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What Can You Draw?

Florian Frick and Fei Peng

Abstract. We address the problem of which planar sets can be drawn with a pencil and eraser. The pencil draws any union of black open unit disks in the plane $\mathbb{R}^2$. The eraser produces any union of white open unit disks. You may switch tools as many times as desired. Our main result is that drawability cannot be characterized by local obstructions: a bounded set can be locally drawable, while not being (globally) drawable. We also show that if drawable sets are defined using closed unit disks, then the cardinality of the collection of drawable sets is strictly larger compared with the definition involving open unit disks.

1. INTRODUCTION. The second author raised the following deceptively simple question: What can you draw? Your canvas is the plane $\mathbb{R}^2$—colored white to begin with—and you are given two tools to draw with: a pencil (or brush), which produces a black unit disk wherever it meets the canvas, and an eraser, which produces a white unit disk. There are no further restrictions on your artistic freedom: you may raise the tool off the canvas, that is, there is no continuity requirement for the centers of disks you draw, and you can switch tools as many times as desired.

More precisely, for a set $A \subset \mathbb{R}^2$ denote its open 1-neighborhood, the union of all open unit disks with center in A, by

$$N(A) = \{x \in \mathbb{R}^2 \ : \ |x - a| < 1 \text{ for some } a \in A\}.$$

A subset of the plane that can be drawn without the use of the eraser is of the form $N(A_1) = D_1$ for some $A_1 \subset \mathbb{R}^2$. We can now "erase" the set $N(A_2)$ for some $A_2 \subset \mathbb{R}^2$ to obtain $D_2 = N(A_1) \setminus N(A_2)$. Using the pencil a second time, we can draw any set of the form $D_3 = (N(A_1) \setminus N(A_2)) \cup N(A_3)$, from which we can erase $N(A_4)$ to produce D_4, and so on. We say that we produced the set D_k after k steps. Denote by $\mathcal{D}_1$ the sets we can draw in one step: the collection of sets $N(A_1)$ for $A_1 \subset \mathbb{R}^2$. Similarly, $\mathcal{D}_2 = \{N(A_1) \setminus N(A_2) : A_1, A_2 \subset \mathbb{R}^2\}$. In general,

$$\mathcal{D}_n = \begin{cases} \{D \cup N(A_n) : D \in \mathcal{D}_{n-1}, A_n \subset \mathbb{R}^2\} & (n \text{ is odd}) \\ \{D \setminus N(A_n) : D \in \mathcal{D}_{n-1}, A_n \subset \mathbb{R}^2\} & (n \text{ is even}) \end{cases}.$$

We are interested in the collection of drawable sets $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$. We will refer to any set in $\mathcal{D}$ as *drawable*; see Figure 1. For $A \in \mathcal{D}$, its presence in $\mathcal{D}$ will be *witnessed* by some $A_1, \dots, A_n$ in the above form, namely

$$A = ((((N(A_1) \setminus N(A_2)) \cup N(A_3)) \setminus N(A_4)) \cdots .$$

Figure 1. Four simple examples of drawable sets.

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The choice that our drawing tools produce open unit disks (instead of closed unit disks) is arbitrary and we will investigate a second model of drawable sets, where open 1-neighborhoods are replaced by their nonstrict counterparts

$$N_{\leq}(A) = \{x \in \mathbb{R}^2 \ : \ |x - a| \leq 1 \text{ for some } a \in A\}.$$

We avoid the terminology "closed 1-neighborhood" since $N_{\leq}(A)$ is not necessarily a closed set, for example if A is an open unit disk. Replacing each $N(A_j)$ by $N_{\leq}(A_j)$ in the definition of $\mathcal{D}$, we get the collection of *closed-disk drawable* sets $\mathcal{D}_{\leq}$.

We can make some observations about drawable sets, such as every closed convex set is drawable and any convex set is closed-disk drawable; see Section 2 for the simple proofs. The purpose of the present article is to derive a more surprising phenomenon, namely that being a drawable set is not a local condition. First, we mention that local obstructions to drawability exist:

Theorem 1. *A* 2×2 *chessboard, that is, the set* $[-1, 0] \times [-1, 0] \cup [0, 1] \times [0, 1]$, *is neither drawable nor closed-disk drawable.*

Call a set $B \subset \mathbb{R}^2$ *locally drawable* if every point $x \in \mathbb{R}^2$ has a neighborhood U such that there is a drawable set $D \in \mathcal{D}$ such that $U \cap B$ is equal to $U \cap D$. That is, if we zoom in close to any point in D, the part of the set we see is indistinguishable from a drawable set. Clearly, any drawable set is locally drawable.

The left image of Figure 3 shows a simple example of a set that is locally drawable, but not drawable: Round off the corners of a 2×2 chessboard to separate the two black squares of the chessboard, thus making it locally drawable. If this smoothing is sufficiently sharp, that is, if we round off with a curve of curvature strictly larger than one, then any unit disk touching the curve from the inside of the black region will extend past the curve. We thus need to use the eraser to achieve this curvature, but the eraser will interfere with the other black region. So neither black region can be drawn last. This is a quick outline of a proof that such a chessboard with rounded corners is not drawable. We find this unsatisfactory, as it feels that we won on a technicality: First, we made the boundary of our drawing so sharp that the pencil does not fit into it; second, the obstruction is still somewhat local, that is, the two black regions need to be close enough so that erasing around one region interferes with the other.

Here we rectify both of these shortcomings. We construct an example of a simple closed curve in the plane with curvature less than one everywhere (so that pencil and eraser can locally approximate it from either side), such that the region bounded by it is not drawable; see Theorem 2. And we identify truly global obstructions to drawability; for given $r > 0$ we construct obstructions to drawability that are found in an annulus of inradius r (and depend on the annulus closing up). We need additional language for a precise statement, which we thus postpone to Theorem 15. The general obstruction we exhibit to prove Theorem 15 is the same as the one used to prove the following:

Theorem 2. *There is a Jordan loop* γ *in the plane, with curvature strictly between* -1 *and* 1, *such that the interior region* R *of* γ *is neither drawable nor closed-disk drawable. However,* R *is locally drawable and locally closed-disk drawable.*

A set bounded by a Jordan loop with curvature strictly between -1 and 1 is locally drawable (and locally closed-disk drawable); see Theorem 12. This is because we may approximate the curve from either side with disk of radius at least one, and thus pencil and eraser "fit into" the curve. This is Blaschke's rolling ball theorem [**2**, p. 114] that a unit disk fits into any convex curve with curvature at most one.

Figure 2. An undrawable 2×2 chessboard and a Jordan curve of curvature less than one that bounds a locally drawable, yet undrawable region, the "snake."

It is almost immediate from the definition that any drawable set is a *Borel set* (see Theorem 5), that is, in the σ-algebra generated by open sets in the plane. Recall that a nonempty set system is called a σ-algebra if it is closed under taking complements and under taking countable unions. Theorem 1 shows the existence of Borel sets that are not drawable. A set $A \subset \mathbb{R}^2$ that differs from a Borel set in a subset of a set of Lebesgue measure zero is called *Lebesgue measurable*. The collection of Lebesgue measurable sets forms a σ-algebra, since countable unions of measure-zero sets have measure zero. Here we show:

Theorem 3. *Any closed-disk drawable set is Lebesgue measurable. Not every Lebesgue measurable subset of* $\mathbb{R}^2$ *is closed-disk drawable, but* $\mathcal{D}_{\leq}$ *has the same cardinality as the set of Lebesgue measurable subsets of* $\mathbb{R}^2$*, and larger cardinality than* $|\mathcal{D}|$*.*

The first part will easily follow from a result of [1]. While the two models of what it means to be a drawable set are very similar—using open unit disks versus closed unit disks—the model where drawing tools leave a closed unit disk produces a larger cardinality of drawable sets.

To the authors' knowledge the notion of drawability has not been investigated earlier. There is, however, the related concept of Dynkin system: A nonempty family of subsets of a set X is called *Dynkin system* if it is closed under taking complements and countable disjoint unions. Keleti [5] showed that the Dynkin system generated by open balls of radius at least one in $\mathbb{R}^d$, $d \geq 3$, does not contain all Borel sets. Keleti and Preiss [6] showed that the Dynkin system generated by all open balls in a separable infinite-dimensional Hilbert space does not contain all Borel sets. Finally, Zelený [8] showed that the Dynkin system generated by balls in $\mathbb{R}^d$ contains all Borel sets.

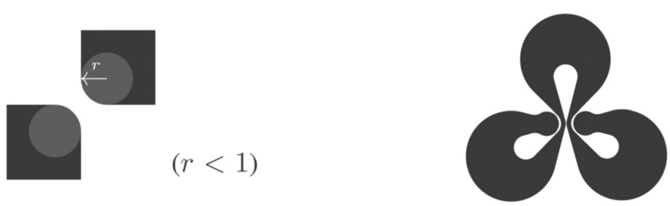

Figure 3. Some locally drawable but undrawable sets. Their nondrawability follows from the general obstruction given in Theorem 15. The construction of the second set—"octopus"—is similar to the snake in Figure 2.

Throughout this article our reasoning will be elementary and geometric. We will focus on geometric insights and encourage the reader to verify any details that may seem nonobvious. Section 5 presents those details.

2. PROPERTIES OF DRAWABLE SETS. In this section we collect some simple properties of drawable and closed-disk drawable sets, and prove Theorem 3. Recall that a set $A \subset \mathbb{R}^2$ is *convex* if for any two $x, y \in A$ the line segment connecting x and y is entirely within A. For two vectors $x, y \in \mathbb{R}^2$ we denote their inner product $x_1 y_1 + x_2 y_2$ by $\langle x, y \rangle$.

Theorem 4. *Any closed convex set in $\mathbb{R}^2$ is drawable.*

Proof. Any open halfplane, that is, any set of the form $H = \{x \in \mathbb{R}^2 \ : \ \langle x, y \rangle > a\}$ for some $y \in \mathbb{R}^2$ of norm 1 and $a \in \mathbb{R}$, is a union of open unit disks. Namely, H is the set $N(A)$, where A is the set of $z + \lambda y$ with $\langle z, y \rangle = a$ and $\lambda \geq 1$; that is, A is the set of points in H at distance at least one to the line $\{x \in \mathbb{R}^2 \ : \ \langle x, y \rangle = a\}$. In a first step we can color the plane black. In a second step we can erase any union of open halfplanes. This means that any intersection of closed halfplanes is drawable. This is precisely the collection of closed convex sets. ■

The condition that the convex set be closed in order to be drawable is indeed needed. In fact, most convex sets are not drawable. We will show this now.

Theorem 5. *Every drawable set is a Borel set.*

Proof. Any set of the form $N(A)$ for $A \subset \mathbb{R}^2$ is open as a union of open disks, and thus every set in $\mathcal{D}_1$ is a Borel set. The claim that every element of $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$ is Borel as well now follows by a simple induction, since sets in $\mathcal{D}_n$ are obtained from sets in $\mathcal{D}_{n-1}$ either by taking complements with open sets or by taking the union with an open set. ■

Corollary 6. *The cardinality of the collection of drawable sets $|\mathcal{D}|$ is strictly less than the cardinality of the collection of convex sets in the plane. In particular, most convex sets are not drawable.*

Proof. There are at most as many drawable sets as there are Borel sets by Theorem 5. The cardinality of the set of Borel sets is $2^{\aleph_0}$, the cardinality of real numbers; see [**7**, Theorem 3.3.18]. However, the set of convex sets in the plane has the same size as the power set of the reals, which is strictly larger than $2^{\aleph_0}$. To see this, observe that any set that fits between the open unit disk centered at the origin and the closed unit disk centered at the origin is convex. That is, let U be any subset of the unit circle S^1. Then $\{x \in \mathbb{R}^2 \ : \ |x| < 1\} \cup U$ is convex. There are as many subsets of S^1 as subsets of the reals. ■

Theorem 7. *Any convex set in $\mathbb{R}^2$ is closed-disk drawable.*

Proof. We begin by showing that any closed convex set is closed-disk drawable. The proof is essentially the same as for Theorem 4, with the difference that now, given some $H = \{x \in \mathbb{R}^2 \ : \ \langle x, y \rangle > a\}$ for $y \in \mathbb{R}^2$ of norm 1 and $a \in \mathbb{R}$, we have to represent it as $N_{\geq}(A)$ for some $A \subset \mathbb{R}^2$, that is, as a union of closed unit disks. The set H is simply the union of closed unit disks centered at $z + \lambda y$ with $\langle z, y \rangle = a$ and $\lambda > 1$; that is, A is the set of points in H at distance strictly greater than one from the line $\{x \in \mathbb{R}^2 \ : \ \langle x, y \rangle = a\}$.

Now given some convex set $C \subset \mathbb{R}^2$, first realize its closure $\overline{C}$ as a closed-disk drawable set. We then have to delete certain boundary points of $\overline{C}$, namely all points in $\overline{C} \setminus C$. The points in $\overline{C} \setminus C$ are contained in the union of closed unit disks that stay entirely within the complement of C. Indeed, for any point $x_0 \in \overline{C} \setminus C$ consider a supporting line ℓ, that is, a line that is disjoint from the interior of C and contains x_0.

If ℓ is defined by the equation $\langle x, y\rangle = a$ for $y \in \mathbb{R}^2$ of norm one and $a \in \mathbb{R}$, then the closed unit disk centered at $x_0 + y$ contains x_0 and is entirely contained within the complement of C. ■

Proof of Theorem 3. Any (not necessarily countable) union of closed unit disks is Lebesgue measurable [**1**, Theorem 1.1]. Since Lebesgue measurable sets form a σ-algebra, this implies that any closed-disk drawable set is Lebesgue measurable. The cardinality of the set of Lebesgue measurable sets is the same as the cardinality of the power set of $\mathbb{R}$, which is equal to the cardinality of convex sets in $\mathbb{R}^2$ by the proof of Corollary 6. All of these sets are closed-disk drawable, showing that there are as many closed-disk drawable sets as Lebesgue measurable sets. Since each drawable set is a Borel set by Theorem 5 and the set of Borel sets has the cardinality $2^{\aleph_0}$ of the reals, we have that $|\mathcal{D}| < |\mathcal{D}_{\geq}|$.

It remains to exhibit an example of a Lebesgue measurable subset of $\mathbb{R}^2$ that is not closed-disk drawable. Observe that for any closed-disk drawable set $A \in \mathcal{D}_{\geq}$ there is a closed unit disk in A or a closed unit disk in the complement of A. This is because every set is finalized in finitely many steps and in the last step we either drew a black unit disk in A or erased a white unit disk. A sufficiently fine checkerboard pattern is an example of a Lebesgue measurable subset A of $\mathbb{R}^2$ such that neither A nor its complement contains a (closed) unit disk. (For a less trivial, bounded example of a Lebesgue measurable set that is not closed-disk drawable—namely a 2×2 chessboard already suffices—see Theorem 1, proved in the next section.) ■

3. NONDRAWABILITY OF THE 2 × 2 CHESSBOARD. In this section, we will show that the 2×2 chessboard is not drawable. In fact, Lemma 8, suggested in this form by an anonymous referee, provides a general way to construct nondrawable sets. It states that if for a set $S \subset \mathbb{R}^2$ we can find a set of black points $X \subset S$ and a set of white points $Y \subset \mathbb{R}^2 \setminus S$ such that every point at distance less than one from a black point is also at distance less than one from a white point and vice versa, then S cannot be drawable. Indeed, in this case, any time that we draw a black point in X we also draw a point in Y, which has to be erased at a later stage, but when erasing a point in Y, we also erase points in X, and so on. Thus we can never simultaneously finalize X and Y. We thus have the following (more details are presented in Section 5):

Lemma 8. *If nonempty sets $X, Y, S \subset \mathbb{R}^2$ satisfy that $N(X) = N(Y)$, $X \subset S$ and $Y \cap S = \emptyset$, then S is not drawable.*

Remark 9. Similarly, a set $S \subset \mathbb{R}^2$ for which there are nonempty $X, Y \subset \mathbb{R}^2$ with $N_{\leq}(X) = N_{\leq}(Y)$, $X \subset S$, and $Y \cap S = \emptyset$ is not closed-disk drawable. The same proof as above, with the obvious changes, works.

We will now prove Theorem 1, which asserts the nondrawability of the 2×2 chessboard. We will need the following simple geometric fact.

Fact 10. If two unit circles $C_1, C_2 \subset \mathbb{R}^2$ intersect in two points v and w, then the shorter arc connecting v to w along C_1 is contained in the disk bounded by C_2, and the longer arc connecting v to w along C_1 is outside of the disk bounded by C_2.

Proof of Theorem 1. Denote the 2×2 chessboard by

$$S = ([-1, 0] \times [-1, 0]) \cup ([0, 1] \times [0, 1]),$$

and denote its interior by

$$\mathrm{int}(S) = ((-1, 0) \times (-1, 0)) \cup ((0, 1) \times (0, 1)).$$

In view of Lemma 8, it suffices to construct nonempty $X, Y \subset \mathbb{R}^2$ such that $N(X) = N(Y)$, $N_{\leq}(X) = N_{\leq}(Y)$, $X \subset S$, and $Y \cap S = \emptyset$.

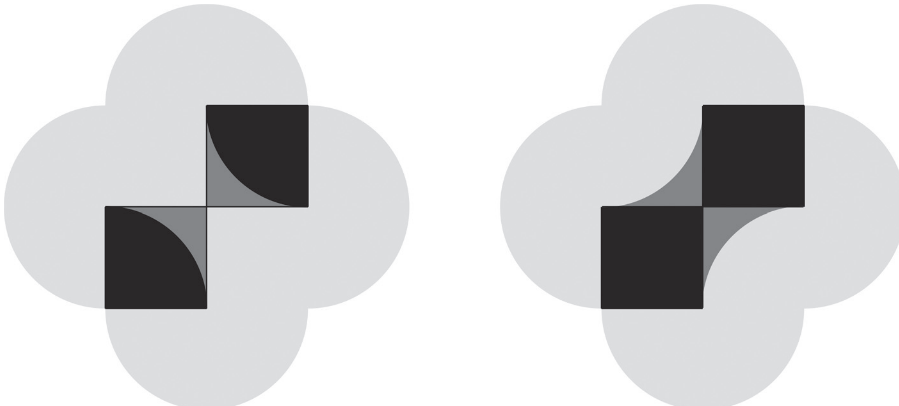

Figure 4. The set X is the dark gray region on the left. The set Y is the dark gray region on the right. Their common 1-neighborhood $N(X) = N(Y)$ is shown in light gray. A unit disk that intersects X must also intersect Y and vice versa.

The set X is $\operatorname{int}(S) \setminus N(\{(1, 1), (-1, -1)\})$, that is, we remove disks of radius one around $(1, 1)$ and $(-1, -1)$ from the interior of the 2×2 chessboard. The set Y is the reflection of the set X over the y-axis. Clearly, $X \subset S$ and $Y \subset \mathbb{R}^2 \setminus S$. This is illustrated in Figure 4.

Let u be in $N(X)$. Assume for contradiction that $u \notin N(Y)$. That is, the unit disk $N(u)$ with center u intersects X but not Y. Then since $N(u)$ is too large to be entirely contained within X, it has to intersect the boundary of X in multiple points. Since $N(u)$ is disjoint from Y it intersects $N(1, 1)$ or $N(-1, -1)$. By symmetry we can assume it intersects $N(1, 1)$. Thus the unit circle C_u bounding $N(u)$ is partially in X and intersects the bounding circle of $N(1, 1)$ in exactly two points v and w. By Fact 10, the longer arc α connecting v and w on C_u is outside of $N(1, 1)$. Since α is outside of $N(1, 1)$ and cannot cross the coordinate axes into Y, it lies entirely in the closure of X. This is clearly absurd: α as a longer arc of a unit circle connects two points at distance two. While the closure of X has two pairs of points at distance two, they are not connected by semi-circles within X. This shows $N(X) \subset N(Y)$.

The inclusion $N(Y) \subset N(X)$ follows from the symmetry of the configuration, reversing the roles of X and Y. Thus $N(X) = N(Y)$ and S is not drawable by Lemma 8. That S is not closed-disk drawable follows in the same way, now using Remark 9 instead of Lemma 8. ∎

4. UNDRAWABLE SETS WITH SMALL CURVATURE AND GLOBAL OBSTRUCTIONS TO DRAWABILITY.

Here we show that if a region is bounded by a curve of small curvature, then it is locally drawable. We construct the *snake*, a region whose boundary has small curvature, but that is not drawable. The obstruction to drawability we exhibit can be phrased in general terms, and this obstruction is "global" instead of "local"; see Theorem 15.

First we recall some basic notions of the differential geometry of planar curves. We refer to do Carmo's book [3] for details. Let γ be a simple smooth closed curve in the plane, parameterized by arclength, that is, $|\gamma'(s)| = 1$ for all s. Here smooth means that γ has well-defined derivatives of all orders. Let $x_0 = \gamma(s_0)$ be a point on the trace of γ. The curve γ has a well-defined tangent line at x_0. Rotate that tangent line by 90° in the positive (i.e., counterclockwise) direction to obtain the unit normal $n(s)$ of $\gamma(s)$. Then since $\gamma'(s)$ is a unit vector, its derivative $\gamma''(s)$ is orthogonal to the tangent $\gamma'(s)$ for every s. Thus $\gamma''(s) = k(s)n(s)$ for some function $k(s)$, called the (signed) curvature of γ. The (unsigned) curvature is $\kappa(s) = |k(s)|$.

The following lemma may be seen as a special case of Blaschke's classical rolling ball theorem [2, p. 114], which states that if two smooth regular (positively oriented) convex curves γ_1 and γ_2 touch at one point x, where they have the same tangent vector, and the curvature of γ_1 is at least equal to the curvature of γ_2, then γ_1 is contained entirely within the region bounded by γ_2. Moreover, if the curvature of γ_1 is strictly less than the curvature of γ_2, then outside of the point x, the curve γ_1 is contained in the interior of the region bounded by γ_2.

We make no assumption on the convexity of curves, but locally every smooth regular curve is convex. Blaschke's theorem shows that we may choose $\varepsilon = 1$ in Lemma 11. Since we do not need a sharp estimate on ε, the lemma follows easily by Taylor expansion. We include the simple argument for the reader's convenience in Section 5.

Lemma 11. *Let $\gamma : I \to \mathbb{R}^2$ be a smooth curve parametrized by arclength, defined on some compact interval I, and let $s_0 \in I$. Assume $\kappa(s) < 1$ for all $s \in I$. Then there are two circles C_1 and C_2 of radius one with centers $\gamma(s_0) \pm n(s_0)$, which touch γ at $\gamma(s_0)$ but do not contain $\gamma(s)$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ for some $\varepsilon > 0$. Moreover, this ε can be chosen independent of s_0.*

In some sense, our notion of drawability may be seen as a sequential version of Blaschke's rolling ball theorem. We can now show that regions bounded by curves of small curvature are locally drawable.

Theorem 12. *Let $\gamma : I \to \mathbb{R}^2$ be a simple, smooth, closed curve, that is, γ is a smooth embedding of a circle into the plane. Assume $|k(s)| < 1$ for all $s \in I$. Then the closed region bounded by γ is locally drawable and locally closed-disk drawable.*

Proof. Denote the closed region bounded by γ by R. Suppose γ is positively oriented, so that $\gamma(s) + \lambda n(s)$ is in R for all $\lambda \in [0, \delta)$ for some sufficiently small $\delta > 0$ and every s. Around any point x in the interior of R, the set R is easily seen to be locally drawable; after all, there is a small open set containing x that is entirely contained in R. By the same reasoning, R is locally drawable around any $x \notin R$.

For x on the boundary of R, say $x = \gamma(s_0)$, choose $\varepsilon > 0$ according to Lemma 11 (and independent of s_0). By perhaps decreasing ε such that $\varepsilon < \delta$, the ε-disk around x intersects γ only in points of the form $\gamma(s)$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. Now the unit disks centered at $\gamma(s) + n(s)$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ witness the local closed-disk drawability of R around x by Lemma 11. To see the local (open-disk) drawability, we erase the unit disks centered at $\gamma(s) - n(s)$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. ∎

We will now derive global obstructions to drawability. The following presentation and the proof of Theorem 15 have benefited from several insightful suggestions from an anonymous referee.

Definition 13. Let $S \subset \mathbb{R}^2$ be a set, $\ell \subset \mathbb{R}^2$ a ray emanating from a point O, and P and Q two points on ℓ at distance a and b from O, respectively. Suppose that P is closer to O than Q, that is, $a < b$. Let d be a positive real number. Consider the two rectangles (on either side of ℓ) with base the segment PQ, where the other side length is d. If the interior of one of these rectangles is contained in S, while the interior of

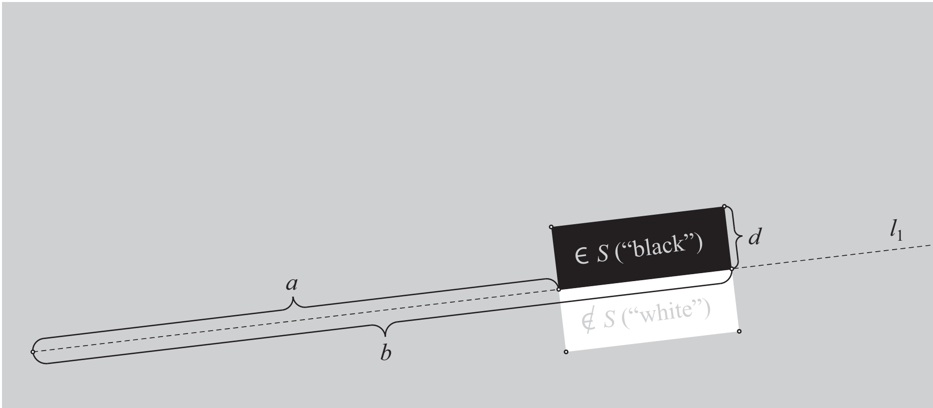

Figure 5. Here S is dissected by ℓ at (a, b) with thickness d.

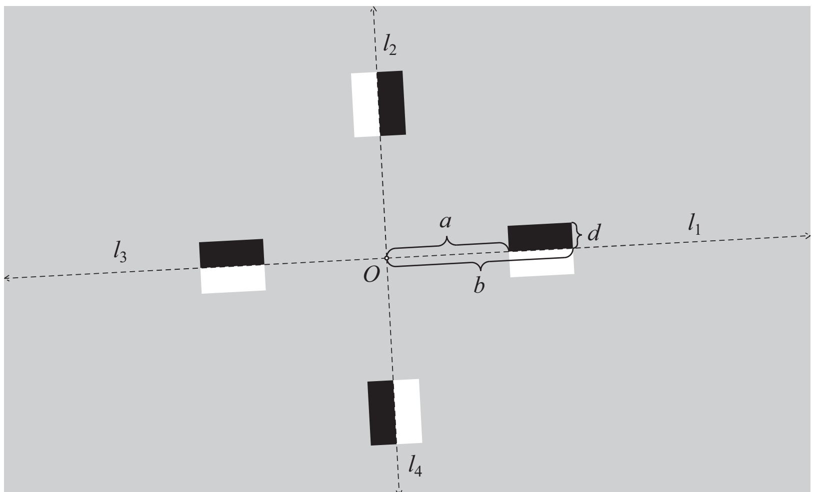

Figure 6. Here S is totally 4-dissected at (a, b) with thickness d.

the other rectangle does not intersect S, we say that S is *dissected by ℓ at the interval (a, b) with thickness d*. An example is illustrated in Figure 5.

Definition 14. For a positive even integer n, a set $S \subset \mathbb{R}^2$ is *totally n-dissected at interval (a, b) with thickness d* if there are rays $\ell_1, \ell_2, \ldots, \ell_n$ emanating from the same point O that divide the plane evenly (i.e., into equal angles), such that S is dissected by ℓ_i at (a, b) with thickness d for every i. See Figure 6 for an example.

Theorem 15. *If $S \subset \mathbb{R}^2$ is totally n-dissected at (a, b) with thickness $d > 0$, then if $a < \cot(\pi/n)$, the set S is not drawable and not closed-disk drawable.*

The bound in Theorem 15 is sharp. This is illustrated in Figure 7.

We will prove Theorem 15 by exhibiting two sets $X \subset S$ and $Y \subset \mathbb{R}^2 \setminus S$ with $N(X) = N(Y)$ and using Lemma 8. Both sets X and Y will consist of a collection of circular arcs that connect the points at distance a and b along each dissecting ray ℓ_i. We may assume that b is sufficiently close to a that these open circular arcs stay entirely

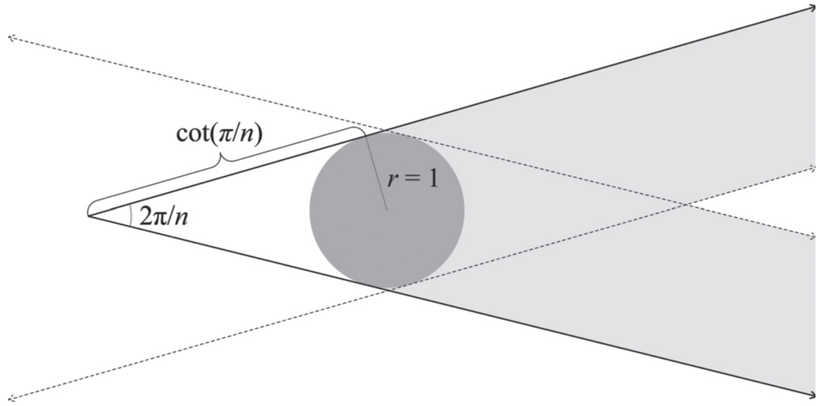

Figure 7. When $a \geq \cot(\pi/n)$, the set could be drawable.

within the rectangles indicated in Figure 6, so that they are indeed contained in S and disjoint from S, respectively. To show that the same points are at distance less than one from X and from Y, we need another simple fact about the geometry of planar unit circles, Lemma 16.

For small $\delta > 0$ denote by $C_-(\delta)$ the unit circle in $\mathbb{R}^2$ with center

$$\left(\delta/2, -\sqrt{1-\frac{\delta^2}{4}}\right),$$

that is, the unit circle that passes through $(0, 0)$ and $(\delta, 0)$ that is mostly below the x-axis. Similarly, the other unit circle passing through $(0, 0)$ and $(\delta, 0)$, which has its center at

$$\left(\delta/2, \sqrt{1-\frac{\delta^2}{4}}\right),$$

will be denoted by $C_+(\delta)$. Let $\alpha_-(\delta)$ denote the shorter arc on $C_-(\delta)$ that connects $(0, 0)$ to $(\delta, 0)$, and define $\alpha_+(\delta)$ in the same way for $C_+(\delta)$.

The circles $C_-(\delta)$ and $C_+(\delta)$ are symmetric over the x-axis. If we reflect $C_-(\delta)$ in a line close to the x-axis, then the center of the reflected circle is still close to $(0, 1)$ by continuity. In particular, the unit circle distinct from $C_-(\delta)$ determined by two points on $\alpha_-(\delta)$ has its center close to $(0, 1)$. Here we give a precise statement (details can be found in Section 5):

Lemma 16. *For any $\varepsilon > 0$ there is a $\delta > 0$ such that if u_1 and u_2 both lie on $\alpha_+(\delta)$ or both lie on $\alpha_-(\delta)$ then the center c of a unit circle in the plane that passes through u_1 and u_2 satisfies $|(0, 1) - c| < \varepsilon$ or $|(0, -1) - c| < \varepsilon$. In the case $u_1 = u_2$ we require that the unit circle passing through $u_1 = u_2$ is tangent to $\alpha_+(\delta)$ or $\alpha_-(\delta)$.*

Proof of Theorem 15. By Lemma 8 we have to exhibit sets $X \subset S$ and $Y \subset \mathbb{R}^2 \setminus S$ such that $N(X) = N(Y)$. Since S is totally n-dissected, there are n rays $\ell_1, \ldots, \ell_n$ emanating from O that at distance a to distance b from O separate S from its complement. We may assume that $b - a$ is sufficiently small. Denote the point at distance a from O on ℓ_i by P_i, and denote the point at distance b from O on ℓ_i by Q_i. On either side of the segment $P_i Q_i$ are rectangles of height d, one whose interior is entirely contained in S, and one whose interior is disjoint from S.

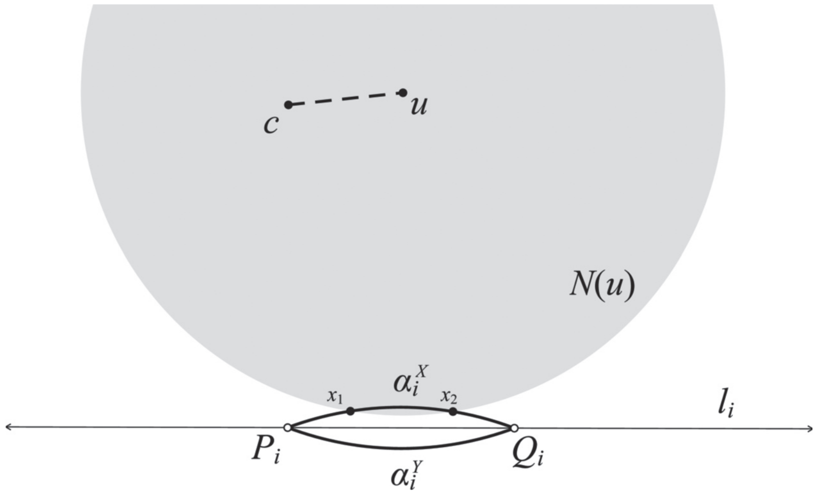

Figure 8. If P_i, Q_i are outside of $N(u)$, then $|c - u| < \varepsilon$, implying that some other P_j is inside $N(u)$.

We first claim that both unit circles that are tangent to ℓ_i at P_i contain one of the points $P_1, \ldots, P_n$ in their interior. To prove this claim, let C be a unit circle that passes through the point P_i and is tangent to the line ℓ_i. Then C is also tangent to a line ℓ emanating from O that makes an angle of $2\mathrm{arccot}(a)$ with ℓ_i. Thus the arc α along the circle of radius a with center O connects the two points of tangency and lies in the interior of C. Since $a < \cot\left(\frac{\pi}{n}\right)$, the arc α intersects one of the lines ℓ_j at the point P_j, which thus is inside of C.

Observe that this claim implies that any unit circle C' whose center c' is sufficiently close to the center c of C also contains at least one of the points $P_1, \ldots, P_n$ in its interior. Say $\varepsilon > 0$ is chosen such that whenever $|c - c'| < \varepsilon$ then some point $P_1, \ldots, P_n$ is in the disk bounded by C'. For this ε, let $\delta > 0$ be chosen according to Lemma 16.

We may assume that $|Q_i - P_i| = b - a < \delta$. We may also possibly decrease δ such that the shorter arc on a unit circle connecting P_i to Q_i stays within distance d of the straight-line segment P_iQ_i. In particular, each such circular arc minus its endpoints is either entirely in S or entirely in $\mathbb{R}^2 \setminus S$. For each $i \in \{1, 2, \ldots, n\}$, let α_i^X be the open circular arc on a unit circle that connects Q_i to P_i and is entirely contained in S. Similarly, let α_i^Y be the open circular arc on a unit circle that connects Q_i to P_i and is entirely contained in $\mathbb{R}^2 \setminus S$. Now let X be the union of all α_i^X, and let Y be the union of all α_i^Y.

Let $u \in N(X)$, that is, the open unit disk $N(u)$ with center u intersects some arc α_i^X. We want to show that some point P_j or Q_j is in $N(u)$. If the endpoints P_i and Q_i of α_i^X are outside of $N(u)$, then the unit circle C_u bounding $N(u)$ intersects the arc with its endpoints $\alpha_i^X \cup \{P_i, Q_i\}$ in two points x_1 and x_2; see Figure 8. By translating the configuration such that P_i becomes the origin and the line ℓ_i becomes the x-axis, we are in the situation of Lemma 16. In this translated and rotated picture a unit circle tangent to ℓ_i at P_i has its center at $(0, 1)$ or $(0, -1)$. Thus, by our claim above and Lemma 16, we see that some P_j is in $N(u)$. Thus $u \in N(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$.

For the reverse inclusion, let $u \in N(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$. Then $|u - P_i| < 1$ or $|u - Q_i| < 1$ for some i. This immediately implies that there is some point x on the open arc α_i^X connecting P_i and Q_i with $|u - x| < 1$. Thus $u \in N(X)$.

This implies $N(X) = N(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$. The set Y was defined by reflecting each arc $\alpha_i^X \subset X$ over the line ℓ_i, so in the same way we see that $N(Y) =$

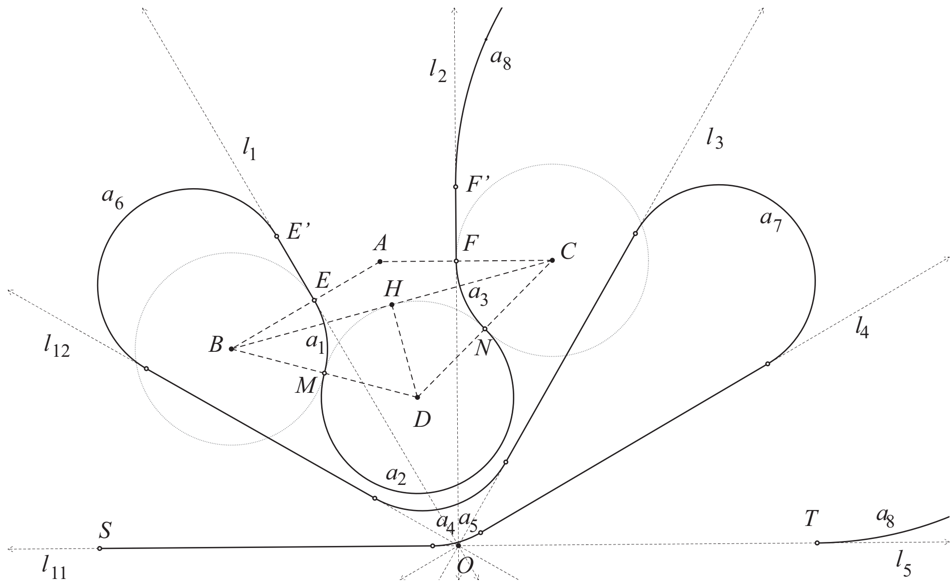

Figure 9. Construction of the *snake*.

$N(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$. We have thus found sets $X \subset S$ and $Y \subset \mathbb{R}^2 \setminus S$ with $N(X) = N(Y)$, and hence S is not drawable by Lemma 8.

To see that S is also not closed-disk drawable, we can verify that the reasoning above actually implies $N_{\leq}(X) = N_{\leq}(Y)$. ■

We now construct the "snake" in Figure 2 that is enclosed by a Jordan curve of curvature less than one, but is undrawable. The *snake* is undrawable by Theorem 15, since we will construct it to be totally 12-dissected.

We will choose $\kappa_0 = 1/1.001 < 1$ as the maximum curvature in the boundary, that is, the smallest osculating circle will have radius $r := 1.001$. We start by constructing a kite $ABDC$, symmetric about the line segment AD, such that $\angle ABC = 15°$, $\angle CBD = 30°$ (thus $\angle ABD = 45°$), and the line segment BD has length $2r$. Next construct three circles with radius r, centered at B, D, and C, respectively. The circle centered at B is tangent to the circle centered at D in point M. Similarly, the circle centered at C is tangent to the circle centered at D in point N. Denote the intersection of AB and the circle centered at B by E, and denote the intersection of AC and the circle centered at C by F. Let a_1 be the shorter arc from E to M along the circle centered at B, a_2 the longer arc from M to N along the circle centered at D, and a_3 the shorter arc from N to F along the circle centered at C.

Construct the point O such that $OE \perp AB$ and $OF \perp AC$. Then $\angle EOF = 30°$. Extend OE and OF as rays ℓ_1, ℓ_2, and construct $\ell_3, \ldots, \ell_{12}$ (all starting at O) so that they together divide the space evenly into twelve parts in clockwise order. Let a_4 be the minor r-arc (i.e., the circular arc with radius r) that is tangent to ℓ_{12} and ℓ_3, and let a_5 be the minor r-arc tangent to ℓ_{11} and ℓ_4. Let a_6, a_7 be the major r-arcs tangent to ℓ_{12} and ℓ_1, and ℓ_3 and ℓ_4, respectively.

Let E' be the tangent point of a_6 closer to A. Let F', S, and T be on ℓ_2, ℓ_{11}, and ℓ_5 such that $|OE'| = |OF'| = |OS| = |OT|$. Let a_8 be the major arc that is tangent to ℓ_2 at F' and tangent to ℓ_5.

As in Figure 9, a curved path connects $S, a_5, a_7, a_4, a_6, a_1, a_2, a_3, a_8$ and T by line segments. Rotate this path by 180° around O, and they together form a simple, closed curve. The *snake* is defined as the region enclosed by this curve.

Remark 17. The boundary of the *snake* is smooth everywhere except at the junctions between arcs and line segments (or arcs and arcs). At points where the curve is smooth, its curvature is always at most κ_0. From the construction of the *snake* it is easily seen that it satisfies Lemma 11, that is, for every point x on the curve there are two unit disks on either side of the curve that intersect it in x, and are otherwise disjoint from the curve in a neighborhood of x. Thus the *snake* is locally drawable (and locally closed-disk drawable) with the same proof as in Theorem 12.

If we wanted a smooth version of the *snake* with the same curvature bounds, we have to fuse the individual segments more carefully, continuously changing the curvature from zero along straight line segments to κ_0 along circle segments. This is precisely the "track transition problem" (or "spiral easement") encountered by railroad and highway engineers. See [**4**] for details.

Next, we show that the *snake* is neither drawable nor closed-disk drawable.

Proof of Theorem 2. By the construction of the *snake* (recall that $r := 1.001$),

$$\begin{aligned}
|BC| &= 2r\tan 30^\circ \;=\; 2\sqrt{3}r,\\
|AE| &= \frac{|BC|}{2\cos 15^\circ} - |BE| \;=\; \frac{4\sqrt{3}-\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}r \approx 0.793\dots,\\
|OE| &= |AE|\cot 15^\circ \;=\; \frac{4\sqrt{3}-\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}r \approx 2.963\dots,\\
|OE'| &= r\cot 15^\circ \;=\; \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}r \approx 3.735\dots.
\end{aligned}$$

We note that the *snake* is totally 12-dissected at (2.964, 3.735) with thickness 0.793. In particular, $\cot(\pi/12) = 2+\sqrt{3} \approx 3.732\dots$, so the *snake* is neither drawable nor closed-disk drawable by Theorem 15. On the other hand, the *snake* is locally drawable (and locally closed-disk drawable) by Remark 17. ■

Remark 18. Theorem 15 can also be used to prove Theorem 1: the 2×2 chessboard, $[-1, 0] \times [-1, 0] \cup [0, 1] \times [0, 1]$, is totally 4-dissected at $(0, 1)$ with thickness 1. As $0 < \cot(\pi/4)$, it is not drawable.

5. ADDITIONAL DETAILS FOR SOME PROOFS.

Proof of Lemma 8. Suppose S is drawable, and this drawability is witnessed by $A_1, \dots, A_n \subset \mathbb{R}^2$. Call the set produced after k steps S_k, that is, $S_1 = N(A_1)$, $S_2 = N(A_1) \setminus N(A_2)$, and so on. Let k be the last step, where either the pencil or the eraser touches the set $X \cup Y$, that is, $N(A_k) \cap (X \cup Y) \neq \emptyset$ and $N(A_\ell) \cap (X \cup Y) = \emptyset$ for $k < \ell \leq n$. Because X is nonempty, some step draws some point in X, so k is well-defined and positive.

Since in the end $X \subset S$ and $Y \cap S = \emptyset$, the same is already true at step k: $X \subset S_k$ and $Y \cap S_k = \emptyset$, because after the kth step, we do not erase points that are in X nor do we draw points that are in Y.

The condition $N(X) = N(Y)$ means that the pencil or eraser touches the set X if and only if it touches the set Y. The sets of points at distance less than one from either X or Y are the same. Since $N(A_k) \cap (X \cup Y) \neq \emptyset$ means some point in A_k is at distance less than one from X or Y, it is at distance less than one from X and Y. Thus, $N(A_k) \cap X \neq \emptyset$ and $N(A_k) \cap Y \neq \emptyset$.

Depending on the parity of k, we either drew $N(A_k)$, that is, $S_k = S_{k-1} \cup N(A_k)$, or we erased $N(A_k)$, that is, $S_k = S_{k-1} \setminus N(A_k)$. In the first case $S_k \cap Y \supset N(A_k) \cap Y \neq \emptyset$; in the second case any $x \in N(A_k) \cap X$ is not in S_k. Both result in a contradiction. ■

Proof of Lemma 11. To simplify the notation we translate I so that $s_0 = 0$. By applying an appropriate rigid motion we may assume that $\gamma(0) = (0, 1)$ and $\gamma'(0) = \pm(1, 0)$. By perhaps reversing orientation we may additionally assume that $\gamma'(0) = (1, 0)$ and thus $n(0) = (0, 1)$. There is a $\delta > 0$ such that the trace of γ restricted to $s \in (-\delta, \delta)$ is the graph of a smooth function, say, $(s, f(s))$ is on the trace of γ for $s \in (-\delta, \delta)$. We note that by the inverse function theorem $\gamma|_{[-\delta,\delta]}$ is the graph of a smooth function as long as the derivative of the first coordinate γ_x' is nonzero everywhere. Since $\gamma_x'(0) = 1$ and $|\gamma_x''(s)| \leq |\gamma''(s)| = \kappa(s) < 1$, we may choose $\delta > 0$ independently of s_0. We chose the coordinate system in such a way that $f(0) = 1$ and $f'(0) = 0$.

The signed curvature of γ at $(s, f(s))$ is

$$k(s) = \frac{f''(s)}{(1 + f'(s)^2)^{3/2}}.$$

Thus $f''(s) = k(s) \cdot (1 + f'(s)^2)^{3/2}$, which is approximately $k(s)$ for small s. By expanding f in a Taylor series, we see that $f(s) = f(0) + f'(0)s + \frac{1}{2} f''(\xi)s^2 = 1 + \frac{1}{2} f''(\xi)s^2$ for some ξ between 0 and s.

The relevant part of the circle of radius one with center $\gamma(0) - n(0) = (0, 0)$ is the trace of the curve $C_1(s) = (s, \sqrt{1 - s^2})$. Similarly, for the circle of radius one with center $\gamma(0) + n(0) = (0, 2)$, we consider the curve $C_2(s) = (s, 2 - \sqrt{1 - s^2})$. We need to show that for small s we have $C_1(s) \leq f(s) \leq C_2(s)$. Equivalently, for small s we need to show that

$$\sqrt{1 - s^2} \leq 1 + \frac{f''(\xi)s^2}{2} \leq 2 - \sqrt{1 - s^2}.$$

This holds with equality for $s = 0$, so we may assume $s \neq 0$ from now on. These two inequalities can equivalently be expressed as

$$\sqrt{1 - s^2} \leq 1 \pm \frac{f''(\xi)s^2}{2}.$$

Squaring this and collecting all terms on the right we have to show that

$$0 \leq (1 \pm f''(\xi))s^2 + \frac{1}{4} f''(\xi)^2 s^4.$$

This is equivalent to $0 \leq 1 \pm f''(\xi) + \frac{1}{4} f''(\xi)^2 s^2$, which is evidently true for s close to 0 since $|k(s)| = \kappa(s) < 1$ and $f''(s) = k(s) \cdot (1 + f'(s)^2)^{3/2} \approx k(s)$. Moreover, $0 \leq 1 \pm f''(\xi) + \frac{1}{4} f''(\xi)^2 s^2$ is a strict inequality for small but nonzero s. We note that since the maximum unsigned curvature in the curve is less than 1 (by compactness), the threshold can be chosen independent of s_0. ■

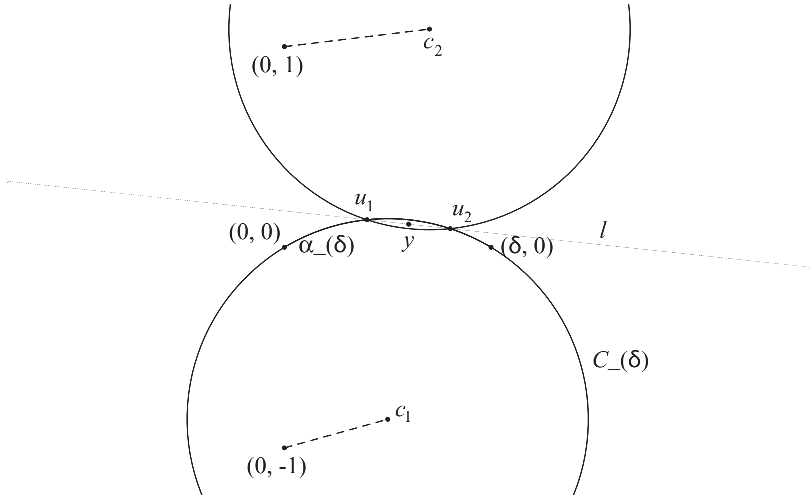

Figure 10. Two unit circles passing through u_1 and u_2. Note that $|y| \le \max(|u_1|, |u_2|) \le \delta$.

Proof of Lemma 16. Let $\delta > 0$ be sufficiently small. We will derive a precise estimate later. Suppose that u_1 and u_2 lie on $\alpha_-(\delta)$. The case that u_1 and u_2 are on $\alpha_+(\delta)$ follows by reflecting over the x-axis.

There are two unit circles that pass through u_1 and u_2: the circle $C_-(\delta)$ and the unit circle obtained from $C_-(\delta)$ by reflecting it in the line ℓ through u_1 and u_2; see Figure 10. If $u_1 = u_2$ we require ℓ to be tangent to $C_-(\delta)$. Note that when δ is small, the center of $C_-(\delta)$, which is at

$$\left(\delta/2, -\sqrt{1 - \frac{\delta^2}{4}}\right),$$

can be arbitrarily close to $(0, -1)$. It is also easily verified that for small δ the line ℓ is close to the x-axis, and thus the reflection of the center of $C_-(\delta)$ over ℓ is as close to $(0, 1)$ as desired.

More precisely, let c_1 be the center of $C_-(\delta)$, and c_2 denote the center of the unit circle obtained from $C_-(\delta)$ by reflecting it in ℓ. Make δ sufficiently small such that

$$2\delta + |(0, -1) - c_1| < \varepsilon.$$

Note that $c \in \{c_1, c_2\}$, and trivially $|(0, -1) - c_1| < \varepsilon$. As u_1, u_2 are equidistant from c_1, the center c_2 is obtained from c_1 by reflecting in the midpoint $y = \frac{1}{2}(u_1 + u_2)$ of u_1 and u_2. Thus $c_2 = 2y - c_1$, and so

$$\begin{aligned} |c_2 - (0, 1)| = |(2y - c_1) - (0, 1)| &\le 2|y| + |(0, -1) - c_1| \\ &\le 2\delta + |(0, -1) - c_1| < \varepsilon. \end{aligned}$$

Hence we must have $|(0, 1) - c| < \varepsilon$ or $|(0, -1) - c| < \varepsilon$. ■

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100 Years Ago This Month in *The American Mathematical Monthly*
Edited by Vadim Ponomarenko

It is a strange anomaly that that branch of mathematics, known as the theory of probability, which rests upon the fewest, simplest, and least controvertible fundamental principles, which demands practically no mathematical prerequisites for its pursuit, which throughout occupies itself with innumerable interesting and important problems that even the layman can understand, should be, at the same time, that branch of mathematics which has presented the greatest number of pitfalls to its most illustrious devotees. [. . .]

Cardan, as will be shown presently, may be said to have inaugurated the study with a mistaken solution; Pascal, another pioneer of the subject, committed a fallacy in his problem of points involving three players; Leibnitz fell into error in thinking that a throw of twelve with two dice is as probable as a throw of eleven. D'Alembert stumbled time and again when dealing with probabilities. James Bernoulli, in his *Ars Conjectandi*, recorded two erroneous solutions of his nineteenth problem which occurred to him before he obtained its true solution.

—Excerpted from "Some Curious Fallacies In the Study of Probabilities",
Robert E. Moritz (1923). 30(1): 14–18.

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