$ and for l such that $|\beta_l| \le |\beta_i|$

$$\begin{aligned}\delta_{i,l} = \Lambda_i\left(\mathbf{x}_\xi^{\beta_l}\right) &= \sum_{k=1}^{n}\sum_{|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k}\overline{\int_k}\Lambda_j\left(\mathbf{x}_\xi^{\beta_l}\right)\\ &= \sum_{k=1}^{n}\sum_{|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k}\operatorname{coeff}(\frac{\mathbf{d}^{\beta_l}}{\beta_l!}, \overline{\int_k}\Lambda_j)\end{aligned}$$

Fix k to be the index of the last non-zero entry of β_l. For all other k's $\mathbf{d}^{\beta_l}$ becomes zero when we substitute 0 into $d_{k+1}, \ldots, d_n$ in $\overline{\int_k}\Lambda_j$. Thus,

$$\begin{aligned}\Lambda_i\left(\mathbf{x}_\xi^{\beta_l}\right) &= \sum_{|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k}\operatorname{coeff}(\frac{\mathbf{d}^{\beta_l}}{\beta_l!}, \overline{\int_k}\Lambda_j)\\ &= \sum_{|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k}\operatorname{coeff}(\frac{\mathbf{d}^{\beta_l-\mathbf{e}_k}}{(\beta_l-\mathbf{e}_k)!}, \Lambda_j).\end{aligned}$$

Since E is closed under division, $\beta_l - \mathbf{e}_k = \beta_m \in E$ for some $m < l$. By duality, we have that $\operatorname{coeff}(\frac{\mathbf{d}^{\beta_m}}{(\beta_m)!}, \Lambda_j) = \delta_{m,j}$, so

$$\Lambda_i\left(\mathbf{x}_\xi^{\beta_l}\right) = \mu_{\beta_i,\beta_m+e_k} = \mu_{\beta_i,\beta_l}.$$

To satisfy $\Lambda_i\left(\mathbf{x}_\xi^{\beta_l}\right) = \delta_{i,l}$ we must have

$$\mu_{\beta_i,\beta_m+\mathbf{e}_k} = \begin{cases}1 & \text{if } \beta_i = \beta_m + \mathbf{e}_k\\ 0 & \text{if } \beta_m + \mathbf{e}_k = \beta_l \in E \text{ but } i \ne l.\end{cases}$$

Conversely, by induction on $t = |\beta_i|$ we have that $\deg(\Lambda_i) \le |\beta_i|$. Then $\Lambda_i\left(\mathbf{x}_\xi^{\beta_l}\right) = 0$ when $|\beta_l| > |\beta_i|$. For $|\beta_l| \le |\beta_i|$, relations (10) imply that the coefficient of $\frac{\mathbf{d}^{\beta_l}}{\beta_l!}$ in Λ_i is 0 if $i \ne l$ and 1 if $i = l$. Therefore (B, Λ) is a graded primal-dual basis pair. □

To compute the inverse system $\mathcal{D}$ of $\mathbf{f}$ at a point ξ, we will consider the additional systems of equations in ξ and $\mu = \{\mu_{\beta_i,\alpha}\}$:

$$\Lambda_i(f_j) = 0 \text{ for } 1 \le i \le r, 1 \le j \le N. \tag{12}$$

Throughout the paper we use the following notation:

Notation 2.6. Let $f_1, \ldots, f_N \in \mathbb{C}[\mathbf{x}]$, $\xi \in \mathbb{C}^n$ and fix $t \in \mathbb{N}$. Let $B_{t-1} = \{\mathbf{x}_\xi^{\beta_1}, \ldots, \mathbf{x}_\xi^{\beta_{r_{t-1}}}\} \subset \mathbb{C}[\mathbf{x}_\xi]_{t-1}$ be closed under division and $\mathbf{\Lambda}_{t-1} = \{\Lambda_1, \ldots, \Lambda_{r_{t-1}}\} \subset \mathbb{C}[\mathbf{d}_\xi]_{t-1}$ dual to B_{t-1} with

$$\partial_{d_k}(\Lambda_j) = \sum_{|\beta_s|<|\beta_j|} \mu_{\beta_j,\beta_s+\mathbf{e}_k}\Lambda_s \quad j = 1, \ldots, r_{t-1}, k = 1, \ldots, n.$$

Algorithm 1 Integration Method - Iteration t.

Input: $t>0$, $\mathbf{f}=(f_1,\dots,f_N)\in\mathbb{C}[\mathbf{x}]^N$, $\xi\in\mathbb{C}^n$, $B_{t-1}=\{\mathbf{x}_\xi^{\beta_1},\dots,\mathbf{x}_\xi^{\beta_{r_{t-1}}}\}\subset\mathbb{C}[\mathbf{x}]$ closed under division and $\boldsymbol{\Lambda}_{t-1}=\{\Lambda_1,\dots,\Lambda_{r_{t-1}}\}\subset\mathbb{C}[\mathbf{d}_\xi]$ a basis for $\mathcal{D}_{t-1}$ dual to B_{t-1}, of the form (8).

Output: Either "$\mathcal{D}_t=\mathcal{D}_{t-1}$" or $B_t=\{\mathbf{x}_\xi^{\beta_1},\dots,\mathbf{x}_\xi^{\beta_{r_t}}\}$ for some $r_t>r_{t-1}$ closed under division and $\boldsymbol{\Lambda}_t=\{\Lambda_1,\dots,\Lambda_{r_t}\}$ with Λ_i of the form (8), satisfying (9), (10) and (12).

(1) Set up the coefficient matrix K_t of the homogeneous linear system (13)-(15) in Notation 2.6 in the variables $\{\nu_j^k\}_{j=1,\dots,r_{t-1},\,k=1,\dots,n}$ associated to an element of the form $\Lambda=\sum_{j=1}^{r_{t-1}}\sum_{k=1}^{n}\nu_j^k\overline{\int_k}\Lambda_j$. Let $h_t:=\dim\ker K_t$.

(2) If $h_t=0$ then return "$\mathcal{D}_t=\mathcal{D}_{t-1}$". If $h_t>0$ define $r_t:=r_{t-1}+h_t$. Perform a triangulation of K_t by row reductions with row permutations and column pivoting so that the non-pivoting columns correspond to exponents $\beta_{r_{t-1}+1},\dots,\beta_{r_t}$ with strict divisors in B_{t-1}. Let $B_t=B_{t-1}\cup\{\mathbf{x}_\xi^{\beta_{r_{t-1}+1}},\dots,\mathbf{x}_\xi^{\beta_{r_t}}\}$.

(3) Compute a basis $\Lambda_{r_{t-1}+1},\dots,\Lambda_{r_t}\in\mathbb{C}[\mathbf{d}_\xi]$ of $\ker K_t$ from the triangular reduction of K_t by setting the coefficients of the non-pivoting columns to 0 or 1. This yields a basis $\boldsymbol{\Lambda}_t=\boldsymbol{\Lambda}_{t-1}\cup\{\Lambda_{r_{t-1}+1},\dots,\Lambda_{r_t}\}$ dual to B_t. The coefficients $\nu_{i,j}^k$ of Λ_i are $\mu_{\beta_i,\beta_j+\mathbf{e}_k}$ in (8) so that Eq. (12) are satisfied. Eq. (10) are satisfied, since $\boldsymbol{\Lambda}_t$ is dual to B_t.

Consider the following homogeneous linear system of equations in the variables $\{\nu_j^k\ :\ j=1,\dots,r_{t-1},\ k=1,\dots,n\}$:

$$\sum_{j:|\beta_s|<|\beta_j|<t}\nu_j^k\,\mu_{\beta_j,\beta_s+\mathbf{e}_l}-\nu_j^l\,\mu_{\beta_j,\beta_s+\mathbf{e}_k}=0,\quad 1\le k<l\le n \tag{13}$$

$$\nu_j^k=0\quad\text{if } \beta_j+\mathbf{e}_k=\beta_l \text{ for some } 1\le l\le r_{t-1} \tag{14}$$

$$\left(\sum_{j=1}^{r_{t-1}}\sum_{k=1}^{n}\nu_j^k\overline{\int_k}\Lambda_j\right)(f_l)=0\quad l=1,\dots,N. \tag{15}$$

We will denote by H_t the coefficient matrix of the equations in (13) and (14) and by K_t the coefficient matrix of the equations in (13)-(15).

By Proposition 2.3 and Lemma 2.5, if $K_t\,\nu=0$ where $\nu=[\nu_j^k : j=1,\dots,s,\ k=1,\dots,n]$, then $\Lambda=\sum_{j=1}^{s}\sum_{k=1}^{n}\nu_j^k\overline{\int_k}\Lambda_j\in(\mathbf{f})^\perp\cap\mathbb{C}[\mathbf{d}_\xi]_t=\mathcal{D}_t$. The main loop of the integration method described in Algorithm 1 consists of computing the new basis elements in $\mathcal{D}_t$ and the new basis monomials in B_t of degree t from the primal-dual basis pair $(B_{t-1},\boldsymbol{\Lambda}_{t-1})$ in degree $t-1$.

Algorithm 1 produces incrementally a basis of $\mathcal{D}$, similarly to Macaulay's method. The algorithmic advantage is the smaller matrix size in $O(rn^2+N)$ instead of $N\binom{n+\delta-1}{\delta}$, where δ is the maximal degree (depth) in the dual, cf. (Mantzaflaris and Mourrain, 2011; Hauenstein et al., 2016).

The full Integration Method consists of taking $\Lambda_1:=1_\xi$ for $t=0$, a basis of $\mathcal{D}_0$ and then iterating algorithm Integration Method - Iteration t until we find a value of t when $\mathcal{D}_t=\mathcal{D}_{t-1}$. This implies that the order $\delta=\delta_\xi(\mathbf{f})=t-1$. This leads to the following definition.

Definition 2.7. We say that $\boldsymbol{\Lambda}\subset\mathbb{C}[\mathbf{d}_\xi]$ is *complete* for $\mathbf{f}$ at ξ if for $\delta:=\mathrm{ord}(\boldsymbol{\Lambda})$ we have $\ker K_{\delta+1}=\{0\}$. Here the linear system K_t is as in (13)-(15).

Notice that the full Integration Method constructs a graded primal-dual basis pair $(B,\boldsymbol{\Lambda})$. The basis $\boldsymbol{\Lambda}\subset(\mathbf{f})^\perp$ spans a space stable by derivation and is complete for $\mathbf{f}$, so that we have $\mathrm{span}(\boldsymbol{\Lambda})=(\mathbf{f})^\perp\cap\mathbb{C}[\mathbf{d}_\xi]=Q^\perp$ where Q is the primary component of $(\mathbf{f})$ at ξ.

To guarantee that B_t is closed under division, one could choose a graded monomial ordering $\prec$ of $\mathbb{C}[\mathbf{d}_\xi]$ and compute an auto-reduced basis of $\ker K_t$ such that the initial terms for $\prec$ are $\mathbf{d}_\xi^{\beta_i}$. The set B_t constructed in this way would be closed under division, since $\mathcal{D}_t$ is stable under derivation. In the approach we use in practice, we choose the column pivot taking into account the numerical values of the coefficients and not according to a monomial ordering and we check *a posteriori* that the set of exponents is closed under division (See Example 7.1).

The main property that we will use for the certification of multiplicities is given in the next theorem.

Theorem 2.8. *If ξ^* is an isolated solution of the system $\mathbf{f}(\mathbf{x}) = 0$ and B is a graded primal basis at ξ^* closed under division, then the system $F(\xi, \mu) = 0$ of all equations (9), (10) and (12) admits (ξ^*, μ^*) as an isolated simple root, where μ^* defines the basis $\mathbf{\Lambda}^*$ of the inverse system of $(\mathbf{f})$ at ξ dual to B, due to (8).*

Proof. This is a direct consequence of (Hauenstein et al., 2016, Theorem 4.11), since the system of equations (9)-(12) is equivalent to the system (14) in (Hauenstein et al., 2016, Theorem 4.11). The equations (9) express the commutation of the transposed of the parametric operator of multiplication in B, which are the same as the equations of commutation of the operators. By Lemma 2.5, the equations (10) are equivalent to the fact that $(B, \mathbf{\Lambda}^*)$ is a graded primal-dual basis pair. Finally, the equations (12) are the same as $\mathcal{N}(f_i) = 0$, $i = 1, \dots, s$ where $\mathcal{N}$ is the parametric normal form defined in Hauenstein et al. (2016)[see Definition 4.7 and following remark]. Therefore the two systems are equivalent. By (Hauenstein et al., 2016, Theorem 4.11), they define the simple isolated solution (ξ^*, μ^*), where μ^* defines the basis $\mathbf{\Lambda}^*$ dual to B due to (8). □

3. Punctual Hilbert scheme

The results in Sections 3 and 4 do not depend on the point $\xi \in \mathbb{C}^n$, so to simplify the notation, we assume in these sections that $\xi = \mathbf{0}$. Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the maximal ideal defining $\xi = \mathbf{0} \in \mathbb{C}^n$. Let $\mathbb{C}[\mathbf{d}]$ be the space of polynomials in the variables $\mathbf{d} = (d_1, \dots, d_n)$ and $\mathbb{C}[\mathbf{d}]_t \subset \mathbb{C}[\mathbf{d}]$ the subspace of polynomials in $\mathbf{d}$ of degree $\leq t$.

For a vector space V, let $\mathcal{G}_r(V)$ be the projective variety of the r dimensional linear subspaces of V, also known as the *Grassmannian* of r-spaces of V. The points in $\mathcal{G}_r(V)$ are the projective points of $\mathbb{P}(\wedge^r V)$ of the form $\mathbf{v} = v_1 \wedge \cdots \wedge v_r$ for $v_i \in V$. Fixing a basis $\mathbf{e}_1, \dots, \mathbf{e}_s$ of V, the Plücker coordinates of $\mathbf{v}$ are the coefficients of $\Delta_{i_1,\dots,i_r}(\mathbf{v})$ of $\mathbf{v} = \sum_{i_1<\cdots<i_r} \Delta_{i_1,\dots,i_r}(\mathbf{v})\, \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r}$. When $V = \mathbb{C}[\mathbf{d}]_{r-1}$, a natural basis is the dual monomial basis $(\frac{\mathbf{d}^\alpha}{\alpha!})_{|\alpha|<r}$. The Plücker coordinates of an element $\mathbf{v} \in \mathcal{G}_r(\mathbb{C}[\mathbf{d}]_{r-1})$ for this basis are denoted $\Delta_{\alpha_1,\dots,\alpha_r}(\mathbf{v})$ where $\alpha_i \in \mathbb{N}^n$, $|\alpha_i| < r$.

If $\mathbf{\Lambda} = \{\Lambda_1, \dots, \Lambda_r\}$ is a basis of a r-dimensional space $\mathcal{D}$ in $\mathbb{C}[\mathbf{d}]_{r-1}$ with $\Lambda_i = \sum_{|\alpha|<r} \mu_{i,\alpha} \frac{\mathbf{d}^\alpha}{\alpha!}$, the Plücker coordinates of $\mathcal{D}$ are, up to a scalar, of the form $\Delta_{\alpha_1,\dots,\alpha_r} = \det\left[\mu_{i,\alpha_j}\right]_{1\leq i,j\leq r}$. In particular, a monomial set $B = \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_r}\} \subset \mathbb{C}[\mathbf{x}]_{r-1}$ has a dual basis in $\mathcal{D}$ iff $\Delta_{\beta_1,\dots,\beta_r}(\mathcal{D}) \neq 0$. If $(B = \{\mathbf{x}^{\beta_i}\}_{i=1}^r, \mathbf{\Lambda} = \{\Lambda_i\}_{i=1}^r)$ is a graded primal-dual basis pair, then $\mu_{i,\beta_j} = \delta_{i,j}$. To keep our notation consistent with the previous sections, the coordinates of $\Lambda_i \in \mathbf{\Lambda}$ when $\mathbf{\Lambda}$ is dual to B will be denoted by $\mu_{\beta_i,\alpha}$ instead of $\mu_{i,\alpha}$. By properties of the determinant, the Plücker coordinates of $\mathcal{D}$ are such that

$$\mu_{\beta_i,\alpha} = \frac{\Delta_{\beta_1,\dots,\beta_{i-1},\alpha,\beta_{i+1},\dots,\beta_r}}{\Delta_{\beta_1,\dots,\beta_r}} \quad i = 1, \dots, r. \tag{16}$$

If $\mathcal{D}$ is the dual of an ideal $Q = \mathcal{D}^\perp \subset \mathbb{C}[\mathbf{x}]$ and $B = \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_r}\}$ is a basis of $\mathbb{C}[\mathbf{x}]/Q$ so that $\Delta_{\beta_1,\dots,\beta_r}(\mathcal{D}) \neq 0$, the normal form of $\mathbf{x}^\alpha \in \mathbb{C}[\mathbf{x}]_{r-1}$ modulo $Q = \mathcal{D}^\perp$ in the basis B is

$$N(\mathbf{x}^\alpha) = \sum_{j=1}^r \mu_{\beta_j,\alpha}\, \mathbf{x}^{\beta_j} = \sum_{j=1}^r \frac{\Delta_{\beta_1,\dots,\beta_{j-1},\alpha,\beta_{j+1},\dots,\beta_r}}{\Delta_{\beta_1,\dots,\beta_r}} \mathbf{x}^{\beta_j}$$

(if $\deg(\mathbf{x}^\alpha) \geq r$, then $N(\mathbf{x}^\alpha) = 0$).

Definition 3.1. Let $\mathcal{H}_r \subset \mathcal{G}_r(\mathbb{C}[\mathbf{d}]_{r-1})$ be the set of linear spaces $\mathcal{D}$ of dimension r in $\mathbb{C}[\mathbf{d}]_{r-1}$ which are stable by the derivations ∂_{d_i} with respect to the variables $\mathbf{d}$ (i.e. $\partial_{d_i}\mathcal{D} \subset \mathcal{D}$ for $i = 1, \dots, n$). We called $\mathcal{H}_r$ the *punctual Hilbert scheme* of points of multiplicity r.

If $\mathcal{D} \subset \mathbb{C}[\mathbf{d}]$ is stable by the derivations ∂_{d_i}, then by duality $I = \mathcal{D}^\perp \subset \mathbb{C}[\mathbf{x}]$ is a vector space of $\mathbb{C}[\mathbf{x}]$ stable by multiplication by x_i, i.e. an ideal of $\mathbb{C}[\mathbf{x}]$.

Proposition 3.2. *$\mathcal{D} \in \mathcal{H}_r$ iff $\mathcal{D}^\perp = Q$ is an $\mathfrak{m}$-primary ideal such that* $\dim \mathbb{C}[\mathbf{x}]/Q = r$.

Proof. Let $\mathcal{D} \in \mathcal{H}_r$. We prove that $\mathcal{D}^\perp = Q$ is an $\mathfrak{m}$-primary ideal. As $\mathcal{D}$ is stable by derivation, $Q = \mathcal{D}^\perp$ is an ideal of $\mathbb{C}[\mathbf{x}]$. This also implies that $1 \in \mathcal{D}$, so that $Q \subset \mathfrak{m}$. As $\dim \mathcal{D} = \dim \mathbb{C}[\mathbf{x}]/Q = r$, $\delta = \mathrm{ord}(\mathcal{D})$ is finite and $\mathfrak{m}^{\delta+1} \subset \mathcal{D}^\perp = Q$. Therefore, Q is $\mathfrak{m}$-primary, which shows the first implication.

Conversely, let Q be a $\mathfrak{m}$-primary ideal such that $\dim \mathbb{C}[\mathbf{x}]/Q = r$. Then by Lemma 2.1, $\mathcal{D} = Q^\perp \subset \mathbb{C}[\mathbf{d}]_t$ is stable by derivation and of dimension $r = \dim \mathbb{C}[\mathbf{x}]/Q$. Thus $\mathcal{D} \in \mathcal{H}_r$. This concludes the proof of the proposition. □

For $\mathcal{D} \in \mathcal{H}_r$, for $t \geq 0$ we denote by $\mathcal{D}_t$ the vector space of elements of $\mathcal{D}$ of order $\leq t$. We verify that $\mathcal{D}_t^\perp = \mathcal{D}^\perp + \mathfrak{m}^{t+1}$. The next theorem follows from Proposition 2.3 and Lemma 2.5.

Theorem 3.3. *For $B \subset \mathbb{C}[\mathbf{x}]$ closed under division such that $|B| = r$ and $\delta = \deg(B)$, the following points are equivalent:*

1. *$\mathcal{D} \in \mathcal{H}_r$ and B_t is a basis of $\mathbb{C}[\mathbf{x}]/(\mathcal{D}^\perp + \mathfrak{m}^{t+1})$ for $t = 1, \ldots, \delta$.*
2. *The dual basis $\mathbf{\Lambda} = \{\Lambda_1, \ldots, \Lambda_r\}$ of B satisfies $\Lambda_1 = 1$ and the equations (8), (9) and (10).*

Proof. (1) ⇒ (2) Assume that $\mathcal{D} \in \mathcal{H}_r$ and that B_t is a basis of $\mathbb{C}[\mathbf{x}]/(\mathcal{D}^\perp + \mathfrak{m}^{t+1})$. Let $\mathbf{\Lambda}_t = \{\Lambda_1, \ldots, \Lambda_{r_t}\}$ be a basis of $\mathcal{D}_t$ dual to B_t with $r_t = |B_t|$. Then, for $j = r_{t-1} + 1, \ldots, r_t$, $\Lambda_j \in \mathcal{D}_t$ is such that

$$\partial_{d_k}(\Lambda_j) = \sum_{j=1}^{r_{t-1}} \nu_{i,k} \Lambda_i$$

for $t = 1, \ldots, o$. By Proposition 2.3, Equations (8) and (9) are satisfied. As B_t is dual to $\Lambda_1, \ldots, \Lambda_{r_t}$, Equation (10) are satisfied.

(2) ⇒ (1) Let $\Lambda_i \in \mathbb{C}[\mathbf{d}]_{r-1}$ for $i = 1, \ldots, r$ be elements of $\mathbb{C}[\mathbf{d}]_{r-1}$ dual to B, which satisfies Equations (8), (9) and (10). By induction on $t = 0, \ldots, \delta = \deg(B)$, we prove that if $\mathbf{\Lambda}_t = \{\Lambda_1, \ldots, \Lambda_{r_t}\}$ is dual to B_t, then $\Lambda_1, \ldots, \Lambda_{r_t} \in \mathbb{C}[\mathbf{d}]_t$. The property is true for $t = 0$ since $\Lambda_1 = 1$. If it is true for $t - 1$, for Λ_j with $j = r_{t-1} + 1, \ldots, r_t$ we have by (8), (9) and Proposition 2.3, that $\partial_{d_k}(\Lambda_j) = \sum_{j=1}^{r_{t-1}} \nu_{i,k} \Lambda_i$, $k = 1, \ldots, n$. Thus $\Lambda_j \in \mathbb{C}[\mathbf{d}]_t$. This shows that $\mathcal{D}_t$ is stable by derivation where $\mathcal{D}_t \subset \mathbb{C}[\mathbf{d}]_t$ is the vector space spanned $\Lambda_1, \ldots, \Lambda_{r_t} \in \mathbb{C}[\mathbf{d}]_t$. Let $\mathcal{D} = \mathcal{D}_\delta$. Since, by (10), B_t is dual to $\Lambda_1, \ldots, \Lambda_{r_t} \in \mathbb{C}[\mathbf{d}]_t$, we see that $\mathcal{D} \cap \mathbb{C}[\mathbf{d}]_t = \mathcal{D}_t$. By Proposition 3.2, $Q = \mathcal{D}^\perp$ is a $\mathfrak{m}$-primary ideal such that $\dim \mathbb{C}[\mathbf{x}]/Q = \dim \mathcal{D} = |B| = r$. Moreover, since B_t is dual to the basis $\{\Lambda_1, \ldots, \Lambda_{r_t}\}$ of $\mathcal{D}_t$, B_t is a basis $\mathbb{C}[\mathbf{x}]/(\mathcal{D}^\perp + \mathfrak{m}^{t+1})$. This proves the reverse inclusion. □

For a sequence $\mathbf{h} = (h_0, h_1, \ldots, h_\delta) \in \mathbb{N}_+^{\delta+1}$ and $0 \leq t \leq \delta$, let $\mathbf{h}_t = (h_0, \ldots, h_t)$, $r_t = \sum_{i=0}^t h_i$. For $r \geq 1$ we denote by $\mathbf{S}^r$ the set of sequences $\mathbf{h}$ of some length $\delta < r$ with $h_i \neq 0$, $h_0 = 1$ and $r_\delta = r$. For $\mathbf{h} \in \mathbf{S}^r$, we consider the following subvarieties of $\mathcal{H}_{r_t}$:

$$\mathcal{H}_{\mathbf{h}_t} = \{\mathcal{D} \in \mathcal{H}_{r_t} \mid \dim \mathcal{D}_i = \dim \mathcal{D} \cap \mathbb{C}[\mathbf{d}]_i \leq r_i, i = 0, \ldots, t\}.$$

These are projective varieties in $\mathcal{H}_{r_t}$ defined by rank conditions on the linear spaces $\mathcal{D} \cap \mathbb{C}[\mathbf{d}]_i$ for $\mathcal{D} \in \mathcal{H}_{r_t}$, that can be expressed in terms of homogeneous polynomials in the Plücker coordinates of $\mathcal{D}$. In particular, the varieties $\mathcal{H}_{\mathbf{h}} := \mathcal{H}_{\mathbf{h}_\delta}$ are projective subvarieties of $\mathcal{H}_r$. They may not be irreducible or irreducible components of $\mathcal{H}_r$, but we have $\mathcal{H}_r = \cup_{\mathbf{h} \in \mathbf{S}^r} \mathcal{H}_{\mathbf{h}}$.

We will study a particular component of $\mathcal{H}_{\mathbf{h}}$, that we call the *regular component of* $\mathcal{H}_{\mathbf{h}}$, denoted $\mathcal{H}_{\mathbf{h}}^{reg}$. It is characterized as follows. Let $\mathcal{H}_{\mathbf{h}_0}^{reg} = \{\langle 1 \rangle\} = \{\mathbb{C}[\mathbf{d}]_0\} = \mathcal{G}_1(\mathbb{C}[\mathbf{d}]_0)$ and assume that $\mathcal{H}_{\mathbf{h}_{t-1}}^{reg}$ has been defined as an irreducible component of $\mathcal{H}_{\mathbf{h}_{t-1}}$. Let

$$W_t = \{(\mathcal{D}_{t-1}, \mathcal{E}_t) \mid \mathcal{D}_{t-1} \in \mathcal{H}_{\mathbf{h}_{t-1}}, \mathcal{E}_t \in \mathcal{G}_{r_t}(\mathbb{C}[\mathbf{d}]_t), \mathcal{D}_{t-1} \subset \mathcal{E}_t, \forall i\ \partial_{d_i} \mathcal{E}_t \subset \mathcal{D}_{t-1}\}$$

The constraints $\mathcal{D}_{t-1} \subset \mathcal{E}_t$ and $\partial_{d_i}\mathcal{E}_t \subset \mathcal{D}_{t-1}$ for $i = 1, \ldots, n$ define a linear system of equations in the Plücker coordinates of $\mathcal{E}_t$ (see e.g. Doubilet et al., 1974), corresponding to the equations (5), (6). By construction, the projection of $W_t \subset \mathscr{H}_{\mathbf{h}_{t-1}} \times \mathscr{G}_{r_t}(\mathbb{C}[\mathbf{d}]_t)$ on the second factor $\mathscr{G}_{r_t}(\mathbb{C}[\mathbf{d}]_t)$ is $\pi_2(W_t) = \mathscr{H}_{\mathbf{h}_t}$ and the projection on the first factor is $\pi_1(W_t) = \mathscr{H}_{\mathbf{h}_{t-1}}$.

There exists a dense subset U_{t-1} of the irreducible variety $\mathscr{H}^{reg}_{\mathbf{h}_{t-1}}$ (with $\overline{U_{t-1}} = \mathscr{H}^{reg}_{\mathbf{h}_{t-1}}$) such that the rank of the linear system corresponding to (5) and (6) defining $\mathcal{E}_t$ is maximal. Since $\pi_1^{-1}(\mathcal{D}_{t-1})$ is irreducible (in fact linear) of fixed dimension for $\mathcal{D}_{t-1} \in U_{t-1} \subset \mathscr{H}^{reg}_{\mathbf{h}_{t-1}}$, there is a unique irreducible component $W_{t,reg}$ of W_t such that $\pi_1(W_{t,reg}) = \mathscr{H}^{reg}_{\mathbf{h}_{t-1}}$ (see e.g. Shafarevich, 2013[Theorem 1.26]). We define $\mathscr{H}^{reg}_{\mathbf{h}_t} = \pi_2(W_{t,reg})$. It is an irreducible component of $\mathscr{H}_{\mathbf{h}_t}$, since otherwise $W_{t,reg} = \pi_2^{-1}(\mathscr{H}^{reg}_{\mathbf{h}_t})$ would not be a component of W_t but strictly included in one of the irreducible components of W_t.

Definition 3.4. Let $\pi_t : \mathscr{H}_{\mathbf{h}_t} \to \mathscr{H}_{\mathbf{h}_{t-1}}$, $\mathcal{D} \mapsto \mathcal{D} \cap \mathbb{C}[\mathbf{d}]_{t-1}$ be the projection in degree $t-1$. We define by induction on t, $\mathscr{H}^{reg}_{\mathbf{h}_0} = \{\langle 1 \rangle\}$ and $\mathscr{H}^{reg}_{\mathbf{h}_t}$ is the irreducible component $\pi_t^{-1}(\mathscr{H}^{reg}_{\mathbf{h}_{t-1}})$ of $\mathscr{H}_{\mathbf{h}_t}$ for $t = 1, \ldots, \delta$.

4. Rational parametrization

Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\} \subset \mathbb{C}[\mathbf{x}]_{r-1}$ be a monomial set. In this section we assume that B is closed under division and its monomials are ordered by increasing degree. For $t \in \mathbb{N}$, we denote by $B_t = B \cap \mathbb{C}[\mathbf{x}]_t$, by $B_{[t]}$ the subset of its monomials of degree t. Let $h_t = |B_{[t]}|$, $r_t = \sum_{0 \le i \le t} h_t = |B_t|$ and $\delta = \deg(B)$.

Let

$$\mathscr{H}_B := \{\mathcal{D} \in \mathscr{H}_r \mid B_t \text{ is a basis of } \mathbb{C}[\mathbf{x}]/(\mathcal{D}^\perp + \mathfrak{m}^{t+1}), t = 0, \ldots, \delta\}.$$

By Theorem 3.3, $\mathscr{H}_B$ is the set of linear spaces $\mathcal{D} \in \mathscr{H}_r$ such that $\mathcal{D}_t = \mathcal{D} \cap \mathbb{C}[\mathbf{d}]_t$ satisfy Equations (8) and (9). It is the open subset of $\mathcal{D} \in \mathscr{H}_{\mathbf{h}}$ such that $\Delta_{B_t}(\mathcal{D}_t) \neq 0$ for $t = 1, \ldots, \delta$, where $\Delta_{B_t} := \Delta_{\beta_1, \ldots, \beta_{r_t}}$ denotes the Plücker coordinate for $\mathscr{G}_{r_t}(\mathbb{C}[\mathbf{d}]_t)$ corresponding to the monomials in B_t.

Since for $\mathcal{D} \in \mathscr{H}_B$ we have $\Delta_B(\mathcal{D}) \neq 0$, we can define the affine coordinates of $\mathscr{H}_B$ using the coordinates of the elements of the basis $\mathbf{\Lambda} = \{\Lambda_1, \ldots, \Lambda_r\}$ dual to B:

$$\left\{ \mu_{\beta_j, \alpha} = \frac{\Delta_{\beta_1, \ldots, \beta_{j-1}, \alpha, \beta_{j+1}, \ldots, \beta_r}}{\Delta_B} \ : \ j = 1, \ldots, r, \ |\alpha| < r \right\}.$$

The following lemma shows that the values of the coordinates $\{\mu_{\beta_i, \beta_j + \mathbf{e}_k} \ : \ i, j = 1, \ldots r, |\beta_j| < |\beta_i|, k = 1, \ldots, n\}$ uniquely define $\mathbf{\Lambda}$.

Lemma 4.1. *Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_{r_t}}\}$ closed under division, $\mathcal{D} \in \mathscr{H}_B$ and $\mathbf{\Lambda} = \{\Lambda_1, \ldots, \Lambda_r\}$ be the unique basis of $\mathcal{D}$ dual to B with $\Lambda_i = \sum_{|\alpha| \le |\beta_i|} \mu_{\beta_i, \alpha} \frac{\mathbf{d}^\alpha}{\alpha!}$ for $i = 1, \ldots, r$. Then $\Lambda_1 = 1$ and for $i = 2, \ldots, r$*

$$\Lambda_i = \sum_{|\beta_j| < |\beta_i|} \sum_{k=1}^{n} \mu_{\beta_i, \beta_j + \mathbf{e}_k} \overline{\int_k} \Lambda_j.$$

Thus, $\mu_{\beta_i, \alpha}$ is a polynomial function of $\{\mu_{\beta_s, \beta_j + \mathbf{e}_k} \ : \ |\beta_s| \le |\beta_i|, |\beta_j| < |\beta_s|, k = 1, \ldots, n\}$ for $i = 1, \ldots, r$, $|\alpha| < |\beta_i|$.

Proof. Since $\mathcal{D}$ is closed under derivation, by Proposition 2.3 there exist $c_{i,s,k} \in \mathbb{C}$ such that $\partial_{d_k}(\Lambda_i) = \sum_{|\beta_s| < |\beta_i|} c_{i,s,k} \Lambda_s$. Then

$$\mu_{\beta_i, \beta_j + \mathbf{e}_k} = \Lambda_i(\mathbf{x}^{\beta_j + \mathbf{e}_k}) = \partial_{d_k}(\Lambda_i)(\mathbf{x}^{\beta_j}) = \sum_{|\beta_s| < |\beta_i|} c_{i,s,k} \Lambda_s(\mathbf{x}^{\beta_j}) = c_{i,j,k}.$$

The second claim follows from obtaining the coefficients in Λ recursively from $\Lambda_1 = 1$ and

$$\Lambda_i = \sum_{|\beta_j|<|\beta_i|} \sum_{k=1}^{n} \mu_{\beta_i,\beta_j+\mathbf{e}_k} \overline{\int_k} \Lambda_j \text{ for } i = 2, \ldots, r. \quad \square$$

We define $\mu := \{\mu_{\beta_i,\beta_j+\mathbf{e}_k}\}_{i,j=1,\ldots r,|\beta_j|<|\beta_i|,k=1,\ldots,n}$, $\mu_t := \{\mu_{\beta_i,\beta_j+\mathbf{e}_k} \in \mu \,:\, |\beta_i| \le t\} \subset \mu$ and $\mu_{[t]} := \{\mu_{\beta_i,\beta_j+\mathbf{e}_k} \in \mu \,:\, |\beta_j| = t\} \subset \mu_t$. The next definition uses the fact that Equations (13) and (14) are linear in ν_j^k with coefficients depending on μ_{t-1}:

Definition 4.2. Given $\mathcal{D}_{t-1} \in \mathscr{H}_{B_{t-1}}$ with a unique basis $\mathbf{\Lambda}_{t-1} = \{\Lambda_1, \ldots, \Lambda_{r_{t-1}}\}$ with $\Lambda_i = \sum_{|\alpha|<t} \mu_{\beta_i,\alpha} \frac{\mathbf{d}^\alpha}{\alpha!}$ for $j = 1, \ldots, r_{t-1}$ that is dual to B_{t-1}, uniquely determined by $\mu_{t-1} = \{\mu_{\beta_i,\beta_j+\mathbf{e}_k} : |\beta_i| \le t-1, |\beta_j| < |\beta_i|\}$ as above. Recall from Notation 2.6 that H_t is the coefficient matrix of the homogeneous linear system (13) and (14) in the variables $\{\nu_j^k \,:\, j = 1, \ldots, r_{t-1},\ k = 1, \ldots, n\}$. To emphasize the dependence of its coefficients on $\mathcal{D}_{t-1}$ or μ_{t-1} we use the notation $H_t(\mathcal{D}_{t-1})$ or $H_t(\mu_{t-1})$. For $\mathcal{D} \in \mathscr{H}_{\mathbf{h}}^{reg}$ in an open subset, the rank ρ_t of $H_t(\mathcal{D}_{t-1})$ is maximal.

The next definition describes a property of a monomial set B such that it will allow us to give a rational parametrization of $\mathscr{H}_B$.

Definition 4.3. For $t = 1, \ldots, \delta = \deg(B)$ we say that $\mathcal{D}_t \in \mathscr{G}_{r_t}(\mathbb{C}[\mathbf{d}]_t)$ is regular for B_t if,

- $\dim(\mathcal{D}_t) = r_t = |B_t|$,
- $\operatorname{rank} H_t(\mathcal{D}_{t-1}) = \rho_t$ the generic rank of H_t on $\mathscr{H}_{\mathbf{h}_t}^{reg}$,
- $\Delta_{B_{[t]}}(\mathcal{D}_{[t]}) \neq 0$ where $\Delta_{B_{[t]}}(\mathcal{D}_{[t]})$ is the Plücker coordinate of $\mathcal{D}_{[t]} \in \mathscr{G}_{h_t}(\mathbb{C}[\mathbf{d}]_r)$ corresponding to the monomials in $B_{[t]}$.

Let $U_t := \{\mathcal{D}_t \in \mathscr{H}_{\mathbf{h}_t}^{reg} \,:\, \mathcal{D}_t \text{ is regular for } B_t\}$. Then U_t is either an open dense subset of the irreducible variety $\mathscr{H}_{\mathbf{h}_t}^{reg}$ or empty if $\Delta_{B_{[t]}}(\mathcal{D}_{[t]}) = 0$ for all $\mathcal{D} \in \mathscr{H}_{\mathbf{h}_t}^{reg}$. We say that B is a *regular basis* if $\overline{U_t} = \mathscr{H}_{\mathbf{h}_t}^{reg}$ (or $U_t \neq \emptyset$) for $t = 1, \ldots, \delta$.

We denote by $\gamma_{[t]} = \dim \mathscr{G}_{h_t}(\ker H_t(\mathcal{D}_{t-1}))$ for $\mathcal{D}_{t-1} \in U_{t-1}$ and $\gamma = \sum_{t=0}^{\delta} \gamma_{[t]}$.

If the basis B is regular and closed under division, then $\mathscr{H}_{\mathbf{h}}^{reg}$ can be parametrized by rational functions of free parameters $\overline{\mu}$. We present hereafter Algorithm 2 to compute such a parametrization iteratively.

Proposition 4.4. *Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\} \subset \mathbb{C}[\mathbf{x}]_{r-1}$ be closed under division and assume that B is a regular basis. There exist a subset $\overline{\mu} \subset \mu$ with $|\overline{\mu}| = \gamma$ and rational functions $q_{\beta_j,\alpha}(\overline{\mu}) \in \mathbb{Q}(\overline{\mu})$ for $j = 1, \ldots, r$ and $|\alpha| < r$, such that the map $\Phi : \mathbb{C}^\gamma \to \mathscr{H}_B$ defined by*

$$\Phi : \overline{\mu} \mapsto \left(q_{\beta_j,\alpha}(\overline{\mu})\right)_{j=1,\ldots,r,|\alpha|<r}$$

parametrizes a dense subset of $\mathscr{H}_{\mathbf{h}}^{reg}$.

Proof. Let us define, by induction on t, parameters $\overline{\mu}_t$ with $|\overline{\mu}_t| = \sum_{i=1}^{t} \gamma_{[i]}$, and a rational parametrization of a basis $\Lambda_1(\overline{\mu}_t), \ldots, \Lambda_{r_t}(\overline{\mu}_t)$ of a generic element of $\mathscr{H}_{B_t}^{reg}$. For $t = 0$, we define $\Lambda_1 = 1$ and $\overline{\mu}_0 = \emptyset$. Assume that there exist $\overline{\mu}_{t-1} \subset \mu_{t-1}$ and a rational parametrization $\Lambda_1(\overline{\mu}_{t-1}), \ldots, \Lambda_{r_{t-1}}(\overline{\mu}_{t-1})$ of a basis dual to B_{t-1} for a generic element $\mathscr{H}_{B_{t-1}}$ defined by the map

$$\Phi_{t-1} : \overline{\mu}_{t-1} \mapsto \left(q_{\beta_j,\alpha}(\overline{\mu}_{t-1})\right)_{|\beta_j| \le t-1, |\alpha| < r}.$$

Algorithm 2 Rational Parametrization - Iteration t.

Input: $t>0$, $B_t=\{\mathbf{x}^{\beta_1},\dots,\mathbf{x}^{\beta_{r_t}}\}\subset\mathbb{C}[\mathbf{x}]_t$ closed under division and regular, $\overline{\mu}_{t-1}\subset\mu_{t-1}$ and $\Phi_{t-1}:\overline{\mu}_{t-1}\mapsto\left(q_{\beta_j,\alpha}(\overline{\mu}_{t-1})\right)_{|\beta_j|\le t-1,|\alpha|<r}$ with $q_{\beta_j,\alpha}\in\mathbb{Q}(\overline{\mu}_{t-1})$ parametrizing a dense subset of $\mathscr{H}^{reg}_{\mathbf{h}_{t-1}}$.
Output: $\overline{\mu}_t\subset\mu_t$ and $\Phi_t:\overline{\mu}_t\mapsto\left(q_{\beta_j,\alpha}\right)_{|\beta_j|\le t,|\alpha|<r}$, $q_{\beta_j,\alpha}\in\mathbb{Q}(\overline{\mu}_t)$ extending Φ_{t-1} and parametrizing a dense subset of $\mathscr{H}^{reg}_{\mathbf{h}_t}$.

(1) Let H_t be as in Notation 2.6, $\nu=[\nu_j^k: j=1,\dots,r_{t-1},k=1,\dots,n]^T$. Decompose $H_t(\Phi_{t-1}(\overline{\mu}_{t-1}))\cdot\nu=0$ as

$$\left[\, A(\overline{\mu}_{t-1}) \;\middle|\; B(\overline{\mu}_{t-1}) \;\middle|\; C(\overline{\mu}_{t-1}) \,\right]\begin{bmatrix}\nu'\\ \nu''\\ \overline{\nu}\end{bmatrix}=0, \tag{17}$$

where ν' is associated to a maximal set of independent columns of $H_t(\Phi_{t-1}(\overline{\mu}_{t-1}))$, $\nu''=\{\nu_j^k:\mathbf{x}^{\beta_j+\mathbf{e}_k}\in B_{[t]}\}$ and $\overline{\nu}$ refers to the rest of the columns. If no such decomposition exists, return "B_t is not regular".
(2) For $\nu_j^k\in\nu'$ express $\nu_j^k=\varphi_j^k(\overline{\nu},\nu'')\in\mathbb{Q}(\overline{\mu}_{t-1})[\overline{\nu},\nu'']_1$ as the generic solution of the system $H_t(\Phi_{t-1}(\overline{\mu}_{t-1}))\cdot\nu=0$.
(3) For $i=r_{t-1}+1,\dots,r_t$ do:

(3.1) Define $\overline{\mu}_{[t],i}:=\left\{\mu_{\beta_i,\beta_j+\mathbf{e}_k}\;:\;\nu_{j,k}\in\overline{\nu}\right\}$, $\mu'_{[t],i}=\{\mu_{\beta_i,\beta_j+\mathbf{e}_k}:\nu_j^k\in\nu'\}$, $\mu''_{[t],i}=\{\mu_{\beta_i,\beta_j+\mathbf{e}_k}:\nu_j^k\in\nu''\}$, and $\overline{\mu}_t:=\overline{\mu}_{t-1}\cup\bigcup_{i=r_{t-1}+1}^{r_t}\overline{\mu}_{[t],i}$.
(3.2) For $\mu_{\beta_i,\beta_j+\mathbf{e}_k}\in\mu''_{[t],i}$ set $q_{\beta_i,\beta_j+\mathbf{e}_k}=\mu_{\beta_i,\beta_j+\mathbf{e}_k}=1$ if $\beta_i=\beta_j+\mathbf{e}_k$ and 0 otherwise.
(3.3) For $\mu_{\beta_i,\beta_j+\mathbf{e}_k}\in\mu'_{[t],i}$ define

$$q_{\beta_i,\beta_j+\mathbf{e}_k}:=\varphi_j^k(\overline{\mu}_{[t],i},\mu''_{[t],i})\in\mathbb{Q}(\overline{\mu}_t)$$

(3.4) For $|\alpha|<r$ and $\mu_{\beta_i,\alpha}\notin\mu_t$ find $q_{\beta_i,\alpha}$ using Lemma 4.1.

This means that $\overline{\operatorname{im}\Phi_{t-1}}=\mathscr{H}_{B_{t-1}}$. Denote by $\mathcal{D}_{t-1}(\overline{\mu}_{t-1})\in\mathscr{G}_{r_{t-1}}(\mathbb{Q}(\overline{\mu}_{t-1})[\mathbf{d}]_{t-1})$ the space spanned by $\{\Lambda_1(\overline{\mu}_{t-1}),\dots,\Lambda_{r_{t-1}}(\overline{\mu}_{t-1})\}$ over the fraction field $\mathbb{Q}(\overline{\mu}_{t-1})$.

By Theorem 3.3 and Lemma 2.5, to define $\overline{\mu}_t$ and to extend $\mathcal{D}_{t-1}(\overline{\mu}_{t-1})$ to $\mathcal{D}_t(\overline{\mu}_t)$, we need to find $\Lambda_{r_{t-1}+1},\dots,\Lambda_{r_t}$ of the form

$$\Lambda_i=\sum_{j=1}^{r_{t-1}}\sum_{k=1}^{n}\mu_{\beta_i,\beta_j+\mathbf{e}_k}\int_k\Lambda_j(\overline{\mu}_{t-1})\;\; i=r_{t-1}+1,\dots,r_t,$$

satisfying the system of equations (13) and (14), i.e. such that

$$\Lambda_i\in\ker H_t(\overline{\mu}_{t-1})\text{ for }i=r_{t-1}+1,\dots,r_t,$$

where $H_t(\overline{\mu}_{t-1})=H_t\left(\Phi_{t-1}(\overline{\mu}_{t-1})\right)$ and Equations (12) are satisfied. Since B is a regular basis, the kernel of $H_t(\overline{\mu}_{t-1})$ over $\mathbb{Q}(\overline{\mu}_{t-1})$ contains a subspace $\mathcal{D}_{[t]}$ of dimension $h_t=|B_{[t]}|$ with $\Delta_{B_{[t]}}(\mathcal{D}_{[t]})\neq 0$. Therefore, the systems $H_t(\overline{\mu}_{t-1})\,\nu=0$ with $\nu=[\nu_j^k: j=1,\dots,r_{t-1},k=1,\dots,n]^T$ can be decomposed as

$$\left[\, A(\overline{\mu}_{t-1}) \;\middle|\; B(\overline{\mu}_{t-1}) \;\middle|\; C(\overline{\mu}_{t-1}) \,\right]\begin{bmatrix}\nu'\\ \nu''\\ \overline{\nu}\end{bmatrix}=0, \tag{18}$$

where ν' is associated to a maximal set of independent columns of $H_t(\overline{\mu}_{t-1})$, $\nu''=\{\nu_j^k:\mathbf{x}^{\beta_j+\mathbf{e}_k}\in B_{[t]}\}$ and $\overline{\nu}$ is associated to the remaining set of columns. Note that $|\overline{\nu}|=\dim(\ker H_t(\overline{\mu}_{t-1}))-h_t$. Thus, $\nu''\cup\overline{\nu}$ is the set of free variables of the homogeneous system $H_t(\overline{\mu}_{t-1})\,\nu=0$ and a general solution is such that the variables in ν' are linear functions of the variables in ν'' and $\overline{\nu}$, with rational coefficients in $\overline{\mu}_{t-1}$.

We obtain the coefficients of $\Lambda_{r_{t-1}+1},\dots,\Lambda_{r_t}$ that satisfy equations (13) and (14) and (12) from the general solutions of $H_t(\overline{\mu}_{t-1})\,\nu=0$ by further specializing the variables in ν'' to 0's and 1's, according the duality conditions. Define

$$\overline{\mu}_{[t],i} := \left\{\mu_{\beta_i,\beta_j+\mathbf{e}_k} \,:\, \nu_{j,k} \in \overline{\nu}\right\} \subset \mu_{[t]}\,.$$

Thus, the parameters in $\mu_{[t]}$ are linear functions of $\overline{\mu}_{[t],i}$ with rational coefficients in $\overline{\mu}_{t-1}$. The denominator in these coefficients is a factor of the numerator of a maximal non-zero minor of $A(\overline{\mu}_{t-1})$. Note that the rest of the coefficients of Λ_i are polynomial functions of the parameters $\mu_{t-1} \cup \mu_{[t]}$ by Lemma 4.1. Define

$$\overline{\mu}_t := \overline{\mu}_{t-1} \cup \bigcup_{i=r_{t-1}+1}^{r_t} \overline{\mu}_{[t],i}.$$

Thus, we get a parametrization of the coefficients of $\Lambda_{r_{t-1}+1}(\overline{\mu}_t), \ldots, \Lambda_{r_t}(\overline{\mu}_t)$ in terms of $\overline{\mu}_t$, which defines the degree t part of the map $\Phi_t : \overline{\mu}_t \mapsto (q_{\beta_j,\alpha}(\overline{\mu}_t))_{|\beta_j|\le t, |\alpha|<r}$. For $\mathcal{D}_t \in \mathscr{H}_{B_t}$, the coefficients of its basis dual to B_t can be parametrized by Φ_t for parameter values $\overline{\mu}_t$ such that a maximal non-zero minor of $A(\overline{\mu}_{t-1})$ in $\mathbb{Q}(\overline{\mu}_{t-1})$ does not vanish.

Note that the number of new parameters introduced is

$$|\overline{\mu}_t \setminus \overline{\mu}_{t-1}| = (r_t - r_{t-1}) \cdot |\overline{\mu}_{[t],i}| = h_t \left(\dim \ker H_t(\overline{\mu}_{t-1}) - h_t\right)$$

which is equal to $\gamma_{[t]} = \dim \mathscr{G}_{h_t}(\ker H_t(\overline{\mu}_{t-1})) = \dim \mathscr{G}_{h_t}(\ker H_t(\mathcal{D}_{t-1}))$ for $\mathcal{D}_{t-1}$ generic in U_{t-1} as claimed.

To prove that Φ_t parametrizes a dense subset of the projective variety $\mathscr{H}^{reg}_{\mathbf{h}_t}$, note that the image $\mathrm{im}(\Phi_t)$ of Φ_t is a subset of $\mathscr{H}_{\mathbf{h}_t}$, the Zariski closure V_t of $\mathrm{im}(\Phi_t)$ is an irreducible subvariety of $\mathscr{H}_{\mathbf{h}_t}$. Furthermore, its projection $\pi_{t-1}(V_t) \subset \mathscr{H}_{\mathbf{h}_{t-1}}$ is the closure of the image of $\mathrm{im}(\Phi_{t-1})$ since if $\mathcal{D}_t = \mathrm{im}\Phi_t(\overline{\mu}^*_t)$ then $\mathcal{D}_{t-1} = \mathcal{D}_t \cap \mathbb{C}[\mathbf{d}]_{t-1} = \Phi_{t-1}(\overline{\mu}^*_{t-1})$. By induction hypothesis,

$$\pi_{t-1}(V_t) = \overline{\mathrm{im}\Phi_{t-1}} = \mathscr{H}^{reg}_{\mathbf{h}_{t-1}}\,.$$

Thus, V_t is the irreducible component of $\mathscr{H}_{\mathbf{h}_t}$ which projects onto $\mathscr{H}^{reg}_{\mathbf{h}_{t-1}}$, that is $\mathscr{H}^{reg}_{\mathbf{h}_t}$. □

Definition 4.5. We denote by $\mathrm{H}_t(\mu)$ *a* maximal square submatrix of A in (17) such that $\det(\mathrm{H}_t(\overline{\mu}_{t-1})) \neq 0$.

The size of $\mathrm{H}_t(\mu)$ is the size of ν' in (17), that is the maximal number of independent columns in $H_t(\overline{\mu}_{t-1})$. Given an element $\mathcal{D} = \Lambda_1 \wedge \cdots \wedge \Lambda_r \in \mathscr{G}_r(\mathbb{C}[\mathbf{d}]_{r-1})$, in order to check that $\mathcal{D}$ is regular for B, it is sufficient to check first that $\Delta_B(\mathcal{D}) \neq 0$ and secondly that $|\mathrm{H}_t(\mu)| \neq 0$ for all $t = 0, \ldots, \delta$, where $\mu = (\mu_{\beta,\alpha})$ is the ratio of Plücker coordinates of $\mathcal{D}$ defined by the formula (16).

5. Newton's iterations

In this section we describe the extraction of a square, deflated system that allows for a Newton's method with quadratic convergence. We assume that the sole input is the equations $\mathbf{f} = (f_1, \ldots, f_N) \in \mathbb{C}[\mathbf{x}]^N$, an approximate point $\xi \in \mathbb{C}^n$ and a tolerance $\varepsilon > 0$.

5.1. Extracting a square system

Using this input we first compute an approximate primal-dual pair $(B, \mathbf{\Lambda})$ by applying the iterative Algorithm 1. The rank and kernel vectors of the matrices K_t (see Algorithm 1) are computed numerically within tolerance ε, using SVD. Note that here and in Section 6 we do not need to certify the SVD computation but we are only using SVD to certify that some matrices are full rank by checking that the distance to the variety of singular matrices is bigger than the perturbation of the matrix. Thus we need a weaker test, which relies only on a lower bound of the smallest singular value.

The algorithm returns a basis $B = \{\mathbf{x}_\xi^{\beta_1}, \ldots, \mathbf{x}_\xi^{\beta_r}\}$ with exponent vectors $E = \{\beta_1, \ldots, \beta_r\}$, as well as approximate values for the parameters $\mu = \{\mu_{\beta_i,\beta_j+\mathbf{e}_k} \,:\, |\beta_j| < |\beta_i| \in E,\ k = 1, \ldots, n\}$. These parameters will be used as a starting point for Newton's iteration. Note that, by looking at B, we can also deduce the multiplicity r, the maximal order δ of dual differentials, the sequences $r_t = |B_t|$, and $h_t = |B_{[t]}|$ for $t = 0, \ldots, \delta$.

Let F be the deflated system with variables $(\mathbf{x}, \mu)$ defined by the relations (8) and Equations (9), (10) and (12) i.e.

$$F(\mathbf{x},\mu)=\begin{cases} \sum\limits_{|\beta_s|<|\beta_j|<|\beta_i|} \mu_{\beta_i,\beta_j+\mathbf{e}_k}\mu_{\beta_j,\beta_s+\mathbf{e}_l} - \mu_{\beta_i,\beta_j+\mathbf{e}_l}\mu_{\beta_j,\beta_s+\mathbf{e}_k}=0 & (a)\\ \quad \text{for all } i=1,\ldots,r,\ |\beta_s|<|\beta_i|, k\neq l\in\{1,\ldots,n\} & \\ \mu_{\beta_i,\beta_j+\mathbf{e}_k} = \begin{cases} 1 & \text{for } \beta_i = \beta_j+\mathbf{e}_k \\ 0 & \text{for } \beta_j+\mathbf{e}_k \in E,\ \beta_i \neq \beta_j+\mathbf{e}_k, \end{cases} & (b)\\ \Lambda_i(f_j)=0, \quad i=1,\ldots,r,\ j=1,\ldots,N. & (c) \end{cases}$$

Here $\Lambda_1 = 1_{\mathbf{x}}$ and $\Lambda_i = \sum_{|\beta_j|<|\beta_i|}\sum_{k=1}^{n} \mu_{\beta_i,\beta_j+\mathbf{e}_k} \underset{k}{\int} \Lambda_j \in \mathbb{C}[\mu][\mathbf{d}_{\mathbf{x}}]$ denote dual elements with parametric coefficients defined recursively. Also, if $\Lambda_i = \sum_{|\alpha|\le|\beta_i|} \mu_{\beta_i,\alpha} \frac{\mathbf{d}_{\mathbf{x}}^{\alpha}}{\alpha!}$ then

$$\Lambda_i(f_j) = \sum_{|\alpha|\le|\beta_i|} \mu_{\beta_i,\alpha} \frac{\boldsymbol{\partial}^{\alpha}(f_j)(\mathbf{x})}{\alpha!} \tag{19}$$

which is in $\mathbb{C}[\mathbf{x}, \mu]$ by Lemma 4.1. Note, however, that (a) and (b) are polynomials in $\mathbb{C}[\mu]$, only (c) depends on $\mathbf{x}$ and μ. Equations (b) define a simple substitution into some of the parameters μ. Hereafter, we explicitly substitute them and eliminate this part (b) from the equations we consider and reducing the parameter vector μ.

By Theorem 2.8, if B is a graded primal basis for $\mathbf{f}$ at the root ξ^* then the above overdetermined system has a simple root at a point (ξ^*, μ^*).

To extract a square subsystem defining the simple root (ξ^*, μ^*) in order to certify the convergence, we choose a maximal set of equations whose corresponding rows in the Jacobian are linearly independent. This is done by extracting first a maximal set of equations in (a) with linearly independent rows in the Jacobian. For that purpose, we use the rows associated to the maximal invertible matrix $\mathtt{H}_t$ (Definition 4.5) for each new basis element $\Lambda_i \in \mathcal{D}_{[t]}$ and $t = 1, \ldots, r$. We denote by G_0 the subsystem of (a) that correspond to rows of $\mathtt{H}_t$.

We complete the system of independent equations G_0 with equations from (c), using a numerical QR decomposition on the transposed Jacobian matrix of G_0 and (c) at the approximate root. Let us denote by F_0 the resulting square system, whose Jacobian, denoted by J_0, is invertible.

For the remaining equations F_1 of (c), not used to construct the square system F_0, define $\Omega = \{(i, j) : \Lambda_i(f_j) \in F_1\}$. We introduce new parameters $\epsilon_{i,j}$ for $(i, j) \in \Omega$ and we consider the perturbed system

$$f_{i,\epsilon} = f_i - \sum_{j|(i,j)\in\Omega} \epsilon_{i,j}\, \mathbf{x}_{\xi}^{\beta_j}.$$

The perturbed system is $\mathbf{f}_\epsilon = \mathbf{f} - \epsilon\, B$, where ϵ is the $N \times r$ matrix with $[\epsilon]_{i,j} = \epsilon_{i,j}$ if $(i, j) \in \Omega$ and $[\epsilon]_{i,j} = 0$ otherwise. Denote by $F(\mathbf{x}, \mu, \epsilon)$ obtained from $F(\mathbf{x}, \mu)$ by replacing $\Lambda_j(f_i)$ by $\Lambda_j(f_{i,\epsilon})$ for $j = 1, \ldots, r, i = 1, \ldots, N$. Then the equations used to construct the square Jacobian J_0 are unchanged. The remaining equations are of the form

$$\Lambda_j(f_{i,\epsilon}) = \Lambda_j(f_i) - \epsilon_{i,j} = 0 \quad (i, j) \in \Omega.$$

Therefore the Jacobian of the complete system $F(\mathbf{x}, \mu, \epsilon)$ is a square invertible matrix of the form

$$J_\epsilon := \begin{pmatrix} J_0 & 0 \\ J_1 & \mathrm{Id} \end{pmatrix}$$

where J_1 is the Jacobian of the system F_1 of polynomials $\Lambda_j(f_i) \in \mathbb{C}[\mathbf{x}, \mu]$ with $(i, j) \in \Omega$.

Since J_ϵ is invertible, the square extended system $F(\mathbf{x},\mu,\epsilon)$ has an isolated root (ξ^*,μ^*,ϵ^*) corresponding to the isolated root (ξ^*,μ^*) of the square system F_0. Furthermore, $\Lambda_j^*(f_i)=\epsilon_{i,j}^*=0$ for $(i,j)\in\Omega$. Here $\Lambda_1^*,\ldots,\Lambda_r^*\in\mathbb{C}[\mathbf{d}_{\xi^*}]$ are defined from (ξ^*,μ^*) recursively by

$$\Lambda_1^*=1_{\xi^*} \text{ and } \Lambda_i^*=\sum_{|\beta_j|<|\beta_i|}\sum_{k=1}^{n}\mu^*_{\beta_i,\beta_j+\mathbf{e}_k}\overline{\int_k}\Lambda_j^*. \tag{20}$$

We have the following property:

Theorem 5.1. *If the Newton iteration*

$$(\xi_{k+1},\mu_{k+1})=(\xi_k,\mu_k)-J_0(\xi_k,\mu_k)^{-1}F_0(\xi_k,\mu_k),$$

starting from a point (ξ_0,μ_0) converges when $k\to\infty$, to a point (ξ^*,μ^*) such that B is a regular basis for the inverse system $\mathcal{D}^*$ associated to (ξ^*,μ^*) and $\mathcal{D}^*$ is complete for $\mathbf{f}$, then there exists a perturbed system $f_{i,\epsilon^*}=f_i-\sum_{j|(i,j)\in\Omega}\epsilon^*_{i,j}\mathbf{x}_{\xi^*}^{\beta_j}$ with $\epsilon^*_{i,j}=\Lambda_j^*(f_i)$ such that ξ^* is a multiple root of f_{i,ϵ^*} with the multiplicity structure defined by μ^*.

Proof. If the sequence (ξ_k,μ_k) converges to the fixed point (ξ^*,μ^*), then we have $F_0(\xi^*,\mu^*)=0$ and in particular, $G_0(\xi^*,\mu^*)=0$ where $G_0(\xi^*,\mu^*)=0$ is the subset of equations selected from (a).

As μ^* is regular for B, if it satisfies $G_0(\xi^*,\mu^*)=0$, it must satisfy all equations (a). Therefore μ^* defines a point $\mathcal{D}^*=\Lambda_1^*\wedge\cdots\wedge\Lambda_r^*\in\mathscr{H}_B^{\mathrm{reg}}$.

As (Λ_i^*) is a basis of $\mathcal{D}^*$ dual to B and $f_{i,\epsilon^*}=f_i-\sum_{j:(i,j)\in\Omega}\epsilon^*_{i,j}\mathbf{x}_{\xi^*}^{\beta_j}$ with $\epsilon^*_{i,j}=\Lambda_j^*(f_i)$ for $(i,j)\in\Omega$, we have that if $(i,j)\in\Omega$ then $\Lambda_j^*(f_{i,\epsilon^*})=\Lambda_j^*(f_i)-\epsilon^*_{i,j}=0$. Otherwise $\Lambda_j^*(f_{i,\epsilon^*})=\Lambda_j^*(f_i)$, since it is one of the equations selected in (c) to construct the system F_0 and $F_0(\xi^*,\mu^*)=0$. This shows that

$$\mathbf{f}_{\epsilon^*}=(f_{i,\epsilon^*})_{i=1}^N\subset(\mathcal{D}^*)^\perp.$$

Since $\mathbf{f}_{\epsilon^*}$ is obtained from $\mathbf{f}$ by adding elements in B, the system (c), at order $\delta+1$ for $\mathbf{f}_{\epsilon^*}$ and $\mathbf{f}$ are equivalent. Thus $\mathcal{D}^*$ is complete for $\mathbf{f}$ and $\mathbf{f}_\epsilon$ and $\mathcal{D}^*=(\mathbf{f}_{\epsilon^*})^\perp\cap\mathbb{C}[\mathbf{d}_{\xi^*}]$ is the inverse system at ξ^* of the system $\mathbf{f}_{\epsilon^*}$. □

5.2. Numerical Newton iteration

We describe now how Newton iterations can be performed efficiently on the (point, dual basis) pair, without resorting to the symbolic expression of the dual basis Λ in terms of the parameters μ. We assume that $\xi\in\mathbb{C}^n$ is an approximate singular point, that $B=\{b_1,\ldots,b_r\}\subset\mathbb{C}[\mathbf{x}]$ with $b_i=\mathbf{x}_\xi^{\beta_i}$ is the primal basis and that $\boldsymbol{\Lambda}=\{\Lambda_1,\ldots,\Lambda_r\}\subset\mathbb{C}[\mathbf{d}_\xi]$ is an (approximate) dual bases with

$$\Lambda_k=\sum_{i\in 1:n,\,j<k}\mu_{k,i,j}\overline{\int_i}\Lambda_j$$

where

$$\mu_{k,i,j}:=\langle\Lambda_k|x_{\xi,i}b_j\rangle_\xi=\mu_{\beta_k,\beta_j+e_i}. \tag{21}$$

According to Lemma 2.5, the coefficients $\mu_{k,i,j}$ such that $\beta_j+\mathbf{e}_i=\beta_l$ are fixed and the others are the free parameters μ. The system of equations, on which Newton iteration is applied, is of the form:

$$F_0(\xi,\mu)=\begin{cases}\langle C_{i,i',k}(\mu)|b_l\rangle_\xi &= 0 \quad \text{for } 1\le i<i'\le n,\ 1\le k\le r \text{ and } 1\le l<k,\\ \langle\Lambda_k(\mu)|f_m\rangle_\xi &= 0 \quad \text{for } (k,m)\in I_0,\end{cases}$$

where $C_{i,i',k}(\mu)=\left(\sum_{j<k}\mu_{k,i,j}x_{\xi,i'}\star\Lambda_j\right)-\left(\sum_{j<k}\mu_{k,i',j}x_{\xi,i}\star\Lambda_j\right)$ are the commutation relations.

To perform a Newton step, we need to evaluate F_0 at (ξ, μ), to compute the Jacobian $J_{F_0}(\xi, \mu)$ of F_0 with respect to ξ and the free parameters $\mu = (\mu_{t,u,v})$, to solve the system $J_{F_0}(\xi, \mu)(\delta\xi, \delta\mu) = -F_0(\xi, \mu)$ and to update the pair $(\xi, \mathbf{\Lambda}(\mu))$ to $(\xi + \delta\xi, \mathbf{\Lambda}(\mu + \delta\mu))$.

The evaluation of $F_0(\xi, \mu)$ requires the evaluation of

$$\begin{aligned}\langle C_{i,i',k}(\mu)|b_l\rangle_\xi &= \left(\sum_{j<k} \mu_{k,i,j}\langle \Lambda_j | x_{\xi,i'} b_l\rangle_\xi\right) - \left(\sum_{j<k} \mu_{k,i',j}\langle \Lambda_j | x_{\xi,i} b_l\rangle_\xi\right) \\ &= \left(\sum_{j<k} \mu_{k,i,j}\mu_{j,i',l} - \mu_{k,i',j}\mu_{j,i,l}\right),\end{aligned}$$

which can be computed from the coefficients $\mu_{k,i,j} = \langle \Lambda_k | x_{\xi,i} b_j\rangle_\xi$. The evaluation of $\langle \Lambda_k(\mu)|f_m\rangle_\xi$ is done using formula (19) evaluated at ξ.

To compute the Jacobian $J_{F_0}(\xi, \mu)$, we first compute the derivatives of $\mathbf{\Lambda}$ with respect to the free parameters $\mu = (\mu^l_{u,v})$, using the following formula:

$$\partial_{\mu_{t,u,v}} \Lambda_k(\mu) = \begin{cases} \sum_{i\in 1:n, j<k} \mu_{k,i,j} \overline{\int}_i \partial_{\mu_{t,u,v}} \Lambda_j & \text{if } t < k \\ \overline{\int}_u \Lambda_v & \text{if } t = k \\ 0 & \text{if } t > k \end{cases}$$

This shows that $\partial_{\mu_{t,u,v}} \Lambda_k(\mu)$ can be computed by induction from the coefficients $\mu_{k,i,j} = \langle \Lambda_k | x_{\xi,i} b_j\rangle_\xi$, and the integration operators $\Lambda \mapsto \overline{\int}_i \Lambda$ applied to $\partial_{\mu_{t,u,v}} \Lambda_j$ with $j < k$.

Similarly, we have

$$\partial_{\mu_{t,u,v}} C_{i,i',k}(\mu) = \begin{cases} x_{\xi,i'} \star \Lambda_v & \text{if } t = k, u = i, \\ -x_{\xi,i} \star \Lambda_v & \text{if } t = k, u = i', \\ 0 & \text{if } t = k, u \neq i, u \neq i', \\ \left(\sum_{j<k} \mu_{k,i,j} x_{\xi,i'} \star \partial_{\mu_{t,u,v}} \Lambda_j\right) & \text{if } t < k, \\ \quad - \left(\sum_{j<k} \mu_{k,i',j} x_{\xi,i} \star \partial_{\mu_{t,u,v}} \Lambda_j\right) & \\ 0 & \text{otherwise.} \end{cases}$$

It also shows that $\partial_{\mu_{t,u,v}} C_{i,i',k}(\mu)$ can be computed by induction, from the coefficients $\mu_{k,i,j} = \langle \Lambda_k | x_{\xi,i} b_j\rangle_\xi$ and the derivation operators $\Lambda \mapsto x_{\xi,i} \star \Lambda$ applied to $\partial_{\mu_{t,u,v}} \Lambda_j$ with $j < k$.

Using (19), we have

$$\partial_{\xi_i} \langle \Lambda_k(\mu)|f_m\rangle_\xi = \sum_\alpha \frac{1}{\alpha!} \mu_{k,\alpha} \boldsymbol{\partial}^\alpha \partial_i f_m(\xi) = \langle \Lambda_k(\mu)|\partial_{\xi,i} f_m\rangle_\xi. \tag{22}$$

Therefore the differential of $F_0(\xi, \mu)$ is of the form

$$J_{F_0}\begin{pmatrix} d\xi \\ d\mu \end{pmatrix} = \begin{cases} \qquad\qquad\qquad\qquad \sum \langle \partial_{\mu_{t,u,v}} C_{i,i',k} | b_l\rangle_\xi d\mu_{t,u,v} \\ \sum_i \langle \Lambda_k | \partial_{x_i} f_m\rangle_\xi d\xi_i \quad + \quad \sum \langle \partial_{\mu_{t,u,v}} \Lambda_k | f_m\rangle_\xi d\mu_{t,u,v} \end{cases}$$

The Jacobian J_{F_0} can thus be computed from ξ, Λ_k, $\partial_{x_i} f_m$, $\partial_{\mu_{t,u,v}} \Lambda_k$ and $\mu_{k,i,j} = \langle \Lambda_k | x_{\xi,i} b_j\rangle_\xi$.

Solving the Jacobian system, we obtain a new (point, parameter) pair $(\xi', \mu') = (\xi, \mu) - J_{F_0}^{-1}(\xi, \mu) F_0(\xi, \mu)$. To update the new inverse system $\mathbf{\Lambda}'$ corresponding to the parameters μ', we compute

$$\Lambda'_k = \sum_{i\in 1:n, j<k} \mu'_{k,i,j} \overline{\int}_i \Lambda'_j$$

also by induction, since Λ'_k depends on $\mu'_{k,i,j}$ and $\overline{\int_i}\Lambda'_j$ for $j<k$.

This provides an algorithm to perform numerically the Newton iterations from a fixed primal basis $B=\{b_1,\ldots,b_r\}\subset\mathbb{C}[\mathbf{x}]$, an (approximate) singular point $\xi\in\mathbb{C}^n$ and an (approximate) dual bases $\mathbf{\Lambda}=\{\Lambda_1,\ldots,\Lambda_r\}\subset\mathbb{C}[\mathbf{d}_\xi]$. We will illustrate it in the experimentation.

6. Certification

In this section we describe how to certify that the Newton iteration defined in Section 5 quadratically converges to a point that defines an exact root with an exact multiplicity structure of a perturbation of the input polynomial system $\mathbf{f}$. More precisely, we are given $\mathbf{f}=(f_1,\ldots,f_N)\in\mathbb{C}[\mathbf{x}]^N$, $B=\{\mathbf{x}^{\beta_1},\ldots,\mathbf{x}^{\beta_r}\}\subset\mathbb{C}[\mathbf{x}]$ in increasing order of degrees and closed under division, $\delta:=|\beta_r|$. We are also given the deflated systems $F(\mathbf{x},\mu)$, its square subsystem $F_0(\mathbf{x},\mu)$ defined in Section 5 and $F_1(\mathbf{x},\mu)$ the remaining equations in $F(\mathbf{x},\mu)$. Finally, we are given $\xi_0\in\mathbb{C}^n$ and $\mu_0=\{\mu^{(0)}_{\beta_i,\beta_j+\mathbf{e}_k}\in\mathbb{C}: i,j=1,\ldots,r, |\beta_j|<|\beta_i|, k=1,\ldots,n\}$. Our certification will consist of a symbolic and a numeric part:

6.1. Regularity certification

We certify that B is regular (see Definition 4.3). This part of the certification is purely symbolic and inductive on t. Suppose for some $t-1<\delta$ we certified that B_{t-1} is regular and computed the parameters $\overline{\mu}_{t-1}$ and the parametrization

$$\Phi_{t-1}:\overline{\mu}_{t-1}\mapsto\left(q_{\beta_i,\alpha}(\overline{\mu}_{t-1})\right)_{|\beta_i|\le t-1,|\alpha|\le t-1}$$

(Algorithm 2). Then to prove that B_t is regular, we consider the coefficient matrix H_t of equations (13) and (14). We substitute the parametrization Φ_{t-1} to get the matrices $H_t(\overline{\mu}_{t-1})$. We symbolically prove that the rows of $\mathrm{H}_t(\overline{\mu}_{t-1})$ (Definition 4.5) are linearly independent and span all rows of $H_t(\overline{\mu}_{t-1})$ over $\mathbb{Q}(\overline{\mu}_{t-1})$. If that is certified, we compute the parameters $\overline{\mu}_t$ and the parametrization $\Phi_t:\overline{\mu}_t\mapsto\left(q_{\beta_i,\alpha}(\overline{\mu}_t)\right)_{|\beta_i|\le t,|\alpha|\le t}$ as in Algorithm 2 inverting the square submatrix H_t of H_t such that the denominators of $q_{\beta_i,\alpha}$ for $|\beta_i|=t$ divide $\det(\mathrm{H}_t(\overline{\mu}_{t-1}))\neq 0$.

6.2. Singularity certification

(C1) We certify that the Newton iteration for the square system F_0 starting from (ξ_0,μ_0) quadratically converges to some root (ξ^*,μ^*) of F_0, such that $\|(\xi_0,\mu_0)-(\xi^*,\mu^*)\|_2\le\tilde{\beta}$, using α-theory.
(C2) We certify that $\mathcal{D}^*=\mathrm{span}(\mathbf{\Lambda}^*)$ is regular for B (see Definition 4.3), by checking that $|\mathrm{H}_t(\mu^*)|\neq 0$ for $t=1,\ldots,\delta$ (See Definition 4.5), using the Singular Value Decomposition of $\mathrm{H}_t(\mu_0)$ and the distance bound $\tilde{\beta}$ between μ^* and μ_0.
(C3) We certify that $\mathbf{\Lambda}^*$ is complete for $\mathbf{f}$ at ξ^* (see Definition 2.7), where $\mathbf{\Lambda}^*\subset\mathbb{C}[\mathbf{d}_{\xi^*}]$ is the dual systems defined from (ξ^*,μ^*) recursively as in (20). This is done by checking that $\ker K_{\delta+1}(\xi^*,\mu^*)=\{0\}$ (See Definition 2.7), using the Singular Value Decomposition of $K_{\delta+1}(\xi_0,\mu_0)$ and the distance bound $\tilde{\beta}$ between (ξ^*,μ^*) and (ξ_0,μ_0).

Let us now consider for a point-multiplicity structure pair (ξ_0,μ_0) $\tilde{\gamma}:=\sup_{k\ge2}\|DF_0^{-1}(\xi_0,\mu_0)\frac{D^kF_0(\xi_0,\mu_0)}{k!}\|^{\frac{1}{k-1}}$, $\tilde{\beta}:=2\|DF_0^{-1}(\xi_0,\mu_0)\,F_0(\xi_0,\mu_0)\|$, $\tilde{\alpha}:=\tilde{\beta}\,\tilde{\gamma}$ and for a matrix function $A(\xi,\mu)$, let $\mathcal{L}_1(A;\xi_0,\mu_0;b)$ be a bound on its Lipschitz constant in the ball $\mathcal{B}_b(\xi_0,\mu_0)$ of radius b around (ξ_0,μ_0) such that $\|A(\xi,\mu)-A(\xi_0,\mu_0)\|\le\mathcal{L}_1(A;\xi_0,\mu_0;b)\,\|(\xi,\mu)-(\xi_0,\mu_0)\|$ for $(\xi,\mu)\in\mathcal{B}_b(\xi_0,\mu_0)$. For a matrix M, let $\sigma_{\min}(M)$ be its smallest singular value. We have the following result:

Theorem 6.1. *Let $B=\{\mathbf{x}^{\beta_1},\ldots,\mathbf{x}^{\beta_r}\}\subset\mathbb{C}[\mathbf{x}]$ be closed under division and suppose B is regular. Suppose that $\tilde{\alpha}<\tilde{\alpha}_0:=0.26141$, $\mathcal{L}_1(K_{\delta+1};\xi_0,\mu_0;\tilde{\beta})\,\tilde{\beta}<\sigma_{\min}(K_{\delta+1}(\xi_0,\mu_0))$ and for $t=2,\ldots,\delta$ it holds that $\mathcal{L}_1(\mathrm{H}_t;\mu_0;\tilde{\beta})\,\tilde{\beta}<\sigma_{\min}(\mathrm{H}_t(\mu_0))$. Then the Newton iteration on the square system F_0 starting from (ξ_0,μ_0) converges quadratically to a point (ξ^*,μ^*) corresponding to a multiple point ξ^* with multiplicity structure*

μ^* of the perturbed system $\mathbf{f}_{\epsilon^*} = \mathbf{f} - \epsilon^* B_{\xi^*}$ such that $\|\epsilon^*\| \le \|F_1(\xi_0, \mu_0)\| + \mathcal{L}_1(F_1; \xi_0, \mu_0; \tilde\beta)\,\tilde\beta$, where $B_{\xi^*} = \{\mathbf{x}_{\xi^*}^{\beta_1}, \dots, \mathbf{x}_{\xi^*}^{\beta_r}\}$.

Proof. By the α-theorem (Blum et al., 1998)[Chap. 8, Thm. 1], the Newton iteration on F_0 starting from (ξ_0, μ_0) converges quadratically to a point (ξ^*, μ^*) such that

$$\|(\xi^*, \mu^*) - (\xi_0, \mu_0)\| < \tilde\beta\,.$$

We deduce that

$$\begin{aligned} \|K_{\delta+1}(\xi^*, \mu^*) - K_{\delta+1}(\xi_0, \mu_0)\| &\le \mathcal{L}_1(K_{\delta+1}; \xi_0, \mu_0; \tilde\beta)\,\|(\xi^*, \mu^*) - (\xi_0, \mu_0)\| \\ &< \sigma_{\min}(K_{\delta+1}(\xi_0, \mu_0)). \end{aligned}$$

Therefore $K_{\delta+1}(\xi^*, \mu^*)$ is within a ball around $K_{\delta+1}(\xi_0, \mu_0)$ of matrices of maximal rank, since $\sigma_{\min}(K_{\delta+1}(\xi_0, \mu_0))$ is the distance between $K_{\delta+1}(\xi_0, \mu_0)$ and the set of matrices not of maximal rank.

Thus $\ker K_{\delta+1}(\xi^*, \mu^*) = \{0\}$. A similar argument shows that $|\mathrm{H}_t(\mu^*)| \ne 0$ for $t = 1, \dots, \delta$. By Theorem 5.1, (ξ^*, μ^*) defines a multiple root ξ^* with multiplicity structure μ^* for the perturbed system $\mathbf{f}_{\epsilon^*} = \mathbf{f} - \epsilon^* B_{\xi^*}$ with

$$\begin{aligned} \|\epsilon^*\| = \|F_1(\xi^*, \mu^*)\| \le \|F_1(\xi_0, \mu_0)\| &+ \|F_1(\xi^*, \mu^*) - F_1(\xi^*, \mu^*)\| \\ &\le \|F_1(\xi_0, \mu_0)\| + \mathcal{L}_1(F_1; \xi_0, \mu_0; \tilde\beta)\,\|(\xi^*, \mu^*) - (\xi_0^*, \mu_0^*)\| \\ &\le \|F_1(\xi_0, \mu_0)\| + \mathcal{L}_1(F_1; \xi_0, \mu_0; \tilde\beta)\,\tilde\beta. \quad \square \end{aligned}$$

7. Experimentation

In this section we work out some examples with (approximate) singularities. The experiments are carried out using Maple, to get the symbolic expressions of the inverse system and of the Jacobian in terms of the parameters μ. The symbolic stage based on simple algebraic computations is carried out rigorously for all our tests in Maple, computing the parametric inverse system, the commutation rules and the maximal subsystems at each step. For the numerical part, in particular the numerical Newton iterations, we use Julia code. All these codes and examples are publicly available at https://gitlab.inria.fr/AlgebraicGeometricModeling/certified-singularities.[1]

Example 7.1. We consider the equations

$$f_1 = x_1^3 + x_2^2 + x_3^2 - 1,\ f_2 = x_2^3 + x_1^2 + x_3^2 - 1,\ f_3 = x_3^3 + x_1^2 + x_2^2 - 1,$$

the approximate root $\boldsymbol{\xi}_0 = (0.002, 1.003, 0.004)$ and threshold $\varepsilon = 0.01$. In the following we use 32-digit arithmetic for all computations.

We shall first compute a primal basis using Algorithm 1. In the first iteration we produce the 3×3 Jacobian

$$K_1 = K_1(\boldsymbol{\xi}_0) = \begin{matrix} \\ \Lambda(f_1) \\ \Lambda(f_2) \\ \Lambda(f_3) \end{matrix} \begin{matrix} \nu_1^1 & \nu_1^2 & \nu_1^3 \\ \left[\begin{matrix} 0.00001 \\ 0.00400 \\ 0.00400 \end{matrix}\right. & \begin{matrix} 2.00600 \\ 3.01803 \\ 2.00600 \end{matrix} & \left.\begin{matrix} 0.00800 \\ 0.00800 \\ 0.00005 \end{matrix}\right] \end{matrix}.$$

[1] Software Heritage permalink: https://archive.softwareheritage.org/swh:1:dir:d35ad5db291637bf71a532aff334d9e8dfe70838;origin=https://gitlab.inria.fr/AlgebraicGeometricModeling/certified-singularities;visit=swh:1:snp:256e9f0fb2921c983d685e98b199a819ef4be9e8;anchor=swh:1:rev:9bb034e298b17a6cce1dd945c12522d8f386c5dc.

The elements in the kernel of this matrix are of the form $\Lambda = \nu_1^1 d_1 + \nu_1^2 d_2 + \nu_1^3 d_3$. The singular values of $K_1(\boldsymbol{\xi}_0)$ are $(4.1421, 0.0064, 0.0012)$, which implies a two-dimensional kernel, since two of them are below threshold ε. The (normalized) elements in the kernel are $\tilde{\Lambda}_2 = d_1 - 0.00117 d_2$ and $\tilde{\Lambda}_3 = d_3 - 0.00235 d_2$. Note that d_2 was not chosen as a leading term. This is due to pivoting used in the numeric process, in order to avoid leading terms with coefficients below the tolerance ε. The resulting primal basis $B_1 = \{1, x_1, x_3\}$ turns out to be closed under derivation.

Similarly, in degree 2 we compute the matrix

$$K_2 = \begin{array}{c} \\ (9) \\ (9) \\ (9) \\ \Lambda(f_1) \\ \Lambda(f_2) \\ \Lambda(f_3) \end{array} \begin{array}{c} \begin{array}{ccccccc} \nu_1^2 & \nu_2^1 & \nu_2^2 & \nu_2^3 & \nu_3^1 & \nu_3^2 & \nu_3^3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0.00117 & 0 & 1.00000 & 0.00235 \\ 0 & 0 & 0 & -1.00000 & 1.00000 & 0 & 0 \\ 0 & -0.00117 & -1.00000 & 0 & -0.00235 & 0 & 0 \\ 2.00600 & 0.00600 & -0.00117 & 0 & 0. & -0.00235 & 1.00000 \\ 3.01803 & 1.00000 & -0.00353 & 0 & 0. & -0.00707 & 1.00000 \\ 2.00600 & 1.00000 & -0.00117 & 0 & 0. & -0.00235 & 0.01200 \end{bmatrix} \end{array},$$

and we obtain one element in its kernel $\tilde{\Lambda}_4 = d_1 d_3 - 0.00002 d_1^2 - 0.00235 d_1 d_2 + 5.5 \cdot 10^{-6} d_2^2 - 0.00117 \cdot d_2 d_3 - 0.00002 d_3^2 + 5.9 \cdot 10^{-6} d_2$. In the final step we produce a matrix K_3 of size 12×9. We stop the iteration due to assuming $\ker K_3 = \{0\}$, since the minimum singular value is $\sigma_{\min} = 0.21549$, therefore we stop the process, since the computed dual is approximately complete (cf. Definition 2.7). We derive that the approximate multiple point has multiplicity $r = 4$ and one primal basis is $B = \{1, x_1, x_3, x_1 x_3\}$.

The full parametric form of a basis of $\mathcal{D}_1$ is $\ker K_1 = \langle \Lambda_2 = d_1 + \mu_{2,2,1} d_2, \Lambda_3 = d_3 + \mu_{3,2,1} d_2 \rangle$. Here we incorporated (10), thus fixing some of the parameters according to primal monomials x_1 and x_3.

The parametric form of the matrix $K_2(\boldsymbol{\xi}, \boldsymbol{\mu})$ of the integration method in degree 2 is

$$\begin{array}{c} \\ (9) \\ (9) \\ (9) \\ \Lambda(f_1) \\ \Lambda(f_2) \\ \Lambda(f_3) \end{array} \begin{array}{c} \begin{array}{ccccccccc} \nu_1^1 & \nu_1^2 & \nu_1^3 & \nu_2^1 & \nu_2^2 & \nu_2^3 & \nu_3^1 & \nu_3^2 & \nu_3^3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{0} & -\mu_{2,2,1} & \mathbf{0} & \mathbf{1} & -\mu_{3,2,1} \\ 0 & 0 & 0 & 0 & \mathbf{0} & -1 & \mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mu_{2,2,1} & \mathbf{-1} & 0 & \boldsymbol{\mu_{3,2,1}} & \mathbf{0} & 0 \\ 3\xi_1^2 & 2\xi_2 & 2\xi_3 & 3\xi_1 & \mu_{2,2,1} & 0 & 3\xi_1 & \mu_{3,2,1} & 1 \\ 2\xi_1 & 3\xi_2^2 & 2\xi_3 & 1 & 3\mu_{2,2,1}\xi_2 & 0 & 0 & 3\mu_{3,2,1}\xi_2 & 1 \\ 2\xi_1 & 2\xi_2 & 3\xi_3^2 & 1 & \mu_{2,2,1} & 0 & 0 & \mu_{3,2,1} & 3\xi_3 \end{bmatrix} \end{array},$$

where the columns correspond to the parameters in the expansion (5):

$$\begin{aligned} \Lambda_4 = {} & \nu_1^1 d_1 + \nu_1^2 d_2 + \nu_1^3 d_3 + \nu_2^1 d_1^2 + \nu_2^2 (d_1 d_2 + \mu_{2,2,1} d_2^2) + \nu_2^3 (d_1 d_3 \\ & + \mu_{2,2,1} d_3 d_2) + \nu_3^1 (\mu_{3,2,1} d_1 d_2) + \nu_3^2 (\mu_{3,2,1} d_2^2) + \nu_3^3 (d_3^2 + \mu_{3,2,1} d_2 d_3) \end{aligned}$$

Setting $\Lambda_4(x_1 x_3) = 1$ and $\Lambda_4(x_1) = \Lambda_4(x_3) = \Lambda_4(1) = 0$, we obtain $\nu_1^1 = \nu_1^3 = 0$ and $\nu_2^3 = 1$. Note that ν_1^1 and ν_1^3 are removed in advance from the numeric version of K_2 above. The dual element of order 2 is has the parametric form

$$\begin{aligned} \Lambda_4 = {} & d_1 d_3 + \mu_{4,2,1} d_2 + \mu_{4,1,2} d_1^2 + \mu_{4,2,2} d_1 d_2 + \mu_{4,3,3} d_3^2 \\ & + (\mu_{2,2,1} + \mu_{3,2,1} \mu_{4,3,3}) d_2 d_3 + (\mu_{2,2,1} \mu_{4,1,3} + \mu_{3,2,1} \mu_{4,2,3}) d_2^2 \end{aligned}$$

($\nu_1^2 = \mu_{4,2,1}$, $\nu_2^1 = \mu_{4,1,2}$, $\nu_2^2 = \mu_{4,2,2}$, $\nu_3^1 = \mu_{4,1,3}$, $\nu_3^2 = \mu_{4,2,3}$, $\nu_3^3 = \mu_{4,3,3}$). Overall 8 parameters are used in the representation of $\mathcal{D}_2$. The highlighted entries of $K_2(\boldsymbol{\xi}, \boldsymbol{\mu})$ form the non-singular matrix H_2 in Definition 4.5, therefore $\mathcal{D}_2$ is regular for B (cf. Definition 4.3). We obtain the polynomial parameterization $\mu_{4,2,2} = \mu_{2,2,1} \mu_{4,1,2} + \mu_{3,2,1}$, $\mu_{4,1,3} = 1$, $\mu_{4,2,3} = \mu_{2,2,1} + \mu_{3,2,1} \mu_{4,3,3}$ with the free parameters $\bar{\mu} = (\mu_{2,2,1}, \mu_{3,2,1}, \mu_{4,2,1}, \mu_{4,1,2}, \mu_{4,3,3})$. There is no denominator since $\det \mathrm{H}_2 = 1$.

We now setup the numerical scheme. The overdetermined and deflated system $F(\mathbf{x}, \boldsymbol{\mu})$ consists of 15 polynomials in the variables $\mathbf{x}, \boldsymbol{\mu}$:

$$F(\mathbf{x}, \boldsymbol{\mu}) = \begin{cases} \mu_{2,2,1}\mu_{4,1,2} + \mu_{3,2,1} - \mu_{4,2,2}\,, -\mu_{4,1,3} + 1\,, -\mu_{2,2,1}\mu_{4,1,3} \\ \quad - \mu_{3,2,1}\mu_{4,3,3} + \mu_{4,2,3}, \\ \Lambda_1(f_1) = x_1^3 + x_2^2 + x_3^2 - 1,\ \Lambda_1(f_2) = x_2^3 + x_1^2 + x_3^2 - 1,\ \Lambda_1(f_3) \\ \quad = x_3^3 + x_1^2 + x_2^2 - 1,\ \Lambda_2(f_1) = 2\mu_{2,2,1}x_2 + 3x_1^2, \\ \Lambda_2(f_2) = 3\mu_{2,2,1}x_2^2 + 2x_1, \Lambda_2(f_3) = 2\mu_{2,2,1}x_2 + 2x_1, \Lambda_3(f_1) = 2\mu_{3,2,1}x_2 + 2x_3, \\ \Lambda_3(f_2) = 3\mu_{3,2,1}x_2^2 + 2x_3,\ \Lambda_3(f_3) = 2\mu_{3,2,1}x_2 + 3x_3^2, \\ \Lambda_4(f_1) = \mu_{2,2,1}\mu_{4,2,2} + \mu_{3,2,1}\mu_{4,2,3} + 2\mu_{4,2,1}x_2 + 3\mu_{4,1,2}x_1 + \mu_{4,3,3}, \\ \Lambda_4(f_2) = 3\mu_{2,2,1}\mu_{4,2,2}x_2 + 3\mu_{3,2,1}\mu_{4,2,3}x_2 + 3\mu_{4,2,1}x_2^2 + \mu_{4,1,2} + \mu_{4,3,3}, \\ \Lambda_4(f_3) = \mu_{2,2,1}\mu_{4,2,2} + \mu_{3,2,1}\mu_{4,2,3} + 2\mu_{4,2,1}x_2 + 3\mu_{4,3,3}x_3 + \mu_{4,1,2} \end{cases}$$

We now consider $J_F(\boldsymbol{\xi}_0, \boldsymbol{\mu}_0)$. This Jacobian is of full rank, and we can obtain a maximal minor by removing $\Lambda_1(f_2), \Lambda_1(f_3), \Lambda_2(f_3)$ and $\Lambda_3(f_3)$ from F. We obtain the square 11×11 system denoted by F_0. The general form of $J_{F_0}(\boldsymbol{x}, \boldsymbol{\mu})$ is

$$\begin{array}{c} \\ (9) \\ (9) \\ (9) \\ \Lambda_1(f_1) \\ \Lambda_2(f_1) \\ \Lambda_2(f_2) \\ \Lambda_3(f_1) \\ \Lambda_3(f_2) \\ \Lambda_4(f_1) \\ \Lambda_4(f_2) \\ \Lambda_4(f_3) \end{array} \begin{array}{cccccccc|ccc} \partial\mu_{2,2,1} & \partial\mu_{3,2,1} & \partial\mu_{4,2,1} & \partial\mu_{4,1,2} & \partial\mu_{4,2,2} & \partial\mu_{4,1,3} & \partial\mu_{4,2,3} & \partial\mu_{4,3,3} & \partial x_1 & \partial x_2 & \partial x_3 \\ \mu_{4,1,2} & 1 & 0 & \mu_{2,2,1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\mu_{4,1,3} & -\mu_{4,3,3} & 0 & 0 & 0 & -\mu_{2,2,1} & 1 & -\mu_{3,2,1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3x_1^2 & 2x_2 & 2x_3 \\ 2x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6x_1 & 2\mu_{2,2,1} & 0 \\ 3x_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6\mu_{2,2,1}x_2 & 0 \\ 0 & 2x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu_{3,2,1} & 2 \\ 0 & 3x_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6\mu_{3,2,1}x_2 & 2 \\ \mu_{4,2,2} & \mu_{4,2,3} & 2x_2 & 3x_1 & \mu_{2,2,1} & 0 & \mu_{3,2,1} & 1 & 3\mu_{4,1,2} & 2\mu_{4,2,1} & 0 \\ 3\mu_{4,2,2}x_2 & 3\mu_{4,2,3}x_2 & 3x_2^2 & 1 & 3\mu_{2,2,1}x_2 & 0 & 3\mu_{3,2,1}x_2 & 1 & 0 & \partial_{x_2}\Lambda_4(f_2) & 0 \\ \mu_{4,2,2} & \mu_{4,2,3} & 2x_2 & 1 & \mu_{2,2,1} & 0 & \mu_{3,2,1} & 3x_3 & 0 & 2\mu_{4,2,1} & 3\mu_{4,3,3} \end{array}$$

where $\partial_{x_2}\Lambda_4(f_2) = 3(\mu_{2,2,1}\mu_{4,2,2} + \mu_{3,2,1}\mu_{4,2,3} + 2\mu_{4,2,1}x_2)$. The blocks in this matrix can be computed recursively using the formulas in Section 5.2. The initial point of the Newton iterations is $\boldsymbol{\xi}_0 = (0.002, 1.003, 0.004)$ and the approximation of the variables $\mu_{i,j}$ provided by the numerical integration method: $\boldsymbol{\mu}_0 = (-0.00117, -0.00235, 5.9 \cdot 10^{-6}, -0.00002, -0.00235, 1.0, -0.00117, -0.00002)$.

We use Theorem 6.1 to certify the convergence to a singular system. We can compute (see e.g. Hauenstein and Sottile, 2012) for $(\boldsymbol{\xi}_0, \boldsymbol{\mu}_0)$:

$$\tilde{\beta} \approx 0.01301544\,, \quad \gamma \leq 18.58366113\,,$$

which leads to $\tilde{\beta}\gamma < \tilde{\alpha}_0$. The Lipschitz constants can be estimated by means of interval arithmetic as:

$$\mathcal{L}_1(K_3; \xi_0, \boldsymbol{\mu_0}; \tilde{\beta}) \leq 9.66542\,, \quad \mathcal{L}_1(\mathrm{H}_2; \boldsymbol{\mu_0}; \tilde{\beta}) \leq 4.08068\,, \quad \mathcal{L}_1(\mathrm{H}_3; \boldsymbol{\mu_0}; \tilde{\beta}) \leq 9.66542$$

as well as the singular values:

$$\sigma_{\min}(K_3(\boldsymbol{\xi}_0, \boldsymbol{\mu}_0)) = 0.21549..\,, \quad \sigma_{\min}(\mathrm{H}_2(\boldsymbol{\mu_0})) = 0.21550..\,, \quad \sigma_{\min}(\mathrm{H}_3(\boldsymbol{\mu_0})) = 0.21550..$$

The assumptions of Theorem 6.1 are satisfied for guaranteed convergence. In the next iterations we observe that the sequence of $\tilde{\beta}$'s tends to zero $(0.01302, 1.1 \cdot 10^{-4}, 7 \cdot 10^{-9}, 2.4 \cdot 10^{-17})$, which confirms that we are in the region of convergence: Indeed, the successive residuals for 4 iterations are $0.00603, 4.0 \cdot 10^{-5}, 2.07 \cdot 10^{-9}, 8.6 \cdot 10^{-18}, 3.55 \cdot 10^{-35}$. Clearly, the residual shrinks with a quadratic rate.[2] We obtain $\boldsymbol{\xi}_4 = (1.8 \cdot 10^{-37}, 1.0, 2.8 \cdot 10^{-36})$ and the overdetermined system is satisfied by this point: $\|F(\boldsymbol{\xi}_4, \boldsymbol{\mu}_4)\|_\infty = 8 \cdot 10^{-35}$; the resulting dual structure is $\mathcal{D}_2^* = \{1, d_1, d_3, d_1 d_3\}$.

[2] The convergence is seen up to machine error. If we increase the accuracy to 150 digits the rate remains quadratic for 7 iterations: $\ldots 3.55 \cdot 10^{-35}, 6.78 \cdot 10^{-70}, 4.15 \cdot 10^{-140}, 5.1 \cdot 10^{-281}$.

Table 1
Size of required matrices and parameters for deflation.

System	r/n	IM	SC	$\#\mu$	OS	r_0	r	N
cmbs1	11/3	27×23	75	74	108×77	8.814e-01	2.361e-31	3
cmbs2	8/3	21×17	21	33	45×36	3.630e-01	1.464e-16	5
mth191	4/3	10×9	3	9	15×12	2.344e-01	3.181e-31	4
decker2	4/2	5×5	4	8	12×10	7.958e-02	1.033e-22	3
Ojika2	2/3	6×5	0	2	6×5	3.954e-01	2.025e-17	5
Ojika3	4/3	12×9	15	14	27×17	3.198e-01	2.238e-16	6
KSS	16/5	155×65	510	362	590×367	4.364e-01	2.914e-11	4
Caprasse	4/4	22×13	6	15	22×19	1.528e+02	1.410e-05	5
Cyclic-9	4/9	104×33	36	40	72×49	3.958e+00	9.058e-15	4

Example 7.2. We demonstrate how our method handles inaccuracies in the input, and recovers a nearby system with a true multiple point. Let

$$f_1 = {x_1}^2 + x_1 - x_2 + 0.003 \quad , \quad f_2 = {x_2}^2 + 1.004x_1 - x_2 .$$

There is a cluster of three roots around $\boldsymbol{\xi}_0 = (0.001, -0.002)$. Our goal is to squeeze the cluster down to a three-fold real root. We use 32 digits for the computation. Starting with $\boldsymbol{\xi}_0$, and a tolerance equal to 10^{-2} Algorithm 1 produces an approximate dual $\mathbf{1}$, $d_1 + 1.00099651 d_2$, $d_1^2 + 1.00099651 d_1 d_2 + 1.00266222 d_2^2 + 0.99933134 d_2$ and identifies the primal basis $B = \{1, x_1, x_1^2\}$ using pivoting on the integration matrix. The sole condition of type (13) reads $\mu_{2,1,2} - \mu_{3,2,2} = 0$, and $\Lambda_1 = \mathbf{1}$, $\Lambda_2 = d_1 + \mu_{2,1,2} d_2$, $\Lambda_3 = d_1^2 + \mu_{2,1,2} d_1 d_2 + \mu_{3,2,1} d_2 + \mu_{3,2,2} \mu_{2,1,2} d_2^2$.

The nearby system that we shall obtain is deduced by the residue in Newton's method. In particular, starting from $\boldsymbol{\xi}_0$, we consider the square system given by removing the equations $\Lambda_1(f_1) = 0$ and $\Lambda_2(f_2) = 0$. The rank of the corresponding Jacobian matrix remains maximal, therefore such a choice is valid. Newton's iterations converge quadratically to the point $(\boldsymbol{\xi}_5, \boldsymbol{\mu}_5) = (1.1 \cdot 10^{-33}, 1.2 \cdot 10^{-33}, 1, 1, 1)$. The full residual is now

$$F(\boldsymbol{\xi}_5, \boldsymbol{\mu}_5) = (0, 0.003, -10^{-32}, 10^{-32}, 0.004, 0, 0) .$$

This yields a perturbation $\tilde{f}_1 \approx f_1 - 0.003$ and $\tilde{f}_2 \approx f_2 - 0.004(x_1 - \xi_1^*)$ to obtain a system with an exact multiple root at the origin (cf. Th. 6.1). Of course, this choice of the square sub-system is not unique. By selecting to remove equations $\Lambda_1(f_1) = 0$ and $\Lambda_1(f_2) = 0$ instead, we obtain $(\boldsymbol{\xi}_5, \boldsymbol{\mu}_5) = (0.00066578, -0.00133245, 1.001, 1.0, 1.001)$ and the residual $F(\boldsymbol{\xi}_5, \boldsymbol{\mu}_5) = (0, 0.005, 0.002, 0, 0, 0, 0)$, so that the nearby system

$$f_1^* \approx {x_1}^2 + x_1 - x_2 + 0.008, \quad f_2^* \approx {x_2}^2 + 1.004x_1 - x_2 + 0.002$$

has a singularity at the limit point $\boldsymbol{\xi}^* \approx (0.00066578, -0.00133245)$ described locally by the coefficients $\boldsymbol{\mu}^* \approx (1.001, 1.0, 1.001)$.

Finally, consider the two square sub-systems as above, after changing f_1, f_2 to define an *exact three-fold root* at the origin (i.e. $f_1 = {x_1}^2 + x_1 - x_2$, $f_2 = {x_2}^2 + x_1 - x_2$). Newton's iteration with initial point $\boldsymbol{\xi}_0$ on either deflated system converges quadratically to $(\boldsymbol{\xi}, \boldsymbol{\mu}) = (\mathbf{0}, \mathbf{1})$. This is a general property of the method: exact multiple roots and their structure are recovered by this process if $\boldsymbol{\xi}_0$ is a sufficiently good initial approximation (cf. Section 5).

Example 7.3. We show some execution details on a set of benchmark examples in taken from Dayton and Zeng (2005), see also Mantzaflaris and Mourrain (2014). For this benchmark, we are given systems and approximate singular points. We compute the approximate inverse system using the integration method, analyze the matrices involved in the computation and apply Newton iterations to obtain better approximation of the singular points and its multiplicity structure.

In Table 1, "IM" is the maximal size of the (numeric) integration matrix that is computed to obtain the multiplicity, "$\#\mu$" is the number of new parameters that are needed for certified deflation, "SC" is the number of constraints of type (13) that were computed and "OS" stands for the size of the overdetermined system (equations $\times$ variables). This is the size of the Jacobian matrix that must be computed and inverted in each Newton's iteration. r_0 and r stand respectively for the norm of residual vector at the initial approximate solution and approximate inverse system and after the last Newton iteration, N is the number of iterations, where the iterations are stopped when the ratio of the norm of two consecutive Newton steps is less than 10. The initial inverse system is computed from the given approximate singular point, using the integration method with an adapted threshold for the numerical rank of the matrices H_t. For the numerical Newton iterations, we apply the formulas given in Section 5.2. The computations are performed with `double` (64 bit) arithmetic. For Caprasse and Cyclic-9 examples, the singular root has complex non-real coordinates.

We can observe that the number of parameters required can grow significantly. Moreover, these parameters induce non-trivial denominators in the rational functions $q_{\beta_j,\alpha}(\overline{\mu})$ of Proposition 4.4, for the instances cmbs1, cmbs2 and KSS. The quadratic convergence of Newton method is observed on all the examples. For Caprasse example, we observe a high initial residual error and a final residual which is not close to the machine precision, due to an early stop of Newton iterations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This research was partly supported by the H2020-MSCA-ITN projects GRAPES (GA 860843) and POEMA (GA 813211) and the NSF grant CCF-1813340.

References

Ayyildiz Akoglu, T., Hauenstein, J.D., Szanto, A., 2018. Certifying solutions to overdetermined and singular polynomial systems over Q. J. Symb. Comput. 84, 147–171.

Bejleri, D., Stapleton, D., 2017. The tangent space of the punctual Hilbert scheme. Mich. Math. J. 66 (3), 595–610.

Blum, L., Cucker, F., Shub, M., Smale, S., 1998. Complexity and Real Computation. Springer, NY.

Briançon, J., 1977. Description de $Hilb^n\mathbb{C}\{x, y\}$. Invent. Math. 41, 45–90.

Briançon, J., Iarrobino, A., 1978. Dimension of the punctual Hilbert scheme. J. Algebra 55 (2), 536–544.

Dayton, B.H., Li, T.-Y., Zeng, Z., 2011. Multiple zeros of nonlinear systems. Math. Comput. 80 (276), 2143–2168.

Dayton, B.H., Zeng, Z., 2005. Computing the multiplicity structure in solving polynomial systems. In: Proc. of ISSAC '05. NY, USA, 2005. ACM, pp. 116–123.

Dedieu, J.-P., Shub, M., 2001. On simple double zeros and badly conditioned zeros of analytic functions of n variables. Math. Comput. 70 (233), 319–327.

Doubilet, P., Rota, G.-C., Stein, J., 1974. On the foundations of combinatorial theory. Stud. Appl. Math. 53 (3), 185–216.

Emiris, I.Z., Mourrain, B., Tsigaridas, E., 2010. The DMM bound: multivariate (aggregate) separation bounds. In: Watt, S. (Ed.), Proceedings of the ISSAC'10. Munich, Germany, July 2010. ACM, pp. 243–250.

Giusti, M., Lecerf, G., Salvy, B., Yakoubsohn, J.-C., 2005. On location and approximation of clusters of zeros of analytic functions. Found. Comput. Math. 5 (3), 257–311.

Giusti, M., Lecerf, G., Salvy, B., Yakoubsohn, J.-C., 2007. On location and approximation of clusters of zeros: case of embedding dimension one. Found. Comput. Math. 7, 1–58.

Giusti, M., Yakoubsohn, J.-C., 2013. Multiplicity hunting and approximating multiple roots of polynomial systems. Contemp. Math. 604, 105–128.

Giusti, M., Yakoubsohn, J.-C., 2020. Approximation numérique de racines isolées multiples de systèmes analytiques. Ann. Henri Lebesgue 3, 901–957. https://doi.org/10.5802/ahl.49.

Hao, W., Sommese, A.J., Zeng, Z., 2013. Algorithm 931: an algorithm and software for computing multiplicity structures at zeros of nonlinear systems. ACM Trans. Math. Softw. 40 (1), 5.

Hauenstein, J.D., Mourrain, B., Szanto, A., 2016. On deflation and multiplicity structure. J. Symb. Comput. 83, 228–253.

Hauenstein, J.D., Sottile, F., 2012. Algorithm 921: alphaCertified: certifying solutions to polynomial systems. ACM Trans. Math. Softw. 38 (4), 28.

Iarrobino, A.A., 1977. Punctual Hilbert Schemes. Memoirs of the American Mathematical Society, vol. 188. AMS, Providence.

Kanzawa, Y., Oishi, S., 1997. Singular solutions of nonlinear equations and a numerical method of proving their existence. In: Proc. of the '97 International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics, pp. 29–32.

Lee, K., Li, N., Zhi, L., 2019. On isolation of singular zeros of multivariate analytic systems. arXiv:1904.0793.

Leykin, A., Verschelde, J., Zhao, A., 2006. Newton's method with deflation for isolated singularities of polynomial systems. Theor. Comput. Sci. 359 (1–3), 111–122.

Leykin, A., Verschelde, J., Zhao, A., 2008. Higher-order deflation for polynomial systems with isolated singular solutions. In: Dickenstein, A., Schreyer, F.-O., Sommese, A. (Eds.), Algorithms in Algebraic Geometry. In: The IMA Volumes in Mathematics and Its Applications, vol. 146. Springer, pp. 79–97.

Li, N., Zhi, L., 2013. Verified error bounds for isolated singular solutions of polynomial systems: case of breadth one. Theor. Comput. Sci. 479, 163–173.

Li, N., Zhi, L., 2014. Verified error bounds for isolated singular solutions of polynomial systems. SIAM J. Numer. Anal. 52 (4), 1623–1640.

Li, Z., Sang, H., 2015. Verified error bounds for singular solutions of nonlinear systems. Numer. Algorithms 70 (2), 309–331.

Mantzaflaris, A., Mourrain, B., 2011. Deflation and certified isolation of singular zeros of polynomial systems. In: Proc. of ISSAC '11. ACM, pp. 249–256.

Mantzaflaris, A., Mourrain, B., 2014. Singular zeros of polynomial systems. In: Dokken, T., Muntingh, G. (Eds.), Advances in Shapes, Geometry, and Algebra. In: Geometry and Computing, vol. 10. Springer, pp. 77–103.

Mantzaflaris, A., Mourrain, B., Punctual, Szanto A., 2020. Hilbert scheme and certified approximate singularities. In: Proc. of ISSAC '20. ACM, pp. 336–343.

Mourrain, B., 1997. Isolated points, duality and residues. J. Pure Appl. Algebra 117–118, 469–493.

Rump, S., Graillat, S., 2010. Verified error bounds for multiple roots of systems of nonlinear equations. Numer. Algorithms 54, 359–377.

Shafarevich, I.R., 2013. Basic Algebraic Geometry 1: Varieties in Projective Space, 3rd edition. Springer, New York.

Wu, X., Zhi, L., 2008a. Computing the multiplicity structure from geometric involutive form. In: Proceedings of ISSAC '08. NY, USA, 2008. ACM, pp. 325–332.

Wu, X., Zhi, L., 2008b. Determining singular solutions of polynomial systems via symbolic-numeric reduction to geometric involutive form. J. Symb. Comput. 27, 104–122.

Yakoubsohn, J.-C., 2000. Finding a cluster of zeros of univariate polynomials. In: Complexity Theory, Real Machines, and Homotopy, vol. 16. Oxford, 1999, pp. 603–638.

Yakoubsohn, J.-C., 2002. Simultaneous computation of all the zero-clusters of a univariate polynomial. In: Foundations of Computational Mathematics. Hong Kong, 2000. World Sci. Publ., River Edge, NJ, pp. 433–455.

Zeng, Z., 2009. The Closedness Subspace Method for Computing the Multiplicity Structure of a Polynomial System. Contemp. Math., vol. 496. Amer. Math. Soc., Providence, RI, pp. 347–362.