Smooth Points on Semi-algebraic Sets

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1 Introduction

Many algorithms for determining properties of semi-algebraic sets rely upon the ability to compute smooth points [1]. We present a simple procedure based on computing the critical points of some well-chosen function that guarantees the computation of smooth points in each connected bounded component of a real atomic semi-algebraic set. Our technique is intuitive in principal, performs well on previously difficult examples, and is straightforward to implement using existing numerical algebraic geometry software. The practical efficiency of our approach is demonstrated by solving a conjecture on the number of equilibria of the Kuramoto model for the $n = 4$ case. We also apply our method to design an efficient algorithm to compute the real dimension of algebraic sets, the original motivation for this research.

2 Computation of Real Smooth Points

2.1 Equidimensional case

Theorem 2.1. *Let $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ and assume that*
$V := V(f_1, \ldots, f_s) \subset \mathbb{C}^n$ is equidimensional of dimension $n-s$. Suppose that $g \in \mathbb{R}[x_1, \ldots, x_n]$ satisfies the following conditions:

1. *$\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$;*

2. *$\dim(V \cap V(g)) < n - s$.*

The set of points where g restricted to $V \cap \mathbb{R}^n$ attains its extreme values intersects each bounded connected component of $(V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n$.

The proof of this theorem is based on the following lemma.

Lemma 2.2. *Let V be as in Theorem 2.1. Let $g \in \mathbb{R}[x_1, \ldots, x_n]$ such that $\dim(V \cap V(g)) < n - s$. Then, either $(V \setminus V(g)) \cap \mathbb{R}^n = \emptyset$ or g restricted to $V \cap \mathbb{R}^n$ attains a non-zero extreme value on each bounded connected component of $(V \setminus V(g)) \cap \mathbb{R}^n$.*

2.2 Application to Kuramoto model

The Kuramoto model [8] is a dynamical system used to model synchronization amongst n coupled oscillators. The maximum number of equilibria (i.e. real solutions to steady-state equations) for $n \geq 4$ remains an open problem [4]. The following confirms the conjecture in [12] for $n = 4$.

Theorem 2.3. *The maximum number of equilibria for the Kuramoto model with $n = 4$ oscillators is 10.*

Let $D(\omega)$ be the discriminant polynomial of the system F, a polynomial in ω of degree 48. The number of real solutions of F is constant in each connected component of $\mathbb{R}^3 \setminus V(D)$, so we need to compute at least one interior point in each of the bounded connected components of $\mathbb{R}^3 \setminus V(D)$. Applying Lemma 2.2 with $F = 0$ and $g = D$ accomplishes this task. Exploiting symmetry and utilizing `Bertini` [2], `alphaCertified` [6], and `Macaulay2` [5] all solutions have been found and certified. In fact, this computation showed that all real critical points of D arose, up to symmetry, along two slices shown in Figure 1. A similar computation then counted the number of real solutions to $F = 0$ showing that the maximum number of equilibria is 10.

2.3 Non-equidimensional algorithm

We now consider the case when $V(f_1, \ldots, f_s)$ is not equidimensional, i.e., it has some components of dimension greater than $n - s$. To handle this case, we perturb the polynomials by constants and take limits. We present an algorithm that computes real smooth points on this limit.

Definition 2.4. For polynomials $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ and a point $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Q}^s$, we say that $f_1, \ldots f_s$ and $\mathbf{a}$ satisfy Assumption (A) if

(A): There exists $e_0 > 0$ such that for all $0 < e \le e_0$, the polynomials $f_1 - ea_1, \ldots, f_s - ea_s$ generate a radical equidimensional ideal and $V_e^{\mathbf{a}} := V(f_1 - ea_1, \ldots, f_s - ea_s)$ is smooth and has dimension $n - s$.

One key aspect of ALGORITHM 1 is a polynomial g that satisfies the conditions of Theorem 2.1, i.e. $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset$
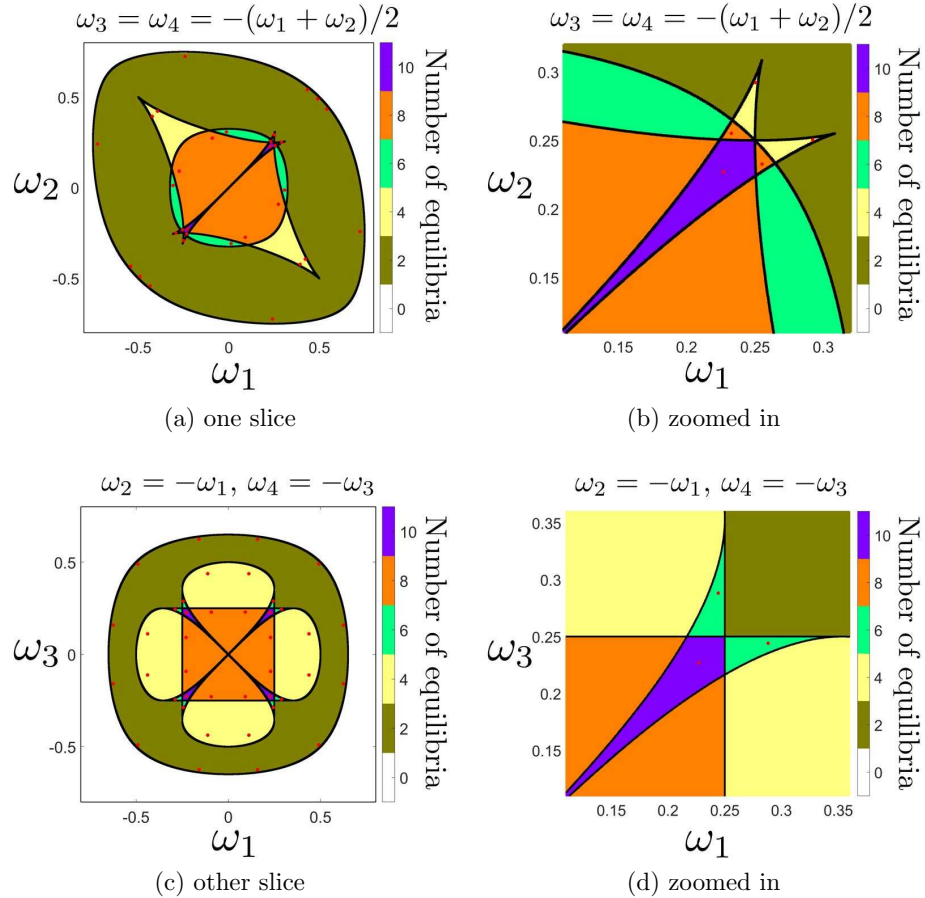

Figure 1: Compact connected regions and critical points for the Kuramoto model with $n = 4$

Algorithm 1 REAL SMOOTH POINT

Input: $n \geq 2$, $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ and $\mathbf{a} \in \mathbb{Q}^s$ satisfying Assumption (A). Let
$V_e^{\mathbf{a}} := V(f_1 - ea_1, \ldots, f_s - ea_s)$ and $V := \lim_{e \to 0^+} V_e^{\mathbf{a}}$.
Output: A finite set $S \subset \mathbb{R}^n$ containing smooth points in each bounded connected component of $V \cap \mathbb{R}^n$ that has dimension $n - s$.
(1) Using isosingular deflation, obtain $\{(g_j, (G_j, L, W_j)) \ : \ j = 1, ..., r\}$ such that g_j is in $\mathbb{R}[x_1, \ldots, x_n]$ and (G_j, L, W_j) is a deflated witness set for some $V_j \subset V$ that is a union of irreducible components of V.
For each $j = 1, \ldots, r$:
 (2) Set up the polynomial system

$$L_e^{(j)} \quad := \quad \left\{ \frac{\partial g_j}{\partial x_i} + \sum_{t=1}^{s} \lambda_t \frac{\partial f_t}{\partial x_i} \ : \ i = 1, \ldots, n \right\} \tag{1}$$
$$\cup \{f_1 - ea_1, f_2 - ea_2, \ldots, f_s - ea_s\}.$$

in the variables $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_s$ and parameter e. For the projection $\pi_x : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C}^n$, compute the finite set $U_j := \lim_{e \to 0} \pi_x(V(L_e^{(j)})) \subset \mathbb{C}^n$ using limits of witness points. Define $T_j := U_j \setminus V(g_j) \cap \mathbb{R}^n$.
 (3) For each $p \in T_j$, use the Membership Test of [3, Sec. 8.4] with input p and (G_j, L, W_j) to find
$S_j := T_j \cap V_j$.
(4) Return $S := \bigcup_{j=1}^{r} S_j$.

$V(g)$ and $\dim(V \cap V(g)) < \dim(V)$. There exist symbolic methods to compute such a g for an irreducible variety V. For example, [11, Lemma 4.3] computes the defining equation w of a generic projection $\overline{\pi(V)}$ that is a hypersurface. Then, g can be taken to be one of the partial derivatives of w. We use a new approach based on isosingular deflation, which computes several g's depending on the isosingular deflation sequence of the irreducible components, as in [7].

3   Application to Real Dimension via Polar Varieties

By [9, Theorem 12.6.1], if we find a real smooth point, we find the real dimension to be the same as the complex one. If there are no real smooth points, we conclude that the real dimension is smaller than the complex dimension. In that case, we need to lower the complex dimension in a way that we do not lose any real points inside the variety. One approach is to replace the variety by its singular set; however, recursively adding minors of the Jacobian matrix for higher codimension varieties can cause a drastic increase in the degree of the polynomials utilized. Here we instead use a sequence of *polar varieties*, following the notation in [10].

Definition 3.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be square-free and $V = V(f) \subset \mathbb{C}^n$. Consider the projections $\pi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i)$ for $i = 1, \ldots, n$. The *polar variety associated to π_i* of V is defined as

$$\mathrm{crit}(V, \pi_i) := V\left(f, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset \mathbb{C}^n \quad i = 1, \ldots, n.$$

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Algorithm 2 NUMERICAL REAL DIMENSION

Input: $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ such that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is compact where $n \geq 2$.
Output: The real dimension of $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$.
(1) Choose a generic $A \in \mathrm{GL}_n(\mathbb{R})$ and define

$$f(\mathbf{x}) := \sum_{i=1}^{s} f_i^A(\mathbf{x})^2 \in \mathbb{R}[x_1, \ldots, x_n].$$

Assume that $\left(f, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n}\right)$ and $\mathbf{a} := \mathbf{e}_1$ satisfy Assumption (**A**). Let $i := n$.

(2) Using the REAL SMOOTH POINT ALGORITHM 1 with input $\left(f, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n}\right)$ and $\mathbf{a} := \mathbf{e}_1$, compute $S \subset \mathbb{R}^n$ that contains smooth points in $V \cap \mathbb{R}^n$, where $V := \lim_{e \to 0} \mathrm{crit}\left(V(f - e), \pi_i\right)$.
(3) If $S \neq \emptyset$ then return $i - 1$.
(4) Set $i := i - 1$. If $i = 0$ then return -1. If $i > 0$ go to Step 2.

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